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COVER SHEET FOR TECHNICAL MEMORANDUM

TITLE- A Computer Method for the Determination of Rational Functions

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AUTHOR(S)- S. Y. Lee

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Rational Function Approximations
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ABSTRACT

To ensure physical realizability in the synthesis of a system, it is frequently necessary to represent certain of the system's attributes for approximations other than algebraic polynomials. Such is the case, for example, in the determination of optimum and suboptimum demodulators for FM signals in the Apollo USB communication system where it is convenient to approximate emperical or analytical spectra of random processes by readily factorized rational functions. This memorandum describes a computer method for the determination of rational functions using sums of exponential approximations.

This method depends upon taking 2n equidistant samples of the input data, where n is the number of poles of the approximated rational function. These samples are used to compute an (n+1)st order determinant and thereby the coefficients and the roots of an nth order polynomial. The roots of this nth order polynomial are further processed to obtain the poles of the rational function. Once the location of the poles has been obtained, a very simple routine determines the residues of these poles and thereby the zeros and the gain constant. An advantage of this technique is that it determines the poles independently of the zeros reducing the number of independent equations by a factor of two, thus eliminating many numerical difficulties and simplifying calculations. Another advantage is that this method minimizes the complexity of the rational function for a given approximation error by allowing the poles and the zeros to be complex as well as real.

The method outlined in this memorandum has been programmed for IBM 7040, IBM 360-65 and UNIVAC 1108 computers. A large number of runs have been made which yielded very good approximations to the input data under various error criteria.

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TECHNICAL MEMORANDUM

I. INTRODUCTION

To ensure physical realizability in the synthesis of a system, it is frequently necessary to represent certain of the system's attributes by approximations other than algebraic polynomials. Such is the case, for instance, in the determination of optimum and suboptimum demodulators for FM signals in the Apollo USB communication system where it is convenient to approximate emperical or analytical spectra of random processes by readily factorized rational functions.[1] The utility of exponential functions is well known in the design of linear networks for a prescribed impulse response. Similarly, sums of exponentials often afford the most suitable approximations to cross-correlation measurements, to radioactive decay and gas absorption data, to mass spectrographs, and to analysis of various curves.

The present method of obtaining rational and exponential function representations depends upon taking 2n equidistant samples of the given data, where n is the number of poles of the rational function. These samples are used to compute an (n+1)st order determinant and therby the coefficients and the roots of an nth order polynomial. The roots of this nth order polynomial are further processed to obtain the poles of the rational function. Once the location of the poles has been obtained, a very simple routine determines the residues of these poles and thereby the zeros and the gain constants. An advantage of this technique is that it determines the poles independently of the zeros reducing the number of independent equations by a factor of two, thus eliminating many numerical difficulties and simplifying calculations. Other advantages are that this method can be applied to experiments in which the number of samples is limited, and minimizes the complexity of the rational function for a given approximation error by allowing the poles and zeros to the complex as well as real.

II. OUTLINE OF ANALYSIS

Any function f(t) which can be approximated by y(t) which is in terms of the sum of exponentials can be written in the form of

$$y(t) = \sum_{i=1}^{n} R_i e^{p_i t} \doteq f(t) \tag{1}$$

or equivalently, of the form

$$y(t) = \sum_{i=1}^{n} R_i q_i^t \tag{2}$$

where $q_i = e^{p_i}$ and where R_i and p_i are in general complex.

Suppose that a linear change variable has been introduced in advance in such a way that the values of y(t) are specified at N equally spaced points at t=0, 1, 2, ..., N-1. Therefore, if (2) were to hold for these values of t, the equations

$$\begin{aligned}
R_1 &+ R_2 &+ \dots + R_n &= y_0 \\
R_1 q_1 &+ R_2 q_2 &+ \dots + R_n q_n &= y_1 \\
R_1 q_1^2 &+ R_2 q_2^2 &+ \dots + R_n q_n^2 &= y_2 \\
\vdots & \quad \vdots & \vdots \quad & \\
R_1 q_1^{N-1} &+ R_2 q_2^{N-1} &+ \dots + R_n q_n^{N-1} &= y_{N-1}
\end{aligned} \tag{3}$$

must be satisfied. Hence, if the constants $q_1, \ldots, q_n$ were known, the equations in (3) would comprise N linear equations in the unknowns $R_1, \ldots, R_n$ and could be solved exactly, if N=n or approximately, by the least-squares method if N>n.

However, if the q's are also to be determined, at least 2n equations are needed, and the difficulty consists of the fact that the equations are nonlinear in the q's. This difficulty can be minimized by the following analysis.

Putting (3) in matrix notation and letting N=2n, we have

$$[Y] = [Q]\ [R] \tag{4}$$

where

$$[Y] = \begin{bmatrix} y_0 \\ y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_{2n-1} \end{bmatrix}, \quad [R] = \begin{bmatrix} R_1 \\ R_2 \\ \cdot \\ \cdot \\ \cdot \\ R_n \end{bmatrix}, \quad [Q] = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ q_1 & q_2 & \cdots & q_n \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ q_1^{2n-1} & q_2^{2n-1} & \cdots & q_n^{2n-1} \end{bmatrix} \tag{5}$$

The first n rows of the (2n x n) [Q] matrix form a Vandermonde matrix. The various q_k's can be considered as the roots of the nth degree polynomial,

$$P(q) = (q-q_1)(q-q_2) \ldots (q-q_n) = 0 \tag{6}$$

Expanding,

$$P(q) = q^n + (-1)(q_1+q_2+\ldots+q_n)q^{n-1} + (-1)^2(q_1q_2+q_1q_3+\ldots+q_nq_{n-1})q^{n-2}$$
$$+ \ldots + (-1)^n q_1q_2\ldots q_n = 0 \qquad (7)$$

Now let u's be the well known symmetric functons

$$u_1 = (-1)(q_1+q_2+\ldots+q_n)$$
$$u_2 = (-1)^2(q_1q_2+q_1q_3+\ldots+q_2q_3+q_nq_{n-1})$$
$$\vdots \qquad (8)$$
$$u_n = (-1)^n(q_1q_2\ldots q_n)$$

then (7) becomes

$$P(q) = q^n + u_1q^{n-1} + u_2q^{n-2} + \ldots + u_n = 0 \qquad (9)$$

Furthermore, let us define an n x 2n matrix [U] such that

$$[U] = \begin{bmatrix} u_n & u_{n-1} & \ldots & u_2 & u_1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & u_n & \ldots & u_3 & u_2 & u_1 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 & u_n & u_{n-1} & u_{n-2} & u_{n-3} & \ldots & 1 \end{bmatrix} \qquad (10)$$

Premultiplying (4) by [U], yields

$$[U][Y] = [U][Q][R] \tag{11}$$

Each element of the matrix [U][Q] is of the form:

$$q_k^{i-1}[u_n+u_{n-1}q_k+u_{n-2}q_k^2+\ldots+q_k^n] \tag{12}$$

But, looking back at (9), it is apparent that (12) is equal to P(q) with q replaced by q_k's and since q_k's are roots of the polynomial P(q), expression (12) must be equal to zero. Thus, all the elements of [U][Q] are zero. Hence, (11) becomes

$$[U][Y] = 0 \tag{13}$$

Looking at the individual elements of this products matrix, we have

$$\begin{array}{lllllllll} u_n y_0 & + & u_{n-1}y_1 & + \cdots + & u_1 y_{n-1} & + & y_n & = & 0 \\ u_n y_1 & + & u_{n-1}y_2 & + \cdots + & u_1 y_n & + & y_{n+1} & = & 0 \\ \cdot & & & & & & & & \\ \cdot & & & & & & & & \\ \cdot & & & & & & & & \\ u_n y_{n-1} & + & u_{n-1}y_n & + \cdots + & u_1 y_{2n-2} & + & y_{2n-1} & = & 0 \end{array} \tag{14}$$

Thus, the parameters in equation (1) are completely determined by equations (3), (9) and (14). It should be noted that an alternative method for obtaining the q_k's to be used in the array (3), involves the inclusion of (9) as the (n+1)th row of

(14). Thus,

$$
\begin{array}{llllllll}
u_n y_0 & + \; u_{n-1} y_1 & + & \cdots & + \; u_1 y_{n-1} & + \; y_n & = & 0 \\
u_n y_1 & + \; u_{n-1} y_2 & + & \cdots & + \; u_1 y_n & + \; y_{n+1} & = & 0 \\
\vdots & & & & & & & \\
u_n y_{n-1} & + \; u_{n-1} y_n & + & \cdots & + \; u_1 y_{2n-2} & + \; y_{2n-1} & = & 0 \\
u_n & + \; u_{n-1} q & + & \cdots & + \; u_1 q^{n-1} & + \; q^n & = & 0
\end{array}
\qquad (15)
$$

Equation (15) is an array of (n+1) equations and only n unknowns. Therefore:

$$
D = \begin{vmatrix}
y_0 & y_1 & \cdots & y_n \\
y_1 & y_2 & \cdots & y_{n+1} \\
\vdots & \vdots & & \vdots \\
y_{n-1} & y_n & \cdots & y_{2n-1} \\
1 & q & \cdots & q^n
\end{vmatrix} = 0 \qquad (16)
$$

By expanding this determinant in terms of the last row yields a polynomial P(q) of degree n in q whose roots are $q_1, q_2, \ldots, q_n$,

$$
P(q) = \Delta_0 + \Delta_1 q + \Delta_2 q^2 + \ldots + \Delta_n q^n = 0 \qquad (17)
$$

where the Δ's are the co-factors corresponding to the last row of the determinant in (16). If all $\Delta_k = 0$, a lower order approximation can be obtained.

Once the q_i have been obtained it is a simple matter to find the time constants of (1), which involves taking the natural logarithm.

The coefficients R_k's can be obtained by considering the n equations of (3). In matrix notation

$$[Y_1] = [V][R] \tag{18}$$

or

$$[R] = [V]^{-1}[Y_1] \tag{19}$$

where

$$[Y_1] = \begin{bmatrix} y_0 \\ y_2 \\ y_4 \\ \vdots \\ y_{2n-2} \end{bmatrix}, \quad [V] = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ q_1^2 & q_2^2 & \cdots & q_n^2 \\ q_1^4 & q_2^4 & \cdots & q_n^4 \\ \vdots & \vdots & & \vdots \\ q_1^{2n-2} & q_2^{2n-2} & \cdots & q_n^{2n-2} \end{bmatrix}, \quad [R] = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_n \end{bmatrix} \tag{20}$$

The matrix [V] is well known Vandermonde matrix whose inverse is easy to compute. It should be noted that one of the advantages of this technique is that it determines the poles independent of the zeros reducing the number of independent equations by a factor of two, thus eliminating many numerical difficulties and simplifying calculations.

III. THE ASSUMPTIONS

The preceding theory was derived under the assumption that the approximation is of the form

$$y(t) = \sum_{i=1}^{n} R_i e^{p_i t} \qquad (21)$$

where each R_i and p_i may be either real or complex, but $y(t)$ must be a real function; hence, the complex R_i's and the complex p_i's must occur in conjugate pairs. Equation (21) implies that the approximation consists only of real exponential, cosine and sine terms.

Taking the Laplace transform of $y(t)$, $Y(s)$ can be written as a rational function,

$$Y(s) = \frac{N(s)}{D(s)} = \frac{K(s-z_1)(s-z_2)\ \ldots\ (s-z_m)}{(s-p_1)(s-p_2)\ \ldots\ (s-p_n)} \qquad (22)$$

The following assumptions are made regarding $N(s)$ and $D(s)$:

a. The degree m of the numerator polynomial $N(s)$ is considered to be at least one degree less than the degree n of the denominator polynomial $D(s)$.

b. The denominator polynomial $D(s)$ does not possess a zero root. This is not a serious restriction, as the zero root in the denominator implies a constant steady value of the output which is easily subtracted.

c. The denominator polynomial $D(s)$ has only simple roots. This restriction can be removed when the analysis is extended to include repeated roots.*

---

*Note that for the analysis of repeated roots the expression (20) is more complicated.

IV. ILLUSTRATIVE EXAMPLES

The purpose of this section is to construct examples to illustrate the application of the method developed in previous sections. These examples were done on a computer using Fortran IV Language.

(A) Example 1:

Consider that the data given in Table I is the impulse response of a system $f(t) = 2e^{-0.5t}\sin t$ or the corresponding frequency response $F(s) = \frac{2}{(s+0.5)^2+1}$ over the time interval $t=0$ to 2π. We will assume that only these data are given and required to determine the rational function.

Using n=2, the computer program chooses samples numbered 0, 30, 60 and 90 given in Table I and obtains for the polynomial of equation (17):

$$0.1683 + 0.00981q + 0.8205q^2 = 0 \qquad (23)$$

Thus, we can determine the roots q_i as:

$$q_{1,2} = -0.00598 \mp j0.4529 \qquad (24)$$

The resulting poles are:

$$p_{1,2} = -0.504 \mp j1.0084 \quad . \qquad (25)$$

Taking samples 0 and 30 for element y_i, and substituting q_1 and q_2 into (19), the R's are obtained as

$$R_{1,2} = \pm j \quad . \qquad (26)$$

Finally, the computed impulse response $\hat{f}(t)$ is

$$\hat{f}(t) = 2e^{-0.504t} \sin 1.0084\, t \qquad (27)$$

SPECTRUM ANALYSIS USING LEES PROGRAM

TABLE I - GENERATED DATA VALUES FOR $f(t) = 2e^{-0.5t}\sin t$

.000000	.102801	.199965	.291454	.377249	.457351	.531781	.600579	.663801	.721519
.773819	.820804	.862588	.899295	.931063	.958037	.980373	.998234	1.011788	1.021212
1.026586	1.028395	1.026526	1.021270	1.012818	1.001364	.987101	.970222	.950919	.929382
.905799	.880355	.853232	.824608	.794659	.763553	.731456	.698528	.664922	.630789
.596271	.561504	.526620	.491744	.456992	.422477	.388303	.354569	.321367	.288782
.256893	.225772	.195486	.166094	.137650	.110201	.083791	.058454	.034222	.011121
-.010831	-.031617	-.051226	-.069653	-.086896	-.102957	-.117843	-.131563	-.144132	-.155565
-.165882	-.175105	-.183260	-.190373	-.196475	-.201596	-.205770	-.209031	-.211415	-.212958
-.213700	-.213679	-.212933	-.211503	-.209429	-.206751	-.203509	-.199744	-.195494	-.190799
-.185698	-.180230	-.174431	-.168338	-.161988	-.155415	-.148652	-.141732	-.134688	-.127549
-.120344	-.113101	-.105047	-.098607	-.091404	-.084262	-.077201	-.070242	-.063402	-.056698
-.050147	-.043763	-.037559	-.031546	-.025735	-.020136	-.014757	-.009604	-.004683	-.000000

and its corresponding rational function $\hat{F}(s)$ is

$$\hat{F}(s) = \frac{2.016}{(s+0.504)^2+(1.0084)^2} \qquad (28)$$

Notice these functions are identical to those given, except for small round errors.

(B) Example II:

Consider that the data given in Table II is the impulse response of a system $f(t) = 2e^{-0.5t}\sin(t) - 1.5e^{-t}\sin(2t)$ or the corresponding frequency response $F(s) = \dfrac{-s^2+s+6.25}{[(s+0.5)^2+1][(s+1)^2+4]}$ over the time interval t=0 to 2π. We will again assume that only these data are given and required to determine the rational function.

Using n=4, the computer program chooses the samples numbered 0, 15, 30, 45, 60, 75, 90 and 105 given in Table II and obtains for the polynomial of equation (17):

$$2.505\text{x}10^{-3}-5.084\text{x}10^{-3}q + 1.744\text{x}10^{-2}q^2-2.517\text{x}10^{-2}q^3 + 2.696q^4 = 0 \qquad (29)$$

Thus, solving for the roots q_i we have

$$q_{1,2} = -0.00598 \mp j\ 0.4529$$

$$q_{3,4} = 0.4727 \mp j\ 0.479 \qquad (30)$$

The resulting poles are:

$$p_{1,2} = -0.504 \mp j\ 1.0084$$

$$p_{2,3} = -1.0084 \mp j\ 2.0168 \qquad (31)$$

SPECTRUM ANALYSIS USING LEES PROGRAM

TABLE II - GENERATED DATA VALUES FOR $f(t) = 2e^{-0.5t}\sin t - 1.5e^{-t}\sin 2t$

.000000	-.047173	-.082971	-.107382	-.120602	-.123013	-.115159	-.097718	-.071479	-.037323
.003804	.050910	.102974	.158969	.217873	.278688	.340449	.402240	.463196	.522521
.579483	.633430	.683782	.730042	.771793	.808694	.840487	.866985	.888076	.903715
.913922	.918774	.918404	.912991	.902758	.887964	.868901	.845884	.819250	.789350
.756546	.721204	.683692	.644376	.603612	.561750	.519126	.476060	.432856	.389798
.347151	.305157	.264039	.223995	.185201	.147812	.111960	.077756	.045290	.014633
-.014163	-.041063	-.066049	-.089118	-.110281	-.129560	-.146990	-.162614	-.176485	-.188663
-.199212	-.208205	-.215715	-.221819	-.226597	-.230129	-.232496	-.233778	-.234054	-.233403
-.231899	-.229616	-.226626	-.222995	-.218789	-.214069	-.208892	-.203315	-.197388	-.191159
-.184674	-.177972	-.171094	-.164074	-.156946	-.149739	-.142482	-.135198	-.127913	-.120646
-.113418	-.106246	-.099146	-.092133	-.085220	-.078421	-.071745	-.065205	-.058808	-.052564
-.046462	-.040567	-.034828	-.029269	-.023898	-.018718	-.013734	-.008951	-.0043[illegible]2	-.000000

Taking samples 0, 15, 30 and 45 for element y_i, and substituting q_1 and q_2, q_3 and q_4 into (19), the R's are obtained as

$$
\begin{aligned}
R_{1,2} &= \mp j \\
R_{3,4} &= \mp j\ 0.75
\end{aligned}
\qquad (32)
$$

Finally, the computed impulse response $\hat{f}(t)$ is

$$\hat{f}(t) = 2e^{-0.504t}\sin(1.0084t) - 1.5e^{-1.0084t}\sin(2.00168t) \qquad (33)$$

and its corresponding rational function is $\hat{F}(s)$ is

$$\hat{F}(s) = \frac{-1.0084s^2 + 1.0181s + 6.41}{[(s+0.504)^2+1.0084^2][(s+1.0084)^2+(2.00168)^2]} \qquad (34)$$

Notice these functions are identical to those given, except for the small round off errors.

(C) Example III:

In the Appendix IV of Reference 1, a method of obtaining optimum filters for the telemetry signal is derived. However, in order to determine these filters, a rational approximation of the signal spectrum must be first obtained. It also pointed out in Reference 1 that if the function

$$v(t) = \begin{cases} 1 - \frac{t}{T} & , \ 0 \le t \le T \\ 0 & , \ \text{all other } t \end{cases} \qquad (35)$$

is approximated by the finite sum of exponentials a rational approximation is found for the signal spectrum $S_\phi(\omega) = T[(\sin\frac{\omega T}{2})/\frac{\omega T}{2}]^2$. To illustrate the usefulness of the present method, we will first in this example obtain an approximation, $\overline{v}(t)$, of $v(t)$ then find the rational approximation, $\overline{S}_\phi(\omega)$, of $S_\phi(\omega)$, where $\overline{S}_\phi(\omega) = F[\overline{v}(|t|)]$.

The data given in Table III is generated from (35). In order to avoid the difficulties of factorization as it was pointed out in Appendix IV of Reference 1, we shall restrict the approximation to be of third degree. For n=3, the computer program chooses samples numbered, 0, 20, 40, 60, 80 and 100 from those in Table III and obtains for the polynomial of equation (17):

$$2.219\text{x}10^{-3} - 7.150\text{x}10^{-3}q + 1.096\text{x}10^{-2}q^2 - 1.339\text{x}10^{-2}q^3 = 0 \quad (36)$$

Thus, solving for the roots q_i we have

$$q_1 = 0.4500$$
$$q_{2,3} = 0.1841 \mp j\ 0.5782 \quad (37)$$

The poles are

$$p_1 = -2.795$$
$$p_{2,3} = -1.749 \mp j\ 4.419 \quad (38)$$

Taking samples 0, 20 and 40 for elements y_i and substituting q_1, q_2 and q_3 into (19), the R's are obtained

$$R_1 = 1.301$$
$$R_{2,3} = -0.151 \pm j\ 0.156 \quad (39)$$

SPECTRUM ANALYSIS USING LEES PROGRAM

TABLE III - GENERATED DATA VALUES FOR $v(t) = \begin{cases} 1 - \frac{t}{T}, & 0 \leq t \leq T \\ 0, & \text{ALL OTHER } t \end{cases}$

1.000000	.985507	.971014	.956522	.942029	.927536	.913043	.898551	.884058	.869565
.855072	.840580	.826087	.811594	.797101	.782609	.768116	.753623	.739130	.724638
.710145	.695652	.681159	.666667	.652174	.637681	.623188	.608696	.594203	.579710
.565217	.550725	.536232	.521739	.507246	.492754	.478261	.463768	.449275	.434783
.420290	.405797	.391304	.376812	.362319	.347826	.333333	.318841	.304348	.289855
.275362	.260870	.246377	.231884	.217391	.202899	.188406	.173913	.159420	.144928
.130435	.115942	.101449	.086957	.072464	.057971	.043478	.028986	.014493	.000000
.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000
.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000
.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000
.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000
.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000

Hence,

$$\overline{V}(s) = \frac{1.301}{(s+2.795)} - \frac{(0.302)(s+1.749) - (0.312)(4.419)}{(s+1.749)^2 + (4.419)^2} \quad (40)$$

or rewriting

$$\overline{V}(s) = \frac{s^2 + 4.561s + 31.765}{s^3 + 6.293s + 32.364s + 63.131} \quad (41)$$

Thus, the corresponding time function $\overline{v}(t)$ can be easily determined from (40)

$$\overline{v}(t) = 1.301e^{-2.795t} - e^{-1.749t}[0.302\cos(4.419t)-0.312\sin(4.419t)] \quad (42)$$

The plots of $\overline{v}(t)$ and $v(t)$ are shown in Figure 1; it can be seen that $v(t)$ is very closely approximated by $\overline{v}(t)$. The standard deviation of the two functions is calculated by the computer program to be 0.016.

Now to obtain $\overline{S}_\phi(\omega)$ from $\overline{V}(s)$, we are only required to perform the following transformation, since

$$F[\overline{v}(t)] = [\overline{V}(s)]_{s=j\omega} \quad (43)$$

where F denotes the Fourier transform, then

$$F[\overline{v}(|t|)] = [\overline{V}(s)]_{s=j\omega} + [\overline{V}(s)]_{s=-j\omega} \quad (44)$$

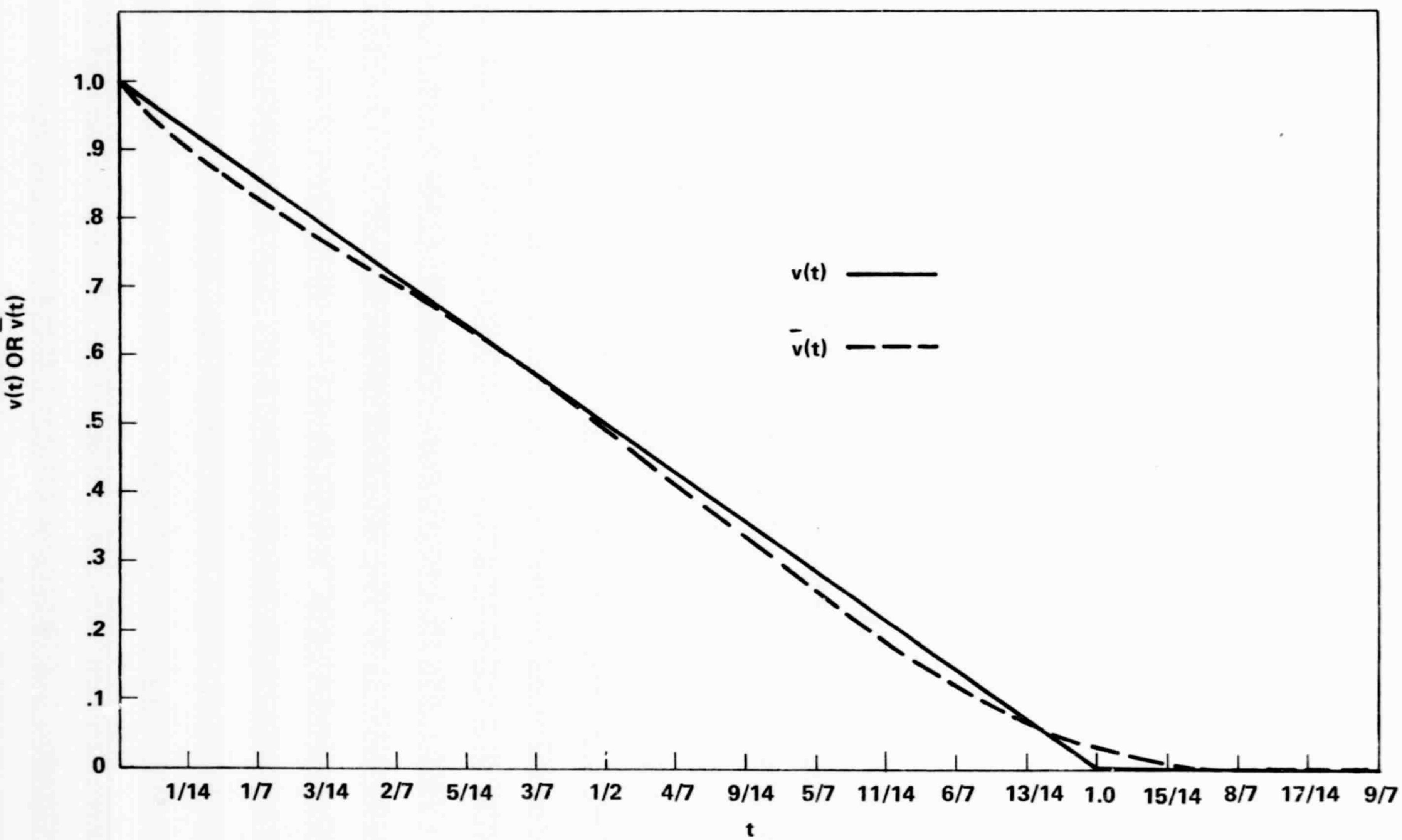

FIGURE 1 - APPROXIMATION OF v(t) WITH THE SUM OF THREE EXPONENTIALS $\bar{v}(t)$

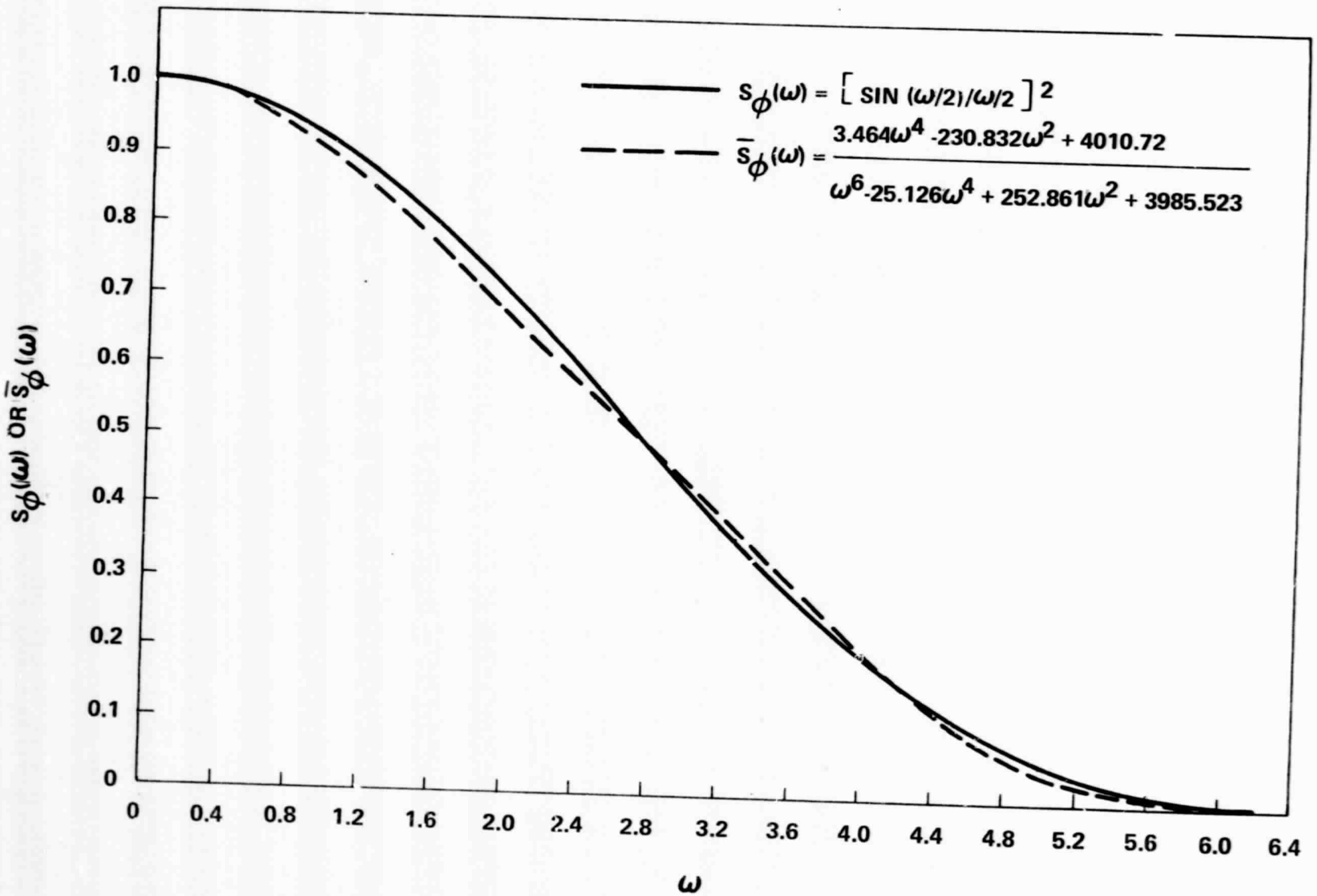

FIGURE 2 - THE PLOTS OF $S_\phi(\omega)$ AND $\bar{S}_\phi(\omega)$

Hence, $\overline{S}_\phi(\omega) = F[\overline{v}(|t|)]$ is obtained from (41) and (44) as

$$\overline{S}_\phi(\omega) = \frac{3.464\omega^4 - 230.832\omega^2 + 4010.712}{\omega^6 - 25.126\omega^4 + 252.861\omega^2 + 3985.523}$$

Figure 2 shows $\overline{S}_\phi(\omega)$ is a good approximation of $S_\phi(\omega)$.

V. REMARKS

The method outlined in this memorandum has been programmed for IBM 7040, IBM 360-65 and UNIVAC 1108 computers. A large number of runs have been made which yielded very good approximations to the input data under various error criteria such as bounds on the root mean square error or on the maximum difference between the function and the approximation.

VI. ACKNOWLEDGEMENT

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Attachment
References

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