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THE EXISTENCE AND STABILITY OF SOLUTIONS TO NONLINEAR OPERATOR DIFFERENTIAL EQUATION

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ABSTRACT

The existence and the stability problem of the operator differential equation $\frac{dx(t)}{dt} = Ax(t) \quad (t \geqq 0)$, where A is a nonlinear operator with domain D(A) and range R(A) both in a complex Hilbert space H, are investigated by using the nonlinear semi-group property. Under the condition R(I - A) = H, A generates a nonlinear contraction (resp. negative contraction) semi-group iff A is dissipative, that is, -A is monotone (resp. strictly dissipative) from which the existence, uniqueness and stability or asymptotic stability of solutions are insured. By the introduction of an equivalent inner product inducing a topologically equivalent Hilbert space, the inner product of H with respect to which A is dissipative can be replaced by an equivalent inner product without affecting the existence and the stability of a solution. This fact makes possible the development of a stability theory by the construction of a "Lyapunov functional" by means of a sesquilinear functional.

1. Introduction

Consider the nonlinear operator differential equation

$$\frac{dx(t)}{dt} = Ax(t) \qquad (t \geqq 0) \tag{1-1}$$

where the unknown $x(t)$ is a vector-valued function defined on $[0, \infty)$ to a complex Hilbert space H and A is a given, in general nonlinear, operator with domain $D(A)$ and range $R(A)$ both contained in H. The object of this paper is to develop criteria for the stability and asymptotic stability as well as the existence and uniqueness of solutions of (1-1). The existence problem of (1-1) has been investigated by Kōmura [6], Kato [5] and by Browder [1]. The results of [5] by Kato have a close connection with this paper.

The stability and asymptotic stability properties of the solutions of (1-1) are developed in terms of nonlinear contraction and nonlinear negative contraction semi-groups (see definition 2.1) since if A is the infitesimal generator (see definition 2.2) of a nonlinear semi-group $\{T_t; t \geqq 0\}$ then a solution of (1-1) starting at t=0 from $x_o \varepsilon D(A)$ is given by $x(t; x_o) = T_t x_o$ for all $t \geqq 0$ with $x(0; x_o) = x_o$, and thus the stability property is ensured by the contraction or negative contraction property of the semi-group $\{T_t;\ t \geqq 0\}$. By the introduction of an equivalent inner product inducing a topologically equivalent Hilbert space (see definition 3.1), the stability property is related to the existence and the construction of a "Lyapunov functional" which

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is defined by means of a sesquilinear functional having certain specific properties. In section 2, conditions on the operator A for the generation of a contraction semi-group or a negative contraction semi-group in a Hilbert space H are estabilshed. The essential condition on A is the dissipativity of A (or equivalently, the monotonicity of -A) with respect to the inner product of H. In section 3, we show that the inner product with respect to which A is dissipative or strictly dissipative can be replaced by an equivalent inner product without affecting the existence and stability property of a solution. We also established the necessary and sufficient conditions for the equivalence between two inner products. In the final section, we developed a stability theory, including the existence of a solution, through the construction of a "Lyapunov functional" which is in parallel to the Lyapunov stability theory of ordinary or partial differential equations.

2. Nonlinear Semi-group and Dissipative Operator

Definition 2.1. Let H be a Hilbert space. The family of nonlinear operators $\{T_t;\ t \geqq 0\}$ is called a nonlinear semi-group on H if and only if the following conditions hold:

(i) for any fixed $t \geqq 0$, T_t is a continuous (nonlinear) operator defined on H into H;

(ii) for any fixed $x \in H$, $T_t x$ is strongly continuous in t;

(iii) $T_s T_t = T_{s+t}$ for $s, t \geqq 0$, and $T_o = I$ (the identity operator);

(iv) $||T_t x - T_t y|| \leqq M||x-y||$ $(M > 0)$ $x, y, \in H$ and $t \geqq 0$.

If (iv) is replaced by

(iv)' $||T_t x - T_t y|| \leqq Me^{-\beta t}||x-y||$ $(\beta>0)$ $x, y, \in H$ and $t \geqq 0$, then it is called a nonlinear negative semi-group on H. If $M \leqq 1$ then $\{T_t;\ t \geqq 0\}$ is called a nonlinear contraction and negative contraction semi-group respectively. The supremum of all the numbers β satisfying (iv)' is called the contractive constant of $\{T_t;\ t \geqq 0\}$. For a subset $\mathcal{D}$ of H, the family $\{T_t;\ t \geqq 0\}$ is said to be a nonlinear contraction (resp., negative contraction) semi-group on $\mathcal{D}$ if the properties (i)-(iv) (resp.,(i)-(iv)') are satisfied for all $x,y \in \mathcal{D}$ (with $M \leqq 1$).

<u>Definition 2.2</u>. The infinitesimal generator A of the nonlinear semi-group $\{T_t;\ t \geqq 0\}$ is defined by

$$Ax = \underset{h\downarrow 0}{\text{w-lim}} \frac{T_h x - x}{h}$$

for all $x \in H$ such that the limit on the right-side exists in the sense of weak convergence.

<u>Definition 2.3</u>. An operator (nonlinear) A with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ both contained in a Hilbert space is said to be dissipative if

$$\text{Re}(Ax - Ay,\ x-y) \leqq 0 \qquad \text{for all } x,y \in D(A); \tag{2-1}$$

and it is said to be strictly dissipative if there exists a real number $\beta>0$ such that

$$\text{Re}(Ax - Ay,\ x-y) \leqq -\beta(x-y,\ x-y) \quad \text{for all } x,y \in D(A). \tag{2-2}$$

The supremum of all the numbers β satisfying (2-2) is called the dissipative constant of A.

It follows from definition 2.3 that A is dissipative if and only if -A is monotone (cf. [8]) and that definition 2.3 coincides with the usual definition of dissipativity when A is a linear operator (cf. [7]). It can be shown that the condition (2-1) implies that $(I - \alpha A)^{-1}$ exists and is Lipschitz continuous for all $\alpha > 0$, where $I - \alpha A$ is an operator with domain $D(A)$ which maps x into $x - \alpha Ax$; and in addition, if the domain of $(I - \alpha A)^{-1}$ is H for some $\alpha > 0$ then the same is true for all $\alpha > 0$ (cf. [5], [9]). Thus for a dissipative operator, the operator $(I - \alpha A)^{-1}$ has domain H either for every $\alpha > 0$ or for no $\alpha > 0$. In the former case A is said to be m-monotone.

The following definition specifies what it meant by a solution in this paper.

Definition 2.4. By a solution $x(t)$ of (1-1) with initial condition $x(0) = x \in D(A)$ in a Hilbert space H (real or complex), we mean the following:

(a) $x(t)$ is uniformly Lipschitz continuous in t for each $t \geqq 0$ with $x(0) = x$.

(b) $x(t) \in D(A)$ for each $t \geqq 0$ and $Ax(t)$ is weakly continuous in t.

(c) The weak derivative of $x(t)$ exists for all $t \geqq 0$ and equals $Ax(t)$.

(d) The strong derivative $dx(t)/dt$ $(= Ax(t))$ exists and is strongly continuous except at a countable number of values t.

The above definition of a solution $x(t)$ is in the sense of a weak solution since $x(t)$ satisfies (1-1) in the weak topology of H. However, by the condition (d), $x(t)$ is an almost everywhere strong solution in the sense that $x(t)$ satisfies (1-1) for almost all values of $t \geqq 0$ in the strong topology of H. The following theorem is essentially due to Kato [5].

Theorem 2.1. Let A be a nonlinear operator with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ both contained in a Hilbert space H such that $\mathcal{R}(I-A)=H$. Then A is the infinitesimal generator of a nonlinear contraction semi-group $\{T_t;\ t \geqq 0\}$ on $\mathcal{D}(A)$ if and only if A is dissipative (i.e. -A is monotone).

Proof. Sufficiency: suppose A is dissipative, (i.e. -A is monotone). Then -A is m-monotone, for by hypothesis, $\mathcal{R}(I+(-A)) = \mathcal{R}(I-A) = H$. The sufficiency follows from the main theorems in [5] since both H and its conjugate space are uniformly convex.

Necessity: Let A be the infinitesimal generator of a non-linear contraction semi-group $\{T_t;\ t \geqq 0\}$ on $\mathcal{D}(A)$. Then for any $x,y \in \mathcal{D}(A)$

$$\begin{aligned} &\mathrm{Re}(h^{-1}(T_h x-x)-h^{-1}(T_h y-y),\ x-y) = h^{-1}\mathrm{Re}[(T_h x-T_h y,\ x-y) - (x-y,\ x-y)] \\ &\leqq h^{-1}[||T_h x - T_h y||\ ||x-y|| - ||x-y||^2] = h^{-1}||x-y||[||T_h x-T_h y|| - \\ &\qquad - ||x-y||] \leqq 0 \end{aligned}$$

for all $h > 0$ since $\{T_t,\ t \geqq 0\}$ is contractive. Letting $h \downarrow 0$ in the above inequality, we have, by the continuity of inner product and by definition 2.2

$$\mathrm{Re}(Ax - Ay,\ x-y) \leqq 0 \qquad \text{for any } x,y \in \mathcal{D}(A).$$

Hence the theorem is proved.

Remark. The nonlinear contraction semi-group $\{T_t;\ t \geqq 0\}$ generated by A in the above theorem has all the properties of a solution in the sense of definition 2.4 (cf. [5]).

It should be noted that in the above theorem, it is not assumed that the domain of A is dense in H. However, if A is a linear operator in a Hilbert space, the dissipativity of A and the condition $R(I - A) = H$ imply that $\mathcal{D}(A)$ is dense in H (cf. [5]), and the above theorem is reduced to the well-known results due to Lumer and Phillips [7]. But it is not known yet whether or not $\mathcal{D}(A)$ is dense in H if A is nonlinear. It can be shown that [10] the nonlinear contraction semi-group $\{T_t;\ t \geqq 0\}$ can be extended to a nonlinear contraction semi-group on $\overline{\mathcal{D}(A)}$, the closure of $\mathcal{D}(A)$. Hence if $\mathcal{D}(A)$ is dense in H, $\{T_t;\ t \geqq 0\}$ can be extended to the whole space H, which is a direct generalization of a strongly continuous semi-group of class C_o [3]. The condition $\mathcal{R}(I-A) = H$ can also be weakened by assuming $\mathcal{R}(I-\alpha_o A) = H$ for some $\alpha_o > 0$ since the dissipativity of A and the condition $\mathcal{R}(I - \alpha_o A) = H$ imply that $\mathcal{R}(I - \alpha A) = H$ for all $\alpha > 0$.

It is clear from the above theorem that if A is dissipative and $\mathcal{R}(I-A) = H$ then an equilibrium solution (or a periodic solution) if it exists, would be stable by the contraction property of the semi-group. However, it is not trivial to relate exponentially asymptotic stability directly to such a property. If A is linear and is the infinitesimal generator of a contraction semi-group $\{T_t;\ t \geqq 0\}$ of class C_o, then the family $\{e^{-\beta t}T_t;\ t \geqq 0\}$ for some $\beta > 0$ is a negative contraction semi-group with the infinitesimal generator $A - \beta I$. But when A is nonlinear, the contraction semi-group $\{T_t;\ t \geqq 0\}$ generated by A is nonlinear and so the family $\{e^{-\beta t}T_t;\ t \geqq 0\}$ is not, in general, a semi-group since property (iii) in definition 2.1 does not hold. In order to extend theorem 2.1 for the generation of a nonlinear negative contraction semi-group, we first prove the following lemma. It is noted that for any two sequences $\{x_n\}$ and $\{y_n\}$ in H such that $x_n \overset{w}{\to} x$ and $y_n \to y$ where $\overset{w}{\to}$ denotes weak convergence then $\lim_{n\to\infty}(x_n, y_n) = (x,y)$. This is due to the fact that a weakly convergent sequence is strongly bounded which implies that

$$\lim_{n\to\infty} |(x_n,y_n) - (x,y)| = \lim_{n\to\infty} |(x_n,y_n-y) + (x_n,y) - (x,y)|$$

$$\leqq \lim_{n\to\infty} ||x_n|| \, ||y_n-y|| + \lim_{n\to\infty} |(x_n,y)-(x,y)| = 0.$$

Lemma 2.1. Let $x(t)$, $y(t)$ be any two solutions of (1-1) (in the sense of definition 2.4). Then $||x(t)-y(t)||^2$ is differentiable in t for each $t \geqq 0$, and

$$\frac{d}{dt} ||x(t)-y(t)||^2 = 2\mathrm{Re}(Ax(t)-Ay(t),\ x(t)-y(t)) \quad \text{for each } t \geqq 0. \tag{2-3}$$

Proof. For any fixed $t > 0$, let $h \neq 0$ and $|h| < t$ so that $x(t+h)$ and $y(t+h)$ are defined. By hypothesis, $h^{-1}(x(t+h)-x(t)) \overset{w}{\to} Ax(t)$ and $h^{-1}(y(t+h)-y(t)) \overset{w}{\to} Ay(t)$ as $h \to 0$. Thus by the continuity of inner product it is easily seen that

$$\lim_{h\to 0} h^{-1}[||x(t+h)-y(t+h)||^2 - ||x(t)-y(t)||^2$$

$$= \lim_{h\to 0} h^{-1}[(x(t+h)-x(t),x(t+h)-y(t+h))-(y(t+h)-y(t),x(t+h)-y(t+h))+$$
$$(x(t)-y(t),x(t+h)-x(t)) - (x(t)-y(t),y(t+h)-y(t))]$$

$$= (Ax(t)-Ay(t),x(t)-y(t)) + (x(t)-y(t),Ax(t)-Ay(t))$$

$$= 2\,\mathrm{Re}(Ax(t)-Ay(t),x(t)-y(t)).$$

Hence, $||x(t)-y(t)||^2$ is differentiable and (2-3) holds for $t > 0$. For $t = 0$, the above is still valid by taking $h > 0$ and $h \downarrow 0$ in place of $h \to 0$ and by defining $\frac{d}{dt} ||x(t)-y(t)||^2$ at $t = 0$ as the right-side limit.

Theorem 2.2. Let A be a nonlinear operator with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ both contained in a Hilbert space H such that $\mathcal{R}(I-A) = H$. Then A is the infinitesimal generator of a nonlinear negative contraction semi-group $\{T_t;\ t \geqq 0\}$ with contractive constant β on $\mathcal{D}(A)$, if and only if A is strictly dissipative with dissipative constant β.

Proof. Necessity: Let A be the infinitesimal generator of $\{T_t;\ t \geqq 0\}$ such that condition (iv)' in definition 2.1 holds for M=1. Then

$$||T_t x - T_t v||^2 \leqq e^{-2\beta\ t}||x-v||^2 \quad \text{for all } t \geqq 0. \qquad (2\text{-}4)$$

Subtracting $||x-v||^2$ and then dividing by $t > 0$ in the above inequality, (2-4) becomes

$$t^{-1}(||T_t x - T_t v||^2 - ||x-v||^2) \leqq t^{-1}(e^{-2\beta t}-1)\ ||x-v||^2 \quad t > 0.$$

As $t \downarrow 0$, we obtain

$$\frac{d}{dt}\ ||T_t x - T_t v||^2_{t=0} \leqq -2\beta||x-v||^2\ .$$

Since for any $x,\ v \in \mathcal{D}(A)$, $T_t x$, $T_t v$ are solutions of (1-1), it follows by lemma 2.1 that

$$\mathrm{Re}(Ax-Av,\ x-v) \leqq -\beta(x-v,\ x-v) \qquad x,\ v \in \mathcal{D}(A).$$

Sufficiency: Let A be strictly dissipative. Then by theorem 2.1, A is the infinitesimal generator of a nonlinear contraction semi-group $\{T_t;\ t \geqq 0\}$ on $\mathcal{D}(A)$. Moreover, by lemma 2.1

$$\frac{d}{dt}\ ||T_t x - T_t v||^2 = 2\mathrm{Re}(AT_t x - AT_t v, T_t x - T_t v) \leqq -2\beta||T_t x - T_t v||^2 \quad t \geqq 0$$

since $T_t x$, $T_t v$ are solutions of (1-1). By integrating the above inequality, we have

$$||T_t x - T_t v||^2 \leqq e^{-2\beta t}\ ||x-v||^2$$

and the result follows.

Theorem 2.2 is a direct generalization of a theorem in [11] when X is a Hilbert space, since the strict dissipativity in theorem 2.2 for a nonlinear operator is a generalization of the strict dissipativity in the sense of [11] for a linear operator and the condition $R((1-\beta)I-A) = H$ is equivalent to $R(I-A) = H$ (cf. [12]).

3. Equivalent Inner Product

The dissipativity in theorems 2.1 and 2.2 is defined with respect to the original inner product of the space. Since the semi-group property is invariant under equivalent norms except possibly the contraction property, the possibility occurs that by defining other inner products inducing equivalent norms on the same vector space the nondissipative operator A could be made dissipative and thus generates a nonlinear contraction semi-group in an equivalent Hilbert space H_1. In the following, we shall show that the contraction semi-group generated by A in H_1 is also a semi-group generated by A in the original space H.

<u>Definition 3.1.</u> Two inner products $(\cdot\ ,\ \cdot)$ and $(\cdot\ ,\ \cdot)_1$ defined on the same vector space H are said to be equivalent if and only if the norms $||\cdot||$ and $||\cdot||_1$ induced by $(\cdot\ ,\ \cdot)$ and $(\cdot\ ,\ \cdot)_1$ respectively are equivalent, that is, there exist constants δ, γ with $0 < \delta \leqq \gamma < \infty$ such that

$$\delta||x|| \leqq ||x||_1 \leqq \gamma||x|| \qquad \text{for all } x \in H. \qquad (3\text{-}1)$$

The Hilbert space H_1 equipped with the inner product $(\cdot\ ,\ \cdot)_1$ is said to be an equivalent Hilbert space to H and is denoted by $(H, (\cdot\ ,\ \cdot)_1)$ or simply by H_1.

Under the equivalent inner product $(\cdot\ ,\ \cdot)_1$, the vector space $(H, (\cdot\ ,\ \cdot)_1)$ is a Hilbert space if and only if the original space $(H, (\cdot\ ,\ \cdot))$ is, since the completeness of one space implies the completeness of the other.

<u>Theorem 3.1.</u> Let A be a nonlinear operator with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ both contained in a Hilbert space $H = (H, (\cdot\ ,\ \cdot))$ such that $\mathcal{R}(I-A) = H$. Then A is the infinitesimal generator of a nonlinear contraction (resp., negative contraction) semi-group $\{T_t;\ t \geqq 0\}$ on

$\mathcal{D}(A)$ in an equivalent Hilbert space $(H, (\cdot\,,\,\cdot)_1)$ if and only if A is dissipative (resp., strictly dissipative) with respect to $(\cdot,\cdot)_1$. In this case the family $\{T_t;\ t \geqq 0\}$ is a nonlinear (resp., nonlinear negative) semi-group $\{T_t;\ t \geqq 0\}$ on $\mathcal{D}(A)$ in the original space H.

<u>Proof</u>. Since the inner product $(\cdot\,,\,\cdot)_1$ is equivalent to $(\cdot\,,\,\cdot)$, the space $H_1 = (H, (\cdot\,,\,\cdot)_1)$ is a Hilbert space and $\mathcal{R}(I-A) = H_1$. Hence by considering H_1 as the underlying space, all the conditions in theorem 2.1 (resp., theorem 2.2) are satisfied, implying the first assertion. To show the second part of the theorem, let $\{T_t;\ t \geqq 0\}$ be the nonlinear contraction (resp., negative contraction) semi-group on $\mathcal{D}(A)$ with respect to the norm $||\cdot||_1$, that is

$$||T_tx-T_ty||_1 \leqq ||x-y||_1 \quad (\text{resp., } ||T_tx-T_ty||_1 \leqq e^{-\beta t}||x-y||_1) \qquad x,y \in \mathcal{D}(A).$$

By the equivalence relation (3-1), we have

$$||T_tx-T_ty|| \leqq \delta^{-1}||T_tx-T_ty||_1 \leqq \delta^{-1}||x-y||_1 \leqq \gamma\delta^{-1}||x-y||$$

$$(\text{resp., } ||T_tx-T_ty|| \leqq \gamma\delta^{-1}\ e^{-\beta t}||x-y||) \qquad x,y \in \mathcal{D}(A).$$

Since the properties of a semi-group in definition 2.1 remain unchanged under equivalent norms except for possibly the contraction property, it follows that $\{T_t;\ t \geqq 0\}$ is a nonlinear (resp., nonlinear negative) semi-group on $\mathcal{D}(A)$ with respect to the original norm (with $M = \gamma\delta^{-1}$).

We next show that an equivalent inner product of a given complex Hilbert space H can be characterized by a positive definite bounded linear operator on H. The following theorem is, in fact, an extension of a theorem in [2].

<u>Theorem 3.2</u>. Let $H_1 = (H, (\cdot,\cdot)_1)$ be a complex Hilbert space. An inner product $(\cdot,\cdot)_2$ defined on the same complex vector space H is

equivalent to the inner product $(\cdot,\cdot)_1$ if and only if there exists a positive definite operator $S \in L(H_1,H_1)$ such that

$$(x,y)_2 = (x, Sy)_1 \qquad \text{for all } x,y \in H, \qquad (3\text{-}2)$$

where $L(H_1,H_1)$ denotes the class of all bounded linear operators on H_1 into H_1.

Proof. Suppose that $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ are equivalent. Define $V(x,y) = (x,y)_2$, then by the definition of equivalent inner product, $V(x,y)$ is a sesquilinear functional defined on $H_1 \times H_1$ and $V(x,y)= \overline{V(y,x)}$. Moreover, by the equivalence relation (3-1) between $||\cdot||_1$ and $||\cdot||_2$

$$|V(x,y)| = |(x,y)_2| \leqq ||x||_2 \; ||y||_2 \leqq \gamma^2 ||x||_1 \; ||y||_1 \quad \text{and}$$

$$V(x,x) = (x,x)_2 \geqq \delta^2 ||x||_1^2 \; .$$

Hence by the Lax-Milgram theorem (cf. [12]) there exists a bounded linear operator S on H_1 such that

$$(x,y)_2 = V(x,y) = (x,Sy)_1 \qquad \text{for all } x,y \in H.$$

The operator S is positive definite on H_1 since

$$(x,Sx)_1 = (x,x)_2 \geqq \delta^2 ||x||_1^2 \qquad \text{for all } x \in H.$$

Conversely, let $S \in L(H_1,H_1)$ be a positive definite operator satisfying (3-2). Then $(x,y)_2 = (x,Sy)_1$ is linear in x. It is known (e.g., see [4]) that a positive definite operator on H is self-adjoint which implies that

$$(x,y)_2 = (x,Sy)_1 = (Sx,y)_1 = \overline{(y,x)_2} \qquad x,y \in H.$$

By the positivity of S, we have

$$(x,x)_2 = (x,Sx)_1 \geqq \delta_1 ||x||_1^2 \qquad \text{for some } \delta_1 > 0,$$

which shows that $(x,x)_2 \neq 0$ if $x \neq 0$. Thus $(\cdot,\cdot)_2$ defines an inner product on H. Moreover, the boundedness of S implies that $||x||_2^2 = (x,Sx)_1 \leqq ||S|| \; ||x||_1^2$. Hence $(\cdot,\cdot)_2$ defines an equivalent inner product of $(\cdot,\cdot)_1$ which proves the theorem.

The nonlinear contraction and negative contraction semi-group $\{T_t; t \geqq 0\}$ generated by A in the equivalent Hilbert space H_1 in theorem 2.1 and theorem 2.2 respectively satisfies for any $x \in \mathcal{D}(A)$ and $t \geqq 0$

$$\left(\frac{dT_t x}{dt}, z\right)_1 = (AT_t x, z)_1 \qquad \text{for every } z \in H_1. \qquad (3\text{-}3)$$

However, it is not obvious that the same equality holds for the inner product $(\cdot,\cdot)$. We shall show with the result of theorem 3.2 that (3-3) holds with respect to $(\cdot,\cdot)$.

<u>Theorem 3.3</u>. Let A be the infinitesimal generator of a nonlinear contraction (resp., negative contraction) semi-group $\{T_t; t \geqq 0\}$ on $\mathcal{D}(A)$ in an equivalent Hilbert space $H_1 = (H, (\cdot,\cdot)_1)$. Then A is the infinitesimal generator of a nonlinear (resp., negative) semi-group $\{T_t; t \geqq 0\}$ on the same domain $\mathcal{D}(A)$ in the original Hilbert space $H = (H, (\cdot,\cdot))$.

<u>Proof</u>. By the equivalence relation between the two inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_1$, the sesquilinear functional $V(x,y) = (x,y)$ defined on the product space $H_1 \times H_1$ satisfies all the hypotheses in the Lax-Milgram theorem. Thus there exists a bounded linear operator S defined on all of H_1 such that

$$(x,y) = V(x,y) = (x,Sy)_1 \qquad \text{for all } x, y \in H. \qquad (3\text{-}4)$$

By hypothesis, A generates the semi-group $\{T_t; t \geqq 0\}$ in H_1 so that

$$\lim_{t \downarrow 0} t^{-1}(T_t x - x, z)_1 = (Ax, z)_1 \qquad \text{for every } z \in H. \qquad (3\text{-}5)$$

It follows from (3-4) and (3-5) that for $x \in \mathcal{D}(A)$ and $z \in H$

$$\lim_{t \downarrow 0} t^{-1}(T_t x - x, z) = \lim_{t \downarrow 0} t^{-1}(T_t x - x, Sz)_1 = (Ax, Sz)_1 = (Ax, z),$$

which shows that A is the infinitesimal generator of the semi-group $\{T_t; t \geqq 0\}$ on $\mathcal{D}(A)$ in the space H. The fact that $\{T_t; t \geqq 0\}$ remains as a semi-group in H is that the semi-group property is invariant under equivalent norms except for possibly the contraction property. Since $\{T_t; t \geq 0\}$ is a contraction semi-group in H_1 and $||\cdot||$ and $||\cdot||_1$ are equivalent, we have by the relation (3-1)

$$||T_t x - T_t y|| \leq \gamma/\delta \; ||x-y|| \qquad x,y \in \mathcal{D}(A)$$

$$(\text{resp., } ||T_t x - T_t y|| \leq \gamma/\delta \; e^{-\beta t}||x-y|| \qquad x,y \in \mathcal{D}(A))$$

and the theorem is proved.

Corollary. Let the operator A appearing in (1-1) be the infinitesimal generator of a nonlinear contraction (resp., negative contraction) semi-group $\{T_t; t \geq 0\}$ on $\mathcal{D}(A)$ in the space $H_1 = (H, (\cdot,\cdot)_1)$ so that for any $x \in \mathcal{D}(A)$, $T_t x$ is the unique solution of (1-1) with $T_0 x = x$. Then $T_t x$ is also the unique solution of (1-1) with $T_0 x = x$ in the space $H=(H, (\cdot,\cdot))$ where $(\cdot,\cdot)_1$ and $(\cdot,\cdot)$ are equivalent.

Proof. Since (3-4) and (3-5) in the proof of the above theorem hold for any $x \in \mathcal{D}(A)$ and $z \in H$, we have for any $x \in \mathcal{D}(A)$ and $t \geq 0$

$$\lim_{h\to 0} h^{-1}(T_{t+h}x - T_t x, z) = \lim_{h\to 0} h^{-1}(T_h T_t x - T_t x, Sz)_1 = (AT_t x, Sz)_1$$

$$= (AT_t x, z) \qquad \text{for every } z \in H,$$

which implies that $T_t x$ is a solution of (1-1) in the space $(H, (\cdot,\cdot))$ since all the other properties in definition 2.4 remain unchanged under equivalent norms.

4. Existence and Stability

In this section, we shall establish some criteria for the existence and the stability of solutions of (1-1) through the construction of a Lyapunov functional.

<u>Definition 4.1</u>. An equilibrium solution of (1-1) is an element x_e in $\mathcal{D}(A)$ satisfying (1-1) (in the weak topology) such that for any solution $x(t)$ of (1-1) with $x(0) = x_e$

$$||x(t)-x_e|| = 0 \qquad \text{for all } t \geqq 0.$$

It follows from the above definition that if $x(t)$ is a solution of (1-1) with $x(0) = x$, then it is an equilibrium solution if and only if $Ax(t) = 0$ for all $t \geqq 0$. To show this, let $Ax(t) = 0$ where $x(t)$ is a solution of (1-1). Then by definition 2.4 the strong derivative $dx(t)/dt = Ax(t) = 0$ exists and is trongly continuous except at a countable number of values t. But $x(0) = x$ and since any solution of (1-1) is strongly continuous it follows that $x(t) = x$ for all $t \geqq 0$. Conversely, let $x(t)$ be an equilibrium solution of (1-1). Then

$$(Ax(t),z) = (dx(t)/dt,z) = \lim_{h\to 0} h^{-1}(x(t+h)-x(t),z) = \lim_{h\to 0} h^{-1}(0,z) = 0$$

for every $z \in H$ and every $t \geqq 0$. Since $x(t)$ is a solution of (1-1), $x(t) \in \mathcal{D}(A)$ and $Ax(t) \in H$ for each $t \geqq 0$; thus the orthogonality of $Ax(t)$ to every z in H implies that for each $t \geqq 0$, $Ax(t) = 0$.

<u>Definition 4.2</u>. An equilibrium solution (or any unperturbed solution) is said to be stable (with respect to initial perturbations) if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$||x-x_e|| < \delta \quad \text{implies} \quad ||x(t)-x_e|| < \varepsilon \qquad \text{for all } t \geqq 0;$$

x_e is said to be asymptotically stable if

(i) it is stable; and

(ii) $\lim_{t\to\infty} ||x(t)-x_e|| = 0$

where $x(t)$ is any solution of (1-1) with $x(0) = x \in \mathcal{D}(A)$. If there exists positive constants M and β such that

(ii)' $||x(t)-x_e|| \leqq Me^{-\beta t}||x-x_e||$ for all $t \geqq 0$,

then x_e is called exponentially asymptotically stable.

Definition 4.3. Let $x(t)$ be a solution of (1-1) with $x(0)=x$. A subset $\mathcal{D}$ of H is said to be a stability region of the equilibrium solution x_e if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $x \in \mathcal{D}$ and $||x-x_e|| < \delta$ imply $||x(t)-x_e|| < \varepsilon$ for all $t \geq 0$.

Definition 4.4. Let H be a Hilbert space, and let $V(x,y)$ be a complex-valued sesquilinear functional defined on the product space $H \times H$ (i.e. $V(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 V(x_1,y) + \alpha_2 V(x_2,y)$ and $V(x,\beta_1 y_1 + \beta_2 y_2) = \bar{\beta}_1 V(x,y_1) + \bar{\beta}_2 V(x,y_2)$). Then $V(x,y)$ is called a defining sesquilinear functional if it satisfies the following conditions:

(i) $V(x,y) = \overline{V(y,x)}$; (symmetry)

(ii) $|V(x,y)| \leq \gamma||x|| \, ||y||$ for some $\gamma > 0$; (boundedness)

(iii) $V(x,x) \geq \delta||x||^2$ for some $\delta > 0$. (positive definiteness)

Note that condition (ii) implies that $V(x,y)$ is continuous both in x and in y.

Definition 4.5. Let $V(x,y)$ be a defining sesquilinear functional. Then the scalar functional $v(x)$ defined by $v(x) = V(x,x)$ is called a Lyapunov functional.

It follows directly from the above definition that there exist real numbers δ_1, γ_1 with $0 < \delta_1 \leq \gamma_1 < \infty$ such that

$$\delta_1 \, ||x||^2 \leq v(x) \leq \gamma_1 \, ||x||^2 \qquad \text{for all } x \in H.$$

Lemma 4.1. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $H = (H, (\cdot,\cdot))$ such that $x_n \overset{w}{\to} x$ and $y_n \to y$ as $n \to \infty$. Then

$$\lim_{n\to\infty} V(x_n,y_n) = V(x,y) \qquad x,y \in H.$$

Proof. By definition of $V(x,y)$, all the conditions (i.e. sesquilinearity, boundedness and positivity) in the Lax-Milgram theorem are satisfied. Thus, there exists a bounded linear operator S such that

$$V(x,y) = (x, Sy) \qquad \text{for all } x, y \in H. \qquad (4\text{-}1)$$

Since a weakly convergent sequence is strongly bounded so that $||x_n|| < \infty$ for all n, it follows by the sesquilinearity of $V(x,y)$ and by the relation (4-1) that

$$\lim_{n\to\infty} |V(x_n,y_n) - V(x,y)| = \lim_{n\to\infty} |V(x_n,y_n-y) + V(x_n,y)-V(x,y)|$$

$$\leqq \lim_{n\to\infty} |(x_n,S(y_n-y))| + \lim_{n\to\infty} |(x_n,Sy)- (x, Sy)|$$

$$\leqq \lim_{n\to\infty} ||x_n|| \; ||S|| \; ||y_n-y|| = 0.$$

Lemma 4.2. For any pair of solutions $x(t)$, $y(t)$ of (1-1)

$$\dot{v}(x(t) - y(t)) = 2\mathrm{Re}\, V(Ax(t) - Ay(t),\, x(t) - y(t)) \qquad (4\text{-}2)$$

where $\dot{v}(z(t))$ denotes the derivate of $v(z(t))$ with respect to t.

Proof. By the sesquilinearity of $V(x,y)$ it is easily seen that

$$V(x-y,\, x+y) + V(x+y,\, x-y) = 2(V(x,x) - V(y,y)) \quad \text{for any } x,y \in H,$$

and by the symmetry of $V(x,y)$, the above equality implies that

$$v(x) - v(y) = V(x,x) - V(y,y) = \frac{1}{2}\,(V(x-y,x+y) + \overline{V(x-y,x+y)}) = \mathrm{Re}\, V(x-y,x+y).$$

Hence for any fixed $t > 0$ and for any number h

$$h^{-1}[v(x(t+h)-y(t+h))-v(x(t)-y(t))] = \mathrm{Re}\, V(h^{-1}(x(t+h)-x(t))-h^{-1}(y(t+h)-y(t)),$$
$$x(t+h)+x(t)-y(t+h)-y(t)).$$

Since $h^{-1}(x(t+h)-x(t)) \overset{w}{\to} Ax(t)$ and $x(t+h) \to x(t)$ as $h \to 0$, (similarly these two limits hold by replacing x by y) we have by lemma 4.1, as $h \to 0$

$$\frac{d}{dt}\, v(x(t)-y(t)) = 2\mathrm{Re}\, V(Ax(t)-Ay(t),x(t)-y(t)).$$

Thus (4-2) is proved for $t > 0$. For the case of $t = 0$, we take $h > 0$ and let $h \downarrow 0$. Therefore (4-2) holds for all $t \geqq 0$ by defining $\dot{v}(x(0)-y(0))$ as the right-side limit at $t = 0$.

<u>Lemma 4.3</u>. Let $H = (H, (\cdot,\cdot))$ be a Hilbert space and let $V(x,y)$ be a sesquilinear functional defined on $H \times H$. Then $V(x,y)$ defines an equivalent inner product of $(\cdot,\cdot)$ if and only if $V(x,y)$ is a defining sesquilinear functional.

<u>Proof</u>. Let $(x,y)_1 = V(x,y)$ be an equivalent inner product of $(\cdot,\cdot)$. It can easily be shown by the definition of an equivalent inner product that $(x,y)_1$ satisfies all the properties of symmetry, boundedness and positive definiteness. Thus $V(x,y)$ is a defining sesquilinear functional. Conversely, if $V(x,y)$ is a defining sesquilinear functional then the properties of sesquilinearity, symmetry and positive definiteness imply that $(x,y)_1 = V(x,y)$ is an inner product and together with the boundedness of $V(x,y)$, $(\cdot,\cdot)_1$ is equivalent to $(\cdot,\cdot)$.

<u>Theorem 4.1</u>. Let A be a nonlinear operator with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ both contained in a Hilbert space $H = (H,(\cdot,\cdot))$ such that $\mathcal{R}(I-A) = H$. Then A is the infinitesimal generator of a nonlinear contraction (resp., negative contraction) semi-group $\{T_t;\ t \geqq 0\}$ on $\mathcal{D}(A)$ in an equivalent Hilbert space $H_1 = (H,(\cdot,\cdot)_1)$ if and only if there exists a Lyapunov functional $v(x) = V(x,x)$ such that

$$\dot{v}(x-y) = 2\mathrm{Re}\ V(Ax-Ay,\ x-y) \leqq 0 \qquad x,y \in \mathcal{D}(A) \qquad (4\text{-}3)$$

$$(\text{resp., } \dot{v}(x-y) = 2\mathrm{Re}\ V(Ax-Ay,\ x-y) \leqq -2\beta||x-y||^2 \quad \text{for some } \beta > 0)$$

where $V(x,y)$ is the defining sesquilinear functional of $v(x)$ on $H \times H$.

<u>Proof</u>. We prove the negative contraction case, the contraction case follows by taking $\beta = 0$. Let A be the infinitesimal generator in the equivalent Hilbert space H_1. Then by theorem 3.1, A is strictly dissipative with respect to $(\cdot,\cdot)_1$, that is

$$\mathrm{Re}(Ax-Ay,\ x-y)_1 \leqq -\beta_1||x-y||_1^2 \ (\beta_1 > 0) \qquad x,y \in \mathcal{D}(A).$$

Define $V(x,y) = (x,y)_1$, then by lemma 4.3 $V(x,y)$ is a defining sesquilinear functional defined on H x H. Hence the scalar functional $v(x) = V(x,x) = (x,x)_1$ is a Lyapunov functional on the space H. By lemma 4.2, for any $x, y \in \mathcal{D}(A)$

$$\dot{v}(T_t x - T_t y) = 2\mathrm{Re}V(AT_t x - AT_t y,\ T_t x - T_t y) \quad (t \geqq 0).$$

In particular, for t = 0

$$\dot{v}(x-y) = 2\mathrm{Re}V(Ax-Ay,\ x-y) \qquad x, y \in \mathcal{D}(A).$$

Thus the strict dissipativity of A with respect to $(\cdot,\cdot)_1$ and the equivalence relation (3-1) imply that

$$\dot{v}(x-y) = 2\mathrm{Re}(Ax-Ay,\ x-y)_1 \leqq -2\beta_1 ||x-y||_1^2 \leqq -2\beta||x-y||^2$$

where $\beta = \beta_1 \delta^2$.

Conversely, suppose that there exists a Lyapunov functional $v(x) = V(x,x)$ such that (4-3) holds, where $V(x,y)$ is a defining sesquilinear functional defined on H x H. By lemma 4.3, the functional $(x,y)_1 = V(x,y)$ defines an equivalent inner product to $(\cdot,\cdot)$. Hence, by hypothesis (4-3) and the equivalence relation (3-1)

$$\mathrm{Re}(Ax-Ay,\ x-y)_1 = \mathrm{Re}V(Ax-Ay,\ x-y) \leqq -\beta||x-y||^2 \leqq -\beta\gamma^{-2}||x-y||_1^2$$

which implies that A is strictly dissipative with respect to $(\cdot,\cdot)_1$. The result follows by applying theorem 3.1.

Theorem 4.2. Let the nonlinear operator A appearing in (1-1) be such that $\mathcal{R}(I-A) = H$. If there exists a Lyapunov functional $v(x) = V(x,x)$, where $V(x,y)$ is a defining sesquilinear functional defined on H x H such that for any $x, y \in \mathcal{D}(A)$

(i) $\dot{v}(x-y) = 2\mathrm{Re}V(Ax-Ay,\ x-y) \leqq 0$ or

(ii) $\dot{v}(x-y) = 2\mathrm{Re}V(Ax-Ay,\ x-y) \leqq -2\beta||x-y||^2 \qquad (\beta > 0)$

Then, (a) for any $x \in \mathcal{D}(A)$ there exists a unique solution $x(t)$ of (1-1) with $x(0) = x$, (b) any equilibrium solution x_e (or any unperturbed solution such as periodic solution), if it exists, is stable under the condition (i) and is asymptotically stable under the condition (ii), and (c) a stability region of x_e is $D(A)$ which can be extended to $\overline{D(A)}$, the closure of $\mathcal{D}(A)$. If, in addition, $0 \in \mathcal{D}(A)$ and $A0 = 0$, then the zero vector is an equilibrium solution, called the null solution, of (1-1) which is stable or asymptotically stable according to (i) or (ii), respectively.

Proof. By hypothesis and applying theorem 4.1, A is the infinitesimal generator of a nonlinear contraction semi-group on $\mathcal{D}(A)$ in an equivalent space $H_1 = (H, (\cdot,\cdot)_1)$ under the condition (i) and is the infinitesimal generator of a nonlinear negative contraction semi-group on $\mathcal{D}(A)$ in H_1 under the condition (ii), where the norm $||\cdot||_1$ induced by $(\cdot,\cdot)_1$ satisfies relation (3-1). By theorem 3.3, A is the infinitesimal generator of a nonlinear semi-group $\{T_t;\ t \geqq 0\}$ on $\mathcal{D}(A)$ in H such that under the condition (i)

$$||T_t x - T_t y|| \leqq \gamma\, \delta^{-1} ||x-y|| \qquad x,\ y \in \mathcal{D}(A)$$

and under the condition (ii)

$$||T_t x - T_t y|| \leqq \gamma\, \delta^{-1}\, e^{-\beta t}\, ||x-y|| \qquad x, y \in \mathcal{D}(A) \qquad (t \geqq 0).$$

Since for any $x \in \mathcal{D}(A)$, $T_t x$ is the unique solution in H_1 with $T_o x = x$, it follows from the corollary of theorem 3.3 that $T_t x$ is also the unique solution in H with $T_o x = x$. By the semi-group property of $\{T_t;\ t \geqq 0\}$ in H, we have under the conditions (i) or (ii)

$$||T_t x - x_e|| \leqq \gamma\, \delta^{-1} ||x - x_e|| \qquad (t \geqq 0)$$

or

$$||T_t x - x_e|| \leqq \gamma\, \delta^{-1} e^{-\beta t} ||x - x_e|| \qquad (t \geqq 0),$$

which shows that the equilibrium solution x_e, if it exists, is stable and asymptotically stable, respectively. Note that $T_t x_e = x_e$ for all

$t \geq 0$. Since the contraction semi-group $\{T_t;\ t \geq 0\}$ on $\mathcal{D}(A)$ in the space H_1 can be extended to $\overline{\mathcal{D}(A)}$ in the $||\cdot||_1$-topology (cf. [10]), the same is true for the semi-group $\{T_t;\ t \geq 0\}$ on $\mathcal{D}(A)$ in the space H. This is due to the fact that the closure of $\mathcal{D}(A)$ in the $||\cdot||_1$-topology is the closure of $\mathcal{D}(A)$ in the $||\cdot||$-topology because of the equivalence relation between these two norms. Hence the results (a), (b) and (c) are proved. The stability property of the null solution follows from (b).

The purpose for the construction of a Lyapunov functional can be demonstrated as follows: Let $v(x) = V(x,x)$ be a Lyapunov functional such that for some $\alpha \geq 0$

$$\dot{v}(x(t)-y(t)) \leq -\alpha\ ||x(t)-y(t)||^2 \qquad (t \geq 0) \tag{4-4}$$

for any two solutions $x(t)$, $y(t)$ of (1-1), where $V(x,y)$ is a defining sesquilinear functional. By lemma 4-3, the functional

$$(x,y)_1 = V(x,y) \qquad x,\ y \in H$$

defines an equivalent inner product to $(\cdot,\cdot)$. Since

$$v(x) = V(x,x) = (x,x)_1 \leq \gamma||x||^2 \quad \text{for all } x \in H,$$

it follows from (4-4) that

$$\dot{v}(x(t)-y(t)) \leq -\alpha/\gamma\ \ v(x(t)-y(t)) = -2\lambda v(x(t)-y(t)) \qquad (2\lambda \equiv \alpha/\gamma).$$

Integrating the above inequality with respect to t and note that $v(x) = ||x||_1^2$,

$$||x(t)-y(t)||_1^2 \leq ||x(0)-y(0)||_1^2\ \ e^{-2\lambda t} \qquad (t \geq 0).$$

By the equivalence relation (3-1), the above inequality implies that

$$||x(t)-y(t)||^2 \leq 1/\delta^2\ \ ||x(t)-y(t)||_1^2 \leq (\gamma/\delta)^2 e^{-2\lambda t}||x(0)-y(0)||^2$$

which is the same as

$$||x(t)-y(t)|| \leq \gamma/\delta\ \ e^{-\lambda t}||x(0)-y(0)|| \qquad \text{for } t \geq 0.$$

Hence, if an equilibrium solution x_e (or any unperturbed solution) exists then by choosing $y(0) = x_e$ in the above inequality, we have

$$||x(t)-x_e|| \leqq \gamma/\delta\ e^{-\lambda t}||x(0)-x_e|| \qquad \text{for all } t \geqq 0$$

which shows that the equilibrium solution x_e is exponentially asymptotically stable if $\alpha > 0$, and is stable if $\alpha = 0$.

The importance of theorems 4.1 and 4.2 is the fact that the existence of a Lyapunov functional satisfying (4-3) alone does not guarantee the existence of a solution to (1-1) and in general, it is rather complicated to prove such solutions exist. However under the additional assumption that $\mathcal{R}(I-A) = H$ the existence of a solution with any initial element $x \in \mathcal{D}(A)$ is assured. This assurance makes the stability of solutions of (1-1) meaningful.

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List of Symbols

A	nonlinear operator
$D(A)$	Domain of A
$\overline{D(A)}$	closure of $D(A)$
H	Hilbert space with inner product $(\cdot,\cdot)$
H_1	Hilbert space with inner product $(\cdot,\cdot)_1$
I	identity operator
M	positive number
$R(A)$	range of A
S	bounded linear operator
$\{T_t;\ t \geqq 0\}$	family of nonlinear semi-group
$V(x,y)$	defining sesquilinear functional
$\dot{v}(x(t))$	derivative of $v(x(t))$ at point $t \geqq 0$
x, y, z	elements of H
x_e	equilibrium solution
α, β, γ, δ, λ	real numbers
$\rightarrow$	strong convergence
$\overset{w}{\rightarrow}$	weak convergence
$(\cdot,\cdot)_1$	equivalent inner product to $(\cdot,\cdot)$