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The Existence and Stability of Nonlinear Wave Equations

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1. Introduction

The object of the present paper is to establish some criteria for the existence, the uniqueness and the stability or asymptotic stability of a solution of the second order wave equation

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} = Lu + f(x,u,\frac{\partial u}{\partial x_i},\frac{\partial u}{\partial t})$$
$$\equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) + c(x)u + f(x,u,\frac{\partial u}{\partial x_i},\frac{\partial u}{\partial t}) \quad (x \in \Omega) \tag{1-1}$$

with the boundary condition

$$u(t,x)\Big|_{\partial\Omega} = h(x') \qquad x' \in \partial\Omega \tag{1-2}$$

where $a \geqq 0$, Ω is a bounded domain in R^n, $\partial\Omega$ is the boundary of Ω and f is a (nonlinear) function defined on some suitable space. By choosing a suitable function $g(x)$ defined on $\bar{\Omega}$ such that $g(x')=h(x')$ on $\partial\Omega$ and replacing u by u-g, equation (1-1) remains the same from with a homogeneous boundary condition except with a different function f. Thus we shall assume that the boundary condition (1-2) is homogeneous. Let $u_1=u$, $u_2 = \dot{u} = \frac{\partial u}{\partial t}$, then (1-1) is reduced to a system of equations of the form

$$\frac{\partial \underline{u}}{\partial t} = \begin{bmatrix} 0 & 1 \\ L & -a \end{bmatrix} \underline{u} + \underline{f}(\underline{u}) \tag{1-3}$$

where

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \underline{f}(\underline{u}) = \begin{bmatrix} 0 \\ f(x,u_1,\frac{\partial u_1}{\partial x_i}, u_2) \end{bmatrix} .$$

By a suitable choice of a function space, equation (1-3) together with the zero boundary condition may be considered as an abstract operator differential equation of the form

$$\frac{d\underline{u}(t)}{dt} = A\,\underline{u}(t) + \underline{f}(\underline{u}(t)) \qquad t \in R^{+} = [0,\infty) \qquad (1\text{-}4)$$

where A is an abstract linear operator from some function space into itself. In this paper, we shall formulate A as a linear operator mapping part of the Hilbert space $H \equiv H_o^1(\Omega) \times L^2(\Omega)$ into itself and $\underline{f}$ is defined on H into H, and then apply the results developed for operator differential equations in [1] and [2] (which are based on the work by Kato [3]) to establish criteria on the coefficients of the partial differential operator L and on the function f so that the existence and uniqueness of a solution and the stability or asymptotic stability of an equilibrium solution (or any unperturbed solution) can be ensured. We shall first discuss the equation (1-1) with $f \equiv 0$ and then consider the more general nonlinear problem which is closely related to the linear form.

2. Background

<u>Definition 2.1.</u> By a solution $\underline{u}(t)$ of (1-4) with initial condition $\underline{u}(0) = \underline{u}_o \in D(A)$ in a Hilbert space H, we mean the following:

(a) $\underline{u}(t)$ is uniformly Lipschitz continuous in t for each $t \geqq 0$ with $\underline{u}(0) = \underline{u}_o$;

(b) $\underline{u}(t) \in D(A)$ for all $t \geqq 0$ and the strong derivative of $\underline{u}(t)$ exists for almost all values of $t \geqq 0$ and satisfies (1-4) a.e. in R^{+}.

<u>Definition 2.2.</u> An equilibrium solution of (1-1) is a solution $\underline{u}_e$ such that for any solution $\underline{u}(t)$ of (1-1) with $\underline{u}(0) = \underline{u}_e$

$$||\underline{u}(t)-\underline{u}_e|| = 0 \qquad \text{for all } t \geqq 0.$$

It can be shown that if u(t) is a solution of (1-4) then it is an equilibrium solution if and only if $A\underline{u}(t) + f(u(t)) = 0$ for all $t \geqq 0$ (cf [1]).

Definition 2.3. An equilibrium solution (or any unperturbed solution) $\underline{u}_e$ of (1-4) is said to be stable (with respect to initial perturbations) if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$||\underline{u}-\underline{u}_e|| < \delta \quad \text{implies} \quad ||\underline{u}(t)-\underline{u}_e|| < \varepsilon \qquad \text{for all } t \geqq 0;$$

$\underline{u}_e$ is said to be asymptotically stable if

(i) it is stable; and

(ii) $\lim_{t\to\infty} ||\underline{u}(t)-\underline{u}_e|| = 0$

where $\underline{u}(t)$ is any solution of (1-4) with $\underline{u}(0)=\underline{u}\varepsilon D(A)$. If there exist positive constants M and β such that

$$\text{(ii)}' \quad ||\underline{u}(t)-\underline{u}_e|| \leqq Me^{-\beta t} ||\underline{u}-\underline{u}_e|| \qquad \text{for all } t \geqq 0$$

then $\underline{u}_e$ is called exponentially asymptotically stable.

Definition 2.4. Let u(t) be a solution to (1-4) with $\underline{u}(0) = \underline{u}$. A subset D of H is said to be a stability region of the equilibrium solution u_e (or any unperturbed solution) if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\underline{u} \in D \quad \text{and} \quad ||\underline{u}-\underline{u}_e|| < \delta \quad \text{imply} \quad ||\underline{u}(t)-\underline{u}_e|| < \varepsilon \qquad \text{for all } t \geqq 0.$$

Definition 2.5. Two inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_e$ defined on the same vector space H are said to be equivalent if there exist some positive constants δ, γ, such that

$$\delta||\underline{u}|| \leqq ||\underline{u}||_e \leqq \gamma||\underline{u}|| \qquad \text{for all } \underline{u} \in H. \tag{2-1}$$

The following two theorems are from [1] and [2] by the authors.

Theorem 2.1. Let A be a linear operator with domain D(A) and range R(A) both contained in a Hilbert space H such that D(A) is dense in H and R(I-A)=H. If A satisfies

$$(u, Au) \leqq -\beta ||u||^2 \qquad (\beta \geqq 0) \qquad \text{for all } u \in D(A) \tag{2-2}$$

and f is defined on all of H and satisfies

$$||f(u) - f(v)|| \leqq k||u-v|| \qquad u, v \in H \tag{2-3}$$

for some $k \leqq \beta$, then (a) for any $u_o \in D(A)$ there exists a unique solution u(t) of (1-4) with $u(0) = u_o$; (b) any unperturbed solution (e.g., equilibrium solution or periodic solution) is asymptotically stable if $k < \beta$ and is stable if $k = \beta$; (c) a stability region is D(A) which can be extended to the whole space H.

Remark: (a) An operator A satisfying (2-2) is called strictly dissipative if $\beta>0$ and is called dissipative if $\beta=0$. The number β is called a dissipative constant of A. (b) Weaker condition on f can be found in [2].

Theorem 2.2. In theorem 2.1, if A is strictly dissipative with a dissipative constant β with respect to an equivalent inner product $(.,.)_e$ and (2-3) is replaced by

$$||f(u)-f(v)||_e \leqq k||u-v||_e \qquad u, v \in H \tag{2-4}$$

for some $k \leqq \beta$, then all the results in theorem 2.1 hold.

3. Formulation of Abstract Operators

Throughout this paper, the following conventional notations will be used: $C^m(\Omega)$ denotes the class of all m-times $(0 \leqq m \leqq \infty)$ continuously differentiable real-valued functions on Ω; the subset $C^m_o(\Omega)$ of $C^m(\Omega)$ consists of functions in $C^m(\Omega)$ with compact support in Ω; we denote

$$(u,v)_o = \int_\Omega u(x)\ v(x)\ dx \qquad u,v \in C^o(\Omega)$$

$$(u,v)_1 = \int_\Omega (u(x)v(x) + \sum_{i=1}^{n} \frac{\partial u(x)}{\partial x_i} \cdot \frac{\partial v(x)}{\partial x_i})dx \qquad u,v \in C^1(\Omega)$$

$$||u||_o = (u,u)_o^{1/2}, \quad ||u||_1 = (u,u)_1^{1/2}$$

where $dx = dx_1 \cdots dx_n$ is Lebesque measure in R^n; for $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, and $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in $C^1(\Omega) \times C^o(\Omega)$, we define

$$(\underline{u}, \underline{v})_o = (u_1, v_1)_o + (u_2, u_2)_o;$$

$$(\underline{u}, v)_H = (u_1, v_1)_1 + (u_2, v_2)_o;$$

$$||u||_H = (\underline{u}, \underline{u})_H^{1/2} .$$

The linear space $C_o^1(\Omega)$ equipped with the inner product $(u, v)_1$ is an inner product space. The closure in the norm $||\cdot||_1$ of $C_o^1(\Omega)$ is a Hilbert space and is denoted by $H_o^1(\Omega)$. The product space $H = H_o^1(\Omega) \times L^2(\Omega)$, where $L^2(\Omega)$ is the class of Lebesque square-integrable functions with its usual inner product, is a linear space with addition and scalar multiplication defined by

$$\underline{u} + \underline{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}, \quad \alpha\ \underline{u} = \begin{pmatrix} \alpha\ u_1 \\ \alpha\ u_2 \end{pmatrix}$$

and it is a Hilbert space with the inner product $(\underline{u}, \underline{v})_H$.

<u>Definition 3.1.</u> Let Ω_o be an open domain in the Euclidean space R^n. The operator

$$L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x) \quad (x \in \Omega_o)$$

is called a formal partial differential operator if $a_{ij}(x) = a_{ji}(x)$

and together with $b_i(x)$, $c(x)$ are all infinitely differentiable functions in Ω_o; and L is called strongly elliptic if there exists a positive constant α such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\ \xi_i \xi_j \geq \alpha \sum_{i=1}^{n} \xi_i^2 \qquad (x \in \Omega_o) \tag{3-1}$$

on R^n for any real vector $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$.

Consider the strongly elliptic formal partial differential operator in the divergence form

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial}{\partial x_j}) + c(x) \qquad x \in \Omega_o. \tag{3-2}$$

Let Ω be a bounded subdomain whose closure $\bar{\Omega}$ is contained in Ω_o and whose boundary is a smooth surface $\partial\Omega$ with no point in $\partial\Omega$ interior to $\bar{\Omega}$. Assume that $c(x) < 0$ for $x \in \bar{\Omega}$ and write $c_m = \min_{x\in\bar{\Omega}}(-c(x))$ and $c_M = \max_{x\in\bar{\Omega}}(-c(x))$. Define T_o as a linear operator from $L^2(\Omega)$ into itself by the equations:

$$\begin{aligned} D(T_o) &= \{u \in C^\infty(\bar{\Omega});\ u(x) = 0 \quad \text{for} \quad x \in \partial\Omega\} \\ T_o u &= Lu \qquad u \in D(T_o) \end{aligned} \tag{3-3}$$

then T_o is a linear operator with domain $D(T_o)$ dense in $L^2(\Omega)$ since $D(T_o) \supset C_o^\infty(\Omega)$ which is dense in $L^2(\Omega)$. We denote the closure of T_o (i.e., the smallest closed extension of T_o) by T.

Lemma 3.1. Let T_o be defined in (3-3). Then for any $u,v \in D(T_o)$ $(T_o u, v) = (u, T_o v)$. Moreover, the closure T of T_o is strictly dissipative with respect to the inner product $(\cdot,\cdot)_o$.

Proof. For any $u, v \in D(T_o)$, integration by parts twice and notice that $a_{ij}(x) = a_{ji}(x)$, it can easily be shown that $(T_o u, v) = (u, T_o v)$. Integration by parts once and using the strong ellipticity of L, we have for any $u \in D(T_o)$

$$(u, T_o u)_o = \int_\Omega [-\sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + c(x)u^2] dx$$

$$\leqq -\int_\Omega [\alpha \sum_{i=1}^{n} (\frac{\partial u}{\partial x_i})^2 - c(x)\ u^2] dx \leqq -c_m ||u||_o^2$$

which shows that T_o is strictly dissipative. For any $u \in D(T)$, there exists a sequence $\{u_n\}$ in $D(T_o)$ such that $u_n \to u$ and $T_o u_n \to T u$ since T is the closure of T_o. It follows by the continuity of inner product that

$$(u, Tu)_o = \lim_{n\to\infty} (u_n, T_o u_n)_o \leqq \lim_{n\to\infty} (-c_m ||u_n||_o^2) = -c_m ||u||_o^2$$

which proves the strict dissipativity of T.

<u>Lemma 3.2.</u> T is a linear operator with $D(T) \subset H_o^1(\Omega)$ and $R(T) \subset L^2(\Omega)$, and for any $\alpha > 0$, $(\alpha I - T)^{-1}$ is an everywhere defined continuous operator on $L^2(\Omega)$ into itself.

<u>Proof.</u> By the definition of T, $D(T) \subset H_o^1(\Omega)$ and the spectrum $\sigma(T)$ of T consists of a countable discrete set of points with no finite limit point (cf. Dunford and Schwartz [4] theorem XIV. 6.23). Hence there exists a real number $\alpha_o > 0$ such that $\alpha_o \notin \sigma(T)$ which implies that the resolvent $R(\alpha_o; T) = (\alpha_o I - T)^{-1}$ is an everywhere defined continuous linear operator on $L^2(\Omega)$. Moreover the dissipativity of T implies that $(\alpha I - T)^{-1}$ exists and is continuous for every $\alpha > 0$ and together with the condition $D((\alpha_o I - T)^{-1}) = L^2(\Omega)$ for some $\alpha_o > 0$, we have $D((\alpha I - T)^{-1}) = L^2(\Omega)$ for every $\alpha > 0$ (cf. Kato [5]).

In order to formulate the partial differential equation (1-3) as an operator differential equation of the form (1-4), we define the operator A by the following equations:

$$D(A) = D(T) \times H_o^1(\Omega) \quad \text{and} \quad A = \begin{pmatrix} 0 & I \\ T & -aI \end{pmatrix} \tag{3-4}$$

where $a \geqq 0$ is the constant appearing in (1-1).

Lemma 3.3. The operator A is a linear operator with domain $D(A)$ dense in $H = H_o^1(\Omega) \times L^2(\Omega)$ and range $R(A)$ in H. Moreover, $R(I-A) = H$.

Proof. It is obvious that A is linear with domain $D(A)$ dense in H since $D(T) \supset D(T_o) \supset C_o^\infty(\Omega)$ which is dense in $H_o^1(\Omega)$, and $H_o^1(\Omega)$, when considered as a subset of $L^2(\Omega)$, is dense in $L^2(\Omega)$. It is also clear that $R(A) \subset H$ since $R(T) \subset L^2(\Omega)$. To show $R(I-A) = H$, let $\underline{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \epsilon H$ and show that there exists an $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \epsilon D(A)$ such that $(I-A)\underline{u} = \underline{w}$, which is equivalent to find an $u_1 \epsilon D(T)$ and $u_2 \epsilon H_o^1(\Omega)$ satisfying the system of equations

$$u_1 - u_2 = w_1 \quad \text{and} \quad -Tu_1 + (1+a)u_2 = w_2.$$

By substituting $u_2 = u_1 - w_1$ into the second equation yields

$$((1+a)I - T)u_1 = (1 + a)\, w_1 + w_2.$$

By lemma 3.2, $R(\alpha I-T) = L^2(\Omega)$ for every $\alpha > 0$ which insures the existence of an $u_1 \epsilon D(T)$ satisfying the above equality since $(1+a)w_1 + w_2$ is in $L^2(\Omega)$. The fact that $D(T) \subset H_o^1(\Omega)$ implies $u_2 = u_1 - w_1 \epsilon H_o^1(\Omega)$ which completes the proof of the lemma.

4. Equivalent Inner Product

In order to prove our main results, we shall introduce an equivalent inner product on H with respect to which the operator A is dissipative or strictly dissipative. Following the same idea used by Buis (cf. [6]), we introduce a linear operator S defined by:

$$D(S) = D(T_o) \times L^2(\Omega)$$

$$S\,\underline{u} = \begin{bmatrix} -2T_o + a^2 I & aI \\ aI & 2I \end{bmatrix} \underline{u} \qquad \underline{u} \;\epsilon\; D(S) \qquad (4\text{-}1)$$

and show the following lemmas.

Lemma 4.1. The functional $V(\underline{u}, \underline{v})$ defined by

$$V(\underline{u}, \underline{v}) = (\underline{u}, S\,\underline{v})_o \quad \underline{u}, \underline{v} \in D(S)$$

is a continuous bilinear functional on D(S) (in the topology of H).

Proof. It is easily seen that $V(\underline{u}, \underline{v})$ is bilinear and that for any $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in D(S)$

$$V(\underline{u},\underline{v}) = -2(u_1, T_o v_1)_o + a^2(u_1, v_1)_o + a(u_1, v_2)_o + a(u_2, v_1)_o + 2(u_2, v_2)_o.$$

Integration by parts of the first term on the right and using the Schwartz inequality, we have

$$|-2(u_1, T_o v_1)_o| = 2\left| \int_\Omega [\sum_{i,j=1}^n a_{ij} \frac{\partial u_1}{\partial x_i} \frac{\partial v_1}{\partial x_j} - c(x) u_1 v_1] dx \right|$$

$$\leqq 2M[(\int_\Omega (\sum_{i=1}^n |\frac{\partial u_1}{\partial x_i}|)^2 dx)^{1/2} (\int_\Omega (\sum_{j=1}^n |\frac{\partial v_1}{\partial x_j}|^2 dx)^{1/2}] + 2c_M ||u_1||_o ||v_1||_o$$

$$\leqq 2Mn[(\sum_{i=1}^n \int_\Omega (\frac{\partial u_1}{\partial x_i})^2 dx)^{1/2} (\sum_{j=1}^n \int_\Omega (\frac{\partial v_1}{\partial x_j})^2 dx)^{1/2}] + 2\, c_M ||u_1||_o ||v_1||_o$$

where $M = \max_{i,j} (\max_{x \in \bar{\Omega}} |a_{ij}(x)|)$ and $c_M = \max_{x\in\bar{\Omega}} (-c(s))$. It follows that

$$|V(\underline{u},\underline{v})| \leqq k(||u_1||_1\, ||v_1||_1 + ||u_1||_o\, ||v_2||_o + ||u_2||_o\, ||v_1||_o +$$
$$+ ||u_2||_o\, ||v_2||_o)$$

$$\leqq k\, (||u_1||_1 + ||u_2||_o)(||v_1||_1 + ||v_2||_o)$$

where $k = \max\,(2Mn,\ 2C_M + a^2,\ 2)$. On squaring both sides of the above inequality and notice that $(\alpha+\beta)^2 \leqq 2(\alpha^2 + \beta^2)$ for any real numbers α, β we have

$$|V(\underline{u},\underline{v})|^2 \leqq 4k^2(||u_1||_1^2 + ||u_2||_o^2)(||v_1||_1^2 + ||v_2||_o^2)$$

which is equivalent to

$$|V(\underline{u},\underline{v})| \leqq k_1 ||\underline{u}||_H ||\underline{v}||_H \qquad (4\text{-}2)$$

where

$$k_1 = 2 \max (2Mn, 2c_M + a^2, 2). \tag{4-3}$$

Thus $V(\underline{u}, \underline{v})$ is continuous in the topology of H which completes the proof of the lemma.

Lemma 4.2. The bilinear functional $V(\underline{u}, \underline{v})$ defines an equivalent inner product on D(S) in H and the extension $\bar{V}(\underline{u}, \underline{v})$ of $V(\underline{u}, \underline{v})$ to H defines an equivalent inner product on the whole space H.

Proof. Define $(\underline{u}, \underline{v})_s = V(\underline{u}, \underline{v})$, then $(\underline{u}, \underline{v})_s$ is bilinear and for any $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \varepsilon$ D(S) we have by lemma 3.1

$$\begin{aligned}(\underline{u}, \underline{v})_s &= -2(u_1, T_o v_1)_o + a^2(u_1, v_1)_o + a(u_1, v_2)_o + a(u_2, v_1)_o + 2(u_2, v_2)_o \\ &= -2(v_1, T_o u_1)_o + a^2(v_1, u_1)_o + a(v_2, u_1)_o + a(v_1, u_2)_o + 2(v_2, u_2)_o \\ &= (\underline{v}, \underline{u})_s\end{aligned}$$

which shows that $(\underline{u}, \underline{v})_s$ is symmetric. Integration by parts and using the strong ellipticity condition of L, we have

$$\begin{aligned}(\underline{u}, \underline{u})_s &= \int_\Omega [2 \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u_1}{\partial x_i} \frac{\partial u_1}{\partial x_j} - 2c(x)\, u_1^2 + (au_1 + u_2)^2 + u_2^2] dx \\ &\geqq \int_\Omega [2\, \alpha \sum_{i=1}^{n} (\frac{\partial u_1}{\partial x_i})^2 + 2c_m\, u_1^2 + u_2^2] dx \geqq k_2 ||\underline{u}||_H^2\end{aligned} \tag{4-4}$$

where

$$k_2 = \min (2\alpha, 2c_m, 1). \tag{4-5}$$

Hence $V(\underline{u}, \underline{v})$ is positive definite on D(S) and that $(\underline{u}, \underline{u})_s \neq 0$ iff $\underline{u} \neq 0$. It follows that $(\underline{u}, \underline{v})_s = V(\underline{u}, \underline{v})$ defines an inner product on D(S). From (4-2) and (4-4), the inner product $(\underline{u}, \underline{v})_s$ is equivalent to $(\underline{u}, \underline{v})_H$ on D(S). Let $\bar{V}(\underline{u}, \underline{v})$ be the extension of $V(\underline{u}, \underline{v})$ from D(S) to H and define

$$(\underline{u}, \underline{v})_e = \bar{V}(\underline{u}, \underline{v}) \qquad \text{for } \underline{u}, \underline{v} \in H \tag{4-6}$$

then by the continuity of $V(\underline{u}, \underline{v})$ on D(S), $\bar{V}(\underline{u}, \underline{v})$ possesses all the properties of bilinearity, symmetry, boundedness and positivity on H.

Moreover, from (4-2) and (4-4)

$$k_2||u||_H^2 \leq ||\underline{u}||_e^2 \leq k_1||u||_H^2 \qquad u \in H. \qquad (4\text{-}7)$$

Note that $(\underline{u},\underline{v})_e$ coincides with $(\underline{u},\underline{v})_s$ for u, v ε D(S). Therefore $(\cdot,\cdot)_e$ is an equivalent inner product of $(\cdot,\cdot)_H$ which completes the proof.

Lemma 4.3. The operator A is strictly dissipative with respect to $(\cdot,\cdot)_e$ if a > 0 and is dissipative if a = 0 where a is the constant appearing in (1-1).

Proof. We first show that A is (strictly) dissipative on $D(T_o)\times D(T_o)$. For any $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \varepsilon\ D(T_o) \times D(T_o)$

$$A\ \underline{u} = \begin{bmatrix} u_2 \\ \\ Tu_1 - au_2 \end{bmatrix} = \begin{bmatrix} u_2 \\ \\ T_o u_1 - au_2 \end{bmatrix} \varepsilon\ D(T_o) \times L^2(\Omega) = D(S).$$

Since $\bar{V}(\underline{u},\underline{v})$ is an extension of $V(\underline{u},\underline{v})$ we have, by the definition of S and by lemma 3.1,

$$(\underline{u},A\underline{u})_e = (\underline{u},S\ A\underline{u})_o = -2(u_1,T_o u_2)_o + a(u_1,T_o u_1)_o + 2(u_2,T_o u_1)_o -$$

$$-a(u_2,u_2)_o = a(u_1,T_o u_1)_o - a(u_2,u_2)_o.$$

It follows by the strong ellipticity of L and the relation (4-7) that

$$(\underline{u},A\underline{u})_e = a \int_\Omega [-\sum_{i,j=1}^n a_{ij}(x) \frac{\partial u_1}{\partial x_i} \frac{\partial u_1}{\partial x_j} + c(x)u_1^2 - u_2^2]dx$$

$$\leq -a \int_\Omega [\alpha \sum_{i=1}^n (\frac{\partial u_1}{\partial x_i})^2 + c_m u_1^2 + u_2^2]dx \leq -a\lambda||\underline{u}||_H^2 \leq -\beta||\underline{u}||_e^2 \quad (4\text{-}8)$$

where

$$\lambda = \min(\alpha, c_m, 1), \quad \beta = \frac{a\lambda}{k_1} \quad . \qquad (4\text{-}9)$$

Thus A is dissipative on $D(T_o) \times D(T_o)$ if a = 0 and is strictly dissipative if $a > 0$. Next we show that A is (strictly) dissipative on $D(T)xD(T_o)$. Let $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \varepsilon D(T) \times D(T_o)$ then $A\,\underline{u} = \begin{bmatrix} u_2 \\ Tu_1 - au_2 \end{bmatrix} \varepsilon D(S)$. Hence

$$(\underline{u}, A\,\underline{u})_e = (\underline{u}, S\,A\underline{u})_o = -2(u_1, T_o u_2)_o + a(u_1, Tu_1)_o + 2(u_2, Tu_1)_o - a(u_2, u_2)_o.$$

Since T is the closure of T_o in $L^2(\Omega)$, there exists a sequence $\{v_n\}$ in $D(T_o)$ such that $v_n \to u_1$ and $T_o v_n \to Tu_1$ in the topology of $L^2(\Omega)$ which implies by lemma 3.1 that

$$(u_2, Tu_1)_o = \lim(u_2, T_o v_n)_o = \lim(T_o u_2, v_n)_o = (T_o u_2, u_1)_o.$$

Moreover by Garding's Inequality, there exist constant c_1, c_2 such that

$$||v_n||_1^2 \leqq c_1(Tv_n, v_n)_o + c_2||v_n||_o^2 \leqq c_1||Tv_n||_o||v_n||_o + c_2||v_n||_o^2$$

we have $v_n \to u_1$ in $H_o^1(\Omega)$. Let $\underline{u}_n = \begin{bmatrix} v_n \\ u_2 \end{bmatrix}$ then $\underline{u}_n \to \underline{u}$ in H and thus $\underline{u}_n \to \underline{u}$ with respect to $||\cdot||_e$. It follows from (4-8) that

$$(\underline{u}, A\,\underline{u})_e = a(u_1, Tu_1)_o - a(u_2, u_2)_o = \lim_{n\to\infty}[a(v_n, T_o v_n)_o - a(u_2, u_2)_o]$$

$$\leqq \lim_{n\to\infty} [-\beta||\underline{u}_n||_e^2] = -\beta||\underline{u}||_e^2 \qquad \underline{u} \;\varepsilon\; D(T)xD(T_o). \qquad (4\text{-}10)$$

To show the dissipativity of A on D(A), let $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \varepsilon D(A)$, that is, $u_1 \;\varepsilon\; D(T)$ and $u_2 \;\varepsilon\; H_o^1(\Omega)$. Then there exists a sequence $\{w_n\}$ in $D(T_o)$ such that $w_n \to u_2$ in the topology of $H_o^1(\Omega)$ since $D(T_o) \supset C_o^\infty(\Omega)$ which is dense in $H_o^1(\Omega)$. Let $\underline{u}_n = \begin{bmatrix} u_1 \\ w_n \end{bmatrix} \varepsilon D(T)xD(T_o)$ then $\lim_{n\to\infty} \underline{u}_n = \underline{u}$ and

$$\lim_{n\to\infty} A\,\underline{u}_n = \lim_{n\to\infty} \begin{bmatrix} w_n \\ Tu_1 - aw_n \end{bmatrix} = \begin{bmatrix} u_2 \\ Tu_1 - au_2 \end{bmatrix} = A\,\underline{u}$$

where the convergence of both $\{\underline{u}_n\}$ and $\{A\underline{u}_n\}$ hold in the topology of H. Hence by the equivalence between $||\cdot||_H$ and $||\cdot||_e$, $\underline{u}_n \to \underline{u}$ and $A\underline{u}_n \to A\underline{u}$ in the topology of H_e. It follows from (4-10) that

$$(\underline{u}, A\underline{u})_e = \lim_{n\to\infty}(\underline{u}_n, A\underline{u}_n)_e \leqq -\beta||\underline{u}||_e^2 \qquad \underline{u} \;\varepsilon\; D(A) \qquad (4\text{-}11)$$

which shows that A is dissipative if $\beta=0$ (i.e., $a=0$) and is strictly dissipative if $\beta>0$ (i.e. $a>0$).

5. The Main Results

Theorem 5.1. Let the operator L in (1-1) be a strongly elliptic formal partial differential operator with $c(x)<0$ for $x \in \bar{\Omega}$ where Ω is a bounded domain in R^n with sufficiently smooth surface $\partial\Omega$. Then for any initial data $u_o(x)$ in D(T) and $v_o(x)$ in $H_o^1(\Omega)$ there exists a unique strong solution u(t,x) (in the sense of definition 2.1 with u(t,x) strongly differentiable in t for all $t \geq 0$) of the linear equation (1-1) (i.e., $f \equiv 0$) and the homogeneous boundary condition such that $u(0,x)=u_o(x)$ and $\dot{u}(0,x)=v_o(x)$. Moreover, the null solution $u(t,x) \equiv 0$ is stable if $a=0$ and is asymptotically stable if $a>0$.

Proof. By lemma 3.3 A is a linear operator with D(A) dense in H and R(A) contained in H such that R(I-A)=H. By lemma 4.3, A is strictly dissipative (resp., dissipative) with a dissipative constant β with respect to the equivalent inner product $(\cdot,\cdot)_e$. It follows by applying theorem 2.2 with $f \equiv 0$ that all the results stated in the theorem hold, where the underlying Hilbert space is H. The strong differentiability of u(t,x) for all $t \geq 0$ follows from a theorem due to Lumer and Phillips [7] (see also [3]).

Theorem 5.2. Let the operator L in (1-1) be the same as in theorem 5.1. Assume that f is a function defined on all of H into $L^2(\Omega)$. If there exists a number $k \geq 0$ such that for any $u,v \in H_o^1(\Omega)$ and $u', v' \in L^2(\Omega)$

$$||f(x,u,u_{x_i},u')-f(x,v,v_{x_i},v')||_o \leq k(||u-v||_o^2+\sum_{i=1}^{n}||u_{x_i}-v_{x_i}||_o^2+||u'-v'||_o^2)^{1/2} \tag{5-1}$$

Then for any initial element $u_o(x)$ in D(T) and $v_o(x)$ in $H_o^1(\Omega)$ there exists a unique solution u(t,x) of (1-1) satisfying the homogeneous boundary condition such that $u(0,x)=u_o(x)$ and $u_t(0,x)=v_o(x)$. Moreover, any unperturbed solution such as equilibrium solution or periodic solution (if any), is stable if $k=(k_2/k_1)^{1/2}\beta$ and is exponentially asymptotically stable if $k<(k_2/k_1)^{1/2}\beta$ where k_1, k_2 are given by (4-3), (4-5) respectively and β is given by (4-9).

Remark. (a) The initial elements $u_o(x)$ and $v_o(x)$ in theorems 5.1 and 5.2 can be in the spaces $H_o^1(\Omega)$ and $L^2(\Omega)$ respectively since the initial element in $D(A)=D(T)\times H_o^1(\Omega)$ can be extended to $\overline{D(A)}$, the closure of D(A), which is equal to H; (b) The condition (5-1) can be weakened to some extent (see [2]); (c) For $k>(k_2/k_1)^{1/2}\beta$, the solution in theorem 5.2 is a weak solution.

Proof. It suffices to show that $\underset{\sim}{f}(\underset{\sim}{u})$ satisfies the conditions in theorem 2.2 in the equivalent Hilbert space H_e since by hypotheses all the assumptions on A are satisfied as is shown in theorem 5.1. By hypothesis, $\underset{\sim}{f}=\begin{pmatrix}0\\ f\end{pmatrix}$ is defined on all of $H=H_o^1(\Omega)\times L^2(\Omega)$ into H. By the relation (5-1) for any $\underset{\sim}{u}=\begin{pmatrix}u\\ u'\end{pmatrix}$, $\underset{\sim}{v}=\begin{pmatrix}v\\ v'\end{pmatrix}$ in H, we have

$$||\underset{\sim}{f}(\underset{\sim}{u})-\underset{\sim}{f}(\underset{\sim}{v})||_e \leqq k_1^{1/2}\ ||\underset{\sim}{f}(\underset{\sim}{u})-\underset{\sim}{f}(\underset{\sim}{v})||_H=k_1^{1/2}||f(x,u,u_{x_i},u')-f(x,v,v_{x_i},v')||_o$$

$$\leqq k_1^{1/2}\ k(||u-v||_1^2 + ||u'-v'||_o^2)^{1/2} \leqq (k_1/k_2)^{1/2}k||\underset{\sim}{u}-\underset{\sim}{v}||_e \qquad (5\text{-}2)$$

where we have used the equivalence relation (4-7). Hence if $(k_1/k_2)^{1/2}\ k\leqq\beta$, that is, $k \leqq (k_2/k_1)^{1/2}\beta$, all the results stated in the theorem follows directly from theorem 2.2. In case $k > (k_2/k_1)^{1/2}\beta$, the existence and uniqueness of a solution in H_e satisfying the properties in the theorem follow from [8] due to Browder. It has been shown in [1] that if u(t,x) is a weak solution in an equivalent Hilbert space H_e, it is also a weak solution in the original Hilbert space H. Therefore the existence and uniqueness of a weak solution is proved for any finite value of k which completes the proof of the theorem.

Corollary. Let the operator L in (1-1) be the same as in theorem 5.1. If $f(x,u,u_{x_i}, u_t) = f(x)$ is in $L^2(\Omega)$ then all the results in theorem 3.2 hold.

The condition (5-1) implies that f is Lipschitz continuous on H. Conversely, if f is continuous on H, not necessarily Lipschitz continuous, we can weaken the condition (5-1) to some extent as in the following theorem which can be proved in a straight forward way.

Theorem 5.3. Let the operator L in (1-1) be the same as in theorem 5.1. If f is defined and continuous on H such that condition (5-1) holds for any dense subset D of H (e.g., $D=C_o^\infty(\Omega)xC_o^\infty(\Omega)$), then all the results in theorem 5.2 are valid.

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