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THE INTERACTION OF AN OBLIQUELY INCIDENT p-POLARIZED PLANE ELECTROMAGNETIC WAVE WITH A HOT PLASMA HALF-SPACE AND PLASMA SLAB

by
Paul E. Bolduc
September 15, 1969

IONOSPHERE RESEARCH LABORATORY

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ABSTRACT

In the foregoing, a relativistic kinetic theory description of the interaction of a plane p-polarized electromagnetic wave obliquely incident on a hot plasma half space and plasma slab is developed. The Laplace transform technique together with the radiation condition and the condition of specular reflection of the electrons at the interface is used to obtain unique linearized solutions for the fields and the particle distribution in a plasma half space. The above procedure is modified to treat the case of the plasma slab. A proof of self-consistency is presented in the treatment of both the slab and the half-space problems. The elementary non-equilibrium thermodynamics of the interaction is discussed in the case of the plasma half space. Power reflection and transmission coefficients are obtained for both problems.

The above theory is then applied to several problems in the limit of large $\beta = \frac{m_o c^2}{KT}$. The critical angle of incidence is obtained and the depths of penetration are computed for the half space. The equivalent Fourier series representation is presented for the slab case with no limitations placed on β. A cursory study of the geometrical resonances in the large β limit is also presented.

I. INTRODUCTION

The kinetic theory description of the interaction of a plane electromagnetic wave with semi-infinite and slab plasmas has been a subject of considerable interest in recent years. In a pair of important papers, Silin,[1] and Silin and Fetisov[2] investigated the reflection and transmission properties of a plasma half space for both perpendicular and oblique incidence using a relativistic treatment to describe the plasma. Both "s" and "p" polarizations* were considered. While the penetration problem was examined in detail, essentially no derivation was given for the field quantities in the medium. Taylor,[3] and later, Comstock,[4] using different mathematical techniques, supplied some of the details on the derivation of electric and magnetic fields within the plasma for the case of normal incidence. Discrepancies between Taylor's and Comstock's results were later resolved by Taylor.[5] Shure[6] and Felderhof[7] also studied the half space problem using a non-relativistic normal mode approach. Felderhof, however, does indicate how to treat the relativistic case for the normal incidence half space problem. Weston[8] extended Felderhof's non-relativistic analysis to the case of oblique incidence of a

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*A wave whose electric field vector is perpendicular to the plane of incidence is said to be s-polarized whereas a p-polarized wave has its electric field vector parallel to the plane of incidence.

p-polarized wave. Weibel[9] computed the analomous skin depth[**] for the case of normal incidence on a non-relativistic plasma half space. The definition of the skin depth which he uses is somewhat different from that used by Silin and the nonrelativistic limit of Silin's results are not in exact agreement with those of Weibel.

Reflection and transmission of a plane electromagnetic wave at the boundaries of a plasma slab has also received considerable attention. While Taylor's relativistic treatment was concerned with normal incidence Kondratenko and Miroshnichenko,[10,11] and Bowman and Weston[12] considered the case of oblique incidence for the non-relativistic problem. The former considered both s- and p-polarizations using a Fourier series expansion while the latter, using a normal mode analysis, considered p-polarization only. Hinton[13] studied the collisionless absorption and emission of an obliquely incident p-polarized wave incident on both sides of a non-relativistic plasma layer.

Ozizmir[14] has recently investigated the oblique incidence of an s-polarized wave on a plasma half space and slab using relativistic kinetic theory. The relativistic treatment is desirable for several reasons. First, it eliminates non-physical results such as the Landau damping of transverse waves. Such damping cannot occur since the phase velocity of the transverse wave is greater than the speed of light and

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[**]The term "anomolous skin depth" is used to characterize non-collisional absorption of electromagnetic waves incident on a medium when the wave's depth of penetration is much smaller than the average distance covered by an electron near the surface during one period of field oscillation.

no particles can be in resonance with this wave. Also, it gives a correct basis for obtaining temperature corrections to cold plasma theory.

Ozizmir's analysis, which is different from that of previous authors, is based on the Laplace transform technique. Assuming specular reflection of the particles at the interface and imposing the "radiation condition" on all field solutions, he determined uniquely the stationary solution to the coupled Maxwell-Vlasov equations. The solution for the slab plasma was obtained by modifying the techniques used in the half space problem.

The present analysis is an extension of Ozizmir's work to the case where the incident electric field lies in the plane of incidence (p-polarization) as shown in Figure 1. In contrast to the s-polarization case, where only transverse waves are set up in the plasma, both longitudinal and transverse waves are found in the medium.

Assuming that the particles are reflected specularly at the interface(s), we obtain rigorous first order solutions for all field quantities valid at arbitrary angles of incidence for both the plasma half space and slab problems. In Section 2, we state the basic equations adopted for the description of the half space problem and obtain explicit expressions for $E_x(x)$, $E_z(x)$, and $B_y(x)$. Making use of these solutions, we determine uniquely the perturbed particle distribution and give a proof of self consistency. Expressions for the reflection and transmission coefficients are also obtained, and the thermodynamics of the interaction is investigated. Section 3 is devoted

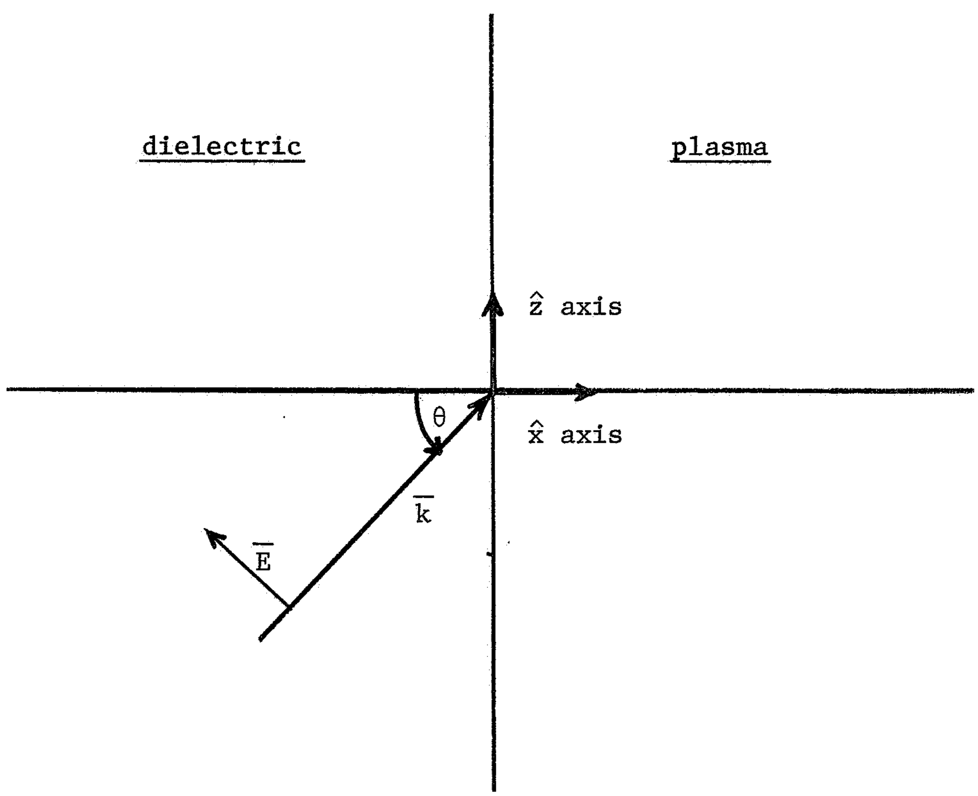


Figure 1. Wave Plasma Interaction

to modifying the above approach to treat the slab case. Expressions for $E_x(x)$, $E_z(x)$, and $B_y(x)$ are given. We obtain the perturbed particle distribution, give a proof of self consistency, and calculate the reflection and transmission coefficients. The approximate evaluation of complicated integral expressions and the physical interpretation of the results are left to Section 4. In the case of the plasma half space, we obtain an expression for the critical angle of incidence and calculate the complex depths of penetration. The concepts of weak and strong spatial dispersion are discussed in some detail. We obtain the equivalent Fourier series solutions for $E_x(x)$, $E_z(x)$, and $B_y(x)$ in the slab geometry and investigate the geometrical resonances. Finally, the Appendices contain all derivations and proofs too lengthy to be included in the body of the text.

2. PLASMA HALF SPACE PROBLEM

2.1 Basic Equations

The interaction of a low intensity plane electromagnetic wave obliquely incident on hot semi-infinite and slab plasmas can be described by the linearized relativistic Vlasov equation coupled to the Maxwell equations. The use of linearized theory is based on the assumption that the energy density of the incident wave is much smaller than the internal energy density of the plasma. The incident wave perturbs the quiescent plasma and sets up electric and magnetic fields within the ionized medium. These field quantities, as well as the reflected fields, can be obtained by using the concepts of self-consistent field theory. In the description of such a boundary value problem, we assume the wave-plasma system has attained a new equilibrium--a quasi-equilibrium since the incident wave heats the plasma, a second order effect. All temporal transient fields are assumed negligibly small. In effect, we are describing the asymptotic time limit of the considerably more difficult "mixed initial value-boundary value" problem.

The ions are assumed to form a uniform neutralizing background. Basically, this assumes that the frequency ω of the incident wave is much greater than the ion plasma frequency ω_{pi} . Ionic effects can be included in a straightforward manner by writing a second relativistic Vlasov equation describing ions and including ionic contributions in the charge and current densities. Finally, we assum the plasma to be in a hot tenuous state, thereby permitting us to neglect collisional effects. For most of the analysis, this is equivalent to assuming that $\omega >> \nu$, where ν is the collision frequency. However, for the discussion of anomalous absorption, a stronger condition is needed. This will be discussed further in Section 4.

Letting $f(x,z,\bar{u},t)$ be the perturbed electron distribution function, the linearized set of equations describing the system can be written as follows:

$$\gamma \frac{\partial f}{\partial t} + u_x \frac{\partial f}{\partial x} + u_z \frac{\partial f}{\partial z} - \frac{n_o|e|}{m_o} \gamma \left[E_x \frac{\partial F_o}{\partial u_x} + E_z \frac{\partial F_o}{\partial u_z} \right] = 0, \quad (1)$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = 4\pi\rho(x,z,t) \quad , \quad (2a)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{1}{c}\frac{\partial B_y}{\partial t} \quad , \quad (2b)$$

$$-\frac{\partial B_y}{\partial z} = \frac{1}{c}\frac{\partial E_x}{\partial t} + \frac{4\pi}{c} j_x \quad , \quad (2c)$$

and

$$\frac{\partial B_y}{\partial x} = \frac{1}{c}\frac{\partial E_z}{\partial t} + \frac{4\pi}{c} j_z \quad , \quad (2d)$$

where $-|e|$ and m_o are the charge and rest mass of an electron, n_o is the electron number density in the unperturbed state, $\bar{u} = \gamma\bar{v}$ where $\bar{v}$ is the electron velocity, and

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \sqrt{1 + u^2/c^2}$$

The perturbed charge and current densities are given by:

$$\rho \equiv -|e| \int f(x,z,\bar{u},t)du^3 \quad , \tag{2e}$$

and

$$\bar{j} \equiv -|e| \int \frac{\bar{u}}{\gamma} f(x,z,\bar{u},t)du^3 \quad . \tag{2f}$$

In writing Equations (1) and (2), we made use of the symmetry of the interaction; i.e., $E_y(x) = B_x(x) = B_z(x) = 0$ everywhere. All quantities are independent of y variations.

The equilibrium state F_o is assumed to be described by the relativistic Maxwell-Boltzmann (Jüttner) distribution; namely,

$$F_o \equiv \frac{\beta e^{-\beta\gamma}}{4\pi c^3 K_2(\beta)} \quad ,$$

where $\beta = \frac{m_o c^2}{KT}$ and $K_2(\beta)$ is the modified Bessel function of the second kind and of order "2".

It is convenient in the following to first analyze the plasma half space $x > 0$ and then modify the basic approach to treat the case of the slab. In order to completely specify the half space problem, we impose a specular reflection boundary condition on $f(0,\bar{u})$; i.e.,

1) $f(0,u_x) = f(o,-u_x)$

and require that

2) $E_x(x)$ and $E_z(x)$ be bounded as $x \to \infty$

and

3) $E_x(x)$ and $E_z(x)$ consist only of waves traveling in the $+x$ direction.

The latter two conditions are usually called the "radiation condition."

We look for solutions of the form $\exp i(k_z z - \omega t)$, where ω and k_z are real positive quantities. All temporal transient fields are assumed negligibly small. In effect, we are describing the asymptotic time limit of the considerably more difficult "mixed initial value-boundary value" problem. Taking the Laplace transform on the x variable, the transformed equations become:

$$i(k_x u_x + k_z u_z - \omega\gamma)\tilde{f} = u_x f(o) + \frac{|e| n_o \gamma}{m_o} \left[\tilde{E}_x \frac{\partial F_o}{\partial u_x} + \tilde{E}_z \frac{\partial F}{\partial u_z} \right] = 0, \tag{3}$$

$$ik_z \tilde{E}_x + E_z(o) - ik_x \tilde{E}_z = \frac{i\omega}{c} \tilde{B}_y \quad , \tag{4}$$

$$- ik_z \tilde{B}_y = - \frac{i\omega}{c} \tilde{E}_x + \frac{4\pi}{c} \tilde{j}_x \quad , \tag{5}$$

$$- B_y(o) + ik_x \tilde{B}_y = - \frac{i\omega}{c} \tilde{E}_z + \frac{4\pi}{c} \tilde{j}_z \quad , \tag{6}$$

where the Laplace transform variable is ik_x with $\mathrm{Im}\, k_x < 0$.

Using Equation (4) to eliminate $\tilde{B}_y(k_x)$ and Equations (2f) and (3) to eliminate the transformed current densities $\tilde{j}_x$ and $\tilde{j}_z$, we obtain:

$$\left[k_x k_z + \frac{\omega\, \omega_p^2}{c^2} \int \frac{u_z \frac{\partial F_o}{\partial u_x} du^3}{(\omega\gamma - \overline{k}\cdot\overline{u})} \right] \tilde{E}_x + \left[\frac{\omega^2}{c^2} - k_x^2 + \frac{\omega\omega_p^2}{c^2} \int \frac{u_z \frac{\partial F_o}{\partial u_z} du^3}{(\omega\gamma - \overline{k}\cdot\overline{u})} \right] \tilde{E}_z =$$

$$= ik_x E_z(o) - \frac{i\omega}{c} B_y(o) - \frac{4\pi|e|\omega}{c^2} \int \frac{u_x u_z f(o) du^3}{\gamma(\omega\gamma - \overline{k}\cdot\overline{u})} \quad , \tag{7}$$

and

$$\left[k_z^2 - \frac{\omega^2}{c^2} - \frac{\omega\omega_p^2}{c^2}\int \frac{u_x \frac{\partial F_o}{\partial u_x} du^3}{(\omega\gamma - \overline{k}\cdot\overline{u})}\right]\tilde{E}_x - \left[k_x k_z + \frac{\omega\omega_p^2}{c^2}\int \frac{u_x \frac{\partial F_o}{\partial u_z} du^3}{(\omega\gamma - \overline{k}\cdot\overline{u})}\right]\tilde{E}_z =$$

$$= ik_z E_z(o) + \frac{4\pi\omega|e|}{c^2}\int \frac{u_x^{\,2} f(o) du^3}{(\gamma\omega - \overline{k}\cdot\overline{u})} \quad , \qquad (8)$$

where $\omega_p^{\,2} \equiv \frac{4\pi n_o |e|^2}{m_o}$ and $\overline{k}\cdot\overline{u} = k_x u_x + k_z u_z$ with k_x complex.

It is now possible to obtain algebraic expressions for $\tilde{E}_x(k_x)$ and $\tilde{E}_z(k_x)$. However, it is very difficult to locate the zeros of the denominators of these expressions and thereby obtain the pole contributions in the inverse Laplace transform integrations. To overcome this difficulty, we adopt an approach outlined in Ozizmir's work and introduce a new coordinate system.

The set of complex base vectors: $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ defined by

$$\hat{e}_1 \equiv \frac{\overline{k}}{k} = \frac{k_x\hat{x} + k_z\hat{z}}{\sqrt{k_x^{\,2} + k_z^{\,2}}} \quad ,$$

$$\hat{e}_2 \equiv \hat{y}$$

and

$$\hat{e}_3 \equiv \hat{e}_1 \times \hat{e}_2 = \frac{k_x\hat{z} - k_z\hat{x}}{\sqrt{k_x^{\,2} + k_z^{\,2}}}$$

defines a complex orthonormal basis. The transformation is a "complex rotation" about the y axis. A 3-vector $\bar{A}$ can be described in this new system and its components related to the original Cartesian system as follows:

$$\bar{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3 \quad ,$$

where

$$A_x = \frac{k_x A_1 - k_z A_3}{k} \qquad A_1 = \frac{k_x A_x + k_z A_z}{k}$$

$$A_y = A_2 \qquad A_2 = A_y$$

$$A_z = \frac{k_z A_1 + k_x A_3}{k} \qquad A_3 = \frac{k_x A_z - k_z A_x}{k}$$

and

$$k \equiv \sqrt{k_x^{\,2} + k_z^{\,2}}$$

It may be readily shown that

$$\hat{\varepsilon}_\alpha \cdot \hat{\varepsilon}_\beta = \delta_{\alpha\beta}$$

and that

$$\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$$

$$\hat{e}_2 \times \hat{e}_3 = \hat{e}_1$$

$$\hat{e}_3 \times \hat{e}_1 = \hat{e}_2 \quad ,$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. As in the case of all rotational transformations, lengths are preserved, i.e., $\bar{A} \cdot \bar{B} = \bar{A}' \cdot \bar{B}'$.

Writing Equations (7) and (8) in this new coordinate system and making use of

$$u_1 \frac{\partial F_o}{\partial u_3} = u_3 \frac{\partial F_o}{\partial u_1}$$

valid for any isotropic velocity distribution, it can be shown that

$$\left[k_z \omega^2 + \omega\omega_p{}^2 \int \frac{(k_z u_1 + k_x u_3)(\partial F_o/\partial u_1) du^3}{(\omega\gamma - ku_1)} \right] \frac{\tilde{E}_1}{kc^2} +$$

$$\left[k_x(\omega^2 - k^2c^2) + \omega\omega_p{}^2 \int \frac{(k_z u_1 + k_x u_3)(\partial F_o/\partial u_3) du^3}{(\omega\gamma - ku_1)} \right] \frac{\tilde{E}_3}{kc^2} =$$

$$= ik_x E_z(o) - \frac{i\omega}{c} B_y(o) - \frac{4\pi|e|\omega}{c^2} \int \frac{u_x u_z f(o) du^3}{\gamma(\gamma\omega - \overline{k}\cdot\overline{u})} \qquad (9)$$

and

$$\left[k_x \omega^2 + \omega\omega_p{}^2 \int \frac{(k_x u_1 - k_z u_3)(\partial F_o/\partial u_1) du^3}{(\gamma\omega - ku_1)} \right] \frac{\tilde{E}_1}{kc^2} +$$

$$\left[k_z(k^2c^2 - \omega^2) + \omega\omega_p{}^2 \int \frac{(k_x u_1 - k_z u_3)(\partial F_o/\partial u_3) du^3}{(\gamma\omega - ku_1)} \right] \frac{\tilde{E}_3}{kc^2} =$$

$$= - ik_z E_z(o) - \frac{4\pi|e|\omega}{c^2} \int \frac{u_x{}^2 f(o) du^3}{\gamma(\gamma\omega - \overline{k}\cdot\overline{u})} , \qquad (10)$$

where the fact that the Jacobian of the transformation is +1 was also used.

The attractiveness of this approach rests on the fact that

$$\int \frac{u_3(\partial F_o/\partial u_1)du^3}{(\gamma\omega - ku_1)} = \int \frac{u_1(\partial F_o/\partial u_3)du^3}{(\gamma\omega - ku_1)} = 0 \quad , \tag{11}$$

which is proved in Appendix A.

When Equation (11) is used in Equations (9) and (10), we obtain

$$k_z\omega\Lambda_L\tilde{E}_1 - k_x\Lambda_T\tilde{E}_3 = kc^2\left[ik_zE_z(o) - \frac{i\omega}{c}B_y(o) - \frac{4\pi|e|\omega}{c^2}\int\frac{u_xu_zf'(o)du^3}{\gamma(\omega\gamma - \overline{k}\cdot\overline{u})}\right] \tag{12}$$

and

$$k_x\omega\Lambda_L\tilde{E}_1 + k_z\Lambda_T\tilde{E}_3 = -kc^2\left[ik_zE_z(o) + \frac{4\pi|e|\omega}{c^2}\int\frac{u_x{}^2f(o)du^3}{\gamma(\omega\gamma - \overline{k}\cdot\overline{u})}\right] \quad , \tag{13}$$

where $\Lambda_L(k_x)$ and $\Lambda_T(k_x)$ are the longitudinal and transverse dispersion functions defined as:

$$\Lambda_L(k_x) \equiv \omega + \omega_p{}^2\int\frac{u_1(\partial F_o/\partial u_1)du^3}{(\gamma\omega - ku_1)} \tag{14}$$

and

$$\Lambda_T(k_x) = (k_x{}^2 + k_z{}^2)c^2 - \omega^2 - \omega\omega_p{}^2\int\frac{u_3(\partial F_o/\partial u_3)du^3}{(\gamma\omega - ku_1)} \quad . \tag{15}$$

Solving Equations (12) and (13) for $\tilde{E}_1$ and $\tilde{E}_3$ and transforming back to the original Cartesian coordinate system, we now obtain:

$$\tilde{E}_x(k_x) = \frac{1}{(k_x^2 + k_z^2)} \left[k_x \frac{G_L(k_x)}{\Lambda_L(k_x)} - k_z \frac{G_T(k_x)}{\Lambda_T(k_x)} \right] \tag{16}$$

and

$$\tilde{E}_z(k_x) = \frac{1}{(k_x^2 + k_z^2)} \left[k_z \frac{G_L(k_x)}{\Lambda_L(k_x)} + k_x \frac{G_T(k_x)}{\Lambda_T(k_x)} \right] , \tag{17}$$

where

$$G_L(k_x) \equiv - \left[ik_z cB_y(o) + 4\pi|e| \int \frac{u_x(k_x u_x + k_z u_z) f(o) du^3}{\gamma(\gamma\omega - \overline{k} \cdot \overline{u})} \right] , \tag{18}$$

and

$$G_T(k_x) = - i(k_x^2 + k_z^2)c^2E_z(o)$$

$$+ \omega \left[ik_x cB_y(o) + 4\pi|e| \int \frac{u_x(k_x u_z - k_z u_x) f(o) du^3}{\gamma(\gamma\omega - \overline{k} \cdot \overline{u})} \right] , \tag{19}$$

$$\Lambda_L(k_x) = \omega + \frac{\omega_p^2}{(k_x^2 + k_z^2)} \int \frac{(k_x u_x + k_z u_z)(k_x \frac{\partial F_o}{\partial u_x} + k_z \frac{\partial F_o}{\partial u_z}) du^3}{(\gamma\omega - k_x u_x - k_z u_z)} \quad (20)$$

and

$$\Lambda_T(k_x) = (k_x^2 + k_z^2)c^2 - \omega^2 - \frac{\omega\omega_p^2}{(k_x^2 + k_z^2)} \int \frac{(k_x u_z - k_z u_x)(k_x \frac{\partial F_o}{\partial u_z} - k_z \frac{\partial F_o}{\partial u_x}) du^3}{(\gamma\omega - k_x u_x - k_z u_z)} . \quad (21)$$

2.2 Field Components Inside the Plasma Half Space

As functions of position x inside the plasma, the electric field components may be written as:

$$E_x(x) = \frac{1}{2\pi} \int_C \tilde{E}_x(k_x) e^{ik_x x} dk_x \quad (22)$$

and

$$E_z(x) = \frac{1}{2\pi} \int_C \tilde{E}_z(k_x) e^{ik_x x} dk_x \quad , \quad (23)$$

where the contour "C" lies in the k_x plane parallel to the real k_x axis and below all singularities of $\tilde{E}_x(k_x)$ and $\tilde{E}_z(k_x)$ as shown in Figure 2. These expressions are deceptively simple "in appearance." Since $\tilde{E}_x(k_x)$ and $\tilde{E}_z(k_x)$ are linearly related to $G_L(k_x)$ and $G_T(k_x)$ and these quantities, related to $B_y(o)$, $E_z(o)$, and an integral over $f(o,\bar{u})$, our answers remain couched in the form of integral equations.

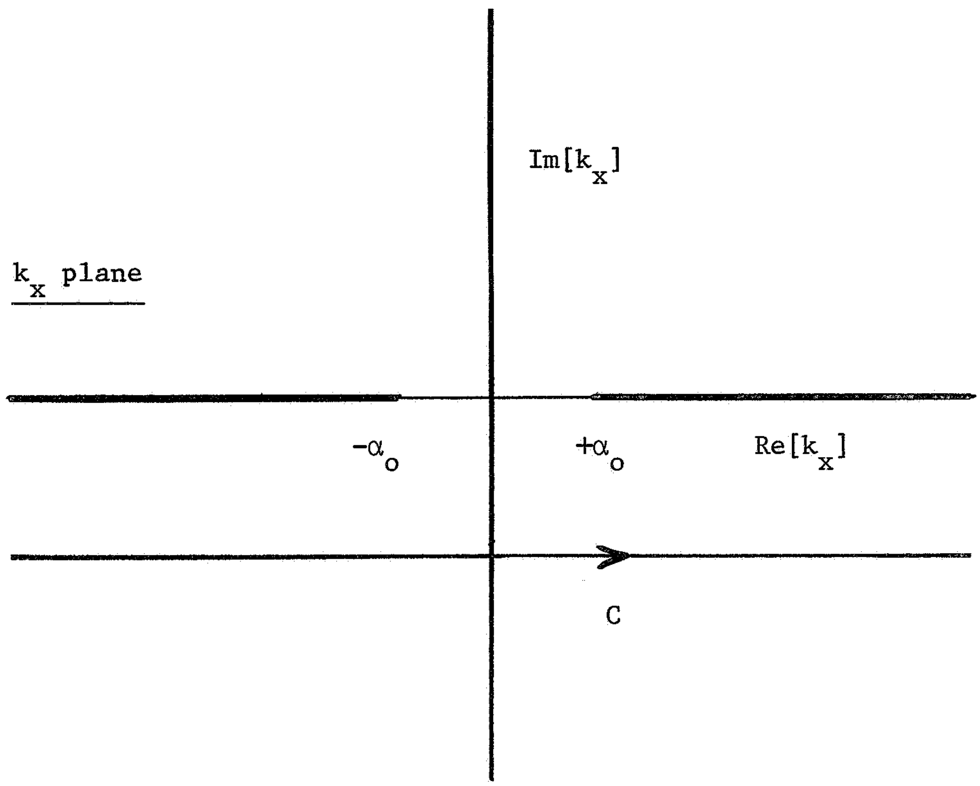


Figure 2. Inverse Contour C in Complex k_x Plane

We overcome this difficulty by making use of the specular reflection boundary condition on $f(o,\bar{u})$ and imposing the "radiation condition" on all field quantities. The application of these boundary conditions imposes strict mathematical conditions on $G_L(k_x)$ and $G_T(k_x)$ and enables us to obtain solutions without having to directly solve the set of integral equations.

The longitudinal and transverse dispersion functions are analyzed in Appendix B. We show that $\Lambda_T(k_x)$ always has two roots, κ_T and $-\kappa_T$, whereas $\Lambda_L(k_x)$ may have two or no roots. Longitudinal roots, when they exist, are designated by κ_L and $-\kappa_L$. In both cases, the roots are either real or pure imaginary. Real roots are always on the open interval $(-\alpha_o,\alpha_o)$, where

$$\alpha_o \equiv \frac{\omega}{c} \cos \theta$$

and θ is the angle of incidence. Longitudinal roots exist only when

$$\frac{\omega^2}{\omega_p^2} < \frac{1}{K_2(\beta)} \left[K_1(\beta) + \frac{2K_0(\beta)}{\beta} \right]$$

which is seen to be independent of the angle of incidence.

When κ_T is real, κ_L is necessarily also real. When κ_L is pure imaginary, κ_T is necessarily also pure imaginary. When κ_T is imaginary, κ_L may be real or pure imaginary depending on θ, β, and ω^2/ω_p^2. The exact conditions are given in Appendix B. Finally, we note that $\Lambda_L(k_x)$ and $\Lambda_T(k_x)$ are analytic everywhere except for a cut which lies along the part of the real k_x axis given by $|k_x| > \alpha_o$. (See Figure 2)

We satisfy part of the radiation condition by requiring that

$$G_L(-\kappa_L) = 0 \tag{24}$$

and

$$G_T(-\kappa_T) = 0 \tag{25}$$

A second part of the radiation condition is satisfied by imposing the condition that:

$$\tilde{E}_x^{\,+}(-k_1) = \tilde{E}_x^{\,-}(-k_1) \tag{26}$$

and

$$\tilde{E}_z^{\,+}(-k_1) = \tilde{E}_z^{\,-}(-k_1) \quad , \tag{27}$$

where $k_1 \equiv \mathrm{Re}\{k_x\} > \alpha_o$ and

$$\tilde{E}_x^{\,\pm}(-k_1) \equiv \lim_{\varepsilon \to 0} \tilde{E}_x(-k_1 \pm i\varepsilon) \tag{28}$$

The functions $\tilde{E}_z^{\,+}(-k_1)$ and $\tilde{E}_z^{\,-}(-k_1)$ are similarly defined.

Equation (24) states that:

$$0 = -\left[ik_z cB_y(o) + 4\pi|e| \int \frac{u_x(-\kappa_L u_x + k_z u_z) f(o) du^3}{\gamma(\gamma\omega + \kappa_L u_x - k_z u_z)} \right] . \tag{29}$$

Changing $u_x \to -u_x$ and using the condition of specular reflection, we obtain:

$$0 = -\left[ik_z cB_y(o) - 4\pi|e| \int \frac{u_x(\kappa_L u_x + {}_z u_z) f(o) du^3}{\gamma(\gamma\omega - \kappa_L u_x - k_z u_z)} \right] . \tag{30}$$

Therefore,

$$G_L(+\kappa_L) = -2ik_z cB_y(o) \tag{31}$$

Similarly, Equation (25) implies that

$$G_T(+\kappa_T) = 2i\omega c\kappa_T B_y(o) \tag{32}$$

The "cut conditions" given by Equations (26) and (27) require that

$$\frac{G_L^{+}(-k_1)}{\Lambda_L^{+}(-k_1)} = \frac{G_L^{-}(-k_1)}{\Lambda_L^{-}(-k_1)} \tag{33}$$

and

$$\frac{G_T^{+}(-k_1)}{\Lambda_T^{+}(-k_1)} = \frac{G_T^{-}(-k_1)}{\Lambda_T^{-}(-k_1)} \tag{34}$$

The last two equations state that the functions $\frac{G_L}{\Lambda_L}$ and $\frac{G_T}{\Lambda_T}$ are continuous across that portion of the cuts of $\Lambda_L(k_x)$ and $\Lambda_T(k_x)$ corresponding to $-k_1 < -\alpha_o$, where $k_1 > 0$. This places stringent restrictions on the functions $G_L(k_x)$ and $G_T(k_x)$. To illustrate these restrictions, we consider the implications of Equation (33). By definition,

$$G_L^{+}(-k_1) \equiv -ik_z cB_y(o) - 4\pi|e| \lim_{\varepsilon\to 0} \int \frac{u_x(-k_1u_x + k_zu_z)f(o)du^3}{\gamma[\gamma\omega - (-k_1 + i\varepsilon)u_x - k_zu_z]}. \tag{35}$$

Letting $u_x \to -u_x$ and using the condition of specular reflection, we obtain:

$$G_L^+(-k_1) = -ik_z cB_y(o) + 4\pi|e| \lim_{\varepsilon\to 0} \int \frac{u_x(k_1 u_x + k_z u_z) f(o) du^3}{\gamma[\gamma\omega - (k_1 - i\varepsilon)u_x - k_z u_z]} \ . \tag{36}$$

Eliminating the integral term in Equation (36) by using the definition of $G_L^-(+k_1)$, we readily find that

$$G_L^+(-k_1) + G_L^-(k_1) = -2ik_z cB_y(o) \tag{37}$$

Similarly, we obtain:

$$G_L^+(k_1) + G_L^-(k_1) = -2ik_z cB_y(o) \tag{38}$$

Using the relations

$$\Lambda_L^+(k_1) = \Lambda_L^-(-k_1) \tag{39}$$

and

$$\Lambda_L^-(k_1) = \Lambda_L^+(-k_1) \tag{40}$$

which are shown in Appendix B and Equations (37) and (38), it can now be shown that

$$\left[\frac{G_L^+(k_1)}{\Lambda_L^+(k_1)} - \frac{G_L^-(k_1)}{\Lambda_L^-(k_1)}\right] = -2ik_z cB_y(o) \left[\frac{1}{\Lambda_L^+(k_1)} - \frac{1}{\Lambda_L^-(k_1)}\right] , \tag{41}$$

where

$$k_1 \geq \alpha_o \equiv \frac{\omega}{c} \cos\theta \quad ; \quad 0 \leq \theta \leq \frac{\pi}{2}$$

In a completely analogous fashion, Equation (34) implies that

$$\left[\frac{G_T^{+}(k_1)}{\Lambda_T^{+}(k_1)} - \frac{G_T^{-}(k_1)}{\Lambda_T^{-}(k_1)}\right] = 2ik_1 c\omega B_y(o)\left[\frac{1}{\Lambda_T^{+}(k_1)} - \frac{1}{\Lambda_T^{-}(k_1)}\right] \tag{42}$$

We may now obtain $E_x(x)$ and $E_z(x)$ by deforming the original contour "C" as shown in Figure 3. The electric field components $E_x(x)$ and $E_z(x)$ are each composed of a longitudinal and a transverse electric field.

To obtain the "longitudinal" electric field contribution to $E_x(x)$ and $E_z(x)$, we deform "C" as shown in Figure 3. We obtain the corresponding "transverse" electric field contributions to $E_x(x)$ and $E_z(x)$ by deforming "C" as shown in Figure 3 and substituting κ_T for κ_L. For illustrative purposes, the longitudinal root was shown as pure imaginary.

In the limit as $R \to \infty$, we obtain from Equation (22):

$$E_x(x) = 2k_z cB_y(o)\left[\frac{\kappa_L e^{i\kappa_L x}}{(\kappa_L^2 + k_z^2)\Lambda_L'(\kappa_L)} + \frac{\omega\kappa_T e^{i\kappa_T x}}{(\kappa_T^2 + k_z^2)\Lambda_T'(\kappa_T)}\right]$$

$$+ \frac{2ik_z cB_y(o)}{2\pi}\int_{\alpha_o}^{\infty}\frac{k_1}{(k_1^2 + k_z^2)}\left[\frac{1}{\Lambda_L^{+}(k_1)} - \frac{1}{\Lambda_L^{-}(k_1)}\right]e^{ik_1 x}dk_1$$

$$+ \frac{2ik_z c\omega B_y(o)}{2\pi}\int_{\alpha_o}^{\infty}\frac{k_1}{(k_1^2+k_z^2)}\left[\frac{1}{\Lambda_T^{+}(k_1)} - \frac{1}{\Lambda_T^{-}(k_1)}\right]e^{ik_1 x}dk_1. \tag{43}$$

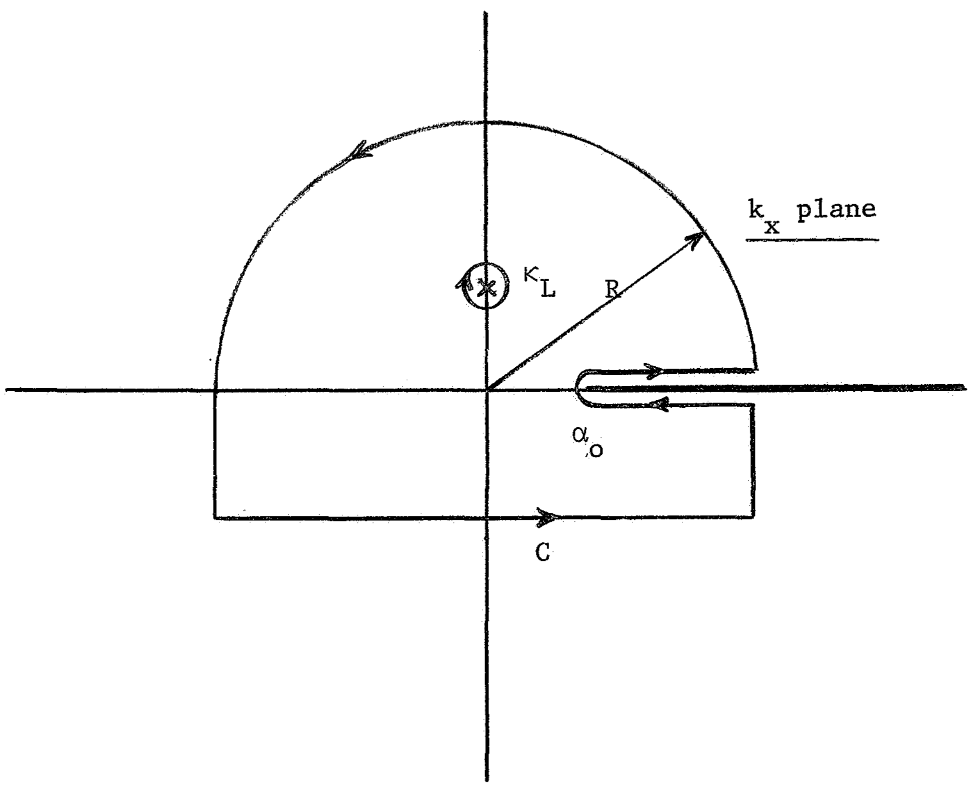


Figure 3. Deformation of the Contour C After Application of the Radiation Condition and the Boundary Condition

where Equations (31), (32), (41), and (42) were used and

$$\Lambda_L'(\kappa_L) \equiv \left.\frac{d\Lambda_L}{dk_x}\right|_{k_x = \kappa_L} \tag{44}$$

and

$$\Lambda_T'(\kappa_T) \equiv \left.\frac{d\Lambda_T}{dk_x}\right|_{k_x = \kappa_T} \tag{45}$$

As shown in Figure 3, $\kappa_L = i|\kappa_L|$.

In obtaining Equation (43), we made use of the fact that

$$\lim_{k_x \to \pm ik_z} (k_x \mp ik_z)\, \tilde{E}_x(k_x) = 0$$

This result, which is also mentioned by Silin and Fetisov,[2] can easily be shown by making use of Equations (B-22) and (B-23) and in no way depends on the assumed boundary condition on $f(o,\bar{u})$.

The z component of the electric field can be obtained in a completely analogous fashion. The results are:

$$E_z(x) = 2cB_y(o)\left[\frac{k_z^2 e^{i\kappa_L x}}{(\kappa_L^2 + k_z^2)\Lambda_L'(\kappa_L)} - \frac{\omega\kappa_T^2 e^{i\kappa_T x}}{(\kappa_T^2 + k_z^2)\Lambda_T'(\kappa_T)}\right]$$

$$+ \frac{2icB_y(o)}{2\pi}\int_{\alpha_o}^{\infty}\frac{k_z^2}{(k_1^2 + k_z^2)}\left[\frac{1}{\Lambda_L^+(k_1)} - \frac{1}{\Lambda_L^-(k_1)}\right]e^{ik_1 x}dk_1$$

$$- \frac{2i\omega cB_y(o)}{2\pi}\int_{\alpha_o}^{\infty}\frac{k_1^2}{(k_1^2 + k_z^2)}\left[\frac{1}{\Lambda_T^+(k_1)} - \frac{1}{\Lambda_T^-(k_1)}\right]e^{ik_1 x}dk_1, \tag{46}$$

where we made use of the fact that the residues of $\tilde{E}_z(k_x)$ at $k_x = \pm ik_z$ are also zero.

In order to obtain Equations (43) and (46), we tacitly assumed that $G_L(k_x)$ and $G_T(k_x)$ are analytic everywhere except for cuts on the real axis defined by $|k_x| > \alpha_o$. These assumptions are justified a posteriori.

It is convenient in what follows to express our solutions for $E_x(x)$ and $E_z(x)$ in a more compact form. Equations (43) and (46) can be rewritten as:

$$E_x(x) = \frac{2k_z cB_y(o)}{2\pi i} \int_{C_L} \frac{k_x e^{ik_x x} dk_x}{(k_x^2+k_z^2)\Lambda_L(k_x)} + \frac{2k_z c\omega B_y(o)}{2\pi i} \int_{C_T} \frac{k_x e^{ik_x x} dk_x}{(k_x^2+k_z^2)\Lambda_T(k_x)} \tag{47}$$

and

$$E_z(x) = \frac{2k_z^2 cB_y(o)}{2\pi i} \int_{C_L} \frac{e^{ik_x x} dk_x}{(k_x^2+k_z^2)\Lambda_L(k_x)} - \frac{2\omega cB_y(o)}{2\pi i} \int_{C_T} \frac{k_x^2 e^{ik_x x} dk_x}{(k_x^2+k_z^2)\Lambda_T(k_x)}, \tag{48}$$

where C_L is the sum of the contours shown in Figure 4. The C_T contour is similar to that shown in Figure 4 with κ_T replacing κ_L . By imposing the radiation condition and making use of specular reflection, we have completely eliminated the difficulty of having our results expressed in terms of integral equations.

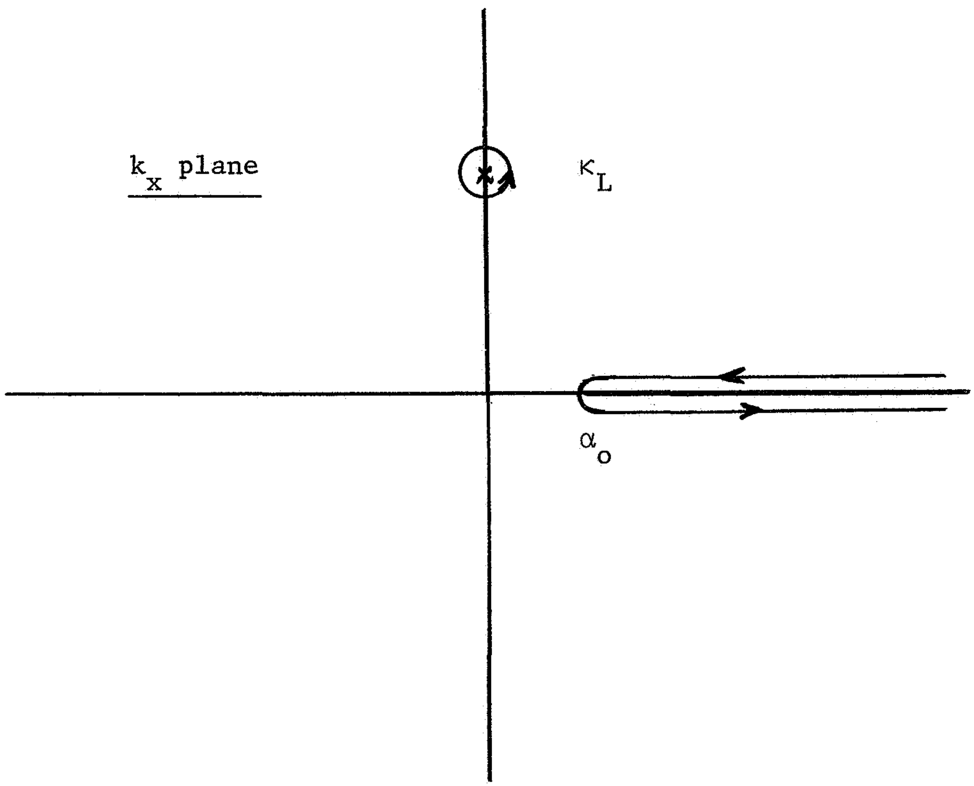


Figure 4. The C_L Contour - A Sum of a Pole and a Cut Contribution

For illustrative purposes, κ_L was assumed to exist and shown as pure imaginary. Since

$$\frac{i\omega}{c} B_y(x) = ik_z E_x(x) - \frac{\partial E_z}{\partial x}(x) \quad , \tag{49}$$

we obtain:

$$B_y(x) = \frac{2c^2 B_y(o)}{2\pi i} \int_{C_T} \frac{k_x e^{ik_x x}}{\Lambda_T(k_x)} dk_x \tag{50}$$

As expected, the magnetic field in the plasma does not depend on any longitudinal effects.

To check the consistency of our results, we take $k_z \to 0$ (normal incidence) and obtain:

$$E_x(x) \to 0 \tag{51}$$

and

$$E_z(x) \to -\frac{2\omega c B_y(o)}{2\pi i} \int_{C_T} \frac{e^{ik_x x}}{\Lambda_T(k_x)} dk_x \quad , \tag{52}$$

where $\Lambda_T(k_x)$ is now independent of k_z and the cut integral is on the interval $(\omega/c,\infty)$. In this special case, there exists no mechanism for exciting longitudinal oscillations. Only transverse fields exist in the plasma medium. We note that Equation (52) corresponds to Equation (24) of Ozizmir for normal incidence if the variable changes $\hat{z} \to \hat{y}$, $\hat{x} \to \hat{z}$, and $\hat{y} \to \hat{x}$ are introduced.

2.3 Determination of the Perturbed Particle Distribution

The perturbed particle distribution satisfies the equation:

$$-i\omega\gamma f + u_x\frac{\partial f}{\partial x} + ik_z u_z f = \frac{|e|n_o\gamma}{m}\left[E_x\frac{\partial F_o}{\partial u_x} + E_z\frac{\partial F_o}{\partial u_z}\right] \quad . \quad (53)$$

We can obtain a formal solution to the above by looking for a solution of the form:

$$f(x,\bar{u},k_z,\omega) = \int_{C_L}\frac{N_L(k_x)e^{ik_x x}dk_x}{(k_x^2+k_z^2)\Lambda_L(k_x)} + \int_{C_T}\frac{N_T(k_x)e^{ik_x x}dk_x}{(k_x^2+k_z^2)\;\Lambda_T(k_x)} . \quad (54)$$

Inserting our solutions for $E_x(x)$ and $E_z(x)$ into Equation (53) and using this assumed form for $f(x,\bar{u})$, we readily obtain $N_L(k_x)$ and $N_T(k_x)$. The perturbed distribution function may be written as:

$$f(x,\bar{u},k_z,\omega) = \frac{2icB_y(o)|e|n_o\gamma}{2\pi i m_o}\left[k_z\int_{C_L}\frac{(k_x\frac{\partial F_o}{\partial u_x} + k_z\frac{\partial F_o}{\partial u_z})e^{ik_x x}dk_x}{(k_x^2 + k_z^2)(\omega\gamma - \bar{k}\cdot\bar{u})\Lambda_L(k_x)}\right.$$

$$\left. + \;\omega\int_{C_T}\frac{k_x(k_z\frac{\partial F_o}{\partial u_x} - k_x\frac{\partial F_o}{\partial u_z})e^{ik_x x}dk_x}{(k_x^2 + k_z^2)(\omega\gamma - \bar{k}\cdot\bar{u})\Lambda_T(k_x)}\right] \quad (55)$$

The function $f(x,\bar{u},k_z,\omega)$, as expressed in Equation (55), is a solution to Equation (53) with $E_x(x)$ and $E_z(x)$ given by Equations (47) and (48). In Appendix C, we prove that it satisfies the condition of specular reflection.

2.4 Uniqueness of the Solution

In order to determine whether or not our solution $f(x, \bar{u}, k_z \omega)$ is unique, we investigate the homogeneous solution $h(x, \bar{u}, k_z, \omega)$ to Equation (53); i.e.,

$$-i(\omega\gamma - k_z u_z)h + u_x \frac{\partial h}{\partial x} = 0 \quad , \tag{56}$$

whose solution may be added to Equation (55). The solution to Equation (56) is:

$$h(x, \bar{u}, k_z, \omega) = A(\bar{u}) e^{i(\omega\gamma - k_z u_z)x/u_x} \quad , \tag{57}$$

where $A(\bar{u})$ is that set of $\bar{u}$ functions which makes $h(x, \bar{u})$ satisfy the condition of specular reflection. The first condition on $A(\bar{u})$ therefore is:

$$A(u_x, u_y, u_z) = A(-u_x, u_y, u_z) \quad , \tag{58}$$

i.e., $A(\bar{u})$ must be an even function of u_x. Since

$$(\omega\gamma - k_z u_z) = \omega\gamma(1 - (v_z/c)\sin\theta) > 0 \quad , \tag{59}$$

we see that Equation (57) corresponds to a wave traveling from infinity to the interface when $u_x < 0$. Imposing the radiation conditions on the homogeneous solution requires that

$$A(u_x, u_y, u_z) = 0 \quad ; \quad u_x < 0 \quad . \tag{60}$$

From Equations (58) and (60), we obtain:

$$h(x, \bar{u}, k_z, \omega) = 0$$

2.5 Determination of $G_L(k_x)$ and $G_T(k_x)$ and Proof of Self-Consistency

In order to obtain expressions for $E_x(x)$ and $E_z(x)$, Equations (47) and (48), we assumed that $G_L(k_x)$ and $G_T(k_x)$ were analytic everywhere except for cuts along the real k_x axis defined by $|k_x| > \alpha_o$. The radiation conditions then imposed the restrictions expressed in Equations (24), (25), (26) and (27) which in turn implied Equations (31), (32), (33), (34), (41), and (42). We now show that $G_L(k_x)$ and $G_T(k_x)$ evaluated from $f(o,\bar{u})$ do indeed satisfy all of these requirements.

We begin by evaluating $G_L(k_x)$. Inserting $f(o,\bar{u})$, Equation (55), into the integral definition of $G_L(k_x)$, Equation (18), and using the fact that

$$\frac{1}{(\gamma\omega - k_x' u_x - k_z u_z)(\gamma\omega - k_x u_x - k_z u_z)} =$$

$$= \frac{1}{(k_x' - k_x)u_x}\left[\frac{1}{(\gamma\omega - k_x' u_x - k_z u_z)} - \frac{1}{(\gamma\omega - k_x u_x - k_z u_z)}\right] \tag{61}$$

where $k_x' \neq k_x$, we obtain:

$$G_1(k_x) = -ik_z cB_y(o)$$

$$-\frac{\omega_p^{\,2}B_y(o)k_x c}{\pi}\int_{C_L}\frac{dk_x'}{(k_x'^2+k_z^{\,2})\Lambda_L(k_x')}\cdot$$

$$\cdot\left[\frac{(k_x'^2+k_z^{\,2})\{\Lambda_L(k_x')-\omega\}}{\omega_p^{\,2}(k_x'-k_x)}-\{k_x'A_{xx}(k_x')+k_zA_{xz}(k_x')\}\right]$$

$$+\frac{\omega_p^{\,2}B_y(o)k_z c}{\pi}\int_{C_L}\frac{dk_x'}{(k_x'^2+k_z^{\,2})\Lambda_L(k_x')}\cdot$$

$$\cdot\left[\frac{(k_x^{\,2}+k_z^{\,2})\{\Lambda_L(k_x)-\omega\}}{\omega_p^{\,2}(k_x'-k_x)}+\{k_xA_{xx}(k_x)+k_zA_{xz}(k_x)\}\right]$$

$$+\frac{\omega_p^{\,2}B_y(o)\omega c}{\pi}\int_{C_T}\frac{k_x'dk_x'}{(k_x'^2+k_z^{\,2})\Lambda_T(k_x')}\{k_zA_{xx}(k_x')-k_x'A_{xz}(k_x')\}$$

$$-\frac{\omega_p^{\,2}B_y(o)\omega c}{\pi}\int_{C_T}\frac{k_x'dk_x'}{(k_x'^2+k_x^{\,2})\Lambda_T(k_x')}\{k_xA_{xz}(k_x)+k_zA_{zz}(k_x)\}\quad, \tag{62}$$

where we made use of the definitions of $\Lambda_L(k_x)$ and $\Lambda_T(k_x)$, Equations (20) and (21), and that of $A_{ij}(k_x)$, Equation (A-4).

Expressing $\Lambda_L(k_x)$ and $\Lambda_T(k_x)$ in terms of the $A_{ij}(k_x)$ functions and using Equation (A-3), where $R(k_x) = 0$, we can simultaneously solve for A_{xx} , A_{xz} , and A_{zz} . We obtain:

$$A_{xx}(k_x) = \frac{k_x^2\omega[\Lambda_L(k_x) - \omega] - k_z^2[\Lambda_T(k_x) - (k_x^2 + k_z^2)c^2 + \omega^2]}{\omega\omega_p^2(k_x^2 + k_z^2)} , \quad (63)$$

$$A_{xz}(k_x) = \frac{k_x k_z[\omega\{\Lambda_L(k_x) - \omega\} + \{\Lambda_T(k_x) - (k_x^2 + k_z^2)c^2 + \omega^2\}]}{\omega\omega_p^2(k_x^2 + k_z^2)} \quad (64)$$

and

$$A_{zz}(k_x) = \frac{k_z^2\omega\{\Lambda_L(k_x) - \omega\} - k_x^2\{\Lambda_T(k_x) - (k_x^2 + k_z^2)c^2 + \omega^2\}}{\omega\omega_p^2(k_x^2 + k_z^2)} . \quad (65)$$

Using Equations (63) - (65) and the definition of $B_y(o)$, Equation (50), in Equation (62), we obtain:

$$G_L(k_x) = \frac{k_z cB_y(o)}{\pi} \int_{C_L} \frac{(k_x'k_x + k_z^2)[\Lambda_L(k_x) - \Lambda_L(k_x')]dk_x'}{(k_x'^2 + k_z^2)\Lambda_L(k_x')(k_x' - k_x)}$$

$$- \frac{k_z c\omega B_y(o)\Lambda_L(k_x)}{\pi} \int_{C_T} \frac{k_x'dk_x'}{(k_x'^2 + k_z^2)\Lambda_T(k_x')}$$

$$- \frac{k_z cB_y(o)}{\pi} \int_{C_T} \frac{k_x'dk_x'}{(k_x'^2 + k_z^2)} \quad (66)$$

The function $G_T(k_x)$ is obtained in an analogous manner. We find:

$$G_T(k_x) = \frac{\omega c B_y(o)}{\pi} \int_{C_T} \frac{k_x'(k_x'k_x + k_z^2)[\Lambda_T(k_x') - \Lambda_T(k_x)]dk_x'}{(k_x'^2 + k_z^2)\Lambda_T(k_x')(k_x' - k_x)}$$

$$- \frac{k_z^2 c B_y(o)\Lambda_T(k_x)}{\pi} \int_{C_L} \frac{dk_x'}{(k_x'^2 + k_z^2)\Lambda_L(k_x')}$$

$$- \frac{\omega k_z^2 c B_y(o)}{\pi} \int_{C_L} \frac{dk_x'}{(k_x'^2 + k_z^2)} \tag{67}$$

It is straightforward to show that $G_L(k_x)$ and $G_T(k_x)$ given by Equations (65) and (67) satisfy all requirements imposed upon them.

2.6 Reflection and Transmission Coefficients

The incident wave's electric ang magnetic fields are given by:

$$\overline{E}^i = (E^i_{ox}\hat{x} + E^i_{oz}\hat{z})e^{i\omega/c(x\cos\theta + z\sin\theta - ct)} \tag{68}$$

and

$$\overline{B}^i = \hat{y}(E^i_{ox}\sin\theta - E^i_{oz}\cos\theta)e^{i\omega/c(x\cos\theta + z\sin\theta - ct)} \quad , \tag{69}$$

where, from Gauss' law,

$$E^i_{ox}\cos\theta + E^i_{oz}\sin\theta = 0 \quad . \tag{70}$$

The reflected wave's electric and magnetic fields are given by:

$$\overline{E}^r = (E^r_{ox}\hat{x} + E^r_{oz}\hat{z})e^{i\omega/c(-x\cos\theta + z\sin\theta - ct)} \tag{71}$$

and

$$\overline{B}^r = \hat{y}(E^r_{ox}\sin\theta + E^r_{oz}\cos\theta)e^{i\omega/c(-x\cos\theta + z\sin\theta - ct)} , \tag{72}$$

where $-E^r_{ox}\cos\theta + E^r_{oz}\sin\theta = 0$ (73)

Applying the boundary conditions on Maxwell's equations and using the condition of specular reflection, we obtain:

$$E^i_{ox} + E^r_{ox} = E_x(o) , \tag{74}$$

$$E^i_{oz} + E^r_{oz} = E_z(o) \tag{75}$$

and

$$\sin\theta(E^i_{ox} + E^r_{ox}) - \cos\theta(E_{oz}{}^i - E_{oz}{}^r) = B_y(o) \tag{76}$$

Equations (74), (75), and (76) may be solved for E^r_{oz} and $B_y(o)$ by making use of Equations (70) and (73) to eliminate E^i_{ox} and E^r_{ox} and using the fact that

$$E_x(o) = B_y(o)\sin\theta , \tag{77}$$

which results from evaluating:

$$-ik_zB_y(x) = -\frac{i\omega}{c}E_x(x) + \frac{4\pi}{c}j_x(x) \tag{78}$$

at $x = 0$. We find

$$E_{oz}^{r} = - E_{oz}^{i} \frac{(\cos\theta + H)}{(\cos\theta - H)} \tag{79}$$

and

$$B_y(o) = \frac{-2E_{oz}^{i}}{(\cos\theta - H)} , \tag{80}$$

where we defined

$$H \equiv \frac{E_z(o)}{B_y(o)} \tag{81}$$

The integral representation of this new parameter can easily be obtained from Equation (48).

The reflection and transmission coefficients are defined as:

$$R \equiv \left| \frac{\mathrm{Re}\ \hat{x} \cdot \langle \overline{S}^{r} \rangle}{\mathrm{Re}\ \hat{x} \cdot \langle \overline{S}^{i} \rangle} \right| \tag{82}$$

and

$$T \equiv \left| \frac{\mathrm{Re}\ \hat{x} \cdot \langle \overline{S}^{t} \rangle}{\mathrm{Re}\ \hat{x} \cdot \langle \overline{S}^{i} \rangle} \right| , \tag{83}$$

where $\overline{S}$ is the time averaged Poynting vector defined by:

$$\langle \overline{S} \rangle \equiv \frac{c}{8\pi} \overline{E} \times \overline{B}^{*} \tag{84}$$

Using Equations (68) through (73), we find:

$$\langle S_x^{\ i}(o) \rangle = \frac{c \ |E_{oz}^{i}|^2}{8\pi \ \cos\theta} \tag{85}$$

and

$$\langle S_x^{\ r}(o) \rangle = - \frac{c \ |E_{oz}^{r}|^2}{8\pi \ \cos\theta} . \tag{86}$$

Consequently,

$$R = \left| \frac{E^r_{oz}}{E^i_{oz}} \right|^2 = \left| \frac{\cos\theta + H}{\cos\theta - H} \right|^2 . \tag{87}$$

The fraction of the total energy which travels across the interface is given by:

$$T(o) = \frac{4 \cos\theta \; |\mathrm{Re} H|}{|\cos\theta - H|^2} \tag{88}$$

Since $\mathrm{Re} H < 0$, we may easily show that

$$R(o) + T(o) = 1 \tag{89}$$

as expected.

The fraction of energy which travels deep into the plasma medium is given by:

$$T(\infty) \equiv \lim_{x \to \infty} T(x) \quad ,$$

where

$$T(x) \equiv \cos\theta \frac{|\mathrm{Re}\, E^t_z(x)\, B^{t*}_y(x)|}{|E^i_{oz}|^2} \tag{90}$$

It is clear from Equations (48), (50) and (90) that $T(\infty)$ is non zero only when $\kappa_T = |\kappa_T|$. There exists two possible cases to consider depending on whether a longitudinal root also exists. When $\kappa_T = |\kappa_T|$ and $\kappa_L = |\kappa_L|$, we obtain:

$$\lim_{x\to\infty} \langle S_x{}^t(x)\rangle =$$

$$= \frac{4\omega c^2 |B_y(o)|^2 \kappa_T (\frac{-c}{8\pi})}{\Lambda_T'(\kappa_T)} \left[\frac{\omega \sin\theta\, e^{i(\kappa_L - \kappa_T)x}}{c(\kappa_L{}^2 + \frac{\omega^2}{c^2}\sin^2\theta)\Lambda_L'(\kappa_L)} - \frac{c\,\kappa_T{}^2}{(\kappa_T{}^2 + \frac{\omega^2}{c^2}\sin^2\theta)\Lambda_T'(\kappa_T)} \right] \tag{91}$$

and therefore,

$$\lim_{x\to\infty} T(x) =$$

$$= \frac{16\omega\kappa_T c^2 \cos\theta}{|\Lambda_T'(\kappa_T)|\,|\cos\theta - H|^2} \left| \frac{\omega \sin\theta \cos(\kappa_L - \kappa_T)x}{c(\kappa_L{}^2 + \frac{\omega^2}{c^2}\sin^2\theta)\Lambda_L'(\kappa_L)} - \frac{\kappa_T{}^2 c}{(\kappa_T^2 + \frac{\omega^2}{c^2}\sin^2\theta)\Lambda_T'(\kappa_T)} \right| \tag{92}$$

The first term represents the interaction of the longitudinal and transverse waves--a second order effect. The electrons, whose density varies harmonically in x in this asymptotic limit, interact with the transverse electric field, thus giving rise to an x dependent energy density. Such a situation arises only when both $\Lambda_L(k_x)$ and $\Lambda_T(k_x)$ have roots on the real k_x axis; i.e., when $\Lambda_T{}^+(o) \leq 0$ and $\Lambda_L{}^+(\alpha_o) < 0$. Evaluating Equations (B-42) and (B-47) in the limit, $\beta \geq 100$, we find as conditions that

$$1 - \frac{\omega_\rho{}^2}{\omega^2}(1 + \frac{1}{2\beta}) < 0$$

and

$$\frac{(1 - 3/2\beta}{\cos^2\theta} - \frac{1}{\beta} \leq \frac{\omega^2}{\omega_\rho^{\;2}}$$

Combining these two inequalities, the existence condition for real longitudinal and transverse roots becomes:

$$\cos^2\theta > \frac{(1 - 3/2\beta)}{(1 + 3/2\beta)}$$

We conclude that real longitudinal and transverse roots exist only for near normal incidence. Since $T(\infty)$ obtained above still contains a term harmonically varying in x, we conclude that a transmission coefficient cannot be properly defined in this case.

When the longitudinal mode does not exist, we obtain:

$$T(\infty) \rightarrow \frac{16\omega(\kappa_T c)^3 \cos\theta}{|\Lambda_T'(\kappa_T)|^2 |\cos\theta - H|^2 (\kappa_T^{\;2} + \omega^2/c^2 \sin^2\theta)} \qquad (93)$$

We note that $H(\theta)$ is finite as $\theta \rightarrow \pi/2$ and consequently, as expected,

$$\lim_{\theta \rightarrow \pi/2} T(\infty,\theta) \rightarrow 0$$

Comparing Equation (88) to Equations (92) and (93), it is clear that some of the incident energy is absorbed by plasma. It is interesting that the energy absorbed by the particles is removed to infinity in the form of a heat flow. This is expected since otherwise strictly stationary solutions (ω real) could not have existed. A model describing an energy exchange mechanism is presented in Section 4.

2.7 Second Order Effects - Heating

To investigate the elementary non-equilibrium thermodynamics of this problem, we follow a procedure outlined by Ozizmir and define two new quantities:

$$U \equiv \mathrm{Re} \int_{\bar{u}} (mc^2\gamma)(n_o F_o + f_1 + f_2) du^3$$

$$= U_o + U_1 + U_2 \tag{94}$$

and

$$\bar{Q} \equiv \mathrm{Re} \int_{\bar{u}} \frac{\bar{u}}{\gamma} (mc^2\gamma)(n_o F_o + f_1 + f_2) du^3$$

$$= \bar{Q}_1 + \bar{Q}_2 \quad , \tag{95}$$

where U is the internal energy, $\bar{Q}$ the "total energy" current density (heat and rest mass), and f_1 and f_2 are the first and second order perturbed distribution functions for electrons.

The internal energy of the equilibrium state is given by:

$$U_o = \frac{m_o c^2 n_o}{K_2(\beta)} \left[K_3(\beta) - \frac{K_2(\beta)}{\beta} \right] \tag{96}$$

Multiplying Equation (1) by $m_o c^2$ and taking the real part of an integration over $\bar{u}$ space, we find:

$$\frac{\partial U_1}{\partial t} + \bar{\nabla}_r \cdot \bar{Q}_1 = 0 \quad . \tag{97}$$

Since U_1 and $\overline{Q}_1$ vary harmonically, it is clear that their time average values vanish.

To obtain an expression describing the heating of the plasma, we investigate the second order Vlasov equation:

$$\gamma \frac{\partial f_2}{\partial t} + \overline{u} \cdot \overline{\nabla}_r f_2 = \frac{|e|\gamma}{m_o}\left[\mathrm{Re}\overline{E}_1 \cdot \mathrm{Re}\overline{\nabla}_u f_1 + n_o \overline{E}_2 \cdot \overline{\nabla}_u F_o \right] + \frac{|e|}{m_o c} \mathrm{Re}(\overline{u} \times \overline{B}_1) \cdot \overline{\nabla}_u \mathrm{Re} f_1 \qquad (98)$$

Multiplying Equation (98) by $m_o c^2$ and integrating over all $\overline{u}$ space, we obtain:

$$\frac{\partial U_2}{\partial t} + \overline{\nabla}_r \cdot \overline{Q}_2 = |e|c^2 \int_{\overline{u}} \gamma(\mathrm{Re}\overline{E}_1 \cdot \mathrm{Re}\overline{\nabla}_u f_1) du^3 \qquad (99)$$

Taking the time average of Equation (99) and using the fact that:

$$\langle \mathrm{Re}\overline{E}_1 \cdot \mathrm{Re}\overline{\nabla}_u f_1 \rangle = \frac{\mathrm{Re}(\overline{E}_1 \cdot \overline{\nabla}_u f_1^*)}{2} , \qquad (100)$$

we find:

$$\left\langle \frac{\partial U_2}{\partial t} \right\rangle + \overline{\nabla}_r \cdot \langle \overline{Q}_2 \rangle = \frac{|e|c^2}{2} \mathrm{Re} \int_{\overline{u}} \gamma(\overline{E}_1 \cdot \overline{\nabla}_u f_1^*) du^3 = \mathrm{Re} \frac{(\overline{E}_1 \cdot \overline{j}_1^*)}{2} \qquad (101)$$

The time averaged Joule heating term may be expressed as a divergence of the Poynting vector by eliminating j_1^* through Maxwell's equations. Equation (101) may then be rewritten as:

$$\left\langle \frac{\partial U_2}{\partial t} \right\rangle + \overline{\nabla}_r \cdot [\langle \overline{Q}_2 \rangle + \mathrm{Re}\langle \overline{S} \rangle] = 0 \tag{102}$$

Following Ozizmir, we note that $f_2(x, \overline{u})$, as defined in Equation (98), contains 0, 2ω , and -2ω frequency components from which we conclude that the bulk heating term in Equation (102) must be zero; i.e.,

$$\left\langle \frac{\partial U_2}{\partial t} \right\rangle = 0 \quad . \tag{103}$$

Making use of Equation (103) in Equation (102) and noting from the symmetry of the interaction that $\langle S_{2z} \rangle$ and $\langle Q_{2z} \rangle$ must be independent of z , we find:

$$\frac{\partial}{\partial x} \left[\langle Q_x \rangle + \mathrm{Re}\langle S_x \rangle \right] = 0 \tag{104}$$

Therefore,

$$\mathrm{Re}\langle S_x(o) \rangle = \langle Q_x(x) \rangle + \mathrm{Re}\langle S_x(x) \rangle \tag{105}$$

and consequently:

$$\langle Q_x(\infty) \rangle = \mathrm{Re}[\langle S_x(o) \rangle - \langle S_x(\infty) \rangle] \tag{106}$$

The fact that the "total energy flow" attains an asymptotic value indicates that the conversion of energy from electromagnetic to a total energy density flow is a surface phenomenon (the skin effect).

Dividing both sides of Equation (106) by $\mathrm{Re}\langle S_x^{\,i}(o)\rangle$, we obtain an expression for the absorption coefficient A , defined as:

$$A \equiv \frac{\langle Q_x(\infty)\rangle}{\mathrm{Re}\langle S_x^{\,i}(o)\rangle} = T(o) - T(\infty) \tag{107}$$

Equation (107) is a statement of the energy conservation.

3. THE SLAB PLASMA

3.1 Modification of Half Space Analysis for Plasma Slab

We now consider a slab plasma whose faces are perpendicular to the $\hat{x}$ axis and situated at $x = 0$ and $x = a$. To avoid repetition, we begin here by indicating the modification necessary for adapting some of the results of the previous analysis to the present case. The functions $\mathcal{E}_x(x)$, $\mathcal{E}_z(x)$ and $\mathcal{B}_y(x)$ are defined to be identical to the functions $E_x(x)$, $E_z(x)$ and $B_y(x)$ within the plasma layer and vanish identically elsewhere. It is then clear that the Laplace transform of $\mathcal{E}_x(x)$ and $\mathcal{E}_z(x)$ are again given by the expressions in Equations (16) and (17) provided we replace all quantities evaluated at $x = 0$, i.e., $f(o)$, $E_x(o)$, $E_z(o)$ and $B_y(o)$ by $f(o) - e^{-ik_x a} f(a)$, $E_x(o) - e^{-ik_x a} E_x(a)$, etc. Making such a substitution, Equations (16) and (17) become:

$$\tilde{E}_x^s(k_x) = \frac{1}{(k_x^2 + k_z^2)} \left[k_x \frac{G_L^s(k_x)}{\Lambda_L(k_x)} - k_z \frac{G_T^s(k_x)}{\Lambda_T(k_x)} \right] \tag{108}$$

and

$$\tilde{E}_z^s(k_x) = \frac{1}{(k_x^2 + k_z^2)} \left[k_z \frac{G_L^s(k_x)}{\Lambda_L(k_x)} + k_x \frac{G_T^s(k_x)}{\Lambda_T(k_x)} \right], \tag{109}$$

where

$$G_L^s(k_x) \equiv G_L^o(k_x) - e^{-ik_x a} G_L^a(k_x) \tag{110}$$

and

$$G_T^{s}(k_x) \equiv G_T^{o}(k_x) - e^{-ik_x a} G_T^{a}(k_x) \tag{111}$$

and the superscripts "s", "o", and "a", respectively, denote a slab quantity, a quantity evaluated at $x = 0$, and a quantity evaluated at $x = a$. The functions $G_L^{o}(k_x)$ and $G_T^{o}(k_x)$ are defined as in Equations (18) and (19). The new functions $G_L^{a}(k_x)$ and $G_T^{a}(k_x)$ are defined as:

$$G_L^{a}(k_x) \equiv - \left[ik_z cB_y(a) + 4\pi|e| \int \frac{u_x(k_x u_x + k_z u_z) f(a) du^3}{\gamma(\gamma\omega - \overline{k} \cdot \overline{u})} \right] \tag{112}$$

and

$$G_T^{a}(k_x) \equiv - i(k_x^2 + k_z^2) c^2 E_z(a)$$

$$+ \omega \left[ik_x cB_y(a) + 4\pi|e| \int \frac{u_x(k_x u_z - k_z u_x) f(a) du^3}{\gamma(\gamma\omega - \overline{k} \cdot \overline{u})} \right] , \tag{113}$$

where $\overline{k} \cdot \overline{u} = k_x u_x + k_z u_z$.

Since $E_x(x)$ and $E_z(x)$ are bounded functions of x , it follows that $\tilde{E}_x(k_x)$ and $\tilde{E}_z(k_x)$ are entire functions of k_x . To make this point clear, we consider:

$$\tilde{E}_x^{s}(k_x) \equiv \int_o^a E_x(x) e^{-ik_x x} dx$$

Given that $|E_x(x)| < M$, and $0 \leq x \leq a < \infty$, it is clear that $\tilde{E}_x{}^s(k_x)$ and all of its derivatives exist for $|k_x| < \infty$; i.e., it is an entire function. The above conclusion depends on $E_x(x)$ being bounded and $a < \infty$. In the case of the half space, the second condition does not hold. The function and its derivatives go to infinity for all k_x values with $\text{Im}k_x > 0$.

Since $\tilde{E}_x{}^s(k_x)$ and $\tilde{E}_x{}^s(k_x)$ are entire functions, we must conclude that:

$$G_L{}^s(\pm\kappa_L) = 0 \tag{114}$$

and

$$G_T{}^s(\pm\kappa_T) = 0 \quad , \tag{115}$$

and that $\tilde{E}_x{}^s(k_x)$ and $\tilde{E}_z{}^s(k_x)$ are continuous across the entire real axis. The latter conditions imply that:

$$\frac{G_L{}^{+s}(k_1)}{\Lambda_L{}^+(k_1)} = \frac{G_L{}^{-s}(k_1)}{\Lambda_L{}^-(k_1)} \tag{116}$$

and

$$\frac{G_T{}^{+s}(k_1)}{\Lambda_T{}^+(k_1)} = \frac{G_T{}^{-s}(k_1)}{\Lambda_T{}^-(k_1)} \tag{117}$$

are valid for all $k_1 \equiv \text{Re}\, k_x$.

3.2 Inverse Transforms

As in the case of the half space, the electric field components inside the plasma slab are given by the inverse Laplace transforms:

$$E_x(x) = \frac{1}{2\pi} \int_C \tilde{E}_x^{\,s}(k_x) e^{ik_x x} dk_x \tag{118}$$

and

$$E_z(x) = \frac{1}{2\pi} \int_C \tilde{E}_z^{\,s}(k_x) e^{ik_x x} dk_x \quad , \tag{119}$$

where C is an open contour parallel to the real k_x axis. Since they do not as yet contain the information of specular reflection, Equations (118) and (119) are not in themselves the two components of the physically meaningful electric field in the plasma.

Before evaluating the electric field components within the plasma, we wish to show that Equations (118) and (119) imply that $E_x(x)$ and $E_z(x)$ are identically zero outside the slab. With this goal in mind, we consider $E_x(x)$ for x values in the two intervals: $x < 0$ and $x > a$. When $x < 0$, we deform the original line contour as shown in Figure 5 and find as $R \to \infty$:

$$E_x(x) + \frac{1}{2\pi} \int_{\Gamma_1} \tilde{E}_x^{\,s}(k_x) e^{ik_x x} dk_x = 0 \tag{120}$$

From the definitions of $G_L^{\,o}(k_x)$, $G_L^{\,a}(k_x)$, $G_T^{\,o}(k_x)$ and $G_T^{\,a}(k_x)$, it is clear that the Γ_1 contribution goes to zero. Thus, we have:

$$E_x(x) = 0 \quad ; \quad x < 0 \quad . \tag{121}$$

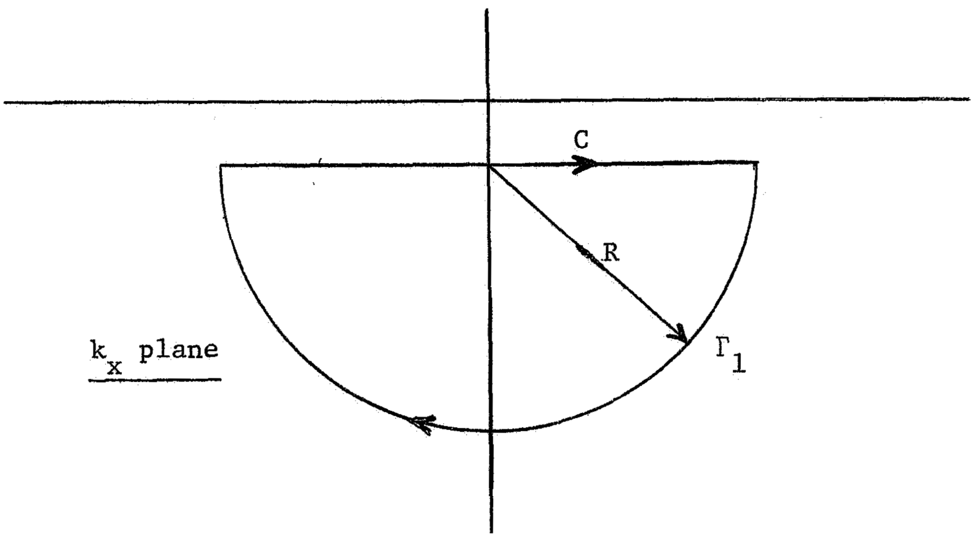


Figure 5. Deformation of Contour C When $x < 0$

When $x > a$, we deform C as shown in Figure 6 and obtain as $R \to \infty$:

$$E_x(x) + \frac{1}{2\pi}\int_{\Gamma_2} \tilde{E}_x^{\,s}(k_x)e^{ik_x x}dk_x = 0 \qquad (122)$$

From the definitions of $G_L^{\,o}(k_x)$, $G_L^{\,a}(k_x)$, $G_T^{\,o}(k_x)$ and $G_T^{\,a}(k_x)$, it is again clear that the Γ_2 contribution goes to zero and, thus, we obtain:

$$E_x(x) = 0 \quad ; \quad x > a \qquad (123)$$

Similarly, we can show that:

$$E_z(x) = 0 \quad ; \quad x < 0 \qquad (124)$$

and

$$E_z(x) = 0 \quad ; \quad x > a \qquad (125)$$

The fact that $B_y(x)$ is zero when $x < 0$ and when $x > a$ follows immediately from Equations (121) through (125).

3.3 Electric Fields Within the Slab

When $0 < x < a$, the x component of the electric field may be written as:

$$E_x(x) = \frac{1}{2\pi}\int_C \frac{1}{(k_x^{\,2} + k_z^{\,2})}\left[k_x\frac{G_L^{\,o}(k_x)}{\Lambda_L(k_x)} - k_z\frac{G_T^{\,o}(k_x)}{\Lambda_T(k_x)}\right]e^{ik_x x}dk_x$$

$$- \frac{1}{2\pi}\int_C \frac{1}{(k_x^{\,2} + k_z^{\,2})}\left[k_x\frac{G_L^{\,a}(k_x)}{\Lambda_L(k_x)} - k_z\frac{G_T^{\,a}(k_x)}{\Lambda_T(k_x)}\right]e^{ik_x(x-a)}dk_x \; . \qquad (126)$$

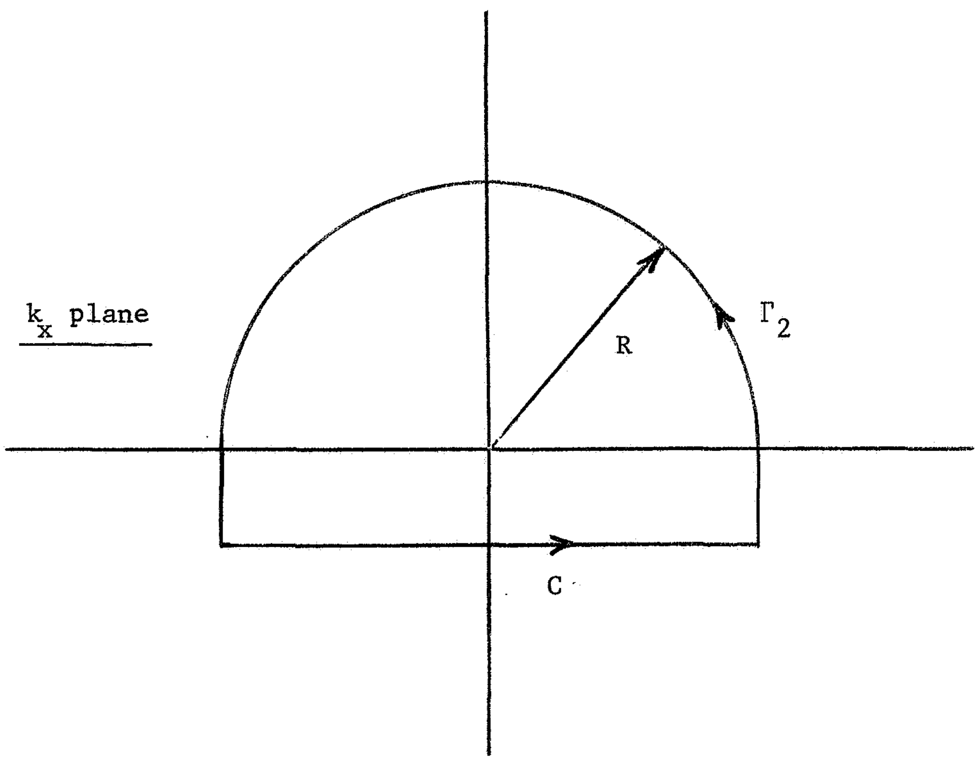


Figure 6. Deformation of Contour C When $x > a$

Since $(x - a) < 0$, we may close the second contour as shown in Figure 7 and obtain:

$$E_x(x) = \frac{1}{2\pi} \int_C \frac{1}{(k_x^2 + k_z^2)} \left[k_x \frac{G_L^o(k_x)}{\Lambda_L(k_x)} - k_z \frac{G_T^o(k_x)}{\Lambda_T(k_x)} \right] e^{ik_x x} dk_x \quad , \tag{127}$$

where C is an open contour parallel to the real k_x axis and <u>below</u> all singularities of the integrand; i.e., the zeros of the $\Lambda_L(k_x)$ and $\Lambda_T(k_x)$ functions. Choosing C to lie below all the singularities of this integrand permitted us to close the second contour in Equation (126) and obtain no contribution.

Similarly, when $0 < x < a$, Equation (119) may be written as:

$$E_z(x) = \frac{1}{2\pi} \int_C \frac{1}{(k_x^2 + k_z^2)} \left[k_z \frac{G_L^o(k_x)}{\Lambda_L(k_x)} + k_x \frac{G_T^o(k_x)}{\Lambda_T(k_x)} \right] e^{ik_x x} dk_x \quad . \tag{128}$$

It is clear that the integrands of the integrals appearing in Equations (127) and (128) are no longer entire functions. These last two equations can be solved by imposing the conditions expressed in Equations (114) through (117) and using the condition of specular reflection at the two interfaces. Since the algebraic manipulations are basically identical to those given for the half space problem, we will simply outline the derivation.

The two relations given in Equation (114) imply:

$$e^{-i\kappa_L a} G_L^o(-\kappa_L) + e^{i\kappa_L a} G_L^o(\kappa_L) = - 2ik_z cB_y(a) \tag{129}$$

Applying specular reflection to $G_L^{\,o}(-\kappa_L)$ and using the definition of $G_L^{\,o}(+\kappa_L)$, we find:

$$G_L^{\,o}(\kappa_L) + G_L^{\,o}(-\kappa_L) = -2ik_z cB_y(o) \tag{130}$$

We obtain from the last two equations:

$$G_L^{\,o}(\kappa_L)\sin\kappa_L a = k_z c[B_y(o)e^{-i\kappa_L a} - B_y(a)] \tag{131}$$

and

$$G_L^{\,o}(-\kappa_L)\sin\kappa_L a = -k_z c[B_y(o)e^{i\kappa_L a} - B_y(a)] \quad , \tag{132}$$

where we notice that Equation (132) can be obtained from Equation (131) by letting $\kappa_L \to -\kappa_L$.

The conditions on the cut may be obtained in a similar fashion. Applying the condition of specular reflection at $x = 0$, we readily obtain:

$$G_L^{\,o+}(k_1) + G_L^{-o}(-k_1) = -2ik_z cB_y(o) \tag{133}$$

and

$$G_L^{\,+o}(-k_1) + G_L^{\,-o}(k_1) = -2ik_z cB_y(o) \quad . \tag{134}$$

Making use of Equation (116) and applying the condition of specular reflection at $x = a$, we find:

$$\left[\frac{G_L^{+o}(k_1)}{\Lambda_L^+(k_1)} - \frac{G_L^{-o}(k_1)}{\Lambda_L^-(k_1)}\right] e^{ik_1 a} - \left[\frac{G_L^{+o}(-k_1)}{\Lambda_L^+(-k_1)} - \frac{G_L^{-o}(-k_1)}{\Lambda_L^-(-k_1)}\right] e^{-ik_1 a} =$$

$$= 2ik_z cB_y(a) \left[\frac{1}{\Lambda_L^-(k_1)} - \frac{1}{\Lambda_L^+(k_1)}\right] \qquad (135)$$

where we also used Equations (39) and (40). The cut conditions on the function $G_L^o(k_x)$ are obtained from Equations (133), (134) and (135). We have:

$$\left[\frac{G_L^{+o}(k_1)}{\Lambda_L^+(k_1)} - \frac{G_L^{-o}(k_1)}{\Lambda_L^-(k_1)}\right] \sin k_1 a =$$

$$= - k_z c \left[B_y(a) - B_y(o)e^{-ik_1 a}\right] \left[\frac{1}{\Lambda_L^+(k_1)} - \frac{1}{\Lambda_L^-(k_1)}\right] , \qquad (136)$$

where $k_1 > 0$. To obtain the cut condition when $k_1 < 0$, we let $k_1 \to - k_1$ in the above expression.

Up to this point, we have satisfied all requirements on $G_L^s(k_x)$ imposed by the "entire" nature of the transformed functions $\tilde{E}_x^s(k_x)$ and $\tilde{E}_z^s(k_x)$. Imposing these requirements on $G_T^s(k_x)$ involves the same tedious procedure outlined above. To avoid repetition, all details will be omitted. Equations (115), (117) and the condition of specular reflection at $x = 0$ and $x = a$ imply:

$$G_T^{o}(\kappa_T)\sin\kappa_T a = -\omega\kappa_T c[B_y(o)e^{-i\kappa_T a} - B_y(a)] \quad , \qquad (137)$$

$$G_T^{o}(-\kappa_T)\sin\kappa_T a = -\omega\kappa_T c[B_y(o)e^{i\kappa_T a} - B_y(a)] \qquad (138)$$

and

$$\left[\frac{G_T^{+o}(k_1)}{\Lambda_T^{+}(k_1)} - \frac{G_T^{-o}(k_1)}{\Lambda_T^{-}(k_1)}\right]\sin k_1 a =$$

$$= k_1 c\omega[B_y(a) - B_y(o)e^{-ik_1 a}]\left[\frac{1}{\Lambda_T^{+}(k_1)} - \frac{1}{\Lambda_T^{-}(k_1)}\right] , \qquad (139)$$

where $k_1 > 0$. To obtain the cut condition when $k_1 < 0$, we let $k_1 \to - k_1$ in the above.

There now arises a certain ambiguity in obtaining the quantities

$$\left[\frac{G_L^{+0}}{\Lambda_L^{+}} - \frac{G_L^{-0}}{\Lambda_L^{-}}\right] \quad \text{and} \quad \left[\frac{G_T^{+0}}{\Lambda_T^{+}} - \frac{G_T^{-0}}{\Lambda_T^{-}}\right]$$

since this involves dividing both sides of Equations (136) and (139) by $\sin k_1 a$, which is not permissible when $|k_1| = |\frac{n\pi}{a}| > \alpha_o = \frac{\omega}{c}\cos\theta$; n = integer. To overcome this difficulty, we follow Landau and consider ω as a complex number with a small positive imaginary part which is let go to zero after the solution is obtained completely. The stationary solutions thus obtained are interpreted as being the asymptotic time limits (solutions) of the mixed initial value-boundary value problem. It is interesting to note that the above ambiguity also arises when Laplace transform techniques are applied to the fluid model description of the same problem. The significance of this procedure ($\omega \to$ complex) in the present analysis is that it completely removes the above mentioned ambiguity. When ω has a small positive imaginary part, the singularities of $[\Lambda_L(k_x)]^{-1}$ and the "cuts" of the C_L contour are shifted as shown in Figure 7. The singularities of $[\Lambda_T(k_x)]^{-1}$ and the cuts of the C_T' contour are shifted as shown in Figure 7 with κ_T replacing κ_L . Expressed in terms of the sum of contours, C_L' and C_T' , Equations (127) and (128) become:

$$E_x(x) = \frac{k_z c}{2\pi} \int_{C_L'} \frac{k_x[B_y(o)e^{-ik_x a} - B_y(a)]e^{ik_x x}}{(k_x^2 + k_z^2)\sin k_x a \Lambda_L(k_x)} dk_x + \frac{k_z c\omega}{2\pi} \int_{C_T'} \frac{k_x[B_y(o)e^{-ik_x a} - B_y(a)]e^{ik_x x}}{(k_x^2 + k_z^2)\sin k_x a \Lambda_T(k_x)} dk_x \qquad (140)$$

and

$$E_z(x) = \frac{k_z^2 c}{2\pi} \int_{C_L'} \frac{[B_y(o)e^{-ik_x a} - B_y(a)]e^{ik_x x}}{(k_x^2 + k_z^2)\sin k_x a \Lambda_L(k_x)} dk_x - \frac{\omega c}{2\pi} \int_{C_T'} \frac{k_x^2[B_y(o)e^{-ik_x a} - B_y(a)]e^{ik_x x}}{(k_x^2 + k_z^2)\sin k_x a \Lambda_T(k_x)} dk_x \ , \qquad (141)$$

where $\sin k_x a \neq 0$ on C_L' and C_T' . In obtaining Equation (140) from Equation (127) and Equation (141) from Equation (128), we took account of the fact that respective integrands have zero residues at the poles, $k_x = \pm ik_z$.

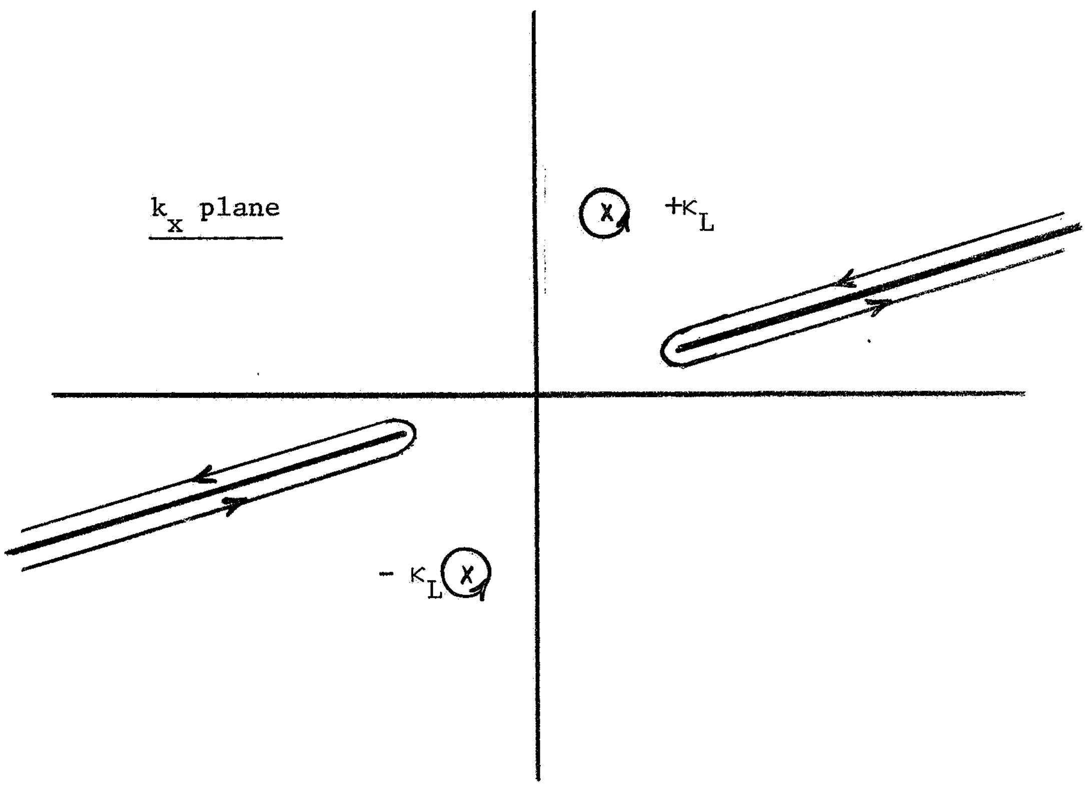


Figure 7. C_L' Contour - Path of Integration When ω Has a Slight Positive Imaginary Part

Using the above equations, we find:

$$B_y(x) = \frac{c^2}{2\pi} \int_{C_T'} \frac{k_x[B_y(o)e^{-ik_x a} - B_y(a)]e^{ik_x x}}{\sin k_x a \Lambda_T(k_x)} dk_x \quad , \tag{142}$$

where, as expected, $B_y(x)$ is independent of all longitudinal effects.

3.4 Determination of the Perturbed Particle Distribution

Inserting Equations (140) and (141) into the linearized Vlasov equation and proceeding as in the case of the half space, we find:

$$f(x,\bar{u},k_z,\omega) =$$

$$= \frac{i|e|n_o\gamma}{m_o}\left(\frac{k_z c}{2\pi}\right) \int_{C_L'} \frac{\left(k_x \frac{\partial F_o}{\partial u_x} + k_z \frac{\partial F_o}{\partial u_z}\right)\left(B_y(o)e^{-ik_x a} - B_y(a)\right)e^{ik_x x}}{(k_x^2 + k_z^2)\sin k_x a \Lambda_L(k_x)(\omega\gamma - \bar{k}\cdot\bar{u})} dk_x$$

$$- \frac{i|e|n_o\gamma}{m_o}\left(\frac{\omega c}{2\pi}\right) \int_{C_T'} \frac{k_x\left(k_x \frac{\partial F_o}{\partial u_z} - k_z \frac{\partial F_o}{\partial u_x}\right)\left(B_y(o)e^{-ik_x a} - B_y(a)\right)e^{ik_x x}}{(k_x^2 + k_z^2)\sin k_x a \Lambda_T(k_x)(\omega\gamma - \bar{k}\cdot\bar{u})} dk_x, \tag{143}$$

where $\bar{k}\cdot\bar{u} = k_x u_x + k_z u_z$. In a straightforward but tedious fashion, we can show that:

$$f(o,u_x) = f(o,-u_x) \tag{144}$$

and

$$f(a,u_x) = f(a,-u_x) \tag{145}$$

The proof follows the same lines as that given for the half space problem.

3.5 Uniqueness

The solution to the homogeneous linearized Vlasov equation, Equation (56), in the case of the slab plasma is given by:

$$h^s(x,\bar{u},k_z,\omega) = A(\bar{u})e^{i(\omega\gamma - k_z u_z)x/u_x} \tag{146}$$

where $\omega \equiv \omega_o + i\nu$; $\nu > 0$. Imposing the condition of specular reflection on this solution at $x = a$, we find:

$$A(u_x, \ldots)e^{i(\omega\gamma - k_z u_z)a/u_x} = A(-u_x, \ldots)e^{-i(\omega\gamma - k_z u_z)a/u_x} \tag{147}$$

For Equation (140) to hold for all $\bar{u}$ values, it is seen that:

$$A(u_x, \ldots) = A(-u_x, \ldots) = 0 \quad , \tag{148}$$

and, therefore,

$$h^s(x,\bar{u},k_z,\omega) = 0 \quad . \tag{149}$$

3.6 Determination of $G_L^o(k_x)$, $G_L^a(k_x)$, $G_T^o(k_x)$ and $G_T^a(k_x)$ and Proof of Self-Consistency

The self-consistency proof for the plasma slab problem follows the same lines as that given for the case of the plasma half space. Defining

$$H(k_x') \equiv \frac{B_y(o)e^{-ik_x'a} - B_y(a)}{\sin k_x'a} \quad ,$$

we can show, after some lengthy algebraic manipulations, that:

$$G_T{}^o(k_x) = \frac{i\omega c}{2\pi} \int_{C_T'} \frac{k_x'(k_x'k_x + k_z{}^2)[\Lambda_T(k_x') - \Lambda_T(k_x)]H(k_x')dk_x'}{(k_x'^2 + k_z{}^2)\Lambda_T(k_x')(k_x' - k_x)}$$

$$- \frac{i\omega k_z{}^2 c}{2\pi} \int_{C_L'} \frac{H(k_x')dk_x'}{(k_x'^2 + k_z{}^2)} - \frac{ik_z{}^2 c\Lambda_T(k_x)}{2\pi} \int_{C_L'} \frac{H(k_x')dk_x'}{(k_x'^2+k_z{}^2)\Lambda_L(k_x')} ,$$

(150)

$$G_T{}^a(k_x) = \frac{i\omega c}{2\pi} \int_{C_T'} \frac{k_x'(k_x'k_x+k_z{}^2)[\Lambda_T(k_x') - \Lambda_T(k_x)]e^{ik_x'a}H(k_x')dk_x'}{(k_x'^2 + k_z{}^2)\Lambda_T(k_x')(k_x' - k_x)}$$

$$- \frac{i\omega k_z{}^2 c}{2\pi} \int_{C_L'} \frac{e^{ik_x'a}H(k_x')}{(k_x'^2 + k_z{}^2)} dk_x'$$

$$- \frac{ik_z{}^2 c\Lambda_T(k_x)}{2\pi} \int_{C_L'} \frac{e^{ik_x'a}H(k_x')dk_x'}{(k_x'^2 + k_z{}^2)\Lambda_L(k_x')} , \qquad (151)$$

$$G_L{}^o(k_x) = \frac{ik_z c}{2\pi} \int_{C_L'} \frac{(k_x'k_x + k_z{}^2)[\Lambda_L(k_x) - \Lambda_L(k_x')]H(k_x')dk_x'}{(k_x'^2 + k_z{}^2)\Lambda_L(k_x')(k_x' - k_x)}$$

$$- \frac{ik_z c}{2\pi} \int_{C_T'} \frac{k_x'H(k_x')dk_x'}{(k_x'^2 + k_z{}^2)}$$

$$- \frac{ik_z c\omega\Lambda_L(k_x)}{2\pi} \int_{C_T'} \frac{k_x'H(k_x')dk_x'}{(k_x'^2 + k_x{}^2)\Lambda_T(k_x')} \qquad (152)$$

and

$$G_L{}^a(k_x) = \frac{ik_z c}{2\pi} \int_{C_L'} \frac{(k_x' k_x + k_z{}^2)[\Lambda_L(k_x) - \Lambda_L(k_x')] e^{ik_x' a} H(k_x')}{(k_x'^2 + k_z{}^2)\Lambda_L(k_x')(k_x' - k_x)} dk_x'$$

$$- \frac{ik_z c}{2\pi} \int_{C_T'} \frac{k_x' e^{ik_x' a} H(k_x') dk_x'}{(k_x'^2 + k_z{}^2)}$$

$$- \frac{ik_z c\omega\Lambda_L(k_x)}{2\pi} \int_{C_T'} \frac{k_x' e^{ik_x' a} H(k_x') dk_x'}{(k_x'^2 + k_z{}^2)\Lambda_T(k_x')} \tag{153}$$

We used the quantities $E_z(o)$, $E_z(a)$, $B_y(o)$ and $B_y(a)$ evaluated from Equations (141) and (142) in deriving Equations (150) to (153). It is clear that Equations (114) through (117) follow from Equations (150) through (153).

3.7 Reflection and Transmission Coefficients--Slab Plasma

As in the case of the half space, the incident and reflected waves' electric and magnetic field vectors are given by Equations (68), (69), (71) and (72). The transmitted wave's electric and magnetic field vectors are given by:

$$\overline{E}^t = (E^t_{ox}\,\hat{x} + E^t_{oz}\,\hat{z}) e^{i\frac{\omega}{c}(x\cos\theta + z\sin\theta - ct)} \tag{154}$$

and

$$\overline{B}^t = \hat{y}(E^t_{ox}\sin\theta - E^t_{oz}\cos\theta) e^{i\frac{\omega}{c}(x\cos\theta + z\sin\theta - ct)} \quad . \tag{155}$$

Applying the boundary conditions on Maxwell's equations and the condition of specular reflection at $x = 0$ and $x = a$, we obtain: Equations (74), (75) and (76) and

$$E_x(a) = E^t_{ox} e^{i\frac{\omega a}{c}\cos\theta} , \tag{156}$$

$$E_z(a) = E^t_{oz} e^{i\frac{\omega a}{c}\cos\theta} \tag{157}$$

and

$$B_y(a) = (E^t_{ox} \sin\theta - E^t_{oz} \cos\theta) e^{i\frac{\omega a}{c}\cos\theta} \tag{158}$$

In order to obtain the reflection and transmission coefficients, it is useful to first express $E_x(o)$, $E_x(a)$, $E_z(o)$, and $E_z(a)$ in terms of $B_y(o)$, $B_y(a)$ and the plasma characteristics ω_p^2/ω^2 and β. Evaluating Equation (78) at $x = 0$ and $x = a$, we find:

$$E_x(o) = B_y(o) \sin\theta \tag{159}$$

and

$$E_x(a) = B_y(a) \sin\theta \tag{160}$$

Similarly, evaluating Equation (141) at $x = 0$ and $x = a$, we obtain:

$$E_x(o) = \mu B_y(o) + \nu B_y(a) \tag{161}$$

and

$$E_z(a) = -\nu B_y(o) - \mu B_y(a) \quad . \tag{162}$$

The integral definitions of μ and ν are easily obtained from Equation (141). Making use of Equations (159) through (162) and of Gauss's law to eliminate E^i_{ox}, E^r_{ox}, and E^t_{ox}, we may simultaneously solve Equations (74) through (76) and Equations (156) through (158). We obtain, after some algebraic manipulations:

$$E^r_{oz} = \frac{(\nu^2 - \mu^2 + \cos^2\theta)E^i_{oz}}{(\nu^2 - \mu^2 + 2\mu\cos\theta - \cos^2\theta)} , \tag{163}$$

$$B_y(o) = \frac{2(\cos\theta - \mu)E^i_{oz}}{(\nu^2 - \mu^2 + 2\mu\cos\theta - \cos^2\theta)} , \tag{164}$$

$$E^t_{oz} = \frac{-2\nu\cos\theta\, e^{-i\frac{\omega a}{c}\cos\theta} E^i_{oz}}{(\nu^2 - \mu^2 + 2\mu\cos\theta - \cos^2\theta)} \tag{165}$$

and

$$B_y(a) = \frac{2\nu\, E^i_{oz}}{(\nu^2 - \mu^2 + 2\mu\cos\theta - \cos^2\theta)} \tag{166}$$

We are now in a position to calculate the power reflection and transmission coefficients. These coefficients are given by:

$$R(o) = \left|\frac{E^r_{oz}}{E^i_{oz}}\right|^2 = \left|\frac{\nu^2 - \mu^2 + \cos^2\theta}{\nu^2 - \mu^2 + 2\mu\cos\theta - \cos^2\theta}\right|^2 \tag{167}$$

and

$$T(a) = \left|\frac{E^t_{oz}}{E^i_{oz}}\right|^2 = \frac{4|\nu|^2\cos^2\theta}{|\nu^2 - \mu^2 + 2\mu\cos\theta - \cos^2\theta|^2} . \tag{168}$$

The above expressions for R(o) and T(a) correspond to those given by Ozizmir in the limit as $\theta \to 0$ (normal incidence).

4. Applications

Up to this point, we have dealt with the most general aspects of the theory describing the interaction of an obliquely incident p-polarized plane electromagnetic wave with a semi-infinite and a slab plasma. Our results are given as complicated mathematical expressions which often tend to obscure the underlying physical phenomena. In this section, we make use of the large β limit of $\Lambda_T(k_x)$ and $\Lambda_L(k_x)$ given in Appendix E to examine three problems in greater detail. By large β, we mean $\beta \equiv \frac{mc^2}{KT} \geq 100$. Physically, this corresponds to a temperature range of $0 \leq T \leq 5.9 \times 10^7 {}^oK$. The upper limit is roughly the core temperature of a white dwarf star. We also note that terms of order β^{-1} are expected to be of importance in controlled thermonuclear devices which should operate at approximately $3.5 \times 10^8 {}^oK$.

The first problem is the determination of a critical angle of incidence θ_c, such that for $\theta < \theta_c$, transmission will not occur. This problem is discussed in Section 4.1. In Sections 4.2 through 4.5, we investigate the penetration of a wave with a frequency $\omega < \omega_p$ into the plasma. For such frequencies, the electromagnetic fields penetrate into the medium, but there is no transmission. In Section 4.2, general expressions are obtained for quantities which characterize the transverse and longitudinal depths of penetration. These expressions contain two parts, viz., the pole contribution associated with the roots of $\Lambda_T(k_x)$ and $\Lambda_L(k_x)$, and, secondly, a cut contribution.

The roots are obtained in Section 4.3. Explicit expressions for the depths of penetration are found in Sections 4.4 and 4.5 for two limiting cases. In Section 4.6, we present a model for non-collisional absorption.

Section 4.7 is devoted to a further development of the penetration through a plasma slab. The fields in the plasma are obtained in a Fourier series representation and the effect of geometrical resonances is discussed.

4.1 Critical Angle of Incidence - Zero Transmission

In discussing the interaction of an obliquely incident plane electromagnetic wave with a plasma medium, the question of a critical angle of incidence θ_c naturally arises. We may obtain an analytic expression for θ_c by studying the condition that gives rise to zero transmission. It is clear from Equations (48), (50), and (90) that $T(\infty) = 0$ when κ_T is an imaginary root. Making use of Equation (B-47) and (E-2), we readily find that κ_T is pure imaginary when

$$\cos^2\theta \leq \frac{\omega_p^2}{\omega^2} \frac{(1 - \frac{3}{2\beta})}{(1 + \frac{\omega_p^2}{\beta\omega^2})}$$

The above condition is satisfied for all θ when $\frac{\omega_p^2}{\omega^2}(1 - \frac{1}{2}\beta) > 1$; i.e., $\omega_p > \omega$. We can also obtain zero tranmission when $\omega_p < \omega$ if

$$\sin\theta \geq \sqrt{1 - \frac{\omega_p^2}{\omega^2} \frac{(1 - 3/2\beta)}{(1 + \omega_p^2/\beta\omega^2)}}$$

The critical angle is given by the equality.

We should indicate that zero transmission does not imply total reflection. The reflection is never total (except at $T = 0$) due to the surface absorption. Finally, we note that the above conditions also apply to the interaction of an obliquely incident s-polarized wave with a plasma half space as can be shown by approximating the integral in Ozizmir's Equation (A.19).

4.2 Depth of Penetration

In dealing with the interaction of an electromagnetic wave with a conducting medium, it is convenient to define a macroscopic length whose magnitude is a measure of the ability to penetrate the medium. The classical skin depth of a metal is a good example of such a length. In the case of normal incidence of a plane electromagnetic wave on a hot tenuous plasma half space, Silin[1] defines a complex depth of penetration λ_p as:

$$\lambda_p = \frac{1}{B_y(o)} \int_0^\infty B_y(x)\, dx = -\frac{E_z(o)}{E_z'(o)} \tag{169}$$

The above definition has intuitive appeal when $B_y(x)$ is an exponentially decaying function of x. The integral is undefined when $B_y(x)$ is a harmonically varying function of x.

A difficulty with Silin's definition is that it is complex and we would prefer to characterize the depth of penetration by a real quantity. One possibility would be to take the real part of Silin's expression. A different approach was used by Weibel. For the case of normal incidence, he defines

$$\delta = -\frac{2}{\left[\frac{d}{dx}\ln(EE^*)\right]_{x=0}} \tag{170}$$

which is equivalent to writing:

$$\frac{1}{\delta} = -\frac{1}{2}\left[\frac{1}{E}\frac{dE}{dx} + \frac{1}{E^*}\frac{dE^*}{dx}\right]_{x=0} = -\mathrm{Re}\left[\frac{1}{E}\frac{dE}{dx}\right]_{x=0}$$

$$= +\mathrm{Re}\left[\frac{1}{\lambda_p}\right] \tag{171}$$

Equation (171) states that Weibel's depth of penetration and that given by Silin for normal incidence are related by:

$$\delta = +\frac{(\mathrm{Re}\lambda_p)^2 + (\mathrm{Im}\lambda_p)^2}{\mathrm{Re}\lambda_p} \tag{172}$$

Finally, we should mention that other definitions of a penetration depth are possible. We could, for example, define an energy depth of penetration as:

$$\langle x\rangle \equiv \frac{\int_0^\infty xT(x)\,dx}{\int_0^\infty T(x)\,dx} \quad ; \quad \omega < \omega_p \quad , \tag{173}$$

where $T(x)$ is the normalized transmitted energy density whose limit as $x \to \infty$ is the transmission coefficient. The number of possible definitions for the penetration depth is indeed unlimited.

The above definition for λ_p , Equation (169), is adequate only when pure transverse waves exist in the plasma medium. We seek a new definition for the case where both longitudinal and transverse waves exist in the plasma. Our choice is not arbitrary, but is a natural consequence based on the plasma impedance. In the discussion on the reflection and transmission coefficients, we introduced a dimensionless quantity H defined as $H \equiv E_z(o)/B_y(o)$. The surface impedance is given by:

$$Z_p \equiv -\frac{4\pi}{c} H \tag{174}$$

Evaluating Equation (48) at $x = 0$, we note that H is a function of ω_p^2/ω^2 , β , and θ , the angle of incidence. The function H may be rewritten in a more instructive form. Evaluating Equation (2-b) at $x = 0$ and eliminating $E_x(o)$ through the use of Equation (77), we find:

$$B_y(o) = \frac{ic}{\omega} \frac{E_z'(o)}{\cos^2\theta} \tag{175}$$

Using this result, we write:

$$H = -\frac{i\omega}{c} \cos^2\theta \frac{E_z(o)}{E_z'(o)} \tag{176}$$

We are now in a position to define a complex depth of penetration as:

$$\lambda_p \equiv - \cos^2\theta \frac{E_z(o)}{E_z'(o)} = \frac{c}{i\omega} H \quad . \tag{177}$$

This definition is identical to that given by Silin and Fetisov[2] in their investigation of the p-polarized interaction. It also reduces to the definition of Equation (169) for the case of normal incidence.

Following Silin and Fetisov, we note that λ_p is made up in additive fashion of λ_T and λ_L, the transverse and longitudinal depths of penetration. From Equation (48), we may write:

$$\lambda_T \equiv \frac{c^2}{\pi} \int_{c_T} \frac{k_x^2 \, dk_x}{(k_x^2 + k_z^2)\Lambda_T(k_x)} \tag{178}$$

and

$$\lambda_L \equiv - \frac{k_z^2 c^2}{\pi\omega} \int_{c_L} \frac{dk_x}{(k_x^2 + k_z^2)\Lambda_L(k_x)} \tag{179}$$

This separation of the depth of penetration into its transverse and longitudinal parts is instructive but can lead to some interpretational difficulties. These will be pointed out in the following discussion.

In Sections (4.4) and (4.5), we obtain the large β limit of Equations (178) and (179). We will restrict our attention to the case where $\omega \leq \omega_p$ since the concept of a penetration depth loses its intuitive significance otherwise.

4.3 Imaginary Transverse and Longitudinal Roots and the Concepts of Weak and Strong Spatial Dispersion

The transverse and longitudinal depths of penetration given by Equations (178) and (179) are each composed of a "pole" and a "cut" contribution. The pole contributions arise from the roots κ_T and κ_L. These roots are functions of β , ω_p^2/ω^2 and θ . In this section, we obtain the large β limit of κ_T and κ_L valid in two significantly different ranges of ω_p^2/ω^2 .

4.3.1 κ_T and κ_L Under Weak Spatial Dispersion.

Imaginary transverse and longitudinal roots can be obtained from the approximate expressions for $\Lambda_T(ik_2)$ and $\Lambda_L(ik_2)$. We first consider the limiting case $k_2^2c^2 << \omega^2(\beta - \cos^2\theta)$. It is shown in Appendix E that for this limit, $\Lambda_T(ik_2)$ and $\Lambda_L(ik_2)$ may be written as:

$$\Lambda_T(ik_2) \doteq -\delta^2 + \frac{\omega_p^2}{K_2(\beta)} \left| K_1(\beta) - \frac{\delta^2}{\omega^2} \frac{K_o(\beta)}{\beta} \right| \tag{180}$$

and

$$\Lambda_T(ik_2) \doteq \omega - \frac{\omega_p^2}{\omega K_2(\beta)} \left| K_1(\beta) + \frac{K_o(\beta)}{\beta} \left\{ 2 - 3\frac{\delta^2}{\omega^2} \right\} \right| , \tag{181}$$

where $\delta \equiv \sqrt{k_2^2c^2 + \omega^2\cos^2\theta}$. The quantities κ_T and κ_L may now be derived from Equations (180) and (181). Using the asymptotic form of the $K_n(\beta)$ functions and keeping terms up to β^{-1} , we find:

$$|\kappa_T| \doteq \frac{\omega}{c}\sqrt{\frac{\omega_p^2}{\omega^2}\left(1 - \frac{3}{2\beta}\right)\left(1 + \frac{\omega_p^2}{\beta\omega^2}\right)^{-1} - \cos^2\theta} \qquad (182)$$

and

$$|\kappa_L| \doteq \frac{\omega}{c}\sqrt{\frac{\beta}{3}\left(1 + \frac{1}{2\beta} - \omega^2/\omega_p^2\right) - \cos^2\theta} \quad . \qquad (183)$$

To understand the meaning of the strong inequality that forms the basis of this approximation, we impose the requirement

$$\omega^2(\beta - \cos^2\theta) \gg k_2^2 c^2$$

on the roots $|\kappa_T|$ and $|\kappa_L|$ and find:

$$\frac{1}{|\kappa_T|}\,,\ \frac{1}{|\kappa_L|} \gg \frac{1}{\omega}\sqrt{\frac{KT}{m}} \quad .$$

The quantity $\frac{1}{\omega}\sqrt{\frac{KT}{m}}$ is the average distance covered by an electron during one period of field oscillation. Since $|\kappa|^{-1}$ characterizes an exponential damping length, we see that the approximation requires the penetration depth to be much larger than the average distance covered by an electron during one period. The electrons experience only a weak gradient in electric fields in one period. Thus, the effect of the electric field on the particle is nearly local. This situation is referred to as weak spatial dispersion. The transverse root gives rise to weak spatial dispersion when $\beta \gg \omega_p^2/\omega^2$; the longitudinal when $\beta \gg \dfrac{\omega_p^2}{2\omega^2(2\omega_p^2/\omega^2 + 1)}$.

Finally, we note that the transverse root has a weak temperature dependence whereas the longitudinal root is strongly temperature dependent. This behavior can be qualitatively explained as follows. There exist two types of waves in the plasma medium. The longitudinal wave describes the organized motion of the electrons whose random internal energy density is proportional to the temperature (large β limit). These waves are strongly temperature dependent. The second type of wave existing in the medium is the transverse electromagnetic wave which does not give rise to charge separation. It is only weakly coupled to the electrons. Its temperature dependence is therefore weak.

4.3.2 κ_T and κ_L Under Strong Spatial Dispersion

When $k_2^2 c^2 >> \omega^2(\beta - \cos^2\theta)$, we show in Appendix E that $\Lambda_T(ik_2)$ and $\Lambda_L(ik_2)$ may be written as:

$$\Lambda_T(ik_2) \doteq -\delta^2 + \sqrt{\frac{\pi\beta}{2}}\,\frac{\omega_p^2\omega}{\delta} \tag{184}$$

and

$$\Lambda_L(ik_2) \doteq \omega\left[1 - \sqrt{\frac{\pi\beta}{2}}\,\frac{\omega_p^2\,\omega}{\delta^3}\right] , \tag{185}$$

where we have made use of the asymptotic form of the $K_n(\beta)$ functions and kept only the largest β contribution. Solving Equations (184) and (185) for $|\kappa_T|$ and $|\kappa_L|$, we obtain:

$$|\kappa_T| = |\kappa_L| \doteq \frac{1}{c} \sqrt{\left[\sqrt{\frac{\pi\beta}{2}}\ \omega_p{}^2\omega\right]^{2/3} - \omega^2\cos^2\theta}$$

$$\doteq \frac{1}{c}\left[\sqrt{\frac{\pi\beta}{2}}\,\omega_p{}^2\omega\right]^{1/3} \tag{186}$$

The condition $k_2{}^2c^2 >> \omega^2(\beta - \cos^2\theta)$ leads to the requirement that:

$$\frac{1}{|\kappa_T|} = \frac{1}{|\kappa_L|} << \frac{1}{\omega}\sqrt{\frac{KT}{m}}$$

In this case, surface electrons experience large spatial gradients of the electric fields during one period of field oscillation. The entire history of the particle near the surface is important. This case is referred to as strong spatial dispersion. Strong spatial dispersion can occur only when $\omega_p{}^2/\omega^2 >> \sqrt{\frac{2}{\pi}}\beta$. Since we assumed that $\omega >> \omega_p i$ and $\omega >> \nu$, the collision frequency, it is clear that we can expect strong spatial dispersion only for relativistic plasmas; i.e., $\beta \geq 100$.

4.4 Depths of Penetration - Weak Spatial Dispersion

We begin our evaluation of the depths of penetration with the physically more common case of weak spatial dispersion. For convenience, we rewrite Equations (178) and (179) as:

$$\lambda_T = \frac{2i\kappa_T^2 c^2}{(\kappa_T^2 + k_z^2)\Lambda_T'(\kappa_T)}$$

$$+ \frac{i\omega\omega_p^2 c}{K_2(\beta)} \int_0^\infty \frac{\sqrt{\sigma^2\cos^2\theta + 1}\; e^{-\beta\sqrt{\sigma^2+1}}\left[1 + \frac{1}{\beta\sqrt{\sigma^2+1}}\right] d\sigma}{\sigma(\sigma^2+1)\left[(\mathrm{Re}\Lambda_T^+)^2 + (\mathrm{Im}\Lambda_T^+)^2\right]} \tag{187}$$

and

$$\lambda_L = \frac{-2ik_z^2 c^2}{\omega(\kappa_L^2 + k_z^2)\Lambda_L'(\kappa_L)}$$

$$+ \frac{ik_z^2 c^3 \omega_p^2}{\omega^3 \beta K_2(\beta)} \int_0^\infty \frac{\sigma^3 e^{-\beta\sqrt{\sigma^2+1}}\left[\beta^2(\sigma^2+1) + 2\beta\sqrt{\sigma^2+1} + 2\right] d\sigma}{(\sigma^2+1)^{5/2}\sqrt{\sigma^2\cos^2\theta+1}\left[(\mathrm{Re}\Lambda_L^+)^2 + (\mathrm{Im}\Lambda_L^+)^2\right]} . \tag{188}$$

4.4.1 Pole Contribution to λ_T .

The pole contribution to λ_T under the condition of weak spatial dispersion is given by:

$$[\lambda_T]_{pole} \doteq \frac{c}{\omega} \frac{\sqrt{\frac{\omega_p^2/\omega^2\,(1 - 3/2\beta)}{(1 + \omega_p^2/\beta\omega^2)} - \cos^2\theta}}{[\omega_p^2/\omega^2\,(1 - 5/2\beta) - 1]} . \tag{189}$$

Under the same conditions, Silin obtains:

$$[\lambda_T(\mathrm{Silin})]_{pole} \doteq \frac{\frac{ic}{\omega}\sqrt{\cos^2\theta - \frac{\omega_p^2/\omega^2(1 + 1/\beta)}{(1 + \omega_p^2/\beta\omega^2)}}}{(1 - \omega_p^2/\omega^2)} \tag{190}$$

In order to compare our results with those given by Šilin, we expand Equation (189) for the case where $\omega_p^2/\omega^2 \gg 1$ and obtain:

$$[\lambda_T]_{pole} \doteq \frac{\frac{c}{\omega}\left[\sqrt{\frac{\omega_p^2}{\omega^2} - \cos^2\theta} - \frac{\omega_p^2}{2\beta\omega^2}\frac{(3/2 + \omega_p^2/\omega^2}{\sqrt{\omega_p^2/\omega^2 - \cos^2\theta}} + \frac{5\omega_p^2}{2\beta\omega^2}\frac{\sqrt{\omega_p^2/\omega^2 - \cos^2\theta}}{(\omega_p^2/\omega^2 - 1)}\right]}{(\omega_p^2/\omega^2 - 1)} . \tag{191}$$

Performing a similar expansion on Equation (190), we find that we disagree with Silin's $1/\beta$ factor. In his case, the $1/\beta$ factor is given by:

$$-\frac{c}{\omega}\frac{\omega_p^2}{2\omega^2}\frac{1}{\sqrt{\omega_p^2/\omega^2 - \cos^2\theta}} .$$

4.4.2 Cut Contribution to λ_T .

The cut contribution to λ_T may be found from Equation (187). Since we are interested in the first β contribution, we make use of Equation (E-27) and approximate Equation (187) as:

$$[\lambda_T]_{cut} \doteq$$

$$\frac{i\omega_p^2 c}{\omega^3 K_2(\beta)} \int_0^\infty \frac{\sigma^3 \sqrt{\sigma^2\cos^2\theta + 1}\; e^{-\beta\sqrt{\sigma^2+1}}\, d\sigma}{(\sigma^2+1)\left\{\left[1 - \frac{\omega_p^2}{\omega^2}\sqrt{\frac{\beta}{2}}\,\sigma^3 \mathrm{Re}Z(\sqrt{\frac{\beta}{2}}\,\sigma)\right]^2 + \pi^2\sigma^6 \frac{\omega_p^4}{\omega^4}\frac{e^{-2\beta\sqrt{\sigma^2+1}}}{4K_2^2(\beta)}\right\}} \,. \tag{192}$$

Since β is large, most of the contribution to the integral occurs for $\sigma^2 << 1$, we may write:

$$[\lambda_T]_{cut} \doteq \frac{4i\omega_p^2 c}{\omega^3}\sqrt{\frac{2}{\pi}}\,\frac{1}{\beta^{3/2}} \int_0^\infty \frac{\mu^3 e^{-\mu^2} d\mu}{\left[1 - \frac{2\omega_p^2}{\beta\omega^2}\mu^3 \mathrm{Re}Z(\mu)\right]^2 + 16\pi \frac{\omega_p^4}{\beta^2\omega^4}\mu^6 e^{-2\mu^2}} \tag{193}$$

where we made the change of variables $\mu^2 \equiv \frac{\beta\sigma^2}{2}$. Noting that $\omega_p^2/\beta\omega^2 << 1$, we may approximate Equation (193) as:

$$[\lambda_T]_{cut} \doteq \frac{4i\omega_p^2 c}{\omega^3}\sqrt{\frac{2}{\pi}}\,\frac{1}{\beta^{3/2}} \int_0^\infty \mu^3 e^{-\mu^2} d\mu$$

$$\doteq \frac{2ic}{\omega}\sqrt{\frac{2}{\pi}}\,\frac{\omega_p^2}{\omega^2}\,\frac{1}{\beta^{3/2}} \tag{194}$$

This is in exact agreement with Silin's result.

4.4.3 Pole Contribution to λ_L .

The pole contribution to λ_L under the condition of weak spatial dispersion is given by:

$$[\lambda_L]_{pole} \doteq \frac{c}{\omega}\frac{\omega^2}{\omega_p^2}\frac{\sin^2\theta}{\sqrt{\beta/3(1-\omega^2/\omega_p^2)+\frac{1}{6}-\cos^2\theta[5/2\beta-(1-\omega^2/\omega_p^2)]}} \qquad (195)$$

where $\beta \gg \dfrac{\omega_p^2/\omega^2}{2(2\omega_p^2/\omega^2+1)}$.

Silin gives as a result:

$$[\lambda_L(\text{Silin})]_{pole} \doteq \frac{ic}{\omega}\frac{\sqrt{\alpha^{\ell}_{nr}}\sin^2\theta}{\varepsilon(\omega)\sqrt{\varepsilon(\omega)-\alpha^{\ell}_{nr}\sin^2\theta}} \quad , \qquad (196)$$

where

$$\varepsilon(\omega) \equiv (1-\omega_p^2/\omega^2)$$

and

$$\alpha^{\ell}_{nr} \equiv \frac{3\omega_p^2}{\beta\omega^2}$$

Equation (196) may be rewritten as:

$$[\lambda_L(\text{Silin})]_{pole} \doteq \frac{ic\sin^2\theta}{\omega(1-\omega_p^2/\omega^2)}\frac{1}{\sqrt{\cos^2\theta+\beta/3(\omega^2/\omega_p^2-1)-1}} \qquad (197)$$

which he claims is valid when $|(1-\omega_p^2/\omega^2)| \ll 1$. Our results do not depend on such a restriction. In the vicinity of the plasma frequency, our results differ significantly from those given by Silin.

4.4.4 Cut Contribution to λ_L .

The cut contribution to λ_L is quite difficult to obtain. We begin by noting that most of the contribution to the integral occurs for $\sigma^2 << 1$ when $\beta \geq 100$. The integral in Equation (188) may be approximated as:

$$[\lambda_L]_{cut} \doteq 4i \frac{c}{\omega} \left(\frac{\omega^2}{\omega_p^2}\right) \sin^2\theta \sqrt{\frac{2}{\pi\beta}} \int_0^\infty \frac{\mu^3 e^{-\mu^2} d\mu}{[\omega^2/\omega_p^2 + G(\mu)]^2 + 4\pi\mu^6 e^{-2\mu^2}} , \quad (198)$$

where

$$G(\mu) \equiv 2\mu^2[1 + \mu \mathrm{Re} Z(\mu)]$$

and where we made the variable change $\mu^2 \equiv \frac{\beta\sigma^2}{2}$. The general behavior of the integrand $R(\mu,\omega^2/\omega_p^2)$ is shown in Figure 8. Since this integral cannot be suitably approximated, it was numerically integrated. Defining the integral as $Q(\omega^2/\omega_p^2)$, we obtain the following list of typical values:

ω^2/ω_p^2	$Q(\omega^2/\omega_p^2)$
1.00	0.41934
0.80	0.33064
0.50	0.31234
0.30	0.33921
0.10	0.41205
0.07	0.43950
0.05	0.48059
0.02	0.55550
0.01	0.66289

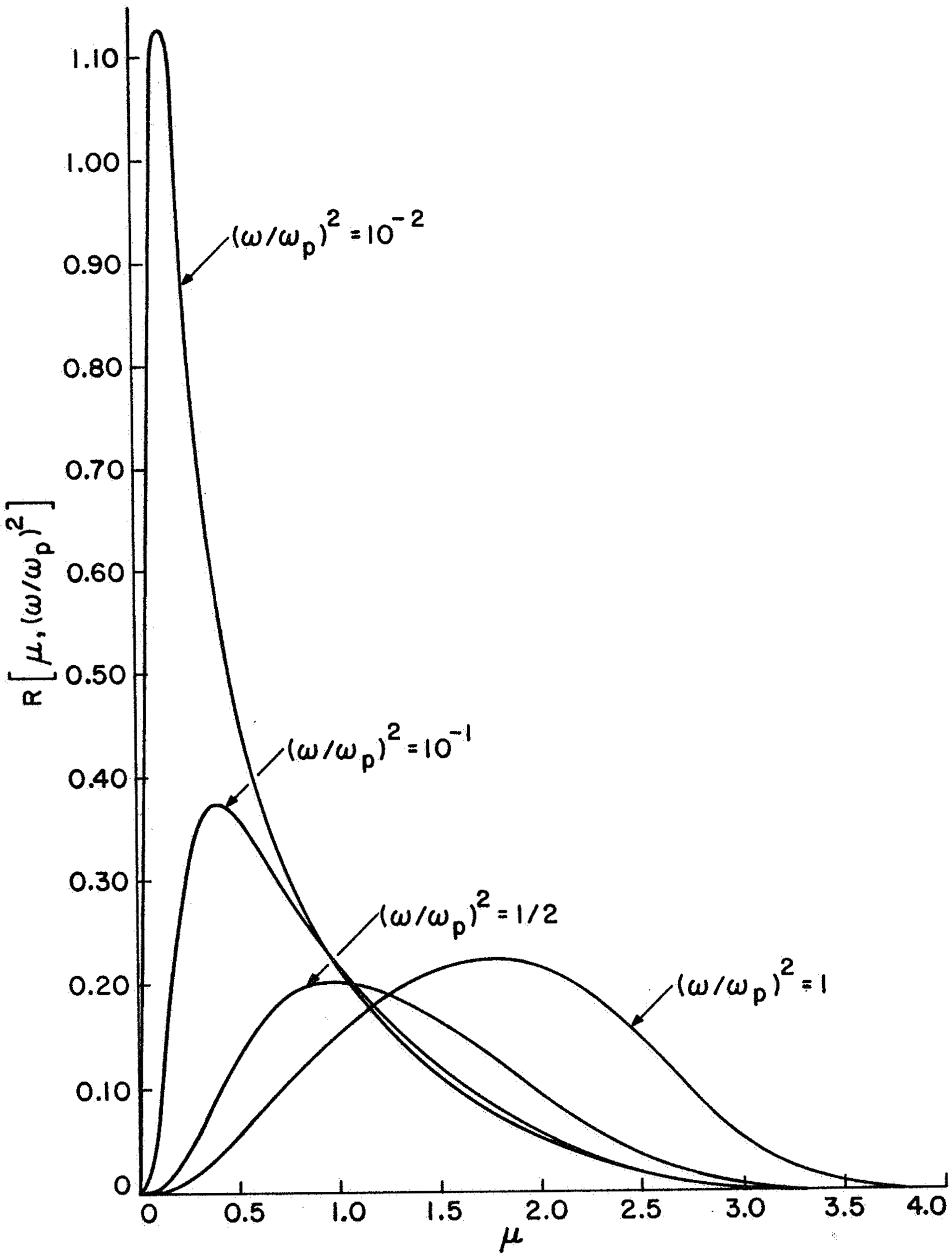


Figure 8. $R(\mu,\omega^2/\omega_p{}^2)$ Versus μ With $\omega^2/\omega_p{}^2$ As a Parameter

Rewriting Equation (198) as:

$$[\lambda_L]_{cut} \doteq 4i \frac{c}{\omega} \left(\frac{\omega^2}{\omega_p^2}\right) \sin^2\theta \sqrt{\frac{2}{\pi\beta}}\ Q(\omega^2/\omega_p^2) \quad , \tag{199}$$

we may compare our results with those given by Silin. When $\omega = \omega_p$, he gives:

$$[\lambda_L(\text{Silin})]_{cut} \doteq 1.7 \frac{ic}{\omega_p} \sin^2\theta \sqrt{\frac{2}{\pi\beta}} \tag{200}$$

We obtain Equation (200) with 1.7 replaced by about 1.67. Silin also claims that the integral can be approximated when $\omega < \omega_p$ and gives the formula:

$$[\lambda_L(\text{Silin})]_{cut} \doteq \frac{ic}{\omega} \sin^2\theta \sqrt{\frac{2}{\pi\beta}} \frac{\omega^2}{\omega_p^2} \left[\frac{1}{2} \ln(1 + \frac{\omega_p^2}{\omega^2}) - \frac{1}{1 + \omega^2/\omega_p^2} \right] \tag{201}$$

Our computer results (see previous page), valid when $\omega < \omega_p$, differ significantly from values obtained by using Silin's approximate formula.

4.5 Depths of Penetration - Strong Spatial Dispersion

We now seek to obtain the characteristic depths of penetration for the case where the electrons experience large spatial gradients of the electric fields during one period of field oscillation. These large spatial gradients give rise to the anomalous skin effect. We begin by considering the transverse case.

4.5.1 Pole Contribution to λ_T .

The pole contribution to λ_T under strong spatial dispersion is expressed by:

$$[\lambda_T]_{pole} \doteq \frac{2c}{3} \frac{\sqrt{\left(\sqrt{\frac{\pi\beta}{2}}\,\omega_p^{\,2}\omega\right)^{2/3} - \omega^2\cos^2\theta}}{\left(\sqrt{\frac{\pi\beta}{2}}\,\omega_p^{\,2}\omega\right)^{2/3} - \omega^2}$$

$$\underset{\sim}{\sim} \frac{2c}{3\left(\sqrt{\frac{\pi\beta}{2}}\,\omega_p^{\,2}\omega\right)^{1/3}} \tag{202}$$

Silin obtains the same expression. Weibel, who considered only the case of normal incidence, gives:

$$\delta = \frac{8\ (\sqrt{2})^{1/3}}{9} \frac{c}{\left(\sqrt{\frac{\pi\beta}{2}}\,\omega_p^{\,2}\omega\right)^{1/3}} \tag{203}$$

His definition of the skin depth is somewhat different, however.

4.5.2 Cut Contribution to λ_T .

The cut contribution to λ_T under strong spatial dispersion may be obtained by first rewriting Equation (193) as:

$$[\lambda_T]_{cut} \doteq \frac{4i\omega_p^{\,2}c}{3\omega^3\beta\alpha^{4/3}} \sqrt{\frac{2}{\pi\beta}} \int_0^\infty \frac{x^{1/3}e^{-(x/\alpha)^{2/3}}dx}{[1 - x\mathrm{Re}Z\{(x/\alpha)^{1/3}\}]^2 + 4\pi x^2 e^{-2(x/\alpha)^2}}, \tag{204}$$

where we defined $\alpha \equiv 2\omega_p^{\,2}/\beta\omega^2$ and made the change of variable $x = \alpha\mu^3$. In the case of strong spatial dispersion, $\alpha >> 1$ from which we conclude that the exponentials play a small role in Equation (204). We may write:

$$[\lambda_T]_{cut} \doteq \frac{4i\omega_p^2 c}{3\omega^3 \beta \alpha^{4/3}} \sqrt{\frac{2}{\pi\beta}} \int_0^\infty \frac{x^{1/3} dx}{1 + 4\pi x^2} \tag{205}$$

Evaluating the integral, we obtain: $\frac{1}{4}\left[\frac{2\sqrt{\pi}}{3}\right]^{2/3}$. Therefore,

$$[\lambda_T]_{cut} \doteq \frac{2}{3} i \left[\frac{1}{3^{2/3} 2^{4/3}} \frac{c}{\left(\sqrt{\frac{\pi\beta}{2}} \omega_p^2 \omega\right)^{1/3}}\right] . \tag{206}$$

Silin also obtains Equation (204) with $\sqrt{3}$ replacing $3^{2/3} 2^{4/3}$ in the denominator.

4.5.3 Pole Contribution to λ_L .

The pole contribution to λ_L under strong spatial dispersion is given by:

$$[\lambda_L]_{pole} \doteq - \frac{2c\sin^2\theta}{3\left(\sqrt{\frac{\pi\beta}{2}} \omega_p^2 \omega\right)^{1/3}} , \tag{207}$$

a negative quantity. This curious result is a direct consequence of our separating the wave's depth of penetration into its transverse and longitudinal parts. The wave's effective damping length is given by $Re[\lambda_T + \lambda_L] = [\lambda_T + \lambda_L]_{pole}$. Making use of Equation (202), we obtain:

$$Re[\lambda_T + \lambda_L] \doteq \frac{2c(1 - \sin^2\theta)}{3\left(\sqrt{\frac{\pi\beta}{2}} \omega_p^2 \omega\right)^{1/3}} \tag{208}$$

This is a case where we must not ascribe physical significance to λ_T and λ_L individually.

4.5.4 Cut Contribution to λ_L .

The cut contribution to λ_L under the condition of strong spatial dispersion is again given by the general expression of Equation (198). Typical values of this integral can be obtained by performing another numerical integration.

4.6 Model for Non-Collisional Absorption Under Strong Spatial Dispersion - Anomalous Skin Effect

It is useful at this point to digress somewhat from the formal development of the theory to discuss an elementary model on non-collisional energy absorption. To simplify our calculations, we assume that the electronic state is described by the non-relativistic Maxwell-Boltzmann distribution function. We also restrict our attention to the case of strong spatial dispersion; i.e., the distance d covered by an average surface electron during one period of field oscillation is much larger than the wave's depth of penetration δ . As a final simplification, we characterize the electric field within the plasma by a step function model; i.e.,

$$\bar{E} = E_x \hat{x} + E_z \hat{z} \quad ; \quad 0 \leq x \leq \delta$$

$$\bar{E} = 0 \quad ; \quad \delta < x < \infty \quad .$$

A surface electron traveling toward the interface with an x-component of velocity v_x enters and leaves the skin depth region in a time $t_o = \dfrac{2\delta}{|v_{x1}|}$. If $t_o << \dfrac{1}{\omega} \equiv \dfrac{T}{2\pi}$, such an electron sees a stationary electric field. Here, we have the mechanism for an energy exchange. The energy of the electron before entering the skin depth region is $E_1 = \frac{m}{2}(v_{x1}^2 + v_y^2 + v_{z2}^2)$. The overall change in v_x is zero since the electron accelerates (decelerates) in the x direction upon entering the δ-region and decelerates (accelerates) in the x direction upon leaving. The change in the v_z component of velocity is given by:

$$(v_{z2} - v_z) = \pm \frac{|e|\mathcal{E}_z}{m} \left| \frac{2\delta}{v_{x1}} \right| , \tag{209}$$

The electron suffers an acceleration or deceleration in the z direction, depending on whether its initial v_z velocity was in the direction of $\mathcal{E}_z$ or not. The corresponding change in the electron's energy is given by:

$$\Delta E = \frac{m}{2} \left[\frac{|e|^2 \mathcal{E}_z^2}{m^2} \left| \frac{2\delta}{v_{x1}} \right|^2 \pm 2v_z \frac{|e|\mathcal{E}_z}{m} \left| \frac{2\delta}{v_{x1}} \right| \right] . \tag{210}$$

All electrons with an x component of velocity directed toward the interface and satisfying the inequality $|v_x| >> 2\delta\omega$ participate in this energy exchange. The change in the plasma's internal energy may be written as:

$$\langle\Delta U\rangle = n_o \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{v_{x1}}^{\infty} (\Delta E) F_o dv^3$$

Since F_o is isotropic in velocity, we obtain:

$$\langle \Delta U \rangle \doteq n_o \left(\frac{m}{2}\right) \frac{|e|^2 E_z^{\,2}}{m^2} (2\delta)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{v_{x1}}^{\infty} \frac{F_o}{v_x^{\,2}}\, dv^3$$

$$= \frac{n_o \omega_p^{\,2} E_z^{\,2} \delta^2}{2\,\pi} \sqrt{\frac{m}{2\pi KT}} \int_{v_{x1}}^{\infty} \frac{e^{-mv_x^{\,2}/2KT}}{v_x^{\,2}}\, dv_x \qquad (211)$$

Similarly, we may obtain an expression for the change in the x component of the energy current density $\overline{Q}_x$. Following the same reasoning as given above, we write:

$$\Delta Q_x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{v_{x1}}^{\infty} v_x (\Delta E) F_o dv^3$$

$$= \frac{m}{2} \left(\frac{|e|^2 E_z^{\,2}}{m^2}\right) (2\delta)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{v_{x1}}^{\infty} \frac{F_o}{v_x}\, dv_x$$

$$= \frac{n_o \omega_p^{\,2} E_z^{\,2} \delta^2}{2\pi} \sqrt{\frac{m}{2\,KT}} \int_{v_{x1}}^{\infty} \frac{e^{-mv_x^{\,2}/2KT}}{v_x}\, dv_x \quad . \qquad (212)$$

We note that although the accelerations imparting energy to the electrons act in the z direction, there results a component of $\Delta\overline{Q}$ in the x direction. This interaction gives rise to a total energy current density propagating to infinity.

4.7 Fourier Series Representation - Slab Problem

The interaction of a p-polarized electromagnetic wave with a plasma slab was investigated in Section 3 and expressions for the electric and magnetic fields set up within the medium were obtained. These general results are given in Equations (142) through (144). They still remain in a form which makes a physical interpretation difficult. In this section, we alleviate this difficulty by finding the equivalent Fourier series representations for these field solutions. The resulting expressions then lend themselves to some physical interpretations. We begin our discussion with Equation (142). The quantity $E_x(x)$ is given in terms of the C_L' and C_T' contours. We recall that the C_L and C_T contours were shifted so as to avoid the zeros of $\sin k_x a$ which lie along the real k_x axis. The equivalent Fourier series representation for $E_x(x)$ can be obtained by simply deforming (individually) the C_L' and C_T' contours such that the resulting closed contours (individually) enclose the entire cut k_x plane and, thus, all the singularities of $[\sin k_x a]^{-1}$. The quantity $E_x(x)$ is, therefore, given by:

$$E_x(x) = -2\pi i \sum \quad \text{residues evaluated at} \quad k_x = \frac{n\pi}{a} .$$

The minus sign is due to the counterclockwise contour deformation. The residues $F(\frac{n\pi}{a})$ are found by evaluating:

$$F(k_x) \equiv \frac{k_x k_z c}{2\pi} e^{ik_x x} \frac{[B_y(o)e^{-ik_x a} - B_y(a)]}{(k_x^2 + k_z^2)} \left[\frac{1}{\Lambda_L(k_x)} + \frac{\omega}{\Lambda_T(k_x)}\right] \tag{213}$$

at $k_x = \frac{n\pi}{a}$. We note that, in the vicinity of a zero,

$$\sin k_x a \doteq \sin n\pi + a \cos k_x a \Big|_{k_x = \frac{n\pi}{a}} (k_x - \frac{n\pi}{a})$$

$$= a(-1)^n(k_x - n\pi/a) \quad .$$

We follow a procedure given by Ozizmir and write:

$$E_x(x) = -(2\pi i)\sum_{n=-\infty}^{\infty}\frac{k_z c}{2\pi}e^{in\pi x/a}\left(\frac{n\pi}{a}\right)\frac{[B_y(o)e^{-in\pi} - B_y(a)]}{(-1)^n a[(\frac{n\pi}{a})^2 + k_z^2]} *$$

$$* \left[\frac{1}{\Lambda_L(\frac{n\pi}{a} - i\delta_n\varepsilon)} + \frac{\omega}{\Lambda_T(\frac{n\pi}{a} - i\delta_n\varepsilon)}\right] \quad , \qquad (214)$$

where

$$\delta_n = 1 \; ; \quad \frac{n\pi}{a} > \delta_o \equiv \frac{\omega}{c}\cos\theta$$

and

$$\delta_n = -1 \; ; \quad \frac{n\pi}{a} < -\alpha_o$$

and

$$\delta_n = 0 \; ; \quad -\alpha_o < \frac{n\pi}{a} < \alpha_o$$

The quantity δ_n is a bookkeeping symbol to remind us that $\Lambda_T(k_x)$ and $\Lambda_L(k_x)$ are discontinuous along the real k_x axis when $|k_x| > \frac{\omega}{c}\cos\theta$.

Letting $\varepsilon \to 0$ and making use of Equations (B-5) and (B-6), we find:

$$E_x(x) = \frac{2k_z c}{a} \sum_{0<n<\frac{a\alpha_o}{\pi}} \frac{n\pi}{a} \sin(\frac{n\pi x}{a}) \frac{[B_y(o) - (-1)^n B_y(a)]}{[(\frac{n\pi}{a})^2 + k_z^2]} \left[\frac{1}{\Lambda_L(\frac{n\pi}{a})} + \frac{\omega}{\Lambda_T(\frac{n\pi}{a})} \right]$$

$$+ \frac{2k_z c}{a} \sum_{n>\frac{a\alpha_o}{\pi}}^{\infty} \frac{n\pi}{a} \sin(\frac{n\pi x}{a}) \frac{[B_y(o) - (-1)^n B_y(a)]}{[(\frac{n\pi}{a})^2 + k_z^2]} \left[\frac{1}{\Lambda_L^-(\frac{n\pi}{a})} + \frac{\omega}{\Lambda_T^-(\frac{n\pi}{a})} \right] . \tag{215}$$

The quantity $E_z(x)$ is found in a similar way. We obtain:

$$E_z(x) = -\frac{2ic}{a} \sum_{0<n<\frac{\alpha_o a}{\pi}} \cos(\frac{n\pi x}{a}) \frac{[B_y(o) - (-1)^n B_y(a)]}{[(\frac{n\pi}{a})^2 + k_z^2]} \left[\frac{k_z^2}{\Lambda_L(\frac{n\pi}{a})} - \frac{\omega(\frac{n\pi}{a})^2}{\Lambda_T(\frac{n\pi}{a})} \right]$$

$$- \frac{2ic}{a} \sum_{n>\frac{\alpha_o a}{\pi}} \cos(\frac{n\pi x}{a}) \frac{[B_y(o) - (-1)^n B_y(a)]}{[(\frac{n\pi}{a})^2 + k_z^2]} \left[\frac{k_z^2}{\Lambda_L^-(\frac{n\pi}{a})} - \frac{\omega(\frac{n\pi}{a})^2}{\Lambda_T^-(\frac{n\pi}{a})} \right] . \tag{216}$$

Finally, we find:

$$B_y(x) = \frac{2c^2}{a} \sum_{n<\frac{a\alpha_o}{\pi}} (\frac{n\pi}{a}) \sin(\frac{n\pi x}{a}) \frac{[B_y(o) - (-1)^n B_y(a)]}{\Lambda_T(n\pi/a)}$$

$$+ \frac{2c^2}{a} \sum_{n>\frac{\alpha_o a}{\pi}} (\frac{n\pi}{a}) \sin(\frac{n\pi x}{a}) \frac{[B_y(o) - (-1)^n B_y(a)]}{\Lambda_T^-(\frac{n\pi}{a})} . \tag{217}$$

We are now in a position to study the transverse and longitudinal geometrical resonances that can be supported by a hot plasma slab of thickness a .

4.7.1 Geometrical Resonances.

In order to make further progress on the slab problem, we note that $E_x(x)$ and $E_z(x)$ become infinitely large as $\Lambda_L(\frac{n\pi}{a})$ and $\Lambda_T(\frac{n\pi}{a})$ go to zero. Similarly, $B_y(x)$ becomes large without bound as $\Lambda_T(\frac{n\pi}{a})$ goes to zero. In order to understand the physical significance of these resonances, we obtain the large β approximation to the roots of $\Lambda_T(\frac{n\pi}{a})$ and $\Lambda_L(\frac{n\pi}{a})$. A complete analysis of these resonances would require a rather detailed study. The following investigation is incomplete in that it only treats the most general aspects of the problem.

4.7.2 Zeros of $\Lambda_T(\frac{n\pi}{a})$.

The approximate zeros of $\Lambda_T(\frac{n\pi}{a})$ may be found by taking the large β limit of Equation (E-3). We find:

$$0 = \frac{n^2\pi^2c^2}{a^2} - \omega^2\cos^2\theta + \omega_p^2\left[(1 - 3/2\beta) - \frac{\cos^2\theta}{\beta} + \frac{n^2\pi^2c^2}{\beta a^2\omega^2}\right] . \tag{218}$$

It is convenient first to express Equation (218) in terms of the incident waves' wavelength. We define a modified wavelength $ƛ$ as $ƛ \equiv \lambda/2\pi = c/\omega$ and obtain from Equation (218):

$$ƛ_n^2 = \frac{\sqrt{A_n^2 + B_n} - A_n}{\dfrac{2n^2\pi^2\omega_p^2}{\beta a^2}} , \tag{219}$$

where

$$A_n \equiv \frac{n^2\pi^2c^2}{a^2} + \omega_p^{\,2}\ \{1 - \frac{(3/2 + \cos^2\theta)}{\beta}\}$$

and

$$B_n \equiv \frac{4n^2\pi^2c^2\omega_p^{\,2}\cos^2\theta}{\beta a^2}$$

and where $0 < n < \frac{a\alpha_o}{\pi} = \frac{a\omega\cos\theta}{\pi c}$.

This expression is considerably simplified by noting that $A_n >> \sqrt{B_n}$; i.e.,

$$\frac{n^2\pi^2c^2}{a^2} + \omega_p^{\,2}\{1 - \frac{(3/2 + \cos^2\theta)}{\beta}\} >> \frac{2n\pi c\omega_p\cos\theta}{\sqrt{\beta}\ a}$$

Basically, this means that

$$\omega_p^{\,2} >> \frac{2n\pi c\omega_p\cos\theta}{\sqrt{\beta}\ a}$$

Since

$$\sqrt{\beta} = \sqrt{\frac{mc^2}{KT}} \doteq \frac{c}{u} = \frac{c}{\omega_p\lambda_D} ,$$

where $u \equiv \sqrt{\frac{KT}{m}}$ and λ_D is the Debye length, the above inequality then reduces to:

$$1 >> 2\pi n\cos\theta\ \frac{\lambda_D}{a}$$

where

$$(n)_{max} \doteq \frac{\omega a\cos\theta}{c}$$

This inequality under its worst condition requires that:

$$\lambda \gg \lambda_D \cos^2\theta \tag{220}$$

which is in essence the condition of a "sharp boundary" that we assumed existed throughout the development of this work. Since the inequality in Equation (220) is always assumed valid, we approximate Equation (219) as:

$$\lambda_n^2 \doteq \frac{c^2\cos^2\theta}{\frac{n^2\pi^2c^2}{a^2} + \omega_p^2\{1 - \frac{(3/2 + \cos^2\theta)}{\beta}\}} \tag{221}$$

A transverse geometrical resonance occurs when the plasma's electrons reradiate in phase with the incident electromagnetic wave.

4.7.3 Zeros of $\Lambda_L(\frac{n\pi}{a})$.

The approximate zeros of $\Lambda_L(\frac{n\pi}{a})$ may be found by taking the large β limit of Equation (E-6) and setting the results to zero. These roots are given by:

$$\lambda_n^2 = \frac{\sqrt{C_n^2 + D_n} - C_n}{\frac{6n^2\pi^2}{\beta a^2}} , \tag{222}$$

where

$$C_n \equiv \left[1 + \frac{(1/2 - 3\cos^2\theta)}{\beta}\right]$$

and

$$D_n \equiv \frac{12\pi^2c^2n^2}{\beta a^2\omega_p^2}$$

Equation (222) can be simplified under certain limiting cases. We will consider one of these. When $C_n >> D_n$; i.e., $\frac{\beta\omega^2}{\omega_p^2} >> 12\pi^2\cos^2\theta$, Equation (221) may be written as:

$$\kappa_n^2 = \frac{c^2}{\omega_p^2}\left[1 - \frac{(1/2 - 3\cos^2\theta + 6\pi^2 c^2 n^2/a^2\omega_p^2)}{\beta}\right] . \qquad (223)$$

As expected, the zero temperature resonance occurs when $\omega = \omega_p$.

5. Conclusions

In the foregoing, a relativistic kinetic theory description of the interaction of a plane p-polarized electromagnetic wave obliquely incident on a hot plasma half space and a plasma slab was developed. The analysis was based on the use of the linearized relativistic collisionless Boltzmann equation. The collisionless approximation for a hot tenuous plasma requires that $\omega \gg \nu_{ei}$ where

$$\nu_{ei} = 3.62 \times 10^{-6}(n/Te^{3/2})\ln\Lambda$$

with $\ln\Lambda \doteq 10$, Te in degrees Kelvin and n in electrons/m^3. The implication of this expression becomes clear from Table 1 where the collision frequency is given for $n = 10^{18}/m^3$ and several temperatures.

Table 1. Values of ν_{ei} versus Te with $n = 10^{18}/m^3$.

Te (^oK)	β	ν_{ei} (sec^{-1})
10^8	5.9×10	36.2
10^6	5.9×10^3	3.62×10^4
10^4	5.9×10^5	3.62×10^7

The corresponding plasma frequency is $\omega_p = 5.65 \times 10^{10}$/sec. Recalling that the strong spatial dispersion analysis requires that $\omega_p^2/\omega^2 \gg \beta$, we find that this part of the analysis should be valid so long as

$$\nu_{ei}^2 < \omega^2 \ll \frac{\omega_p^2}{\beta}$$

For $Te \geq 10^4$ and $n = 10^{18}/m^3$, such a frequency range always exists.

A second condition which must be satisfied in order to use the collisionless theory for skin depth calculations is that the mean free path for collisions ℓ be large compared to the skin depth. For $n = 10^{18}/m^3$ and $Te = 10^{4o}K$, $\ell \doteq 2$ cm. This is approximately the same size as the anomalous skin depth.

Measurements of the anomalous skin effect have recently been reported by Kofoid.[15,16] However, these experiments violate the condition $\omega > \nu_{ei}$ and our theory cannot be used without modification. To the author's knowledge, no experiments have yet been performed when $\omega > \nu$.

In the present investigation, the specular reflection boundary condition was used. This condition played a significant role in simplifying the mathematical development of the theoretical model. Such a condition is conventionally used to describe the reflection of an electron from a plasma sheath as found, for instance, at the walls of a discharge. A different boundary condition that is appropriate for particle generation near the wall is the diffuse boundary condition. Here, particles coming from the body of the plasma are absorbed at the boundary and new particles with a different velocity distribution are emitted. The sharp boundary requires that the incoming wave experience a sharp change in propagation media. This requires that

$$\lambda \gg \left. \frac{n}{\frac{dn}{dx}} \right|_{x=0}$$

where n is the electron density. For a sheath boundary, this would be approximately satisfied by $\lambda \gg \lambda_D$, the Debye length.

Finally, we should mention that the theory makes no provision for the creation and destruction of particles which would occur in thermonuclear reactions, nor does it provide a means of treating the Bremsstrahlung radiation fields expected to be important in high temperature devices.

Several extensions of this work are suggested. Silin and Fetisov[2] investigated the penetration of an obliquely incident s and p-polarized electromagnetic wave on a hot plasma half space for both specular and diffuse boundary conditions, but gave no derivation for their field solutions within the medium. Such a derivation has yet to be given for the case of diffuse reflection of electrons whose equilibrium state is described by the relativistic Maxwell-Boltzmann distribution. Silin[1] claims that the absorption coefficient under weak spatial dispersion is greater by a factor of β for diffuse reflection in the large β limit. Stepanov[17] determined the depth of penetration of a circularly polarized electromagnetic wave normally incident on a plasma half space for the case where the external magnetic field is perpendicular to the plasma boundary. He used the non-relatitivistic kinetic equations to describe ion and electron behavior and used the specular reflection boundary condition for both ions and electrons. Again, no deviation was presented. Considering Silin's claim that the diffuse absorption coefficient can be as large as β times that for the specular case, it would be of interest to apply the diffuse boundary condition to Stepanov's problem. Finally, we should mention that our formulation lends itself to the kinetic theory description of the surface wave-body wave problem.[18] However, the analysis of $\Lambda_T(k_x)$ and $\Lambda_L(k_x)$ is complicated by the fact that $(\omega\gamma - k_z u_z)$ can be negative.

APPENDIX A

In obtaining Equations (12) and (13), we used:

$$S(k_x) \equiv \int \frac{u_1 \frac{\partial F_o}{\partial u_3} du^3}{(\gamma\omega - ku_1)} = \int \frac{u_3 \frac{\partial F_o}{\partial u_1} du^3}{(\gamma\omega - ku_1)} = 0 \quad . \tag{A-1}$$

Since u_1 and u_3 are "complex velocities," it is not evident that Equation (A-1) holds. The proof follows. Rewriting Equation (A-1) in the original Cartesian system, we obtain:

$$S(k_x) = \frac{R(k_x)}{(k_x^2 + k_z^2)} \quad , \tag{A-2}$$

where

$$R(k_x) \equiv \int \frac{(k_x u_x + k_z u_z)(k_x \frac{\partial F_o}{\partial u_z} - k_z \frac{\partial F_o}{\partial u_x})\, du^3}{(\gamma\omega - k_x u_x - k_z u_z)}$$

$$= (k_x^2 - k_z^2)A_{xz} + k_x k_z (A_{zz} - A_{xx}) \tag{A-3}$$

and

$$A_{ij} \equiv \int \frac{u_i \frac{\partial F_o}{\partial u_j} du^3}{(\gamma\omega - k_x u_x - k_z u_z)} \tag{A-4}$$

The properties of A_{ij} are studied in Appendix B. Using Equation (B-17) it is found that

$$R^+(k_1) = R^-(k_1) = 0 \tag{A-5}$$

which holds for all $k_1 \equiv Re(k_x)$.

Rewriting

$$R(k_x) = \omega \int \frac{\gamma(k_x \frac{\partial F_o}{\partial u_z} - k_z \frac{\partial F_o}{\partial u_x})\, du^3}{(\gamma\omega - k_x u_x - k_z u_z)} \quad , \qquad \text{(A-6)}$$

it is clear that

$$\oint_{C_1} R(k_x)\, dk_x = \oint_{C_2} R(k_x)\, dk_x = 0 \quad , \qquad \text{(A-7)}$$

where C_1 and C_2 are the closed contours shown in Figure 9. Taking the limit as $\rho \to \infty$ and letting the horizontal line integrals coalesce onto the real k_x axis, we conclude from Morera's theorem that $R(k_x)$ is everywhere analytic. Since $R(k_x)$ is everywhere bounded, we conclude from Liouville's theorem that

$$R(k_x) = \text{constant} = 0$$

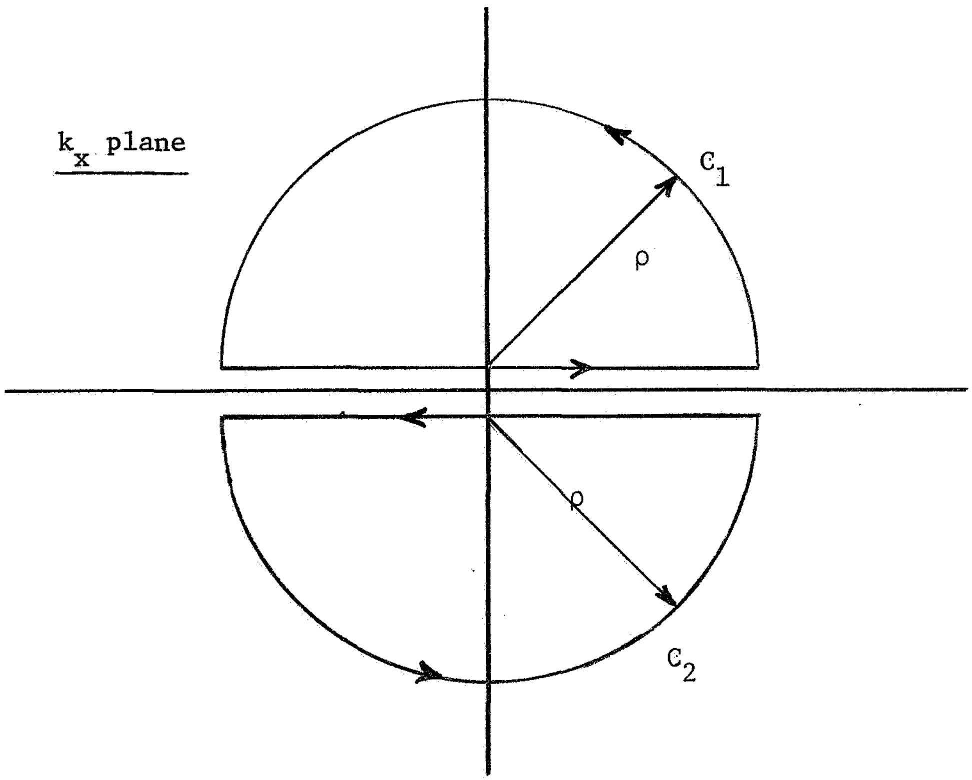


Figure 9. Closed Contours C_1 and C_2 in k_x Plane

APPENDIX B

B.1 Cut Analysis of $\Lambda_L(k_x)$ and $\Lambda_T(k_x)$

It is clear from the definitions of $\Lambda_L(k_x)$ and $\Lambda_T(k_x)$, Equations (20) and (21), that these functions become discontinuous somewhere along the real k_x axis. In order to locate the extent of the corresponding cut(s) in the k_x plane, we focus our attention on a typical integral entering in the definitions of these functions. Making use of the definition of $A_{ij}(k_x)$ given in Equation (A-4) and the Maxwell-Boltzmann-(Jüttner) distribution to describe the equilibrium state, we find:

$$A_{ij}(k_x) = -\frac{\beta^2}{4\pi c^5 K_2(\beta)} \int \frac{u_i u_j e^{-\beta\gamma} du^3}{\gamma(\gamma\omega - k_x u_x - k_z u_z)} , \qquad \text{(B-1)}$$

where u_i and u_j are either u_x or u_z. Calling k_1 the real part of k_x, we define the following useful functions:

$$A_{ij}^{\pm}(\pm k_1) \equiv \lim_{\varepsilon \to o} A_{ij}(\pm k_1 \pm i\varepsilon) , \qquad \text{(B-2)}$$

where $k_1 > 0$ and $\varepsilon > 0$. Making use of the isotropic nature of F_o, we readily obtain:

$$\left.\begin{aligned} A_{xx}^{+}(k_1) &= A_{xx}^{-}(-k_1) \\ A_{xz}^{+}(k_1) &= -A_{xz}^{-}(-k_1) \\ A_{zz}^{+}(k_1) &= A_{zz}^{-}(-k_1) \end{aligned}\right\} \qquad \text{(B-3)}$$

$$\left.\begin{aligned} A_{xx}^-(k_1) &= A_{xx}^+(-k_1) \\ A_{xz}^-(k_1) &= -A_{xz}^+(-k_1) \\ A_{zz}^-(k_1) &= A_{zz}^+(-k_1) \end{aligned}\right\} \tag{B-4}$$

Writing the dispersion functions in terms of the $A_{ij}(k_x)$ functions and using Equations (B-3) and (B-4), we find:

$$\left.\begin{aligned} \Lambda_L^+(k_1) &= \Lambda_L^-(-k_1) \\ \Lambda_L^+(-k_1) &= \Lambda_L^-(k_1) \end{aligned}\right\} , \tag{B-5}$$

and

$$\left.\begin{aligned} \Lambda_T^+(k_1) &= \Lambda_T^-(-k_1) \\ \Lambda_T^+(-k_1) &= \Lambda_T^-(k_1) \end{aligned}\right\} \tag{B-6}$$

Since k_x is the only complex number entering into the definitions of $\Lambda_L(k_x)$ and $\Lambda_T(k_x)$, it is clear that:

$$\left.\begin{aligned} [\Lambda_L^+(k_1)]^* &= \Lambda_L^-(k_1) \\ [\Lambda_L^+(-k_1)]^* &= \Lambda_L^-(-k_1) \end{aligned}\right\} \tag{B-7}$$

and

$$\left.\begin{aligned} [\Lambda_T^+(k_1)]^* &= \Lambda_T^-(k_1) \\ [\Lambda_T^+(-k_1)]^* &= \Lambda_T^-(-k_1) \end{aligned}\right\} \tag{B-8}$$

We conclude from Equations (B-7) and (B-8) that $\Lambda_L(k_x)$ and $\Lambda_T(k_x)$ are discontinuous across the real k_x axis whenever these functions take on complex values for k_x real. To insure single-valuedness, a cut is said to exist along that part of the real k_x axis on which $\Lambda_L(k_x)$ and $\Lambda_T(k_x)$ are complex.

B.2 Roots of the Dispersion Functions

Given that κ_T is a root of the equation $\Lambda_T(k_x) = 0$ and using the isotropic nature of F_o, it follows that $-\kappa_T$, κ_T^*, and $-\kappa_T^*$ are also roots. Similarly, we find that if κ_L is a root of $\Lambda_L(k_x) = 0$, then $-\kappa_L$, κ_L^*, and $-\kappa_L^*$ are also roots.

The location of these roots in the complex k_x plane is determined by mapping $\Lambda_L(k_x)$ and $\Lambda_T(k_x)$ and applying the argument principle to the results.

B.3 Evaluation of $A_{ij}^+(k_1)$

In order to analyze the functions $\Lambda_L(k_x)$ and $\Lambda_T(k_x)$ on the real k_x axis, it is useful to reduce the volume integral definitions of these functions to single integral definitions. By definition,

$$A_{ij}^+(k_1) \equiv -\frac{\beta^2}{4\pi c^5 K_2(\beta)} \lim_{\varepsilon\to o} \int \frac{u_i u_j e^{-\beta\gamma} du^3}{\gamma[\gamma\omega - (k_1 + i\varepsilon)u_x - k_z u_z]} . \quad \text{(B-9)}$$

In this analysis, it is convenient to express the denominator of the above integrand as an integral over a damped plane wave. With this goal in mind, we consider:

$$\frac{1}{\gamma\omega - (k_1+i\varepsilon)u_x - k_z u_z} = \frac{k_1}{(k_1 + i\varepsilon)} \frac{1}{\eta(\omega\gamma - k_z u_z) - k_1 u_x - i\delta}$$

$$= \frac{ik_1}{(k_1 + i\varepsilon)} \int_0^\infty e^{-\delta t} e^{-i[\eta(\omega\gamma - k_z u_z) - k_1 u_x]t} dt \quad , \tag{B-10}$$

where

$$\eta \equiv \frac{k_1^{\,2}}{k_1^{\,2} + \varepsilon^2} > 0 \quad ,$$

$$\delta \equiv \frac{\varepsilon k_1(\omega\gamma - k_z u_z)}{k_1^{\,2} + \varepsilon^2} > 0 \quad ,$$

$$\varepsilon, k_1 > 0 \quad ,$$

and where we used

$$(\omega\gamma - k_z u_z) = \omega\gamma(1 - \frac{v_z}{c}\sin\theta) > 0 \quad .$$

Equation (B-9) may then be rewritten as:

$$A_{ij}^{\ +}(k_1) = -\frac{\beta^2}{4\pi c^5 K_2(\beta)} \int_0^\infty \Phi_{ij}^{\ +}(k_1,t)dt \quad , \tag{B-11}$$

where

$$\Phi_{ij}^{\ +}(k_1,t) = \lim_{\varepsilon\to 0} \frac{ik_1}{(k_1+i\varepsilon)} \int_{\bar{u}} \frac{u_i u_j e^{-\beta\gamma} e^{-\delta t} e^{-i[\eta(\omega\gamma - k_z u_z) - k_1 u_x]t}}{\gamma} du^3$$

$$= i \int_{\bar{u}} \frac{u_i u_j e^{-\beta\gamma} e^{-i(\omega\gamma - \bar{k}\cdot\bar{u})t}}{\gamma} du^3 \quad , \tag{B-12}$$

with $\overline{k}\cdot\overline{u}$ defined as $\lim_{\varepsilon\to o} (k_1 + i\varepsilon)u_x + k_z u_z$. Rewriting Equation (B-12) as:

$$\Phi_{ij}^{+}(k_1,t) = -\frac{i}{t^2}\frac{\partial^2 L}{\partial k_i \partial k_j} \tag{B-13}$$

and going to a polar coordinate system with $\overline{k}$ (real) along the $\hat{u}_z$ axis, we obtain:

$$L = \frac{4\pi c^2}{kt}\int_0^\infty \frac{x \sin xy\ e^{-\alpha\sqrt{1+x^2}}dx}{\sqrt{1+x^2}} \quad , \tag{B-14}$$

where $y \equiv kct$

$\alpha \equiv \beta + i\omega t$

$k \equiv \sqrt{k_1^{\ 2} + k_z^{\ 2}}$

From Vol. 1, p. 75, of the Bateman Manuscript, we find:

$$L(k_1,\omega,t) = \frac{4\pi c^3 K_1(z)}{z} \quad , \tag{B-15}$$

where

$$z \equiv \sqrt{k^2c^2t^2 + (\beta + i\omega t)^2}$$

Inserting this result into Equation (B-13) and performing the differentiations, we obtain:

$$\Phi_{ij}^{+}(k_1,t) = 4\pi i c^5 \left[\delta_{ij}\frac{K_2(z)}{z^2} - k_i k_j c^2 t^2 \frac{K_3(z)}{z^3}\right] \quad . \tag{B-16}$$

Equation (B-11), therefore, becomes:

$$A_{ij}^{+}(k_1) = -\frac{i\beta^2}{K_2(\beta)} \int_0^\infty \left[\delta_{ij} \frac{K_2(z)}{z^2} - k_i k_j c^2 t^2 \frac{K_3(z)}{z^3} \right] dt . \quad \text{(B-17)}$$

This derivation follows that given by Ozizmir.

B.4 Proof that $\Lambda_T(\pm ik_z) + \omega\Lambda_L(\pm ik_z) = 0$

Writing $\Lambda_L(k_x)$ and $\Lambda_T(k_x)$ in terms of the $A_{ij}(k_x)$ functions and using Equation (B-17), we obtain:

$$\Lambda_L^{+}(k_1) = \omega - \frac{i\omega_\rho^2\beta^2}{K_2(\beta)} \left[\int_0^\infty \frac{K_2(z)dt}{z^2} - k^2c^2 \int_0^\infty \frac{t^2K_3(z)dt}{z^3} \right]$$

$$= \omega + \frac{\omega\ \omega_\rho^2\beta^2}{K_2(\beta)} \int_0^\infty t(\beta + i\omega t) \frac{K_3(z)}{z^3}\, dt \quad \text{(B-18)}$$

and

$$\Lambda_T^{+}(k_1) = k^2c^2 - \omega^2 + \frac{i\omega\omega_\rho^2\beta^2}{K_2(\beta)} \int_0^\infty \frac{K_2(z)}{z^2}\, dt \quad . \quad \text{(B-19)}$$

Calling $Q(k_x)$ the analytic continuation of $\int_0^\infty \frac{K_2(z)}{z^2}\, dt$ into the upper k_x plane and $T(k_x)$, the continuation of $\int_0^\infty \frac{t^2K_3(z)}{z^3}\, dt$ into the upper k_x plane, we readily find:

$$\Lambda_L(ik_z) = \omega - \frac{i\omega_\rho^2\beta^2}{K_2(\beta)}\, Q(ik_z) \quad \text{(B-20)}$$

and

$$\Lambda_T(ik_z) = -\,\omega \left[\omega - \frac{i\omega_\rho^2\beta^2}{K_2(\beta)}\, Q(ik_z) \right] \quad \text{(B-21)}$$

Therefore,

$$\Lambda_T(ik_z) + \omega\Lambda_L(ik_z) = 0 \quad . \tag{B-22}$$

Similarly, we can show that:

$$\Lambda_T(-ik_z) + \omega\Lambda_L(-ik_z) = 0 \quad . \tag{B-23}$$

B.5 Analysis of $\Lambda_T^+(k_1)$

In order to locate the cuts of $\Lambda_T^+(k_1)$ along the real k_x axis, it is useful to express $\Lambda_T^+(k_1)$ in terms of its real and imaginary parts. With this goal in mind, we define a new function $I(k)$ as:

$$I(k) \equiv \int_0^\infty \frac{K_2(z)}{z^2}\, dt \tag{B-24}$$

When $(k^2c^2 - \omega^2) > 0$, we introduce a new variable of integration:

$$\psi \equiv \frac{\omega}{\beta\sigma}\left(t + \frac{i\beta\sigma^2}{\omega}\right) \tag{B-25}$$

and rewrite Equation (B-24) as:

$$I(k) = \frac{\sigma}{\beta\omega} \int_{i\sigma}^{\infty+i\sigma} \frac{K_2(\beta\sqrt{\psi^2 + \sigma^2 + 1}\, d\psi}{(\psi^2 + \sigma^2 + 1)} \quad , \tag{B-26}$$

where

$$\sigma \equiv \frac{\omega}{\sqrt{k^2c^2 - \omega^2}} \quad . \tag{B-27}$$

Deforming the original line integral as shown in Figure 10, we obtain as $R \to \infty$:

$$I(k) = \frac{\sigma}{\beta\omega} \int_0^{\infty} \frac{K_2(\beta\sqrt{x^2 + \sigma^2 + 1})}{(x^2 + \sigma^2 + 1)} dx - \frac{i\sigma}{\beta\omega} \int_0^{\sigma} \frac{K_2(\beta\sqrt{\sigma^2 + 1 - y^2})}{(\sigma^2 + 1 - y^2)} dy \tag{B-28}$$

When $(k^2c^2 - \omega^2) < 0$, we introduce a new variable of integration:

$$\eta \equiv \frac{i\omega}{\beta\tilde{\sigma}} (t - \frac{i\beta\tilde{\sigma}^2}{\omega})$$

and rewrite Equation (B-24) as:

$$I(k) = \frac{\tilde{\sigma}}{i\beta\omega} \int_{\tilde{\sigma}}^{\tilde{\sigma}+i\infty} \frac{K_2(\beta\sqrt{\eta^2 + 1 - \tilde{\sigma}^2})}{(\eta^2 + 1 - \tilde{\sigma}^2)} d\eta \quad , \tag{B-29}$$

where

$$\tilde{\sigma} \equiv \frac{\omega}{\sqrt{\omega^2 - k^2c^2}} \tag{B-30}$$

Deforming this line integral as shown in Figure 11 and letting $R \to \infty$, we obtain:

$$I(k) = \frac{\tilde{\sigma}}{i\beta\omega} \int_{\tilde{\sigma}}^{\infty} \frac{K_2(\beta\sqrt{x^2 + 1 - \tilde{\sigma}^2}}{(x^2 + 1 - \tilde{\sigma}^2)} dx \tag{B-31}$$

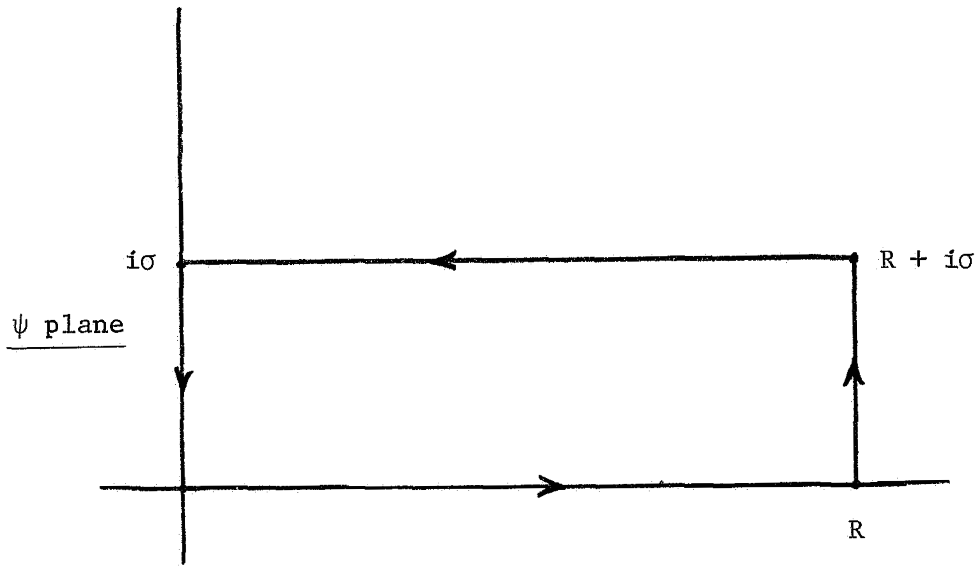


Figure 10. Contour Deformation in Complex ψ Plane When $(k^2c^2 - \omega^2) > 0$

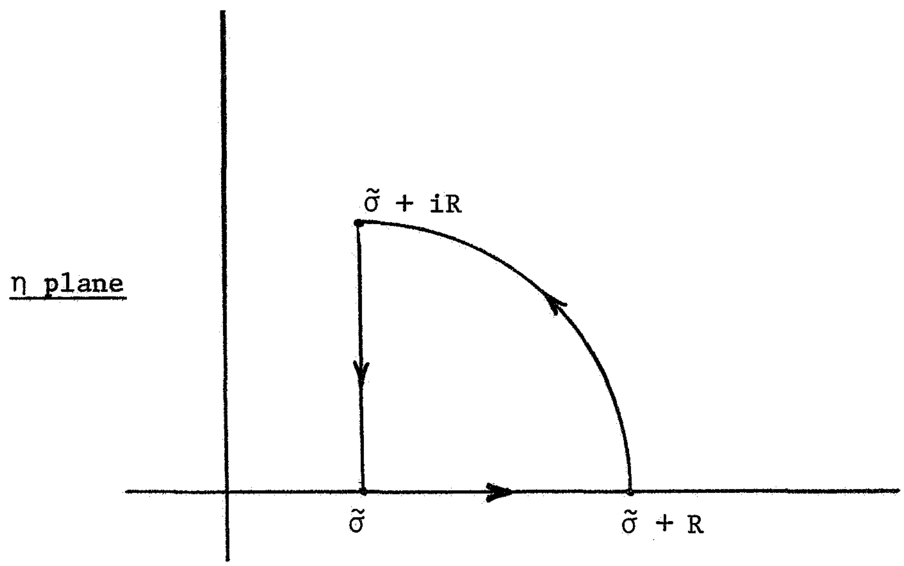


Figure 11. Contour Deformation in Complex η Plane When $(k^2c^2 - \omega^2) < 0$

To summarize, the analysis of $\Lambda_T^+(k_1)$ can be divided into two parts. When $(k^2c^2 - \omega^2) < 0$, we have:

$$\Lambda_T^+(k_1) = (k_1^2 + k_z^2)c^2 - \omega^2 + \frac{\beta\tilde{\sigma}\omega_\rho^2}{K_2(\beta)} \int_{\tilde{\sigma}}^{\infty} \frac{K_2(\beta\sqrt{x^2 + 1 - \tilde{\sigma}^2})}{(x^2 + 1 - \tilde{\sigma}^2)} dx . \tag{B-32}$$

When $(k^2c^2 - \omega^2) > 0$, $\Lambda_T^+(k_1)$ may be written as:

$$\Lambda_T^+(k_1) = (k_1^2 + k_z^2)c^2 - \omega^2 + \frac{\beta\sigma\omega_\rho^2}{K_2(\beta)} \int_0^{\sigma} \frac{K_2(\beta\sqrt{\sigma^2 + 1 - y^2})}{(\sigma^2 + 1 - y^2)} dy$$

$$+ \frac{i\beta\sigma\omega_p^2}{K_2(\beta)} \int_0^{\infty} \frac{K_2(\beta\sqrt{x^2 + \sigma^2 + 1})}{(x^2 + \sigma^2 + 1)} dx \quad , \tag{B-33}$$

where $\text{Im}[\Lambda_T^+(k_1)]$ may be integrated. We obtain:

$$\text{Im}[\Lambda_T^+(k_1)] = \frac{\pi\sigma\omega_\rho^2 e^{-\beta\sqrt{\sigma^2 + 1}}}{2K_2(\beta)} \left[1 + \frac{1}{\beta\sqrt{\sigma^2 + 1}} \right] . \tag{B-34}$$

B.6 Analysis of $\Lambda_L^+(k_1)$

The analysis of $\Lambda_L^+(k_1)$ proceeds along the very same lines as that outlined for $\Lambda_T^+(k_1)$. To avoid repetition, we simply state the results. When $(k^2c^2 - \omega^2) < 0$, $\Lambda_L^+(k_1)$ may be written as:

$$\Lambda_L^+(k_1) = \omega - \frac{\beta\tilde{\sigma}^3\omega_\rho^2}{\omega K_2(\beta)} \int_0^{\infty} \frac{x\, K_2(\beta\sqrt{x^2 + 1})}{(x^2 + \tilde{\sigma}^2)^{3/2}} dx \tag{B-35}$$

When $(k^2c^2 - \omega^2) > 0$, we obtain:

$$\Lambda_L^+(k_1) = \omega + \frac{\omega_\rho^2 \beta^2}{K_2(\beta)} \left(\frac{\sigma^2}{\omega}\right) \left[(2\sigma^2 + 1) \frac{K_2(\beta)}{\beta} - \sigma \int_0^\sigma \frac{(y^2 + \sigma^2 + 1)\, K_3(\beta\sqrt{\sigma^2 + 1 - y^2})}{(\sigma^2 + 1 - y^2)^{3/2}}\, dy \right]$$

$$- \frac{i\beta^2\omega_\rho^2\sigma^3}{\omega K_2(\beta)} \int_0^\infty \frac{(\sigma^2 + 1 - x^2)\, K_3(\beta\sqrt{x^2 + \sigma^2 + 1})}{(x^2 + \sigma^2 + 1)^{3/2}}\, dx \,. \tag{B-36}$$

Integrating the expression for $\mathrm{Im}[\Lambda_L^+(k_1)]$, we obtain:

$$\mathrm{Im}[\Lambda_L^+(k_1)] = - \frac{\pi\omega_\rho^2\sigma^3 e^{-\beta\sqrt{\sigma^2 + 1}}}{2\omega\beta K_2(\beta)(\sigma^2 + 1)^{3/2}} \left[\beta^2(\sigma^2 + 1) + 2\beta\sqrt{\sigma^2 + 1} + 2 \right]. \tag{B-37}$$

It is therefore clear from Equations (B-7), (B-8), (B-33) and (B-36) that $\Lambda_L(k_x)$ and $\Lambda_T(k_x)$ are discontinuous across that part of the real axis defined by $|k_x| > \frac{\omega}{c}\cos\theta \equiv \alpha_o$. The corresponding cuts are shown in Figure 2.

B.7 Mapping of $\Lambda_T(k_x)$ -- Location of the Roots

The number and location of the zeros of the dispersion functions can be found by the argument principle. We first turn our attention to the analysis of the somewhat simpler transverse dispersion function.

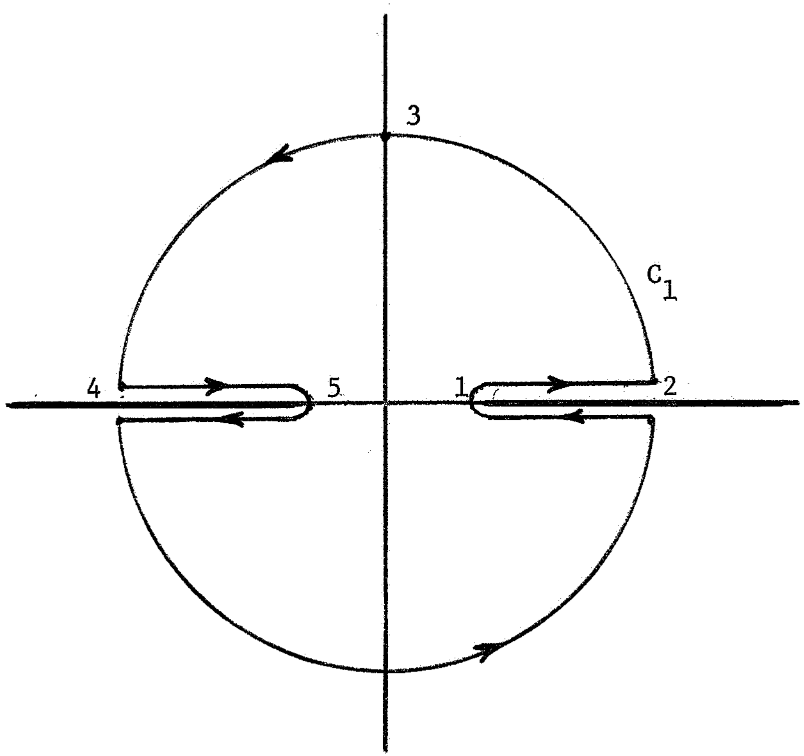
As k_x completes one tour along C_1 (Figure 12), $\Lambda_T(k_x)$ twice traverses the contour C_2 (Figure 13). Since

$$\Lambda_T^+(\alpha_o) \equiv \lim_{\tilde{\sigma}\to\infty} \left[-\frac{\omega^2}{\tilde{\sigma}^2} + \frac{\beta\tilde{\sigma}\omega_\rho^{\ 2}}{K_2(\beta)} \int_{\tilde{\sigma}}^{\infty} \frac{K_2(\beta\sqrt{x^2+1-\tilde{\sigma}^2}}{(x^2+1-\tilde{\sigma}^2)} dx \right]$$

$$= \omega_\rho^{\ 2} \frac{K_1(\beta)}{K_2(\beta)} \quad , \qquad \text{(B-38)}$$

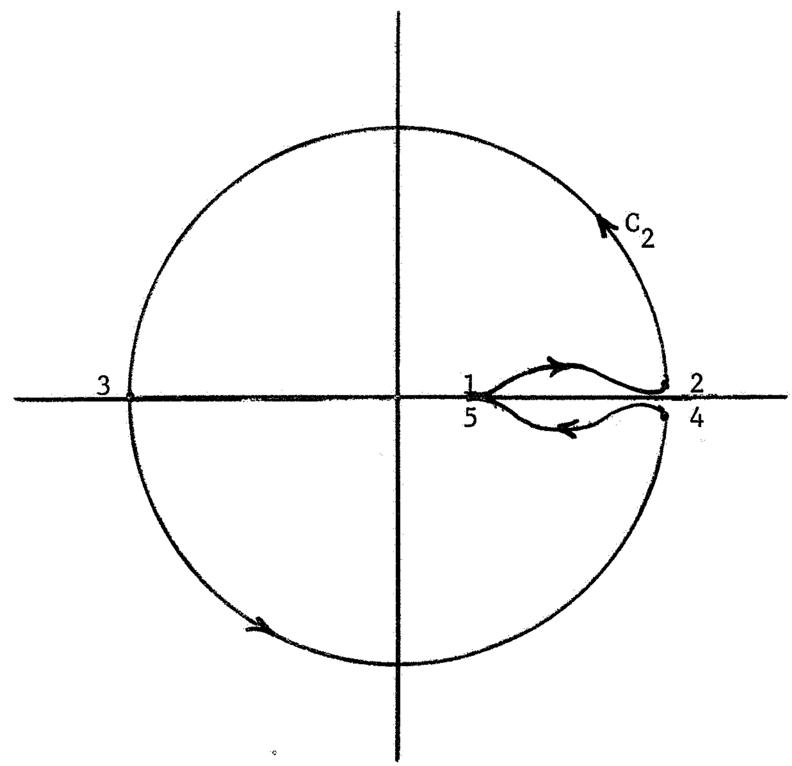
we find that there are always two roots of $\Lambda_T(k_x) = 0$ in the entire cut k_x plane. In mapping $\Lambda_T(k_x)$, we made use of the cut properties of this function given in Equations (B-6) and (B-8). Since $\text{Im}[\Lambda_T^+(k_1)]$ does not change sign as k_1 varies from $\alpha_o \to \infty$, the exact behavior of $\text{Re}[\Lambda_T^+(k_1)]$ is not critical to this analysis. Only the general behavior $\text{Re}[\Lambda_T^+(k_1)]$ was shown in Figure 12. We found that κ_T, $-\kappa_T$, κ_T^* and $-\kappa_T^*$ are all roots of $\Lambda_T(k_x) = 0$. The fact that only two roots exist implies that these roots are either real or pure imaginary. When real, they lie in the open interval $(-\alpha_o, \alpha_o)$.

The location of these two roots can be determined by performing one additional mapping as shown in Figures 14 and 15 where, for illustrative purposes, $\Lambda_T^+(o)$ (point 1 in Figure 15) was shown lying to the right of the origin in the $\Lambda_T(k_x)$ plane.



k_x plane

Figure 12. The C_1 Contour



$\Lambda_T(k_x)$ plane

Figure 13. The C_2 Contour

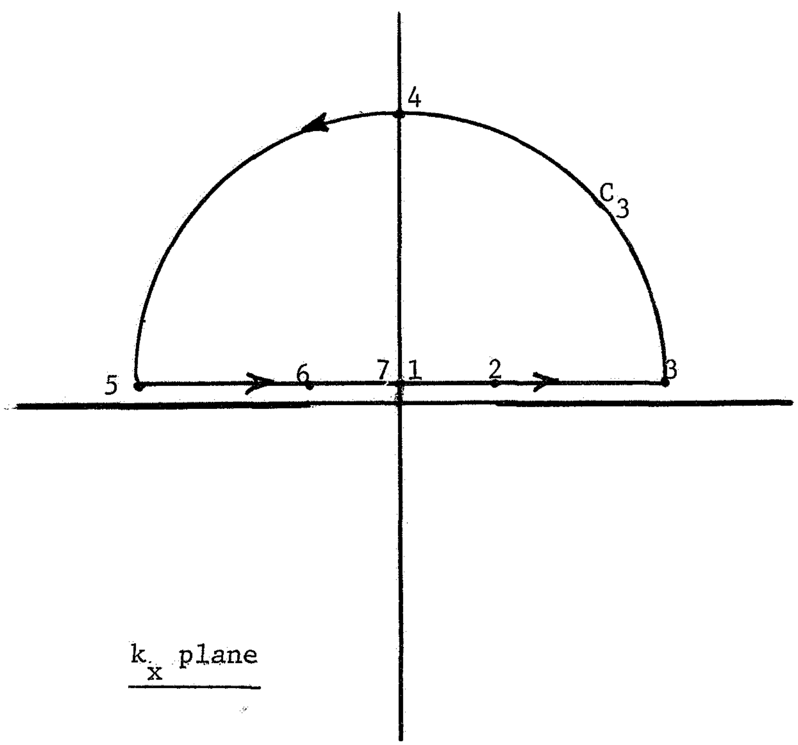


Figure 14. The C_3 Contour

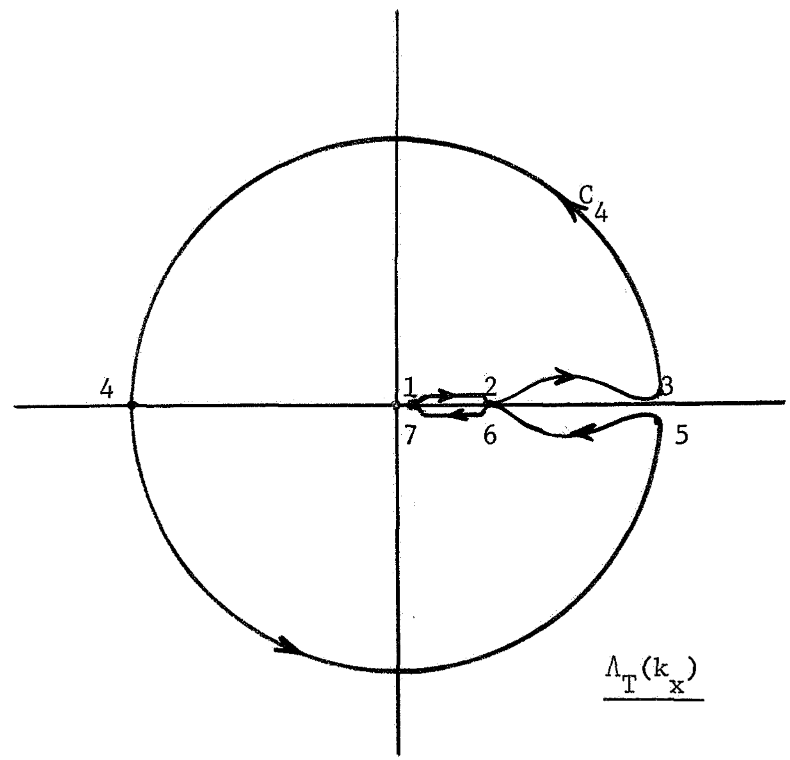


Figure 15. The C_4 Contour

In this second mapping, we made use of the fact that

$$\Lambda_T^{+}(o) \leq \Lambda_T^{+}(k_1) \leq \Lambda_T^{+}(\alpha_o) \; ; \quad 0 \leq k_1 \leq \alpha_o \quad , \qquad \text{(B-39)}$$

where

$$\Lambda_T^{+}(o) = -\frac{\omega^2}{\tilde{\sigma}_o^{\,2}} + \frac{\beta\tilde{\sigma}_o\omega_\rho^{\,2}}{K_2(\beta)} \int_0^\infty \frac{K_2(\beta\sqrt{y^2+1})y}{(y^2+1)\sqrt{y^2+\tilde{\sigma}_o^{\,2}}}\,dy \qquad \text{(B-40)}$$

and

$$\tilde{\sigma}_o = \frac{1}{\cos\,\theta} \qquad \text{(B-41)}$$

As k_x makes one tour of the contour C_3 , $\Lambda_T(k_x)$ traces one circuit of contour C_4 , thus indicating the existence of one pure imaginary root of $\Lambda_T(k_x) = 0$ in the upper k_x plane. A similar analysis shows the presence of a pure imaginary root in lower k_x plane.

B.8 Summary on Roots of $\Lambda_T(k_x) = 0$

I) $\Lambda_T(k_x) = 0$ always has two zeros in the entire "cut" k_x plane.

II) These roots are pure imaginary when $\Lambda_T^{+}(o) > 0$.

III) They are real and in the open interval $(-\alpha_o, \alpha_o)$ otherwise.

B.9 Mapping of $\Lambda_L(k_x)$ -- Location of the Roots

Finding the number and location of the roots of $\Lambda_L(k_x)$ follows the same procedure as that given for $\Lambda_T(k_x)$. As k_x completes one tour along C_1 (Figure 12), $\Lambda_L(k_x)$ describes the contour C_5 shown in Figure 16. In mapping $\Lambda_L(k_x)$, we used the cut properties of this

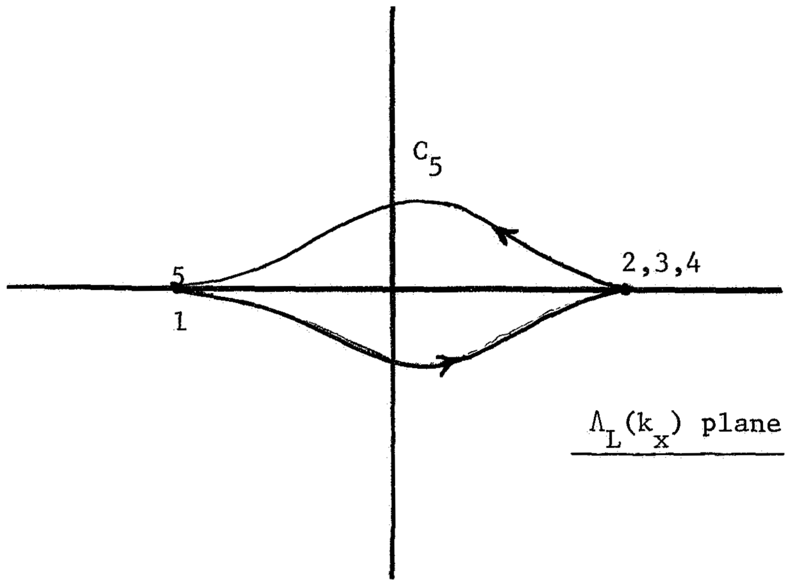


Figure 16. The C_5 Contour

function given in Equations (B-5) and (B-7) and assumed that $\Lambda_L(\alpha_o) < 0$, where

$$\Lambda_L^+(\alpha_o) = \lim_{\tilde{\sigma}\to\infty} \left[\omega - \frac{\omega_\rho^2 \beta \tilde{\sigma}^3}{\omega K_2(\beta)} \int_0^\infty \frac{x\, K_2(\beta \sqrt{x^2+1})}{(x^2+\hat{\sigma}^2)^{3/2}}\, dx \right]$$

$$= \omega - \frac{\omega_\rho^2}{\omega K_2(\beta)} \left[K_1(\beta) + 2\frac{K_o(\beta)}{\beta} \right] \qquad \text{(B-42)}$$

Equation (B-37) was also used.

Since one complete circuit of k_x on contour C_1 corresponds to two encirclements of the origin in the $\Lambda_L(k_x)$ plane, we conclude that there exists two zeros of $\Lambda_L(k_x)$ in the entire cut k_x plane when $\Lambda_L(\alpha_o) < 0$ and none otherwise. From our previous discussion on the zeros of $\Lambda_L(k_x)$, these roots must be real or pure imaginary. When real, they lie in the open interval $(-\alpha_o, \alpha_o)$.

One additional mapping is necessary to determine when these roots are real and when they are imaginary. As k_x makes one tour of the contour C_3 (Figure 14), $\Lambda_L(k_x)$ traces one circuit of contour C_6 (Figure 17), where we used the fact that

$$\Lambda_L^+(\alpha_o) \leq \Lambda_L^+(k_1) \leq \Lambda_L^+(o) \;;\quad 0 \leq k_1 \leq \alpha_o$$

and, for illustrative purposes, assumed $\Lambda_L^+(o) < 0$. There exists one pure imaginary root to $\Lambda_L(k_x) = 0$ in the upper k_x plane (and its complex conjugate in the lower k_x plane) when $\Lambda_L^+(o) < 0$, where

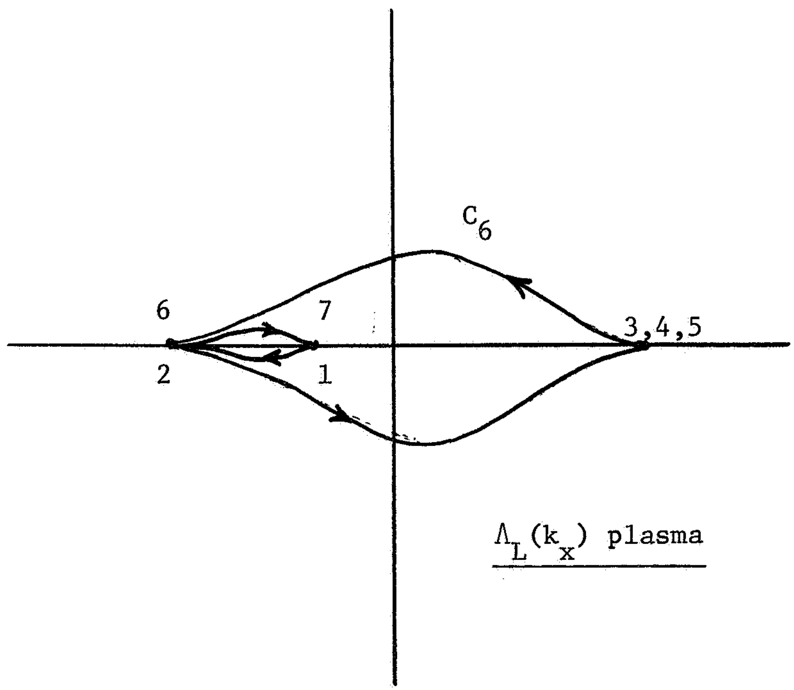


Figure 17. The C_6 Contour

$$\Lambda_L^+(o) = \omega - \frac{\omega_\rho^2 \beta \tilde{\sigma}_o^3}{\omega K_2(\beta)} \int_0^\infty \frac{x\, K_2(\beta \sqrt{x^2+1})}{(x^2 + \tilde{\sigma}_o^2)^{3/2}} dx \qquad \text{(B-43)}$$

and

$$\tilde{\sigma}_o = \frac{1}{\cos\theta}$$

B.10 Summary on Roots of $\Lambda_L(k_x) = 0$

(I) $\Lambda_L(k_x) = 0$ has two zeros in the entire cut k_x plane when $\Lambda_L^+(\alpha_o) < 0$.

(II) Pure imaginary roots exist when $\Lambda_L^+(0) < 0$.

(III) The roots are real when $\Lambda_L^+(0) > 0$.

B.11 Interrelation Between Existence of Longitudinal and Transverse Roots

Given that longitudinal roots exist, we may ask whether the plasma medium can support all four combinations of longitudinal and transverse discrete modes. To answer this question, we define two new functions:

$$A(\beta,\tilde{\sigma}_o) \equiv \frac{\beta \tilde{\sigma}_o^3}{K_2(\beta)} \int_0^\infty \frac{x K_2(\beta \sqrt{x^2+1})\, dx}{(x^2+1)\sqrt{x^2+\sigma_o^2}} \qquad \text{(B-44)}$$

and

$$B(\beta,\tilde{\sigma}_o) \equiv \frac{\beta \tilde{\sigma}_o^3}{K_2(\beta)} \int_0^\infty \frac{x K_2(\beta \sqrt{x^2+1})\, dx}{(x^2+\tilde{\sigma}_o^2)^{3/2}} \qquad \text{(B-45)}$$

and note that since $\tilde{\sigma}_o = \dfrac{1}{\cos\theta}$,

$$A(\beta,\tilde{\sigma}_o) \geq B(\beta,\tilde{\sigma}_o) \tag{B-46}$$

Expressed in terms of these new functions, Equations (B-40) and (B-43) may be rewritten as:

$$\Lambda_T^{+}(0) = \frac{\omega_\rho^{\ 2}}{\tilde{\sigma}_o^{\ 2}} \left[-\frac{\omega^2}{\omega_\rho^{\ 2}} + A \right] \tag{B-47}$$

and

$$\Lambda_L^{+}(0) = \frac{\omega_\rho^{\ 2}}{\omega} \left[\frac{\omega^2}{\omega_\rho^{\ 2}} - B \right] \tag{B-48}$$

Using Equations (B-46), (B-47) and (B-48) and the existence conditions for real or pure imaginary roots of $\Lambda_L(k_x)$ and $\Lambda_T(k_x)$ given in the Summaries, we conclude that:

(I) κ_L is necessarily real if κ_T is real.

(II) κ_T is necessarily pure imaginary if κ_L is pure imaginary.

(III) κ_L may be real when κ_T is pure imaginary.

(IV) The case where κ_L is pure imaginary and κ_T , real, is impossible.

The longitudinal and transverse roots are both pure imaginary when

$$\frac{\omega^2}{\omega_\rho^{\ 2}} < B \leq A$$

The longitudinal root is real and the transverse root, pure imaginary when

$$B < \frac{\omega^2}{\omega_\rho^{\ 2}} < A$$

APPENDIX C

To complete our proof of self consistency for the case of the plasma half space, we must show that $f(o,u_x) = f(o,-u_x)$. With this end in mind, we consider our solution for the perturbed particle distribution evaluated at $x = 0$:

$$f(o,\overline{u}) = \frac{cB_y(o)|e|n_o\gamma}{\pi m}\left[k_z \int_{C_L} \frac{\left(k_x \frac{\partial F_o}{\partial u_x} + k_z \frac{\partial F_o}{\partial u_z}\right)dk_x}{(k_x^2 + k_z^2)(\omega\gamma - \overline{k}\cdot\overline{u})\Lambda_L(k_x)} + \omega \int_{C_T} \frac{k_x\left(k_z \frac{\partial F_o}{\partial u_x} - k_x \frac{\partial F_o}{\partial u_z}\right)dk_x}{(k_x^2 + k_z^2)(\omega\gamma - \overline{k}\cdot\overline{u})\Lambda_T(k_x)}\right] \quad . \tag{C-1}$$

Since $(\gamma\omega - k_x u_x - k_z u_z) \neq 0$ for $-\alpha_o < k_x < \alpha_o$, we may write that:

$$\int_{C_7} \frac{\left(k_x \frac{\partial F_o}{\partial u_x} + k_z \frac{\partial F_o}{\partial u_z}\right)dk_x}{(k_x^2 + k_z^2)(\omega\gamma - \overline{k}\cdot\overline{u})\Lambda_L(k_x)} = 0 \tag{C-2}$$

and

$$\int_{C_8} \frac{k_x\left(k_z \frac{\partial F_o}{\partial u_x} - k_x \frac{\partial F_o}{\partial u_z}\right)dk_x}{(k_x^2 + k_z^2)(\omega\gamma - \overline{k}\cdot\overline{u})\Lambda_T(k_x)} = 0 \quad , \tag{C-3}$$

where C_7 is the closed contour shown in Figure 18. The C_8 contour is

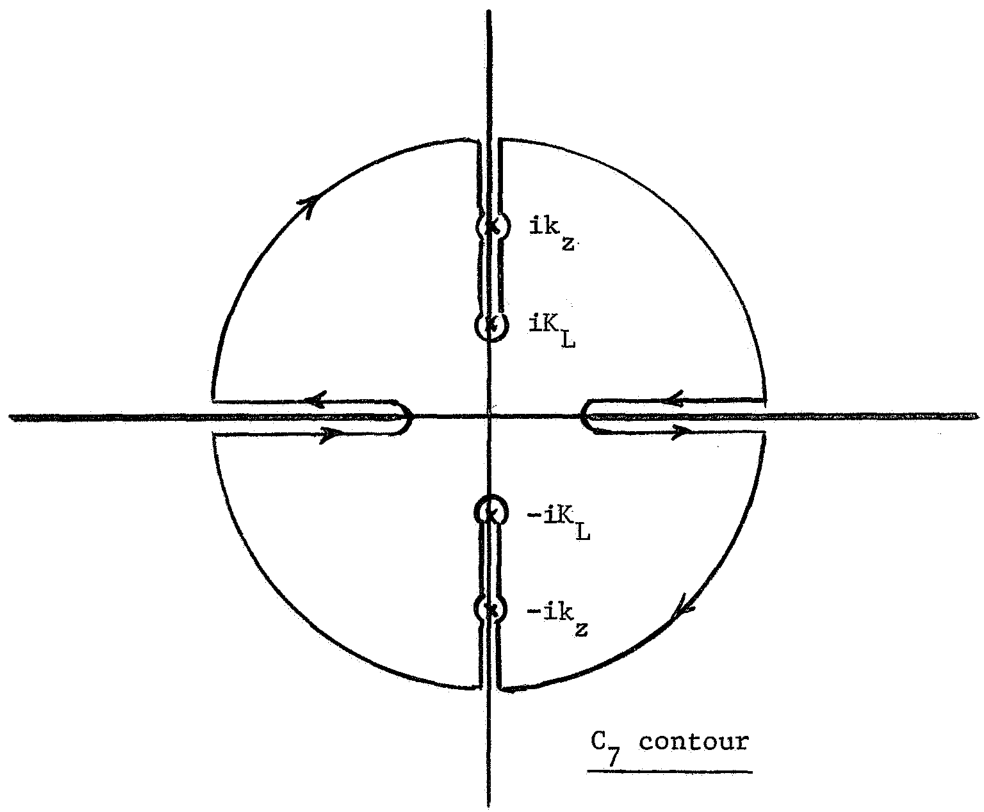


Figure 18. The C_7 Contour

identical to that given in Figure 18 with κ_T replacing κ_L . For illustrative purposes, the longitudinal root was shown as pure imaginary. Equations (C-2) and (C-3) may be rewritten as:

$$\int_{C_L} \frac{(k_x \frac{\partial F_o}{\partial u_x} + k_z \frac{\partial F_o}{\partial u_z}) dk_x}{(k_x^2 + k_z^2)(\gamma\omega - k_x u_x - k_z u_z)\Lambda_L(k_x)}$$

$$- \int_{C_L} \frac{(-k_x \frac{\partial F_o}{\partial u_x} + k_z \frac{\partial F_o}{\partial u_z}) dk_x}{(k_x^2 + k_z^2)(\omega\gamma + k_x u_x - k_z u_z)\Lambda_L(k_x)}$$

$$- \pi \frac{(-i \frac{\partial F_o}{\partial u_x} + \frac{\partial F_o}{\partial u_z})}{(\omega\gamma + ik_z u_x - k_z u_z)\Lambda_L(-ik_z)}$$

$$+ \pi \frac{(i \frac{\partial F_o}{\partial u_x} + \frac{\partial F_o}{\partial u_z})}{(\omega\gamma - ik_z u_x - k_z u_z)\Lambda_L(ik_z)} = 0 \qquad \text{(C-4)}$$

and

$$\int_{C_T} \frac{k_x(k_z \frac{\partial F_o}{\partial u_x} - k_x \frac{\partial F_o}{\partial u_z})\, dk_x}{(k_x^2 + k_z^2)(\omega\gamma - k_x u_x - k_z u_z)\Lambda_T(k_x)}$$

$$- \int_{C_T} \frac{k_x(- k_z \frac{\partial F_o}{\partial u_x} - k_x \frac{\partial F_o}{\partial u_z})\, dk_x}{(k_x^2 + k_z^2)(\omega\gamma + k_x u_x - k_z u_z)\Lambda_T(k_x)}$$

$$+ \frac{i\pi k_z(\frac{\partial F_o}{\partial u_x} + i \frac{\partial F_o}{\partial u_z})}{(\omega\gamma + ik_z u_x - k_z u_z)\Lambda_T(- ik_z)}$$

$$+ \frac{i\pi k_z(\frac{\partial F_o}{\partial u_x} - i \frac{\partial F_o}{\partial u_z})}{(\omega\gamma - ik_z u_x - k_z u_z)\Lambda_T(ik_z)} = 0 \quad , \qquad \text{(C-5)}$$

where we made use of Equations (B-5) and (B-6). Inserting these results in Equation (C-1) and using Equations (B-22) and (B-23), we find:

$$f(o,u_x) - f(o,-u_x) = 0 \quad . \qquad \text{(C-6)}$$

APPENDIX D

$\Lambda_T(k_x)$ and $\Lambda_L(k_x)$ on the Imaginary k_x Axis

In order to complete our analysis of $\Lambda_T(k_x)$ and $\Lambda_L(k_x)$, it is necessary to evaluate these functions on the imaginary k_x axis. The resulting expressions will then be used to obtain approximate values for pure imaginary longitudinal and transverse roots. Following the format of Appendix B, we focus our attention on a typical integral entering in the definitions of $\Lambda_T(k_x)$ and $\Lambda_L(k_x)$ on the imaginary axis. We define

$$N_{ij} \equiv \int \frac{u_i u_j e^{-\beta\gamma} du^3}{\gamma[\gamma\omega - k_z u_z - ik_2 u_x]} , \tag{D-1}$$

where k_x was written as $k_x = ik_2$ and $k_2 \geq 0$. The subscripts i and j may be either x or z.

Rewriting the denominator as an integral over a damped plane wave, we may write:

$$N_{xx} = - \int_0^\infty \frac{1}{t^2} \frac{\partial^2 W(t)}{\partial^2 k_2{}^2} dt \tag{D-2}$$

and

$$N_{xz} = - i \int_0^\infty \frac{1}{t^2} \frac{\partial^2 W(t)}{\partial k_2 \partial k_z} dt \tag{D-3}$$

and

$$N_{zz} = + \int_0^\infty \frac{1}{t^2} \frac{\partial^2 W(t)}{\partial k_z^2} dt \quad , \tag{D-4}$$

where $W(t)$ is defined as:

$$W(t) \equiv \int_{\bar{u}} \frac{e^{-\gamma(\beta + \omega t)} e^{k_z u_z t} \cos k_2 u_x t}{\gamma} du^3 \tag{D-5}$$

Considering the u_x integration first, we define:

$$Q \equiv 2 \int_0^\infty \frac{e^{-\gamma(\beta + \omega t)} \cos k_2 u_x t}{\gamma} du_x \quad , \tag{D-6}$$

where

$$\gamma = \frac{1}{c} \sqrt{c^2 + u_x^2 + u_y^2 + u_z^2}$$

The value of this last integral is given in the Bateman Manuscript, p. 17. We may therefore write:

$$Q = 2c\, K_o \left[\frac{\phi}{c} \sqrt{c^2 + u_y^2 + u_z^2} \right] \quad , \tag{D-7}$$

where

$$\phi \equiv \sqrt{(\beta + \omega t)^2 + k_2^2 c^2 t^2} \quad .$$

Inserting this result into the definition of W, we obtain:

$$W = 2c \iint_{-\infty}^{\infty} K_o\left[\frac{\phi}{c}\sqrt{c^2 + u_y^2 + u_z^2}\right] e^{k_z u_z t} du_y du_z \qquad \text{(D-8)}$$

The u_y integral above is given in the Table of Integrals, Series, and Products, p. 705. Equation (D-8) may therefore be written as:

$$W = \frac{2\pi c^2}{\phi} \int_{-\infty}^{\infty} e^{-\frac{\phi}{c}\sqrt{c^2 + u_z^2}} e^{k_z u_z t} du_z \quad , \qquad \text{(D-9)}$$

where we made use of the half integer properties of $K_\nu(z)$.

The integral in Equation (D-9) may be cast in a more amenable form by making a change of variables. We let $u_z = c \sinh\xi$ and obtain:

$$W = -\frac{2\pi c^3}{\phi} \frac{\partial I}{\partial \phi} \quad , \qquad \text{(D-10)}$$

where

$$I \equiv \int_{-\infty}^{\infty} e^{-\phi \cosh \xi + \eta \sinh \xi} d\xi \qquad \text{(D-11)}$$

and

$$\eta \equiv k_z ct$$

Equation (D-11) may readily be integrated if we make the final change of variables:

$$- \phi \cosh \xi + \eta \sinh \xi = - A \cosh (\xi - b) \quad ,$$

where

$$A = \sqrt{\phi^2 - \eta^2}$$

and

$$b = \tanh^{-1}(\eta/\phi)$$

We obtain:

$$I = 2 \int_0^\infty e^{- A \cosh x} d_x$$

$$= 2 K_o(A). \tag{D-12}$$

Consequently,

$$W = 4\pi c^3 \frac{K_1 \left[\sqrt{\phi^2 - \eta^2}\right]}{\sqrt{\phi^2 - \eta^2}} \quad . \tag{D-13}$$

Inserting this result into Equations (D-2), (D-3), and (D-4), we readily find N_{xx} , N_{xz} , and N_{zz} . These results are then inserted into the definitions of $\Lambda_T(ik_2)$ and $\Lambda_L(ik_2)$ and yield:

$$\Lambda_T(ik_2) = (-k_2^2 + k_z^2)c^2 - \omega^2 + \frac{\beta^2 \omega\omega_p^2}{K_2(\beta)} \int_0^\infty \frac{K_2(z)}{z^2} dt \tag{D-14}$$

and

$$\Lambda_L(ik_2) = \omega - \frac{\beta^2\omega_p^{\ 2}}{K_2(\beta)}\left[\int_0^\infty \frac{K_2(z)}{z^2}\, dt + (-k_2^{\ 2}+k_z^{\ 2})c^2 \int_0^\infty t^2 \frac{K_3(z)}{z^3}\, dt\right] , \qquad \text{(D-15)}$$

where

$$z \equiv \sqrt{(\beta + \omega t)^2 + (k_2^{\ 2} - k_z^{\ 2})c^2t^2}$$

Equation (D-15) may also be rewritten as:

$$\Lambda_L(ik_2) = \omega - \frac{\beta^2\omega_p^{\ 2}}{K_2(\beta)}\left[1 + 2(k_2^{\ 2} - k_z^{\ 2})c^2 \frac{\partial}{\partial\delta^2}\right]\int_0^\infty \frac{K_2(z)}{z^2}\, dt \quad , \qquad \text{(D-16)}$$

where we have rewritten z as:

$$z = \sqrt{\beta^2 + 2\omega\beta t + \delta^2t^2}$$

and defined

$$\delta \equiv \sqrt{\omega^2\cos^2\theta + k_2^{\ 2}c^2}$$

We shall make use of Equation (D-16) in evaluating the approximate expressions for κ_L on the imaginary axis.

APPENDIX E

Appendices B and D were primarily devoted to the evaluation of $\Lambda_T(k_x)$ and $\Lambda_L(k_x)$ on the real and imaginary k_x axes. The results of these calculations are given in Equations (B-32), (B-33), (B-35), and (B-36) and in (D-14) and (D-16). They remain in the form of integrals. In this Appendix, we obtain the large β limit of these functions. The results are used in Section 4 to obtain approximate values for the transverse and longitudinal roots and in the evaluation of the critical angle of incidence. These approximate calculations also play a role in the evaluation of the "cut" contributions to the transverse and longitudinal depths of penetration and in the study of the geometrical resonances that arise in the slab problem. By large β, we mean $\beta \equiv \frac{mc^2}{KT} \geq 100$. Physically, this corresponds to a temperature range of $0 \leq T \leq 5.9 \times 10^7 \,^oK$.

E.1 $\Lambda_T^+(k_1)$ and $\Lambda_L^+(k_1)$ When $(k_1^2c^2 - \omega^2\cos^2\theta) < 0$

Real transverse and longitudinal roots may be found from the approximate evaluations of $\Lambda_T^+(k_1)$ and $\Lambda_L^+(k_1)$ when $(k_1^2c^2 - \omega^2\cos^2\theta) < 0$. The integral appearing in Equation (B-32) may be rewritten as:

$$W \equiv \int_0^\infty \frac{xK_2(\beta\sqrt{x^2+1})\,dx}{(x^2+1)\sqrt{x^2+\tilde{\sigma}^2}} \tag{E-1}$$

where

$$\frac{1}{\cos\theta} \leq \frac{1}{\tilde{\sigma}} < \infty$$

Integrating Equation (E-1) by parts, we obtain:

$$W = \frac{K_1(\beta)}{\beta\tilde{\sigma}} - \frac{K_o(\beta)}{\beta^2\tilde{\sigma}^3} + \frac{3}{\beta^2}\int_0^\infty \frac{xK_o(\beta\sqrt{x^2+1})dx}{(x^2+\tilde{\sigma}^2)^{5/2}} \quad , \tag{E-2}$$

where the last term of Equation (E-2) is of the order $\frac{K_o(\beta)}{\beta^3}$.

Inserting this result into Equation (B-32), we find:

$$\Lambda_T^+(k_1) \doteq -\frac{\omega^2}{\tilde{\sigma}^2} + \frac{\omega_p^2}{K_2(\beta)}\left[K_1(\beta) - \frac{K_o(\beta)}{\beta\,\tilde{\sigma}^2}\right] + \omega_p^2\, 0\left(\frac{1}{\beta^2\tilde{\sigma}^4}\right). \tag{E-3}$$

We note that Equation (E-3) is in agreement with Equation (B-38) as $k_1 \to \alpha_o$ $(\tilde{\sigma} \to \infty)$. This approximate expression for $\Lambda_T^+(k_1)$ is used in calculating the critical angle of incidence.

We apply a similar procedure to obtain an algebraic expression for $\Lambda_L^+(k_1)$. The integral in Equation (B-35) is defined as:

$$T \equiv \int_0^\infty \frac{xK_2(\beta\sqrt{x^2+1})}{(x^2+\tilde{\sigma}^2)^{3/2}}\,dx \tag{E-4}$$

Two integrations by parts yield:

$$T = \frac{K_1(\beta)}{\beta\tilde{\sigma}^3} + (2\tilde{\sigma}^2-3)\frac{K_o(\beta)}{\beta^2\tilde{\sigma}^5} - \frac{3}{\beta^2}\int_0^\infty \frac{x(4\tilde{\sigma}^2 - x^2 - 5)K_2(\beta\sqrt{x^2+1})dx}{(x^2+\tilde{\sigma}^2)^{7/2}} \quad . \tag{E-5}$$

Inserting this result into Equation (B-35), we find:

$$\Lambda_L^+(k_1) \doteq \omega - \frac{\omega_p^2}{\omega K_2(\beta)} \left[K_1(\beta) + \frac{K_o(\beta)}{\beta\tilde{\sigma}^2} (2\tilde{\sigma}^2 - 3) \right] + \frac{\omega_p^2}{\omega} \, 0\left(\frac{1}{\beta^2\tilde{\sigma}^4}\right) \quad . \tag{E-6}$$

As a check, we note that Equation (E-6) agrees with Equation (B-42) as $k_1 \to \alpha_o$. We also see from Equation (B-6) that real longitudinal roots can exist only if $\Lambda_L^+(\alpha_o) < 0$.

The real transverse and longitudinal roots may now be derived from Equations (E-3) and (E-6).

E.2 $\Lambda_T(ik_2)$ and $\Lambda_L(ik_2)$ -- Weak Spatial Dispersion

Imaginary transverse and longitudinal roots may be obtained from the approximate evaluations of $\Lambda_T(ik_2)$ and $\Lambda_L(ik_2)$, Equations (D-14) and (D-16). The integral appearing in Equation (D-14) may be rewritten as:

$$G = \frac{1}{\beta\delta} \int_0^\infty \frac{xK_2(\beta\sqrt{x^2+1})dx}{(x^2+1)\sqrt{x^2+\omega^2/\delta^2}} \quad , \tag{E-7}$$

where $\delta \equiv \sqrt{\omega^2\cos^2\theta + k_2^2c^2}$ and $\omega\cos\theta \leq \delta \leq \infty$. We need to approximate Equation (E-7) under two significantly different conditions; i.e., when ik_2 is near the real k_x axis and when ik_2 is far removed from the real k_x axis. The corresponding physical situations are discussed in Section 4.

When ω^2/δ^2 is much larger than $x_o^{\,2} \doteq 1/\beta$, the x value for which $xK_2(\beta\sqrt{x^2+1})$ attains its maximum, it is clear that $(x^2 + \omega^2/\delta^2)^{-1/2}$ plays a small role in the evaluation of Equation (E-7). Integrating by parts, we obtain:

$$G \doteq \frac{1}{\omega\beta^2}\left[K_1(\beta) - \frac{\delta^2}{\omega^2}\frac{K_o(\beta)}{\beta}\right] + 0\left(\frac{\delta^4}{\omega^4}\frac{K_o(\beta)}{\beta^2}\right) \qquad \text{(E-8)}$$

Inserting this result into Equations (D-14) and (D-16), we find:

$$\Lambda_T(ik_2) \doteq -\delta^2 + \frac{\omega_p^{\,2}}{K_2(\beta)}\left[K_1(\beta) - \frac{\delta^2}{\omega^2}\frac{K_o(\beta)}{\beta}\right] \qquad \text{(E-9)}$$

and

$$\Lambda_L(ik_2) \doteq \omega - \frac{\omega_p^{\,2}}{\omega K_2(\beta)}\left[K_1(\beta) + \frac{K_o(\beta)}{\beta}\left\{2 - 3\frac{\delta^2}{\omega^2}\right\}\right] \quad . \qquad \text{(E-10)}$$

As a check on the internal consistency of these approximations, we note that Equations (E-3) and (E-9) go to the same limit as k_1, $k_2 \to 0$. We also note that Equations (E-6) and (E-10) go to the same limit as $k_1, k_2 \to 0$. $\Lambda_T(ik_2)$ and $\Lambda_L(ik_2)$ given by Equations (E-9) and (E-10) also satisfy Equations (B-22) and (B-23).

The roots to Equations (E-9) and (E-10) lie near the real k_x axis and give rise to weak spatial dispersion.

E.3 $\Lambda_T(ik_2)$ and $\Lambda_L(ik_2)$ -- Strong Spatial Dispersion

Finally, we seek approximate evaluations for $\Lambda_T(ik_2)$ and $\Lambda_L(ik_2)$ which are valid when ik_2 is far removed from the real k_x axis. When $\omega^2/\delta^2 << 1/\beta$, we may approximate Equation (E-7) as:

$$G \doteq \frac{1}{\beta\delta} \int_0^\infty \frac{K_2(\beta\sqrt{x^2+1})}{(x^2+1)}\,dx \tag{E-11}$$

which may be integrated to give:

$$G \doteq \frac{\pi e^{-\beta}}{2\beta^2\delta}\left(1+\frac{1}{\beta}\right) \tag{E-12}$$

Inserting this result into Equations (D-14) and (D-16), we obtain:

$$\Lambda_T(ik_2) \doteq -\delta^2 + \sqrt{\frac{\pi\beta}{2}}\,\frac{\omega_p^2\omega}{\delta} \tag{E-13}$$

and

$$\Lambda_L(ik_2) \doteq \omega\left[1 - \sqrt{\frac{\pi\beta}{2}}\,\frac{\omega_p^2\omega}{\delta^3}\right] . \tag{E-14}$$

The roots to Equations (E-13) and (E-14) are far removed from the real axis and give rise to strong spatial dispersion.

E.4 $\mathrm{Re}\Lambda_L^+(k_1)$ When $(k_1^2c^2 - \omega^2\cos^2\theta) > 0$

In order to evaluate the "cut" contribution to the longitudinal depth of penetration, it is necessary to obtain an approximate expression for $\mathrm{Re}\Lambda_L^+(k_1)$ when $(k_1^2c^2 - \omega^2\cos^2\theta) > 0$. With this goal in mind, we note that the integral appearing in the definition of $\mathrm{Re}\Lambda_L^+(k_1)$, Equation (B-36), may be rewritten as:

$$L(\sigma) \equiv \sigma \int_0^1 \frac{[\sigma^2(1+\xi^2)+1]K_3(\beta\sqrt{\sigma^2(1-\xi^2)+1})d\xi}{[\sigma^2(1-\xi^2)+1]^{3/2}}$$

$$\doteq \sqrt{\frac{\pi}{2\beta}}\,\sigma \int_0^1 \frac{[\sigma^2(1+\xi^2)+1]e^{-\beta\sqrt{\sigma^2(1-\xi^2)+1}}}{[\sigma^2(1-\xi^2)+1]^{7/4}}\,d\xi \quad , \tag{E-15}$$

where we made use of the asymptotic form of $K_3(z)$. It is now convenient to divide our analysis into two parts, depending on the size of σ^2 . In the first region, σ^2 satisfies the two inequalities: $0 \leq \sigma^2 << 1$ and $0 \leq \beta\sigma^2 \leq 10$. Expanding $\sqrt{\sigma^2(1-\xi^2)+1}$ about $\sigma^2 = 0$ and keeping only the first two terms, we are left with:

$$L(\sigma) \doteq \sqrt{\frac{\pi}{2\beta}}\;\sigma e^{-\beta} \int_0^1 e^{-\frac{\beta\sigma^2}{2}(1-\xi^2)}\,d\xi \tag{E-16}$$

Inserting this expression into Equation (B-36) and using the asymptotic expression for $K_2(\beta)$, we may write:

$$\mathrm{Re}\Lambda_L^{+}(k_1) \doteq \omega + \frac{\omega_p^2}{\omega}\,\beta^2\sigma^2\left[\frac{1}{\beta} - \sigma^2\int_0^1 e^{-\frac{\beta\sigma^2}{2}(1-\xi^2)}\,d\xi\right]$$

$$\doteq \omega + 2\mu^2\,\frac{\omega_p^2}{\omega}\left[1 + \mu\mathrm{Re}Z(\mu)\right] \quad , \tag{E-17}$$

where $\mathrm{ReZ}(\mu)$ is the real part of the plasma dispersion function[21] defined as:

$$\mathrm{ReZ}(\mu) \equiv -2\mu e^{-\mu^2} \int_0^\mu \frac{e^{x^2}}{\mu} dx$$

and

$$\mu^2 \equiv \frac{\beta\sigma^2}{2}$$

Equation (E-17) plays an important role in the calculation of the longitudinal depth of penetration.

We now proceed to obtain the large β limit of $\mathrm{Re}\Lambda_L^+(k_1)$ when $(k_1^2c^2 - \omega^2\cos^2\theta) > 0$ and σ^2 is large; i.e., $\beta\sigma^2 \geq 10$. In this case, we write:

$$L(\sigma) = \beta^3\sigma[U(\sigma) + V(\sigma)] \quad , \tag{E-18}$$

where

$$U(\sigma) \equiv \int_0^a \frac{[\sigma^2(1 + \xi^2) + 1]K_3(z)}{z^3} d\xi \tag{E-19}$$

and

$$V(\sigma) \equiv \int_a^1 \frac{[\sigma^2(1 + \xi^2) + 1]K_3(z)}{z^3} d\xi \tag{E-20}$$

and where

$$a \equiv \frac{1}{\sqrt{\beta\sigma^2}}$$

and

$$z \equiv \beta \sqrt{\sigma^2(1 - \xi^2) + 1}$$

Integrating the expression for $V(\sigma)$ several times by parts, we find:

$$V(\sigma) \doteq \frac{1}{\beta^4\sigma^2}\left[(2\sigma^2 + 1)K_2(\beta) + \frac{K_1(\beta)}{\beta\sigma^2} + \frac{(2\sigma^2 + 3)}{\beta^2\sigma^4}K_o(\beta)\right]$$

$$+ O\left[K_n(\beta\sqrt{\sigma^2 + 1} - \frac{1}{\beta})\right] \qquad \text{(E-21)}$$

Noting that

$$U(\sigma) \doteq O\left[K_n(\beta\sqrt{\sigma^2 + 1} - \frac{1}{\beta})\right] \quad , \qquad \text{(E-22)}$$

we may approximate Equation (E-18) as:

$$L(\sigma) \doteq \frac{1}{\beta\sigma}\left[(2\sigma^2 + 1)K_2(\beta) + \frac{K_1(\beta)}{\beta\sigma^2} + \frac{(2\sigma^2 + 3)}{\beta^2\sigma^4}K_o(\beta)\right] \quad , \qquad \text{(E-23)}$$

where we dropped higher order terms. Making use of this result in Equation (B-36), we obtain:

$$\text{Re}[\Lambda_L^+(k_1)] \doteq \omega - \frac{\omega_p^2}{\omega K_2(\beta)}\left[K_1(\beta) + \frac{(2\sigma^2 + 3)}{\beta\sigma^2}K_o(\beta)\right]$$

$$\doteq \omega - \frac{\omega_p^2}{\omega}\left[1 + \frac{1}{2\beta} + \frac{3}{\beta\sigma^2}\right] \qquad \text{(E-24)}$$

which goes to Equation (B-42) in the limit as $\sigma \to \infty$. As a check on the internal consistency of our two expansions for $\mathrm{Re}\Lambda_L^+(k_1)$, we note that asymptotically, Equation (B-17) becomes:

$$\mathrm{Re}\Lambda_L^+(k_1) \doteq \omega - \frac{\omega_p^2}{\omega}\left(1 + \frac{3}{\beta\sigma^2}\right)$$

which is in close agreement with Equation (E-24).

To complete our analysis of $\mathrm{Re}\Lambda_L^+(k_1)$, we obtain the zeros of this function. Equation (E-17) may be rewritten as:

$$\mathrm{Re}\Lambda_L^+(k_1) \doteq 2\mu^2 \frac{\omega_p^2}{\omega}\left[g(\mu,\omega^2/\omega_p^2) - \phi(\mu)\right] , \qquad \text{(E-25)}$$

where we have defined

$$g(\mu,\omega^2/\omega_p^2) \equiv \frac{\omega^2}{2\mu^2\omega_p^2}$$

and

$$\phi(\mu) \equiv -[1 + \mu \mathrm{Re}Z(\mu)]$$

In Figure 19, we plot $g(\mu,\omega^2/\omega_p^2)$ versus μ for various choices of the parameter $\frac{\omega^2}{\omega_p^2}$ and superimpose on the same graph $\phi(\mu)$ versus μ. The intersection of these curves corresponds to a zero of $\mathrm{Re}\Lambda_L^+(k_1)$. We conclude upon examining Figure 19 that the zeros of $\mathrm{Re}\Lambda_L^+(k_1)$ fall near $\mu = 1$ when $\omega^2/\omega_p^2 < 1/2$. Under these conditions, it is easy to show that the root is approximately given by:

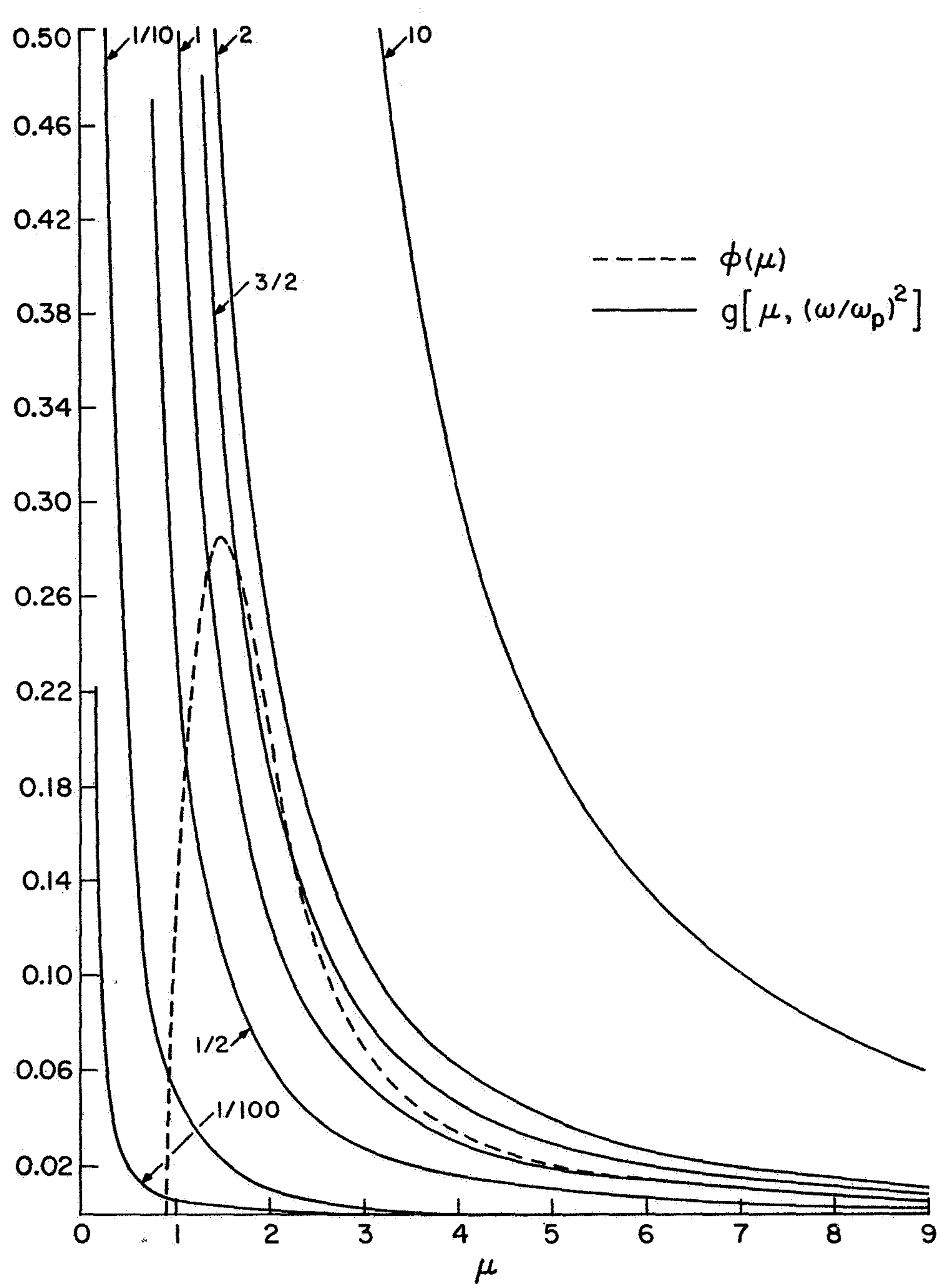


Figure 19. $\phi(\mu)$ Versus μ and $g(\mu,\omega^2/\omega_p{}^2)$ Versus μ With $\omega^2/\omega_p{}^2$ as a Parameter.

$$\mu_o \doteq + \sqrt{\frac{-\left[2 + \text{ReZ}(1) + \frac{\omega^2}{\omega_p^2}\right]}{\text{ReZ}(1)}} \quad , \tag{E-26}$$

where

$$\text{ReZ}(1) \doteq -1.07616 \ .$$

E.5 $\text{Re}\Lambda_T^+(k_1)$ When $(k_1^2c^2 - \omega^2\cos^2\theta) > 0$

In order to evaluate the "cut" contribution to the transverse depth of penetration, we must find an approximate expression for $\text{Re}\Lambda_T^+(k_1)$ when $(k_1^2c^2 - \omega^2\cos^2\theta) > 0$. Since the development follows the same lines as that given for $\text{Re}\Lambda_L^+(k_1)$, we will simply give the results.

When σ^2 satisfies the two inequalities $0 \leq \sigma^2 << 1$ and $0 \leq \beta\sigma^2 < 10$, we find:

$$\text{Re}\Lambda_T^+(k_1) \doteq \omega^2 \left[\frac{\beta}{2\mu^2} - \frac{\omega_p^2}{\omega^2} \mu\text{ReZ}(\mu) \right] \ . \tag{E-27}$$

When $\beta\sigma^2 \geq 10$, we may write

$$\text{Re}\Lambda_T^+(k_1) \doteq \frac{\omega^2}{\sigma^2} + \frac{\omega_p^2}{K_2(\beta)} \left[K_1(\beta) + \frac{K_o(\beta)}{\beta\sigma^2} \right] \tag{E-28}$$

which goes to Equation (B-38) as $\sigma \to \infty$. Since $\text{Re}\Lambda_T^+(k_1) > 0$ for $0 \leq \sigma^2 < \infty$, we have no need for a more detailed study of this function.

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