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Technical Report 32-1394

Propagation in a Planar Inhomogeneous Plasma Medium

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Preface

The work described in this report was performed by the Telecommunications Division of the Jet Propulsion Laboratory.

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Abstract

The theory of wave reflection and refraction in a planar inhomogeneous plasma medium is presented in this report. The first method for determining the reflection and refraction coefficients for various inhomogeneous distributions is to stratify the medium into thin plasma layers, each containing a homogeneous plasma, and satisfy the boundary conditions on the field vectors across each layer. The second method is to examine the change in the reflection and refraction coefficients as the medium is constructed from one edge to the other by the addition of planar layers of infinitesimal thickness. The reflection and refraction coefficients satisfy Riccati equations, which are solved numerically to yield the desired coefficients.

Propagation in a Planar Inhomogeneous Plasma Medium

I. Introduction

The theory of wave reflection and refraction in a planar inhomogeneous plasma medium is presented in this report. The properties of the plasma are assumed to vary continuously along one axis of a rectangular coordinate system but do not change in the planes perpendicular to this axis. The plasma is also assumed to be linear, isotropic, and cold. Of interest are the reflection and refraction coefficients of a lossless or lossy plasma for normal or oblique incidence.

Two methods are used to determine the reflection and refraction coefficients for various inhomogeneous distributions, and the results for both methods are compared. The first method is to stratify the medium into thin planar layers. The properties of the plasma are assumed to change discontinuously across the boundary of each layer, while the plasma within each layer is assumed to be homogeneous. This technique is useful when the inhomogeneity is specified in a point-by-point fashion.

The second method is to examine the change in the reflection and refraction coefficients as the medium is constructed from one edge to the other by the addition of planar layers of infinitesimal thickness. This application of the method of invariant imbedding as derived from Ambarzumian's principle of invariance (Refs. 1 and 2; and C. H. Papas, "Plane Inhomogeneous Dielectric Slab," California Institute of Technology, Antenna Laboratory Note, March 1954) shows that the reflection and refraction coefficients satisfy Riccati equations, which are solved numerically to yield the desired coefficients. This technique is useful when the inhomogeneity is described analytically.

II. Preliminaries

For completeness of presentation, the sections immediately following are devoted to the relatively simple task of finding the constitutive parameters and the propagation constant of the plasma. Also, the concepts of intrinsic and transverse impedance of the plasma are introduced.

A. Constitutive Parameters of the Plasma

A suitable model of the plasma that is consistent with the limitations of this study is that of a certain number n of electrons per unit volume free to move in an applied electromagnetic field but subject to a damping force owing to collisions characterized by the damping constant ω_c. The damping constant ω_c represents the average number of collisions the electrons undergo per unit time. The macroscopic equation of motion of such electrons is

$$nm\frac{d\mathbf{v}}{dt} = nq(\mathscr{E} + \mathbf{v} \wedge \mathscr{B}) - nm\,\omega_c\,\mathbf{v} \tag{1}$$

where the applied electromagnetic field is characterized by the field vectors $\mathscr{E}(\mathbf{r}, t)$ and $\mathscr{B}(\mathbf{r}, t)$. In the present case, the nonlinear $\mathbf{v} \wedge \mathscr{B}$ term is dropped since $|\mathbf{v} \wedge \mathscr{B}| << |\mathscr{E}|$.

In the steady state, with $e^{-i\omega t}$ time-dependence, the equation of motion becomes

$$-i\omega nm\,\mathbf{v} = nq\,\mathbf{E} - nm\,\omega_c\,\mathbf{v} \tag{2}$$

where the applied electromagnetic field in the steady state is characterized by the single field vector $\mathbf{E}(\mathbf{r})$.

A rigorous derivation of the equation of motion using a statistical distribution function to describe the state of the plasma can be found in Ref. 3.

If the complex electric current density $\mathbf{j}(\mathbf{r})$ and the radian plasma frequency ω_p are defined by

$$\mathbf{j} \equiv nq\,\mathbf{v} \tag{3}$$

$$\omega_p^2 \equiv \frac{nq^2}{m\epsilon_0} \tag{4}$$

then solving Eq. (2) for $\mathbf{v}$ and substituting the result into Eq. (3) shows the following relationship between $\mathbf{j}(\mathbf{r})$ and $\mathbf{E}(\mathbf{r})$:

$$\mathbf{j}(\mathbf{r}) = \frac{\epsilon_0\omega_p^2}{-i\omega + \omega_c}\mathbf{E}(\mathbf{r}) = \left(\frac{\epsilon_0\omega_c\omega_p^2}{\omega^2 + \omega_c^2} + i\omega\frac{\epsilon_0\omega_p^2}{\omega^2 + \omega_c^2}\right)\mathbf{E}(\mathbf{r}) \tag{5}$$

The form of Eq. (5) suggests that the plasma has a complex conductivity σ_c given by

$$\sigma_c \equiv \frac{\epsilon_0\omega_p^2}{-i\omega + \omega_c} \tag{6}$$

For the avoidance of complex conductivities, the plasma is considered to be a lossy dielectric. In this case, the constitutive parameters are real and are found by substituting Eq. (5) for $\mathbf{j}(\mathbf{r})$ into Maxwell's equations. By a casting of the resulting equations into the standard form for a lossy dielectric, it follows that the conductivity of the plasma is given by

$$\sigma \equiv \frac{\epsilon_0\omega_c\omega_p^2}{\omega^2 + \omega_c^2} \tag{7}$$

its permittivity is given by

$$\epsilon \equiv \epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2 + \omega_c^2}\right) \tag{8}$$

and its permeability is given by

$$\mu \equiv \mu_0 \tag{9}$$

B. Propagation Constant of the Plasma

Let a plasma be characterized by its electron concentration $n(\mathbf{r}, t)$, its collision frequency $f_c(\mathbf{r}, t)$, and its permeability $\mu = \mu_0$.

If $n(\mathbf{r}, t)$ and $f_c(\mathbf{r}, t)$ are slowly varying functions of position in the interior of the plasma, then the field quantities are continuous and have continuous derivatives and, therefore, satisfy Maxwell's equations.

Let $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ be the complex electric and magnetic field vectors in the plasma defined respectively by

$$\mathbf{E}(\mathbf{r}) \equiv Re\{\mathscr{E}(\mathbf{r}, t)\,e^{-i\omega t}\} \tag{10}$$

$$\mathbf{H}(\mathbf{r}) \equiv Re\{\mathscr{H}(\mathbf{r}, t)\,e^{-i\omega t}\} \tag{11}$$

where $\mathscr{E}(\mathbf{r}, t)$ and $\mathscr{H}(\mathbf{r}, t)$ are the real electric and magnetic field vectors in the plasma, respectively. Consider a point $\mathbf{r}_0$ in the interior of the plasma, and a neighborhood $N(\mathbf{r}_0, \delta)$ of that point, which is sufficiently small to be assumed homogeneous with constants $n(\mathbf{r}_0, t)$ and $f_c(\mathbf{r}_0, t)$. If the plasma is also linear, isotropic, and cold, then for $r < \delta$

$$\mathbf{D}(\mathbf{r}) = \epsilon\mathbf{E}(\mathbf{r}) \tag{12}$$

$$\mathbf{B}(\mathbf{r}) = \mu\mathbf{H}(\mathbf{r}) \tag{13}$$

$$\mathbf{j}(\mathbf{r}) = \sigma\mathbf{E}(\mathbf{r}) \tag{14}$$

where the complex vector $\mathbf{D}(\mathbf{r})$ represents the electric flux density, $\mathbf{B}(\mathbf{r})$ represents the magnetic flux density, and $\mathbf{j}(\mathbf{r})$ represents the electric current density of the plasma; and ϵ, μ, and σ are the constitutive parameters of the plasma as found in the previous section.

With $e^{-i\omega t}$ time-dependence, Maxwell's two independent equations take the following steady state form in the plasma:

$$\nabla \wedge \mathbf{E}(\mathbf{r}) = i\omega\mu_0 \mathbf{H}(\mathbf{r}) \tag{15}$$

$$\nabla \wedge \mathbf{H}(\mathbf{r}) = (\sigma - i\omega\epsilon)\mathbf{E}(\mathbf{r}) \tag{16}$$

The wave equation for $\mathbf{E}(\mathbf{r})$ is found by substituting Eq. (16) into the curl of Eq. (15) and expanding the resulting curl-curl operation. The wave equation can be written in the form

$$(\nabla^2 + \gamma^2)\mathbf{E}(\mathbf{r}) = 0 \tag{17}$$

where the propagation constant γ is given by

$$\gamma^2 \equiv \omega^2\mu_0\left(\epsilon + i\frac{\sigma}{\omega}\right) \tag{18}$$

The constant γ is complex and, accordingly, is written in the form

$$\gamma \equiv \beta + i\alpha \qquad (\alpha, \beta = \text{positive–definite}) \tag{19}$$

which displays the phase factor β and the attenuation factor α. Explicit expressions for β and α in terms of the constitutive parameters are found by substituting Eq. (19) into Eq. (18). It follows that

$$\beta = \omega\sqrt{\frac{\mu_0}{2}}\sqrt{\left[\epsilon + \sqrt{\left(\epsilon^2 + \frac{\sigma^2}{\omega^2}\right)}\right]} \tag{20}$$

$$\alpha = \omega\sqrt{\frac{\mu_0}{2}}\sqrt{\left[-\epsilon + \sqrt{\left(\epsilon^2 + \frac{\sigma^2}{\omega^2}\right)}\right]} \tag{21}$$

The simplest solution of Eq. (17) for $\mathbf{E}(\mathbf{r})$ is a plane wave

$$\mathbf{E} = \mathbf{E}_0 e^{i\boldsymbol{\gamma}\cdot\mathbf{r}} \tag{22}$$

where $\mathbf{E}_0$ is a constant vector defining the direction of polarization and $\boldsymbol{\gamma}$ is a constant vector defining the direction of propagation. The magnitude of $\mathbf{E}_0$ is equal to the modulus of the electric field and the magnitude of $\boldsymbol{\gamma}$ is equal to the value of the propagation constant as defined in the previous section.

The corresponding expression for $\mathbf{H}(\mathbf{r})$ is found from Eq. (15)

$$\mathbf{H} = \frac{1}{i\omega\mu_0}\nabla \wedge \mathbf{E} \tag{23}$$

which, after substitution of Eq. (22), gives

$$\mathbf{H} = \frac{\boldsymbol{\gamma} \wedge \mathbf{E}}{\omega\mu_0} \tag{24}$$

C. Intrinsic and Transverse Impedance of the Plasma

In the investigation of the propagation of plane waves, it is useful to introduce the concept of the intrinsic impedance η of the plasma defined by

$$\eta \equiv \frac{E}{H} = \frac{\omega\mu_0}{\gamma} \tag{25}$$

where E and H are the moduli of the complex electric and magnetic field vectors in the plasma, respectively.

In the case of wave reflection and refraction at plane boundaries, it is also useful to introduce the concept of the transverse impedance of the plasma defined by

$$\zeta \equiv \frac{E_t}{H_t} \tag{26}$$

where H_t and E_t are the moduli of the transverse components of the complex electric and magnetic field vectors in the plasma, respectively.

Consider a plane wave obliquely incident at an angle ϑ on a plane boundary. For the case of polarization perpendicular to the plane of incidence, $E_t = E$, $H_t = H\cos\vartheta$ and, consequently,

$$\zeta = \frac{E}{H\cos\vartheta} = \frac{\eta}{\cos\vartheta} \tag{27}$$

For the case of polarization parallel to the plane of incidence, $E_t = E\cos\vartheta$, $H_t = H$ and, consequently,

$$\zeta = \frac{E\cos\vartheta}{H} = \eta\cos\vartheta \tag{28}$$

With the use of the appropriate ζ, the theory developed in the following section will be equally applicable for either parallel or perpendicular polarization.

III. Stratified Media

The reflection and refraction coefficients of an inhomogeneous plasma medium are derived in this section. In the theoretical treatment of the problem, the medium is assumed to be stratified into a series of homogeneous layers.

A. Geometry of the Problem

Figure 1 is a schematic diagram of the geometry of the problem showing all pertinent parameters and the coordinate system used in the derivation of all equations.

Suppose that between two semi-infinite, homogeneous, dielectric media, characterized by their constant permittivities and conductivities, there is an inhomogeneous plasma medium characterized by its electron concentration and collision frequency. The electron concentration and collision frequency of the plasma are not necessarily constants and are allowed to vary with the coordinate z only. The boundaries formed by the plasma–dielectric interface are assumed to be planar and to lie in the x–y plane.

If the inhomogeneous plasma is replaced with a stratified medium consisting of N layers, each layer containing a homogeneous plasma, a step-by-step approximation to the actual inhomogeneity can be formed. The parameters of each layer are chosen so that the approximation is everywhere within a prescribed error of the actual inhomogeneity.

The N layers, denoted by $1, \cdots, N$, are assumed to be planar, extending to infinity in the x–y plane and extending a finite distance in the z-direction. The pth layer, where p is an integer in the set $\{1, \cdots, N\}$, extends between z_{p-1} and z_p.

If the coordinate system is oriented in such a way that the y-component of the propagation constant in each layer is equal to zero, then the original three-dimensional problem is reduced to an equivalent two-dimensional problem in x and z.

B. Boundary Conditions

By hypothesis, the moduli of the field vectors in each layer are independent of the coordinates and, therefore, the existence of boundary conditions at $z = z_p$ implies that the spatial variation of all field vectors must be the same at $z = z_p$. Consequently,

$$\boldsymbol{\gamma}_{p+} \cdot \mathbf{r} \big|_{z=z_p} = \boldsymbol{\gamma}_{p-} \cdot \mathbf{r} \big|_{z=z_p} = \boldsymbol{\gamma}_{p+1,+} \cdot \mathbf{r} \big|_{z=z_p} \tag{29}$$

where the incident, reflected, and refracted propagation constants are denoted by γ_{p+}, γ_{p-}, and $\gamma_{p+1,+}$, respectively. By introduction of ϑ_{p+}, ϑ_{p-}, and $\vartheta_{p+1,+}$, the angles of incidence, reflection, and refraction, respectively, Eq. (29) can be written as

$$\gamma_{p+} \sin \vartheta_{p+} = \gamma_{p-} \sin \vartheta_{p-} = \gamma_{p+1,+} \sin \vartheta_{p+1,+} \tag{30}$$

Since $\gamma_{p+} \equiv \gamma_{p-} \equiv \gamma_p$ and $\gamma_{p+1,+} \equiv \gamma_{p+1}$, where γ_p and γ_{p+1} are the propagation constants in the respective layers, it follows that

$$\vartheta_{p+} = \vartheta_{p-} = \vartheta_p \tag{31}$$

$$\gamma_p \sin \vartheta_p = \gamma_{p+1} \sin \vartheta_{p+1} \tag{32}$$

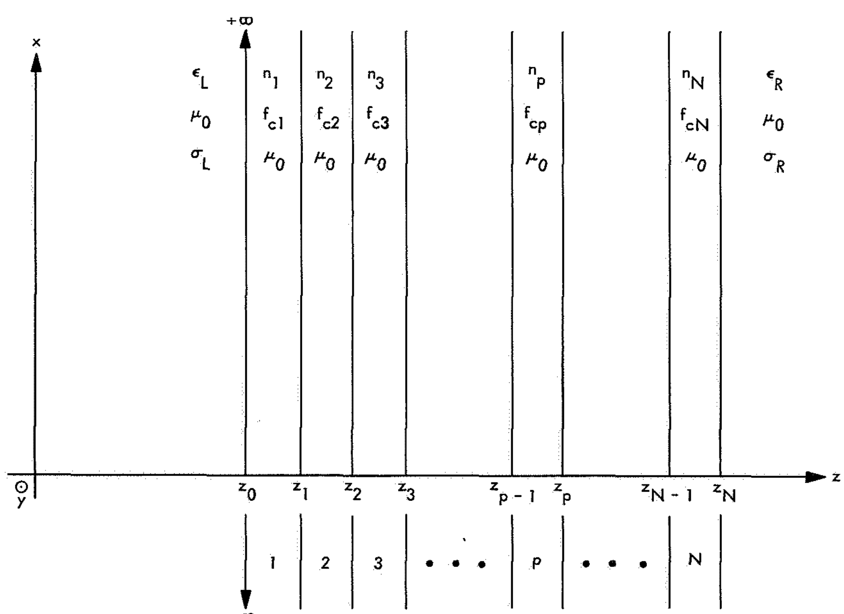


Fig. 1. Plasma model showing parameters and coordinate system

where ϑ_p and ϑ_{p+1} are the angles the propagation constant makes with the normal to the boundaries in the respective layers. Equations (31) and (32) are Snell's laws of reflection and refraction.

The following theory is valid for both types of polarization; however, for definitiveness, consider the reflection and refraction of a plane wave polarized perpendicular to the plane of incidence.

With the aid of Snell's laws, the transverse electric and magnetic fields in the pth layer are given respectively by

$$E_{tp}(z) = e_{p+}\, e^{i\gamma_p (z-z_p)\cos\vartheta_p} + e_{p-}\, e^{-i\gamma_p (z-z_p)\cos\vartheta_p} \tag{33}$$

$$H_{tp}(z) = \frac{1}{\zeta_p}\left[e_{p+}\, e^{i\gamma_p (z-z_p)\cos\vartheta_p} - e_{p-}\, e^{-i\gamma_p (z-z_p)\cos\vartheta_p}\right] \tag{34}$$

$$p = 0, 1, \cdot \cdot \cdot, N$$

where e_{p+} is the modulus of the electric field traveling in the $+z$ direction and e_{p-} is the modulus of the electric field traveling in the $-z$ direction.

The x dependence of all field vectors is given by

$$e^{i\gamma_p x \sin\vartheta_p} \tag{35}$$

and is omitted for brevity.

The expressions for $p=0$, correspond to the fields in the dielectric medium to the left of the plasma. The wave number γ_0 is given by Eq. (19) with the appropriate permittivity and conductivity of the dielectric inserted into the equation. These parameters are assumed to be given.

For $p = N + 1$, i.e., in the dielectric to the right of the plasma, the fields must correspond to waves traveling in the $+z$ direction only, since the radiation condition on the field behavior as $z \to \infty$ requires that the waves originate on the boundary at $z = z_N$. Therefore

$$E_{t,N+1}(z) = e_{N+1,+}\, e^{i\gamma_{N+1}(z-z_N)\cos\vartheta_{N+1}} \tag{36}$$

$$H_{t,N+1}(z) = \frac{e_{N+1,+}}{\zeta_{N+1}}\, e^{i\gamma_{N+1}(z-z_N)\cos\vartheta_{N+1}} \tag{37}$$

The wave number γ_{N+1} is also given by Eq. (19) with the appropriate permittivity and conductivity of the dielectric inserted into the equation. These parameters are also assumed to be given.

The boundary conditions on the transverse components of the field vectors at $z = z_p$ are

$$E_{tp}(z_p) = E_{t,p+1}(z_p) \tag{38}$$

$$H_{tp}(z_p) = H_{t,p+1}(z_p) \tag{39}$$

In terms of positively and negatively traveling waves, the boundary conditions become

$$e_{p+} + e_{p-} = e_{p+1,+}\, e^{i\varphi_{p+1}} + e_{p+1,-}\, e^{-i\varphi_{p+1}} \tag{40}$$

$$\frac{1}{\zeta_p}[e_{p+} - e_{p-}] = \frac{1}{\zeta_{p+1}}[e_{p+1,+}\, e^{i\varphi_{p+1}} - e_{p+1,-}\, e^{-i\varphi_{p+1}}] \tag{41}$$

$$p = 0, 1, \cdot \cdot \cdot, N$$

$$e_{N+1,-} = 0$$

$$\varphi_{N+1} = 0$$

where $\varphi_{p+1} \equiv \gamma_{p+1}(z_p - z_{p+1})\cos\vartheta_{p+1}$ is the phase shift in the layer $(p + 1)$.

In terms of reflection and refraction coefficients, Eqs. (40) and (41) become

$$1 + \rho(z_p) = \tau(z_p) + \frac{e_{p+1,-}}{e_{p+}}\, e^{-i\varphi_{p+1}} \tag{42}$$

$$1 - \rho(z_p) = \frac{\zeta_p}{\zeta_{p+1}}\left[\tau(z_p) - \frac{e_{p+1,-}}{e_{p+}}\, e^{-i\varphi_{p+1}}\right] \tag{43}$$

where $\rho(z_p)$ and $\tau(z_p)$ are the reflection and refraction coefficients defined respectively by

$$\rho(z_p) \equiv \frac{e_{p-}}{e_{p+}} \tag{44}$$

$$\tau(z_p) \equiv \frac{e_{p+1,+}}{e_{p+}}\, e^{i\varphi_{p+1}} \tag{45}$$

With the elimination of

$$\frac{e_{p+1,-}}{e_{p+}}\, e^{-i\varphi_{p+1}}$$

from Eqs. (44) and (45), the following relationship between the reflection coefficient and the refraction coefficient at $z = z_p$ can be found:

$$\tau(z_p) = \frac{1}{2}\left\{1 + \rho(z_p) + \frac{\zeta_{p+1}}{\zeta_p}[1 - \rho(z_p)]\right\} \tag{46}$$

Note that for the case of polarization parallel to the plane of incidence, $\tau(z_p)$ as just given is actually the transverse refraction coefficient at $z = z_p$. The true refraction coefficient can be found by multiplying $\tau(z_p)$ by $\cos\vartheta_p/\cos\vartheta_{p+1}$.

C. Generalized Impedance of the Plasma

At any point z in the interior of the pth layer, let the generalized impedance be defined by

$$\xi_p(z) \equiv \frac{E_{tp}(z)}{H_{tp}(z)} \tag{47}$$

At $z = z_p$, let

$$\xi_p(z_p) = \zeta_p \frac{e_{p+} + e_{p-}}{e_{p+} - e_{p-}} \tag{48}$$

At $z = z_{p-1}$, let

$$\xi_p(z_{p-1}) = \zeta_p \frac{e_{p+}e^{i\varphi_p} + e_{p-}e^{-i\varphi_p}}{e_{p+}e^{i\varphi_p} - e_{p-}e^{-i\varphi_p}} \tag{49}$$

where $\varphi_p \equiv \gamma_p(z_{p-1} - z_p)\cos\vartheta_p$ is the phase shift in the pth layer.

A substitution of Eq. (48) into Eq. (49) gives

$$\xi_p(z_{p-1}) = \zeta_p \frac{\xi_p(z_p) + i\zeta_p \tan\varphi_p}{\zeta_p + i\xi_p(z_p)\tan\varphi_p} \tag{50}$$

An examination of Eqs. (38) and (39) reveals without further calculation that the generalized impedance varies continuously across the boundary of each layer since it is just the ratio of two continuously varying functions. Therefore

$$\xi_{p-1}(z_{p-1}) = \xi_p(z_{p-1}) = \zeta_p \frac{\xi_p(z_p) + i\zeta_p \tan\varphi_p}{\zeta_p + i\xi_p(z_p)\tan\varphi_p} \tag{51}$$

where the second equality is Eq. (50).

The generalized impedance in the layer $N + 1$ is by definition ζ_{N+1}, and from the continuity of the generalized impedance across the boundary $z = z_N$ it follows that

$$\xi_N(z_N) = \xi_{N+1}(z_N) = \zeta_{N+1} \tag{52}$$

Therefore, by successive use of Eq. (51) for $p = N, \cdots, 1$, the generalized impedance at each boundary interface can be found.

D. Reflection and Refraction Coefficients of the Plasma

By the inversion of Eq. (48) for $\rho(z_p) \equiv e_{p-}/e_{p+}$, the following expression for the reflection coefficient at $z = z_p$ is found in terms of the generalized impedance at $z = z_p$:

$$\rho(z_p) = \frac{\xi_p(z_p) - \zeta_p}{\xi_p(z_p) + \zeta_p} \tag{53}$$

By the substitution of Eq. (53) into Eq. (46), the following expression for the refraction coefficient at $z = z_p$ is found in terms of the generalized impedance at $z = z_p$:

$$\tau(z_p) = \frac{\xi_p(z_p) + \zeta_{p+1}}{\xi_p(z_p) + \zeta_p} \tag{54}$$

Therefore, the reflection and refraction coefficients at $z = z_p$ can be found in terms of the generalized impedance at $z = z_p$ as found in the previous section.

Of interest in this present study is the reflection coefficient r evaluated at $z = z_0$ and the refraction coefficient t evaluated between $z = z_0$ and $z = z_N$.

From this theory, it follows that

$$r \equiv \frac{e_{0-}}{e_{0+}} = \rho(z_0) \tag{55}$$

$$t \equiv \frac{e_{N+1,+}}{e_{0+}} = \prod_{p=0}^{N} \tau(z_p)\, e^{-i\varphi_{p+1}} \tag{56}$$

Note that for the case of polarization parallel to the plane of incidence, t is actually the transverse refraction coefficient between $z = z_0$ and $z = z_N$. The true refraction coefficient can be found by multiplying t by $\cos\vartheta_0/\cos\vartheta_{N+1}$.

IV. Invariant Imbedding

The concept of invariant imbedding as derived from Ambarzumian's principle of invariance is now used to derive the differential equations which the reflection and refraction coefficients satisfy (Refs. 1–3).

A. Geometry of the Problem

Although the geometry of the problem is essentially the same as that in the previous section, only one typical layer need now be considered. This typical layer, as shown schematically in Fig. 2, lies in the region $\eta < z < \eta + \delta\eta$ where η satisfies the inequality $\alpha < \eta < \beta$ and $\delta\eta << 1$. The layer is assumed to be homogeneous with constants $\epsilon(\eta)$, μ_0, and $\sigma(\eta)$.

In the notation of the previous section and in the region $z \leqq \alpha$, it follows that

$$E_{t1}(z) = e_{1+} e^{i\gamma_1 (z-\alpha)\cos\vartheta_1} + e_{1-} e^{-i\gamma_1 (z-\alpha)\cos\vartheta_1} = e_0 \left[e^{i\gamma_1 (z-\alpha)\cos\vartheta_1} + r e^{-i\gamma_1 (z-\alpha)\cos\vartheta_1} \right] \quad (57)$$

$$H_{t1}(z) = \frac{1}{\zeta_1} \left[e_{1+} e^{i\gamma_1 (z-\alpha)\cos\vartheta_1} - e_{1-} e^{-i\gamma_1 (z-\alpha)\cos\vartheta_1} \right] = \frac{e_0}{\zeta_1} \left[e^{i\gamma_1 (z-\alpha)\cos\vartheta_1} - r e^{-i\gamma_1 (z-\alpha)\cos\vartheta_1} \right] \quad (58)$$

and in the region $z \geqq \beta$, it follows that

$$E_{t2}(z) = e_{2+} e^{i\gamma_2 (z-\beta)\cos\vartheta_2} = e_0 t e^{i\gamma_2 (z-\beta)\cos\vartheta_2} \quad (59)$$

$$H_{t2}(z) = \frac{e_{2+}}{\zeta_2} e^{i\gamma_2 (z-\beta)\cos\vartheta_2} = \frac{e_0 t}{\zeta_2} e^{i\gamma_2 (z-\beta)\cos\vartheta_2} \quad (60)$$

where r is the reflection coefficient at $z = \alpha$, t is the refraction coefficient between $z = \alpha$ and $z = \beta$, and e_0 is the modulus of the incident electric field vector.

In the region $\alpha < z < \beta$, the medium is described by $\epsilon(z)$, μ_0, and $\sigma(z)$. The constitutive parameters $\epsilon(z)$ and $\sigma(z)$ are not necessarily constants and are allowed to vary with the z coordinate only.

The propagation constant at any point $z = \eta$ is

$$\gamma(\eta) \equiv \beta(\eta) + i\alpha(\eta) \quad (61)$$

where

$$\beta(\eta) = \omega \sqrt{\frac{\mu_0}{2}} \sqrt{\left\{ \epsilon(\eta) + \sqrt{\left[\epsilon^2(\eta) + \frac{\sigma^2(\eta)}{\omega^2} \right]} \right\}} \quad (62)$$

$$\alpha(\eta) = \omega \sqrt{\frac{\mu_0}{2}} \sqrt{\left\{ -\epsilon(\eta) + \sqrt{\left[\epsilon^2(\eta) + \frac{\sigma^2(\eta)}{\omega^2} \right]} \right\}} \quad (63)$$

If $\gamma_t(\eta) \equiv \gamma(\eta)\cos\vartheta(\eta)$, then, by Snell's law,

$$\gamma_t^2(\eta) = \gamma^2(\eta) - \gamma_{t1}^2 \quad (64)$$

where

$$\gamma_{t1} \equiv \gamma_1 \sin\vartheta_1 \quad (65)$$

For the case of polarization perpendicular to the plane of incidence,

$$\zeta \equiv \frac{\eta}{\cos\vartheta} = \frac{\omega\mu_0}{\gamma\cos\vartheta} = \frac{\omega\mu_0}{\gamma_t}$$

For the case of polarization parallel to the plane of incidence,

$$\zeta \equiv \eta\cos\vartheta = \omega\mu_0 \frac{\cos\vartheta}{\gamma} = \omega\mu_0 \frac{\gamma_t}{\gamma^2}$$

B. Reflection Coefficient

The theoretical treatment of the problem of finding the reflection coefficient begins at the interface $z = \beta$. The region $z > \beta$ is homogeneous with constants ϵ_2, μ_0, and σ_2. If the region $z \leqq \beta$ is also assumed to be homogeneous with constants $\epsilon(\beta)$, μ_0, and $\sigma(\beta)$, the reflection coefficient at $z = \beta$ is

$$\rho(\beta) = \frac{\zeta_2 - \zeta(\beta)}{\zeta_2 + \zeta(\beta)} \quad (66)$$

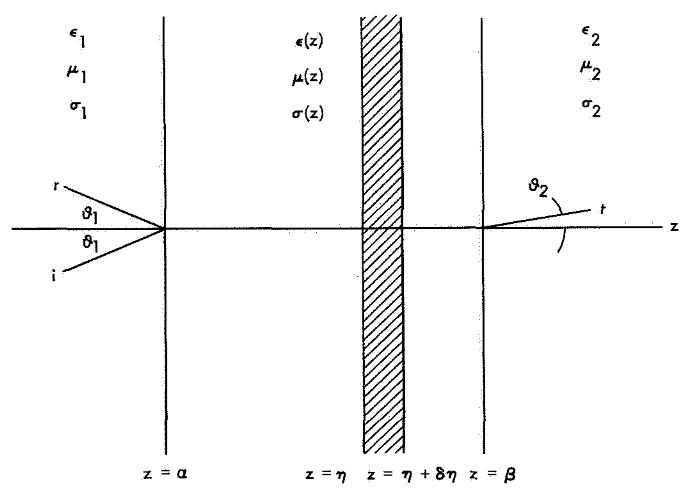


Fig. 2. Plasma model showing typical layer

The change in the reflection coefficient is examined as the medium is constructed from $z = \beta$ to $z = \alpha$ by successively adding layers of infinitesimal thickness. The change in the reflection coefficient because of each additional layer can be determined by examining a typical layer as shown schematically in Fig. 2.

If the reflection coefficient at $z = \eta$ is determined by the method of multiple reflections, it is found that

$$\rho(\eta) = \rho_1(\eta) + \rho_2(\eta) + \rho_3(\eta) + \cdots \tag{67}$$

where the multiple reflections are defined by

$$\begin{aligned} \rho_1(\eta) &\equiv \rho_+ \\ \rho_2(\eta) &\equiv \tau_+ e^{i\varphi(\eta)} \rho(\eta + \delta\eta) e^{i\varphi(\eta)} \tau_- \\ \rho_3(\eta) &\equiv \tau_+ e^{i\varphi(\eta)} \rho(\eta + \delta\eta) e^{i\varphi(\eta)} \\ &\qquad \rho_- e^{i\varphi(\eta)} \rho(\eta + \delta\eta) e^{i\varphi(\eta)} \tau_- \\ &\vdots \end{aligned} \tag{68}$$

and where

$$\rho_+ \equiv \frac{\zeta(\eta + \delta\eta) - \zeta(\eta)}{\zeta(\eta + \delta\eta) + \zeta(\eta)} \equiv -\rho_- \tag{69}$$

$$\tau_\pm = 1 + \rho_\pm \tag{70}$$

$$\varphi(\eta) \equiv \gamma_t(\eta)\, \delta\eta \tag{71}$$

A series representation of $\rho(\eta)$ can be obtained from Eqs. (67) and (68)

$$\rho(\eta) = \rho_+ + \frac{\tau_+ \tau_-}{\rho_-} \sum_{\nu=1}^{\infty} [\rho_- \rho(\eta + \delta\eta) e^{i2\varphi(\eta)}]^\nu \tag{72}$$

This series can be summed explicitly to give

$$\rho(\eta) = \frac{\rho_+ + \rho(\eta + \delta\eta) e^{i2\varphi(\eta)}}{1 + \rho_+ \rho(\eta + \delta\eta) e^{i2\varphi(\eta)}} \tag{73}$$

If $\delta\eta$ is allowed to approach zero, then

$$\rho_+ \simeq \frac{\dfrac{d\zeta(\eta)}{dz}\delta\eta}{2\zeta(\eta)} \tag{74}$$

$$\rho(\eta + \delta\eta) \simeq \rho(\eta) + \frac{d\rho(\eta)}{dz}\delta\eta \tag{75}$$

$$e^{i2\varphi(\eta)} \simeq 1 + i2\gamma_t(\eta)\,\delta\eta \tag{76}$$

If Eqs. (74), (75), and (76) are introduced into Eq. (73) then, by keeping terms to the first order in $\delta\eta$,

$$\frac{d\rho(\eta)}{dz} + [1 - \rho^2(\eta)] \frac{d\zeta(\eta)/dz}{2\zeta(\eta)} + i2\gamma_t(\eta)\rho(\eta) = 0 \tag{77}$$

From the boundary conditions on the field vectors at $z = \alpha$, it can be shown that

$$r = \frac{\xi(\alpha) - \zeta_1}{\xi(\alpha) + \zeta_1} \tag{78}$$

where

$$\xi(\alpha) \equiv \zeta(\alpha) \frac{1 + \rho(\alpha)}{1 - \rho(\alpha)} \tag{79}$$

If the differential Eq. (77) is solved numerically, the reflection coefficient can be found as given by Eq. (78). The numerical solution of this equation is described fully in the appendix.

C. Refraction Coefficient

The theoretical treatment of the problem of finding the refraction coefficient begins at the interface $z = \beta$ and proceeds to the interface $z = \alpha$ in a manner completely analogous to the procedure of finding the reflection coefficient in the previous section. Therefore, only a summary of the results pertinent to the determination of the refraction coefficient is given.

The refraction coefficient at $z = \beta$ is

$$\tau(\beta) = \frac{\alpha\zeta_2}{\zeta_2 + \zeta(\beta)} \tag{80}$$

The refraction coefficient at $z = \eta$ is

$$\tau(\eta) = \tau_1(\eta) + \tau_2(\eta) + \tau_3(\eta) + \cdots \tag{81}$$

where

$$\begin{aligned} \tau_1(\eta) &\equiv \tau_+ e^{i\varphi(\eta)} \tau(\eta + \delta\eta) \\ \tau_2(\eta) &\equiv \tau_+ e^{i\varphi(\eta)} \rho(\eta + \delta\eta) e^{i\varphi(\eta)} \rho_- e^{i\varphi(\eta)} \tau(\eta + \delta\eta) \\ \tau_3(\eta) &\equiv \tau_+ e^{i\varphi(\eta)} \rho(\eta + \delta\eta) e^{i\varphi(\eta)} \rho_- e^{i\varphi(\eta)} \rho(\eta + \delta\eta) e^{i\varphi(\eta)} \\ &\qquad \rho_- e^{i\varphi(\eta)} \tau(\eta + \delta\eta) \end{aligned} \tag{82}$$

A series representation of $\tau(\eta)$ can be obtained from Eqs. (81) and (82)

$$\tau(\eta) = \tau_+ \tau(\eta + \delta\eta) \sum_{\nu=0}^{\infty} \rho_-^{\nu} \rho^{\nu}(\eta + \delta\eta) e^{i(2\nu+1)\varphi(\eta)} \tag{83}$$

This series can be summed explicitly to give

$$\tau(\eta) = \frac{\tau_+ \tau(\eta + \delta\eta) e^{i\varphi(\eta)}}{1 + \rho_+ \rho(\eta + \delta\eta) e^{i2\varphi(\eta)}} \tag{84}$$

If $\delta\eta$ is allowed to approach zero, the above equation becomes to first order in $\delta\eta$

$$\frac{d\tau(\eta)}{dz} + [1 - \rho(\eta)] \frac{d\zeta(\eta)/dz}{2\zeta(\eta)} \tau(\eta) + i\gamma_t(\eta)\tau(\eta) = 0 \tag{85}$$

From the boundary conditions on the field vectors at $z = \alpha$, it can be shown that

$$t = \frac{2\xi(\alpha)}{\xi(\alpha) + \zeta_1} \tag{86}$$

where

$$\xi(\alpha) \equiv \zeta(\alpha) \frac{\tau(\alpha)}{2 - \tau(\alpha)} \tag{87}$$

Again note that for the case of polarization parallel to the plane of incidence, t as given here is actually the transverse refraction coefficient between $z = \alpha$ and $z = \beta$. The true refraction coefficient can be found by multiplying t by $\cos\vartheta_1/\cos\vartheta_2$.

If the differential Eq. (85) is solved numerically, the refraction coefficient can be found as given by Eq. (86). The numerical solution of this equation is described fully in the appendix.

Appendix

The Calculation of Reflection and Refraction Coefficients: A Computer Program

I. Introduction

This program has been specifically developed for use with the study on wave propagation in a planar inhomogeneous plasma medium as described in this report. The source language is FORTRAN II for use on an SDS 930 computer.

A. Program Description

The purpose of this program is to determine the reflection and refraction coefficients of a wave propagating in a planar inhomogeneous plasma medium. The calculations to be performed in determining the reflection and refraction coefficients are derived in the main body of this report.

The Main Program 1 performs the necessary calculations by assuming that the plasma is stratified into a series of homogeneous layers. The Main Program 2 performs the necessary calculations by solving numerically the Riccati equations which the reflection and refraction coefficients satisfy.

In either of the main programs, the plasma is assumed to be linear, isotropic, and cold. The plasma may be lossy or lossless. The properties of the plasma are not necessarily constants and are allowed to vary with the z-coordinate only.

The wave may be incident on the plasma from any angle. If, however, incidence is other than normal, the wave must be separated into its parallel and perpendicular components of polarization relative to the plasma, and each component treated separately.

B. Input Requirements

The required input data to the program are

(1) Signal frequency.

(2) Angle of incidence.

(3) Number of layers.

(4) Separation of layers.

(5) Electron density and collision frequency of each layer.

(6) Endpoints of the plasma.

(7) Step-size and number for electron density and collision frequency profiles.

(8) Step-size and number for numerical solution of Riccati equations.

(9) Degree of polynomial used when interpolating and differentiating numerically.

(10) Permittivities and conductivities of the external dielectrics.

C. Program Limitations

The program has certain limitations, but they are entirely in keeping with the study. The limitations are as follows:

(1) The plasma is assumed to be linear, isotropic, and cold.

(2) The properties of the plasma are allowed to vary with the z-coordinate only.

(3) The two external regions are assumed to be lossy dielectrics.

(4) The permeabilities of all regions are assumed to be that of a vacuum.

(5) Harmonic time-dependence is assumed with $e^{-i\omega t}$ variation.

(6) For the case of a lossless plasma only, the permittivity ϵ is not allowed to change sign.

II. Use

A. Definitions

Use of the program is given in the form of tables (A-1 to A-5) that follow.

Table A-1. Input definitions

Program name	Symbol	Property	Unit
F	f	Signal frequency	Hz
AOID	ϑ_i	Angle of incidence	deg
N	N	Number of layers $N \leqq 100$	—
Z (I), I = 1, N + 1	z_i	z-coordinate of the left edge of each layer, including the right external dielectric	m
ED (I), I = 1, N	n_i	Electron density of the *i*th layer	e^-/cm^3
FC (I), I = 1, N	f_{ci}	Collision frequency of the *i*th layer	Hz
ALPHA	α	Left endpoint of the plasma	m
BETA	β	Right endpoint of the plasma	m
M	m	Number of steps in the electron density and collision frequency profiles $M \leqq 101$	—
SM	s_m	Step-size in the electron density and collision frequency profiles	m
N	n	Number of steps in the numerical solution of the Riccati equations $N \leqq 9999$	—
SN	s_n	Step size in the numerical solution of the Riccati equations	m
NP	np	Degree of the polynomial used when interpolating and differentiating numerically $np \leqq 10$	—
EL	ϵ_l	Permittivity of left external dielectric	farads/m
ER	ϵ_r	Permittivity of right external dielectric	farads/m
OL	σ_l	Conductivity of left external dielectric	mhos/m
OR	σ_r	Conductivity of right external dielectric	mhos/m

Table A-2. Numerical input, Main Program I

Card number	Program name	Format
1	F, AOID, N	E10.0, F10.0, I10
2	Z (I), I = 1, N + 1	8F10.0
	ED (I), I = 1, N	8E10.0
	FC (I), I = 1, N	8E10.0
	EL, ER, OL, OR	4E10.0

Table A-3. Analytical input, Main Program I

Card number	Program name	Format
1	F, AOID, ALPHA, BETA, N, SN	E10.0, 3F10.0, I3, F7.0
2	EL, ER, OL, OR	4E10.0

Table A-4. Analytical input, Main Program 2

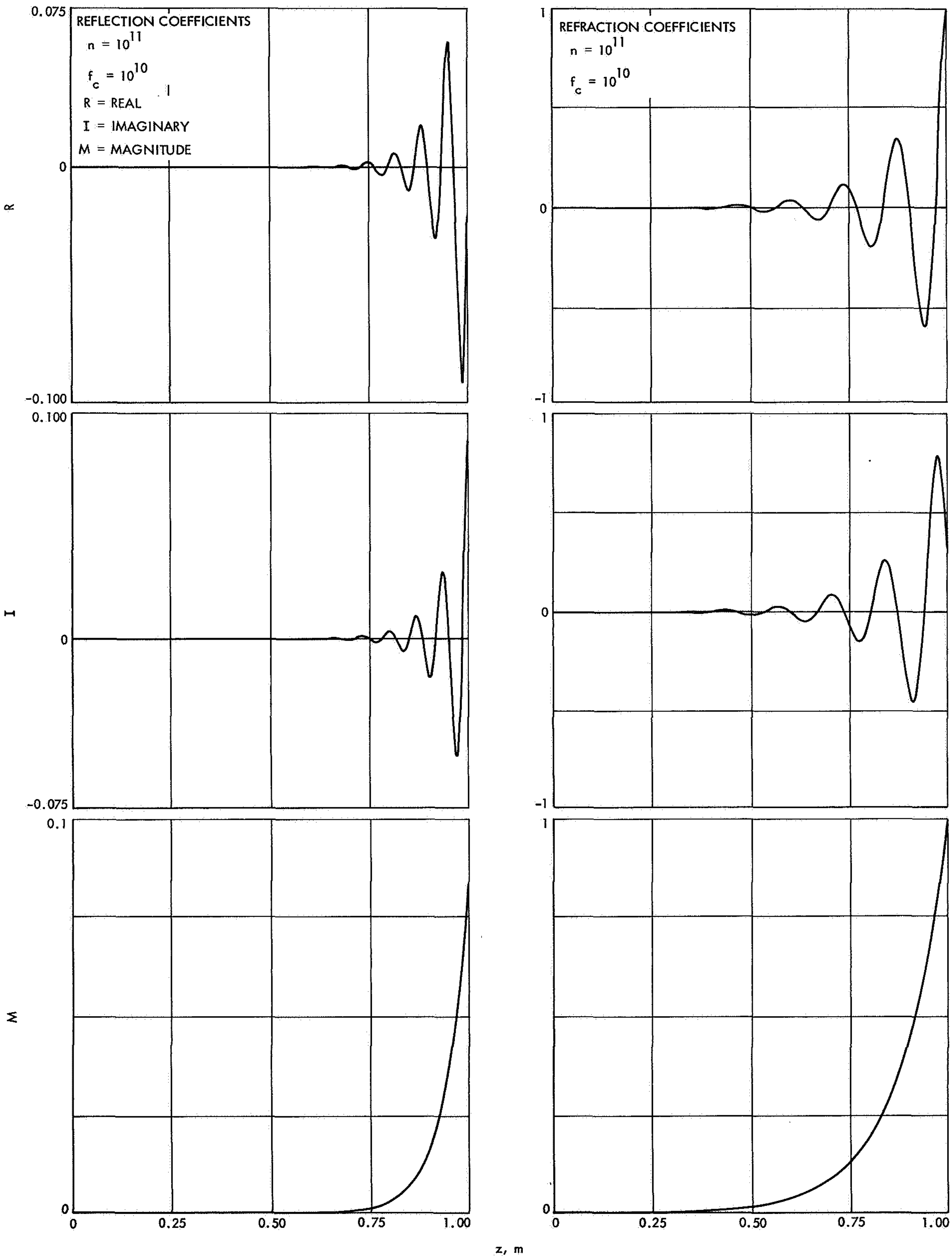
Card number	Program name	Format
1	F, AOID, ALPHA, BETA, M, SM, N, SN, NP	E10.0, 3F10.0, I3, F7.3, I4, F6.3, I2
2	EL, ER, OL, OR	4E10.0

Table A-5. Output definitions

Program name	Symbol	Property	Unit
ED (I)	n_i	Electron density of the *i*th layer	e^-/cm^3
FC (I)	f_{ci}	Collision frequency of the *i*th layer	Hz
FP (I)	f_{pi}	Plasma frequency of the *i*th layer	Hz
FCO (I)	f_{coi}	Cutoff frequency of the *i*th layer	Hz
E (I)	ϵ_i	Permittivity of the *i*th layer	farads/m
U (I)	μ_i	Permeability of the *i*th layer	henrys/m
O (I)	σ_i	Conductivity of the *i*th layer	mhos/m
AOTD	ϑ_t	Angle of refraction	deg
RR	Re $\{\rho\}$	Real part of the reflection coefficient	—
RI	Im $\{\rho\}$	Imaginary part of the reflection coefficient	—
RM	Mag $\{\rho\}$	Magnitude of the reflection coefficient	—
RAD	Arg $\{\rho\}$	Argument of the reflection coefficient	deg
TR	Re $\{\tau\}$	Real part of the refraction coefficient	—
TI	Im $\{\tau\}$	Imaginary part of the refraction coefficient	—
TM	Mag $\{\tau\}$	Magnitude of the refraction coefficient	—
TAD	Arg $\{\tau\}$	Argument of the refraction coefficient	deg
AN	A_n	Attenuation	nepers
ADB	A_{db}	Attenuation	dB

B. Sample Output Data

The following examples show reflection and refraction coefficients for various electron concentration and collision frequency distributions.



REFLECTION COEFFICIENTS
n = 10^11
f_c = 10^10
R = REAL
I = IMAGINARY
M = MAGNITUDE
REFRACTION COEFFICIENTS
n = 10^11
f_c = 10^10
0.075
0
-0.100
0.100
0
-0.075
0.1
0
1
0
-1
1
0
-1
1
0
R
I
M
0
0.25
0.50
0.75
1.00
z, m

$n = 10^{11}$

$f_c = 10^{10}$

SIGNAL FREQUENCY = 2.2950000E 09 HZ

Z-COMPONENT OF SEPARATION

.000	.010	.020	.030	.040	.050	.060	.070	.080	.090
.100	.110	.120	.130	.140	.150	.160	.170	.180	.190
.200	.210	.220	.230	.240	.250	.260	.270	.280	.290
.300	.310	.320	.330	.340	.350	.360	.370	.380	.390
.400	.410	.420	.430	.440	.450	.460	.470	.480	.490
.500	.510	.520	.530	.540	.550	.560	.570	.580	.590
.600	.610	.620	.630	.640	.650	.660	.670	.680	.690
.700	.710	.720	.730	.740	.750	.760	.770	.780	.790
.800	.810	.820	.830	.840	.850	.860	.870	.880	.890
.900	.910	.920	.930	.940	.950	.960	.970	.980	.990
1.000									

PERMITTIVITY OF LEFT EXTERNAL REGION = 8.8540000E-12 FARADS/M
PERMEABILITY OF LEFT EXTERNAL REGION = 1.2566370E-06 HENRYS/M
CONDUCTIVITY OF LEFT EXTERNAL REGION = .0000000E 00 MHOS/M

PERMITTIVITY OF RIGHT EXTERNAL REGION = 8.8540000E-12 FARADS/M
PERMEABILITY OF RIGHT EXTERNAL REGION = 1.2566370E-06 HENRYS/M
CONDUCTIVITY OF RIGHT EXTERNAL REGION = .0000000E 00 MHOS/M

NORMAL INCIDENCE

RE [REFLECTION COEFFICIENT]= .46114149E-02
IM [REFLECTION COEFFICIENT]=-.86903500E-01

MAG[REFLECTION COEFFICIENT]= .87025763E-01
ARG[REFLECTION COEFFICIENT]=-.86962528E 02 [DEGREES]

RE [REFRACTION COEFFICIENT]=-.26651733E-03
IM [REFRACTION COEFFICIENT]= .47607977E-04

MAG[REFRACTION COEFFICIENT]= .27073604E-03
ARG[REFRACTION COEFFICIENT]= .16987208E 03 [DEGREES]

ATTENUATION IN NEPERS = 1.6428732E 01
ATTENUATION IN DB = 7.1349078E 01

SIGNAL FREQUENCY = 2.2950000E 09 HZ

ALPHA = .000
BETA = 1.000

PROFILE NUMBER OF STEPS = 101 STEP SIZE = .010
RUNGE/KUTTA NUMBER OF STEPS = 1001 STEP SIZE = .001 DEGREE = 4

PERMITTIVITY OF LEFT EXTERNAL REGION = 8.8540000E-12 FARADS/M
PERMEABILITY OF LEFT EXTERNAL REGION = 1.2566370E-06 HENRYS/M
CONDUCTIVITY OF LEFT EXTERNAL REGION = .0000000E 00 MHOS/M

PERMITTIVITY OF RIGHT EXTERNAL REGION = 8.8540000E-12 FARADS/M
PERMEABILITY OF RIGHT EXTERNAL REGION = 1.2566370E-06 HENRYS/M
CONDUCTIVITY OF RIGHT EXTERNAL REGION = .0000000E 00 MHOS/M

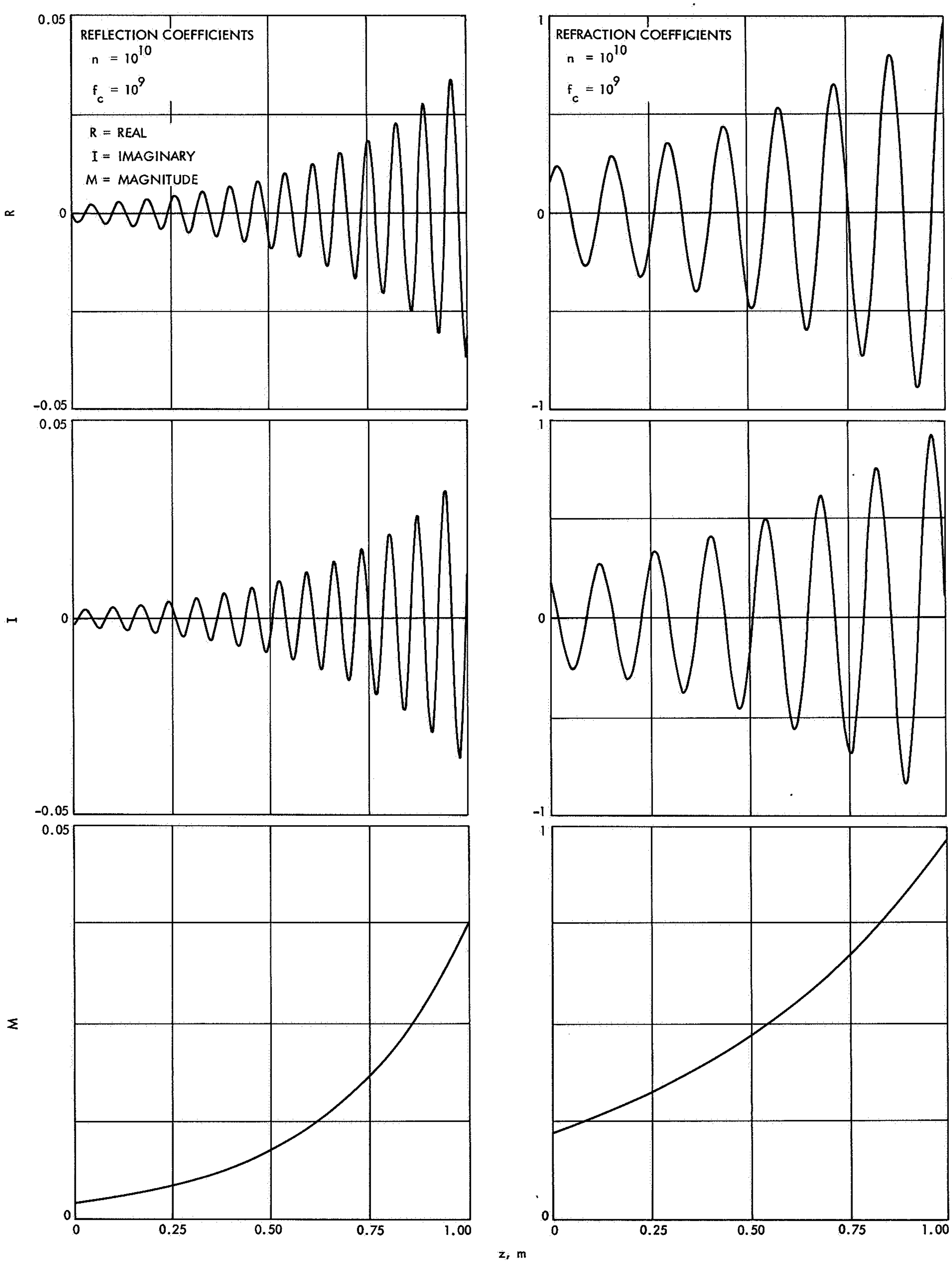
NORMAL INCIDENCE

RE [REFLECTION COEFFICIENT]= .46114149E-02
IM [REFLECTION COEFFICIENT]=-.86903499E-01

MAG[REFLECTION COEFFICIENT]= .87025763E-01
ARG[REFLECTION COEFFICIENT]=-.86962528E 02 [DEGREES]

RE [REFRACTION COEFFICIENT]=-.26158034E-03
IM [REFRACTION COEFFICIENT]= .70672616E-04

MAG[REFRACTION COEFFICIENT]= .27095921E-03
ARG[REFRACTION COEFFICIENT]= .16488105E 03 [DEGREES]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n = 10^10
f_c = 10^9
R = REAL
I = IMAGINARY
M = MAGNITUDE
REFRACTION COEFFICIENTS
n = 10^10
f_c = 10^9
0.05
0
-0.05
1
-1
R
I
M
0
0.25
0.50
0.75
1.00
z, m

$n = 10^{10}$

$f_c = 10^9$

SIGNAL FREQUENCY = 2.2950000E 09 HZ

Z-COMPONENT OF SEPARATION

.000	.010	.020	.030	.040	.050	.060	.070	.080	.090
.100	.110	.120	.130	.140	.150	.160	.170	.180	.190
.200	.210	.220	.230	.240	.250	.260	.270	.280	.290
.300	.310	.320	.330	.340	.350	.360	.370	.380	.390
.400	.410	.420	.430	.440	.450	.460	.470	.480	.490
.500	.510	.520	.530	.540	.550	.560	.570	.580	.590
.600	.610	.620	.630	.640	.650	.660	.670	.680	.690
.700	.710	.720	.730	.740	.750	.760	.770	.780	.790
.800	.810	.820	.830	.840	.850	.860	.870	.880	.890
.900	.910	.920	.930	.940	.950	.960	.970	.980	.990
1.000									

PERMITTIVITY OF LEFT EXTERNAL REGION = 8.8540000E-12 FARADS/M
PERMEABILITY OF LEFT EXTERNAL REGION = 1.2566370E-06 HENRYS/M
CONDUCTIVITY OF LEFT EXTERNAL REGION = .0000000E 00 MHOS/M

PERMITTIVITY OF RIGHT EXTERNAL REGION = 8.8540000E-12 FARADS/M
PERMEABILITY OF RIGHT EXTERNAL REGION = 1.2566370E-06 HENRYS/M
CONDUCTIVITY OF RIGHT EXTERNAL REGION = .0000000E 00 MHOS/M

NORMAL INCIDENCE

RE [REFLECTION COEFFICIENT]= .33627880E-01
IM [REFLECTION COEFFICIENT]=-.18113105E-01

MAG[REFLECTION COEFFICIENT]= .38195797E-01
ARG[REFLECTION COEFFICIENT]=-.28308397E 02 [DEGREES]

RE [REFRACTION COEFFICIENT]= .13892823E 00
IM [REFRACTION COEFFICIENT]= .19072456E 00

MAG[REFRACTION COEFFICIENT]= .23595956E 00
ARG[REFRACTION COEFFICIENT]= .53929532E 02 [DEGREES]

ATTENUATION IN NEPERS = 2.8881897E 00
ATTENUATION IN DB = 1.2543249E 01

SIGNAL FREQUENCY = 2.2950000E 09 HZ

ALPHA = .000
BETA = 1.000

PROFILE NUMBER OF STEPS = 101 STEP SIZE = .010
RUNGE/KUTTA NUMBER OF STEPS = 1001 STEP SIZE = .001 DEGREE = 4

PERMITTIVITY OF LEFT EXTERNAL REGION = 8.8540000E-12 FARADS/M
PERMEABILITY OF LEFT EXTERNAL REGION = 1.2566370E-06 HENRYS/M
CONDUCTIVITY OF LEFT EXTERNAL REGION = .0000000E 00 MHOS/M

PERMITTIVITY OF RIGHT EXTERNAL REGION = 8.8540000E-12 FARADS/M
PERMEABILITY OF RIGHT EXTERNAL REGION = 1.2566370E-06 HENRYS/M
CONDUCTIVITY OF RIGHT EXTERNAL REGION = .0000000E 00 MHOS/M

NORMAL INCIDENCE

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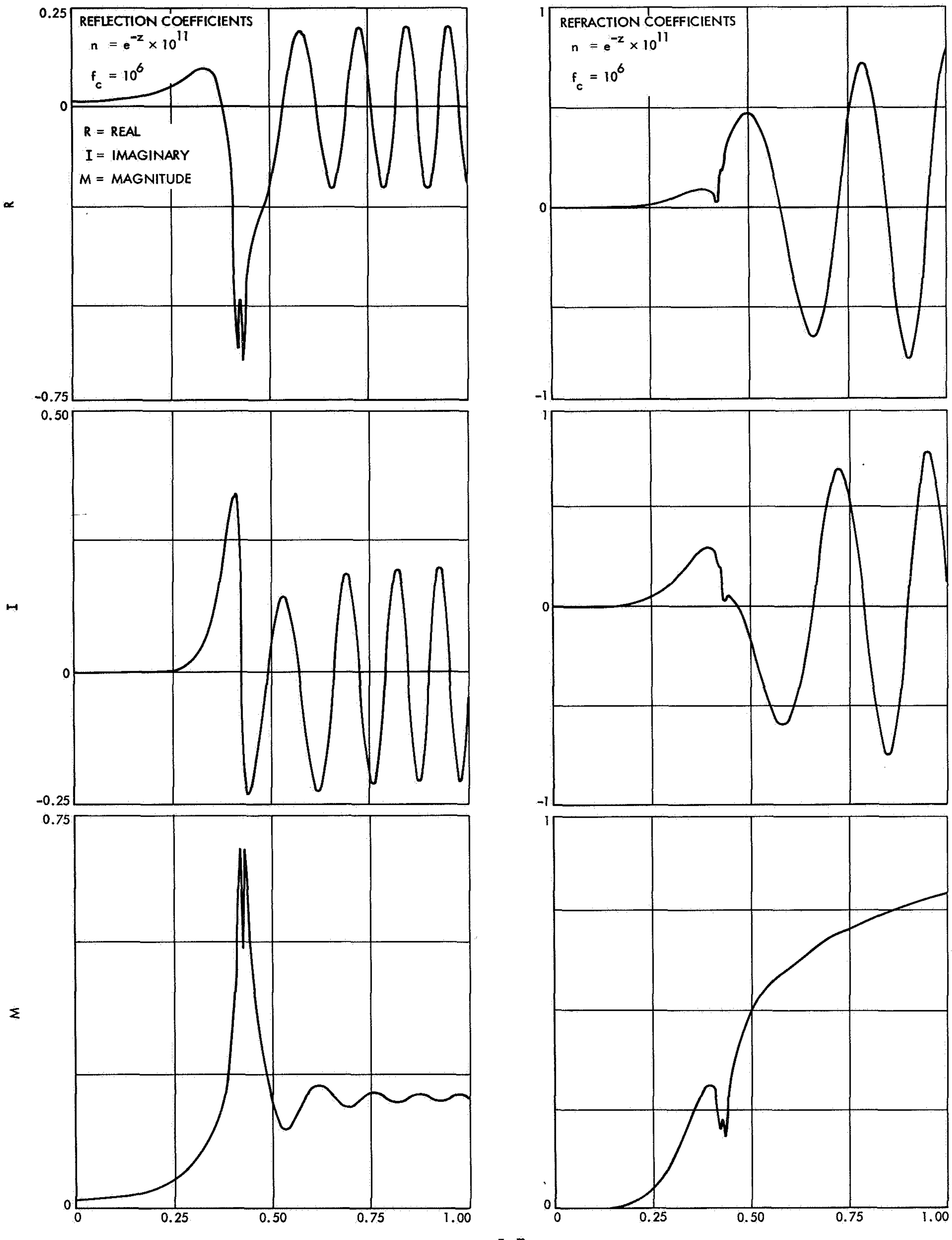
IM [REFLECTION COEFFICIENT]=-.18113101E-01

MAG[REFLECTION COEFFICIENT]= .38195704E-01
ARG[REFLECTION COEFFICIENT]=-.28308464E 02 [DEGREES]

RE [REFRACTION COEFFICIENT]= .14672215E 00
IM [REFRACTION COEFFICIENT]= .19283635E 00

MAG[REFRACTION COEFFICIENT]= .24230816E 00
ARG[REFRACTION COEFFICIENT]= .52733817E 02 [DEGREES]



REFLECTION COEFFICIENTS
$n = e^{-z} \times 10^{11}$
$f_c = 10^6$
R = REAL
I = IMAGINARY
M = MAGNITUDE
REFRACTION COEFFICIENTS
$n = e^{-z} \times 10^{11}$
$f_c = 10^6$
R
I
M
0.25
0
-0.75
0.50
-0.25
0.75
1
-1
0
0.25
0.50
0.75
1.00
z, m

$n = e^{-z} \times 10^{11}$

$f_c = 10^6$

SIGNAL FREQUENCY = 2.2950000E 09 HZ

Z-COMPONENT OF SEPARATION

.000	.010	.020	.030	.040	.050	.060	.070	.080	.090
.100	.110	.120	.130	.140	.150	.160	.170	.180	.190
.200	.210	.220	.230	.240	.250	.260	.270	.280	.290
.300	.310	.320	.330	.340	.350	.360	.370	.380	.390
.400	.410	.420	.430	.440	.450	.460	.470	.480	.490
.500	.510	.520	.530	.540	.550	.560	.570	.580	.590
.600	.610	.620	.630	.640	.650	.660	.670	.680	.690
.700	.710	.720	.730	.740	.750	.760	.770	.780	.790
.800	.810	.820	.830	.840	.850	.860	.870	.880	.890
.900	.910	.920	.930	.940	.950	.960	.970	.980	.990
1.000									

PERMITTIVITY OF LEFT EXTERNAL REGION = 8.8540000E-12 FARADS/M
PERMEABILITY OF LEFT EXTERNAL REGION = 1.2566370E-06 HENRYS/M
CONDUCTIVITY OF LEFT EXTERNAL REGION = .0000000E 00 MHOS/M

PERMITTIVITY OF RIGHT EXTERNAL REGION = 8.8540000E-12 FARADS/M
PERMEABILITY OF RIGHT EXTERNAL REGION = 1.2566370E-06 HENRYS/M
CONDUCTIVITY OF RIGHT EXTERNAL REGION = .0000000E 00 MHOS/M

NORMAL INCIDENCE

RE [REFLECTION COEFFICIENT]= .33361337E 00
IM [REFLECTION COEFFICIENT]=-.94204088E 00

MAG[REFLECTION COEFFICIENT]= .99936925E 00
ARG[REFLECTION COEFFICIENT]=-.70498964E 02 [DEGREES]

RE [REFRACTION COEFFICIENT]= .74962586E-04
IM [REFRACTION COEFFICIENT]= .68426900E-04

MAG[REFRACTION COEFFICIENT]= .10149695E-03
ARG[REFRACTION COEFFICIENT]= .42390268E 02 [DEGREES]

ATTENUATION IN NEPERS = 1.8390964E 01
ATTENUATION IN DB = 7.9870941E 01

SIGNAL FREQUENCY = 2.2950000E 09 HZ

ALPHA = .000
BETA = 1.000

PROFILE NUMBER OF STEPS = 101 STEP SIZE = .010
RUNGE/KUTTA NUMBER OF STEPS = 1001 STEP SIZE = .001 DEGREE = 4

PERMITTIVITY OF LEFT EXTERNAL REGION = 8.8540000E-12 FARADS/M
PERMEABILITY OF LEFT EXTERNAL REGION = 1.2566370E-06 HENRYS/M
CONDUCTIVITY OF LEFT EXTERNAL REGION = .0000000E 00 MHOS/M

PERMITTIVITY OF RIGHT EXTERNAL REGION = 8.8540000E-12 FARADS/M
PERMEABILITY OF RIGHT EXTERNAL REGION = 1.2566370E-06 HENRYS/M
CONDUCTIVITY OF RIGHT EXTERNAL REGION = .0000000E 00 MHOS/M

NORMAL INCIDENCE

RE [REFLECTION COEFFICIENT]= .32605677E 00
IM [REFLECTION COEFFICIENT]=-.94468835E 00

MAG[REFLECTION COEFFICIENT]= .99937435E 00
ARG[REFLECTION COEFFICIENT]=-.70958018E 02 [DEGREES]

RE [REFRACTION COEFFICIENT]= .71345228E-04
IM [REFRACTION COEFFICIENT]=-.24566118E-04

MAG[REFRACTION COEFFICIENT]= .75456185E-04
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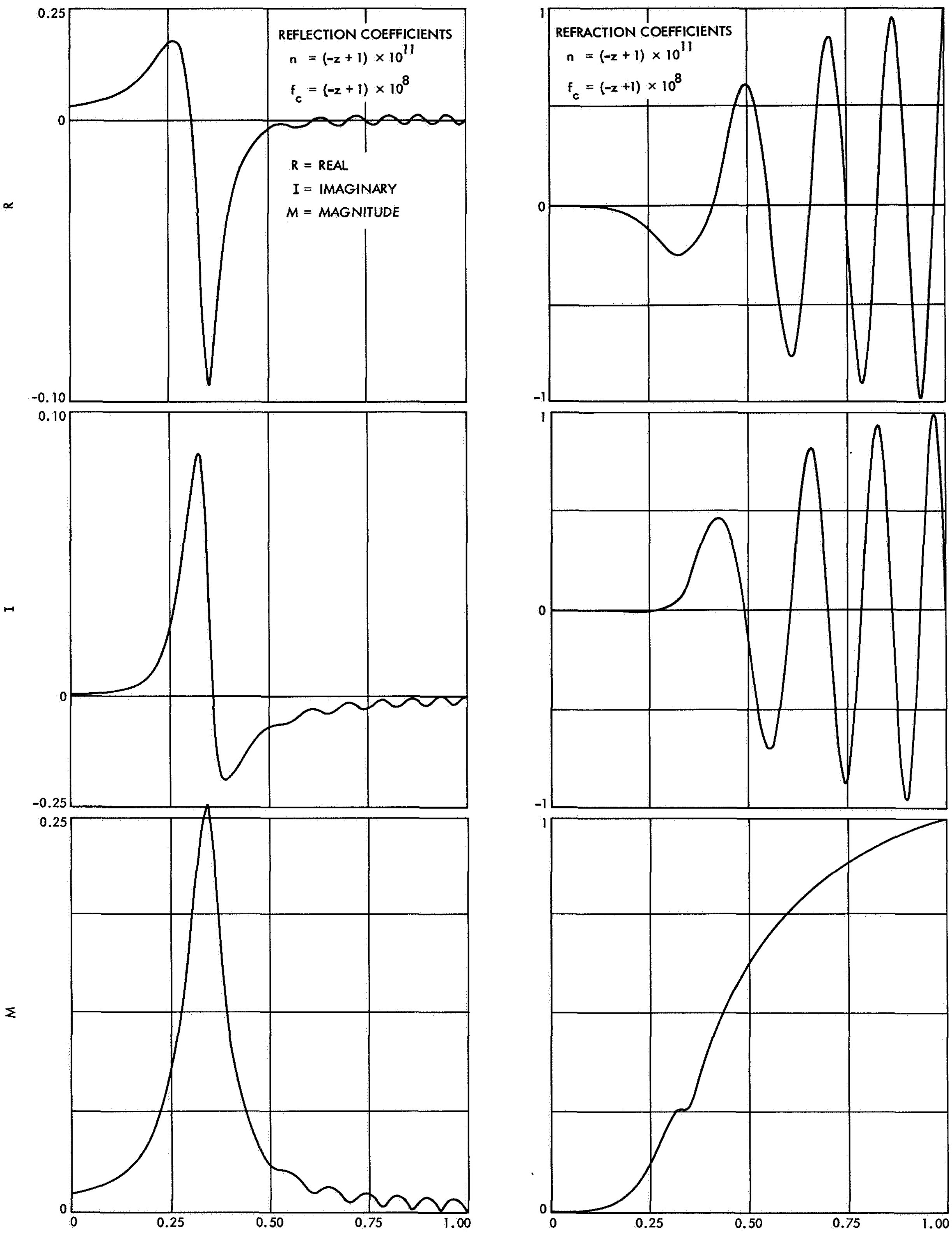


REFLECTION COEFFICIENTS
n = (-z + 1) × 10^11
f_c = (-z + 1) × 10^8
R = REAL
I = IMAGINARY
M = MAGNITUDE
REFRACTION COEFFICIENTS
n = (-z + 1) × 10^11
f_c = (-z +1) × 10^8
R
I
M
0.25
0
-0.10
0.10
0
-0.25
0.25
0
1
0
-1
1
0
-1
1
0
0
0.25
0.50
0.75
1.00
0
0.25
0.50
0.75
1.00
z, m

$n = (-z+1) \times 10^{11}$

$f_c = (-z+1) \times 10^{8}$

SIGNAL FREQUENCY = 2.2950000E 09 HZ

Z-COMPONENT OF SEPARATION

.000	.010	.020	.030	.040	.050	.060	.070	.080	.090
.100	.110	.120	.130	.140	.150	.160	.170	.180	.190
.200	.210	.220	.230	.240	.250	.260	.270	.280	.290
.300	.310	.320	.330	.340	.350	.360	.370	.380	.390
.400	.410	.420	.430	.440	.450	.460	.470	.480	.490
.500	.510	.520	.530	.540	.550	.560	.570	.580	.590
.600	.610	.620	.630	.640	.650	.660	.670	.680	.690
.700	.710	.720	.730	.740	.750	.760	.770	.780	.790
.800	.810	.820	.830	.840	.850	.860	.870	.880	.890
.900	.910	.920	.930	.940	.950	.960	.970	.980	.990
1.000									

PERMITTIVITY OF LEFT EXTERNAL REGION = 8.8540000E-12 FARADS/M
PERMEABILITY OF LEFT EXTERNAL REGION = 1.2566370E-06 HENRYS/M
CONDUCTIVITY OF LEFT EXTERNAL REGION = .0000000E 00 MHOS/M

PERMITTIVITY OF RIGHT EXTERNAL REGION = 8.8540000E-12 FARADS/M
PERMEABILITY OF RIGHT EXTERNAL REGION = 1.2566370E-06 HENRYS/M
CONDUCTIVITY OF RIGHT EXTERNAL REGION = .0000000E 00 MHOS/M

NORMAL INCIDENCE

RE [REFLECTION COEFFICIENT]= .31357184E 00
IM [REFLECTION COEFFICIENT]=-.88621486E 00

MAG[REFLECTION COEFFICIENT]= .94005536E 00
ARG[REFLECTION COEFFICIENT]=-.70514554E 02 [DEGREES]

RE [REFRACTION COEFFICIENT]=-.29560216E-03
IM [REFRACTION COEFFICIENT]=-.77156879E-04

MAG[REFRACTION COEFFICIENT]= .30550584E-03
ARG[REFRACTION COEFFICIENT]=-.16537127E 03 [DEGREES]

ATTENUATION IN NEPERS = 1.6187083E 01
ATTENUATION IN DB = 7.0299610E 01

SIGNAL FREQUENCY = 2.2950000E 09 HZ

ALPHA = .000
BETA = 1.000

PROFILE NUMBER OF STEPS = 101 STEP SIZE = .010
RUNGE/KUTTA NUMBER OF STEPS = 1001 STEP SIZE = .001 DEGREE = 4

PERMITTIVITY OF LEFT EXTERNAL REGION = 8.8540000E-12 FARADS/M
PERMEABILITY OF LEFT EXTERNAL REGION = 1.2566370E-06 HENRYS/M
CONDUCTIVITY OF LEFT EXTERNAL REGION = .0000000E 00 MHOS/M

PERMITTIVITY OF RIGHT EXTERNAL REGION = 8.8540000E-12 FARADS/M
PERMEABILITY OF RIGHT EXTERNAL REGION = 1.2566370E-06 HENRYS/M
CONDUCTIVITY OF RIGHT EXTERNAL REGION = .0000000E 00 MHOS/M

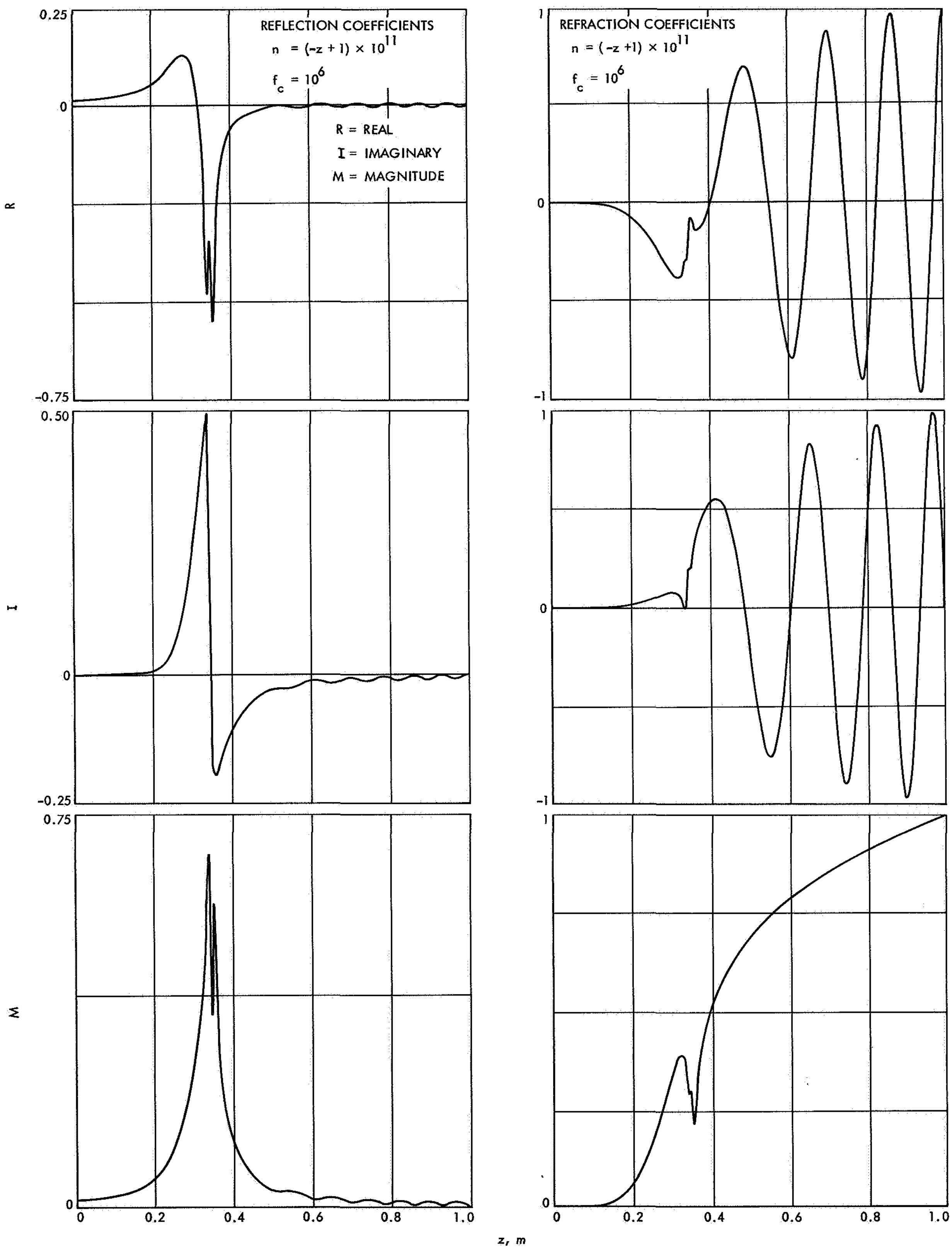
NORMAL INCIDENCE

RE [REFLECTION COEFFICIENT]= .30639859E 00
IM [REFLECTION COEFFICIENT]=-.88888293E 00

MAG[REFLECTION COEFFICIENT]= .94020900E 00
ARG[REFLECTION COEFFICIENT]=-.70980891E 02 [DEGREES]

RE [REFRACTION COEFFICIENT]=-.14226480E-03
IM [REFRACTION COEFFICIENT]= .17478165E-03

MAG[REFRACTION COEFFICIENT]= .22536170E-03
ARG[REFRACTION COEFFICIENT]= .12914412E 03 [DEGREES]



REFLECTION COEFFICIENTS
n = (-z + 1) × 10^11
f_c = 10^6
R = REAL
I = IMAGINARY
M = MAGNITUDE
REFRACTION COEFFICIENTS
n = (-z +1) × 10^11
f_c = 10^6
0.25
0
-0.75
0.50
0
-0.25
0.75
0
R
I
M
1
0
-1
1
0
-1
1
0
0
0.2
0.4
0.6
0.8
1.0
z, m

$n = (-z + 1) \times 10^{11}$

$f_c = 10^6$

SIGNAL FREQUENCY = 2.2950000E 09 HZ

Z.COMPONENT OF SEPARATION

.000	.010	.020	.030	.040	.050	.060	.070	.080	.090
.100	.110	.120	.130	.140	.150	.160	.170	.180	.190
.200	.210	.220	.230	.240	.250	.260	.270	.280	.290
.300	.310	.320	.330	.340	.350	.360	.370	.380	.390
.400	.410	.420	.430	.440	.450	.460	.470	.480	.490
.500	.510	.520	.530	.540	.550	.560	.570	.580	.590
.600	.610	.620	.630	.640	.650	.660	.670	.680	.690
.700	.710	.720	.730	.740	.750	.760	.770	.780	.790
.800	.810	.820	.830	.840	.850	.860	.870	.880	.890
.900	.910	.920	.930	.940	.950	.960	.970	.980	.990
1.000									

PERMITTIVITY OF LEFT EXTERNAL REGION = 8.8540000E-12 FARADS/M
PERMEABILITY OF LEFT EXTERNAL REGION = 1.2566370E-06 HENRYS/M
CONDUCTIVITY OF LEFT EXTERNAL REGION = .0000000E 00 MHOS/M

PERMITTIVITY OF RIGHT EXTERNAL REGION = 8.8540000E-12 FARADS/M
PERMEABILITY OF RIGHT EXTERNAL REGION = 1.2566370E-06 HENRYS/M
CONDUCTIVITY OF RIGHT EXTERNAL REGION = .0000000E 00 MHOS/M

NORMAL INCIDENCE

RE [REFLECTION COEFFICIENT]= .33408635E 00
IM [REFLECTION COEFFICIENT]=-.94187203E 00

MAG[REFLECTION COEFFICIENT]= .99936810E 00
ARG[REFLECTION COEFFICIENT]=-.70470171E 02 [DEGREES]

RE [REFRACTION COEFFICIENT]=-.40182724E-03
IM [REFRACTION COEFFICIENT]= .28698439E-03

MAG[REFRACTION COEFFICIENT]= .49378657E-03
ARG[REFRACTION COEFFICIENT]= .14446563E 03 [DEGREES]

ATTENUATION IN NEPERS = 1.5226814E 01
ATTENUATION IN DB = 6.6129215E 01

SIGNAL FREQUENCY = 2.2950000E 09 HZ

ALPHA = .000
BETA = 1.000

PROFILE NUMBER OF STEPS = 101 STEP SIZE = .010
RUNGE/KUTTA NUMBER OF STEPS = 1001 STEP SIZE = .001 DEGREE = 4

PERMITTIVITY OF LEFT EXTERNAL REGION = 8.8540000E-12 FARADS/M
PERMEABILITY OF LEFT EXTERNAL REGION = 1.2566370E-06 HENRYS/M
CONDUCTIVITY OF LEFT EXTERNAL REGION = .0000000E 00 MHOS/M

PERMITTIVITY OF RIGHT EXTERNAL REGION = 8.8540000E-12 FARADS/M
PERMEABILITY OF RIGHT EXTERNAL REGION = 1.2566370E-06 HENRYS/M
CONDUCTIVITY OF RIGHT EXTERNAL REGION = .0000000E 00 MHOS/M

NORMAL INCIDENCE

RE [REFLECTION COEFFICIENT]= .32638651E 00
IM [REFLECTION COEFFICIENT]=-.94457354E 00

MAG[REFLECTION COEFFICIENT]= .99937347E 00
ARG[REFLECTION COEFFICIENT]=-.70938000E 02 [DEGREES]

RE [REFRACTION COEFFICIENT]= .86000469E-04
IM [REFRACTION COEFFICIENT]= .34373794E-03

MAG[REFRACTION COEFFICIENT]= .35433297E-03
ARG[REFRACTION COEFFICIENT]= .75953407E 02 [DEGREES]

C. Program Listing

1. *Main Program 1*

```
C     PRØPAGATIØN IN A PLANAR INHØMØGENIØUS PLASMA MEDIUM
C     MAIN PRØGRAM 1 SDS 930
     0DIMENSIØN Z(102),ED(100),FC(100),A(102),B(102),CSR(102),CSI(102),
     1           XØ(102),YØ(102),XM(102),YM(102)
    1 FØRMAT (E10.0,F10.0,I10)
    2 FØRMAT (8F10.0)
    3 FØRMAT (8E10.0)
    4 FØRMAT (4E10.0)
    5 FØRMAT (E10.0,3F10.0,I3,F7.0)
   10 FØRMAT (1H1)
   11 FØRMAT (19H0SIGNAL FREQUENCY =,1PE14.7,3H HZ)
   12 FØRMAT (26H0Z-CØMPØNENT ØF SEPARATIØN)
   13 FØRMAT (10F10.3)
  140FØRMAT (39H0PERMITTIVITY ØF LEFT EXTERNAL REGIØN =,
     1          1PE14.7,9H FARADS/M)
  150FØRMAT (39H PERMEABILITY ØF LEFT EXTERNAL REGIØN =,
     1          1PE14.7,9H HENRYS/M)
  160FØRMAT (39H CØNDUCTIVITY ØF LEFT EXTERNAL REGIØN =,
     1          1PE14.7,7H MHØS/M)
  170FØRMAT (40H0PERMITTIVITY ØF RIGHT EXTERNAL REGIØN =,
     1          1PE14.7,9H FARADS/M)
  180FØRMAT (40H PERMEABILITY ØF RIGHT EXTERNAL REGIØN =,
     1          1PE14.7,9H HENRYS/M)
  190FØRMAT (40H CØNDUCTIVITY ØF RIGHT EXTERNAL REGIØN =,
     1    .     1PE14.7,7H MHØS/M)
   22 FØRMAT (13H1INTERFACE Z=,F7.3,5X,16HNØRMAL INCIDENCE)
   23 FØRMAT (13H1INTERFACE Z=,F7.3,5X,26HPERPENDICULAR PØLARIZATIØN)
   24 FØRMAT (13H1INTERFACE Z=,F7.3,5X,21HPARALLEL PØLARIZATIØN)
   25 FØRMAT (17H1NØRMAL INCIDENCE)
   26 FØRMAT (49H1PØLARIZATIØN PERPENDICULAR TØ PLANE ØF INCIDENCE)
   27 FØRMAT (44H1PØLARIZATIØN PARALLEL TØ PLANE ØF INCIDENCE)
   28 FØRMAT (29H0RE (REFLECTIØN CØEFFICIENT)=,E14.8)
   29 FØRMAT (29H IM (REFLECTIØN CØEFFICIENT)=,E14.8)
   30 FØRMAT (29H0MAG(REFLECTIØN CØEFFICIENT)=,E14.8)
   31 FØRMAT (29H ARG(REFLECTIØN CØEFFICIENT)=,E14.8,5X,9H(DEGREES))
   32 FØRMAT (29H0RE (REFRACTIØN CØEFFICIENT)=,E14.8)
   33 FØRMAT (29H IM (REFRACTIØN CØEFFICIENT)=,E14.8)
   34 FØRMAT (29H0MAG(REFRACTIØN CØEFFICIENT)=,E14.8)
   35 FØRMAT (29H ARG(REFRACTIØN CØEFFICIENT)=,E14.8,5X,9H(DEGREES))
  360FØRMAT (34H0REFRACTIØN CØEFFICIENT BETWEEN Z=,F7.3,
     1          19H AND LEFT INTERFACE)
   37 FØRMAT (24H0ATTENUATIØN IN NEPERS =,1PE14.7)
   38 FØRMAT (24H ATTENUATIØN IN DB     =,1PE14.7)
  410FØRMAT (36H SET SENSE SWITCH 1 TØ -ØN- TØ PRINT,
     1          37H ELECTRØN DENSITY/CØLLISIØN FREQUENCY)
  420FØRMAT (36H SET SENSE SWITCH 2 TØ -ØN- TØ PRINT,
     1          34H PLASMA FREQUENCY/CUTØFF FREQUENCY)
  430FØRMAT (36H SET SENSE SWITCH 3 TØ -ØN- TØ PRINT,
     1          39H PERMITTIVITY/PERMEABILITY/CØNDUCTIVITY)
  440FØRMAT (36H SET SENSE SWITCH 1 TØ -ØN- TØ PRINT,
     1          35H REFLECTIØN/REFRACTIØN CØEFFICIENTS)
   45 FØRMAT (6H START)
   46 FØRMAT (36H SENSE SWITCH 1 ØN  NUMERICAL  INPUT)
   47 FØRMAT (36H SENSE SWITCH 1 ØFF ANALYTICAL INPUT)
  510FØRMAT (36H SET SENSE SWITCH 2 TØ -ØN- TØ WRITE,
     1          14H ØUTPUT TAPE 3)
   52 FØRMAT (9(1X,E13.7))
  300 CØNTINUE
      TYPE 46
      TYPE 47
      TYPE 45
      PAUSE
      IF (SENSE SWITCH 1) 201,202
  201 READ 1,F,AØID,N
      NP1=N+1
      NP2=N+2
      READ 2,( Z(I),I=1,NP1)
      READ 3,(ED(I),I=1,N)
      READ 3,(FC(I),I=1,N)
      READ 4,EL,ER,ØL,ØR
      GØ TØ 203
  202 READ 5,F,AØID,ALPHA,BETA,N,SN
      READ 4,EL,ER,ØL,ØR
      NP1=N+1
```

```
      NP2=N+2
      Z(1)=ALPHA
      DO 211 J=1,N
      K=J+1
      Z(K)=Z(J)+SN
      CALL EDFCP(Z(K),ED(J),FC(J))
211   CONTINUE
203   Z(NP2)=Z(NP1)
      PI=3.1415926
      W=F*2.*PI
      U0=PI*4.E-07
      UL=U0
      UR=U0
      PRINT 10
      PRINT 11,F
      PRINT 12
      PRINT 13,(Z(I),I=1,NP1)
      PRINT 14,EL
      PRINT 15,UL
      PRINT 16,0L
      PRINT 17,ER
      PRINT 18,UR
      PRINT 19,0R
      AX=0.
      BX=0.
      CALL PC(EL,UL,0L,F,AX,BX,A(1),B(1),AY,BY,AD)
      CALL II(F,UL,A(1),B(1),A(1),B(1),X0(1),Y0(1))
      TYPE 41
      TYPE 42
      TYPE 43
      TYPE 45
      PAUSE
      DO 101 J=1,N
      K=J+1
      CALL PPC(ED(J),FC(J),F,E,U,0,J)
      CALL PC(E,U,0,F,AX,BX,A(K),B(K),AY,BY,AD)
      CALL II(F,U,A(K),B(K),A(K),B(K),X0(K),Y0(K))
101   CONTINUE
      CALL PC(ER,UR,0R,F,AX,BX,A(NP2),B(NP2),AY,BY,AD)
      CALL II(F,UR,A(NP2),B(NP2),A(NP2),B(NP2),X0(NP2),Y0(NP2))
      IF (A0ID) 110,111,110
111   M=1
      DO 121 I=1,NP2
      XM(I)=X0(I)
      YM(I)=Y0(I)
121   CONTINUE
      GO TO 114
110   CALL ARR(A0ID,A,B,CSR,CSI,A0TD,N)
112   M=2
      DO 122 I=1,NP2
      CM2=CSR(I)*CSR(I)+CSI(I)*CSI(I)
      XM(I)=(X0(I)*CSR(I)+Y0(I)*CSI(I))/CM2
      YM(I)=(Y0(I)*CSR(I)-X0(I)*CSI(I))/CM2
122   CONTINUE
      GO TO 114
113   M=3
      DO 123 I=1,NP2
      XM(I)=X0(I)*CSR(I)-Y0(I)*CSI(I)
      YM(I)=X0(I)*CSI(I)+Y0(I)*CSR(I)
123   CONTINUE
114   PTR=1.
      PTI=0.
      PXS=1.
      XN=XM(NP2)
      YN=YM(NP2)
      TYPE 44
      TYPE 51
      TYPE 45
      PAUSE
      IF (SENSE SWITCH 2) 301,303
301   GO TO (302,302,303),M
302   REWIND 3
303   CONTINUE
      DO 131 J=1,NP1
```

```
      K=NP2-J+1
      KM1=K-1
      D=Z(KM1)-Z(K)
      IF (AØID) 141,140,141
140   FR=B(K)*D
      FI=A(K)*D
      GØ TØ 142
141   FR=(B(K)*CSR(K)-A(K)*CSI(K))*D
      FI=(B(K)*CSI(K)+A(K)*CSR(K))*D
142   SN=SINF(FR)
      CS=CØSF(FR)
      CALL SCH(FI,SNH,CSH)
      SR=+SN*CSH
      SI=+CS*SNH
      CR=+CS*CSH
      CI=-SN*SNH
      ZNCR=XN*CR-YN*CI
      ZNCI=XN*CI+YN*CR
      ZMSR=XM(K)*SR-YM(K)*SI
      ZMSI=XM(K)*SI+YM(K)*SR
      ZMCR=XM(K)*CR-YM(K)*CI
      ZMCI=XM(K)*CI+YM(K)*CR
      ZNSR=XN*SR-YN*SI
      ZNSI=XN*SI+YN*SR
      SNR=ZNCR-ZMSI
      SNI=ZNCI+ZMSR
      SDR=ZMCR-ZNSI
      SDI=ZMCI+ZNSR
      SNDR=SNR*SDR+SNI*SDI
      SNDI=SNI*SDR-SNR*SDI
      SNDM2=SDR*SDR+SDI*SDI
      XN=(XM(K)*SNDR-YM(K)*SNDI)/SNDM2
      YN=(XM(K)*SNDI+YM(K)*SNDR)/SNDM2
      RSNR=XN-XM(KM1)
      RSNI=YN-YM(KM1)
      TSNR=XN+XM(K)
      TSNI=YN+YM(K)
      SDR=XN+XM(KM1)
      SDI=YN+YM(KM1)
      SDM2=SDR*SDR+SDI*SDI
      RR=(RSNR*SDR+RSNI*SDI)/SDM2
      RI=(RSNI*SDR-RSNR*SDI)/SDM2
      CALL RTP(RR,RI,RM,RAD)
      TR=(TSNR*SDR+TSNI*SDI)/SDM2
      TI=(TSNI*SDR-TSNR*SDI)/SDM2
      CALL RTP(TR,TI,TM,TAD)
      XSR=CR+SI
      XSI=CI-SR
      XTR=TR*XSR-TI*XSI
      XTI=TR*XSI+TI*XSR
      XPTR=PTR*XTR-PTI*XTI
      XPTI=PTR*XTI+PTI*XTR
      PTR=XPTR
      PTI=XPTI
      CALL RTP(PTR,PTI,PTM,PTAD)
      XYNM2=XØ(K)*XØ(K)+YØ(K)*YØ(K)
      XYDM2=XØ(KM1)*XØ(KM1)+YØ(KM1)*YØ(KM1)
      TM2=TM*TM
      AF=SQRTF(XYNM2/XYDM2)/TM2
      XAF=EXPF(-2.*FI)*AF
      PXS=PXS*XAF
      IF (SENSE SWITCH 1) 151,352
151   GØ TØ (161,162,163),M
161   PRINT 22,Z(KM1)
      GØ TØ 164
162   PRINT 23,Z(KM1)
      GØ TØ 164
163   PRINT 24,Z(KM1)
164   PRINT 28,RR
      PRINT 29,RI
      PRINT 30,RM
      PRINT 31,RAD
      PRINT 32,TR
      PRINT 33,TI
```

```
      PRINT 34,TM
      PRINT 35,TAD
      PRINT 36,Z(KM1)
      PRINT 32,PTR
      PRINT 33,PTI
      PRINT 34,PTM
      PRINT 35,PTAD
352   IF (SENSE SWITCH 2) 311,131
311   WRITE OUTPUT TAPE 3,52,Z(KM1),RR,RI,RM,RAD,PTR,PTI,PTM,PTAD
131   CONTINUE
      CRR=RR
      CRI=RI
      CALL RTP(CRR,CRI,CRM,CRAD)
      CTR=PTR
      CTI=PTI
      GO TO (171,171,172),M
172   CTR=CTR*CSR(1)/CSR(NP2)
      CTI=CTI*CSR(1)/CSR(NP2)
171   CALL RTP(CTR,CTI,CTM,CTAD)
      AN=ELOGF(PXS)
      ADB=AN*10./ELOGF(10.)
      GO TO (181,182,183),M
181   PRINT 25
      GO TO 184
182   PRINT 26
      GO TO 184
183   PRINT 27
184   PRINT 28,CRR
      PRINT 29,CRI
      PRINT 30,CRM
      PRINT 31,CRAD
      PRINT 32,CTR
      PRINT 33,CTI
      PRINT 34,CTM
      PRINT 35,CTAD
      PRINT 37,AN
      PRINT 38,ADB
      IF (SENSE SWITCH 2) 321,322
321   ENDFILE 3
322   GO TO (191,113,191),M
191   GO TO 300
      S=ABSF(0.)
      S=ATANF(0.)
      END
```

2. *Main Program 2*

```
C     PROPAGATION IN A PLANAR INHOMOGENIOUS PLASMA MEDIUM
C     MAIN PROGRAM 2 SDS 930
      DIMENSION A(101),B(101),AY(101),BY(101),ARG(101),XO(101),YO(101)
    1 FORMAT (E10.0,3F10.0,I3,F7.0,I4,F6.0,I2)
    2 FORMAT (4E10.0)
   10 FORMAT (1H1)
   11 FORMAT (19HOSIGNAL FREQUENCY =,1PE14.7,3H HZ)
   12 FORMAT (8HOALPHA =,F10.3)
   13 FORMAT (8H BETA   =,F10.3)
  140FORMAT (12HOPROFILE    ,5X,17HNUMBER OF STEPS =,I5,5X,
    1        11HSTEP SIZE =,F7.3)
  150FORMAT (12H RUNGE/KUTTA,5X,17HNUMBER OF STEPS =,I5,5X,
    1        11HSTEP SIZE =,F7.3,5X,8HDEGREE =,I3)
  160FORMAT (39HOPERMITTIVITY OF LEFT EXTERNAL REGION =,
    1        1PE14.7,9H FARADS/M)
  170FORMAT (39H PERMEABILITY OF LEFT EXTERNAL REGION =,
    1        1PE14.7,9H HENRYS/M)
  180FORMAT (39H CONDUCTIVITY OF LEFT EXTERNAL REGION =,
    1        1PE14.7,7H MHOS/M)
  190FORMAT (40HOPERMITTIVITY OF RIGHT EXTERNAL REGION =,
    1        1PE14.7,9H FARADS/M)
  200FORMAT (40H PERMEABILITY OF RIGHT EXTERNAL REGION =,
    1        1PE14.7,9H HENRYS/M)
  210FORMAT (40H CONDUCTIVITY OF RIGHT EXTERNAL REGION =,
    1        1PE14.7,7H MHOS/M)
   22 FORMAT (8H1INDEX (,I4,1H),5X,2HZ=,F7.3,5X,16HNORMAL INCIDENCE)
  230FORMAT (8H1INDEX (,I4,1H),5X,2HZ=,F7.3,5X,
    1        26HPERPENDICULAR POLARIZATION)
  240FORMAT (8H1INDEX (,I4,1H),5X,2HZ=,F7.3,5X,
    1        21HPARALLEL POLARIZATION)
   25 FORMAT (17H1NORMAL INCIDENCE)
   26 FORMAT (49H1POLARIZATION PERPENDICULAR TO PLANE OF INCIDENCE)
   27 FORMAT (44H1POLARIZATION PARALLEL TO PLANE OF INCIDENCE)
   28 FORMAT (29HORE (REFLECTION COEFFICIENT)=,E14.8)
   29 FORMAT (29H IM (REFLECTION COEFFICIENT)=,E14.8)
   30 FORMAT (29HOMAG(REFLECTION COEFFICIENT)=,E14.8)
   31 FORMAT (29H ARG(REFLECTION COEFFICIENT)=,E14.8,5X,9H(DEGREES))
   32 FORMAT (29HORE (REFRACTION COEFFICIENT)=,E14.8)
   33 FORMAT (29H IM (REFRACTION COEFFICIENT)=,E14.8)
   34 FORMAT (29HOMAG(REFRACTION COEFFICIENT)=,E14.8)
   35 FORMAT (29H ARG(REFRACTION COEFFICIENT)=,E14.8,5X,9H(DEGREES))
   36 FORMAT (33H1ANGLE OF INCIDENCE  IN DEGREES =,F10.3)
   37 FORMAT (33HOANGLE OF REFRACTION IN DEGREES =,F10.3)
  410FORMAT (36H SET SENSE SWITCH 1 TO -ON- TO PRINT,
    1        37H ELECTRON DENSITY/COLLISION FREQUENCY)
  420FORMAT (36H SET SENSE SWITCH 2 TO -ON- TO PRINT,
    1        34H PLASMA FREQUENCY/CUTOFF FREQUENCY)
  430FORMAT (36H SET SENSE SWITCH 3 TO -ON- TO PRINT,
    1        39H PERMITTIVITY/PERMEABILITY/CONDUCTIVITY)
  440FORMAT (36H SET SENSE SWITCH 1 TO -ON- TO PRINT,
    1        35H REFLECTION/REFRACTION COEFFICIENTS)
   45 FORMAT (6H START)
  510FORMAT (36H SET SENSE SWITCH 2 TO -ON- TO WRITE,
    1        14H OUTPUT TAPE 3)
   52 FORMAT (9(1X,E13.7))
  200 CONTINUE
      READ 1,F,AOID,ALPHA,BETA,M,SM,N,SN,NP
      READ 2,EL,ER,OL,OR
      PI=3.1415926
      W=F*2.*PI
      AOIR=AOID*PI/180.
      UO=PI*4.E-07
      UL=UO
      UR=UO
      PRINT 10
      PRINT 11,F
      PRINT 12,ALPHA
      PRINT 13,BETA
      PRINT 14,M,SM
      PRINT 15,N,SN,NP
      PRINT 16,EL
      PRINT 17,UL
      PRINT 18,OL
      PRINT 19,ER
```

```
      PRINT 20,UR
      PRINT 21,ØR
      BXL=0.
      AXL=0.
      CALL PC(EL,UL,ØL,F,AXL,BXL,AL,BL,AYL,BYL,AD)
      CALL II(F,UL,AL,BL,AL,BL,XØL,YØL)
      BXL=BL*SINF(AØIR)
      AXL=AL*SINF(AØIR)
      BYL=BL*CØSF(AØIR)
      AYL=AL*CØSF(AØIR)
      TYPE 41
      TYPE 42
      TYPE 43
      TYPE 45
      PAUSE
      AM=ALPHA
      DØ 101 I=1,M
      CALL EDFCP(AM,EDP,FCP)
      CALL PPC(EDP,FCP,F,E,U,Ø,I)
      CALL PC(E,U,Ø,F,AXL,BXL,A(I),B(I),AY(I),BY(I),AD)
      CALL II(F,U,A(I),B(I),A(I),B(I),XØ(I),YØ(I))
      ARG(I)=AM
      AM=AM+SM
  101 CØNTINUE
      CALL PC(ER,UR,ØR,F,AXL,BXL,AR,BR,AYR,BYR,AØTD)
      CALL II(F,UR,AR,BR,AR,BR,XØR,YØR)
      L=1
      IF (AØID) 111,110,111
  111 L=2
      PRINT 36,AØID
      PRINT 37,AØTD
      AØTR=AØTD*PI/180.
      CALL II(F,UL,AYL,BYL,AYL,BYL,XØL,YØL)
      DØ 121 I=1,M
      CALL II(F,U,AY(I),BY(I),AY(I),BY(I),XØ(I),YØ(I))
  121 CØNTINUE
      CALL II(F,UR,AYR,BYR,AYR,BYR,XØR,YØR)
      GØ TØ 110
  112 L=3
      CALL II(F,UL,AL,BL,AYL,BYL,XØL,YØL)
      DØ 122 I=1,M
      CALL II(F,U,A(I),B(I),AY(I),BY(I),XØ(I),YØ(I))
  122 CØNTINUE
      CALL II(F,UR,AR,BR,AYR,BYR,XØR,YØR)
  110 SNR=XØR-XØ(M)
      SNI=YØR-YØ(M)
      SDR=XØR+XØ(M)
      SDI=YØR+YØ(M)
      SDM2=SDR*SDR+SDI*SDI
      RR=(SNR*SDR+SNI*SDI)/SDM2
      RI=(SNI*SDR-SNR*SDI)/SDM2
      CALL RTP(RR,RI,RM,RAD)
      TR=2.*(XØR*SDR+YØR*SDI)/SDM2
      TI=2.*(YØR*SDR-XØR*SDI)/SDM2
      CALL RTP(TR,TI,TM,TAD)
      TYPE 44
      TYPE 51
      TYPE 45
      PAUSE
      IF (SENSE SWITCH 2) 201,203
  201 GØ TØ (202,202,203),L
  202 REWIND 3
  203 CØNTINUE
      AN=BETA
      IF (SENSE SWITCH 1) 131,232
  131 GØ TØ (141,142,143),L
  141 PRINT 22,N,AN
      GØ TØ 144
  142 PRINT 23,N,AN
      GØ TØ 144
  143 PRINT 24,N,AN
  144 PRINT 28,RR
      PRINT 29,RI
      PRINT 30,RM
```

```
      PRINT 31,RAD
      PRINT 32,TR
      PRINT 33,TI
      PRINT 34,TM
      PRINT 35,TAD
232   IF (SENSE SWITCH 2) 211,212
211   WRITE OUTPUT TAPE 3,52,AN,RR,RI,RM,RAD,TR,TI,TM,TAD
212   CONTINUE
132   DO 151 J=2,N
      NP2=N+2
      K=NP2-J
      KM1=K-1
      CALL RK(ARG,X0,Y0,AY,BY,M,NP,-SN,AN,RR,RI,TR,TI)
      CALL RTP(RR,RI,RM,RAD)
      CALL RTP(TR,TI,TM,TAD)
      AN=AN-SN
      IF (SENSE SWITCH 1) 161,251
161   GO TO (171,172,173),L
171   PRINT 22,KM1,AN
      GO TO 174
172   PRINT 23,KM1,AN
      GO TO 174
173   PRINT 24,KM1,AN
174   PRINT 28,RR
      PRINT 29,RI
      PRINT 30,RM
      PRINT 31,RAD
      PRINT 32,TR
      PRINT 33,TI
      PRINT 34,TM
      PRINT 35,TAD
251   IF (SENSE SWITCH 2) 221,151
221   WRITE OUTPUT TAPE 3,52,AN,RR,RI,RM,RAD,TR,TI,TM,TAD
151   CONTINUE
      RM2=RR*RR+RI*RI
      SR=1.-RM2
      SI=2.*RI
      SM2=1.-2.*RR+RM2
      CR=(X0(1)*SR-Y0(1)*SI)/SM2
      CI=(X0(1)*SI+Y0(1)*SR)/SM2
      SNR=CR-X0L
      SNI=CI-Y0L
      SDR=CR+X0L
      SDI=CI+Y0L
      SDM2=SDR*SDR+SDI*SDI
      CRR=(SNR*SDR+SNI*SDI)/SDM2
      CRI=(SNI*SDR-SNR*SDI)/SDM2
      CALL RTP(CRR,CRI,CRM,CRAD)
      TM2=TR*TR+TI*TI
      SR=2.*TR-TM2
      SI=2.*TI
      SM2=4.*(1.-TR)+TM2
      CR=(X0(1)*SR-Y0(1)*SI)/SM2
      CI=(X0(1)*SI+Y0(1)*SR)/SM2
      SNR=2.*CR
      SNI=2.*CI
      SDR=CR+X0L
      SDI=CI+Y0L
      SDM2=SDR*SDR+SDI*SDI
      CTR=(SNR*SDR+SNI*SDI)/SDM2
      CTI=(SNI*SDR-SNR*SDI)/SDM2
      GO TO (181,181,182),L
182   CTR=CTR*COSF(A0IR)/COSF(A0TR)
      CTI=CTI*COSF(A0IR)/COSF(A0TR)
181   CALL RTP(CTR,CTI,CTM,CTAD)
      GO TO (191,192,193),L
191   PRINT 25
      GO TO 194
192   PRINT 26
      GO TO 194
193   PRINT 27
194   PRINT 28,CRR
      PRINT 29,CRI
      PRINT 30,CRM
      PRINT 31,CRAD
```

```
      PRINT 32,CTR
      PRINT 33,CTI
      PRINT 34,CTM
      PRINT 35,CTAD
      IF (SENSE SWITCH 2) 241,242
  241 ENDFILE 3
  242 GO TO (199,112,199),L
  199 GO TO 200
      S=ABSF(0.)
      S=ATANF(0.)
      S=EXPF(0.)
      S=ELOGF(0.)
      S=SQRTF(0.)
      END
```

3. Subroutines

```
      SUBROUTINE EDFCP(ARG,EDP,FCP)
C     ELECTRON DENSITY/COLLISION FREQUENCY PROFILE
C     INSERT THE ELECTRON DENSITY PROFILE EDP
C     AS A FUNCTION OF THE ARGUEMENT ARG
      EDP=    ...EDP(ARG)...
C     INSERT THE COLLISION FREQUENCY PROFILE FCP
C     AS A FUNCTION OF THE ARGUEMENT ARG
      FCP=    ...FCP(ARG)...
      RETURN
      END
```

```
      SUBROUTINE PPC(ED,FC,F,E,U,O,I)
C     INPUT   ELECTRON DENSITY/COLLISION FREQUENCY/SIGNAL FREQUENCY
C     OUTPUT  PERMITTIVITY/PERMEABILITY/CONDUCTIVITY
    1 FORMAT (22H1ELECTRON DENSITY      (,I3,2H)=,1PE14.7,6H EL/CC)
    2 FORMAT (22H COLLISION FREQUENCY   (,I3,2H)=,1PE14.7,3H HZ)
    3 FORMAT (22H0PLASMA FREQUENCY      (,I3,2H)=,1PE14.7,3H HZ)
    4 FORMAT (22H0NO CUTOFF FREQUENCY   (,I3,8H) EXISTS)
    5 FORMAT (22H CUTOFF FREQUENCY      (,I3,2H)=,1PE14.7,3H HZ)
    6 FORMAT (22H0PERMITTIVITY          (,I3,2H)=,1PE14.7,9H FARADS/M)
    7 FORMAT (22H PERMEABILITY          (,I3,2H)=,1PE14.7,9H HENRYS/M)
    8 FORMAT (22H CONDUCTIVITY          (,I3,2H)=,1PE14.7,7H MHOS/M)
      PI=3.1415926
      EQ=1.602E-19
      EM=9.108E-31
      E0=8.854E-12
      U0=PI*4.E-07
      W=F*2.*PI
      IF (SENSE SWITCH 1) 101,102
  101 PRINT 1,I,ED
      PRINT 2,I,FC
  102 EDX=ED*1.E+06
      WP2=EDX*EQ/EM*EQ/E0
      FP=SQRTF(WP2)/(2.*PI)
      IF (SENSE SWITCH 2) 111,112
  111 PRINT 3,I,FP
  112 WC=FC*2.*PI
      WEF2=W*W+WC*WC
      WC02=WP2-WC*WC
      IF (WC02) 121,122,122
  121 IF (SENSE SWITCH 2) 131,123
  131 PRINT 4,I
      GO TO 123
  122 FC0=SQRTF(WC02)/(2.*PI)
      IF (SENSE SWITCH 2) 141,123
  141 PRINT 5,I,FC0
  123 E=E0*(1.-WP2/WEF2)
      U=U0
      O=E0*WC*WP2/WEF2
      IF (SENSE SWITCH 3) 151,152
  151 PRINT 6,I,E
      PRINT 7,I,U
      PRINT 8,I,O
  152 RETURN
      END
```

```
      SUBROUTINE PC(E,U,O,F,AX,BX,A,B,AY,BY,AD)
C     INPUT   PERMITTIVITY/PERMEABILITY/CONDUCTIVITY/SIGNAL FREQUENCY
C     OUTPUT  PROPAGATION CONSTANT
      PI=3.1415926
      W=F*2.*PI
      XLF=1.+(O/W*O/W)/(E*E)
      ABSP=+E+ABSF(E)*SQRTF(XLF)
      ABSM=-E+ABSF(E)*SQRTF(XLF)
      B=W*SQRTF(U/2.)*SQRTF(ABSP)
      A=W*SQRTF(U/2.)*SQRTF(ABSM)
      GM2=B*B+A*A
      XPR=B*B-A*A-BX*BX+AX*AX
      XPI=2.*(B*A-BX*AX)
      XQN=XPI*XPI
      XQD=XPR*XPR
      XLF=1.+XQN/XQD
      ABSP=+XPR+ABSF(XPR)*SQRTF(XLF)
      ABSM=-XPR+ABSF(XPR)*SQRTF(XLF)
      BY=SQRTF(ABSP/2.)
      AY=SQRTF(ABSM/2.)
      GYM2=BY*BY+AY*AY
      CS2=GYM2/GM2
      SN2=1.-CS2
      TN2=SN2/CS2
      TN=SQRTF(TN2)
      AR=ATANF(TN)
      AD=AR*180./PI
      RETURN
      END

      SUBROUTINE II(F,U,A,B,AY,BY,XO,YO)
C     INPUT  SIGNAL FREQUENCY/PERMEABILITY/PROPAGATION CONSTANT
C     OUTPUT INTRINSIC IMPEDANCE
      PI=3.1415926
      W=F*2.*PI
      DR=B*B-A*A
      DI=2.*B*A
      DM2=DR*DR+DI*DI
      GYDR=BY*DR+AY*DI
      GYDI=AY*DR-BY*DI
      XO=W*U*GYDR/DM2
      YO=W*U*GYDI/DM2
      RETURN
      END

      SUBROUTINE RTP(XRE,XIM,XMAG,ARGD)
C     RECTANGULAR TO POLAR
      PI=3.1415926
      XMAG2=XRE*XRE+XIM*XIM
      XMAG=SQRTF(XMAG2)
      XTAN=XIM/XRE
      XARGR=ATANF(XTAN)
      XARGD=XARGR*180./PI
      IF (XRE) 101,102,102
  101 ARGDA=ABSF(XARGD)
      IF (XIM) 103,104,104
  103 ARGD=-180.+ARGDA
      GO TO 105
  104 ARGD=+180.-ARGDA
      GO TO 105
  102 ARGD=XARGD
  105 RETURN
      END
```

```
      SUBROUTINE ARR(AOID,A,B,CSR,CSI,AOTD,N)
C     INPUT  ANGLE OF INCIDENCE/PROPAGATION CONSTANT
C     OUTPUT ANGLE OF REFLECTION/REFRACTION
      DIMENSION A(51),B(51),CSR(51),CSI(51)
    1 FORMAT (33H1ANGLE OF INCIDENCE  IN DEGREES =,F10.3)
    2 FORMAT (33H0ANGLE OF REFRACTION IN DEGREES =,F10.3)
      PRINT 1,AOID
      NP1=N+1
      NP2=N+2
      PI=3.1415926
      AOIR=AOID*PI/180.
      SNR=SINF(AOIR)
      SNI=0.
      CSR(1)=COSF(AOIR)
      CSI(1)=0.
      DO 101 I=1,NP1
      IP1=I+1
      GM2=B(IP1)*B(IP1)+A(IP1)*A(IP1)
      SLR=(B(I)*B(IP1)+A(I)*A(IP1))/GM2
      SLI=(A(I)*B(IP1)-B(I)*A(IP1))/GM2
      XSNR=SLR*SNR-SLI*SNI
      XSNI=SLR*SNI+SLI*SNR
      SNR=XSNR
      SNI=XSNI
      XPR=1.-SNR*SNR+SNI*SNI
      XPI=-2.*SNR*SNI
      XQN=XPI*XPI
      XQD=XPR*XPR
      XLF=1.+XQN/XQD
      ABSP=+XPR+ABSF(XPR)*SQRTF(XLF)
      ABSM=-XPR+ABSF(XPR)*SQRTF(XLF)
      CSR(IP1)=SQRTF(ABSP/2.)
      CSI(IP1)=SQRTF(ABSM/2.)
  101 CONTINUE
      XTAN=SNR/CSR(NP2)
      AOTR=ATANF(XTAN)
      AOTD=AOTR*180./PI
      PRINT 2,AOTD
      RETURN
      END
```

```
      SUBROUTINE SCH(ARG,SNH,CSH)
C     HYPERBOLIC SINE/COSINE
      XI=1.
      FCTRL=1.
      SS=0.
      CS=0.
      IF (ARG) 100,102,100
  100 ARGA=ABSF(ARG)
  103 ST=(ARGA**XI)/FCTRL
      XI=XI+1.
      FCTRL=FCTRL*XI
      CT=(ARGA**XI)/FCTRL
      XI=XI+1.
      FCTRL=FCTRL*XI
      SS=SS+ST
      CS=CS+CT
      STSS=ST/SS-1.0E-08
      CTCS=CT/CS-1.0E-08
      IF (STSS) 101,101,103
  101 IF (CTCS) 102,102,103
  102 SNHA=SS
      CSHA=1.+CS
      IF (ARG) 111,112,112
  111 SNH=-SNHA
      CSH=+CSHA
      GO TO 113
  112 SNH=SNHA
      CSH=CSHA
  113 RETURN
      END
```

```
      SUBRØUTINE TLU(TX,TY,NT,NP,X,Y,YD)
C     TABLE LØØK UP  (AITKENS METHØD)
C     INTERPØLATIØN/DIFFERENTIATIØN
      DIMENSIØN TX(101),TY(101),P(10),Q(10),QD(10)
      NLØW=1
      NHIGH=NT
      NPH=NP/2
  112 N=(NLØW+NHIGH)/2
      IF (X-TX(N)) 101,100,102
  103 NTD=NHIGH-NLØW
      IF (NTD-1) 111,111,112
  101 NHIGH=N
      GØ TØ 103
  102 NLØW=N
      GØ TØ 103
  100 Y=TY(N)
      NLØW=N
  111 NL=NLØW-NPH+1
      IF (NL-1) 121,122,122
  121 NL=1
      GØ TØ 124
  122 NU=NL+NP-1
      IF (NT-NU) 123,124,124
  123 NL=NT-NP+1
  124 DØ 131 J=1,NP
      K=NL+J-1
      P(J)=TX(K)
      Q(J)=TY(K)
      QD(J)=0.
  131 CØNTINUE
      I=NP-1
      DØ 141 J=1,I
      L=J+1
      DØ 141 K=L,NP
      PKMJ=P(K)-P(J)
      QKMJ=Q(K)-Q(J)
      XMPJ=X-P(J)
      XMPK=X-P(K)
      Q(K)=(XMPJ*Q(K)-XMPK*Q(J))/PKMJ
      QD(K)=(QKMJ+XMPJ*QD(K)-XMPK*QD(J))/PKMJ
  141 CØNTINUE
      IF (X-TX(N)) 151,152,151
  151 Y=Q(NP)
  152 YD=QD(NP)
      RETURN
      END
```

```
      SUBRØUTINE XDRR(ARG,XØ,YØ,AY,BY,M,NP,
     1                AN,RR,RI,TR,TI,DRR,DRI,DTR,DTI)
C     DERIVATIVE REFLECTIØN/REFRACTIØN CØEFFICIENTS
      DIMENSIØN ARG(101),XØ(101),YØ(101),AY(101),BY(101)
      CALL TLU(ARG,XØ,M,NP,AN,X,XD)
      CALL TLU(ARG,YØ,M,NP,AN,Y,YD)
      CALL TLU(ARG,AY,M,NP,AN,A,AD)
      CALL TLU(ARG,BY,M,NP,AN,B,BD)
      XYM2=X*X+Y*Y
      SNR=(XD*X+YD*Y)/(2.*XYM2)
      SNI=(YD*X-XD*Y)/(2.*XYM2)
      SR=RR*RR-RI*RI-1.
      SI=2.*RR*RI
      DPR=SR*SNR-SI*SNI
      DPI=SR*SNI+SI*SNR
      DQR=2.*(B*RR-A*RI)
      DQI=2.*(B*RI+A*RR)
      DRR=DPR+DQI
      DRI=DPI-DQR
      SR=RR-1.
      SI=RI
      DPR=SR*SNR-SI*SNI
      DPI=SR*SNI+SI*SNR
      DQR=DPR+A
      DQI=DPI-B
      DTR=DQR*TR-DQI*TI
      DTI=DQR*TI+DQI*TR
      RETURN
      END
```

```
      SUBROUTINE RK(ARG,X0,Y0,AY,BY,M,NP,H,AN,RR,RI,TR,TI)
C     RUNGE/KUTTA 4 TH ORDER CLASSICAL
      DIMENSION ARG(101),X0(101),Y0(101),AY(101),BY(101)
     0CALL XDRR(ARG,X0,Y0,AY,BY,M,NP,
     1          AN,RR,RI,TR,TI,DRR,DRI,DTR,DTI)
      F1RR=H*DRR
      F1RI=H*DRI
      F1TR=H*DTR
      F1TI=H*DTI
      AN1=AN+H/2.
      RR1=RR+F1RR/2.
      RI1=RI+F1RI/2.
      TR1=TR+F1TR/2.
      TI1=TI+F1TI/2.
     0CALL XDRR(ARG,X0,Y0,AY,BY,M,NP,
     1          AN1,RR1,RI1,TR1,TI1,DRR,DRI,DTR,DTI)
      F2RR=H*DRR
      F2RI=H*DRI
      F2TR=H*DTR
      F2TI=H*DTI
      AN2=AN+H/2.
      RR2=RR+F2RR/2.
      RI2=RI+F2RI/2.
      TR2=TR+F2TR/2.
      TI2=TI+F2TI/2.
     0CALL XDRR(ARG,X0,Y0,AY,BY,M,NP,
     1          AN2,RR2,RI2,TR2,TI2,DRR,DRI,DTR,DTI)
      F3RR=H*DRR
      F3RI=H*DRI
      F3TR=H*DTR
      F3TI=H*DTI
      AN3=AN+H
      RR3=RR+F3RR
      RI3=RI+F3RI
      TR3=TR+F3TR
      TI3=TI+F3TI
     0CALL XDRR(ARG,X0,Y0,AY,BY,M,NP,
     1          AN3,RR3,RI3,TR3,TI3,DRR,DRI,DTR,DTI)
      F4RR=H*DRR
      F4RI=H*DRI
      F4TR=H*DTR
      F4TI=H*DTI
      RR=RR+(F1RR+2.*(F2RR+F3RR)+F4RR)/6.
      RI=RI+(F1RI+2.*(F2RI+F3RI)+F4RI)/6.
      TR=TR+(F1TR+2.*(F2TR+F3TR)+F4TR)/6.
      TI=TI+(F1TI+2.*(F2TI+F3TI)+F4TI)/6.
      RETURN
      END
```

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