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MINIMUM FUEL ATTITUDE CONTROL OF A NONLINEAR SATELLITE SYSTEM WITH BOUNDED CONTROL BY A METHOD BASED ON LINEAR PROGRAMMING

by Gary D. Wolske and I. Flügge-Lotz

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ABSTRACT

The problem treated here is the computation of fuel optimal controls for the large angle attitude motions of a satellite system in which the control is obtained by three sets of gas jets with bounded thrusts, each generating torques about one of the principal axes of inertia. Using a result from optimal control theory, an algorithm is developed that iteratively improves on an initial guess (nominal) for the control history which does not meet terminal constraints and/or does not minimize the fuel cost. In using the algorithm, which is based on linear programming, it is necessary to express the variation of the fuel cost and variations of the components of the terminal state constraint vector as linear functions of variations in the control.

The algorithm is tested on two sets of satellite differential equations. In one case, all dynamical effects are considered. In the other case, because control torque bounds are large enough, it is possible to neglect gravity gradient torque effects and orbital motion effects.

A method to recursively approach minimum time control solutions by using this minimum fuel algorithm is described and illustrated. Numerical results are compared with the results of others who have worked identical examples.

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LIST OF SYMBOLS

a_{ji}	see expression (3-12)
δA_{area}	see expression (4-4)
A_i	see expression (4-17)
$C(\underline{x},t)$	see expressions (2-9) and (3-10)
d_j	weighting coefficients for fuel cost- see expression (2-10)
$\underline{f}(\underline{x},\underline{u},t)$	time derivatives of state vector see expression (2-1)
f_i	components of state vector derivative
f_o	integrand of cost functional - see expression (2-5)
F	see expression (3-4)
g_i	see expression (2-9)
H	Hamiltonian - see expressions (2-7)
J	cost functional - see expressions (2-7)
J_n	augmented cost functional - see expression (2-6)
δJ	variation in fuel cost - see expression (4-3)
$\delta J'$	see expression (4-19)
n	dimension of state vector
N_1, N_2, N	see Figure 4-1
p	dimension of the control vector
δQ_i	see expression 4-17
r	dimension of ψ
t_o	initial time
t_f	final time
t	independent variable (time)

τ	scaled independent variable - see expression (7-1)
T_i	switching times - see Figure 3-1
$T_{i_{old}}, T_{i_{new}}$	see expression (4-10)
δT_i	variations T_i
$\underline{u}$	control vector
U_j	bound on $j^{\underline{th}}$ control vector component
u_{1s}, u_{2s}, u_{3s}	scaled values of control - see expression 7-1
$\underline{X}$	satellite state vector
$\underline{x}$	state vector and scaled state vector for satellite problem
$\underline{x_o}$	initial state
$\delta \underline{x}$	variation of state vector - see expression (3-2)
$\underline{x}_A, \underline{x}_B$	see expression (3-1)
α_i	see expression (4-6)
$\underline{\psi}$	see expression (2-4)
$\underline{\lambda}^T(t)$	Lagrange Multiplier (Adjoint) vector - see expression (2-6)
$\Phi(t_2, t_1)$	transition matrix
$\underline{\delta\psi}$	variations $\underline{\psi}$
$\underline{\psi}_A, \underline{\psi}_B$	see expression (3-7)
ψ_x	see expression (3-8)
a	a dimension of elliptical orbit - see Figure A-1
b_j	see Appendix B
c_i	see Appendix B
d_{ij}	see expression (A-12a)

G	gravitational constant
I_x, I_y, I_z	principal moments of inertia of the satellite
k_x, k_y, k_z	inertia parameters - see expression (A-9)
m, M	mass of satellite and earth, respectively
$\underline{n}_{xr}$	unit vector along x_r axis
N_x, N_y, N_z	active torque components
N_{xg}, N_{yg}, N_{zg}	components of gravity gradient torque
P, P^*	see Figure A-1
r	distance from center of earth to satellite
v	derivative of r with respect to time
x_b, y_b, z_b	satellite fixed reference frame
x_e, y_e, z_e	inertially fixed reference frame
x_r, y_r, z_r	orbitally fixed reference frame
T	period of satellite orbit about earth
W_1, W_2, W_3, W_4	Euler Parameters
z_i	see Appendix B
β	see expression (A-11)
ϵ	eccentricity of satellite orbit
θ	see Figure A-1
λ	see Appendix B
$\underline{\omega}^R$	angular velocity of orbital reference frame
$\underline{\omega}^B$	angular velocity of the satellite
$\underline{\omega}^{B/R}$	see expression (A-4)
$\omega_x, \omega_y, \omega_z$	see expression (A-2)

I. INTRODUCTION

A. OUTLINE AND MOTIVATION FOR THE PROBLEM

This report gives a method by which a satellite or general system may be controlled such that a minimum amount of fuel is consumed. The nonlinear satellite equations of motion are used in this report. The satellite is assumed to have an arbitrary initial orientation and tumble rate. The solution consists of a scheme by which an active torque device may be actuated to position the satellite to another given orientation and tumble rate at a given time in the future. The active torque device is three sets of cold gas jets located orthogonally on the spacecraft. The magnitude of torque generated by these devices is directly proportional to the time rate of fuel consumption. The "minimum fuel problem" consists of accomplishing the orientation mission while expending a minimum amount of fuel. The satellite to be considered is in elliptic orbit about the earth, but this assumption does not critically influence the solution of the problem. The specific orbit used only modifies the dynamical equations of the system.

There are, however, some assumptions and restrictions to be imposed on the problem. The first of these restrictions concerns the control torque. The torque levels are bounded in magnitude. Since no device can generate arbitrarily large torques, this assumption is reasonable. However, by bounding the torque, it is possible to request a mission which is impossible to accomplish in the allowed time. If such is the case, one must either equip the satellite with larger torque generating gas jets or accept the longer time necessary to accomplish the mission with the smaller jets.

The next assumption is that the control torque enters the dynamics equations linearly. What this means is more specifically defined in Chapter II, but it is not a very severe restriction because for many space vehicle systems one prefers to design controls which enter in the dynamics equations linearly. We shall arbitrarily limit the total rotation of the satellite in seeking a new orientation to less than 180 degrees. This excludes the possibility of orienting the satellite in a position diametrically opposite from the original position. However, if one did desire to turn the satellite 180 degrees, it could be accomplished in two missions. The first mission would spin the satellite β_1 degrees ($o < \beta_1, < 180$) and the second mission would spin the satellite $180-\beta_1$ degrees. Another approach is to just let the satellite drift a little and then start the control from there. The reason for limiting the spin to 180 degrees has to do with ambiguities which arise in the dynamics equations. This is discussed more fully in Chapter II in the section on indifference regions.

In the event of an elliptical orbit, another control scheme must be used for maintaining the position once the new orientation has been reached.

Although the work on optimization problems is well justified by what is learned in studying them, there are important practical contributions to be gained from optimization. Even if the optimal control scheme is not used, it provides valuable insight into just how good other more practical control schemes are. Since pioneering work in the theory of optimal control and the advent of Breakwell's computational technique using large digital computers for optimization calculations, optimization has evolved to the point of becoming practical to implement in the actual control of some systems. This report will point to the possibility of applying the following algorithm of optimization in actually controlling a satellite.

B. RÉSUMÉ OF RELATED WORK

In this section will be discussed briefly some other reports which are related in either the problem statement or method of solution to the problem in this report. There are numerous articles which deal only with low order, linear systems and no mention of these articles will be made.

The motivation for this report comes principally from work done in 1966 by K. A. Hales and I. Flügge-Lotz in reference 1. Their project was to compute minimum fuel controls for the same satellite acquisition control system as in this report. The approach used was an iterative procedure of "steepest descent". A nominal control was improved each iteration by minimizing the integral over the time interval $[t_o, t_f]$ of a weighted sum of the squares of the variations of the control components. This minimization was subject to the constraint of the dynamics of the system. This constraint was imposed on the minimization by the Lagrange Multiplier Technique. The cost is introduced as an additional state variable and is then treated as just another terminal constraint. The minimum fuel control which is arrived at by this technique does satisfy the terminal constraints on the state and gives a cost which is considerably lower than the cost associated with a good "classical" feedback design. The method has the advantage of being quite insensitive to the initial arbitrary choice of control. It has the disadvantages of often requiring many iterations to converge to a solution and of seldom converging to a true minimum fuel control. The reason why the solution seldom converges to a true minimum is connected with the idea of introducing the cost as another state. In doing this, Hales not only had to choose the final state to which the solution must converge, but also the final cost. Since one does not know the minimum cost apriori, chances of randomly picking the true minimum cost as the cost to which the solution should converge are quite remote. Hales did use a techique of picking this cost, though, which normally gave a solution of control resulting in a cost only 10 to 15 per cent above the true optimal cost.

In 1962, L. A. Zadeh and B. H. Whalen (reference 2) proposed a method for solving linear discrete optimal control problems using linear programming. They proposed solutions for optimization with respect to either time or fuel consumption. In both cases, the linearity of the system is an important assumption, since this results in one of the sets of linear programming constraint equations. For continuous time plants, the time interval must be discretized. Discretizing usually necessitates solving a linear programming problem of many variables, particularly if the system is of high order.

Linear programming has also been applied to minimax problems. In reference 3, G. Lack and M. Enns maximize the closest approach of a trajectory to a "danger region" in state space. This is directly applicable to the area of nuclear reactors. The minimax problem is converted to a linear programming problem by defining a dummy variable which is less than the minimum of the distance from the trajectory to the danger region in state space. The problem is to then maximize this variable while obeying the dynamics equations and terminal constraints.

In reference 4, H. C. Torng works the time optimal problem for a discrete linear system. His approach varies from that used by Zadeh and Whalen in the following way. Both reports are concerned with bounded control magnitudes. Torng chooses a certain time interval and calculates to see if there is a feasible control foı this time interval such that the control magnitude remains under a given upper bound. Initially this is usually not the case for the chosen time interval. The time interval is then increased and the procedure is repeated until a feasible solution for the control is found for a new time interval, $[t_o, t_f]$. This smallest time interval for which a feasible solution for the control exists is then the minimum time and the feasible control is a minimum time control for the problem. Zadeh's and Whalen's approach also involves an iterative technique. However, they minimize the largest absolute value which the control must take such that a control is feasible. If at any instant in the time interval, the control magnitude must be larger than the given upper bound on the control magnitude, then a longer time interval must be chosen. Repeating the procedure for longer time intervals should eventually lead to a control solution which remains within specified magnitude bounds.

In reference 5, M. O'Hagen uses a gradient projection method to compute optimal trajectories for both linear and nonlinear systems. For nonlinear systems, a technique is used in which optimization for the nonlinear system is done by optimizing recursively for a linear, time varying system. Although the method is quite general in the range of problems it can solve, convergence difficulties were encountered for some nonlinear problems. Furthermore, because the gradient of the cost functional is required, no work was done for problems in which the cost functional was the time intergral of the absolute value of the control. The gradient projection method worked best for cost functionals which are quadratic forms in the state and/or control.

T. E. Bullock and G. F. Franklin treat the computation of optimal controls by a second-order feedback method in reference 6. As opposed to ordinary gradient methods, they minimize the cost (augmented with the state equations by the Lagrange Multiplier Technique) by minimizing its expansion to second-order terms in the variation of the control. By linearizing a given nonlinear system about a nominal trajectory, the minimization process to find the variation of the control can be handled by solving a linear quadratic loss problem. Many systems may be solved using this technique and convergence to an optimal is usually rapid. However, the method is not suitable for minimizing the fuel from cold gas jets. In order for the method to work, the first and second partial derivatives of the integrand of the cost functional with respect to the control and state must exist. In the principal problems considered in this report, these derivatives do not exist.

In reference 7, Dyer and McReynolds develop an algorithm of computing optimal controls by extending the successive sweep method. The dynamic programming equation is expanded to second order and strong variations in control are considered to join solutions of the return function on either side of the discontinuity of the control. From necessary conditions, one arrives at an algorithm for changing switching times of the control. The method treats terminal constraints on the state with penalty functions. The control of the satellite system of this and Hales' report is solved in their report. The method, however, is very sensitive to the initial guess for the control history. In fact, when this extended successive sweep method was performed by Dyer and McReynolds using for their nominal the control which Hales and Flügge-Lotz had found as optimal (with cost approximately 10 per cent above true optimal), the solution did not converge. The report does include sufficiency conditions for checking optimal controls.

C. CONTRIBUTIONS

An iterative technique incorporating linear programming is developed such that high-order nonlinear systems with magnitude bounded controls entering the state equations linearly and entering the performance index linearly in the absolute value of the control can be optimized effectively.

The technique, which gives (locally) optimal open-loop controls and meets terminal constraints "exactly", is shown to be relatively insensitive to the nominal control history and is shown to converge rapidly through tests performed on a satellite system described by Euler Parameters. When compard with identical examples to those worked by Hales and Flügge-Lotz, the costs obtained by this method are between 10 and 15 per cent lower. While both methods take approximately 20 seconds per iteration on a modern computer, Hales' method takes 20 or more iterations to give a solution while the method of this report takes only about five iterations. Although O'Hagen's gradient projection method can optimize nonlinear dynamical systems with respect to several different performance indices, it is not capable of solving the minimum fuel problem. The second order method of Dyer and McReynolds is much more sensitive to the choice of the initial control history than is the method of this report. Because certain necessary partial derivatives do not exist, the second-order method of Bullock and Franklin can not be used when fuel cost is the performance index.

Computer sub-programs to do linear programming are quite standard and readily available, making it easy to implement the algorithm of this report.

An approach to solving time optimal problems is also described in Chapter 6.

II. DEVELOPMENT OF THE OPTIMAL CONTROL

In this chapter, the optimal control of a general non-linear, time-varying system will be discussed. Although this report is concerned with optimizing such a system with respect to the fuel consumed by a control consisting of gas jets, initially the discussion will be more general. Of the entire class of piecewise continuous functions of time which constitute acceptable candidates for an optimal fuel control, all but those satisfying a rather restrictive form as a function of time will be eliminated. This is done by applying a criterion developed by L. S. Pontryagin which imposes a necessary condition on the form which an optimal solution may have. The chapter ends with a discussion of the construction of the control and the possibilities of a "singular" control and "indifference" regions in the state space.

A. PROBLEM FORMULATION AND THE REGULAR SOLUTION

The dynamical system satisfies a set of differential equations denoted as:

$$\underline{\dot{x}} = \frac{d\underline{x}}{dt} = \underline{f}(\underline{x},\underline{u},t) \tag{2-1}$$

where $\underline{x}$ is an n- dimensional vector referred to as the "state". The independent (scalar) variable is time, t. The p- dimensional vector, $\underline{u}$, is the control and is the variable for which a solution is to be found. Each component of $\underline{u}$ is constrained in magnitude by inequality 2-2.

$$|u_j(t)| \leq U_j; \quad U_j > 0, \quad j=1,2,\ldots p \tag{2-2}$$

The time, t, is constrained to satisfy

$$t_o \leq t \leq t_f$$

where t_o and t_f are given. The state at $t=t_o$ is given as

$$x(t_o)=x_o \tag{2-3}$$

The state at $t=t_f$ is constrained to satisfy the following given r-dimensional vector relationship:

$$\underline{\psi}\ [\underline{x}(t_f)] = 0 \tag{2-4}$$

That is, there are r relationships between the n components of the state at the final time. The problem is to find $\underline{u}(t)$ for all $t(t_o \leq t \leq t_f)$ such that all of the above relationships are satisfied while at the same time minimizing the scalar J, where J is defined as

$$J = \int_{t_o}^{t_f} f_o(\underline{x},\underline{u},t)dt \tag{2-5}$$

The technique of Lagrange can be used to minimize J. Minimizing J subject to equation 2-1 is equilvalent to minimizing J_n, where

$$J_n = \int_{t_o}^{t_f} \quad f_o(\underline{x},\underline{u},t) + \underline{\lambda}^T(t)\ [\dot{\underline{x}} - \underline{f}(\underline{x},\underline{u},t)] \quad dt \tag{2-6}$$

This follows because the second term of equation 2-6 is identically zero from equation 2-1. The first variation of J_n with respect to small variations in $\underline{u}$ must be zero is the control under consideration is to be a candidate for minimizing J_n. Pontryagin has shown that this necessary condition on the first variation is equivalent to maximizing a function commonly referred to as the Hamiltonian and defined as

$$\begin{aligned} H &= \underline{\lambda}^T(t)\underline{f}(\underline{x},\underline{u},t,) - f_o(\underline{x},\underline{u},t) \\ &= \sum_{i=1}^{n} \lambda_i(t) f_i(\underline{x},\underline{u},t) - f_o(\underline{x},\underline{u},t) \end{aligned} \tag{2-7}$$

Maximization of H is with respect to $\underline{u}$. The $\lambda^T(t)$ introduced in equations 2-6 and 2-7 is the transpose of an n- dimensional vector whose components are referred to as adjoint variables, sensitivity variables, or Lagrange Multipliers. Components of the adjoint vector satisfy:

$$\dot{\lambda}_i(t) = -\frac{\partial H}{\partial x_i}, \quad i = 1,2,\ldots n \tag{2-8}$$

In this report, the control vector, $\underline{u}$, will be assumed to enter the state equations linearly as in equation 2-9.

$$f_i(\underline{x},\underline{u},t) = g_i(\underline{x},t) + \sum_{j=1}^{p} C_{ij}(\underline{x},t)u_j(t) \quad i = 1,2,\ldots n \tag{2-9}$$

It will be assumed from now on that the p- dimensional control vector has 3 components. The performance index to be minimized is total fuel used by the gas jets. The system (satellite) will have three sets of gas jets---each set applying torque about one of the principal axes of inertia. Therefore, the fuel consumption is given as:

$$J = \int_{t_o}^{t_f} f_o(\underline{x},\underline{u},t)dt = \int_{t_o}^{t_f} \sum_{j=1}^{3} d_j|u_j|dt \tag{2-10}$$

Substituting the specific form of the state equations and cost functional, equations 2-9 and 2-10, into equation 2-7 gives

$$H = \sum_{i=1}^{n} \left\{ \lambda_i(t)[g_i(\underline{x},t) + \sum_{j=1}^{3} C_{ij}(\underline{x},t)u_j(t)] - \sum_{j=1}^{3} d_j|u_j(t)| \right\} \tag{2-11}$$

Expanding H in a form more appropriate for applying the Pontryagin Principle leads to

$$H = \sum_{i=1}^{n} \lambda_i(t) \sum_{j=1}^{3} C_{ij}(\underline{x},t)u_j(t) - \sum_{j=1}^{3} d_j|u_j| + \text{terms not involving } \underline{u} \tag{2-12}$$

Maximizing H with respect to each control component gives

$$u_{j\ \text{optimal}} = \arg\max_{u_j} \left\{ \sum_{i=1}^{n} \lambda_i(t)C_{ij}(\underline{x},t) \quad u_j(t) \right.$$

(continued)

$$- d_j |u_j(t)| \Big\} \quad j = 1,2,3 \tag{2-13}$$

Because the $u_j(t)$ are bounded by inequality 2-2, the form of the control which maximizes H is

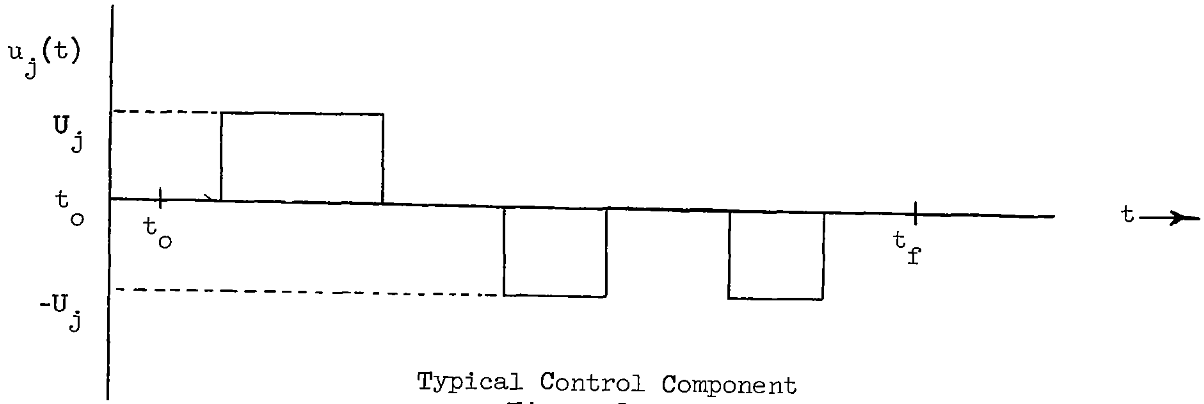
$$u_j(t) = U_j \frac{\sum_{i=1}^{n} \lambda_i(t) C_{ij}(\underline{x},t)}{\left|\sum_{i=1}^{n} \lambda_i(t) C_{ij}(\underline{x},t)\right|} = U_j S_{gn} \left\{ \sum_{i=1}^{n} \lambda_i(t) C_{ij}(\underline{x},t) \right\}$$

$$\text{for } \left|\sum_{i=1}^{n} \lambda_i(t) C_{ij}(\underline{x},t)\right| \geq d_i \tag{2-14}$$

$$u_j(t) = 0 \quad \text{for } \left|\sum_{i=1}^{n} \lambda_i(t) C_{ij}(\underline{x},t)\right| < d_i \quad j = 1,2,3$$

This function for the components of $\underline{u}$ is referred to as a "coast function" and gives the control as a series of pulses with intervals of zero control between the pulses. (See Fig. 2-1). The intervals of zero control must be non-zero in duration between pulses of opposite polarity. This is clear from equation 2-14 and noting that the adjoint variables are continuous functions in time.

From expression 2-14 it is seen that the optimal solution would be trivial if the components of $\underline{\lambda}(t)$ were known. Unfortunately, $\underline{\lambda}(t)$ must be found by the simultaneous solution of the n system differential equations (eqn. 2-1) and the n adjoint differential equations (eqn. 2-8). This is difficult because the boundary conditions for the 2n differential equations are given at two different times, t_o and t_f. (More about this two point boundary value is given in Chapter 2 of reference 19.) There have been some attempts to relate the terminal constraints (eqn. 2-4) to the solution of the adjoint equations at $t = t_o$. This is essentially an attempt to convert the two point boundary value problem to an initial condition problem. Simple methods for determining the adjoint variables



Typical Control Component
Figure 2-1

at $t = t_o$ as a function of the state constraints at $t = t_f$ (eqn. 2-4) do not in general exist.

The important point derived from the Pontryagin Principle and to be used in this report is the form of equation 2-14. That is, an optimal control for the problem has each of its components in the form of a "coast function", (figure 2-1). By limiting the search for optimal control solutions to this narrow class of functions the problem becomes very much easier, since it is reduced to a minimization over a finite dimensional parameter space rather than over a function space.

B. SINGULAR CONTROL

Although it was stated in the last section that a necessary condition in order for a given control to be a fuel optimal control was that it be in the form of a "coast" function, there are exceptions to this. These exceptions are classified as singular controls. Mathematically, this means that under certain circumstances, the Hamiltonian has a maximized value which is independent of the control for certain values of the control over a certain time. In particular, for the problem being considered here, it is seen from equation 2-13 (and remembering that $d_j > 0$, $j = 1,2,3$) that if

$$\sum_{i=1}^{n} \lambda_i(t) c_{ij}(\underline{x},t) \equiv d_j \qquad j = 1,2, \underline{\underline{\text{or}}}\ 3$$

then H is maximized for all u_j, where $0 \le u_j(t) \le U_j$, $j = 1,2,$ <u>or</u> 3. Likewise, if

$$\sum_{i=1}^{n} \lambda_i(t) c_{ij}(x,t) \equiv -d_j \qquad j = 1,2, \underline{\underline{\text{or}}}\ 3$$

then the H is maximized for all $u_j(t)$ where $-U_j \le u_j(t) \le 0$, $j = 1,2,$ <u>or</u> 3.

Johnson and Gibson (reference 7) have investigated these problems. As their work rather pessimistically concludes, singular problems are best treated very specifically, since few generalizations are available even for the simple linear, time-invariant plants. This study will not treat singular controls since, for the vehicles being considered, the control devices (such as gas jets) are either completely on or completely off and, hence, have no provisions to generate intermediate control levels.

C. CONSTRUCTION OF THE CONTROL

The most desirable solution of the problem is to be able to give it in feedback form. That is, to find functions of the state variables,

$$u_j^*(\underline{x},t), \qquad j = 1,2,3$$

such that

$$u_j(t)_{\text{optimal}} = u_j^*(\underline{x},t) \qquad j = 1,2,3$$

would mean that the instantaneous control to apply would be known from the instantaneous state. This is presently impossible except in very elementary problems. There are some techniques which attempt to give a feedback form of solution to the optimal control problem by choosing a form of the feedback function, $u_j(\underline{x},t)$, with several free parameters. These free parameters are chosen in a manner to optimize the system. In classical frequency domain analysis, this would essentially mean picking a filter of certain order and then adjusting the "poles and zeros" to optimize the system. The method is really just another sub-optimal scheme, since it depends on a somewhat arbitrary choice of the filter dynamics.

Here we shall give only the "open-loop" control program, i.e. for a given set of initial and terminal conditions on the state, a given initial and terminal time, and a given set of control bounds, a time function for each control variable will be found which meets all constraints and minimizes the cost in a "local" sense. This optimal set of time functions for the control will be arrived at by an iterative procedure in which an initial guess (nominal) for the control, which neither meets the terminal state constraints nor minimizes the cost functional, evolves to the optimal solution. As mentioned above the solution will minimize the cost "locally", as opposed to "globally". This is because the algorithm improves upon the arbitrary nominal and will converge to a local minimum. The whole space of possible solutions is not searched. One can be reasonably certain to obtain the global optimal solution by repeating the problem for several radically different "nominal" controls and observing that they do converge to the same optimal, but no claim of global results is made.

D. REGIONS OF INDIFFERENCE IN STATE SPACE

In certain problems, such as a spinning satellite, a desired physical terminal constraint may have several mathematical equivalents. When the terminal constraint to be met is that all motion be stopped and a certain orientation be met at $t = t_f$, it may make no physical difference whether one adds $2\pi m$, $m = \pm 1, \pm 2, \ldots$ to the state variables specifying the orientation. Mathematically, however, whether one considers $\underline{x}(t_f) = x_f$ as the constraint or $\underline{x}(t_f) = x_f \pm 2\pi m$ as the constraint may make considerable difference in the solution to the control problem. There is no investigation of this situation in this report. In order to eliminate the possibility of this difficulty, initial conditions will be chosen small enough. The initial conditions will, however, be much too large to allow one to get meaningful answers by linearizing the dynamical equations.

III. SENSITIVITY RELATIONS FOR THE TERMINAL CONSTRAINTS

In chapter II, it was noted that there are a set of terminal constraints (eqn. 2-4) which must be satisfied by the state variables at $t = t_f$. In this chapter, relations will be derived which show how variations in the control affect those terminal constraints.

As was previously derived, the control will be in the form of positive and negative effort pulses with periods of zero control between them. Because the control is structured as pulses, it can be determined for each instant of time, t, by just declaring the value of a finite number of "switching times", i.e. values which give the time at which the control changes from "on" to "off" or vice versa. There will be N (an even number) switching times, which means there are $N/2$ pulses of control effort. The switching times have values $t = T_i$, $i = 1,2,...N$, with $T_i \leq T_{i+1}$. This is illustrated in Figure 3-1.

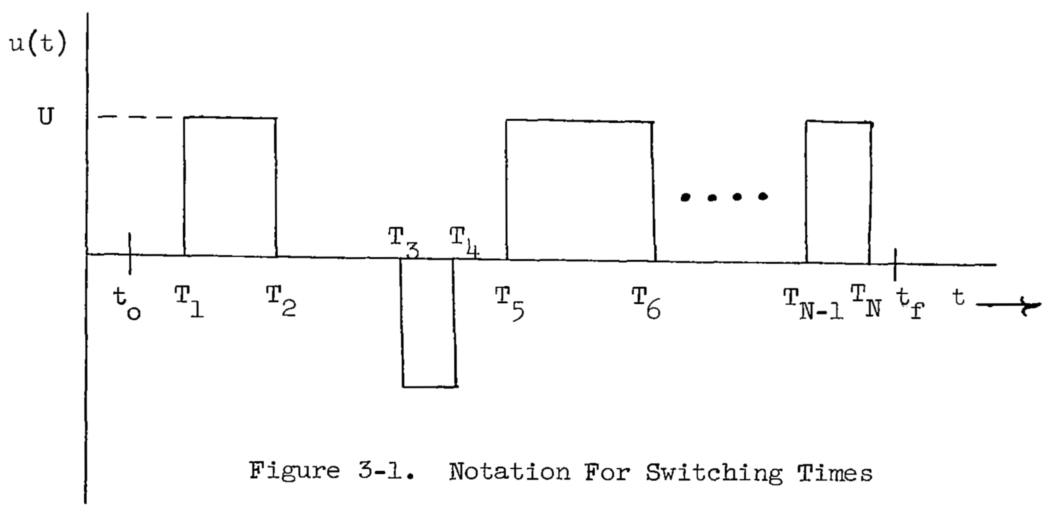


Figure 3-1. Notation For Switching Times

For this chapter, we shall treat the control vector as having only one component. This is done to make matters simpler to express and simpler to understand.

In determining how small variations in control affect the terminal constraints, the solution to an equivalent problem will suffice: how small variations in the switching times affect terminal state conditions. The approach used here is to find how a small variation in a single arbitrary switching time affects the terminal state. Because variations in switching times are to be small, the "Principle of Superposition", applies approximately. Hence, the net variation in terminal states due to variations in all the switching times is approximately the sume of the individual variations in the states caused by the variations in each switching time.

To find the approximate variations in the state at $t = t_f$ due to a small change in an arbitrary switching time, T_i, consider the two state trajectories shown in Figure 3-2.

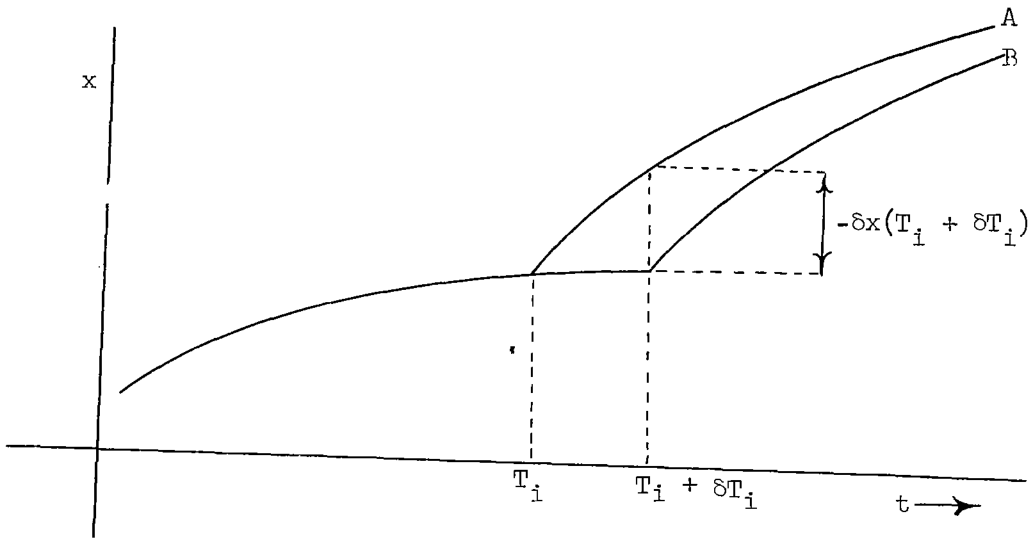


Figure 3-2. Variation in State Due to Strong Variation in Control

Curve A is generated by the system as the result of a control with switching time at $t = T_i$. Curve B is identical to Curve A with the exception that the switching time originally at $t = T_i$ now occurs at $t = T_i + \delta T_i$. By fitting linear approximations to the trajectories A and B, it is seen that

$$\delta \underline{x}(T_i + \delta T_i) = \underline{x}_B - \underline{x}_A \approx \underline{f}(\underline{x},\underline{u}(T_i^-),T_i)\delta T_i - \underline{f}(\underline{x},\underline{u}(T_i^+),T_i)\delta T_i \qquad (3\text{-}1)$$

$\underline{f}(\underline{x},\underline{u}(T_i^-),T_i)$ and $\underline{f}(\underline{x},\underline{u}(T_i^+),T_i)$ are the right side of equation 2-1 evaluated immediately before and after the switch at $t = T_i$, respectively.

$\underline{x}_A$ and $\underline{x}_B$ is a notation for the state along trajectories A and B respectively. Since for small δT_i the left side of equation 3-1 can be written approximately as $\delta x(T_i)$ one obtains

$$\delta \underline{x}(T_i) \approx (\underline{f}^-(T_i) - \underline{f}^+(T_i))\delta T_i \tag{3-2}$$

Notation for $\underline{f}(\underline{x},\underline{u}(T_i^{\pm}),T_i)$ has been reduced to $\underline{f}^{\pm}(T_i)$.

The variation in the state at the final time, $\delta \underline{x}(t_f)$, is related to the variation in the state at $t = T_i, \delta \underline{x}(T_i)$, by equation 3-3.

$$\delta \underline{x}(t_f) = \Phi(t_f,T_i)\delta \underline{x}(T_i) \tag{3-3}$$

$\Phi(t_f,T_i)$ is called the "transition matrix" or the "fundamental matrix". It is an n x n matrix which satisfies the following vector differential equation and boundary equation.

$$\begin{aligned} \dot{\Phi}(t_f,t) &= -\Phi(t_f,t)F(t) \\ \Phi(t_f,t_f) &= I \end{aligned} \tag{3-4}$$

$F(t)$ is a matrix of the various partial derivatives of $\underline{f}(\underline{x},\underline{u},t)$ with respect to $\underline{x}$. I is the identity matrix. More about equation 3-4 will be found in Chapter 5, where it will be needed as part of the solution to the satellite problem. More material on transition matrices is available in reference [11].

Substitution of equation 3-2 into equation 3-3 gives:

$$\delta \underline{x}(t_f) = \Phi(t_f,T_i) \left\{ \underline{f}^-(T_i) - \underline{f}^+(T_i) \right\} \delta T_i \tag{3-5}$$

When there is more than one switching time which has a change associated with it, the change in the final state is the composite effect of the changes in all of the switching times. That is, if each switching time, T_i, $i = 1,2,\ldots N$ undergoes a variation, δT_i, $i = 1,2,\ldots N$ then $\delta \underline{x}(t_f)$ is formed as

$$\delta \underline{x}(t_f) = \sum_{i=1}^{N} \Phi(t_f,T_i) \left\{ \underline{f}^-(T_i) - \underline{f}^+(T_i) \right\} \delta T_i \tag{3-6}$$

This is merely a summation of terms appearing in equation 3-5.

For the trajectories given in Figure 3-2, it is not necessarily true that $\underline{\psi}[\underline{x}(t_f)]$ (eqn. 2-4) is equal zero. For each of the trajectories, we will subscript $\underline{\psi}[\underline{x}(t_f)]$ to indicate on which trajectory it is evaluated at $t = t_f$. Then, by definition,

$$\delta\underline{\psi}[\underline{x}(t_f)] \equiv \underline{\psi}[\underline{x}(t_f)]_B - \underline{\psi}[\underline{x}(t_f)]_A \tag{3-7}$$

To a first order approximation,

$$\underline{\psi}[\underline{x}(t_f)]_B - \underline{\psi}[\underline{x}(t_f)]_A = \psi_x[\underline{x}(t_f)]\delta\underline{x}(t_f) \tag{3-8}$$

where $\psi_x[x(t_f)]$ is an $r \times n$ matrix of partial derivatives. In particular, the element in the i^{th} row and j^{th} column of ψ_x is $\frac{\partial\psi_i}{\partial x_j}$ (evaluated along trajectory A). From equation 3-6, 3-7, and 3-8 we determine by appropriate substitution that the variation in ψ at $t = t_f$, $\delta\underline{\psi}[\underline{x}(t_f)]$, is given as in equation 3-9.

$$\delta\underline{\psi}[\underline{x}(t_f)] = \psi_x[\underline{x}(t_f)]\left[\sum_{i=1}^{N} \Phi(t_f,T_i)\left\{\underline{f}^-(T_i) - \underline{f}^+(T_i)\right\}\delta T_i\right] \tag{3-9}$$

If the specific form of the state equations, equation 2-9, is substituted in equation 3-9, the expression for $\delta\underline{\psi}[\underline{x}(t_f)]$ is modified to

$$\delta\underline{\psi}[\underline{x}(t_f)] = \psi_x[\underline{x}(t_f)]\left[\sum_{i=1}^{N} \Phi(t_f,T_i)\left\{C(\underline{x},T_i)\Big(\underline{u}^-(T_i) - \underline{u}^+(T_i)\Big)\right\}\delta T_i\right]. \tag{3-10}$$

$C(\underline{x},t)$ is the $n \times p$ dimension matrix whose components are the $C_{ij}(\underline{x},t)$ of equation 2-9.

Expression 3-10 is rather long and contains notation which, though necessary, could induce the reader to miss an important point. Therefore, equation 3-10 will be rewritten using coefficients a_{ji}, where a_{ji} is the j^{th} element in the vector which is formed as the product of the matrices and vectors in equation 3-10. Specifically,

a_{ji} = the $j^{\underline{th}}$ element of the vector

$$\left\{ \psi_x[\underline{x}(t_f)]\Phi(t_f,T_i)\ C(\underline{x}(T_i),T_i)\ \left[\ \underline{u}^-(T_i) - u^+(T_i)\right]\right\} \qquad (3\text{-}11)$$

$$j = 1,2,\ldots r$$

Equation 3-10 can be expanded to scalar form to obtain

$$\delta\underline{\psi}_j[x(t_f)] = \sum_{i=1}^{N} a_{ji}\ \delta T_i \qquad j = 1,2,\ldots r \qquad (3\text{-}12)$$

In summary, one sees from equation 3-12 that the variation in the $j^{\underline{th}}$ component of the terminal constraint, $\delta\underline{\psi}[\underline{x}(t_f)]$, due to small variations in all of the switching times in Figure 3-1 is just a linear combination of the variations in the switching times.

IV. AN ITERATIVE TECHNIQUE OF IMPROVING THE CONTROL SOLUTION BASED ON LINEAR PROGRAMMING

In chapter II, it was mentioned that the optimal solution for the control would evolve from a nominal (guess) for the control which neither satisfied the terminal state requirement nor minimized the cost functional. In this chapter, the details for this iterative technique will be developed. The results of Chapter III will be used as an integral part of the following discussion.

In Figure 4-1, a typical control history for all three components of the control vector is shown. As in Chapter III, there are still N (even number) switching times, leading to N/2 pulses divided arbitrarily between the three control components, with $N_1/2$ $(N_2-N_1)/2$, and and $(N-N_2)/2$ pulses associated with $u_1(t)$, $u_2(t)$ and $u_3(t)$ respectively. The initial pulse associated with each control component may be either positive or negative. The basic configuration of Figure 4-1 will be used for the control throughout the remainder of this report.

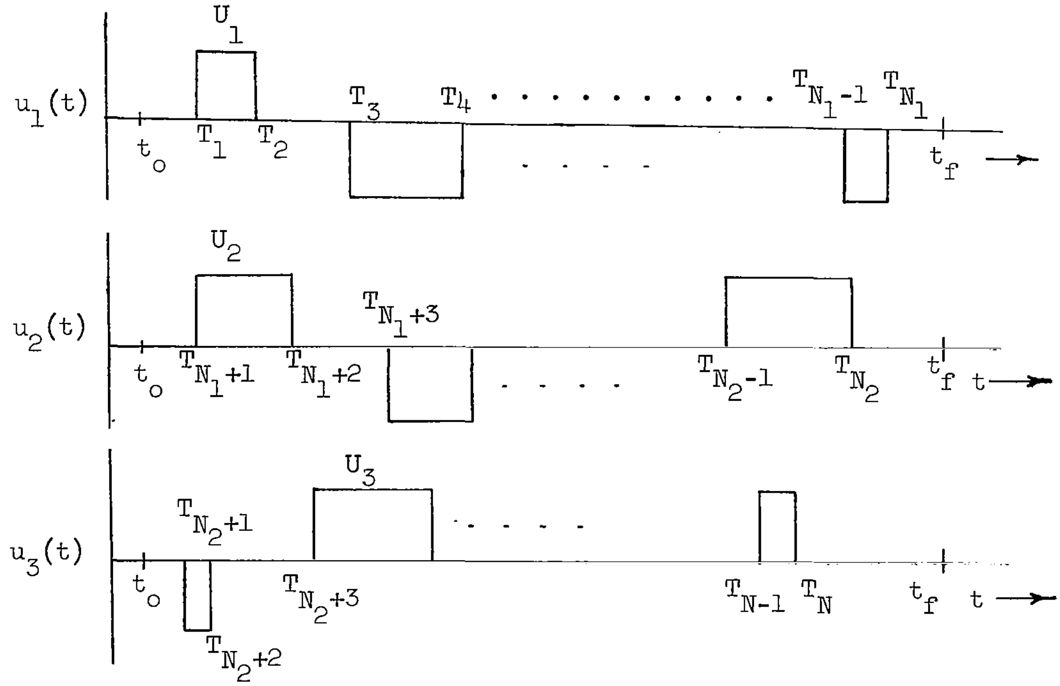


Figure 4-1. Configuration For 3- Component Control

A. THE FUEL COST

In this section the cost, J, and the variation of the cost, δJ, will be written in a form appropriate to be used with linear programming.

From the second chapter, one recalls that the fuel consumption was given as

$$J = \int_{t_o}^{t_f} (d_1|u_1| + d_2|u_2| + d_3|u_3|)dt \tag{4-1}$$

This cost in fuel, J, can be expressed as a summation over the switching times as in equation 4-2.

$$J = \sum_{i=1}^{N_1/2} d_1U_1(T_{2i} - T_{2i-1}) + \sum_{i=N_1/2+1}^{N_2/2} d_2U_2(T_{2i} - T_{2i-1}) + \sum_{i=N_2/2+1}^{N/2} d_3U_3(T_{2i} - T_{2i-1}) \tag{4-2}$$

Equation 4-2 follows from expression 4-1 easily if one refers to Figure 4-1 and observes that the integrand is constant between switching times.

The next point to consider is the variation, δJ, in the fuel cost. From equation 4-2, the variation in fuel, δJ, can be seen to be a sum of the variations in switching times δT_i, $i = 1,2,\ldots N$

$$\delta J = \sum_{i=1}^{N_1/2} d_1U_1(\delta T_{2i} - \delta T_{2i-1}) + \sum_{i=N_1/2+1}^{N_2/2} d_2U_2(\delta T_{2i} - \delta T_{2i-1}) + \sum_{i=N_2/2+1}^{N/2} d_3U_3(\delta T_{2i} - \delta T_{2i-1}) \tag{4-3}$$

Geometrically, this is represented in Figure 4-2, where the area (proportional to fuel) under the original pulse is $U_j(T_{2i}-T_{2i-1})$ and is represented by vertical lines. The area under the new pulse (after varying the switching times) is $U_j \left| (T_{2i}+\delta T_{2i}) - (T_{2i-1}+\delta T_{2i-1}) \right|$ and is represented by horizontal lines.

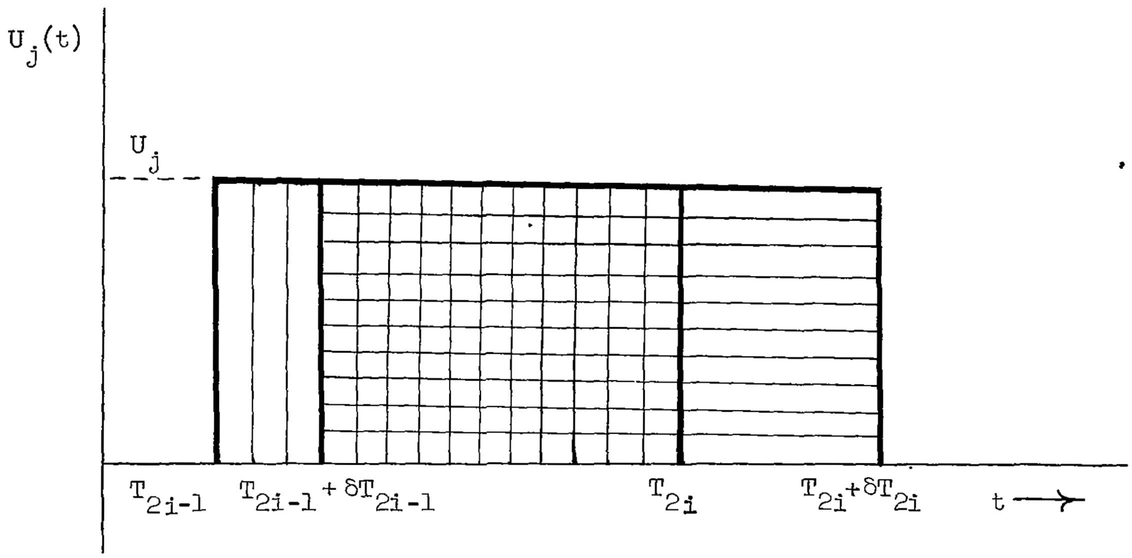


Figure 4-2. Variation in Fuel Due to Variations in Switching Times

Hence the change in area (fuel), δA_{area}, is the difference between the areas and is given as

$$\delta A_{area} = U_j(\delta T_{2i} - \delta T_{2i-1}) \tag{4-4}$$

Equation 4-3 is made of the terms of equation 4-4, but weighted with d_1, d_2 or d_3.

Equation 4-3 is the main result of this section. It gives the variation in the fuel consumption as a linear combination in the switching times.

The approach to finding the optimal control is to lower the cost J, (eqn. 4-2) in a step by step fashion. This can be done by minimizing the variation, δJ, of equation 4-3. However, one does not want to indiscriminately minimize δJ. Minimization of δJ without constraints on the independent variables would often be meaningless because solutions for the δT_i, $i=1,2,\ldots N$ and δJ would be unbounded. Therefore, several constraints will now be imposed.

B. FINAL VALUE CONSTRAINTS

The first constraint has to do with the variations in the states at $t=t_f$ which were developed in the last chapter. Equation 3-12 is rewritten here (in expanded form) as equation 4-5 so that it can be put in proper context with the solution to the optimal control problem.

$$\begin{aligned}
\delta\psi_1 &= a_{11}\delta T_1 + a_{12}\delta T_2 + \ldots a_{1N}\delta T_N \\
&\vdots \\
\delta\psi_r &= a_{r1}\delta T_1 + a_{r2}\delta T_2 + \ldots a_{rN}\delta T_N
\end{aligned} \tag{4-5}$$

If $\delta\psi_1,\ldots\delta\psi_r$ are specified, then variations of δT_i, $i=1\ldots N$ are limited so that equation 4-5 is satisfied.

C. LINEARITY CONSTRAINT

The next constraint in the problem solution is called the linearity constraint. The validity of equation 4-5 depends on the variations of the switching times, δT_i, $i=1,2,\ldots N$ being small. For equation 4-5 to be strictly true, the δT_i should be infintesimally small. This follows because the coefficients of δT_i, i.e. a_{ji}, contain terms from the transition matrix which involved a linearization in equation 3-4. Equation 3-2 and its inherent linearization is another reason which invalidates equation 4-5 for large δT_i.

In this report, the δT_i, $i=1\ldots N$ are limited to remain small by the simple magnitude inequality on δT_i as given in inequality 4-6.

$$|\delta T_i| \leq \alpha_i, \quad \alpha_i > 0 \quad i=1\ldots N \tag{4-6}$$

Mathematically, inequality 4-7 is equvalent to 4-6, but simplifies matters computationally.

$$\begin{array}{l} \delta T_i \le \alpha_i \\ \delta T_i \ge -\alpha_i \end{array} \qquad \alpha_i > 0, \quad i=1\ldots N \tag{4-7}$$

The 2N inequalities of 4-7 constitute what is referred to at the beginning of this section as the linearity constraint.

D. SWITCHING SEQUENCE CONSTRAINT

If, in the course of varying the switching times of the control in Figure 4-1, one were to move T_1, such that $T_1 > T_2$, the result would be ambiguous. In this section on switching time sequences, the mathematical constraints which prevent such situations from occuring will be developed.

By referring to Figure 4-1 and applying the idea of the last paragraph, inequality 4-8 follows.

$$\begin{array}{ll} T_{i-1} \le T_i & i=2,3,4,\ldots N_1 \\ T_{j-1} \le T_j & j=N_1+2,\ N_1+3\ldots N_2 \\ T_{k-1} \le T_k & k=N_2+2,\ N_2+3\ldots N \end{array} \tag{4-8}$$

Furthermore, one does not want any switching time to be shifted outside of the time interval of the problem, $[t_o, t_f]$. This is formalized by inequality 4-9.

$$\begin{array}{l} T_1 \ge t_o \\ T_{N_1+1} \ge t_o \\ T_{N_2+1} \ge t_o \\ T_{N_1} \le t_F \\ T_{N_2} \le t_F \\ T_N \le t_F \end{array} \tag{4-9}$$

All of the remaining switching times are implicitly constrained to be in the interval of time, $[t_o, t_f]$. This can be reasoned by applying inequality 4-8 in conjunction with 4-9.

The next step is to convert relations 4-8 and 4-9 into relations among the variations of the switching times, δT_i. To accomplish this, additional notation is introduced. Imagine that the switching times, T_i, i=1...N are assigned values $T_{i_{old}}$, i=1,2,...N and that the system equations, 2-1, are then solved for the state, $\underline{x}(t)$, using the control resulting from these switching times. After this computation, the switching times may later be shifted to new values, called $T_{i_{new}}$, i=1...N. The amount that each switching time is varied is δT_i, i=1...N. Equation 4-10 then relates the old switching times to the new switching times.

$$T_{i_{new}} = T_{i_{old}} + \delta T_i \quad i=1,2,\ldots N \tag{4-10}$$

It will be assumed that inequalities 4-8 and 4-9 hold for the old switching times, $T_{i_{old}}$, i=1...N. Presently, requirements on the variations of the switching times such that relations 4-8 and 4-9 hold for the new switching times, $T_{i_{new}}$, i=1,2,...N will be found.

Because the sequencing constraint of this section requires that relations 4-8 and 4-9 are valid for $T_{i_{new}}$, i=1,2,...N inequalities 4-11 and 4-12 follow.

$$\begin{aligned}
T_{i-1_{new}} &\le T_{i_{new}} \;; \quad i=2\ldots N_1 \\
T_{j-1_{new}} &\le T_{j_{new}} \;; \quad j=N_1+1 \;\; N_1+3,\ldots N_2 \\
T_{k-1_{new}} &\le T_{k_{new}} \;; \quad k=N_2+2,\; N_2+3,\ldots N
\end{aligned} \tag{4-11}$$

$$\begin{aligned}
T_{1_{new}} &\ge t_o \\
T_{N_1+1_{new}} &\ge t_o
\end{aligned} \tag{4-12}$$

(continued)

$$
\begin{aligned}
T_{{N_2+1}_{new}} &\geq t_o \\
T_{{N_1}_{new}} &\leq t_f \\
T_{{N_2}_{new}} &\leq t_f \\
T_{N_{new}} &\leq t_f
\end{aligned}
\qquad (4\text{-}12)
$$

Relations 4-10 is now substituted into 4-11 and 4-12 to obtain expressions 4-13 and 4-14.

$$
\begin{aligned}
T_{{i-1}_{old}} + \delta T_{i-1} &\leq T_{i_{old}} + \delta T_i; \quad i=2\ldots N \\
T_{{j-1}_{old}} + \delta T_{j-1} &\leq T_{j_{old}} + \delta T_j; \quad j=N_1+2,\ N_1+3,\ldots N_2 \\
T_{{k-1}_{old}} + \delta T_{k-1} &\leq T_{k_{old}} + \delta T_k; \quad k=N_2+2,\ldots N
\end{aligned}
\qquad (4\text{-}13)
$$

$$
\begin{aligned}
T_{1_{old}} + \delta T_1 &\geq t_o \\
T_{{N_1+1}_{old}} + \delta T_{N_1+1} &\geq t_o \\
T_{{N_2+1}_{old}} + \delta T_{N_2+1} &\geq t_o \\
T_{{N_1}_{old}} + \delta T_{N_1} &\leq t_f \\
T_{{N_2}_{old}} + \delta T_{N_2} &\leq t_f \\
T_{N_{old}} + \delta T_N &\leq t_f
\end{aligned}
\qquad (4\text{-}14)
$$

Relations 4-13 and 4-14 can now be rearranged so that the independent variables, δT_i, appear on the left side. If this is done, inequalities 4-15 and 4-16 result.

$$\delta T_{i-1} - \delta T_i \le T_{i_{old}} - T_{i-1_{old}}\ ; \quad i=2,3,\ldots N_1$$

$$\delta T_{j-1} - \delta T_j \le T_{j_{old}} - T_{j-1_{old}}\ ; \quad j=N_1+2,\ldots N_2 \qquad (4\text{-}15)$$

$$\delta T_{k-1} - \delta T_k \le T_{k_{old}} - T_{k-1_{old}}\ ; \quad k=N_2+2,\ldots N$$

$$-\delta T_1 \le T_{1_{old}} - t_o$$

$$-\delta T_{N_1+1} \le T_{N_1+1_{old}} - t_o$$

$$-\delta T_{N_2+1} \le T_{N_2+1_{old}} - t_o \qquad (4\text{-}16)$$

$$\delta T_{N_1} \le t_f - T_{N_{2_{old}}}$$

$$\delta T_{N_2} \le t_f - T_{N_{2_{old}}}$$

$$\delta T_N \le t_f - T_{N_{old}}$$

Inequality 4-15 and 4-16, similar to equation 4-5 and inequality 4-7, constitute the last of the necessary constraints to make the solution to the control problem meaningful.

E. TRANSLATION OF VARIABLES

In the linear programming algorithm discussed in Appendix B, all of the variables for which a solution is being sought are constrained to be non-negative. In the control problem, it is necessary to consider both positive and negative values of the independent variables, δT_i, $i=1\ldots N$. To fit the control problem into the context of the Simplex linear programming algorithm, it is necessary to define new variables related to the δT_i, $i=1...N$ such that these new variables are non-negative. A simple and successful approach is to just add constants to each δT_i. Define δQ_i, $i=1...N$ by equation 4-17.

$$\delta Q_i = \delta T_i + A_i; \; A_i > 0 \quad i=1...N \tag{4-17}$$

By choosing each A_i such that $A_i \geq \text{Max}|\delta T_i|$, δQ_i is constrained to be non-negative.

In this section the results of the previous sections are converted into equivalent statements about the new variables δQ_i, $i=1...N$.

The variational cost, δJ in equation 4-3 can be represented as in equation 4-18 if equation 4-17 is substituted in equation 4-3.

$$\begin{aligned}\delta J = &\sum_{i=1}^{N_1/2} d_1U_1([\delta Q_{2i} - A_{2i}] - [\delta Q_{2i-1} - A_{2i-1}]) \\ &+\sum_{i=(N_1/2)+1}^{N_2/2} d_2U_2([\delta Q_{2i} - A_{2i}] - [\delta Q_{2i-1} - A_{2i-1}]) \\ &+\sum_{i=(N_2/2)+1}^{N/2} d_3U_3([\delta Q_{2i} - A_{2i}] - [\delta Q_{2i-1} - A_{2i-1}])\end{aligned} \tag{4-18}$$

The δQ_i, $i=1,2,...N$ which minimize δJ in equation 4-18 also minimize $\delta J'$ in equation 4-19 because the δJ and $\delta J'$ differ only by an additive constant which is not a function of the δQ_i.

$$\begin{aligned}\delta J' = &\sum_{i=1}^{N_1/2} d_1U_1(\delta Q_{2i} - \delta Q_{2i-1}) + \sum_{i=(N_1/2)+1}^{N_2/2} d_2U_2(\delta Q_{2i} - \delta Q_{2i-1}) \\ &+\sum_{i=(N_2/2)+1}^{N/2} d_3U_3(\delta Q_{2i} - \delta Q_{2i-1})\end{aligned} \tag{4-19}$$

Hence, the procedure of solution now involves minimization of $\delta J'$ with respect to the δQ_i, i=1,2...N. The constraints (expressions 4-5, 4-7, 4-15, and 4-16) to which this minimization is subject will now be converted into equivalent statements involving δQ_i rather than δT_i. Only the results will be given. Details may be easily verified by the reader by substitution of equation 4-17 into expressions 4-5, 4-7, 4-15, and 4-16.

Equation 4-5 becomes equation 4-20.

$$\begin{aligned} a_{11}\delta Q_1 + a_{12}\delta Q_2 + \dots a_{1N}\delta Q_N &= \sum_{i=1}^{N} a_{1i}A_i + \delta\psi_1 \\ &\vdots \\ a_{r1}\delta Q_1 + a_{r2}\delta Q_2 + \dots a_{rN}\delta Q_N &= \sum_{i=1}^{N} a_{ri}A_i + \delta\psi_r \end{aligned} \qquad (4\text{-}20)$$

Inequality 4-7 becomes 4-21.

$$\begin{aligned} \delta Q_i &\le \alpha_i + A_i \\ & \qquad\qquad i=1,2,\dots N \\ \delta Q_i &\ge -\alpha_i + A_i \end{aligned} \qquad (4\text{-}21)$$

And inequalities 4-15 and 4-16 become inequality 4-22.

$$\begin{aligned} \delta Q_{i-1} - \delta Q_i &\le (T_{i_{old}} - T_{i-1_{old}}) + A_{i-1} - A_i; \quad i=2,3\dots N_1 \\ \delta Q_{j-1} - \delta Q_j &\le (T_{j_{old}} - T_{j-1_{old}}) + A_{j-1} - A_j; \quad j=N_1+2,\ N_1 \dots N_2 \\ \delta Q_{k-1} - \delta Q_k &\le (T_{k_{old}} - T_{k-1_{old}}) + A_{k-1} - A_k; \quad k=N_2+2,\ N_2 \dots N \\ -\delta Q_1 &\le T_{1_{old}} - t_o - A_1 \\ -\delta Q_{N_1+1} &\le T_{N_1+1_{old}} - t_o - A_{N_1+1} \\ -\delta Q_{N_2+1} &\le T_{N_{2_{old}}} - t_o - A_{N_2+1} \end{aligned} \qquad (4\text{-}22)$$

(continued)

$$\delta Q_{N_1} \leq t_f - T_{N_{1_{old}}} + A_{N_1}$$

$$\delta Q_{N_2} \leq t_f - T_{N_{2_{old}}} + A_{N_2} \qquad (4\text{-}22)$$

$$\delta Q_N \leq t_f - T_{N_{old}} + A_N$$

The control problem has now been reduced to finding the minimization of $\delta J'$ (equation 4-19) subject to the constraints of expressions 4-20, 4-21, and 4-22. This is the precise form of the linear programming problem which is outlined in Appendix B. In the next section, a general discussion will give the overall pictures of how this computational algorithm is implemented in the solution to the control problem.

F. COMPUTATIONAL CONSIDERATIONS

This section describes how the ideas developed so far in this report may be used to compute minimum fuel controls for satellites.

One first chooses a control which is structured as in Figure 4-1. The switching times, T_i, $i=1...N$, are chosen arbitrarily. but, as will be elaborated in the next chapter when the actual satellite problem is solved, discriminate choice normally guarantees a faster solution. Next the system equations, 2-1, are integrated from $t = t_o$ to $t = t_f$. During this integration, there must be provisions for storing the time history of the state vector. Next, the transition matrix, $\Phi(t_f, T_i)$ is evaluated at each switching time. This is accomplished by integrating equations 3-4 backward from $t = t_f$ to $t = t_o$. With these integrations performed, the a_{ij}, $i=1,2...N$, $j=1,2,...r$ in equation 4-20 can be evaluated.

The next step is the selection of the $\delta\underline{\psi}_j$, $j = 1,2,...r$ in equation 4-20. Since the desired terminal state in the control problem is that $\underline{\psi}[\underline{x}(t_f)] = 0$ and because we normally will not satisfy $\underline{\psi}[\underline{x}(t_f)] = 0$ with an arbitrarily picked control, one chooses $\delta\underline{\psi}_j$, $j = 1,2,...r$ such that the constraint is more nearly satisfied on the next iteration. In particular, one usually sets $\delta\underline{\psi}_j$ such that $\delta\underline{\psi}_j = -\underline{\psi}_j[\underline{x}(t_f)]$, $j = 1,2,...r$.

Values must now be chosen for α_i and A_i, $i = 1,2,\ldots N$. When choosing these parameters, one simplifies the problem considerably by choosing A_i such that

$$A_i = \alpha_i \quad i=1,2,\ldots N$$

From the second part of expression 4-21, it becomes clear why the A_i are chosen equal to the α_i. The second part of equation 4-21 can then be written as

$$\delta Q_i = 0; \quad i=1,2,\ldots N$$

This constraint is now eliminated from the linear programming problem because it is a restriction which is implicitly incorporated in the linear programming algorithm, i.e., all independent variables are non-negative. The linear programming problem statement is thus shortened by N equations and N slack variables. (Slack variable are discussed briefly in Appendix B.)

With all coefficients evaluated, $\delta J'$ of equation 4-19 is minimized subject to expressions 4-20, 4-21, and 4-22 by using a standard linear programming technique. The solution is given as non-negative values for the δQ_i $i=1,2,\ldots N$. The δT_i are found as

$$\delta T_i = \delta Q_i - A_i, \quad i=1,2,\ldots N$$

The switching times are updated as

$$T_{i_{new}} = T_{i_{old}} + \delta T_i; \quad i=1,2,\ldots N \tag{4-23}$$

The whole process is repeated using the new switching times for the control. Normally, J_{new} will be less than J_{old}. Ideally, the terminal constraints on the state, $\underline{\psi}[\underline{x}(t_f)]$ should be satisfied. Because of the inaccuracies introduced by linearizing the sensitivity equations, obtaining $\underline{\psi}[\underline{x}(t_f)] \equiv 0$ would be exceptional on the first iteration of the total solution.

Normally after a few iterations the control converges to a solution which minimizes the fuel cost (locally) and satisfies the state constraints at $t = t_f$. When the cost can no longer be improved significantly, the procedure is terminated.

Although one is not here confronted with the common problem of taking the inverse of matrices which tend toward singularity as the time t approaches t_f, a certain amount of care must be taken against the possibility of "infeasible solutions". Infeasibility means that no solution for the δQ_i may exist which satisfies the constraints of expressions 4-20, 4-21, and 4-22. What this usually implies is that the A_i, $i=1,2,...N$ have been chosen so small that equation 4-20 can not be satisfied for the given $\delta \underline{\psi}_j$, $j=1,2,...r$. This is only a minor problem and was only rarely observed in simulation. It can be corrected by proper compensation in the choice of the A_i, $i=1,2,...N$ and also $\delta \underline{\psi}_j$, $j=1,2,...r$.

In the next chapter this material will be applied to the general satellite system.

V. HIGH TORQUE ACQUISITION PROBLEM

In this chapter, the procedure discussed in the last chapter will be applied to a system of differential equations describing the attitude motion of a satellite in orbit about a fixed body such as the earth. Because of the high control torque levels, the effects of the gravity gradient and orbital motion are neglected from the dynamics equations. The purpose is to construct an on-off time history for the gas jet attitude controllers such that the satellite acquires a desired orientation and spin rate at a given time, t_f, in the future. Several examples are illustrated along with comparisons to similar results of other people.

A. HIGH TORQUE DYNAMICS AND SENSITIVITY EQUATIONS

The dynamics equations to be used for the examples of this chapter are given by expression 5-1.

$$
\begin{aligned}
\dot{X}_1 &= u_1 - K_x X_2 X_3 \\
\dot{X}_2 &= u_2 - K_y X_1 X_3 \\
\dot{X}_3 &= u_3 - K_z X_1 X_2 \\
\dot{X}_4 &= (X_5 X_3 - X_6 X_2 + X_7 X_1)/2 \\
\dot{X}_5 &= (X_2 X_7 - X_4 X_3 + X_1 X_6)/2 \\
\dot{X}_6 &= (X_4 X_2 - X_5 X_1 + X_7 X_3)/2 \\
\dot{X}_7 &= (-X_4 X_1 - X_5 X_2 - X_6 X_3)/2
\end{aligned}
\qquad (5\text{-}1)
$$

Equation 5-1 follows from equations A-25 in Appendix A if the gravity gradient terms and terms involving the rotation of the orbital reference frame are dropped and if the W_i are defined as X_{i+3}, $i=1,\ldots 4$. Because the control torque levels are high relative to the terms involving gravity gradient and because the rotation of the orbital reference frame is negligible in the time interval of control, $t_f - t_o$, these omissions are reasonable. Since the orbital parameters do not enter in these abbreviated high torque equations, the last two differential equations in equation A-26 are unnecessary.

Although there are seven differential equations in equation 5-1, there are only six independent states. X_7 can be expressed in terms of the other components of the state by equation 5-2.

$$X_7 = (4 - X_4{}^2 - X_5{}^2 - X_6{}^2)^{1/2} \qquad (5\text{-}2)$$

From equation 3-4, one of the steps in applying the algorithm of Chapter 4 involves the integration of a matrix differential equation. The expanded version of this equation, in component form, is:

$$\mathring{\Phi}_{ij}(t_f,t) = -\sum_{k=1}^{6} \Phi_{ik}(t_f,t)F_{kj}(\underline{X}(t),t) \quad \begin{array}{l} i=1,2,\ldots 6 \\ j=1,2,\ldots 6 \end{array} \qquad (5\text{-}3)$$

The boundary conditions for equation 5-3 are

$$\Phi_{ij}(t_f,t_f) = \begin{array}{ll} 1, & \text{when } i=j \\ 0, & \text{when } i\neq j. \end{array}$$

The coefficients of equation 5-3 are given as

$$F_{kj}(\underline{X}(t),t) = \frac{\partial}{\partial X_j} f_k(\underline{X}(t),\underline{u},t)$$

where $f_k(X(t),\underline{u},t)$ is the $k^{\underline{th}}$ component of equation 2-1. Applying this to the specific dynamics of equation 5-1, the $F_{kj}(\underline{X}(t),t)$, $k=1,2,\ldots 6$ $j=1,2,\ldots 6$ are given by matrix equation 5-4.

$F_{kj}(\underline{X}(t),t) =$

(continued)

k \ j	1	2	3	4	5	6
1	0	$-K_x X_3$	$-K_x X_2$	0	0	0
2	$-K_y X_3$	0	$-K_y X_1$	0	0	0
3	$-K_z X_2$	$-K_z X_1$	0	0	0	0
4	X_7	$-X_6$	X_5	$\frac{-X_1 X_4}{X_7}$	$X_3 - \frac{X_1 X_5}{X_7}$	$-X_2 - \frac{X_1 X_6}{X_7}$
5	X_6	X_7	$-X_4$	$-X_3 - \frac{X_2 X_4}{X_7}$	$\frac{-X_2 X_5}{X_7}$	$X_1 - \frac{X_2 X_6}{X_7}$
6	$-X_5$	X_4	X_7	$X_2 - \frac{X_3 X_4}{X_7}$	$-X_1 - \frac{X_3 X_5}{X_7}$	$\frac{-X_3 X_6}{X_7}$

(5-4)

The examples worked in the following sections of this chapter use satellite parameters based on a preliminary model of the OGO spacecraft described in (reference 12). The moments of inertia of

$$I_x = 800 \text{ slug-ft}^2$$

$$I_y = 581 \text{ slug-ft}^2$$

$$I_z = 300 \text{ slug-ft}^2$$

are equivalent to the inertia parameters

$$K_x = -.351$$

$$K_y = .860$$

$$K_z = -.730$$

B. NUMERICAL EXAMPLES

In this section, optimal fuel controls for the satellite system described by equation 5-1 will be determined by the algorithm described in Chapter 4 for different sets of initial conditions on the state (x_o), time intervals (t_f-t_o), and control angular acceleration levels, (U_i). Comparisons will frequently be made between these results and the corresponding results of examples worked by Hales and Flügge-Lotz in (reference 1).

The examples in this chapter have final value constraints on all six of the states*. With the exception of the last example, the final value constraints are $\underline{x}(t_f) = 0$. This means that in equation 2-4, the r-dimensional $\underline{\psi}[x(t_f)]$ becomes the 6 dimensional vector, $\underline{x}(t_f)$. Likewise, $\psi_x = -I$ and $\delta\underline{\psi}$ of equation 4-5 becomes $\delta\underline{x}(t_f)$.

Table 5-1 gives three sets of initial conditions used in the examples along with other pertinent information. $\beta(t_o)$ gives the "equivalent rotation" defined by the Euler Parameters (Appendix A).

In the first example, the initial conditions R-1 in Table 5-1 are used. The final time, t_f, is taken as 60 seconds and the control acceleration levels are set at .412 degrees/sec.^{2} for each component of the control. In Figure 5-1a and 5-1b, the state trajectories are illustrated for the nominal control with four pulses for each control variable. Figures 5-2a and 5-2b depict another nominal control history with six pulses for each control variable along with the corresponding state trajectories. In both cases, the linear programming procedure described in Chapter 4 yields the optimal control and trajectories of Figures 5-3a and 5-3b after five iterations of computation. Only two pulses for each control variable are needed for the optimal control history in Figure 5-3a; the other pulses tended to zero width and hence, give no contribution to the cost. The cost of the fuel in this example is .131 sec^{-1} as compared to a cost of .162 sec.$^{-1}$ computed by Hales and Flügge-Lotz for the identical situation. In (reference 7), Dyer and McReynolds work this example and get a solution identical to the one of this report. Their method also gives sufficiency conditions to guarantee local optimality,

*Strictly speaking, there are only six states because X_7 is an "integral of motion" by equation 5-2.

Run Number	R-1	R-2	R-3	R-4
$X_1(t_o)$, deg./sec.	1	.5	0	1
$X_2(t_o)$, deg./sec.	1	.5	0	1
$X_3(t_o)$, deg./sec.	1	.5	0	1
$X_4(t_o)$	.4	.5	0	.4
$X_5(t_o)$	.8	.5	0	.8
$X_6(t_o)$	.8	.5	0	.8
$X_7(t_o)$	1.6	1.8	2.0	1.6
$X_i(t_f)$ i=1,...6	0	0	0	0.3
$X_7(t_f)$	2	2	2	1.93
$\beta(t_o)$ degrees	73.8	51.8	0	73.8
t_o	0	0	0	0
t_f = 45 seconds	Fig. 5-5			
t_f = 60 seconds	Fig. 5-1 5-2 5-3 5-4	Fig. 5-7	Fig. 5-8 Fig. 5-9	Fig. 5-10
t_f = 120 seconds	Fig. 5-6			

Table 5-1 Summary of Boundary Conditions for High Torque Examples

but unfortunately, its computational success is very sensitive to the nominal control chosen initially.

In the second example the initial conditions of run R-1 in Table 5-1 are again used, but the control level of each jet is halved to .206 degrees/sec^2. Four pulse nominal controls similar to Figure 5-1b are used for most of the remaining examples, including this example. After four iterations, the controls and state trajectories have evolved to those pictured in Figures 5-4a and 5-4b. As one might expect, since the maximum thrust has been lowered from the previous example (with all other parameters remaining the same), the duration of the pulses is longer than in the previous example. The cost has now risen to .142 sec.$^{-1}$ as composed with a value of .159 sec.$^{-1}$ found by Hales.

The two examples illustrated in Figures 5-5a, 5-5b, 5-6a, and 5-6b are identical with the second example except that final times are 45 seconds and 120 seconds, respectively. In the third example, after 4 iterations, a cost in fuel of .154 sec.$^{-1}$ was obtained as compard with an approxiamtely 20% higher cost of .1969 sec.$^{-1}$ obtained by Hales. In the example of Figure 5-6, where the final time is 120 seconds, an optimal solution yielding a cost of .0924 sec.$^{-1}$ was obtained in five iterations. This was only 10% below the .1024 sec.$^{-1}$ cost obtained by Hales.

For the fifth example the optimal solution is illustrated in Figure 5-7a and 5-7b. Here the initial conditions on the state were given by R-2 in Table 5-1. The cost of the optimal solution is .08 sec.$^{-1}$.

If $\underline{x}(t_o) = 0$, it can easily be shown that the optimal fuel control is given as $u_1(t) = u_2(t) = u_3(t) \equiv 0$ for $t_o \leq t \leq t_f$. This above example, for which the analytical answer is known, will be solved presently to see whether or not the algorithm gives the correct solution for the control. The nominal control, similar to other nominal controls, and associated state trajectories are illustrated in Figures 5-8a and 5-8b. After three iterations, the control and trajectories are given by Figure 5-9a and 5-9b. Only the non-zero pulses are shown. The cost of this almost zero control effort is .00041 sec.$^{-1}$ or less than 1/2 of one per cent of the cost of previous examples.

In the final example of this chapter, the terminal state constraint function, $\underline{\psi}[\underline{x}(t_f)]$ of equation 2-4, which heretofore has been identical to $\underline{x}(t_f)$ is changed to $\underline{x}(t_f)-\underline{c}$ where $\underline{c}$ is a constant. In the example of Figures 5-10a and 5-10b, the initial conditions of R-4 are used to generate the control for an example in which the final value of $\underline{x}(t_f)$ is selected to be $.\underline{3}$ deg/sec for the angular velocity components and .3 for the first three Euler Parameters. The cost of the fuel for example was .1145 $sec.^{-1}$.

It can be shown (Appendix A) that the positioning of a body from any attitude to a new attitude can be accomplished by a single rotation about some fixed axis. This axis of rotation is related closely to the Euler Parameters. For the example of Figure 5-3, calculations on the optimal control history and trajectory revealed that the first pulses in the control: (1), aligned the angular velocity vector, $\underline{\omega}$, approximately parallel to this axis of rotation, and (2) set the approximate mean magnitude of the angular velocity to the proper value to cause the satellite to rotate the proper angle for a single rotation of β degrees in the time interval $(t_f - t_o)$. The pulses of control at the end of the time interval stop the rotation of the satellite. There is reason to believe that for a certain class of problems, the optimal fuel control in general has the characteristics described above. Generalizations are difficult (even for the examples of this chapter in which gravity gradient and orbital effects do not enter) because the angular velocity for a body is not in general constant for torque free motion.

C. COMPUTATIONAL CONSIDERATIONS

Simulation of the system dynamics was done by numerical integration on a digital computer. The time interval, t_o to t_f, was divided into approximately 100 increments and the state was stored for each time increment. The intervals of storage are not necessarily all equal, since it is occasionally necessary to change the step size in integration (when a switch in the control is imminent) to be sure that the state is stored at times exactly equal to the switching times. It is also important to integrate the system equations exactly up to the time at which the control switches. If a switch is to occur within a given integration step, the step must be reduced appropriately.

There are several ways in which the control may be programmed into the differential equations. The method used in this report was to declare the magnitude and sign of the first pulse of each component of the control along with nominal switching times. The program included logic which determined between which switching times T_i, the independent variable t was during each step in the integration of the septum equations. It then assigned the control, $u_i(t)$, in a $\pm U_i$, 0, $\mp U_i$, 0, $\pm U_i$,...fashion. The updating of the control after each iteration was done by changing the switching times.

The linear programming was done by an algorithm commonly called the "Simplex Method" described briefly in Appendix B. The α_i equation 4-7 were set equal to one second initially, but feasible solutions for variations in the switching times often did not exist for such small α_i's before a nearly optimal solution was obtained. The α_i were then set equal to seven seconds for all examples in this chapter. The components of $\underline{\delta\psi}$ in equation 4-5 were assigned the value of $-kx(t_f)$ for the regulator problem with $0 \leq k \leq 1$. $k=1$ was found to give the best results in the later interations of a problem.

Because the convergence of numerical methods of computing optimal controls depends to a certain extent on the choice of a nominal control, the following rule from (reference 1) was usually employed to assign the polarity of the first pulse of each component of the nominal control.

$$\text{Sgn(first pulse of } u_i) = \begin{cases} -\text{Sgn}[X_i(t_o)] & \text{if } X_i(t_o) \neq 0 \\ -\text{Sgn}[X_{i+3}(t_o)] & \text{otherwise; } i=1,2,3 \end{cases}$$

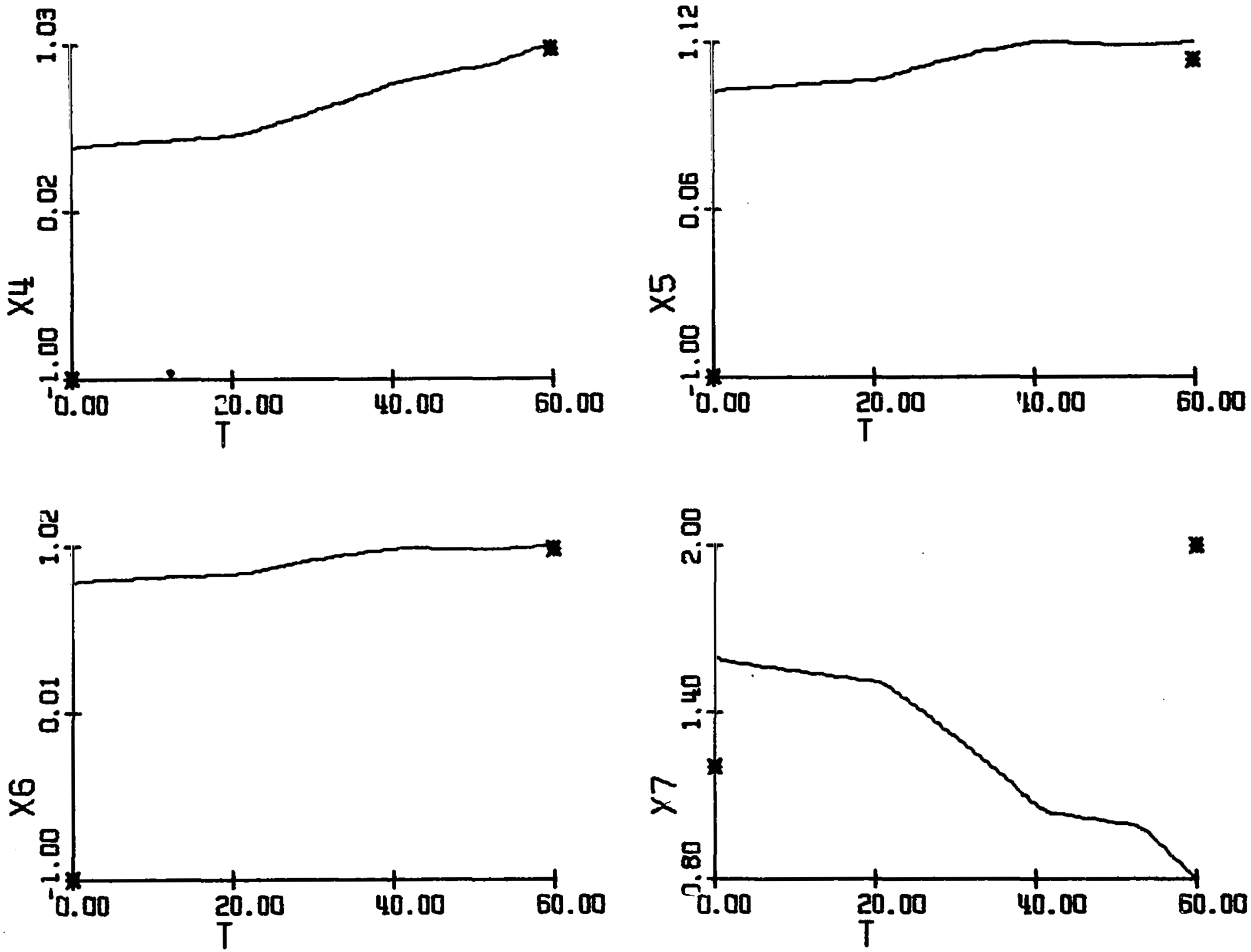


Figure 5-1a Response to 12-Pulse Nominal Control Applied
to Initial Conditions of Run R-1

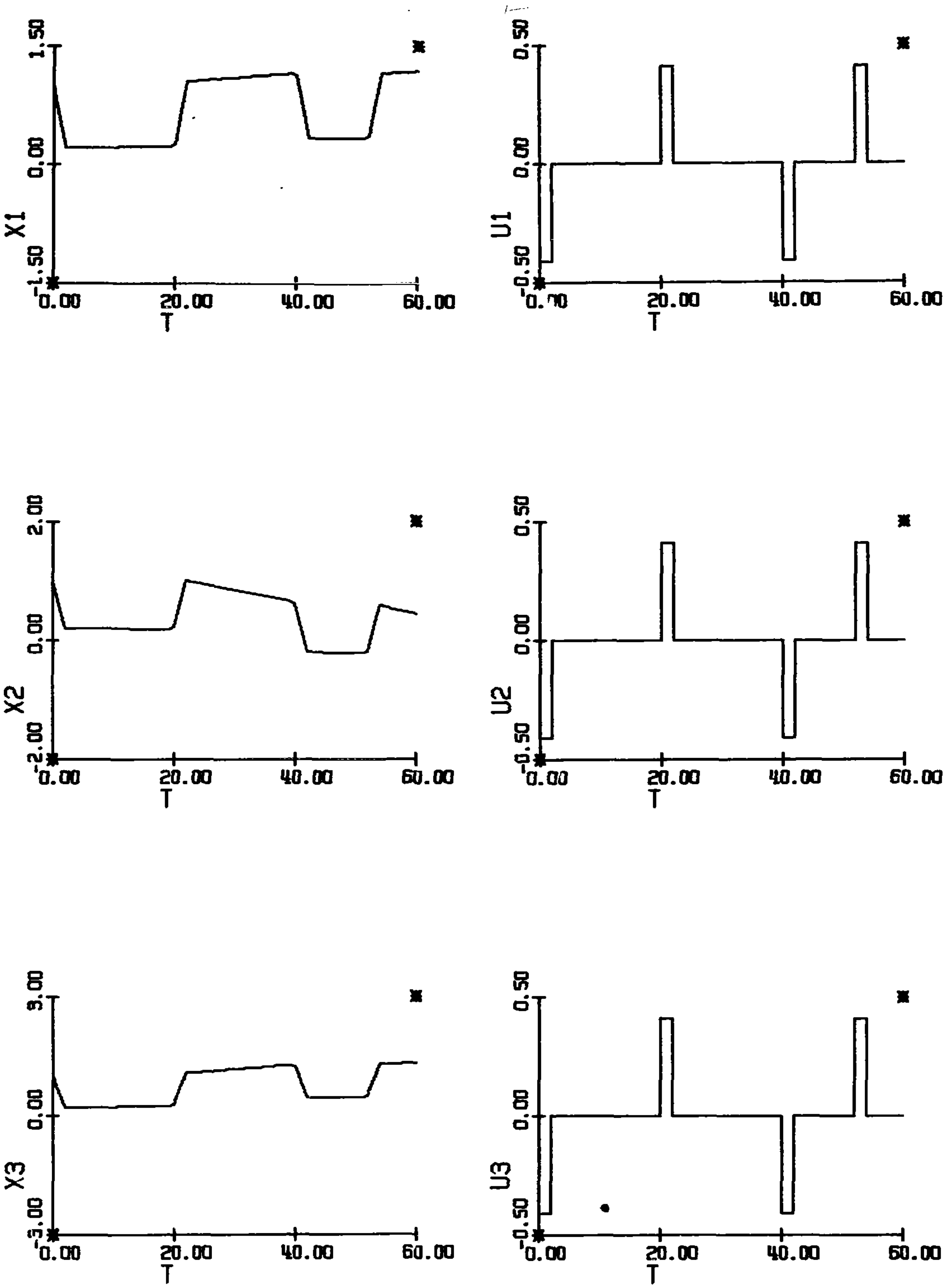


Figure 5-1b Response to 12-Pulse Nominal Control Applied to Initial Conditions of Run R-1

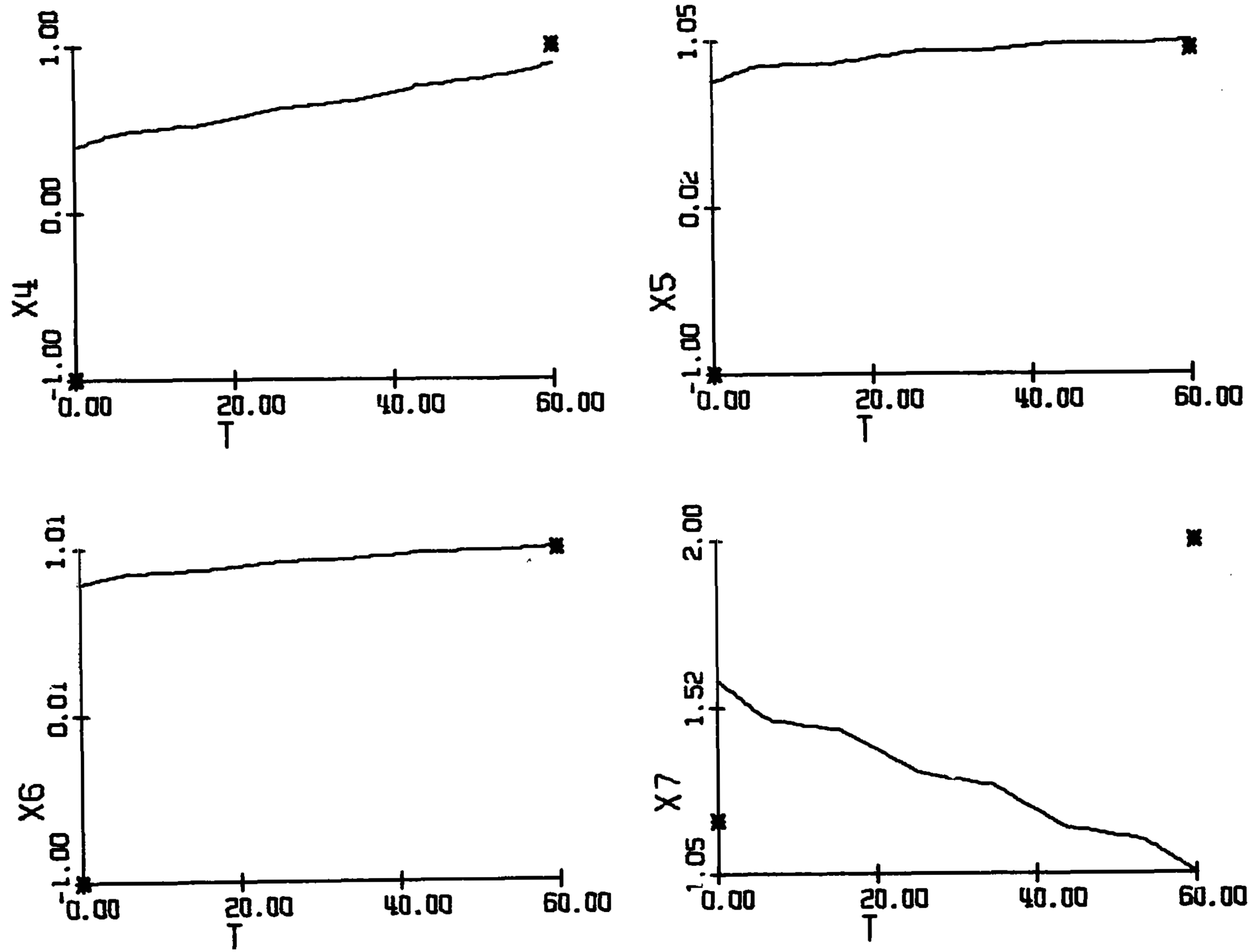


Figure 5-2a Response to 18-Pulse Nominal Control Applied to Initial Conditions of Run R-1

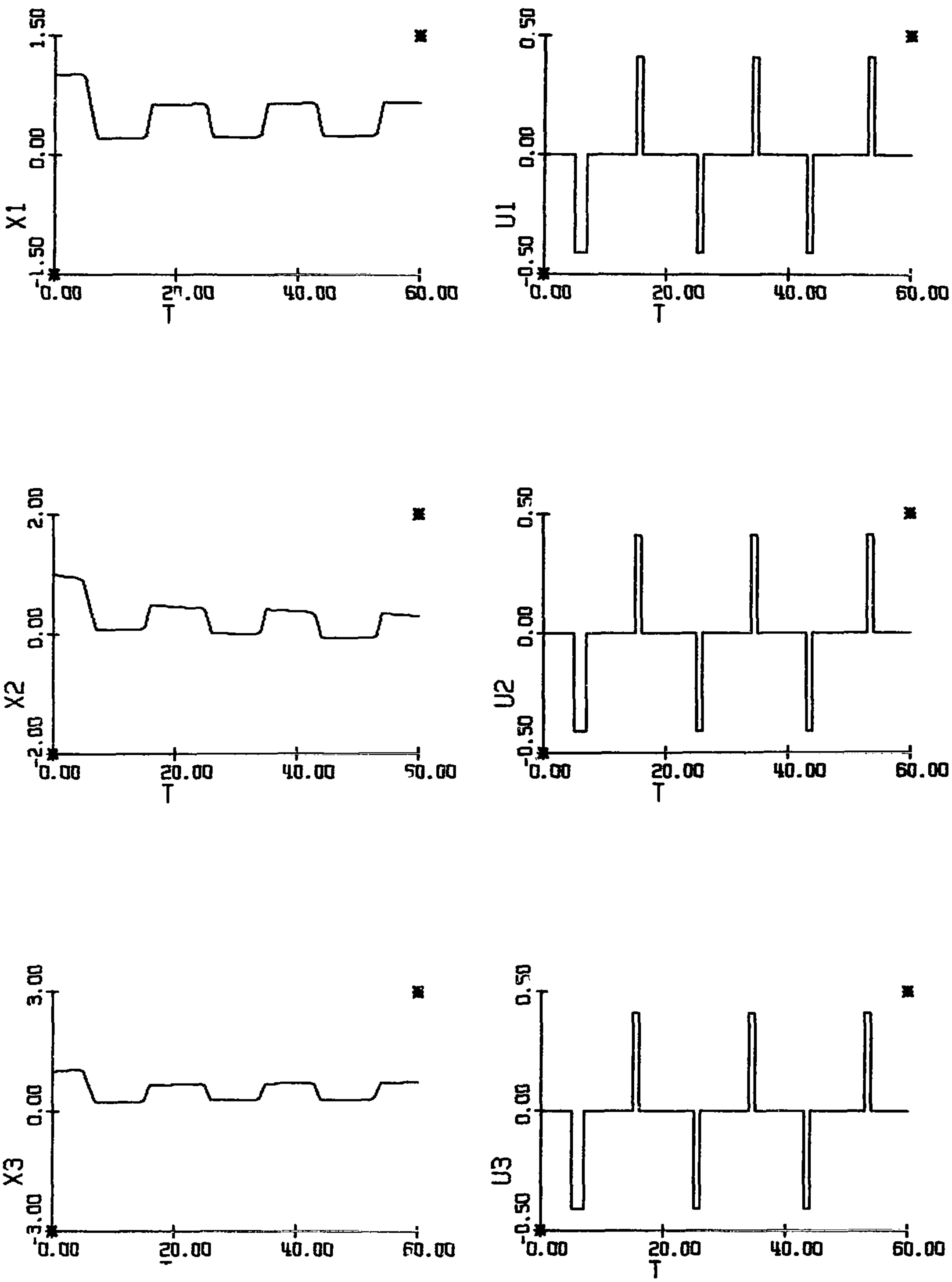


Figure 5-2b Response to 18-Pulse Nominal Control Applied to Initial Conditions of Run R-1

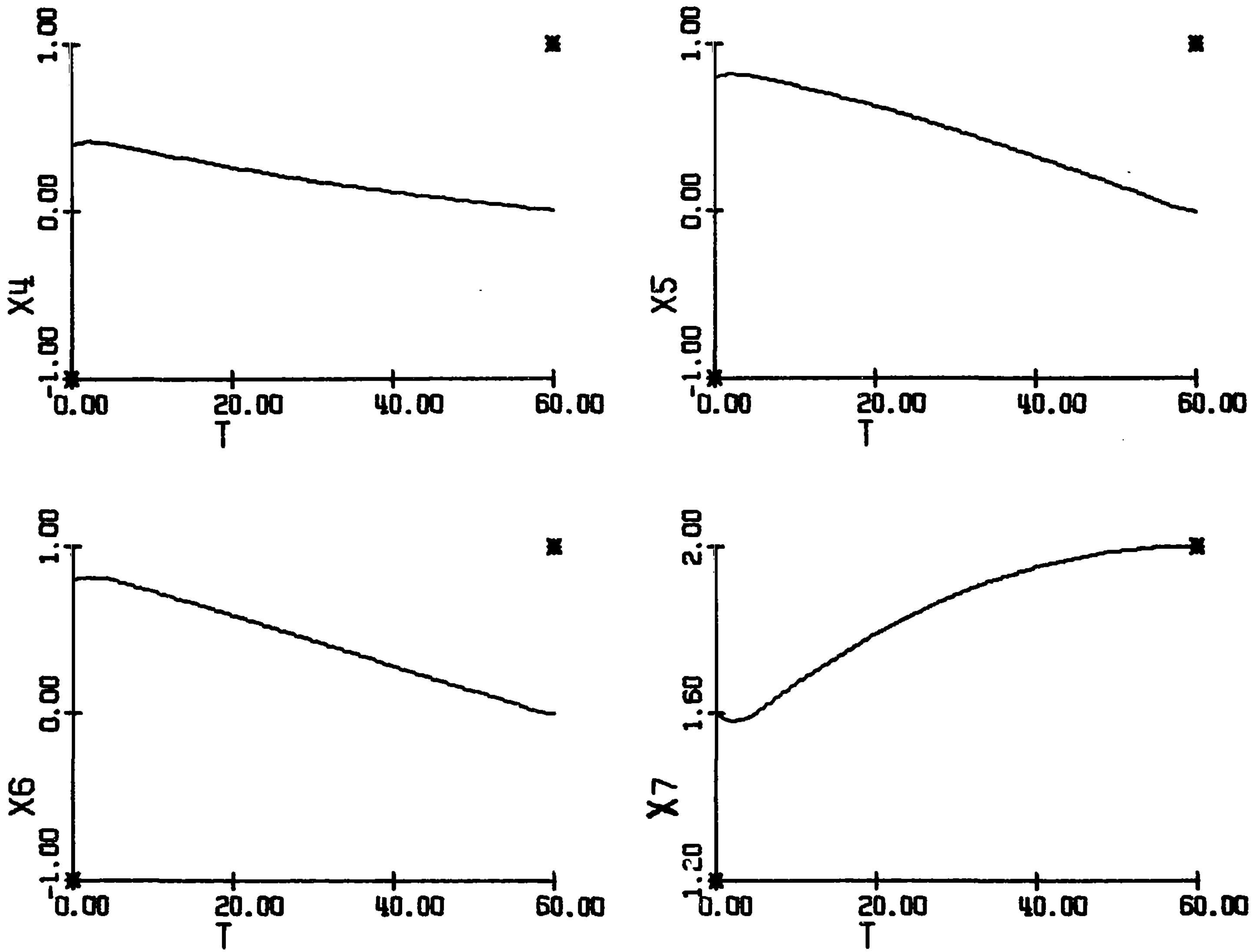


Figure 5-3a Optimal Response For Initial Conditions of Run R-1; $U_i = .412$ deg./sec^2; $t_f = 60$ sec

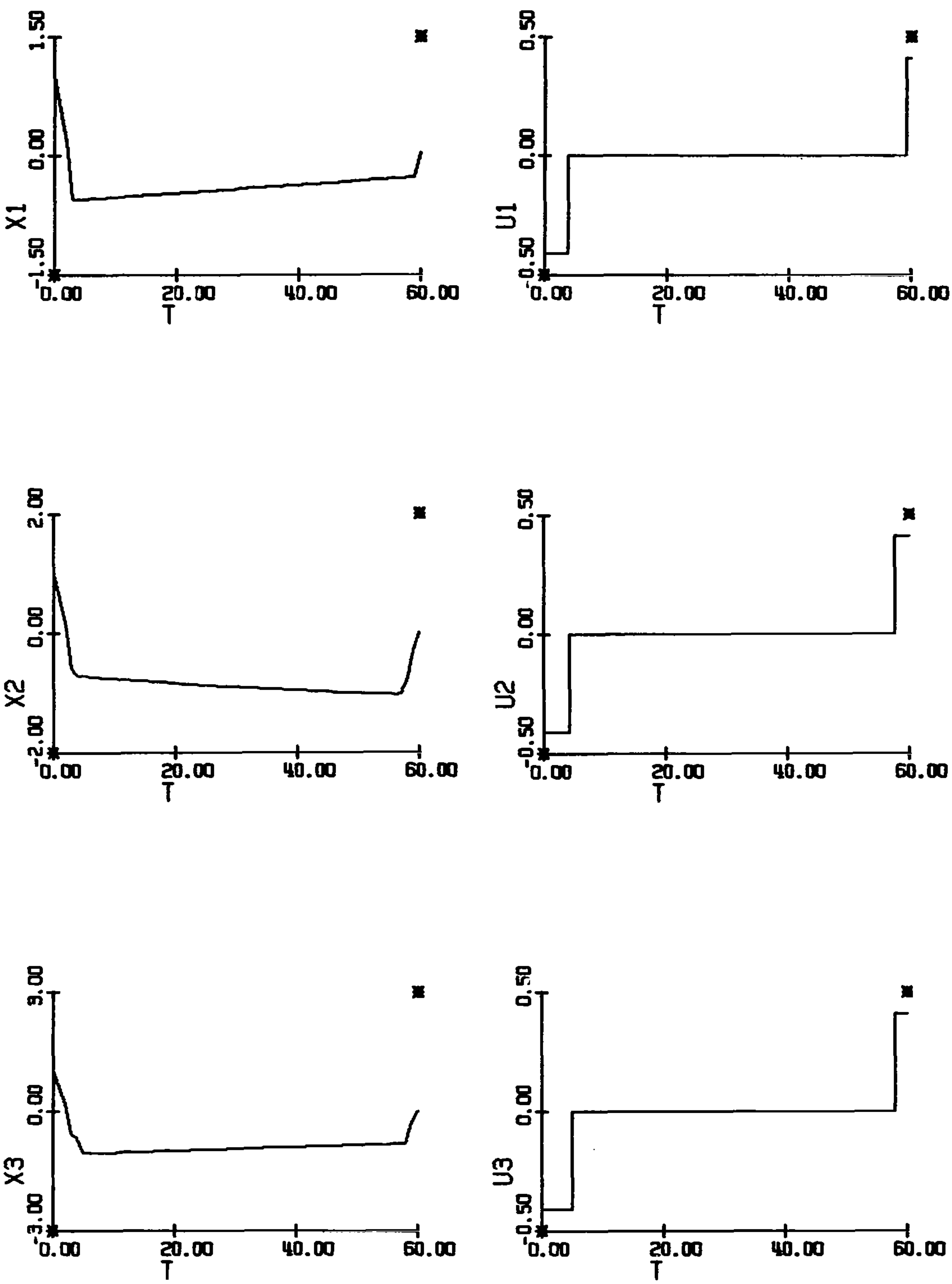


Figure 5-3b Optimal Response For Initial Conditions of Run R-1; U_i=.412 deg./sec^2; t_f=60 sec

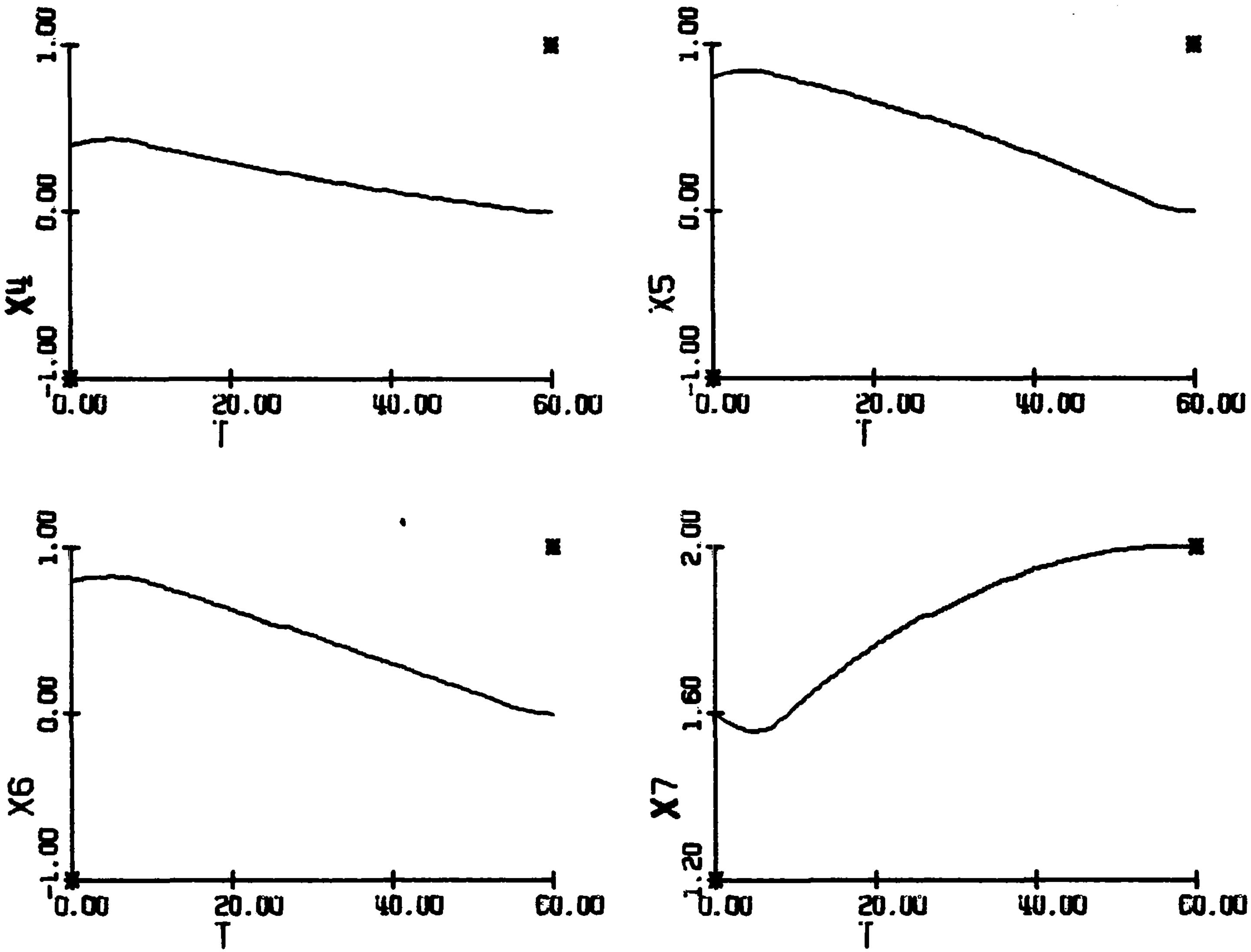


Figure 5-4a Optimal Response For Initial Conditions of Run R-1; U_i=.206 deg./sec^2; t_f=60 sec

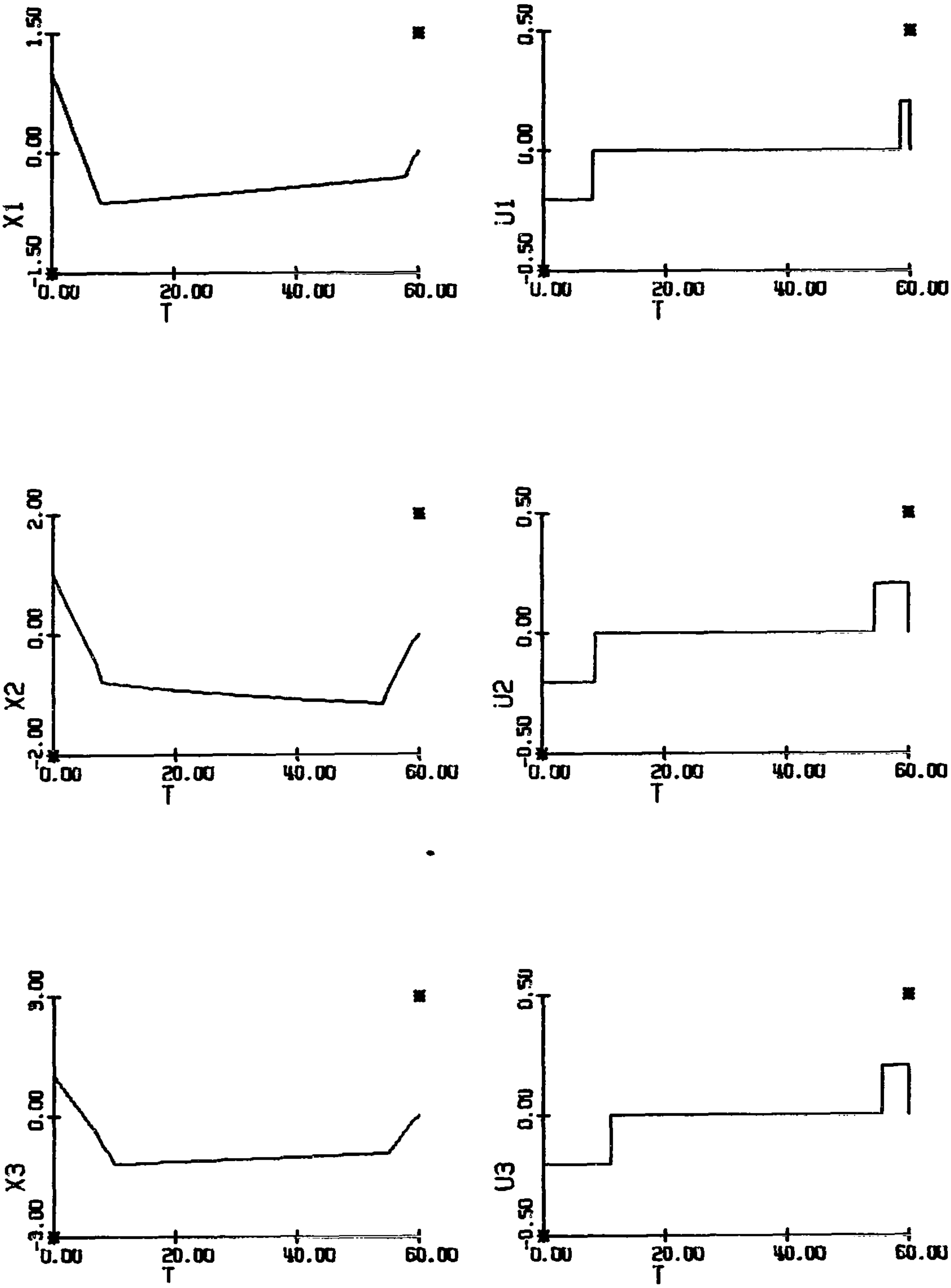


Figure 5-4b Optimal Response For Initial Conditions of Run R-1; U_i=.206 deg./sec^2; t_f=60 sec

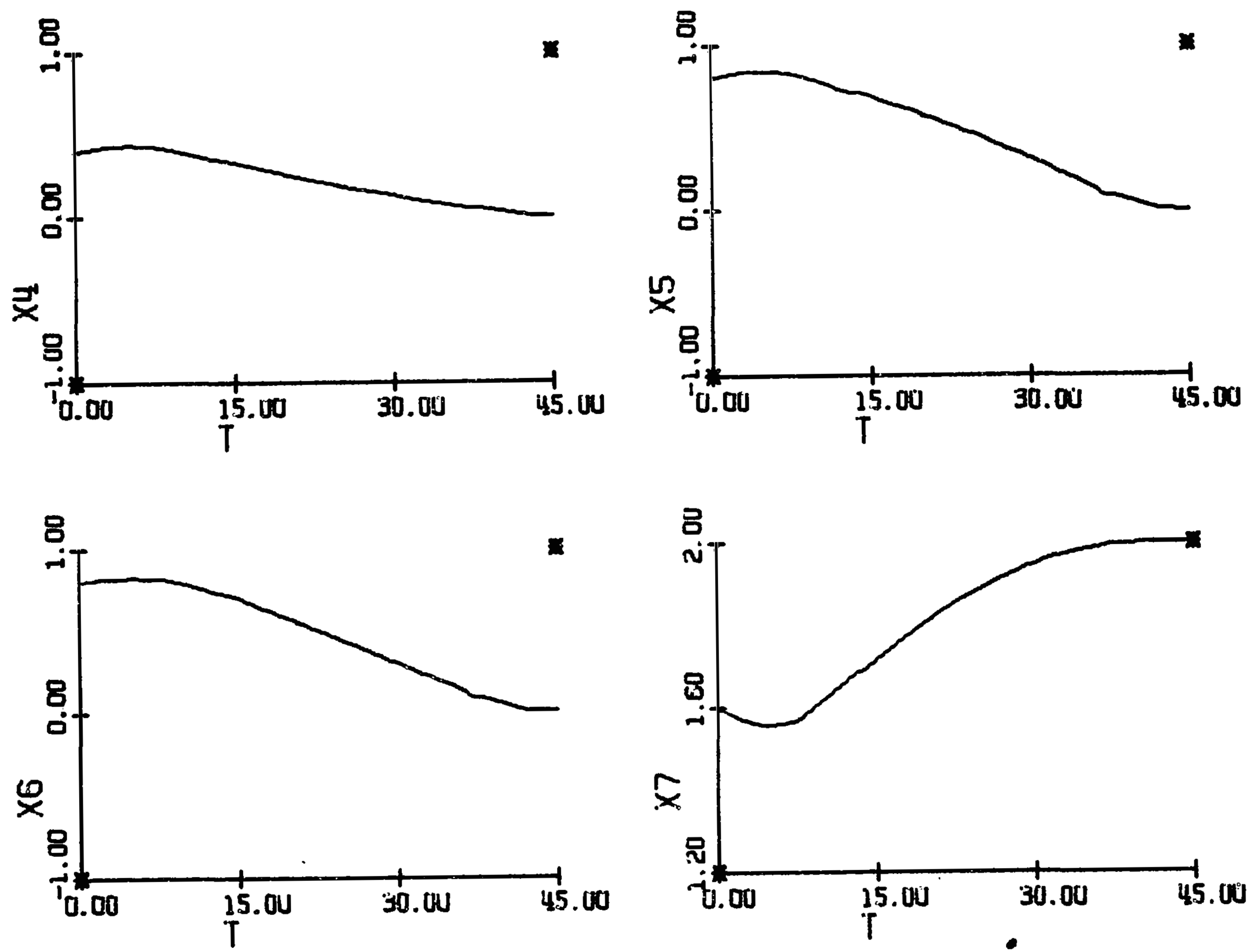


Figure 5-5a Optimal Response for Run R-1 with
U_i=.206 deg./sec^2 and with t_f=45 sec

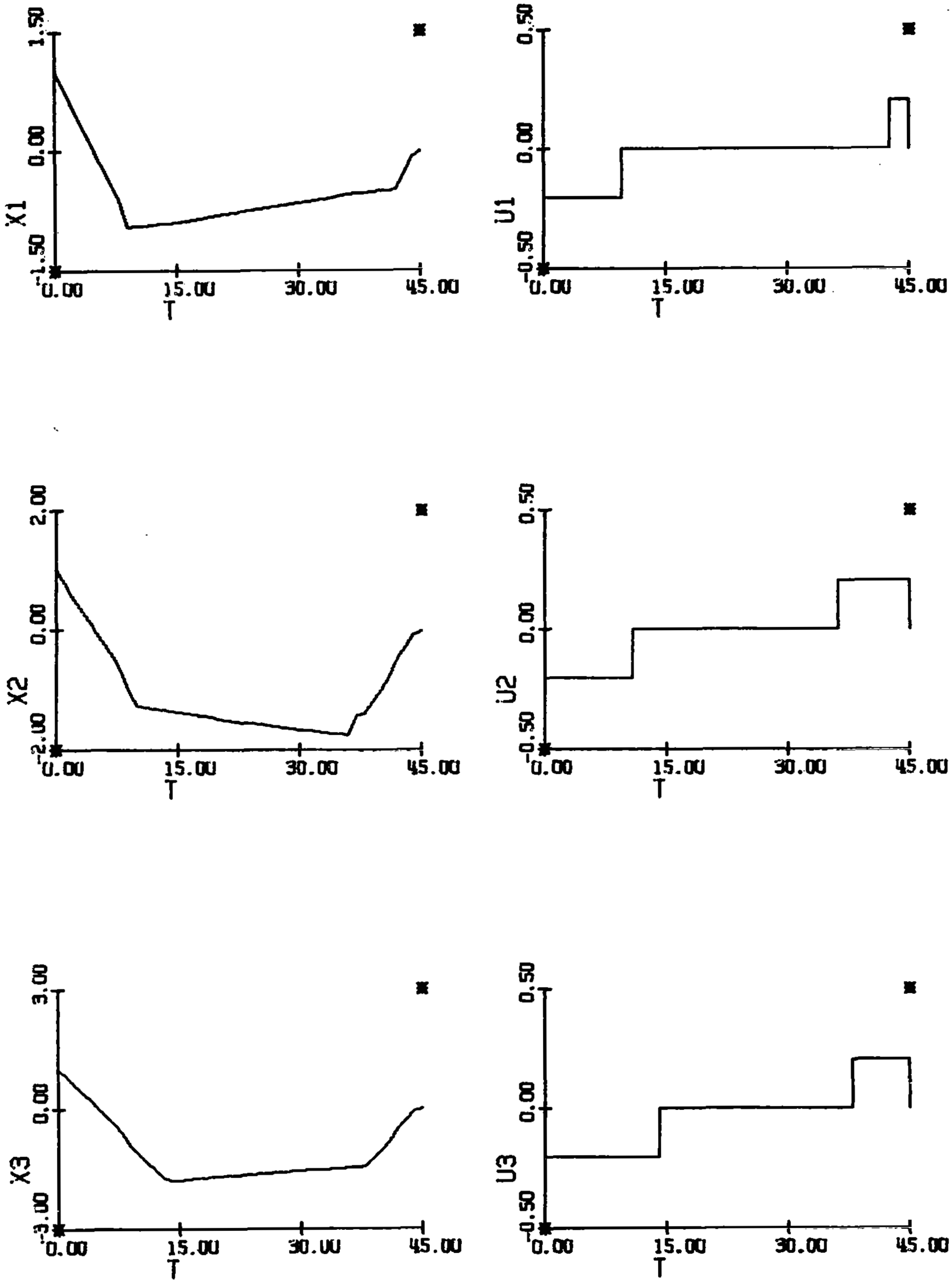


Figure 5-5b Optimal Response for Run R-1 with U_i=.206 deg./sec^2 and with t_f=45 sec

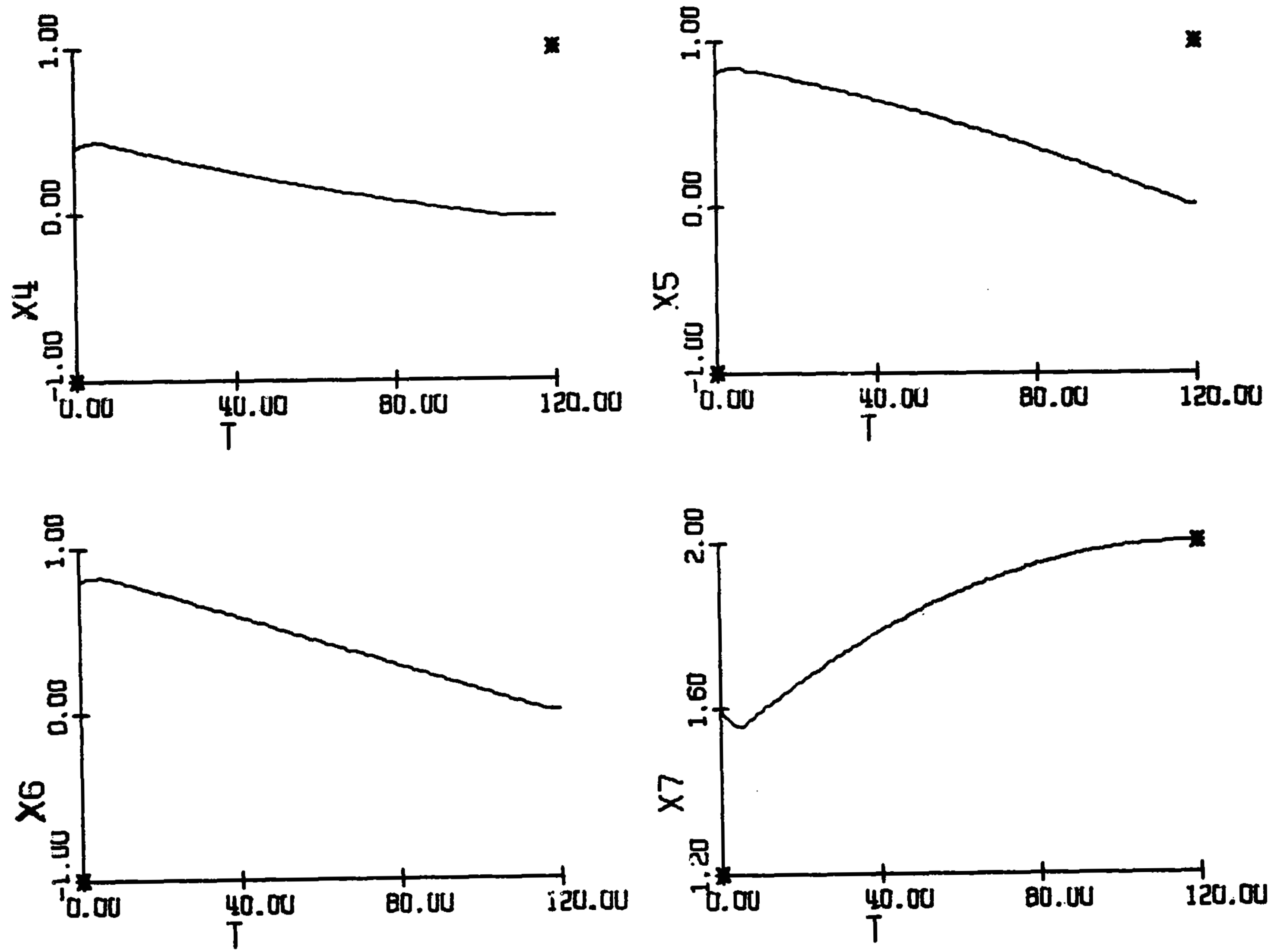


Figure 5-6a Optimal Response for Run R-1 with
U_i=.206 deg./sec^2 and with t_f=120 sec

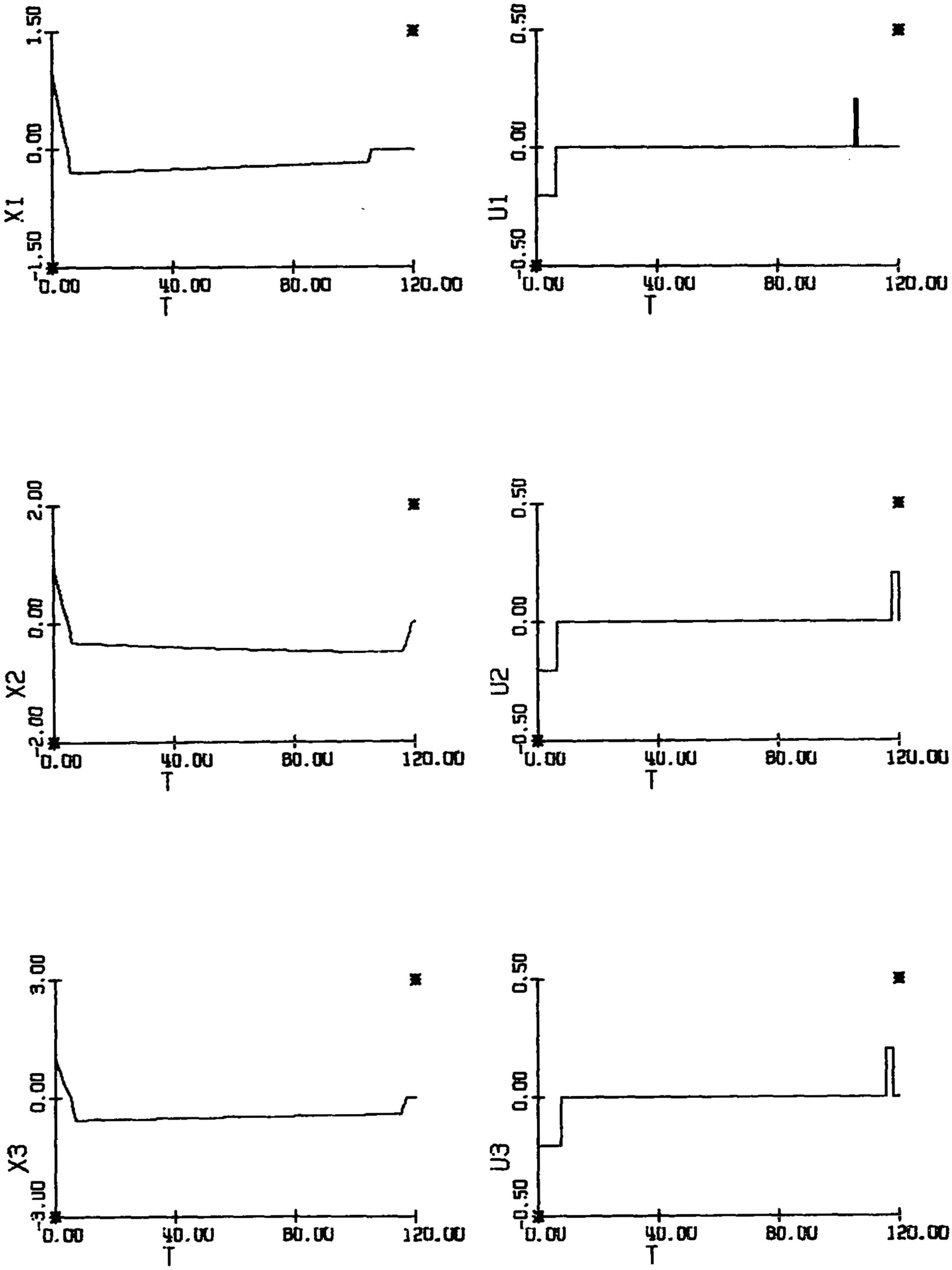


Figure 5-6b Optimal Response for Run R-1 with U_i=.206 deg./sec^2 and with t_f=120 sec

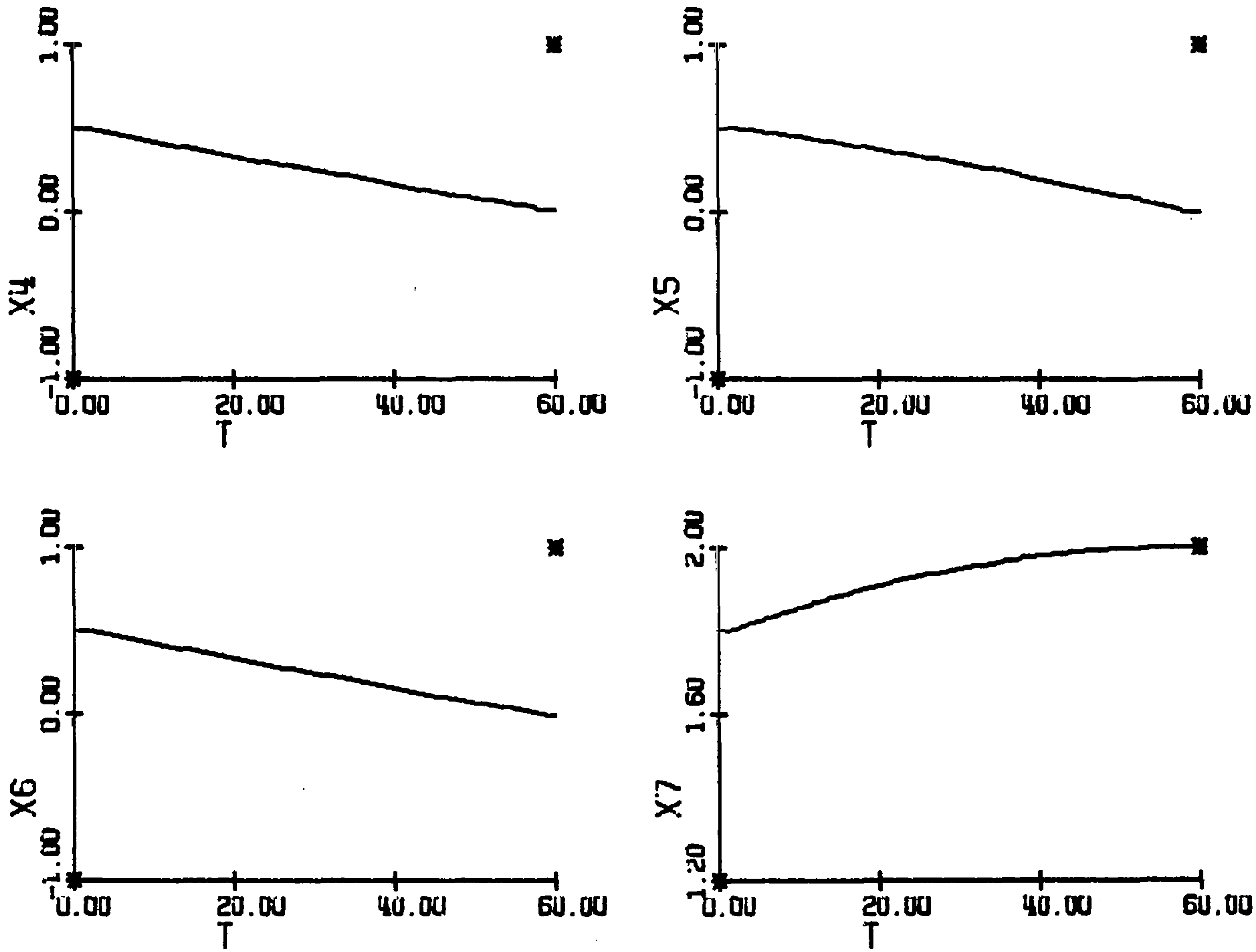


Figure 5-7a Optimal Response Using Initial Conditions of Run R-2

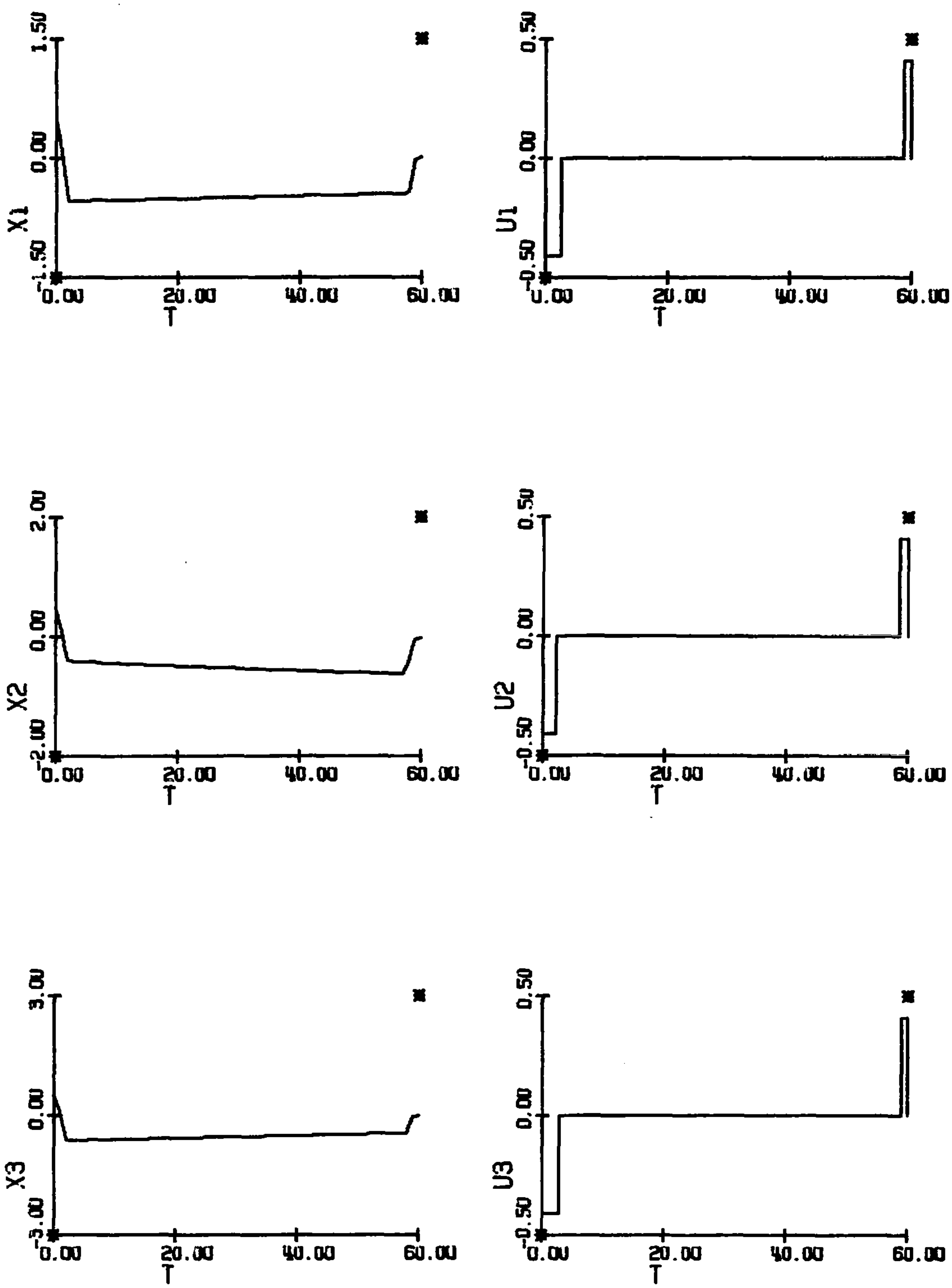


Figure 5-7b Optimal Response Using Initial Conditions of Run R-2

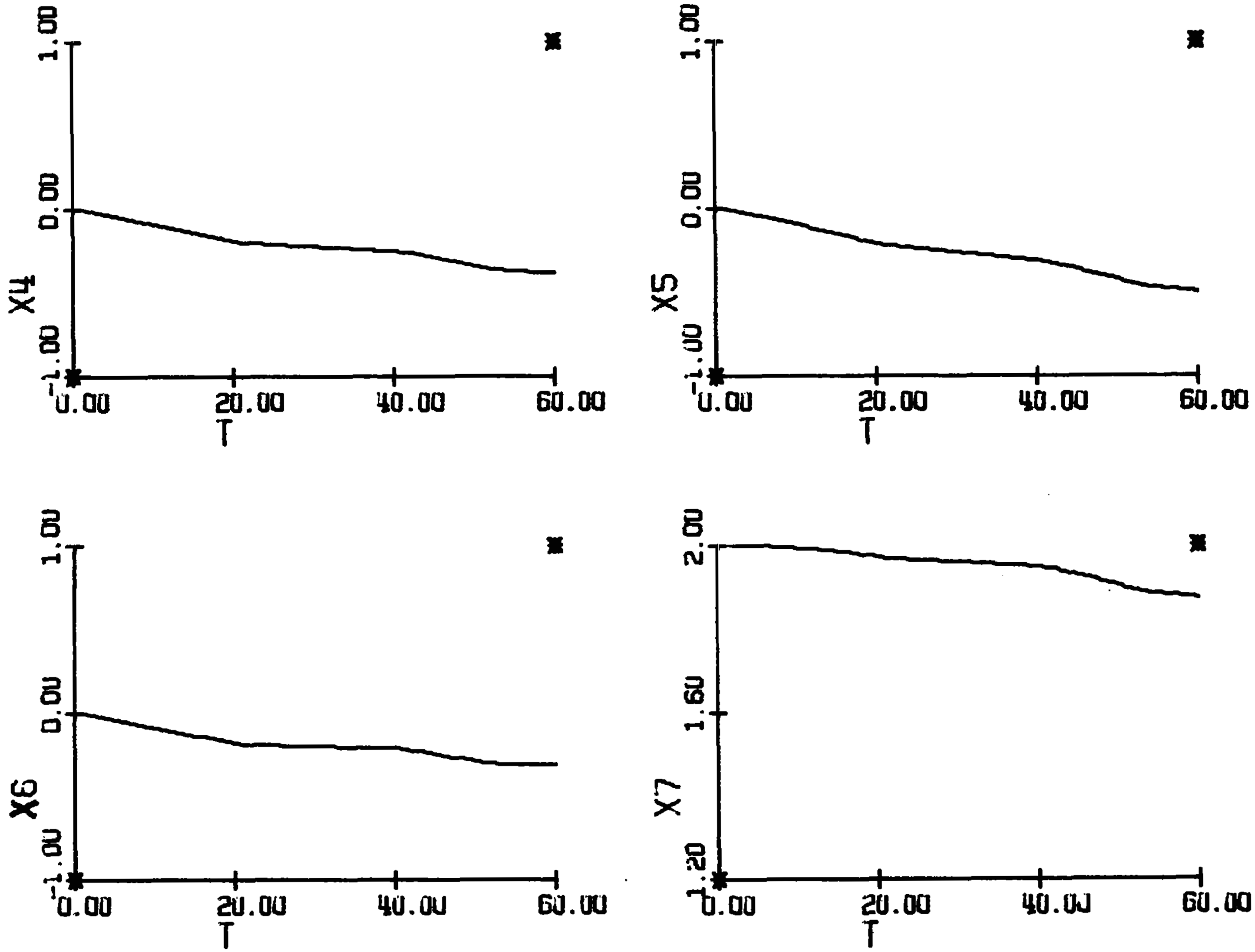


Figure 5-8a Response to Nominal Control Using Initial Conditions of Run R-3, i.e. $\underline{x}(t_o)=\underline{0}$

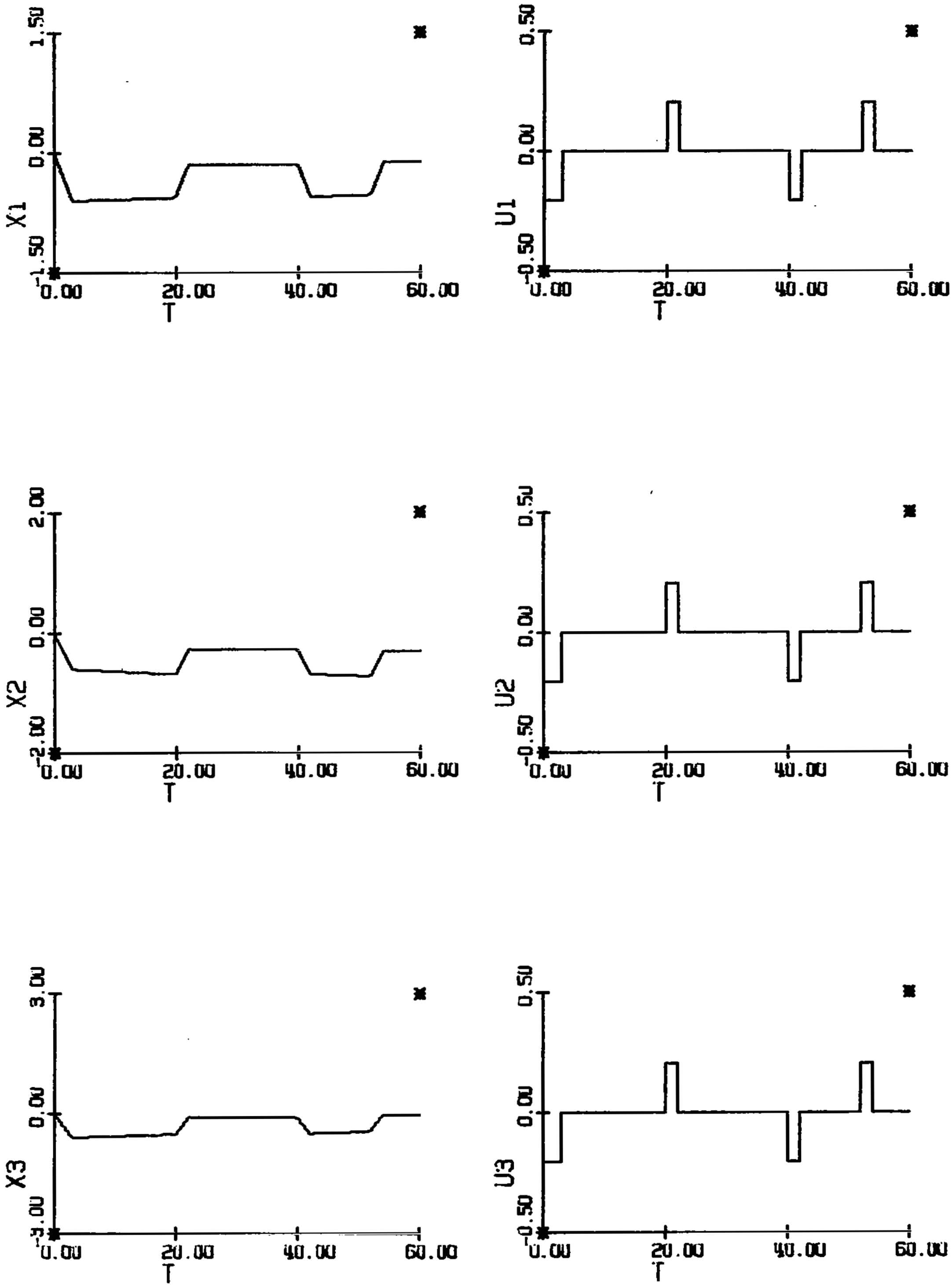


Figure 5-8b Response to Nominal Control Using Initial Conditions of Run R-3, i.e. $\underline{x}(t_o)=\underline{0}$

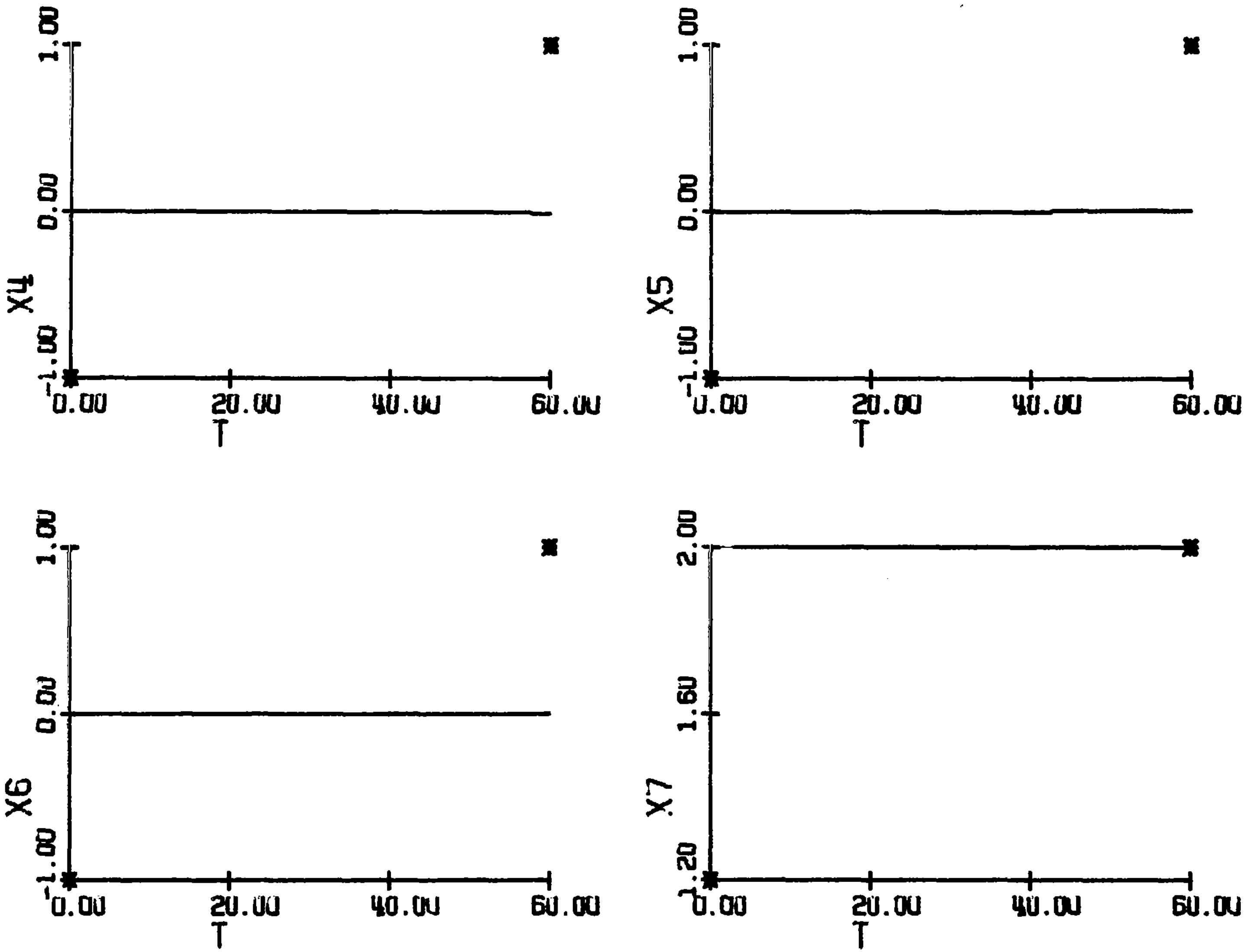


Figure 5-9a Optimal Response for Case in which $\underline{x}(t_o)=\underline{0}$

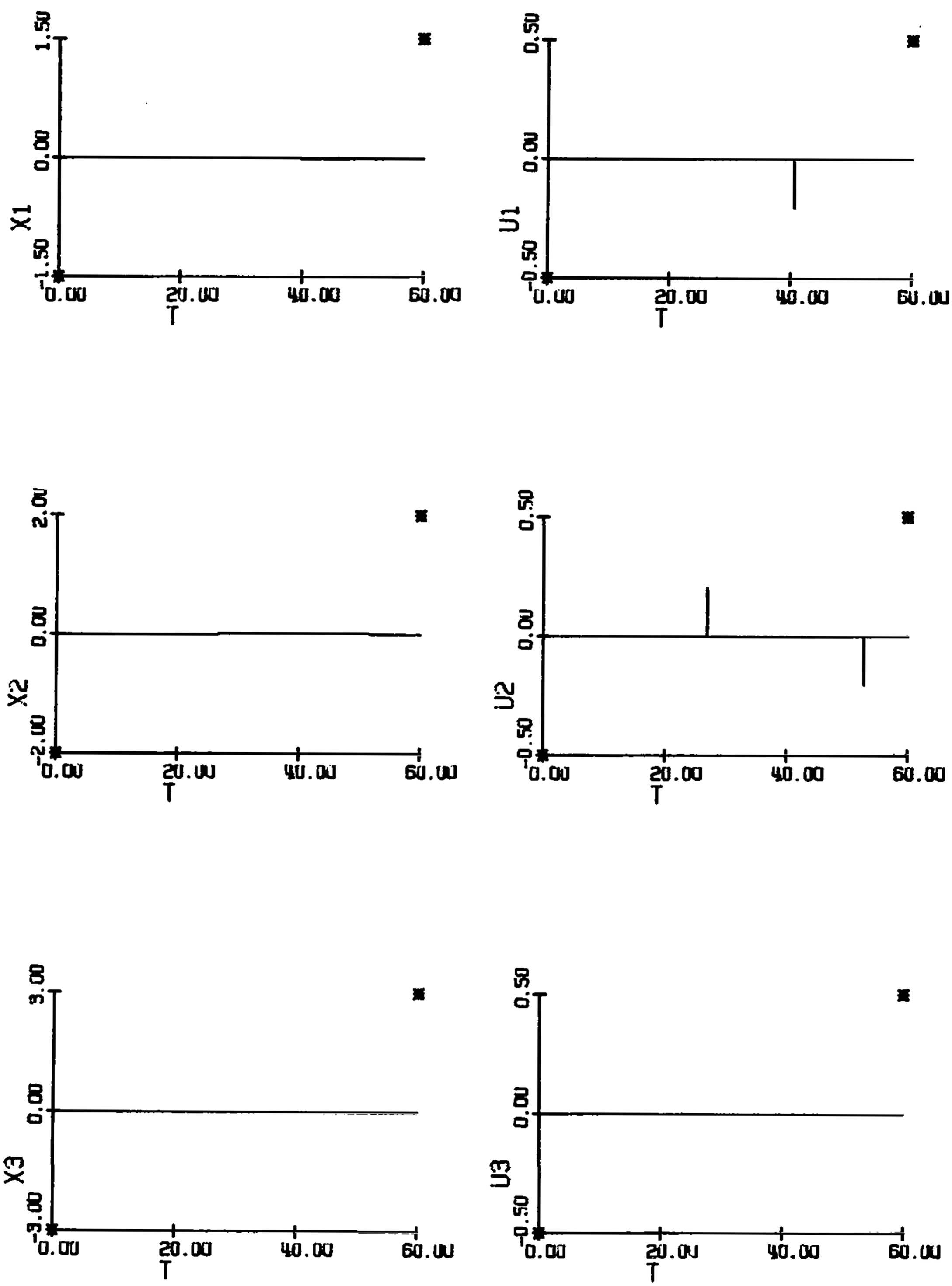


Figure 5-9b Optimal Response for Case in which $\underline{x}(t_o)=\underline{0}$

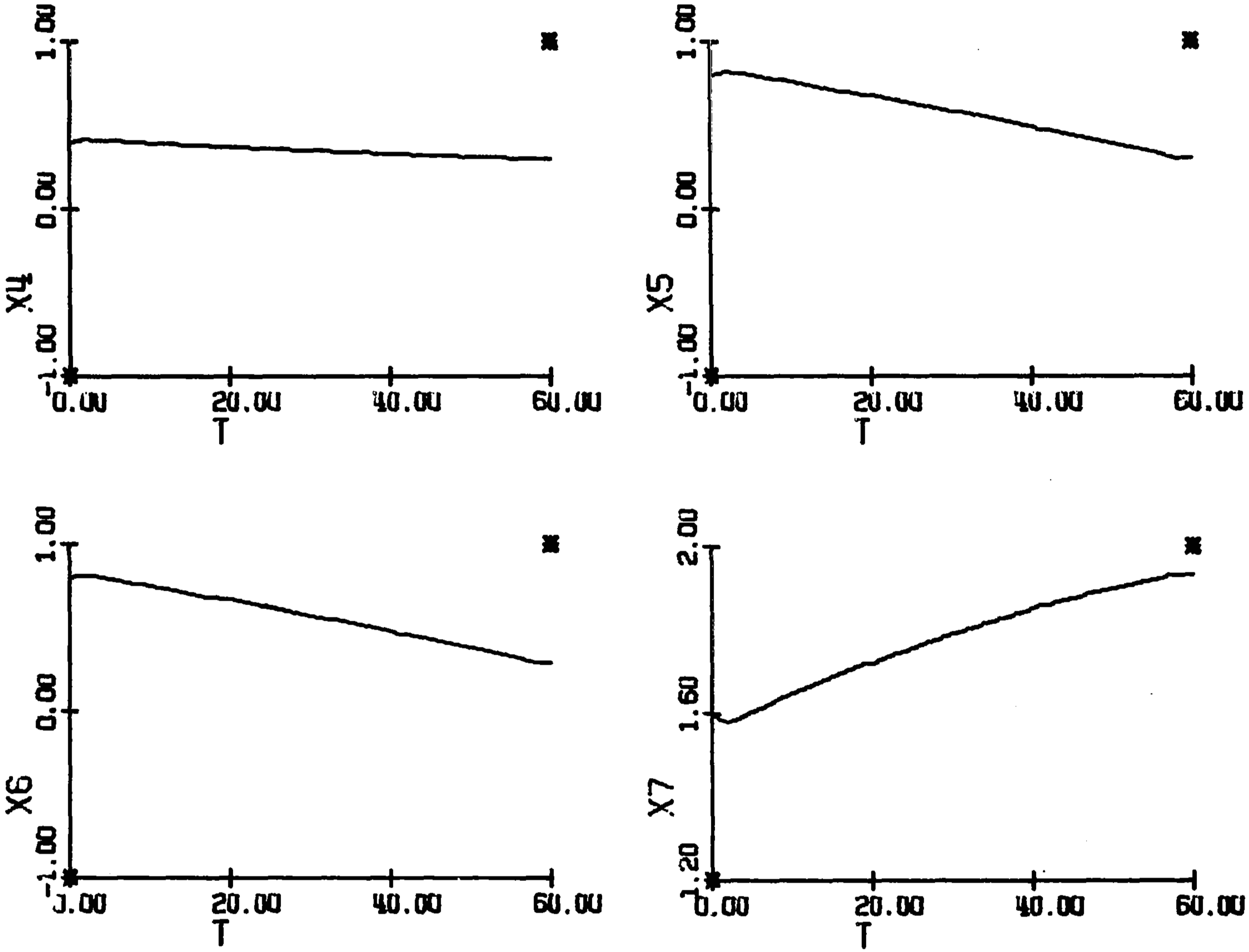


Figure 5-10a Optimal Control and State for Example in which Terminal State Constraints are Nonzero, i.e. $\underline{x}(t_f)=.\underline{3}$

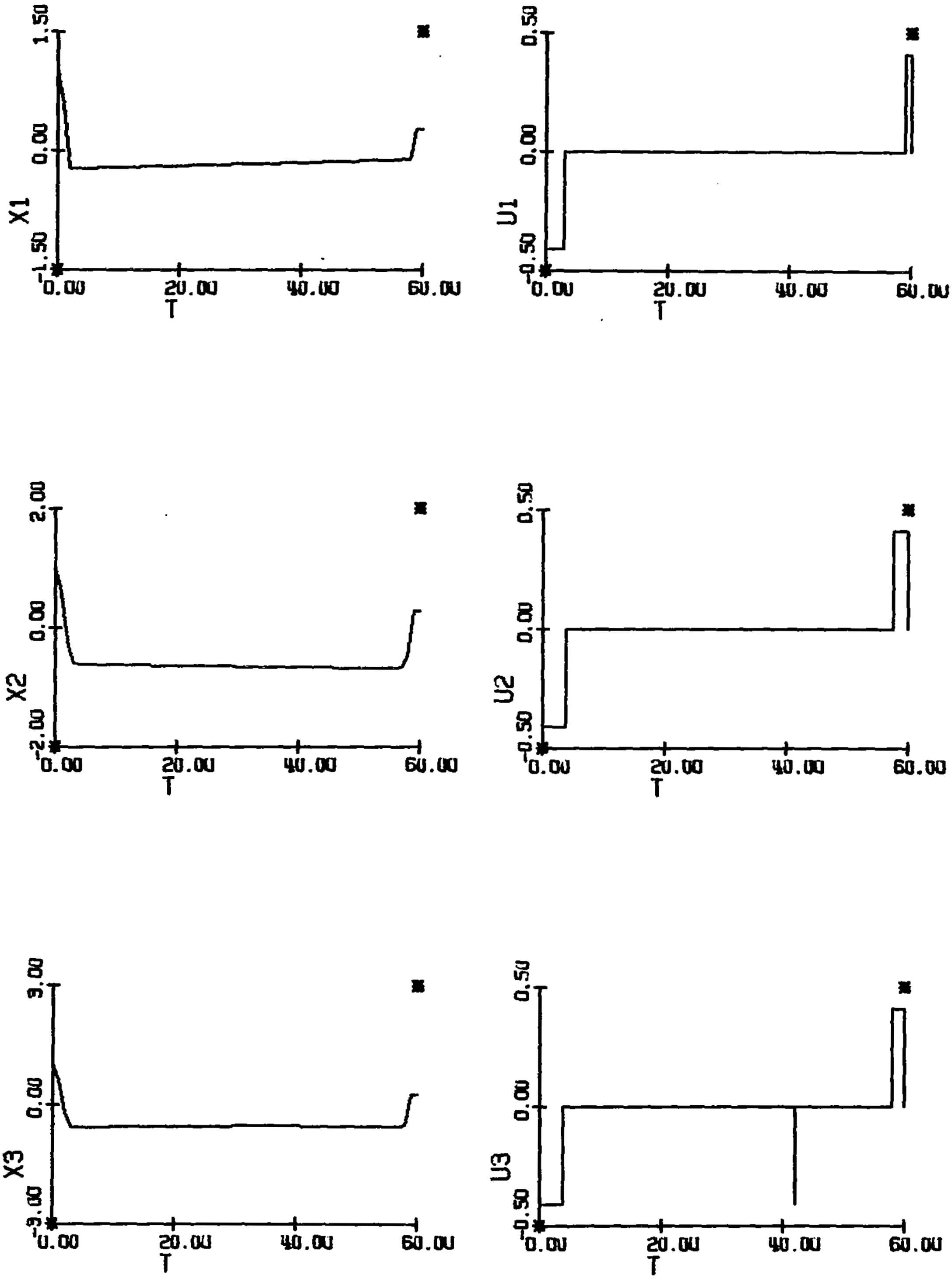


Figure 5-10b Optimal Control and State for Example in which Terminal State Constraints are Nonzero, i.e. $\underline{x}(t_f)=.\underline{3}$

VI. MINIMUM TIME ACQUISITION PROBLEM

An indirect approach using the method of Chapter 4 can be used to solve the minimum time attitude control problem.

In this problem, the objective is to determine the control to reach a given terminal state, $\underline{x}(t_f)$ from a given initial state, $\underline{x}(t_o)$, in minimum time. In the context of Chapter 2, the integrand of the functional defined in equation 2-10 becomes

$$f_o(\underline{x}(t),\underline{u}(t),t) = 1 \qquad (6\text{-}4)$$

for the minimum time problem. All other aspects are identical to the minimum fuel problem as presented in Chapter 2. If equation 6-1, is substituted into equation 2-7, the Hamiltonian, H, is given as

$$H = \sum_{i=1}^{n} \lambda_i(t) \sum_{j=1}^{3} c_{ij}(\underline{x},t)u_j(t) \quad + \text{ terms not involving } \underline{u} \qquad (6\text{-}2)$$

Maximization of H with respect to the control, bearing in mind that the control magnitude is bounded, gives

$$u_j(t) = U_j \text{ Sgn } \sum_{i=1}^{n} \lambda_i(t)C_{ij}(\underline{x}(t),t) \qquad j=1,2,3 \qquad (6\text{-}3)$$

A control of this nature is referred to as "bang-bang" and is always "on" at its maximum value with the polarity being either positive or negative.

The dynamics equations used to work the example of this chapter are the time invariant high torque equations of Chapter 5.

A. INDIRECT ALGORITHM FOR TIME OPTIMAL PROBLEMS

Before giving the indirect algorithm used here to solve the minimum time problem we consider two ideas which will clarify the reasoning for this "indirect" algorithm.

The first idea is best presented by referring to the control histories presented in Figures 5-5b and 5-6b. These examples were identical with the exception that $t_f = 45$ seconds for one case and $t_f = 120$ seconds for the other. In both examples, $u_3(t)$ is the control component which is "on" for the longest time. The time during which $u_3(t)$ is "off" is much shorter for the case in which $t_f = 45$ seconds than it is for the case in which $t_f = 120$ seconds. The reasons for this are two-fold. In the first place, the cost of the optimal fuel solution for $t_f = 120$ seconds can be no higher than the optimal fuel solution for the case in which $t_f = 45$ seconds, hence trivially, the time in which the control is off is larger for the case in which the final time is larger. The second reason is that, in general, as the parameter t_f is lowered while leaving everything else fixed, the cost usually increases--and increases rapidly as t_f approaches the minimum value for which a feasible solution exists (this gains credence if one considers the well known analytical fuel optimal solution for the classical "$1/s^2$ problem).

The second idea is presented as a proposition. This proposition concerns the fuel optimal problem as presented in Chapter 2.

Proposition: If the optimal fuel control for a given arbitrary set of initial conditions, has the property that all of the components are "on" for all time t, then that control is also the time optimal control. (The final value constraints on the state must be an equilibrium point).

Proof of Proposition: Assume the fuel optimal solution is not time optimal. Then there exists a t_1 such that $t_1 < t_f$ for which a feasible solution to the problem may be found. Because the final value constraint on the state is an equilibrium point, it follows that this solution is a feasible solution for the problem in which the final time is t_f rather than t_1. But since the assumed optimal fuel control has its control completely "on" for a time t_f with $t_f > t_1$, its fuel cost is larger than the fuel cost associated with the other feasible solutions ending t_1 and it is therefore not a fuel optimal solution.

This proposition guarantees that if a fuel problem is solved and has the property that all of the components of the control are "on" for all of the time, then a minimum time solution has also been found. In working examples, we will assume that if we find a minimum fuel problem for which the control is "on" most of the time, then the final time for this problem is a good approximation to the minimum time problem.

Using the two ideas presented above, the following scheme to compute time optimal controls is proposed. Solve the minimum fuel problem for a value of final time which is larger than the minimum time since the minimum time is not known, it may be necessary to increase the value of the final time and repeat the computation. After a minimum fuel solution is found for some value of t_f, a smaller value of t_f is taken and a new minimum fuel solution is computed for the new (smaller) value of t_f. This is repeated until a solution is found for which the control is "on" for the entire period from t_o to t_f. Then, by the proposition, it is known that this control is time optimal. Since converging to the exact time optimal would be difficult and coincidental, one would normally only continue lowering t_f and repeating the computational scheme until the control was on for almost all of the time. The question of how much to lower t_f each time a new minimum fuel solution is computed has not been mentioned. It must be remembered from the first idea above that the period during which the controllers are off may be drastically shortened by lowering the final time by just a small value-especially when the final time is near its minimum possible value. In general, one might lower the final time, t_f, by about one-sixth of the "off" period of the component of the control with the least "off" period.

B. NUMERICAL EXAMPLE OF MINIMUM TIME PROBLEM

In Chapter 5, Figures 5-6, 5-4, and 5-5 give the fuel optimal solutions for the same example for values of t_f of 120 seconds, 60 seconds, and 45 seconds respectively. A solution which is relatively close to the minimum time solution will be computed for that example in this section. Lowering the final time to 39 seconds yields the solution of Figures 6-1a and 6-1b. Although there is a fairly large period (about 10 seconds) during which the third control component $u_3(t)$, is "off", even in lowering t_f by only two seconds one can see by Figures 6-2a and 6-2b that the control cost has increased and that $u_3(t)$ is on for almost the entire time interval $[t_o, t_f]$. The procedure was attempted for a case in which $t_f = 35$ seconds, but as could be expected from what has been stated in the last section, a feasible solution could not even be found. Hence, $t_f = 37$ seconds (Figures 6-2a and 6-2b) is taken as the approximate time optimal solution.

The necessary condition on the time optimal control in equation 6-3 stated that each component of the control is "on" with either a positive or negative polarity for the entire time interval, $[t_o, t_f]$. Yet in the time optimal solution of Figure 6-2a, large gaps of zero control are indicated for $u_1(t)$ and $u_2(t)$. One expects from the result in eqn. 6-3 that the proper bang-bang control (with no intervals of coasting) for $u_1(t)$ and $u_2(t)$ would be able to lower t_f below the 37 seconds of the above example. As mentioned earlier, though, the approach used in this chapter to solve the minimum time problem is not exact.

There do exist cases, however, in which time optimal control histories for the class of problems considered here may have intervals of coasting. The following sixth order system is such an example.

$$\dot{x}_1 = x_2 \qquad x_1(0) = 1$$

$$\dot{x}_2 = u_1 \qquad x_i(0) = 0;\ i=2,3,\ldots 6$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = u_2$$

$$\dot{x}_5 = x_6$$

$$\dot{x}_6 = u_3$$

The constraints at $t = t_f$ are that $\underline{x}(t_f) = 0$ and $|u_j| \leq 1$, $j=1,2,3$. It follows that the minimum time solution is $t_f = 2$ with one of many possible optimal controls given as:

$$u_1(t) = -\mathrm{Sgn}(1-t)$$

$$u_2(t) = 0 \qquad\qquad 0 \le t \le 2$$

$$u_3(t) = 0$$

Hence it is seen that in the family of time optimal controls, one of the many possible solutions is that two of the components of the control are zero for the entire interval $[t_o, t_f]$. Other time optimal solutions can easily be obtained for this linear example in which some of the control components have non-zero periods of pulse control with periods of zero control between the pulses.

The results of the example of this chapter along with those Figures 5-4, 5-5, and 5-6 are tabulated in Table 6-1. The fuel cost is plotted against the terminal time, t_f, in Figure 6-3. The initial time, t_o, was zero in all cases. The costs obtained by Hales for these same examples are also shown. In general, Hale's costs were higher than those obtained in this report. In examples worked by both methods, it is apparent from this graph how the fuel cost rises as the terminal time parameter is lowered.

In addition to Euler Parameters, another way of describing the three dimensional orientation of a body with respect to a reference frame is the three-axis Euler Angles description. Definitions and illustrations of three-axis Euler Angles are given in (reference 13). Figure 6-4 gives the Euler Angle description of examples 6-1 and 6-2. E-1, E-2, and E-3 (expressed on a scale from -100 degrees to +100 degrees), correspond to X_4, X_5, and X_6, respectively.

Although the minimum time problem is solved in this chapter only approximately, the method has a certain practical merit in view of the fact that one should be able to make a judicious choice for the "nominal" control each time (after the initial fuel optimal computation) that t_f is lowered, and hence, save computation time.

Final Time, t_f	Cost By Hales Extended Method Of Steepest Descent	Cost By Method Based On Linear Programming	See Figure:
120 Sec.	.1024 $Sec.^{-1}$	.093 $Sec.^{-1}$	5-6-a 5-6-b
60 Sec.	.1595	.142 $Sec.^{-1}$	5-4-a 5-4-b
45 Sec.	.1969	.154 $Sec.^{-1}$	5-5-a 5-5-b
39 Sec.	-	.241 $Sec.^{-1}$	6-1-a 6-1-b
37 Sec.	-	.258 $Sec.^{-1}$	6-2-a 6-2-b

Data For Figure 6-3

Table 6-1

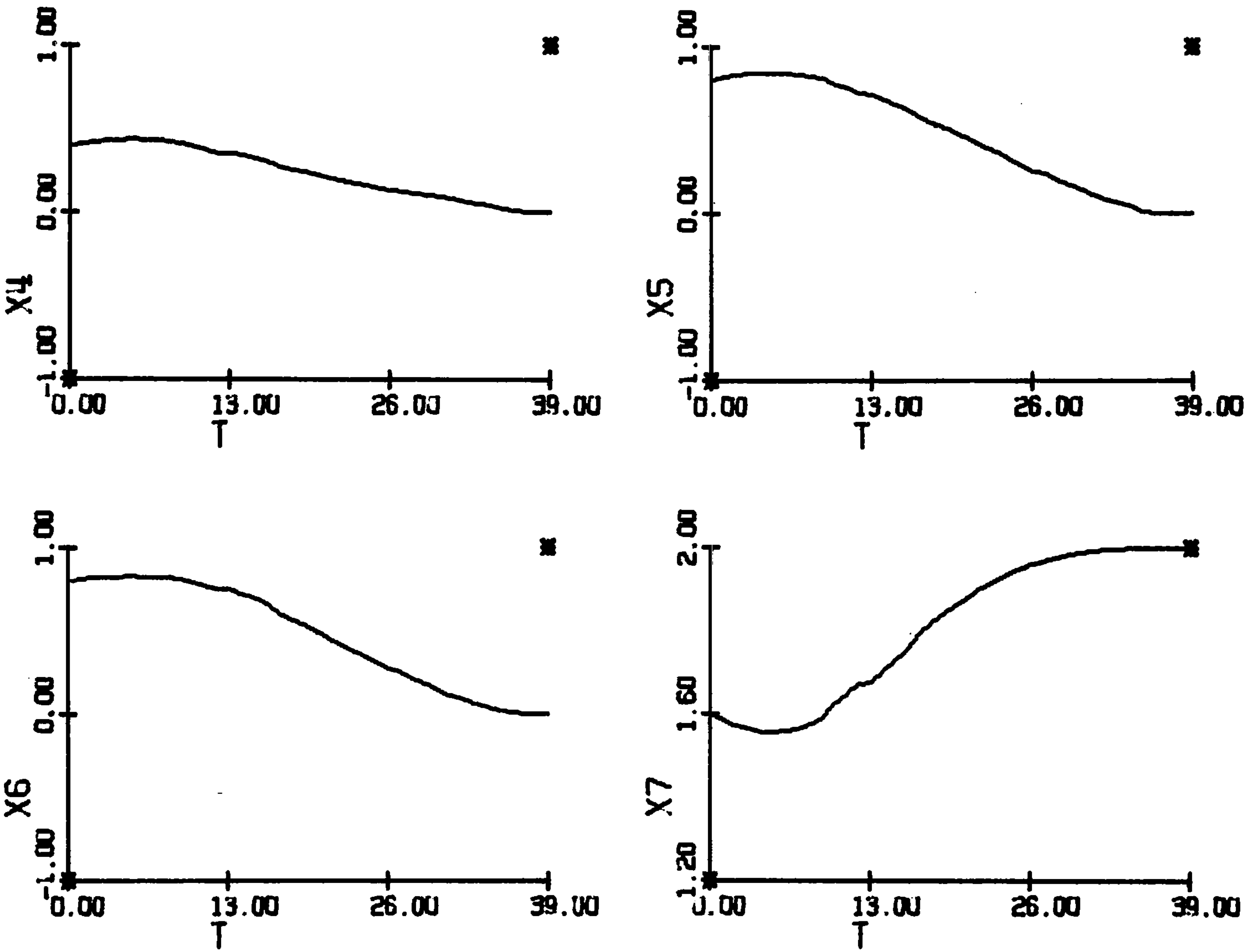


Figure 6-1a Optimal Response for Run R-1 with U_i=.206 deg./sec^2 and t_f=39 sec

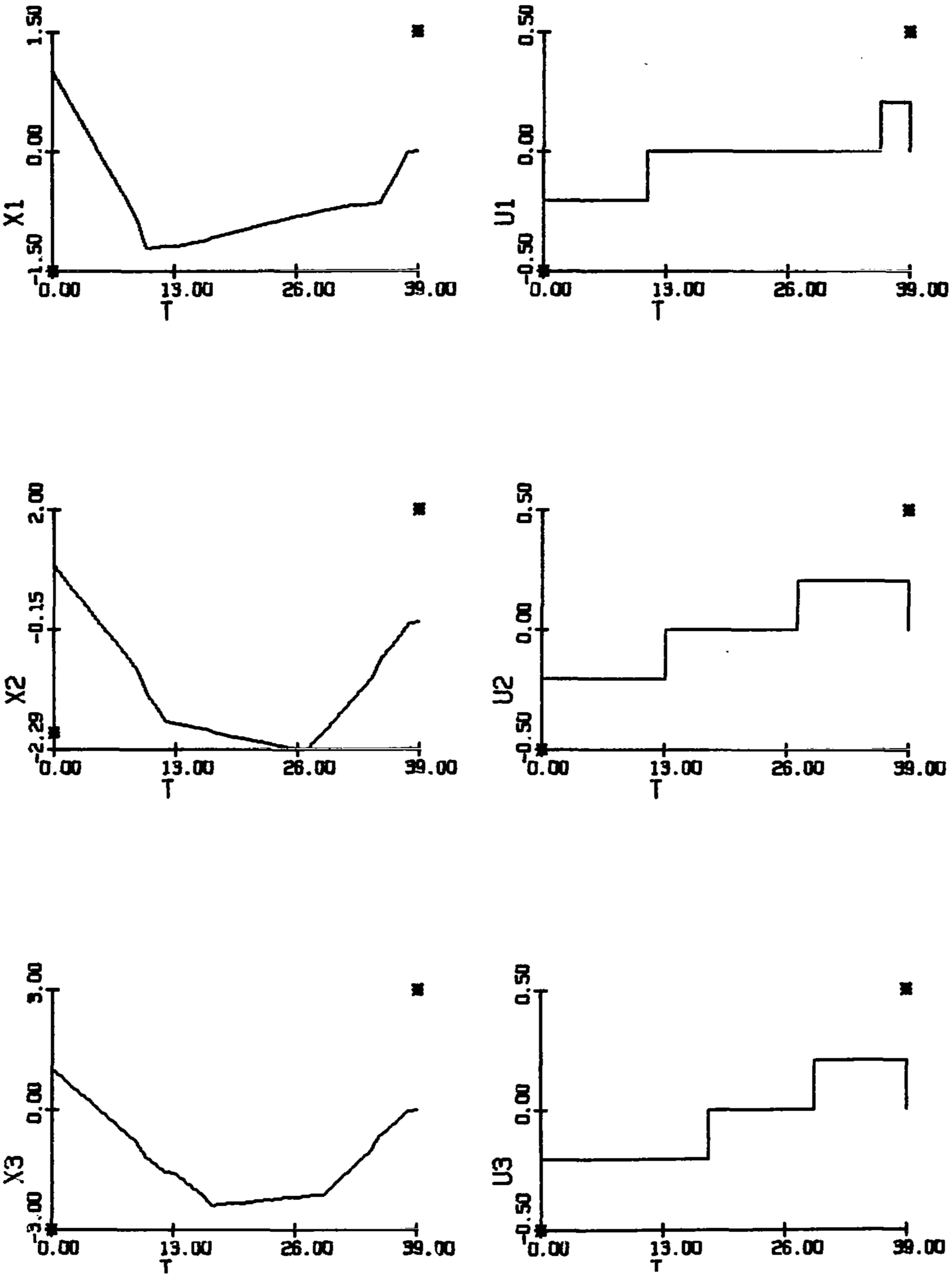


Figure 6-1b Optimal Response for Run R-1 with U_i=.206 deg./sec^2 and t_f=39 sec

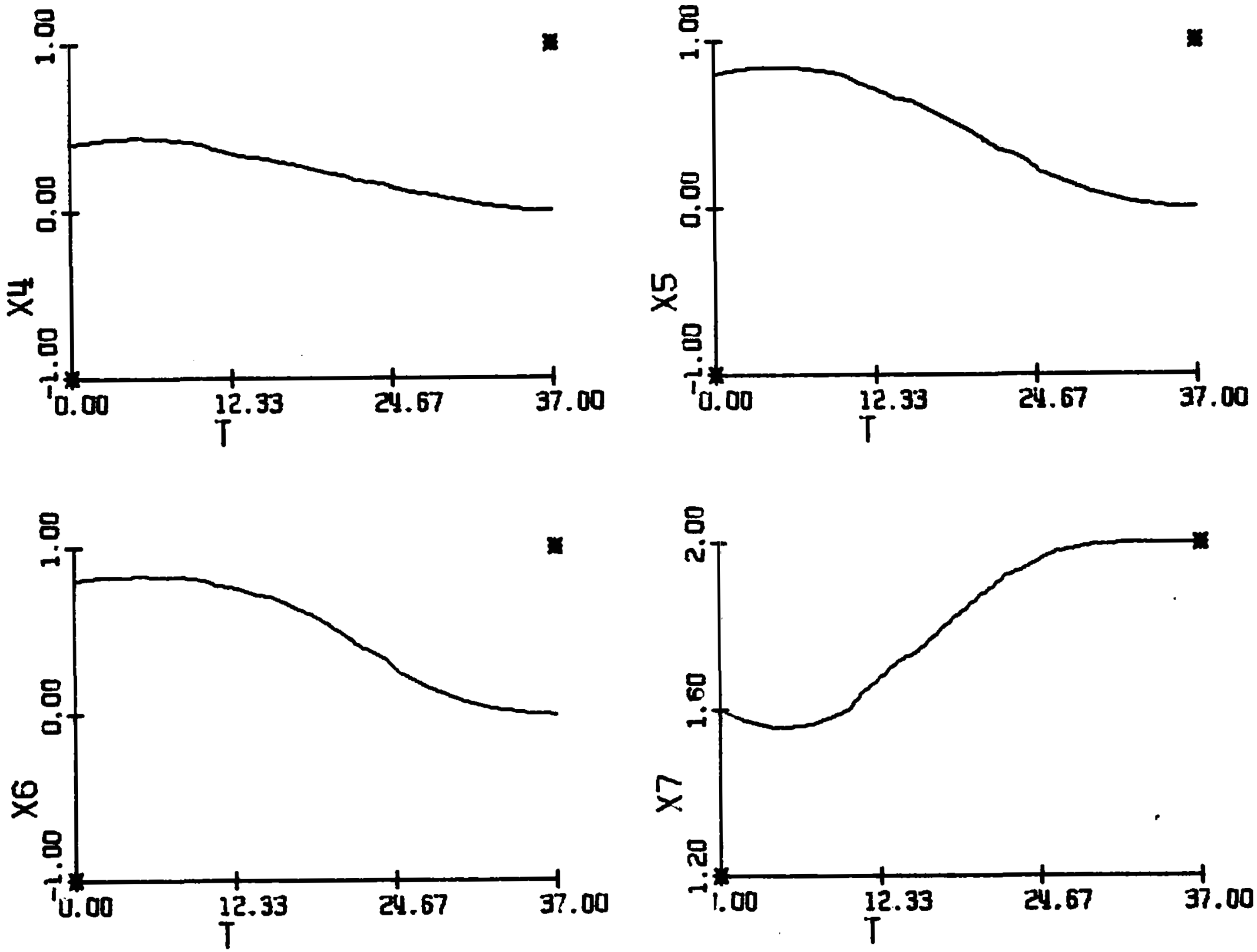


Figure 6-2a Approximate Time Optimal; t_f=37 sec

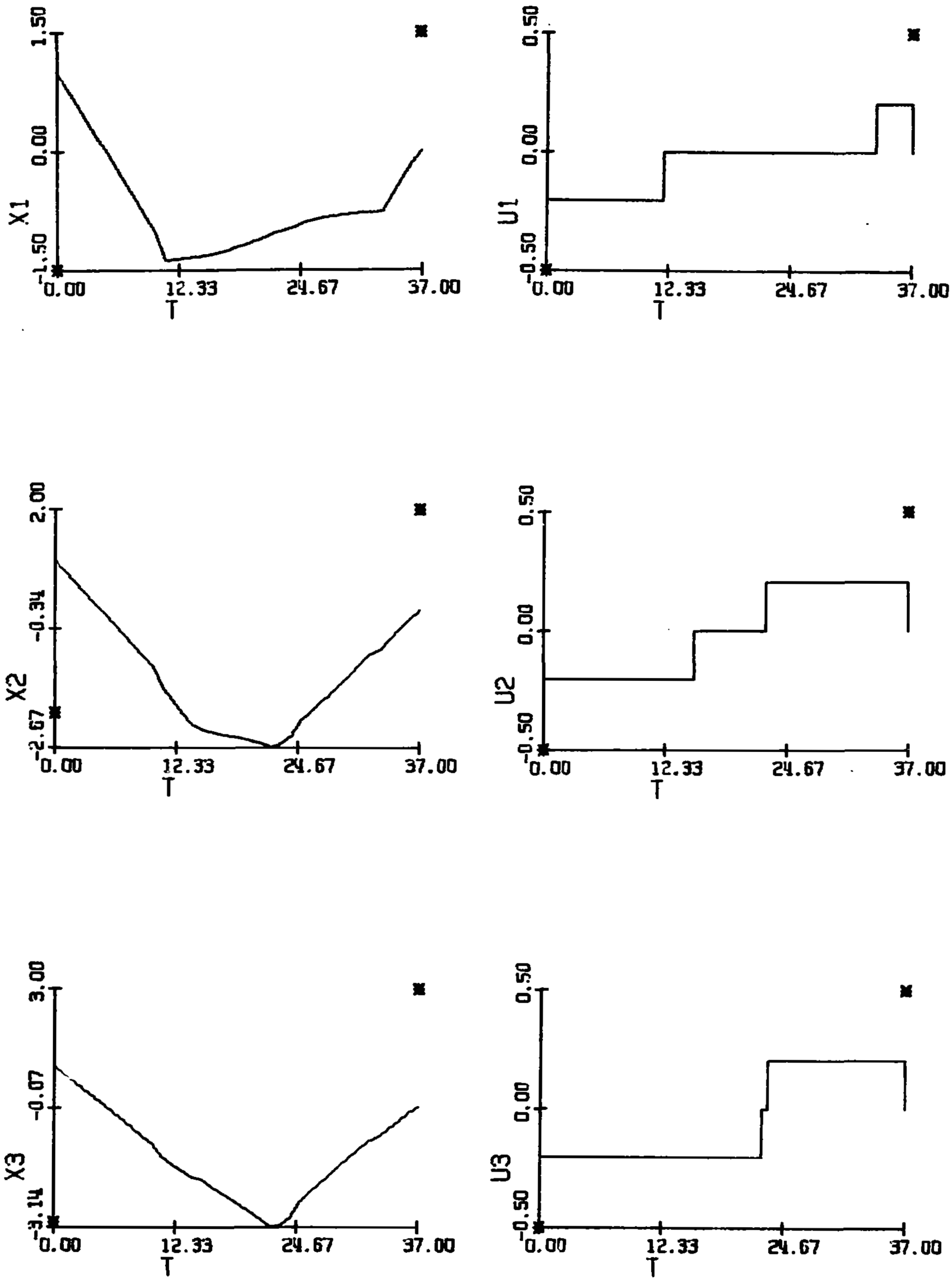


Figure 6-2b Approximate Time Optimal; t_f=37 sec

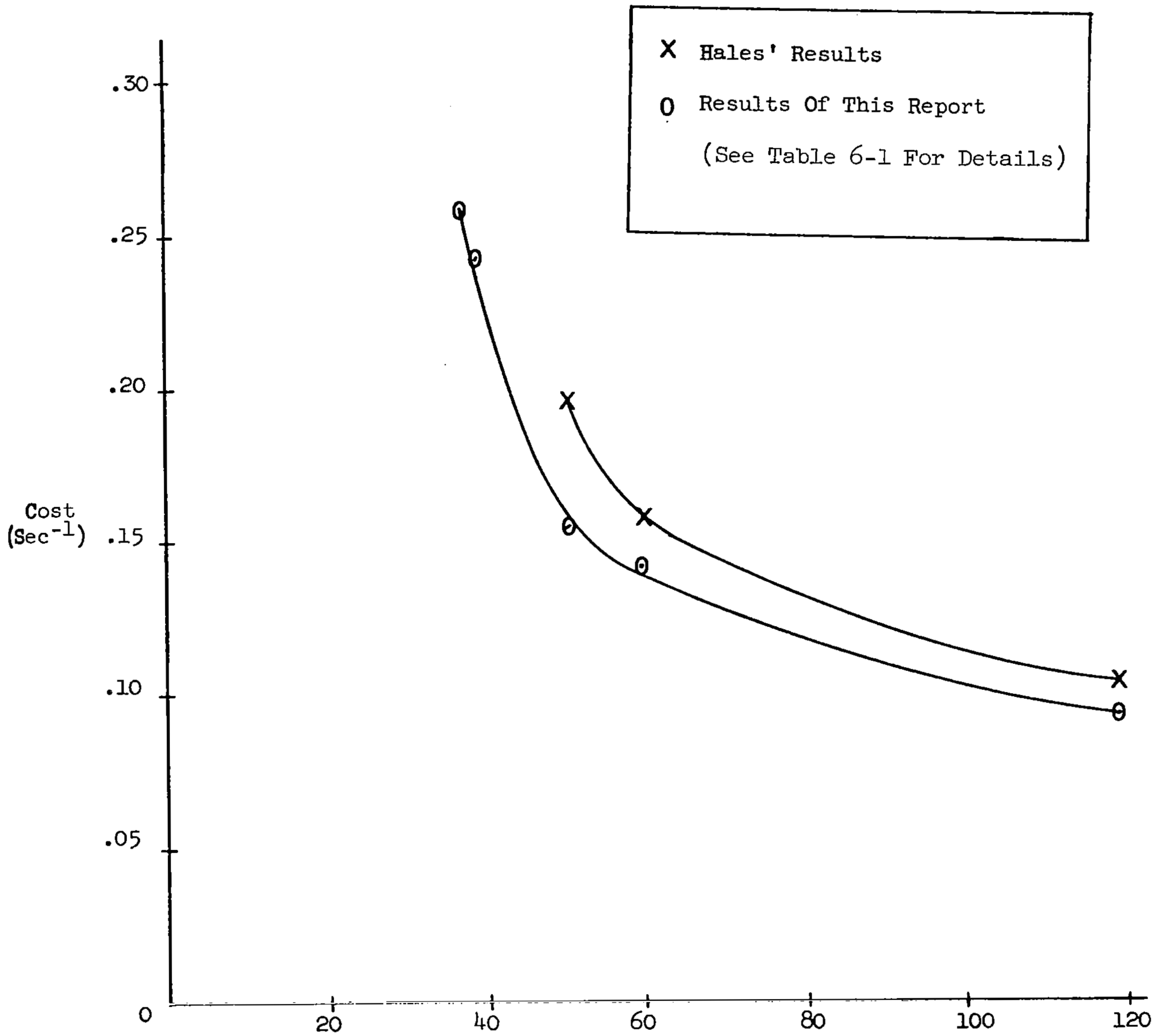


Figure 6-3. Fuel Cost vs. Final Time with Initial Conditions of Run R-1; $U_i(t) = .206 \frac{\text{deg.}}{\text{sec.}^2}$

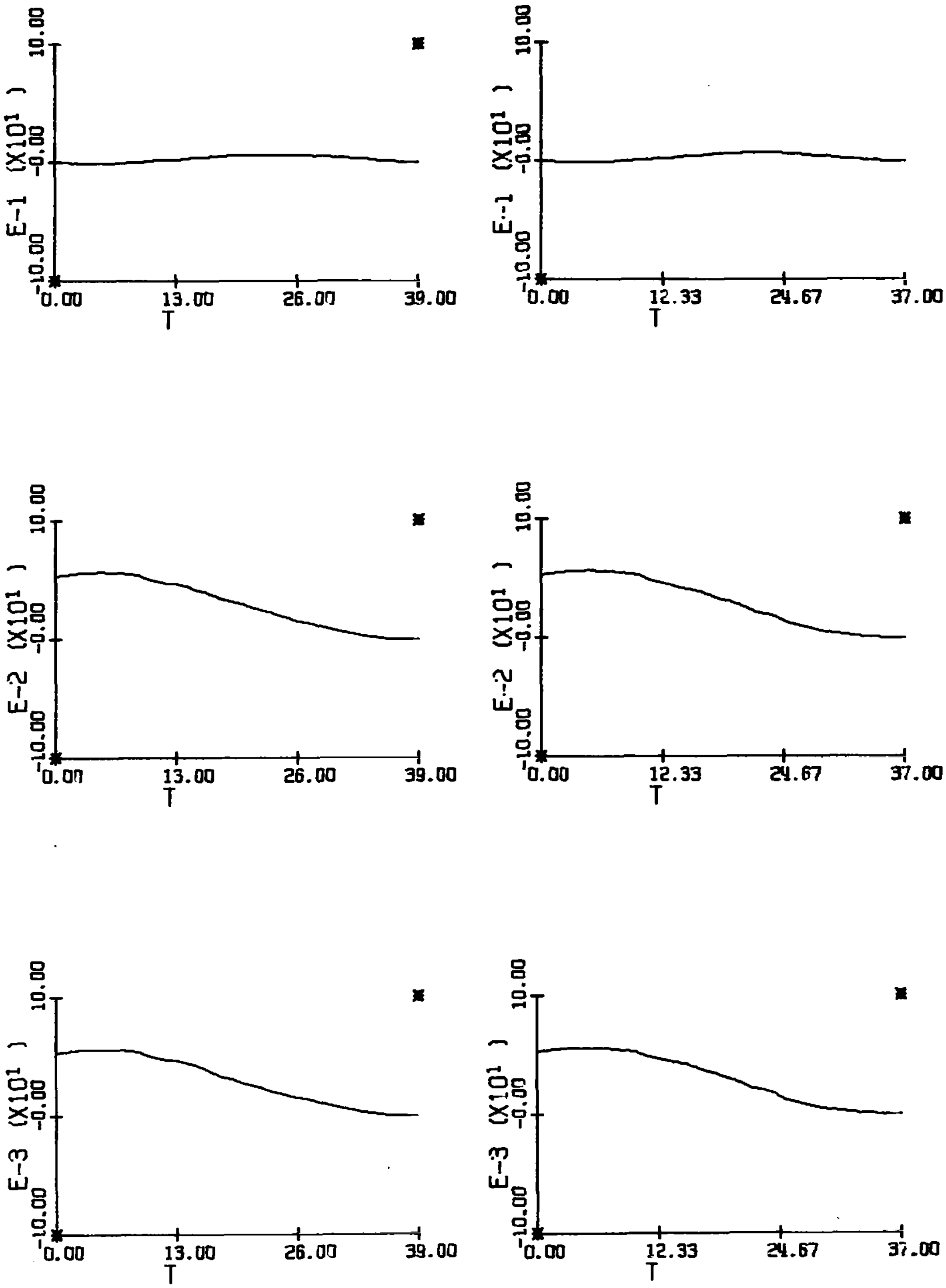


Figure 6-4 3-Axis Euler Angles for Euler Parameters of Figures 6-1 and 6-2

VII. LOW TORQUE ACQUISITION PROBLEM

In this chapter, fuel optimal controls are computed for the same satellite system considered in Chapter 5. In this case, however, the torque levels of the control jets are low enough to preclude the possibility of omitting the effects in the dynamics equations of the gravity gradient torque and the orbital motion of the satellite about the earth.

A. LOW TORQUE DYNAMICS AND SENSITIVITY EQUATIONS

The complete equations of motion of the satellite used for this chapter are given by equation A-25 (Appendix A). These equations will subsequently be time and magnitude scaled for convenience.

The following definitions are given in equation 7-1 for τ, x_1, x_2, x_3, X_4, X_5, X_6, X_7, x_8, x_9, u_{1s}, u_{2s}, and u_{3s}.

$$
\begin{aligned}
\tau &= t \left(\frac{GM}{a^3}\right)^{\frac{1}{2}} \\
x_1 &= X_1 / \left(\frac{GM}{a^3}\right)^{\frac{1}{2}} \\
x_2 &= X_2 / \left(\frac{GM}{a^3}\right)^{\frac{1}{2}} \\
x_3 &= X_3 / \left(\frac{GM}{a^3}\right)^{\frac{1}{2}} \\
X_4 &= W_1 \\
X_5 &= W_2 \\
X_6 &= W_3 \\
X_7 &= W_4
\end{aligned}
\qquad (7\text{-}1)
$$

(continued)

$$X_8 = v/\left(\sqrt{\frac{GM}{a^3}} \cdot a\right)$$

$$X_9 = r/a$$

$$u_{1s} = u_1/\left(\frac{GM}{a^3}\right)$$

$$u_{2s} = u_2/\left(\frac{GM}{a^3}\right) \qquad (7\text{-}1)$$

$$u_{3s} = u_3/\left(\frac{GM}{a^3}\right)$$

The notation for differentiation with respect to time, t, and scaled time, τ, are given in equation 7-2.

$$(\dot{\ }) = \frac{d}{dt}\ (\ ) \ ; \qquad (\ )' = \frac{d}{d\tau} \qquad (7\text{-}2)$$

Using equation 7-1 and 7-2, the equations of motion (A-25) are given as follows:

$$x_1' = u_{1s} + \frac{3}{(x_9)^3} k_x a_{21} a_{31} - \theta'' a_{13} + \theta'(a_{33} x_2 - a_{23} x_3)$$

$$- k_x (x_2 + \theta' a_{23})(x_3 + \theta' a_{33})$$

$$x_2' = u_{2s} + \frac{3}{(x_9)^3} k_y a_{11} a_{31} - \theta'' a_{23} + \theta'(a_{13} x_3 - a_{33} x_1)$$

$$- k_y (x_3 + \theta' a_{33})(x_1 + \theta' a_{13}) \qquad (7\text{-}3)$$

$$x_3' = u_{3s} + \frac{3}{(x_9)^3} k_z a_{11} a_{21} - \theta'' a_{33} + \theta'(a_{23} x_1 - a_{13} x_2)$$

$$- k_z (x_1 + \theta' a_{13})(x_2 + \theta' a_{23})$$

$$X_4' = 1/2(x_1 X_7 - x_2 X_6 + x_3 X_5)$$

(continued)

$$X_5' = 1/2(x_1X_6 + x_2X_7 - x_3X_4)$$

$$X_6' = 1/2(-x_1X_5 + x_2X_4 + x_3X_7)$$

$$X_7' = 1/2(-x_1X_4 - x_2X_5 - x_3X_6) \tag{7-3}$$

$$x_8' = (1-\epsilon^2)/x_9^3 - 1/x_9^2$$

$$x_9' = x_8$$

The F_{ij}; $i=1,2,\ldots 6$; $j=1,2,\ldots 6$ to be used in the backward integration of equation 5-3 are given for the low torque dynamics by equation 7-4,

$$F_{11} = 0$$

$$F_{12} = \theta' a_{33} - k_x(x_3 + \theta' a_{33})$$

$$F_{13} = \theta' a_{23} - k_x(x_2 + \theta' a_{23})$$

$$\begin{aligned} F_{14} = & -\frac{\theta''}{2} E_2 - \theta' x_2 X_4(1 - k_x) - x_3\theta' E_1(1 + k_x)/2 \\ & - k_x\theta'(\theta' E_1 a_{33} - 2\theta' a_{23} X_4)/2 \\ & + Sk_x(a_{21}E_5 + a_{31}E_3)/2 \end{aligned} \tag{7-4}$$

$$\begin{aligned} F_{15} = & \frac{-\theta''}{2} E_4 - \theta' x_2 X_5(1 - k_x) - x_3\theta' E_5(1 + k_x)/2 \\ & - k_x\theta'(\theta' E_5 a_{33} - 2\theta' a_{23} X_5)/2 \\ & + Sk_x(a_{31}E_6 - a_{21}E_4)/2 \end{aligned}$$

$$\begin{aligned} F_{16} = & \frac{-\theta''}{2} E_6 - x_3\theta' E_9(1 + k_x)/2 - k_x(\theta')^2 E_9 a_{33}/2 \\ & + Sk_x(a_{21}E_8 + a_{31}E_7)/2 \end{aligned}$$

$$F_{21} = -\theta' a_{33} - k_y(x_3 + \theta' a_{33})$$

$$F_{22} = 0$$

(continued)

$$F_{23} = \theta' a_{13} - k_y(x_1 + \theta' a_{13})$$

$$F_{24} = \frac{-\theta''}{2} E_1 + x_3\theta' E_2(1 - k_y)/2 + \theta' x_1 X_4 (1 + k_y)$$

$$- k_y\theta'(\theta' E_2 a_{33} - 2\theta' a_{13} X_4)/2 + S k_y a_{11} E_5/2$$

$$F_{25} = -\theta'' E_5/2 + x_3\theta' E_4(1 - k_y)/2 + \theta' x_1 X_5(1 + k_y)$$

$$- k_y\theta'(\theta' E_4 a_{33} - 2\theta' a_{13} X_5)/2 - S k_y(a_{11} E_4 + 2a_{31} X_5)/2$$

$$F_{26} = -\theta'' E_9/2 + x_3\theta' E_6(1 - k_y)/2 - k_y(\theta')^2 E_6 a_{33}/2$$

$$+ S k_y(a_{11} E_8 - 2a_{31} X_6)/2$$

$$F_{31} = \theta' a_{23} - k_z(x_2 + \theta' a_{23})$$

$$F_{32} = -\theta' a_{13} - k_z(x_1 + \theta' a_{13}) \qquad (7\text{-}4)$$

$$F_{33} = 0$$

$$F_{34} = \theta'' X_4 + x_1\theta' E_1(1 - k_z)/2 - x_2\theta' E_2(1 + k_z)/2$$

$$- k_z\theta'(\theta' E_1 a_{13} + \theta' E_2 a_{23})/2 + S k_z a_{11} E_3/2$$

$$F_{35} = \theta'' X_5 + x_1\theta' E_5(1 - k_z)/2 - x_2\theta' E_4(1 + k_z)/2$$

$$- k_z\theta'(\theta' E_5 a_{13} + \theta' E_4 a_{23})/2 + S k_z(a_{11} E_6 - 2a_{21} X_5)/2$$

$$F_{36} = x_1\theta' E_9 \frac{(1 - k_z)}{2} - x_2\theta' E_6(1 + k_z)/2 - k_z\theta'(\theta' E_9 a_{13} + \theta' E_6 a_{23})/2$$

$$+ S k_z(a_{11} E_7 - 2a_{21} X_6)/2$$

The F_{ij} $i=4,5,6;\ j=1,2,\ldots 6$ are the same as in Chapter 5.
Values for S, E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8, and E_9 are defined as follows:

$$S = 3/(x_9)^3$$

$$E_1 = X_7 - X_4{}^2/X_7 \qquad \text{(continued)}$$

$E_2 = X_6 + X_4X_5/X_7$

$E_3 = X_5 + X_4X_6/X_7$

$E_4 = -X_7 + X_5^2/X_7$

$E_5 = X_6 - X_4X_5/X_7$

$E_6 = X_4 + X_5X_6/X_7$

$E_7 = -X_7 + X_6^2/X_7$

$E_8 = X_4 - X_5X_6/X_7$

$E_9 = X_5 - X_4X_6/X_7$

It must be remembered from equation 5-2 and A-12a that X_7 is dependent on X_4, X_5, and X_6 and that the $d_{ij}(i,j=1,2,3)$ are dependent on X_4, X_5, X_6 and X_7. Because of this, one must often make single or double application of the "chain rule" of differentiation in evaluating the F_{ij}. For example, F_{14} is given as follows:

$$F_{14} = \frac{\partial f_1}{\partial X_4} + \sum_{j=1}^{3} \sum_{k=1}^{3} \left\{ \frac{\partial f_1}{\partial d_{jk}} \left(\frac{\partial d_{jk}}{\partial X_4} + \frac{\partial d_{jk}}{\partial X_7} \frac{\partial X_7}{\partial X_4} \right) \right\} + \frac{\partial f_1}{\partial X_7} \frac{\partial X_7}{\partial X_4} \qquad (7\text{-}5)$$

B. LOW TORQUE NUMERICAL EXAMPLE

Orbital parameters for this example are given as follows: eccentricity of the elliptical orbit, .0521; apogee of orbit, 4651 miles; perigee of orbit, 4190 miles; orbital period, 99 minutes. Using 3960 miles and 4.11×10^{23} slugs as the radius and mass of the earth, the scaling factor used in equation 7-1 is easily computed as:

$$\frac{GM}{a^3} = 1.1046 \times 10^{-6} \text{sec.}^{-2}$$

In this report will be computed the fuel optimal control for an example identical to one done by Hales and Flügge-Lotz in (reference 1). The moments of inertia of the satellite are the same as those used in Chapter 5 of this report. The initial and final time (t_o and t_f) are given as 0 and 1196 seconds, respectively. The initial values of the state, $\underline{x}(t_o)$, are given as 3.8×10^{-2} degrees/sec., $- 7.27 \times 10^{-2}$ degrees/sec., 3.44×10^{-2} degrees/sec., .218, .638, .104, and 1.88, respectively. Values for x_8 and x_9 are -5.1×10^{-2} and .986. Thrust acceleration bounds are lowered to only 1.905×10^{-4} degrees/sec.$^{-2}$ Scaled values (see equation 7-1) of t_o and t_f are 0 and 1.255. The first three (angular velocity) components of the initial state have scaled values of .63, -1.21, and .57. And the thrust acceleration scales to the value of 3.03 for each component.

By using the method described in Chapter 4 for this example, the optimal control and trajectory of Figures 7-1 and 7-2 are obtained in five iterations. The fuel cost is 1.52×10^{-3} sec.$^{-1}$ as opposed to 1.70×10^{-3} sec.$^{-1}$ obtained by Hales with his "extended method steepest descent." The cost of 1.52×10^{-3} sec.$^{-1}$ agrees to within less than one percent of the true optimal cost of this trajectory which was initially generated by backward integration of the adjoint and state differential equations. The second pulse for u_3 (shown in Figure 7-1) drifted between the position shown and a position at the terminal time for each new iteration without affecting the cost. Apparently the positioning of this pulse is not critical to the cost or terminal state constraints.

In the simulation of this example, it is necessary to evaluate θ' and θ'' (from scaled versions of equations A-22 and A-24) for substitution into the right hand sides of equation 7-3.

Although the time varying dynamics of equation 7-3 do not conceptually alter this method of solving for fuel optimal controls, approximately 25% more computing time is required per iteration because of the bulky right hand sides of equations 7-3 and 7-4. One would probably only be interested in using such small thrust levels as those in this chapter if the satellite were to be engaged in long term experimentation with the necessity of using the acquisition system many times.

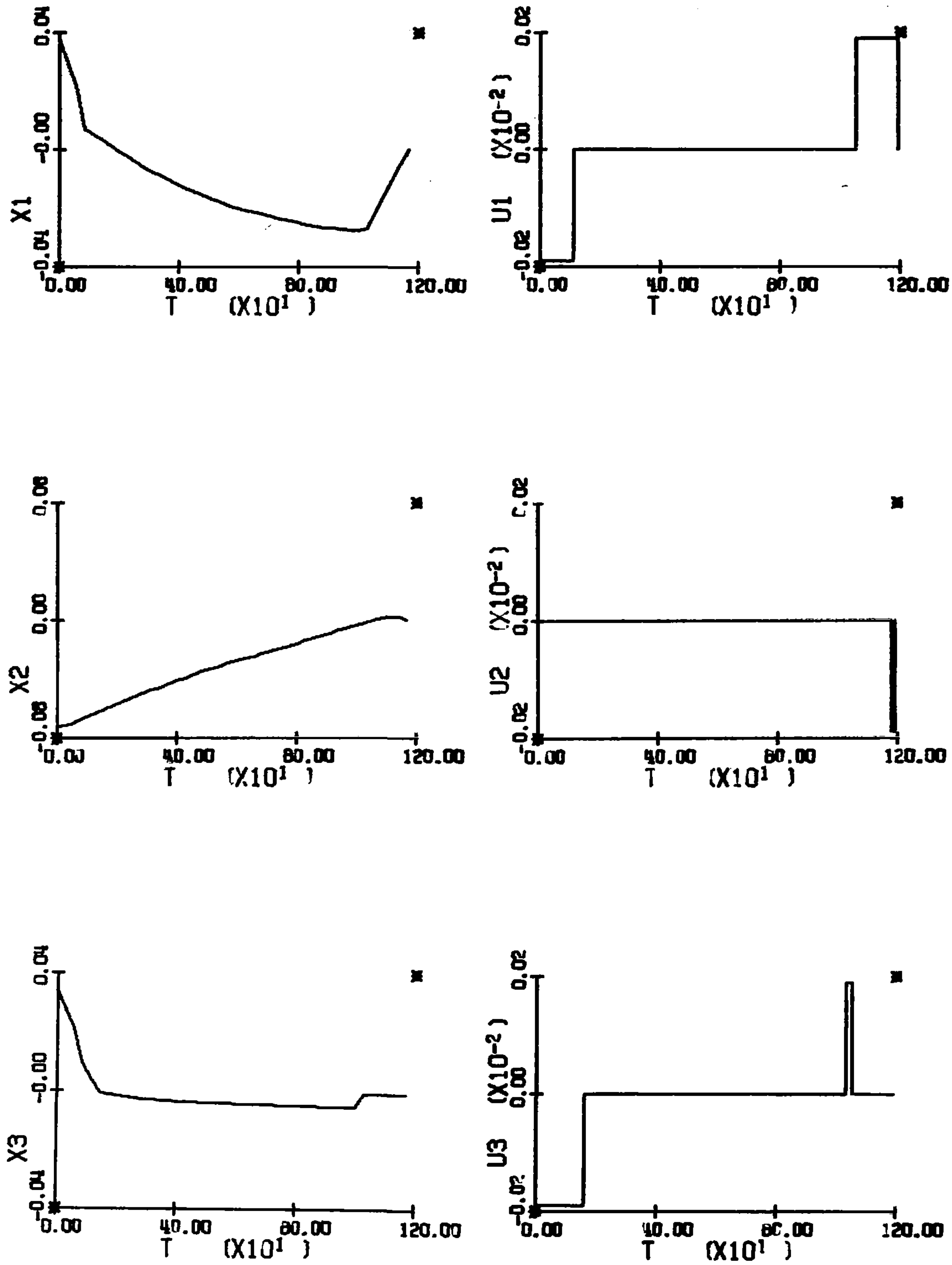


Figure 7-1 Optimal Angular Velocity and Control Responses for Low Torque Example

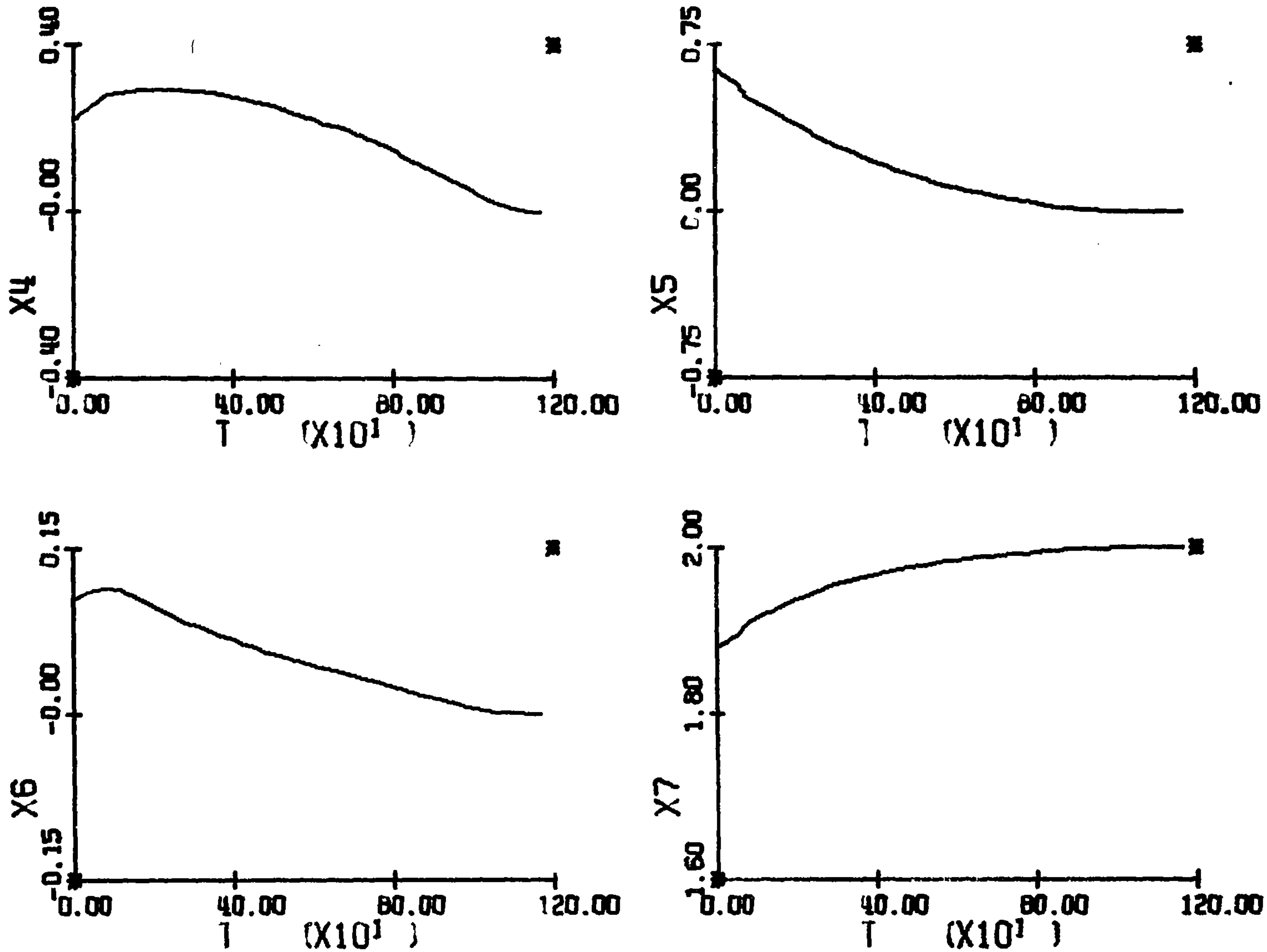


Figure 7-2 Optimal Euler Parameters for Low Torque Example

VIII. CONCLUSION

By utilizing the fact that the optimal fuel control history for the nonlinear control problems described in this report must necessarily be of a "bang-coast-bang" nature, an algorithm utilizing linear programming has been developed which iteratively improves on a nominal control history. The algorithm is based on expressing the variation of the fuel cost and the variations of the components of the terminal state constraints as linear functions of variations of the "switching times" of the control. In using the algorithm, the nominal control is expressed as a series of alternately positive and negative pulses of control with intervals of zero control between each pulse. The magnitude of the pulses is equal to the bound on the magnitude of the control.

The algorithm was tested on a nonlinear system of differential equations describing (by Euler Parameters) the complete attitude motion of a satellite in elliptical orbit about the earth. In the case where the control level was high relative to other terms in the dynamical equations, simplifications were made, but the basic nonlinearities were retained. The algorithm gave solutions which compared well with solutions to identical examples obtained by other methods.

This algorithm has the advantage that it is quite insensitive to choices in the nominal control compared to other methods. In the event the optimal control has several pulses for each control component, the α_i of equation 4-7 must be reduced toward zero (as the terminal constraints are being met) to guarantee convergence. If this is not done, the control will oscillate around the optimal solution without being exactly optimal. In situations such as this, it may be advantageous to switch to a second-order method (which usually depend on being initially at a nearly optimal solution) such as described in (reference 7) to complete the convergence to an optimal solution. As in many other optimization techniques, there is no known way to verify that the solution obtained by this linear programming algorithm is globally optimal.

This technique of solving for open-loop fuel optimal controls could be actually applied to a satellite system as follows: Measure the present state of the satellite and extrapolate to what the state will be (by integrating the dynamics equations) at some suitably distant time in the future if no control is being applied. The optimal control could then be calculated in the interim time before the calculated state is reached and applied to the satellite when the predicted state is reached. In doing this, however, it is to be noted that there is a possibility (remote) of noise disturbing the extrapolated state.

For future investigations, effort might be directed toward developing similar algorithms for other cost criteria (such as minimum time) and toward developing a minimum fuel feedback control law. Although this report suggests an experimental approach to the minimum time problem it does not treat this problem completely. It is also to be noted that the optimal feedback control problem where the control is known to have a bang-coast-bang character is still essentially unsolved for dynamical systems with three or more state variables.

APPENDIX A. EQUATIONS OF MOTION

The equations of motion of a satellite in orbit are derived in this appendix by us ng Euler Parameters. Hales and Flügge-Lotz have given a rather complete derivation of the attitude dynamical equations of a rotating body in (reference 1), but have only stated the form of the orbital equations. In view of their work, the attitude equations will be discussed only by pointing out the more salient features in the derivation. The orbital equations, however, are derived in a more detailed manner.

In deriving the orbital and attitude equations of a satellite in orbit about a fixed mass, three reference frames will be used. Figure A-1 indicates two of the three coordinate systems to be used in deriving the equations of motion. The earth (or other fixed attracting body), about which the orbit exists, is designated by P and is assumed to be an inertially fixed point mass. The center of mass of the satellite, P*, moves in an elliptical orbit about P. The origin of the (x_e, y_e, z_e) axes is inertially fixed at P, with z_e perpendicular to the orbital plane and x_e and y_e of arbitrary orientation. The orbital reference frame, denoted by (x_r, y_r, z_r) is centered at P* with z_r parallel to z_e. x_r is either directed along the line from P to P* or remains parallel to x_e. A third reference frame, a body fixed reference denoted by (x_b, y_b, z_b), is centered at P* and fixed parallel to the satellite's principal moments of inertia.

Unit vectors parallel to each of the above axes will be denoted by the vector $\underline{n}$ with appropriate subscripts. For example, $\underline{n}_{x_r}$ denotes the unit vector parallel to the x_r axis.

1. ATTITUDE DYNAMICAL EQUATIONS

Euler's dynamical equations are given A-1.

$$
\begin{aligned}
I_x \dot{\omega}_x^B - (I_y - I_z)\omega_y^B \omega_z^B &= N_x \\
I_y \dot{\omega}_y^B - (I_z - I_x)\omega_z^B \omega_x^B &= N_y \qquad \text{(A-1)} \\
I_z \dot{\omega}_z^B - (I_x - I_y)\omega_x^B \omega_y^B &= N_z
\end{aligned}
$$

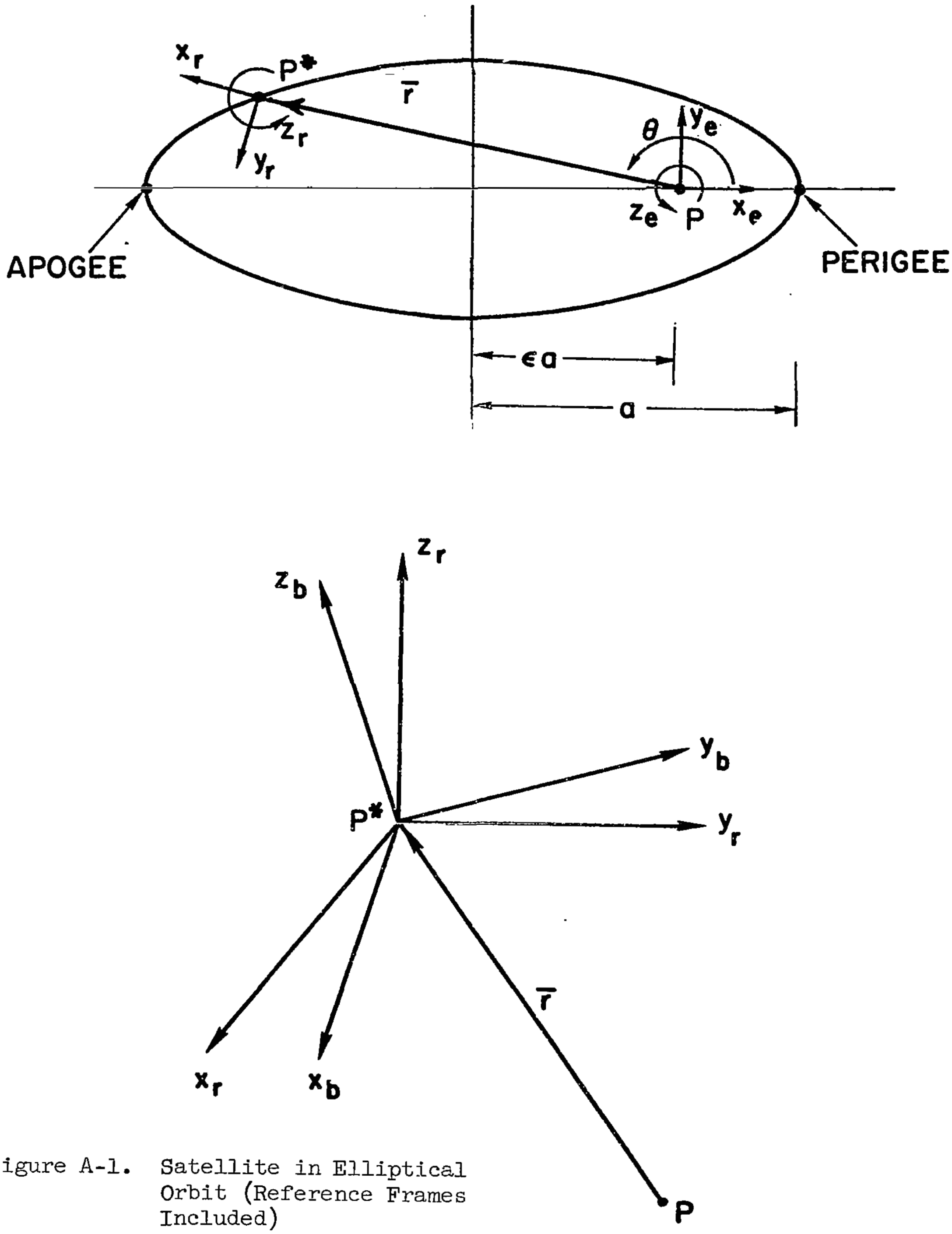


Figure A-1. Satellite in Elliptical Orbit (Reference Frames Included)

I_x, I_y and I_z are the centroidal moments of inertia about the principal axes of the body and N_x, N_y, and N_z are components of the total active torque exerted on the body and resolved along the respective body fixed axes. ω_x^B, ω_y^B, and ω_z^B are defined in equation A-2, where $\underline{\omega}^B$ is the total angular velocity of the satellite in the inertial reference frame.

$$\underline{\omega}^B = \omega_x^B \underline{n}_{xb} + \omega_y^B \underline{n}_{yb} + \omega_z^B \underline{n}_{zb} \tag{A-2}$$

The angular velocity of the orbital reference frame with respect to the inertial reference frame is given by equation A-3.

$$\underline{\omega}^R = \dot{\theta}\underline{n}_{zr} \tag{A-3}$$

The angular velocity of the satellite in the orbital reference frame is defined by $\underline{\omega}^{B/R}$ and given in equation A-4.

$$\underline{\omega}^{B/R} = \underline{\omega}^B - \underline{\omega}^R \tag{A-4}$$

The components of $\underline{\omega}^{B/R}$ are defined by X_1, X_2, and X_3; hence equation A-5 follows.

$$\underline{\omega}^{B/R} = X_1\underline{n}_{xb} + X_2\underline{n}_{yb} + X_3\underline{n}_{zb} \tag{A-5}$$

The unit vectors $\underline{n}_{xr}$, $\underline{n}_{yr}$, $\underline{n}_{zr}$ of the orbital reference frame are related to the unit vectors $\underline{n}_{xb}$, $\underline{n}_{yb}$, $\underline{n}_{zb}$ of the body fixed reference frame by a direction cosine transformation matrix, D, defined in equation A-6.

$$\begin{Bmatrix} \underline{n}_{xb} \\ \underline{n}_{yb} \\ \underline{n}_{zb} \end{Bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \begin{Bmatrix} \underline{n}_{xr} \\ \underline{n}_{yr} \\ \underline{n}_{zr} \end{Bmatrix} = D \begin{Bmatrix} \underline{n}_{xr} \\ \underline{n}_{yr} \\ \underline{n}_{zr} \end{Bmatrix} \tag{A-6}$$

The dynamical equations of A-1 can now be expressed in terms of the relative angular velocities of equation A-5 and the direction cosines of equation A-6. Combining equations A-2 through A-6 appropriately gives the expressions for the total angular velocity components as:

$$\omega_x^{B} = X_1 + \dot{\theta} d_{13}$$

$$\omega_y^{B} = X_2 + \dot{\theta} d_{23} \qquad \text{(A-7)}$$

$$\omega_z^{B} = X_3 + \dot{\theta} d_{33}$$

The time derivatives of (A-7) give

$$\dot{\omega}_x^{B} = \dot{X}_1 + \dot{\theta} \dot{d}_{13} + \ddot{\theta} d_{13}$$

$$\dot{\omega}_x^{B} = \dot{X}_2 + \dot{\theta} \dot{d}_{23} + \ddot{\theta} d_{23} \qquad \text{(A-8)}$$

$$\dot{\omega}_z^{B} = \dot{X}_3 + \dot{\theta} \dot{d}_{33} + \ddot{\theta} d_{33}$$

Normalized inertia parameters k_x, k_y, k_z are defined by equation A-9.

$$k_x = \frac{I_z - I_y}{I_x} ; \quad k_y = \frac{I_x - I_z}{I_y} ; \quad k_z = \frac{I_y - I_x}{I_z} \qquad \text{(A-9)}$$

The Euler dynamical equations of A-1 can now be reduced by use of equations A-7, A-8, and A-9 as follows:

$$\dot{X}_1 = \frac{N_x}{I_x} - \dot{\theta} \dot{d}_{13} - \ddot{\theta} d_{13} - k_x (X_2 + \dot{\theta} d_{23})(X_3 + \dot{\theta} d_{33})$$

$$\dot{X}_2 = \frac{N_y}{I_y} - \dot{\theta} \dot{d}_{23} - \ddot{\theta} d_{23} - k_y (X_1 + \dot{\theta} d_{13})(X_3 + \dot{\theta} d_{33}) \qquad \text{(A-10)}$$

$$\dot{X}_3 = \frac{N_z}{I_z} - \dot{\theta} \dot{d}_{33} - \ddot{\theta} d_{33} - k_z (X_1 + \dot{\theta} d_{13})(X_2 + \dot{\theta} d_{23})$$

The differential equations of A-10 are not yet complete. Expressions for the directional cosine components and time derivatives of these components which appear in equation A-10 will be discussed in the following section. In this section 3 of this appendix, expressions for the active torque components of equation A-10 will be expanded, and in section 4 the orbital considerations will give expressions for $\dot{\theta}$ and $\ddot{\theta}$.

2. KINEMATICAL EQUATIONS

In (reference 14), the relative merits of various schemes of computing and describing spacial rotations of a rigid body are described. This reference concludes that Euler Parameters provide the most useful characteristics for analysis and simulation of problems dealing with large angle maneuvers of unsymmetrical bodies. Although the Euler Angle description of rotation lends itself to easier geometric interpretation, there are singularities in the equations at rotations of 90 degrees. The Euler Parameter description does not encounter a singularity in the equations until the rotation is 180°.

The purpose of this section is to briefly describe Euler Parameters and to state first-order differential equations for the Euler Parameters in terms of the components of the relative angular velocity (X_1, X_2, X_3) and in terms of the Euler Parameters. Then, the relations which express the components of the matrix D (in equation A-6) in terms of the Euler Parameters are given.

From kinematical considerations, it can be shown two sets of orthogonal axes $(\underline{n}_{xr}, \underline{n}_{yr}, \underline{n}_{zr},$ and $\underline{n}_{xb}, \underline{n}_{yb}, \underline{n}_{zb})$ with the same vertex can be made coincidental (except in special cases involving a singularity) by a single rotation about some fixed unit vector, $\underline{k}$. The components of this vector are invariant to expression in either of the two reference frames. If the components of $\underline{k}$ are given as e_x, e_y, and e_z, the Euler Parameters will be defined as in Equation A-11.

$$
\begin{aligned}
W_1 &= 2 \sin \beta/2 \, e_x \\
W_2 &= 2 \sin \beta/2 \, e_y \\
W_3 &= 2 \sin \beta/2 \, e_z \\
W_4 &= 2 \cos \beta/2
\end{aligned}
\qquad \text{(A-11)}
$$

β is the magnitude of the rotation. Physically, one would expect only three independent Euler Parameters. From trigonometric considerations of equation A-11 it can be seen that the expected redundancy is given by equation A-12.

$$\sum_{i=1}^{4} W_i^2 = 4 \qquad \text{(A-12)}$$

In (reference 15), the direction cosines of equation A-6 are given in terms of the Euler Parameters as:

$$\begin{aligned}
d_{11} &= \tfrac{1}{4} (W_1^2 + W_4^2 - W_2^2 - W_3^2) \\
d_{12} &= \tfrac{1}{2} (W_1W_2 + W_3W_4) \\
d_{13} &= \tfrac{1}{2} (W_3W_1 - W_2W_4) \\
d_{21} &= \tfrac{1}{2} (W_1W_2 - W_3W_4) \\
d_{22} &= \tfrac{1}{4} (W_2^2 + W_4^2 - W_1^2 - W_3^2) \qquad \text{(A-12a)} \\
d_{23} &= \tfrac{1}{2} (W_2W_3 + W_1W_4) \\
d_{31} &= \tfrac{1}{2} (W_1W_3 + W_2W_4) \\
d_{32} &= \tfrac{1}{2} (W_2W_3 - W_1W_4) \\
d_{33} &= \tfrac{1}{4} (W_3^2 + W_4^2 - W_1^2 - W_2^2)
\end{aligned}$$

And differential equations for the Euler Parameters are given as

$$\begin{aligned}
\dot{W}_1 &= \tfrac{1}{2} (W_2X_3 - W_3X_2 + W_4X_1) \\
\dot{W}_2 &= \tfrac{1}{2} (-W_1X_3 + W_3X_1 + W_4X_2)
\end{aligned} \qquad \text{(A-13)}$$

(continued)

$$\dot{W}_3 = \frac{1}{2} (W_1X_2 - W_2X_1 + W_4X_3)$$

$$\dot{W}_4 = \frac{1}{2} (-W_1X_1 - W_2X_2 - W_3X_3) \qquad \text{(A-13)}$$

Differentiating A-12a with respect to time and using A-13 gives the expressions of A-14 which will later be used in equation A-10.

$$\dot{d}_{13} = X_3d_{23} - X_2d_{33}$$

$$\dot{d}_{23} = X_1d_{33} - X_3d_{13} \qquad \text{(A-14)}$$

$$\dot{d}_{33} = X_2d_{13} - X_1d_{23}$$

3. ACTIVE TORQUES

The active torque applied to the satellite consists of an external torque due to gravity gradient from the earth and a control torque generated by the gas jets on the satellite. The three sets of control are assumed to be mounted such that the torque from each contributes torque about only one principal axis of inertia. Therefore, the active torque terms in equation A-10 may be expressed as in equation A-15. Note that the control torque terms are written as products in the respective moments of inertia so that the equations may later be normalized to angular acceleration.

$$N_x = I_xu_1 + N_{xg}$$

$$N_y = I_yu_2 + N_{yg} \qquad \text{(A-15)}$$

$$N_z = I_zu_3 + N_{zg}$$

The N_{xg}, N_{yg}, N_{zg} are gravity gradient terms and are given in (reference 18) as

$$N_{xg} = \frac{3GM}{r^3} (I_z - I_y)d_{31}d_{21} \qquad \text{(A-16)}$$

(continued)

$$N_{yg} = \frac{3GM}{r^3} (I_x - I_z) d_{11} d_{31}$$

$$N_{zg} = \frac{3GM}{r^3} (I_y - I_x) d_{11} d_{21} \quad \text{(A-16)}$$

where G is the universal gravity constant, M is the mass of point P (earth), and r is the distance from P to P* (satellite). Combining the above two sets of expressions and normalizing with respect to the moments of inertia yields A-17 for the active angular accelerations.

$$\frac{N_x}{I_x} = u_1 + \frac{3GM}{r^3} k_x d_{21} d_{31}$$

$$\frac{N_y}{I_y} = u_2 + \frac{3GM}{r^3} k_y d_{11} d_{31} \quad \text{(A-17)}$$

$$\frac{N_z}{I_z} = u_3 + \frac{3GM}{r^3} k_z d_{11} d_{21}$$

4. ORBITAL EQUATIONS

The orbital characteristics of the satellite (see ref. 16) will be considered in this section. If the mass of the satellite is denoted by M, the force exerted on it by gravity may be used to determine two differential equations.

$$\frac{-GMm}{r^2} \underline{n}_{xr} = \underline{F} = m\, \underline{a}_s \quad \text{(A-18)}$$

$\underline{a}_s$ is the acceleration of the mass center of the satellite. From Figure A-1 and kinematical considerations, the velocity, $\underline{v}_s$, of the satellite is given as follows:

$$\underline{v}_s = \dot{r}\underline{n}_{xr} + r\dot{\theta}\underline{n}_{yr}$$

It follows by time differentiation of $\underline{v}_s$ that $\underline{a}_s$ is given by equation A-19.

$$\underline{a}_s = (\ddot{r} - r\dot{\theta}^2)\underline{n}_{xr} + (\ddot{r}\theta + 2\dot{r}\dot{\theta})\underline{n}_{yr} \qquad \text{(A-19)}$$

Substitution of (A-19) into (A-18) and equating coefficients of respective unit vectors yields differential equations (A-20).

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \qquad \text{(A-20)}$$

The second differential equation can be solved easily to yield

$$\frac{1}{2}\, r\dot{\theta}^2 = \text{constant} \qquad \text{(A-21)}$$

From "Keplers's Law", the constant in equation (A-2) can be evaluated as

$$2\pi a^2 \frac{(1-\epsilon^2)^{1/2}}{T}$$

where a and ϵ describe the geometry of the ellipse (see Figure (A-1)) and T is the orbital time period. In (reference 17), T is given as

$$T = 2\pi a^{3/2}\left(\frac{1}{GM}\right)^{1/2}$$

In view of equation (A-21) and the above constants, (A-21) can be written as follows:

$$\dot{\theta} = a^{1/2}(1-\epsilon^2)^{1/2}\left(GM\,\frac{1}{r^2}\right)^{1/2} \qquad \text{(A-22)}$$

Substitution of (A-22) into the first equation of (A-20) yields:

$$\ddot{r} = a(1-\epsilon^2)^{1/2}\, GM\,\frac{1}{r^3} - \frac{GM}{r^2} \qquad \text{(A-23)}$$

$\ddot{\theta}$ can be evaluated by differentiating equation (A-22) to yield:

$$\ddot{\theta} = -2a^{1/2}(1-\epsilon^2)^{1/2}\left(GM \frac{1}{r^3}\dot{r}\right)^{1/2} \qquad \text{(A-24)}$$

5. COMPLETE SATELLITE EQUATIONS OF MOTION

The results of the previous sections will be combined presently to give a complete set of dynamical state equations. Complete equations for X_1, X_2, X_3 are obtained by substitution of equations A-14 and A-17 into A-10. The differential equations of the Euler Parameters (equation A-13) are repeated below. Equation A-23 may be written as two first order differential equations by defining v as $v = \dot{r}$. The results are given in equation A-25.

$$\begin{aligned}
\dot{X}_1 &= u_1 + \frac{3GM}{r^3} k_x d_{21} d_{31} - \ddot{\theta} d_{13} - \dot{\theta}(X_3 d_{23} - X_2 d_{33}) \\
&\quad - k_x(X_2 + \dot{\theta} d_{23})(X_3 + \dot{\theta} d_{33}) \\
\dot{X}_2 &= u_2 + \frac{3GM}{r^3} k_y d_{11} d_{31} - \ddot{\theta} d_{23} - \dot{\theta}(X_1 d_{33} - X_3 d_{13}) \\
&\quad - k_y(X_1 + \dot{\theta} d_{13})(X_3 + \dot{\theta} d_{33}) \\
\dot{X}_3 &= u_3 + \frac{3GM}{r^3} k_z d_{11} d_{21} - \ddot{\theta} d_{33} - \dot{\theta}(X_2 d_{13} - X_1 d_{23}) \\
&\quad - k_z(X_1 + \dot{\theta} d_{13})(X_2 + \dot{\theta} d_{23}) \\
\dot{W}_1 &= \tfrac{1}{2}(W_2 X_3 - W_3 X_2 + W_4 X_1) \qquad \text{(A-25)} \\
\dot{W}_2 &= \tfrac{1}{2}(-W_1 X_3 + W_3 X_1 + W_4 X_2) \\
\dot{W}_3 &= \tfrac{1}{2}(W_1 X_2 - W_2 X_1 + W_4 X_3) \\
\dot{W}_4 &= \tfrac{1}{2}(-W_1 X_1 - W_2 X_2 - W_3 X_3)
\end{aligned}$$

(continued)

$$\dot{v} = a(1 - \epsilon^2)\ GM \frac{1}{r^3} - GM \frac{1}{r^2}$$

$$\dot{r} = v$$

$\dot{\theta}$ and $\ddot{\theta}$ are evaluated algebraically in the above state equations by equations A-22 and A-24. $\dot{\theta}$ and $\ddot{\theta}$ are repeated as equation A-26 for convenience.

$$\dot{\theta} = a^{1/2}\ (1-\epsilon^2)^{1/2} (GM \frac{1}{r^2})^{1/2}$$

(A-26)

$$\ddot{\theta} = -2a^{1/2}(1-\epsilon^2)^{1/2} (GM \frac{1}{r^3}\ v)^{1/2}$$

Equation A-25 may now be solved (numerically) for X_1, X_2, X_3, W_1, W_2, W_3, W_4, v and r if initial conditions and values for the control variables (u_1, u_2, u_3) are given and if A-26 is used to evalute $\dot{\theta}$ and $\ddot{\theta}$ algebraically for substitution into A-25. The differential equations of A-25 may either be integrated as they stand or they may be integrated after deleting one of the differential equations for the Euler Parameters and using the algebraic equation A-12 for solving for the deleted Euler Parameter. The former method is used in the report.

APPENDIX B: LINEAR PROGRAMMING

This is intended to be only a brief discussion to mention a few important ideas in linear programming. The reader is referred to (references 19 and/or 20) for further discussion either on linear programming in general or on the powerful Simplex Method of solving linear programming problems.

The linear programming problem suitable for solution by the Simplex Method can be stated as:

$$\text{Minimize} \quad c_1 z_1 + c_2 z_2 \ldots c_n z_n \tag{B-1}$$

subject to: $z_j \geq 0 \quad j=1,2,\ldots n$

$$\begin{aligned} \text{and} \quad & a_{11} z_1 + a_{12} z_2 \ldots a_{12} z_n = b_1 \\ & \vdots \\ & a_{m1} z_1 + a_{m2} z_2 + \ldots a_{mn} z_n = b_m \end{aligned} \tag{B-2}$$

The a_{ij}, b_i, c_j $(i=1,2,\ldots m;\ j=1,2,\ldots n)$ are given.

Some of the constraint equations of B-2 may be given as constraint inequalities, but the inequalities can easily be reduced to equalities by the appropriate introduction of non-negative dummy variables. Hence, no generality is lost by considering only equality constraints in expression B-2.

The following definitions lead to valuable considerations.

Definition:

A set, S is said to be <u>convex</u> if, given any two points $\underline{z}_a$ and $\underline{z}_b$, both elements of S, then every point $\underline{z}_c$ satisfying

$$\underline{z}_c = \lambda \underline{z}_a + (1 - \lambda) \underline{z}_b \qquad 0 \leq \lambda \leq 1$$

is also an element of S.

Definition:

A point, $\underline{z}_c$, which is an element of a convex set, S, is said to be an <u>extreme point</u> of S if it can <u>not</u> be expressed as

$$z_c = \lambda \underline{z}_a + (1 - \lambda)\underline{z}_b \qquad 0 \le \lambda \le 1$$

for any $\underline{z}_a$ and $\underline{z}_b$ (excluding $\underline{z}_c$) in S.

It can be proved that the solution (if it exists) for $\underline{z}$ ($\underline{z} = z_1, z_2, z_3, \ldots z_n$) which minimizes the functional defined in expression B-1 occurs at an extreme point of the convex set defined by expression B-2.

The next step is to relate the extreme points of the convex set of feasible solutions of expression B-2 to the a_{ij} of expression B-2. Before doing this, A_j is defined as the column vector whose components are $a_{1j}, a_{2j}, a_{3j}, \ldots a_{mj}$. B is similarly defined as the column vector with components $b_1, b_2, \ldots b_m$. Expression B-2 can thus be written as

$$z_1 A_1 + z_2 A_2 + \ldots z_n A_n = B \qquad \text{(B-3)}$$

The theorem which relates the extreme points of the convex set of feasible solutions to the A_j of expression B-3 can be stated as:

$$\underline{z} = (z_1, z_2, z_3, \ldots z_n)$$

is an extreme point of the convex set of feasible solutions of expression B-2 if and only if the positive z_j are coefficients of linearly independent vectors, A_j in expression B-3.

From all of this, it is seen that in solving a linear programming program, only feasible solutions generated by m linearly independent vectors need be investigated. This would still be an enormous task for linear programming problems of the dimension encountered in this report were it not for the Simplex Method. The Simplex Method finds an extreme point and determine whether or not it minimizes expression B-1. If not, it continues to find new neighboring extreme points by a process using the previously stated theorem which give values for the functional of expression B-1 not greater than the value associated with the preceeding extreme point. In a finite number of steps (usually less than 2m) a minimum solution is found. The method also is able to identify problems with no finite minimum solutions and problems with no feasible solutions.

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