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GRADIENT METHODS IN CONTROL THEORY
PART 2 - SEQUENTIAL GRADIENT-RESTORATION ALGORITHM

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Gradient Methods in Control Theory

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Abstract. This paper considers the problem of minimizing a functional I which depends on the state x(t), the control u(t), and a parameter π. Here, I is a scalar, x an n-vector, u an m-vector, and π a p-vector. At the initial point, the state x is prescribed. At the final point, the state x and the parameter π are required to satisfy q scalar relations. Along the interval of integration, the state, the control, and the parameter are required to satisfy n scalar differential equations. A sequential algorithm composed of the alternate succession of gradient phases and restoration phases is presented.

In the gradient phase, nominal functions x(t), u(t), π satisfying all the differential equations and boundary conditions are assumed. Variations $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ leading to varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ are determined so that the value of the functional is decreased. These variations are obtained by minimizing the first-order change of the functional subject to the linearized differential equations, the linearized boundary conditions, and a quadratic constraint on the variations of the control and the parameter.

Since the constraints are satisfied only to first order during the gradient phase, the functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ may violate the differential equations and/or the boundary conditions. This being the case, a restoration phase is needed prior to starting the next

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gradient phase. In this restoration phase, the functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ are assumed to be the nominal functions. Variations $\Delta\tilde{x}(t)$, $\Delta\tilde{u}(t)$, $\Delta\tilde{\pi}$ leading to varied functions $\hat{x}(t)$, $\hat{u}(t)$, $\hat{\pi}$ consistent with all the differential equations and boundary conditions are determined. These variations are obtained by requiring the least-square change of the control and the parameter subject to the linearized differential equations and the linearized boundary conditions. Of course, the restoration phase must be performed iteratively until the cumulative error in the differential equations and boundary conditions becomes smaller than some preselected value.

If the gradient stepsize is α, an order of magnitude analysis shows that the gradient corrections are $\Delta x = O(\alpha)$, $\Delta u = O(\alpha)$, $\Delta\pi = O(\alpha)$, while the restoration corrections are $\Delta\tilde{x} = O(\alpha^2)$, $\Delta\tilde{u} = O(\alpha^2)$, $\Delta\tilde{\pi} = O(\alpha^2)$. Hence, for α sufficiently small, the restoration phase preserves the descent property of the gradient phase: the functional I decreases between any two successive restoration phases.

To obtain a reasonable convergence rate, the gradient stepsize α must be determined in an optimal fashion. In this connection, two methods are presented: one is based on information available at the end of the gradient phase and one is based on information available at the end of the restoration phase.

1. Introduction

Over the past several years, considerable work has been done on the application of gradient methods to control theory. Among the possible approaches, the method of penalty functions must be mentioned. The advantage of this approach is that the constrained minimal problem is replaced by a mathematically simpler, unconstrained minimal problem. The disadvantages are these: no clear-cut method exists for choosing the penalty constants; the algorithm must be repeated several times for increasing values of the penalty constants; the values of the functional between iterations are not comparable, since the constraints are not satisfied; and even when the algorithm is terminated, the constraints are satisfied only approximately.

Penalty functions were avoided in the approach employed by Bryson in Ref. 1. In this approach, nominal conditions satisfying the differential equations and the initial conditions, but not the final conditions, are used. Therefore, the differences between the desired final conditions and the nominal final conditions appear as forcing terms in the descent process. The drawback is that the values of the functional between iterations are not comparable, since the final conditions are not satisfied; also, no clear-cut method for choosing the stepsize exists.

In the light of the previous remarks, a simple and clear way to implement gradient algorithms is to make sure that all the equations and boundary conditions are satisfied at the beginning of each gradient phase. Since the differential equations and boundary conditions are considered only in linearized form during the gradient phase, some degree of constraint dissatisfaction exists at the end of the gradient phase. Therefore, prior to starting the next gradient phase, a restoration phase must be inserted: small perturbations, leading to the satisfaction of all the differential equations and boundary conditions, are introduced into the system.

In Ref. 2, Kelley considered a restoration cycle at the end of the gradient phase. He superimposed to the control change $\Delta u(t)$ associated with the gradient phase a perturbation $\Delta\tilde{u}(t)$ obtained by combining linearly some arbitrarily prescribed functions $f_1(t)$, $f_2(t)$, ..., $f_q(t)$. The constants of the combination were determined so as to satisfy the prescribed final conditions. The advantage of this approach is that the values of the functional between iterations are comparable. The disadvantage is that, owing to the arbitrariness of the functions $f_1(t)$, $f_2(t)$, ..., $f_q(t)$, one cannot ensure that the perturbation $\Delta\tilde{u}(t)$ is small.

Philosophically speaking, one must assume that the conditions obtained at the end of the gradient phase are a reasonable approximation to the desired optimum. Therefore, in the authors' opinion, the restoration phase should be performed <u>without rocking the boat too much</u>, that is, <u>causing the least overall disturbance to the system</u>. To this effect, a least-square criterion should be adopted (Ref. 3), and the logical choice is the least-square change of the control $u(t)$ and the parameter π. This point of view is taken in this paper.

For the gradient phase, both Bryson (Ref. 1) and Kelley (Ref. 2) performed a preliminary integration of the linearized differential equations in order to obtain the state change $\Delta x(t)$ in terms of the control change $\Delta u(t)$. This integration was performed prior to optimizing the control change. Although the approach is correct, this preliminary integration is neither necessary nor desirable. Indeed, a simpler derivation of the gradient algorithm is possible if one avoids integrating the state change in terms of the control change and views the minimal problem as a variational problem of the Bolza type with an added isoperimetric constraint of the quadratic type on the variations of the control and the parameter (for a particular case, the problem with fixed final time and free final state, see Ref. 4).

For those optimization problems where the final time is free, the actual interval of integration varies from iteration to iteration. From a computational point of view, this is not a desirable characteristic. Therefore, it is convenient to normalize the actual running time T in terms of the actual final time T_f: this is done by introducing the new independent variable $t = T/T_f$, where $0 \leq t \leq 1$ (for the implementation of this idea in quasilinearization, see Long, Ref. 5). In this way, the interval of integration is kept constant throughout the algorithm and the actual final time T_f becomes a parameter to be optimized. This is an additional difference between the present formulation and that of Refs. 1-2.

1.1. Sequential Gradient-Restoration Algorithm. In the light of the previous discussion, this paper considers the problem of minimizing a functional I which depends on the state x(t), the control u(t), and a parameter π (Section 2). Here, I is a scalar, x an n-vector, u an m-vector, and π a p-vector. At the initial time $t = 0$, the state x is prescribed. At the final time $t = 1$, the state x and the parameter π are required to satisfy q scalar relations. Along the interval of integration, the state, the control, and the parameter are required to satisfy n scalar differential equations. A sequential algorithm composed of the alternate succession of gradient phases and restoration phases is presented.

In the gradient phase (Section 5), nominal functions x(t), u(t), π satisfying all the differential equations and boundary conditions are assumed. Variations $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$

leading to varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ are determined so that the value of the functional is decreased. These variations are obtained by minimizing the first-order change of the functional subject to the linearized differential equations, the linearized boundary conditions, and a quadratic constraint on the variations of the control and the parameter.

In the restoration phase (Section 6), the functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ are assumed to be the nominal functions. Variations $\Delta\tilde{x}(t)$, $\Delta\tilde{u}(t)$, $\Delta\tilde{\pi}$ leading to varied functions $\hat{x}(t)$, $\hat{u}(t)$, $\hat{\pi}$ consistent with all the differential equations and boundary conditions are determined. These variations are obtained by requiring the least-square change of the control and the parameter subject to the linearized differential equations and the linearized boundary conditions. Of course, the restoration phase must be performed iteratively until the cumulative error in the differential equations and boundary conditions becomes smaller than some preselected value.

Both the gradient phase and the restoration phase are treated as variational problems of the Bolza type. Since the resulting Euler equations are linear, the differential system describing the optimum corrections is linear for both the gradient phase and the restoration phase. Hence, any of the known techniques for solving linear, two-point boundary value problems can be employed. To the authors' knowledge, the simplest technique is the method of particular solutions developed by Miele in Ref. 6. This method is employed systematically throughout the paper. The applicability of the method of particular solutions to iterative problems has been demonstrated by Heideman in Ref. 7.

1.2. Notation. Throughout the paper, vector-matrix notation is used for conciseness. The following table shows the dimensions of the matrices employed in the sequential gradient-restoration algorithm.

Table 1

Quantity	Dimensions	Quantity	Dimensions
f	1 x 1	x	n x 1
f_x	n x 1	$\dot{x}$	n x 1
f_u	m x 1	u	m x 1
f_π	p x 1	π	p x 1
F	1 x 1	φ	n x 1
F_x	n x 1	φ_x	n x n
$F_{\dot{x}}$	n x 1	φ_u	m x n
F_u	m x 1	φ_π	p x n
F_π	p x 1		
g	1 x 1	ψ	q x 1
g_x	n x 1	ψ_x	n x q
g_π	p x 1	ψ_π	p x q
G	1 x 1	λ	n x 1
G_x	n x 1	ν	q x 1
G_π	p x 1		

2. Statement of the Problem

The purpose of this paper is to study the minimization of the functional

$$I = \int_0^1 f(x, u, \pi, t)dt + [g(x, \pi)]_1 \tag{1}$$

with respect to the functions $x(t)$, $u(t)$ and the parameter π which satisfy the differential constraint

$$\dot{x} - \varphi(x, u, \pi, t) = 0 \tag{2}$$

the initial condition

$$(x)_0 = \text{given} \tag{3}$$

and the final condition

$$[\psi(x, \pi)]_1 = 0 \tag{4}$$

In the above equations, the functions f and g are scalar, the function φ is an n-vector, and the function ψ is a q-vector. The symbol x, an n-vector, denotes the state variable; the symbol u, an m-vector, denotes the control variable; and the symbol π, a p-vector, denotes the parameter. The time t, a scalar, is the independent variable; without loss of generality, the prescribed initial time is $t = 0$ and the prescribed final time is $t = 1$.

At the initial point, all the components of the state vector are given, so that $(x)_0$ is known. At the final point, q scalar relations are specified, where $0 \le q \le n+p$. Problems where the final time is other than unity can be reduced to the form (1)-(4)

by normalizing the time with respect to the final time and by regarding the final time, if it is free, as one of the components of the parameter π.

3. Exact First-Order Conditions

From calculus of variations (see, for instance, Refs. 8-9), it is known that the previous problem is one of the Bolza type. It can be recast as that of minimizing the functional

$$J = \int_0^1 F dt + (G)_1 \tag{5}$$

subject to (2)-(4). In the above expression, the functions F and G are given by

$$F = f + \lambda^T(\dot{x} - \varphi) , \qquad G = g + \mu^T\psi \tag{6}$$

where λ, an n-vector, is a variable Lagrange multiplier and μ, a q-vector, is a constant Lagrange multiplier.

The optimum solutions x(t), u(t), π must satisfy (2)-(4), the Euler equations

$$(d/dt)F_{\dot{x}} = F_x , \quad 0 = F_u , \quad \int_0^1 F_\pi dt + (G_\pi)_1 = 0 \tag{7}$$

and the following natural condition arising from the transversality condition:

$$(F_{\dot{x}} + G_x)_1 = 0 \tag{8}$$

On account of (6), the explicit form of Eqs. (7)-(8) is the following:

$$\begin{aligned} \dot{\lambda} &= f_x - \varphi_x \lambda \\ 0 &= f_u - \varphi_u \lambda \\ 0 &= \int_0^1 (f_\pi - \varphi_\pi \lambda) dt + (g_\pi + \psi_\pi \mu)_1 \end{aligned} \tag{9}$$

and

$$(\lambda + g_x + \psi_x \mu)_1 = 0 \tag{10}$$

Summarizing, we seek the functions $x(t)$, $u(t)$, $\lambda(t)$ and the parameters π, μ which satisfy Eqs. (2) and (9) subject to the boundary conditions (3), (4), (10).

4. Approximate Methods

In general, the differential system (2)-(4) and (9)-(10) is nonlinear; consequently, approximate methods must be employed. These methods are of two kinds: first-order methods and second-order methods.

Within the context of this paper, let the norm of a vector a be defined as

$$N(a) = a^T a \tag{11}$$

where the superscript T denotes the transpose of a matrix. Let the functionals P and Q be defined as

$$P = \int_0^1 N(\dot{x} - \varphi)dt + N(\psi)_1 \tag{12}$$

and

$$Q = \int_0^1 N(\dot{\lambda} - f_x + \varphi_x \lambda)dt + \int_0^1 N(f_u - \varphi_u \lambda)\, dt + N\left[\int_0^1 (f_\pi - \varphi_\pi \lambda)dt + (g_\pi + \psi_\pi \mu)_1\right] + N(\lambda + g_x + \psi_x \mu)_1 \tag{13}$$

These functionals measure the cumulative errors in the constraints and optimum conditions, respectively. We observe that $P = 0$ and $Q = 0$ for the exact variational solution, while $P > 0$ and $Q > 0$ for any approximation to the variational solution.

When approximate methods are used, they must ultimately lead to functions x(t), u(t), $\lambda(t)$ and parameters π, μ such that

$$P \leq \epsilon_1 \tag{14}$$

and

$$Q \leq \epsilon_2 \tag{15}$$

where ϵ_1 and ϵ_2 are small, preselected numbers. In a first-order method, one tries to decrease sequentially the functional (1) while satisfying Ineq. (14) at each iteration. In a second-order method, one tries to decrease sequentially the overall cumulative error $P + Q$ in the constraints and optimum conditions.

5. Gradient Phase

Suppose that nominal functions $x(t)$, $u(t)$, π satisfying the differential equation (2), the initial condition (3), and the final condition (4) are available. Let $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ denote varied functions satisfying Eqs. (2)-(4) to first order. These varied functions are related to the nominal functions as follows:

$$\tilde{x}(t) = x(t) + \Delta x(t), \quad \tilde{u}(t) = u(t) + \Delta u(t), \quad \tilde{\pi} = \pi + \Delta\pi \tag{16}$$

where $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ denote the perturbations of x, u, π about the nominal values.

To first order, the values of the varied functional $\tilde{I}$ and the nominal functional I are related by

$$\tilde{I} = I + \delta I \tag{17}$$

where the first variation δI is given by

$$\delta I = \int_0^1 (f_x^T \Delta x + f_u^T \Delta u + f_\pi^T \Delta\pi)\, dt + (g_x^T \Delta x + g_\pi^T \Delta\pi)_1 \tag{18}$$

Also to first order, Eq. (2) can be approximated by

$$\Delta\dot{x} - \varphi_x^T \Delta x - \varphi_u^T \Delta u - \varphi_\pi^T \Delta\pi = 0 \tag{19}$$

while the boundary conditions (3)-(4) are written as

$$(\Delta x)_0 = 0 \tag{20}$$

and

$$(\psi_x^T \Delta x + \psi_\pi^T \Delta\pi)_1 = 0 \tag{21}$$

To first order, the minimum of the functional $\tilde{I}$ is achieved if the first variation (18) is minimized subject to (19)-(21). To make the problem meaningful, we require the variations $\Delta u(t)$, $\Delta\pi$ to satisfy the quadratic isoperimetric constraint

$$K = \int_0^1 \Delta u^T \Delta u \, dt + \Delta\pi^T \Delta\pi \tag{22}$$

where K is a prescribed positive constant.

5.1. Variational Approach. From calculus of variations (see, for instance, Refs. 8-9), it is known that the previous problem is one of the Bolza type with an added isoperimetric condition on the variations of the control and the parameter. It can be recast as that of minimizing the augmented functional

$$J = \int_0^1 F \, dt + (G)_1 \tag{23}$$

subject to (19)-(22). In the above expression, the functions F and G are given by

$$\begin{aligned} F &= f_x^T \Delta x + f_u^T \Delta u + f_\pi^T \Delta\pi + \lambda^T(\Delta\dot{x} - \varphi_x^T \Delta x - \varphi_u^T \Delta u - \varphi_\pi^T \Delta\pi) + (1/2\alpha)\Delta u^T \Delta u \\ G &= g_x^T \Delta x + g_\pi^T \Delta\pi + \mu^T(\psi_x^T \Delta x + \psi_\pi^T \Delta\pi) + (1/2\alpha)\Delta\pi^T \Delta\pi \end{aligned} \tag{24}$$

where the n-vector λ is a variable Lagrange multiplier, the q-vector μ is a constant Lagrange multiplier, and the scalar $1/2\alpha$ is a constant Lagrange multiplier. The quantity α is called the stepsize of the gradient phase.

The optimum solutions $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ must satisfy Eqs. (19)-(22), the Euler equations

$$(d/dt)F_{\Delta\dot{x}} = F_{\Delta x}, \quad 0 = F_{\Delta u}, \quad \int_0^1 F_{\Delta\pi} dt + (G_{\Delta\pi})_1 = 0 \tag{25}$$

and the following natural condition arising from the transversality condition:

$$(F_{\Delta\dot{x}} + G_{\Delta x})_1 = 0 \tag{26}$$

On account of (24), the explicit form of Eqs. (25)-(26) is the following:

$$\begin{aligned} \dot{\lambda} &= f_x - \varphi_x \lambda \\ 0 &= f_u - \varphi_u \lambda + \Delta u/\alpha \\ 0 &= \int_0^1 (f_\pi - \varphi_\pi \lambda) dt + (g_\pi + \psi_\pi \mu)_1 + \Delta\pi/\alpha \end{aligned} \tag{27}$$

and

$$(\lambda + g_x + \psi_x \mu)_1 = 0 \tag{28}$$

5.2. Coordinate Transformation. To simplify the problem, we introduce the auxiliary variables

$$A = \Delta x/\alpha, \quad B = \Delta u/\alpha, \quad C = \Delta\pi/\alpha \tag{29}$$

where A denotes an n-vector proportional to the state change, B denotes an m-vector proportional to the control change, and C denotes a p-vector proportional to the parameter change. With these variables, Eqs. (19) and (27) become

$$\begin{aligned} \dot{A} &= \varphi_x^T A + \varphi_u^T B + \varphi_\pi^T C \\ \dot{\lambda} &= f_x - \varphi_x \lambda \\ B &= -f_u + \varphi_u \lambda \\ C &= -\int_0^1 (f_\pi - \varphi_\pi \lambda)\, dt - (g_\pi + \psi_\pi \mu)_1 \end{aligned} \tag{30}$$

and the boundary conditions (20), (21), (28) are written as

$$(A)_0 = 0 \tag{31}$$

and

$$(\psi_x^T A + \psi_\pi^T C)_1 = 0\,, \quad (\lambda + g_x + \psi_x \mu)_1 = 0 \tag{32}$$

Finally, the isoperimetric condition (22) becomes

$$K = \alpha^2 \left[\int_0^1 B^T B dt + C^T C \right] \tag{33}$$

We note that the differential system (30)-(32) is linear and nonhomogeneous in the functions $A(t)$, $B(t)$, $\lambda(t)$ and the parameters C, μ and can be solved without assigning a value to the gradient stepsize α. Once the system (30)-(32) has been solved, the stepsize α can be determined from Eq. (33), since (33) establishes a correspondence between the values of the isoperimetric constant K and the values of the stepsize α. However, there is no way to determine _a priori_ convenient values for the isoperimetric constant K; therefore, the implementation of the algorithm becomes simpler if one avoids evaluating α in terms of K and assigns values to α directly.

5.3. _Integration Technique_. We integrate the differential system (30)-(32) $q+1$ times using a backward-forward integration scheme in combination with the method of particular solutions (Ref. 6). In each integration (subscript i), we assign a different set of values to the components of the multiplier μ, for instance,

$$\mu_i = \begin{bmatrix} \delta_{i1} \\ \delta_{i2} \\ \vdots \\ \delta_{iq} \end{bmatrix}, \qquad i = 1, 2, \ldots, q+1 \tag{34}$$

where the Kronecker delta δ_{ij} is such that

$$\begin{aligned} \delta_{ij} &= 1 , \quad i = j \\ \delta_{ij} &= 0 , \quad i \neq j \end{aligned} \tag{35}$$

With μ_i specified, the corresponding multiplier λ_i at the final point is obtained from (32-2), that is, from

$$(\lambda_i + g_x + \psi_x \mu_i)_1 = 0 , \qquad i = 1, 2, \ldots, q+1 \tag{36}$$

Next, Eq. (30-2) is integrated backward q+1 times to yield the functions

$$\lambda_i = \lambda_i(t) , \qquad i = 1, 2, \ldots, q+1 \tag{37}$$

Then, the functions

$$B_i = B_i(t), \qquad i = 1, 2, \ldots, q+1 \tag{38}$$

are computed from (30-3) and the parameters

$$C_i , \qquad i = 1, 2, \ldots, q+1 \tag{39}$$

are computed from (30-4). Subsequently, Eq. (30-1) is integrated forward q+1 times subject to the initial condition

$$(A_i)_0 = 0 , \qquad i = 1, 2, \ldots, q+1 \tag{40}$$

In this way, we obtain the functions

$$A_i = A_i(t)\,, \qquad i = 1, 2, \ldots, q+1 \tag{41}$$

which are characterized by final values generally not consistent with (32-1). Summarizing, the q+1 particular solutions thus obtained satisfy Eqs. (30), (31), (32-2) but not (32-1).

Next, we introduce the q + 1 undetermined, scalar constants k_i and form the linear combinations

$$A(t) = \sum_{i=1}^{q+1} k_i A_i(t)\,, \quad B(t) = \sum_{i=1}^{q+1} k_i B_i(t)\,, \quad \lambda(t) = \sum_{i=1}^{q+1} k_i \lambda_i(t) \tag{42}$$

and

$$C = \sum_{i=1}^{q+1} k_i C_i\,, \quad \mu = \sum_{i=1}^{q+1} k_i \mu_i \tag{43}$$

Then, we inquire whether, by an appropriate choice of the constants, these linear combinations can satisfy all the differential equations and boundary conditions. By simple substitution, it can be verified that (42)-(43) satisfy the differential equations (30), the initial condition (31), and the final condition (32-2) providing the constants k_i are such that

$$\sum_{i=1}^{q+1} k_i = 1 \tag{44}$$

Finally, the functions (42)-(43) satisfy the final condition (32-1) providing

$$\left[\sum_{i=1}^{q+1} k_i(\psi_x^T A_i + \psi_\pi^T C_i)\right]_1 = 0 \tag{45}$$

The linear system (44)-(45) is equivalent to q+1 scalar equations: the unknowns are the q+1 constants k_i. In this way, the two-point boundary-value problem is solved. After the quantities $A(t)$, $B(t)$, C have been determined and after a stepsize α has been selected (see Section 8), the variations $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ can be computed from (29) and the varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ from (16).

5.4. Descent Property. After suitable manipulations, omitted for the sake of brevity, the first variation (18) can be written in the form

$$\delta I = -\alpha\left[\int_0^1 B^T B dt + C^T C\right] \tag{46}$$

Since the quantity within the brackets is positive, Eq. (46) shows that the first variation δI is negative for $\alpha > 0$. Therefore, if α is sufficiently small, the functional I decreases during the gradient phase.

5.5. Summary of the Gradient Algorithm. In the light of previous discussion, we summarize the gradient algorithm as follows:

(a) Assume nominal functions $x(t)$, $u(t)$, π satisfying the differential equations and boundary conditions (2)-(4).

(b) For the nominal functions, compute the vectors f_x, f_u, f_π and the matrices φ_x, φ_u, φ_π along the interval of integration. At the final point, evaluate the vectors g_x, g_π and the matrices ψ_x, ψ_π.

(c) Integrate the differential system (30)-(32) q+1 times using a backward-forward integration scheme in combination with the method of particular solutions. Obtain the functions $A_i(t)$, $B_i(t)$, $\lambda_i(t)$ and the parameters C_i, μ_i, where $i = 1, 2, \ldots, q+1$.

(d) Solve Eqs. (44)-(45) to obtain the constants k_i, $i = 1, 2, \ldots, q+1$.

(e) Using (42)-(43), combine the particular solutions linearly and obtain the functions $A(t)$, $B(t)$, $\lambda(t)$ and the parameters C, μ.

(f) For a given stepsize α (see Section 8), compute the gradient corrections $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ using Eqs. (29). Then, compute the varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ using Eqs. (16).

6. Restoration Phase

At the end of the gradient phase, the functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ are known. If the differential equation (2) and the final condition (4) are linear, the cumulative constraint error (12) is $\tilde{P} = 0$. On the other hand, if the differential constraint (2) is nonlinear and/or the final condition (4) is nonlinear, the relation $\tilde{P} \neq 0$ holds, which means that some degree of constraint dissatisfaction exists. Therefore, a restoration phase is needed prior to starting the next gradient phase. Specifically, one has to apply small variations $\Delta\tilde{x}(t)$, $\Delta\tilde{u}(t)$, $\Delta\tilde{\pi}$ to $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ to generate new functions

$$\hat{x}(t) = \tilde{x}(t) + \Delta\tilde{x}(t)\ , \quad \hat{u}(t) = \tilde{u}(t) + \Delta\tilde{u}(t)\ , \quad \hat{\pi} = \tilde{\pi} + \Delta\tilde{\pi} \tag{47}$$

such that $\hat{P} = 0$. While there are infinite ways to perform the restoration, the most logical is that developed in Ref. 3: the differential equation and the final condition are restored to a preselected degree of accuracy subject to the least-square change of the control and the parameter.

If quasilinearization is employed, Eqs. (2)-(4) are approximated by[4]

$$\Delta\dot{\tilde{x}} - \tilde{\varphi}_x^T \Delta\tilde{x} - \tilde{\varphi}_u^T \Delta\tilde{u} - \tilde{\varphi}_\pi^T \Delta\tilde{\pi} + (\dot{\tilde{x}} - \tilde{\varphi}) = 0 \tag{48}$$

$$(\Delta\tilde{x})_0 = 0 \tag{49}$$

$$(\tilde{\psi} + \tilde{\psi}_x^T \Delta\tilde{x} + \tilde{\psi}_\pi^T \Delta\tilde{\pi})_1 = 0 \tag{50}$$

In order to prevent the variations $\Delta\tilde{x}(t)$, $\Delta\tilde{u}(t)$, $\Delta\pi$ from becoming too large, we imbed

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[4] The tilde superimposed on the functions φ, ψ and their derivatives denotes evaluation of these quantities at the end of the gradient phase.

Eqs. (48)-(50) in the one-parameter family

$$\Delta\dot{\tilde{x}} - \tilde{\varphi}_x^T \Delta\tilde{x} - \tilde{\varphi}_u^T \Delta\tilde{u} - \tilde{\varphi}_\pi^T \Delta\tilde{\pi} + \tilde{\alpha}(\dot{\tilde{x}} - \tilde{\varphi}) = 0 \tag{51}$$

$$(\Delta\tilde{x})_0 = 0 \tag{52}$$

$$(\tilde{\alpha}\tilde{\psi} + \tilde{\psi}_x^T \Delta\tilde{x} + \tilde{\psi}_\pi^T \Delta\tilde{\pi})_1 = 0 \tag{53}$$

where

$$0 \le \tilde{\alpha} \le 1 \tag{54}$$

denotes a scaling factor, a prescribed constant. In the light of the previous discussion, we seek the minimum of the quadratic functional

$$\tilde{K} = (1/2\tilde{\alpha})\left[\int_0^1 \Delta\tilde{u}^T \Delta\tilde{u}dt + \Delta\tilde{\pi}^T \Delta\tilde{\pi}\right] \tag{55}$$

subject to (51)-(53).

6.1. Variational Approach. From calculus of variations (see, for instance, Refs. 8-9), it is known that the previous problem is one of the Bolza type. It can be recast as that of minimizing the augmented functional

$$J = \int_0^1 Fdt + (G)_1 \tag{56}$$

subject to (51)-(53). In the above expression, the functions F and G are given by

$$\begin{aligned} F &= (1/2\tilde{\alpha})\Delta\tilde{u}^T \Delta\tilde{u} + \tilde{\lambda}^T [\Delta\dot{\tilde{x}} - \tilde{\varphi}_x^T \Delta\tilde{x} - \tilde{\varphi}_u^T \Delta\tilde{u} - \tilde{\varphi}_\pi^T \Delta\tilde{\pi} + \tilde{\alpha}(\dot{\tilde{x}} - \tilde{\varphi})] \\ G &= (1/2\tilde{\alpha})\Delta\tilde{\pi}^T \Delta\tilde{\pi} + \tilde{\mu}^T (\tilde{\alpha}\tilde{\psi} + \tilde{\psi}_x^T \Delta\tilde{x} + \tilde{\psi}_\pi^T \Delta\tilde{\pi}) \end{aligned} \tag{57}$$

where the n-vector $\tilde{\lambda}$ is a variable Lagrange multiplier and the q-vector $\tilde{\mu}$ is a constant Lagrange multiplier. The quantity $\tilde{\alpha}$ is called the stepsize of the restoration phase.

The optimum solutions $\Delta\tilde{x}(t)$, $\Delta\tilde{u}(t)$, $\Delta\tilde{\pi}$ must satisfy Eqs. (51)-(53), the Euler equations

$$(d/dt)F_{\Delta\dot{\tilde{x}}} = F_{\Delta\tilde{x}}, \quad 0 = F_{\Delta\tilde{u}}, \quad \int_0^1 F_{\Delta\tilde{\pi}}\,dt + (G_{\Delta\tilde{\pi}})_1 = 0 \tag{58}$$

and the following natural condition arising from the transversality condition:

$$(F_{\Delta\dot{\tilde{x}}} + G_{\Delta\tilde{x}})_1 = 0 \tag{59}$$

On account of (57), the explicit form of Eqs. (58)-(59) is the following:

$$\begin{aligned} \dot{\tilde{\lambda}} &= -\tilde{\varphi}_x \tilde{\lambda} \\ 0 &= -\tilde{\varphi}_u \tilde{\lambda} + \Delta\tilde{u}/\tilde{\alpha} \\ 0 &= -\int_0^1 \tilde{\varphi}_\pi \tilde{\lambda}\,dt + (\tilde{\psi}_\pi \tilde{\mu})_1 + \Delta\tilde{\pi}/\tilde{\alpha} \end{aligned} \tag{60}$$

and

$$(\tilde{\lambda} + \tilde{\psi}_x \tilde{\mu})_1 = 0 \tag{61}$$

6.2. Coordinate Transformation. To simplify the problem, we introduce the auxiliary variables

$$\tilde{A} = \Delta\tilde{x}/\tilde{\alpha}, \quad \tilde{B} = \Delta\tilde{u}/\tilde{\alpha}, \quad \tilde{C} = \Delta\tilde{\pi}/\tilde{\alpha} \tag{62}$$

where $\tilde{A}$ denotes an n-vector proportional to the state change, $\tilde{B}$ denotes an m-vector proportional to the control change, and $\tilde{C}$ denotes a p-vector proportional to the parameter change. With these variables, Eqs. (51) and (60) become

$$\dot{\tilde{A}} = \tilde{\varphi}_x^T \tilde{A} + \tilde{\varphi}_u^T \tilde{B} + \tilde{\varphi}_\pi^T \tilde{C} - (\dot{\tilde{x}} - \tilde{\varphi})$$

$$\dot{\tilde{\lambda}} = - \tilde{\varphi}_x \tilde{\lambda}$$

$$\tilde{B} = \tilde{\varphi}_u \tilde{\lambda} \tag{63}$$

$$\tilde{C} = \int_0^1 \tilde{\varphi}_\pi \tilde{\lambda}\, dt - (\tilde{\psi}_\pi \tilde{\mu})_1$$

and the boundary conditions (52), (53), (61) are written as

$$(\tilde{A})_0 = 0 \tag{64}$$

and

$$(\tilde{\psi} + \tilde{\psi}_x^T \tilde{A} + \tilde{\psi}_\pi^T \tilde{C})_1 = 0 \ , \quad (\tilde{\lambda} + \tilde{\psi}_x \tilde{\mu})_1 = 0 \tag{65}$$

We note that the differential system (63)-(65) is linear and nonhomogeneous in the functions $\tilde{A}(t)$, $\tilde{B}(t)$, $\tilde{\lambda}(t)$ and the parameters $\tilde{C}, \tilde{\mu}$ and can be solved without assigning a value to the restoration stepsize $\tilde{\alpha}$. Once the system (63)-(65) has been solved, the stepsize $\tilde{\alpha}$ must be determined so as to reduce the cumulative constraint error (12).

6.3. Integration Technique. We integrate the differential system (63)-(65) q+1 times using a backward-forward integration scheme in combination with the method of particular solutions (Ref. 6). In each integration (subscript i), we assign a different set of values to the components of the multiplier $\tilde{\mu}$, for instance,

$$\tilde{\mu}_i = \begin{bmatrix} \delta_{i1} \\ \delta_{i2} \\ \vdots \\ \delta_{iq} \end{bmatrix} \ , \qquad i = 1, 2, \ldots, q+1 \tag{66}$$

where the Kronecker delta δ_{ij} is such that

$$\begin{aligned} \delta_{ij} &= 1\,, \quad i = j \\ \delta_{ij} &= 0\,, \quad i \neq j \end{aligned} \tag{67}$$

With $\tilde{\mu}_i$ specified, the corresponding multiplier $\tilde{\lambda}_i$ at the final point is obtained from (65-2), that is, from

$$(\tilde{\lambda}_i + \tilde{\psi}_x \tilde{\mu}_i)_1 = 0\,, \qquad i = 1, 2, \ldots, q+1 \tag{68}$$

Next, Eq. (63-2) is integrated backward q+1 times to yield the functions

$$\tilde{\lambda}_i = \tilde{\lambda}_i(t)\,, \qquad i = 1, 2, \ldots, q+1 \tag{69}$$

Then, the functions

$$\tilde{B}_i = \tilde{B}_i(t)\,, \qquad i = 1, 2, \ldots, q+1 \tag{70}$$

are computed from (63-3) and the parameters

$$\tilde{C}_i\,, \qquad i = 1, 2, \ldots, q+1 \tag{71}$$

are computed from (63-4). Subsequently, Eq. (63-1) is integrated forward q+1 times subject to the initial condition

$$(\tilde{A}_i)_0 = 0\,, \qquad i = 1, 2, \ldots, q+1 \tag{72}$$

In this way, we obtain the functions

$$\tilde{A}_i = \tilde{A}_i(t)\,, \qquad i = 1, 2, \ldots, q+1 \tag{73}$$

which are characterized by final values generally not consistent with (65-1). Summarizing, the q+1 particular solutions thus obtained satisfy Eqs. (63), (64), (65-2) but not (65-1).

Next, we introduce the q+1 undetermined, scalar constants $\tilde{k}_i$ and form the linear combinations

$$\tilde{A}(t) = \sum_{i=1}^{q+1} \tilde{k}_i \tilde{A}_i(t), \quad \tilde{B}(t) = \sum_{i=1}^{q+1} \tilde{k}_i \tilde{B}_i(t), \quad \tilde{\lambda}(t) = \sum_{i=1}^{q+1} \tilde{k}_i \tilde{\lambda}_i(t) \tag{74}$$

and

$$\tilde{C} = \sum_{i=1}^{q+1} \tilde{k}_i \tilde{C}_i, \quad \tilde{\mu} = \sum_{i=1}^{q+1} \tilde{k}_i \tilde{\mu}_i \tag{75}$$

Then, we inquire whether, by an appropriate choice of the constants, these linear combinations can satisfy all the differential equations and boundary conditions. By simple substitution, it can be verified that (74)-(75) satisfy the differential equations (63), the initial condition (64), and the final condition (65-2) providing the constants $\tilde{k}_i$ are such that

$$\sum_{i=1}^{q+1} \tilde{k}_i = 1 \tag{76}$$

Finally, the functions (74)-(75) satisfy the final condition (65-1) providing

$$\left[\tilde{\psi} + \sum_{i=1}^{q+1} \tilde{k}_i (\tilde{\psi}_x^T \tilde{A}_i + \tilde{\psi}_\pi^T \tilde{C}_i)\right]_1 = 0 \tag{77}$$

The linear system (76)-(77) is equivalent to q+1 scalar equations: the unknowns are the q+1 constants $\tilde{k}_i$. In this way, the two-point boundary-value problem is solved. After the quantities $\tilde{A}(t)$, $\tilde{B}(t)$, $\tilde{C}$ have been determined and after a stepsize $\tilde{\alpha}$ has been selected, the variations $\Delta\tilde{x}(t)$, $\Delta\tilde{u}(t)$, $\Delta\tilde{\pi}$ can be computed from (62) and the varied functions $\hat{x}(t)$, $\hat{u}(t)$, $\hat{\pi}$ from (47). Of course, the restoration phase must be performed iteratively until a desired degree of accuracy is obtained, that is, until the cumulative constraint error satisfies the Ineq. (14).

6.4. Descent Property. After suitable manipulations, omitted for the sake of brevity, the first variation of the cumulative constraint error can be written in the form

$$\delta\tilde{P} = -2\tilde{\alpha}\tilde{P} \tag{78}$$

Since $\tilde{P} > 0$, Eq. (78) shows that the first variation $\delta\tilde{P}$ is negative for $\tilde{\alpha} > 0$. Therefore, if $\tilde{\alpha}$ is sufficiently small, the cumulative constraint error (12) decreases during the restoration phase.

6.5. Summary of the Restoration Algorithm. In the light of the previous discussion, we summarize the restoration algorithm as follows:

(a) Assume the functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ obtained at the end of the gradient phase as the nominal functions.

(b) For the nominal functions, compute the vector $\dot{x} - \varphi$ and the matrices $\varphi_x, \varphi_u, \varphi_\pi$ along the interval of integration. At the final point, evaluate the vector ψ and the matrices ψ_x, ψ_π.

(c) Integrate the differential system (63)-(65) q+1 times using a backward-forward integration scheme in combination with the method of particular solutions. Obtain the functions $\tilde{A}_i(t)$, $\tilde{B}_i(t)$, $\tilde{\lambda}_i(t)$ and the parameters $\tilde{C}_i, \tilde{\mu}_i$, where $i = 1, 2, \ldots, q+1$.

(d) Solve Eqs. (76)-(77) to obtain the constants $\tilde{k}_i$, $i = 1, 2, \ldots, q+1$.

(e) Using (74)-(75), combine the particular solutions linearly and obtain the functions $\tilde{A}(t)$, $\tilde{B}(t)$, $\tilde{\lambda}(t)$ and the parameters $\tilde{C}$, $\tilde{\mu}$.

(f) Assuming $\tilde{\alpha} = 1$, compute the restoration corrections $\Delta\tilde{x}(t)$, $\Delta\tilde{u}(t)$, $\Delta\tilde{\pi}$ using Eqs. (62). Then, compute the varied functions $\hat{x}(t)$, $\hat{u}(t)$, $\hat{\pi}$ using Eqs. (47).

(g) For the varied functions $\hat{x}(t)$, $\hat{u}(t)$, $\hat{\pi}$, compute the cumulative constraint error $\hat{P}$ using Eq. (12). If $\hat{P} < \tilde{P}$, the stepsize $\tilde{\alpha} = 1$ is acceptable. If $\hat{P} > \tilde{P}$, the previous value of $\tilde{\alpha}$ must be replaced by some smaller value in the range (54) until the condition $\hat{P} < \tilde{P}$ is met. This can be achieved through successive bisections of $\tilde{\alpha}$.

(h) After a value of $\tilde{\alpha}$ in the range (54) has been found such that $\hat{P} < \tilde{P}$, the first cycle of the restoration phase is completed. Next, the functions $\hat{x}(t)$, $\hat{u}(t)$, $\hat{\pi}$ given by Eqs. (47) are employed as the nominal functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ for the second iteration, and the procedure is repeated until a desired degree of accuracy is obtained, that is, until the cumulative constraint error satisfies Ineq. (14).

(i) Once the restoration algorithm is completed, verify the inequality

$$\hat{I} < I \tag{79}$$

If Ineq. (79) is satisfied, start the next gradient phase. If Ineq. (79) is violated, return to the previous gradient phase and reduce the gradient stepsize α until, after restoration, Ineq. (79) is satisfied.

7. Order of Magnitude Analysis

The functions at the end of the restoration phase and the functions at the beginning of the gradient phase are related by

$$\hat{x}(t) = x(t) + \Delta x(t) + \Delta\tilde{x}(t)$$

$$\hat{u}(t) = u(t) + \Delta u(t) + \Delta\tilde{u}(t) \tag{80}$$

$$\hat{\pi} = \pi + \Delta\pi + \Delta\tilde{\pi}$$

where $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ are the gradient corrections and $\Delta\tilde{x}(t)$, $\Delta\tilde{u}(t)$, $\Delta\tilde{\pi}$ are the restoration corrections.

For the gradient phase, the solutions $A(t)$, $B(t)$, C of the system (30)-(32) are independent of the gradient stepsize α. Therefore, the gradient corrections $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ have the order

$$\Delta x(t) = O(\alpha)\ , \quad \Delta u(t) = O(\alpha)\ , \quad \Delta\pi = O(\alpha) \tag{81}$$

For the restoration phase, the magnitude of the solutions $\tilde{A}(t)$, $\tilde{B}(t)$, $\tilde{C}$ of the system (63)-(65) depends on the magnitude of the forcing terms appearing in Eqs. (63-1) and (65-1). Since the constraints (2)-(4) are satisfied exactly at the beginning of the gradient phase and to first order at the end of the gradient phase, a Taylor expansion shows that the forcing terms are

$$\dot{\tilde{x}} - \tilde{\varphi} = O(\alpha^2)\ , \qquad (\tilde{\psi})_1 = O(\alpha^2) \tag{82}$$

Therefore, the solutions of the system (63)-(65) are

$$\tilde{A}(t) = O(\alpha^2)\,, \quad \tilde{B}(t) = O(\alpha^2)\,, \quad \tilde{C} = O(\alpha^2) \tag{83}$$

If one assumes the restoration stepsize $\tilde{\alpha} = O(1)$, the restoration corrections $\Delta\tilde{x}(t)$, $\Delta\tilde{u}(t)$, $\Delta\tilde{\pi}$ have the order

$$\Delta\tilde{x}(t) = O(\alpha^2)\,, \quad \Delta\tilde{u}(t) = O(\alpha^2)\,, \quad \Delta\tilde{\pi} = O(\alpha^2) \tag{84}$$

It follows from (81) and (84) that, for sufficiently small values of the gradient stepsize α,

$$|\Delta\tilde{x}(t)| \ll |\Delta x(t)|, \quad |\Delta\tilde{u}(t)| \ll |\Delta u(t)|, \quad |\Delta\tilde{\pi}| \ll |\Delta\pi| \tag{85}$$

7.1. <u>Descent Property of the Algorithm.</u> Finally, we consider functions $x(t)$, $u(t)$, π and $\hat{x}(t)$, $\hat{u}(t)$, $\hat{\pi}$ satisfying the constraints (2)-(4) within the preselected degree of accuracy (14). To first order, the difference of the values of the functional I is given by

$$\begin{aligned}\hat{I} - I \cong \int_0^1 [f_x^T(\Delta x + \Delta\tilde{x}) + f_u^T(\Delta u + \Delta\tilde{u}) + f_\pi^T(\Delta\pi + \Delta\tilde{\pi})]\, dt \\ + [g_x^T(\Delta x + \Delta\tilde{x}) + g_\pi^T(\Delta\pi + \Delta\tilde{\pi})]_1\end{aligned} \tag{86}$$

On account of (85), Eq. (86) can be approximated by

$$\hat{I} - I \cong \int_0^1 [f_x^T\Delta x + f_u^T\Delta u + f_\pi^T\Delta\pi]dt + (g_x^T\Delta x + g_\pi^T\Delta\pi)_1 = -\alpha\left[\int_0^1 B^T B dt + C^T C\right] \tag{87}$$

Therefore, for α sufficiently small, the restoration algorithm preserves the descent property of the gradient algorithm: the functional I decreases between any two successive restoration phases.

8. Stepsize Determination

In order to obtain a reasonable convergence rate, it is essential that the gradient stepsize α be determined in an optimal fashion. The choice of α can be made on the basis of information available during the gradient phase (Method 1) or on the basis of information available at the end of the restoration phase (Method 2).

Method 1. At the end of the gradient phase, the functions $A(t)$, $B(t)$, C which solve Eqs. (30)-(32) are available. With these functions, one can form the one-parameter family

$$\tilde{x}(t) = x(t) + \alpha A(t), \quad \tilde{u}(t) = u(t) + \alpha B(t), \quad \tilde{\pi} = \pi + \alpha C \tag{88}$$

and explore the behavior of the functional (1) with respect to the parameter α. For the family (88), the functional (1) becomes

$$\tilde{I} = \int_0^1 f(x + \alpha A, u + \alpha B, \pi + \alpha C, t)\,dt + [g(x + \alpha A, \pi + \alpha C)]_1 \tag{89}$$

Since the nominal functions $x(t)$, $u(t)$, π and the correction functions $A(t)$, $B(t)$, C are known, Eq. (89) has the form

$$\tilde{I} = \tilde{I}(\alpha) \tag{90}$$

The slope of this function at the origin is negative and is given by

$$\dot{\tilde{I}}(0) = -\left[\int_0^1 B^T B\,dt + C^T C\right] \tag{91}$$

Assuming that a minimum of $\tilde{I}(\alpha)$ exists, one can employ a quadratic interpolation scheme or a cubic interpolation scheme to determine the optimum value of the gradient stepsize α, that is, that value for which

$$\dot{\tilde{I}}(\alpha) = 0 \tag{92}$$

If necessary, these procedures can be used iteratively until the modulus of the slope becomes such that

$$|\dot{\tilde{I}}(\alpha)| \leq \epsilon_3 \tag{93}$$

where ϵ_3 is a small, preselected number.

At any rate, the value of α supplied by the quadratic or cubic interpolation scheme is acceptable only if

$$\tilde{I}(\alpha) < \tilde{I}(0) \tag{94}$$

Otherwise, α must be replaced by some smaller value (for example, with a bisection process) until Ineq. (94) is met.

In order to limit the constraint violation, one may require the solution of Eq. (92) to be subordinated to the inequalities

$$\alpha \leq \epsilon_4 \text{ and/or } \tilde{P} \leq \epsilon_5 \tag{95}$$

where ϵ_4 and ϵ_5 are small, preselected numbers. Incidentally, Ineqs. (95) are of fundamental importance in cases where the function $\tilde{I}(\alpha)$ is monotonically decreasing, that is, where Eq. (92) has no real solution.

Remark 8.1. Both the quadratic interpolation scheme and the cubic interpolation scheme are first-order techniques in that they employ the function $\tilde{I}(\alpha)$ and its first derivative

$\dot{\tilde{I}}(\alpha)$. Alternatively, one can solve Eq. (92) by quasilinearization, as done, for example in Ref. 4. Since quasilinearization employs the function $\tilde{I}(\alpha)$ and its first and second derivatives $\dot{\tilde{I}}(\alpha)$ and $\ddot{\tilde{I}}(\alpha)$, the resulting algorithm is a hybrid: this is due to the fact that the basic system of variations is obtained from first-order considerations, while the gradient stepsize is obtained from second-order considerations.

Remark 8.2. Within the general frame of Method 1, the optimum gradient stepsize can also be determined by replacing the functional I with any of the following functionals (Ref. 10):

$$I_1 = I + L \ , \quad I_2 = I + kP, \quad I_3 = I + L + kP \tag{96}$$

where

$$\begin{aligned} L &= \int_0^1 [\lambda^T(\dot{x} - \varphi)]dt + (\mu^T \psi)_1 \\ P &= \int_0^1 N(\dot{x} - \varphi)dt + N(\psi)_1 \end{aligned} \tag{97}$$

and where k is a positive constant. Note that L is linear in the constraints and P is quadratic in the constraints; also note that (96-1) is called the augmented functional, (96-2) the penalty functional, and (96-3) the augmented penalty functional. For the one-parameter family (88), the functionals (97) have the form

$$\tilde{L} = \tilde{L}(\alpha) \ , \qquad \tilde{P} = \tilde{P}(\alpha) \tag{98}$$

so that

$$\tilde{I}_1 = \tilde{I}_1(\alpha), \quad \tilde{I}_2 = \tilde{I}_2(\alpha) \ , \quad \tilde{I}_3 = \tilde{I}_3(\alpha) \tag{99}$$

If the gradient stepsize is $\alpha = 0$, we have

$$\tilde{L}(0) = \tilde{P}(0) = 0 , \qquad \dot{\tilde{L}}(0) = \dot{\tilde{P}}(0) = 0 \tag{100}$$

so that

$$\begin{aligned} \tilde{I}_1(0) &= \tilde{I}_2(0) = \tilde{I}_3(0) = \tilde{I}(0) \\ \dot{\tilde{I}}_1(0) &= \dot{\tilde{I}}_2(0) = \dot{\tilde{I}}_3(0) = \dot{\tilde{I}}(0) \end{aligned} \tag{101}$$

Since the functionals (96) have the same basic descent property as the functional I, any of them can be employed to determine an appropriate value for the gradient stepsize α.

<u>Method 2</u>. This method makes use of information available at the end of the restoration phase. To each gradient stepsize α corresponds a restored curve $\hat{x}(t)$, $\hat{u}(t)$, $\hat{\pi}$. The totality of curves obtained with the combined gradient-restoration algorithm constitute the one-parameter family

$$\hat{x}(t) = h_1(\alpha, t) , \quad \hat{u}(t) = h_2(\alpha, t), \quad \hat{\pi} = h_3(\alpha) \tag{102}$$

For this family, the functional (1) has the form

$$\hat{I} = \hat{I}(\alpha) \tag{103}$$

In particular, for $\alpha = 0$, the ordinate and the slope of the curve $\hat{I}(\alpha)$ are identical with the ordinate and the slope of the curve $\tilde{I}(\alpha)$.

If the minimal problem (1)-(4) is well posed, the function (103) exhibits a relative minimum with respect to the gradient stepsize α. Therefore, a quadratic interpolation scheme or a cubic interpolation scheme can be employed to determine the optimum value of the gradient stepsize α, that is, that value for which

$$\dot{\hat{I}}(\alpha) = 0 \tag{104}$$

approximately. This value of α is acceptable providing

$$\hat{I}(\alpha) < \hat{I}(0) \tag{105}$$

Otherwise, α must be replaced by some smaller value (for example, with a bisection process) until Ineq. (105) is met.

9. Remarks

The following remarks are pertinent to the previous theoretical development:

Remark 9.1. The restoration algorithm employs quasilinearization at its best due to the fact that the corrections $\Delta\tilde{u}(t)$, $\Delta\tilde{\pi}$ are kept at the smallest average value compatible with the linearized constraints and boundary conditions. Nevertheless, situations may arise where, because of the nonlinearity of the constraints, even the present minimum corrections are too large for the cumulative constraint error (12) to decrease. This is why it is necessary to include the scaling factor $\tilde{\alpha}$ in Eqs. (51)-(53) and, consequently, in the solutions for $\Delta\tilde{x}(t)$, $\Delta\tilde{u}(t)$, $\Delta\tilde{\pi}$.

Remark 9.2. At the end of the gradient phase, the state, the control and the parameter are updated by superimposing on $x(t)$, $u(t)$, π the corrections $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ computed by solving Eqs. (2)-(4) and (27)-(28). An alternate way is to update the control and the parameter from $\tilde{u}(t) = u(t) + \Delta u(t)$ and $\tilde{\pi} = \pi + \Delta\pi$ and determine the new state $\tilde{x}(t)$ by forward integration of Eq. (2) subject to the initial condition (3). This procedure is especially convenient in problems where the final time is given while the final state is free: the differential constraints and boundary conditions are automatically restored, and the restoration phase can be bypassed.

Remark 9.3. A remark analogous to 9.2 holds for updating the state, the control, and the parameter at the end of each restoration cycle. One may compute the new control and the new parameter from $\hat{u}(t) = \tilde{u}(t) + \Delta\tilde{u}(t)$ and $\hat{\pi} = \tilde{\pi} + \Delta\tilde{\pi}$ and determine the new state $\hat{x}(t)$ by forward integration of Eq. (2) subject to the initial condition (3).

Remark 9.4. At the end of the gradient phase, the cumulative constraint error (12) must be computed. If P violates Ineq. (14), the restoration phase is started. If P satisfies Ineq. (14), the restoration phase is bypassed and the next gradient phase is started.

Remark 9.5. Numerical experiments indicate that the degree of dissatisfaction of the constraints occurring at the end of the gradient phase decreases rapidly as successive gradient phases are performed. Consequently, the number of restoration cycles decreases as the sequential gradient-restoration algorithm progresses toward termination. In practice, after several gradient phases have been executed, Ineq. (14) is satisfied at the end of the gradient phase, in which case the restoration algorithm is bypassed.

Remark 9.6. The present algorithm can be started even if the nominal curve $x(t)$, $u(t)$, π is not consistent with the differential equation (2) and the final condition (4). In this case, the first gradient phase must be preceded by a restoration phase performed in accordance with Section 6.

Remark 9.7. The sequential gradient-restoration algorithm is terminated either when Ineq. (15) is satisfied or when

$$|\hat{I} - I| \leq \epsilon_6 \tag{106}$$

where ϵ_6 denotes a small, preselected number.

Remark 9.8. An important characteristic of the sequential gradient-restoration algorithm is that it yields a physically possible solution $\hat{x}(t)$, $\hat{u}(t)$, $\hat{\pi}$ at the end of each iteration. Sometimes, the behavior of some other functional I' is of interest in addition to that of the functional I. If both I and I' are computed at the end of each iteration, one can realistically evaluate the sequence of solutions obtained and decide whether a solution less than optimal from the point of view of the functional I is actually more desirable from the point of view of the functional I'.

Remark 9.9. Another positive characteristic of the present formulation is that it is suitable for design studies; this is due to the inclusion of the parameter π in Eqs. (1)-(4). Consider a configuration which depends on r scalar design parameters. If these scalar design parameters are regarded as components of the vector parameter π, the sequential gradient-restoration algorithm can be employed to yield the optimum values of the r scalar design parameters simultaneously with the optimum trajectory of the system.

10. Discussion and Conclusions

In this paper, the problem of minimizing a functional I involving the state x(t), the control u(t), and the parameter π is considered. The admissible state, control, and parameter are required to satisfy a vector differential equation, a vector initial condition, and a vector final condition.

A sequential algorithm composed of the alternate succession of gradient phases and restoration phases is presented. In the gradient phase, the first-order change of the functional is minimized subject to the linearized differential equation, the linearized boundary conditions, and a quadratic constraint on the variations of the control and the parameter. In the restoration phase, a functional quadratic in the variations of the control and the parameter is minimized subject to the linearized differential equation and the linearized boundary conditions. For both the gradient phase and the restoration phase, the differential system describing the optimum corrections is linear. Its solution is obtained using a backward-forward integration scheme in combination with the method of particular solutions.

Criteria are presented to determine the gradient stepsize α from either conditions at the end of the gradient phase or conditions at the end of the restoration phase. It is shown that, if α is the gradient stepsize, the gradient corrections are of order α and the restoration corrections are of order α^2. Therefore, for α sufficiently small, the restoration phase preserves the descent property of the gradient phase: the functional I decreases between any two successive restoration phases.

At this time, several numerical examples are under way for problems with final coordinates given, free, or partly given and partly free. From the preliminary results, it appears that the sequential gradient-restoration algorithm exhibits fast convergence to the desired solution. A full description of these computer experiments will be given in subsequent papers.

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