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A Green's Function Approach to the Natural Vibration of Thin Spherical Shell Segments (A Numerical Method)

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SUMMARY

This report presents a development of a Green's function approach to the analysis of linear, undamped vibration of thin spherical shell segments. The segment may be of arbitrary contour, simply or multiply connected and may have mixed arbitrary boundary conditions specified along the bounding contour. As a demonstration, the developed method is applied to the shallow shell symmetric vibration problem and results are compared with a published solution.

INTRODUCTION

Thin spherical shells and spherical shell segments frequently find application in space vehicle structures. As such vehicles are likely to experience dynamic excitation from any of several sources, a knowledge of the vibrational behavior and response to excitation of spherical shells is essential to rational design and to trouble shooting analysis.

Other investigations have dealt with symmetric spherical segments and with symmetric boundary conditions. Both ani-symmetric and unsymmetric vibration modes have been treated. Excepting for finite element techniques, insufficient attention has been directed to the spherical shell segment of general contour and with arbitrary boundary conditions.

In what follows, an approximate method is developed for obtaining frequencies and mode shapes of thin spherical shell segments. The class of problems considered is limited by the following assumptions:

1. The vibration is undamped.

2. The vibration is of small amplitude and hence a linear elastic law holds for the spherical shell.

3. The shell is thin, permitting use of the simplified elastic law in which the thickness-to-radius ratio is ignored when compared with unity.

It should be noted, however, that although subject to the above limitations, the developed approach has application to spherical shell segments with arbitrary boundary conditions and of arbitrary shape, including multiply connected shapes (within a spherical surface).

GREEN'S FUNCTION FORMULATION

The vibration problem to be considered is first replaced by an equivalent static problem. A static load proportional to displacement is substituted for the inertial loading of the vibrating shell. Additionally, the artifice of an elastic foundation is introduced such that the foundation reaction is proportional to displacement but in opposite sense. Thus, if the applied load is proportional to displacement, then also the net load on the shell (applied load together with foundation reaction) adheres to this proportionality. This manner of reacting applied loads permits a simplifying symmetry for the required fundamental problems and yet does not disturb the proportionality of load to displacement which is necessary to simulate the vibration problem.

The relationship between the posed vibration problem and the equivalent static problem may be expressed symbolically:*

(a) For the vibration problem

$$\vec{q} = \mu\, \omega^2\, \vec{u}$$

where,

$\vec{q}$ = inertial shell force per unit surface area

μ = shell mass per unit surface area

ω = natural angular frequency

$\vec{u}$ = displacement vector of the middle surface

(b) For the shell on the elastic foundation

$$\vec{q} = \frac{1}{\lambda}\,\vec{u} - k\vec{u} = \left(\frac{1}{\lambda} - k\right)\vec{u}$$

where

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*Symbols are listed in Appendix I.

$\vec{q}$ = net load on shell (per unit surface area)

λ = proportionality factor between applied load and displacement

k = foundation modulus

Consequently, if $(\frac{1}{\lambda} - k)$ is made equal to $\mu\,\omega^2$, the static problem is equivalent to the vibration problem.

The equivalent static problem is next formulated in terms of influence functions, that is, response functions to unit stimuli applied to the complete sphere on the elastic foundation. In brief, if the displacement and stress fields are known for a unit load (and a unit couple) applied to a point on the complete spherical shell then through superposition, the required relationships may be written satisfying boundary conditions (specified along the bounding contour) as well as the condition that applied load must be proportional to displacement. In addition to the distributed surface load over the spherical segment of interest, a line load system is applied along the bounding contour to assist in meeting the specified boundary conditions.

At this point, it is convenient to define four "vector" quantities (that is, four sets of associated components). If we introduce a global reference coordinate system, surface points may be located by the polar coordinates ϕ and θ, the usual latitudinal and meridinal angles (ϕ measured from the pole).

Let two three-dimensional vectors, q_i and u_i be defined such that:

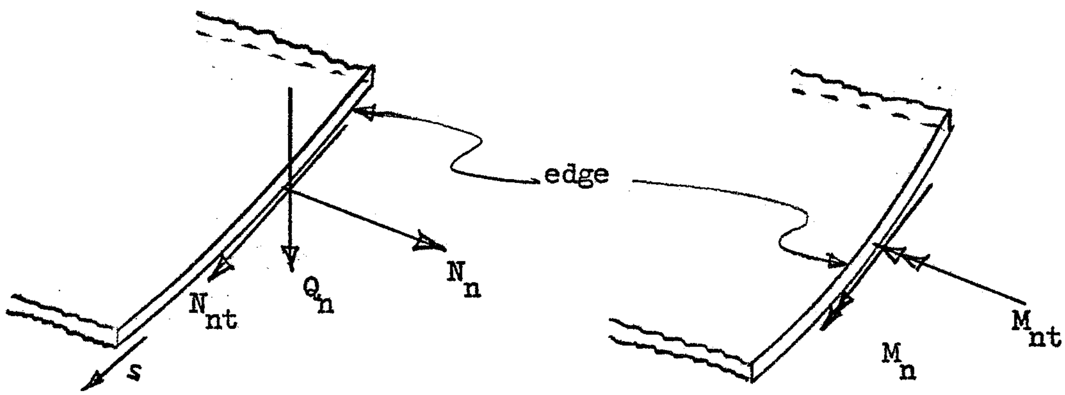
q_1, q_2, q_3 are physical components of the applied surface force (per unit surface area) in three orthogonal directions: respectively -- normal to the surface, tangential to the meridian circle (the increasing ϕ direction), and tangential to the latitude circle (the increasing θ direction).

u_1, u_2, u_3 are the displacement components in the normal and two tangential directions, respectively.

Consider next two four-dimensional "vectors", R_J and L_J* associated with the spherical segment boundary. The vector R_J represents the four boundary condition "residuals", that is, quantities that must vanish to meet the prescribed boundary conditions. As an example, if a free boundary

*Upper case subscripts refer to four-dimensional quantities, lower case Latin indices are used for three-dimensional quantities and Greek indices are reserved for the two-dimensional surface tensors to be introduced later.

is specified, the edge resultants (per unit length) must vanish at the boundary. Referring to the sketch and the introduced symbols below, the static equivalent edge force system consists of the quantities V_n, S_{nt}, N_n and M_n.



Stress Resultants   Stress Couple Resultants

N_n and N_{nt} are membrane stress resultants

Q_n is normal shear resultant

M_{nt} is the twisting moment

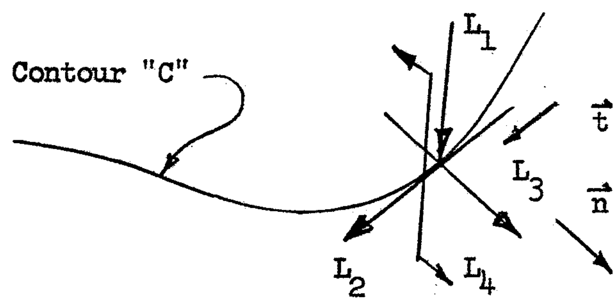
$V_n = \left(Q_n + \frac{\partial M_{nt}}{\partial s}\right)$, a static equivalent normal edge reaction

$S_{nt} = \left(N_{nt} - \frac{M_{nt}}{a}\right)$, a static equivalent tangential edge reaction

a = radius of sphere.

For the free edge boundary condition, we let R_1, R_2, R_3, R_4 be V_n, S_{nt}, N_n and M_n, respectively. Then, $R_J = 0$, establishes the free edge condition. Other possible components for the "residual vector", R_J, include w, $\bar{u}_n$, $\bar{u}_t$ and $\frac{\partial \bar{w}}{\partial n}$, which are components of displacement and the normal derivative of $\bar{w}$. These latter quantities would be involved if boundary conditions concern displacements.

The "vector," L_J, has as components line loads applied to the complete shell along the segment boundary contour, curve "C".



Referring to the above sketch, let L_J be:

L_1, L_2, L_3, the physical components of force (per unit contour length), respectively: normal to the surface, tangential to curve C and in the direction of the unit surface vector $\vec{n}$.

L_4 is the force-couple (per unit length) with axis tangential to curve C.

Let p_i and F_M be "base" load vectors with components defined similarly to vectors q_i and L_M, respectively, except that p_i and F_M are for concentrated loads and have components of unit magnitude.

Next we define the influence matrix functions:

$A_{ij}(\phi,\theta,\bar{\phi},\bar{\theta})$ as the displacement u_i (at point ϕ,θ) due to a load vector component, p_j, applied at point $\bar{\phi},\bar{\theta}$.

$B_{Kj}(s,\bar{\phi},\bar{\theta})$ as the residual component, R_K, (at point s on C) due to the load component, p_j, applied at point $\bar{\phi},\bar{\theta}$.

$C_{KM}(s,t)$ as the residual component, R_K, (at point s on C) due to a load component, F_M, applied at point t on contour C.

$D_{iM}(\phi,\theta,t)$ as the displacement component, u_i (at point ϕ,θ) due to a load component, F_M, at point t on contour C.

Then for any applied surface load vector of components q_i, acting over the surface element $\underline{d\sigma}$, the contribution to residual, R_K, is:

$$dR_K = \sum_{j=1}^{3} q_j d\sigma \, B_{Kj}$$

or employing the summation convention (for repeated indices):

$$dR_K = B_{Kj} \, q_j \, d\sigma$$

Similarly, for the line load system of components L_M acting over arc length $\underline{ds}$ of the contour $\underline{C}$, the contribution to residual, R_K, is:

$$dR_K = C_{KM} \, L_M \, ds$$

The requirement that all residuals vanish along contour $\underline{C}$ then leads to the integral equation:

$$\iint_S B_{Kj} \, q_j \, d\sigma + \oint_C C_{KM} \, L_M \, ds = 0 \tag{1}$$

In a similar fashion contributions to displacements, u_i, from the applied surface loads q_i and line load system L_M lead to the relationship:

$$\iint_S A_{ij} \, q_j \, d\sigma + \oint_C D_{iM} \, L_M \, ds = u_i \tag{2}$$

Next consider the vector function L_M (t) along contour $\underline{C}$ to be eliminated between equations (1) and (2) (employing appropriate inverse operators). The resulting equation could be symbolized by:

$$\iint_S G_{ij} \, q_j \, d\sigma = u_i \tag{3}$$

Finally, the requirement that the surface load vector field be proportional to the displacement vector field, which proportionality may be expressed as,

$$q_i = \frac{1}{\lambda} u_i$$

leads to the eigenvalue problem:

$$\iint G_{ij} \; u_j^{d\tau} = \lambda \, u_i \tag{4}$$

Given the matrix function, G_{ij} $(\phi, \theta, \bar{\phi}, \bar{\theta})$, the "Green's Function" of the problem, it is possible by numerical methods to obtain eigenvalues and eigenfunctions for the equation (4). These, in turn, provide frequencies and mode shapes for the original vibration problem.

FUNDAMENTAL PROBLEMS

The Green's Function, G_{ij}, (of the eigenvalue problem, equation 4) may be obtained from the influence functions, A_{ij}, B_{Kj}, C_{KM} and D_{iM} as introduced and defined in the foregoing discussion.* The influence functions, in turn, are seen to be response functions (stress resultants or displacements) to unit load stimuli. The stimuli include: the unit normal load, unit tangential load and the unit force-couple with tangential axis. We then define three fundamental problems associated with each of the three stimuli:

Fundamental Problem I consists of a complete spherical shell on an elastic foundation subjected to a unit normal load.

Fundamental Problem II is similar to Problem I except that the unit load is tangential to the spherical surface.

Fundamental Problem III is again the same except that the shell is subjected to a unit force couple.

The solutions of each of the three fundamental problems is developed in the discussion that follows:

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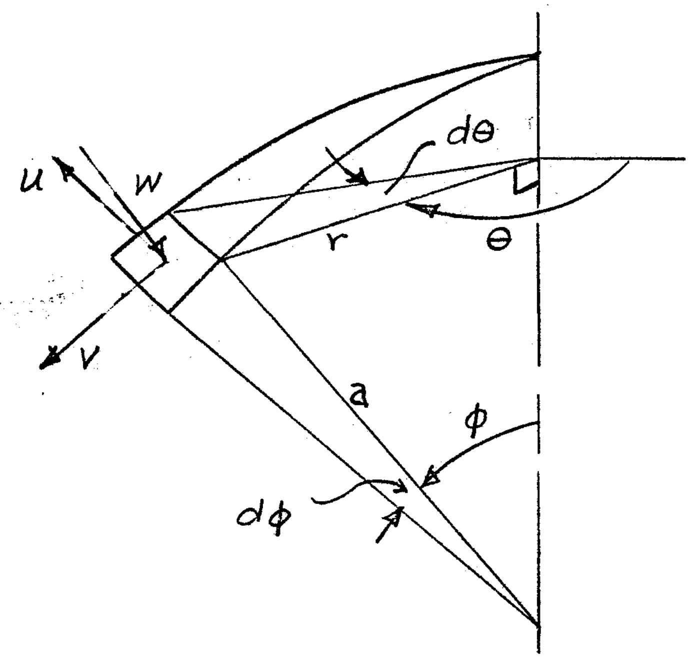
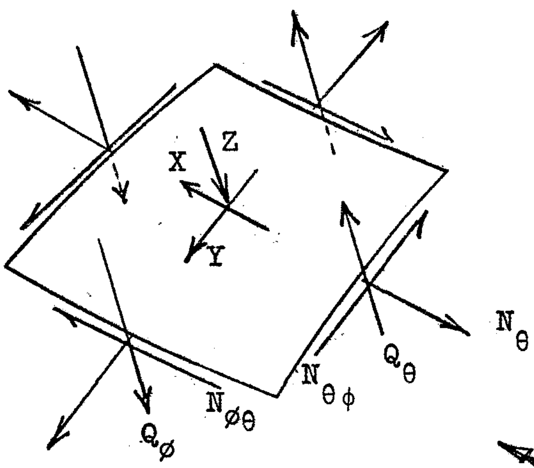
* The mechanics of obtaining the Green's Function from the influence functions is treated in discussion to follow.

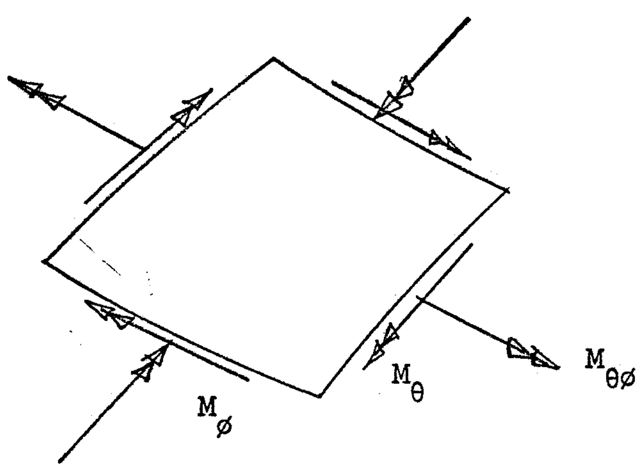
Unit Normal Load Problem

Consistent with the symbols and conventions employed by Timoshenko in "Theory of Plates and Shells", the stress resultants and positive directions for displacements are taken as shown in the following sketches:





Stress Resultants and Loads



Force Couple Resultants

Equilibrium of an element of the spherical shell leads to:

$$
\begin{aligned}
&(r\,N_{\phi})^{\circ} + a\,N'_{\theta\phi} - a\,N_{\theta}\cos\phi - rQ_{\phi} = -ra\,Y \\
&(r\,N_{\phi\theta})^{\circ} + a\,N'_{\theta} + a\,N_{\theta\phi}\cos\phi - a\,Q_{\theta}\sin\phi = -ra\,X \\
&a\,N_{\theta}\sin\phi + r\,N_{\phi} + a\,Q'_{\theta} + (r\,Q_{\phi})^{\circ} = -ra\,Z \\
&(r\,M_{\phi})^{\circ} + a\,M'_{\theta\phi} - a\,M_{\theta}\cos\phi = ra\,Q_{\phi} \\
&(r\,M_{\phi\theta})^{\circ} + a\,M'_{\theta} + a\,M_{\theta\phi}\cos\phi = ra\,Q_{\theta}
\end{aligned}
\qquad (5\text{ a-e})
$$

where

$$(\;\;)^{\circ} \equiv \frac{\partial(\;\;)}{\partial\phi} \quad \text{and} \quad (\;\;)' = \frac{\partial(\;\;)}{\partial\theta} \qquad r = a\sin\phi$$

From strain-displacement relationships and Hooke's Law, the thin spherical shell elastic law reduces to the following:

$$
\begin{aligned}
&N_{\phi} = K\,(\epsilon_{\phi} + \nu\,\epsilon_{\theta}) \\
&N_{\theta} = K\,(\epsilon_{\theta} + \nu\,\epsilon_{\phi}) \\
&N_{\phi\theta} = N_{\theta\phi} = K\left(\frac{1-\nu}{2}\right)\epsilon_{\phi\theta} \\
&M_{\phi} = -D\,(\kappa_{\phi} + \nu\,\kappa_{\theta}) \\
&M_{\theta} = -D\,(\kappa_{\theta} + \nu\kappa_{\phi}) \\
&M_{\phi\theta} = M_{\theta\phi} = -D\left(\frac{1-\nu}{2}\right)\kappa_{\phi\theta}
\end{aligned}
\qquad (6\text{ a-f})
$$

where

$$\epsilon_{\phi} = (\bar{v}^{\circ} - \bar{w})\,\frac{1}{a} \qquad \epsilon_{\theta} = \left(\frac{\bar{u}'}{\sin\phi} + \bar{v}\cot\phi - \bar{w}\right)\frac{1}{a}$$

$$\epsilon_{\phi\theta} = \left(\bar{u}^{\circ} - \bar{u}\cot\phi + \frac{\bar{v}'}{\sin\phi}\right)\frac{1}{a}$$

$$\kappa_{\phi} = \frac{(\bar{v} + \bar{w}^{\circ})^{\circ}}{a^2} \qquad \kappa_{\theta} = \frac{\bar{u}'}{a^2\sin\phi} + \frac{(\bar{v} + \bar{w}^{\circ})}{a^2}\cot\phi + \frac{\bar{w}''}{a^2\sin^2\phi}$$

$$\kappa_{\phi\theta} = \frac{1}{a^2}\left(\bar{u}^{\circ} - \bar{u}\cot\phi + \frac{\bar{v}'}{\sin\phi} - \frac{2\,\bar{w}'}{\sin\phi}\cot\phi + \frac{2\,\bar{w}^{\circ\prime}}{\sin\phi}\right)$$

$$K = \frac{Eh}{1-\nu^2} \quad , \quad D = \frac{Eh^3}{12(1-\nu^2)}$$

The elastic foundation reaction loads are:

$$X = - k\overline{u} \quad , \quad Y = - k\overline{v} \quad , \quad Z = - k\overline{w}$$

Since Fundamental Problem I possesses polar symmetry, equations (5) and (6) may be simplified accordingly. For polar symmetry, the displacement $\overline{u}$ is zero and derivatives with respect to the variable, θ, vanish. However, several steps are still required to obtain the governing differential equations for Problem I in the simplest form.

We start by solving equations (6a) and (6b) for the strains ϵ_ϕ and ϵ_θ. This leads to:*

$$(v^\circ - w) = \frac{a}{K(1-\nu^2)} \left(N_\phi - \nu N_\theta\right) \tag{7a}$$

$$(v \cot \phi - w) = \frac{a}{K(1-\nu^2)} \left(N_\theta - \nu N_\phi\right) \tag{7b}$$

Differentiating the latter equation, we obtain,

$$v^\circ \cot \phi - \frac{v}{\sin^2 \phi} - w^\circ = \frac{a}{K(1-\nu^2)} \left(N^\circ_\theta - \nu N^\circ_\phi\right) \tag{7c}$$

The functions v° and w may be eliminated from the three equations (7 a-c) to yield:

$$(v + w^\circ) = \frac{a}{K(1-\nu^2)} \left[\nu N^\circ_\phi - N^\circ_\theta + (1 + \nu)(N_\phi - N_\theta) \cot\phi\right]$$

or introducing $\chi \equiv \dfrac{v+w^\circ}{a}$,

$$K\,(1-\nu^2)\,\chi = \left[\nu N^\circ_\phi - N^\circ_\phi + (1+\nu)(N_\phi - N_\theta) \cot \phi\right] \tag{8}$$

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*The symbols u, v, w (without bars) denote functions of ϕ only.

Consider next that portion of the shell lying above the latitudinal plane, ϕ = constant. Acting upon this free body are the edge forces, N_ϕ and Q_ϕ, the foundation reaction forces, and the applied unit load at the pole. Equilibrium then leads to:

$$N_\phi = - Q_\phi \cot\phi - \frac{1}{2\pi a \sin^2\phi} + \frac{ka}{\sin^2\phi} \int_0^\phi \left[v \sin\phi + w \cos\phi\right] \sin\phi \, d\phi \quad (9a)$$

Upon substitution of this expression into the equilibrium equation (5c) we obtain:

$$N_\theta = - Q_\phi^\circ + \frac{1}{2\pi a \sin^2\phi} - \frac{ka}{\sin^2\phi} \int_0^\phi \left[v \sin\phi + w \cos\phi\right] \sin\phi \, d\phi + ka\, w \quad (9b)$$

When expressions (9) are introduced into equation (8) the latter reduces to:

$$\left[K\,(1-\nu^2) + ka^2\right] \chi = L\,(Q_\phi) + \nu\, Q_\phi + ka(2 + \nu)v \qquad (10)$$

where the operator L is defined by:

$$L\,(\ \ ) = (\ \ )^{\circ\circ} + (\ \ )^\circ \cot\phi - (\ \ ) \cot^2\phi$$

Substitution of the elastic law equations (6d,e) into the equilibrium equation (5d) yields an expression for Q_ϕ in terms of χ:

$$- \frac{a^2}{D} Q_\phi + L\,(\chi) - \nu\chi \qquad (11)$$

To obtain a third second order differential equation for Problem I, we first add elastic law equations (6a) and (6b):

$$v^\circ + v \cot\phi = \frac{a}{K(1+\nu)} (N_\phi + N_\theta) + 2w$$

Then we substitute expressions (9) into the right hand side to obtain:

$$v^\circ + v \cot\phi = \frac{-a}{K(1+\nu)} \left[Q_\phi^\circ + Q_\phi \cot\phi\right] + \left[2 + \frac{\beta}{1+\nu}\right] w \qquad (12)$$

Differentiation of this equation and combination with equation (10) yields:

$$L\,(v) - v = (1+\nu)\,(a\chi) + \left(\frac{aQ_\phi}{K}\right) + (\beta-2)v \qquad (13)$$

where:

$$\beta = \frac{ka^2}{K}$$

We now define the symbols:

$$Y_1 = v^\circ + v \cot\phi \qquad Y_2 = v$$

$$Y_3 = \frac{a}{K}\left[Q_\phi{}^\circ + Q_\phi \cot\phi\right] \qquad Y_4 = \frac{aQ_\phi}{K} \tag{14}$$

$$Y_5 = a\left[\chi^\circ + \chi \cot\phi\right] \qquad Y_6 = a\chi$$

Then the three second order equations (10) (11) and (13) may be written as a set of six first order linear differential equations in the six functions Y_1 through Y_6. In matrix form these are:

$$\frac{d}{d\phi}\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{bmatrix} = \begin{bmatrix} 0 & (\beta-2) & 0 & 1 & 0 & (1+\nu) \\ 1 & -\cot\phi & 0 & 0 & 0 & 0 \\ 0 & -(2+\nu)\beta & 0 & -(1+\nu) & 0 & (1-\nu^2+\beta) \\ 0 & 0 & 1 & -\cot\phi & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\alpha} & 0 & -(1-\nu) \\ 0 & 0 & 0 & 0 & 1 & -\cot\phi \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{bmatrix} \tag{15}$$

where

$$\alpha = \frac{D}{Ka^2} = \frac{h^2}{12a^2}$$

With suitable sets of initial values assigned the functions $Y_1,\ldots,Y_6$ solutions may be obtained numerically for this set of equations. (The Kutta-Merson technique proves to be an efficient and accurate means of accomplishing this integration).* Subsequently, through superposition of solutions, boundary conditions may be satisfied both at the origin and at a remote boundary, $\phi = \phi_n$.

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*See L. Fox, Numerical Solution of Ordinary and Partial Differential Equations (1962), Pergamon Press, (Addison-Wesley), pp. 24-26.

For the very thin spherical shell a special difficulty arises in the numerical integration of equations (15). If the parameter $\underline{\alpha}$ is extremely small an instability arises in the numerical integration due to the element, $1/\alpha$, in the coefficient matrix of equations (15). For almost any set of initial values the magnitudes of the functions Y_i grow rapidly with increasing ϕ while in the physical problem a rapid decay actually occurs of the functions, Y_1 through Y_6. This instability condition produces a serious accumulation of loss in significant figures in the superposition process, should the numerical integration extend over a large range in ϕ .

The difficulty may be circumvented through the use of a "segmental" numerical integration technique as described in Appendix II.

Directing attention now to the task of obtaining consistent initial values for functions Y_i near the apex, $\phi = 0$, to provide a starting solution we proceed to simplify the governing equations (10) and (11) based upon the small magnitude of the independent variable ϕ . First, we assume the displacement v (near the origin) to be negligible in comparison with the quantity, $a\chi$ (the validity of which assumption is readily established from the resulting solution). Then equations (10) and (11) may be combined to obtain:

$$L^2(\chi) + \varkappa^4\chi = 0 \tag{16}$$

where

$$\varkappa^4 = \frac{1-\nu^2 + \beta - \nu^2\alpha}{\alpha}$$

Equation (16) is factorable into two second order equations:

$$L\,(\chi) \pm i\,\varkappa^2\chi = 0 \tag{17}$$

We further simplify these for small ϕ by substitution of $1/\phi$ for $\cot\phi$. The operator L then becomes:

$$L = (\quad)^{\circ\circ} + \frac{1}{\phi}(\quad)^{\circ} - \frac{1}{\phi^2}(\quad)$$

Equations (17) are then transformed into Bessel's equations by appropriate variable substitutions. The resulting equations have solutions in terms of the Kelvin (or Thompson) functions.* Combining solutions, we obtain:

$$\chi = A_1 \mathrm{ker}'\, x + A_2 \mathrm{kei}'\, x \qquad \text{where } x = \varkappa\phi$$

From the polar symmetry condition, χ must vanish at the origin. Therefore,

$$\chi = A_2 \mathrm{kei}'\, x \, .$$

Using the additional boundary conditions that the shear Q_ϕ must satisfy the load singularity requirement,

$$\lim_{\phi \to 0} (2\pi\phi Q_\phi a) = -1$$

and the condition that w, as evaluated from equation (12), must be finite, the following solution results:

$$K\,Y_4 = \frac{\varkappa}{2\pi}\left[\mathrm{ker}'\ x - \frac{1+\nu}{\varkappa^2}\,\mathrm{kei}'\ x\right]$$

$$K\,Y_6 = -\frac{1}{2\pi\varkappa\alpha}\,\mathrm{kei}'\ x$$

$$K\,Y_2 = \frac{1}{2\pi\varkappa}\left[\mathrm{kei}'\ x + \frac{1+\nu}{\varkappa^2\alpha}\left(\mathrm{ker}'\ x + \frac{1}{x}\right)\right]$$

$$K\,w = \frac{-1}{2\pi\varkappa^2\alpha}\,\mathrm{kei}\ x$$

These values together with the function definitions (13) then provide a starting solution, or set of initial values, for the cap particular solution. The actual numerical integration could not start exactly at the origin as singularities prevent this, but the solution (18) permits a start at an aribitrarily small value of ϕ.

For the three "complementary" cap solutions (associated with no unit load singularity) starting solutions for initial conditions are readily

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*See Flugge, Stresses in Shells, pp. 345.

obtained by substitution of polynomials in ϕ for the functions Y_2, Y_4 and Y_6 in the differential equations (15) and retention of terms up to third order in ϕ. Three linearly independent solutions result:

$$Y_2 = \frac{\phi}{2} \quad \frac{\beta-2}{16}\,\phi^3$$
$$Y_4 = -(2+\nu)\beta\,\frac{\phi^3}{16}$$
$$Y_6 = 0$$

and

$$Y_2 = \frac{\phi^3}{16}$$
$$Y_4 = \frac{\phi}{2} - \frac{1+\nu}{16}\,\phi^3$$
$$Y_6 = -\frac{\phi^3}{16\alpha}$$

and

$$Y_2 = \frac{1+\nu}{16}\,\phi^3$$
$$Y_4 = (1-\nu^2+\beta)\,\frac{\phi^3}{16}$$
$$Y_6 = \frac{\phi}{2} - \frac{1-\nu}{16}\,\phi^3$$

Employing the foregoing starting solutions for the initial values in the first spherical cap, the segmental integration technique (as discussed) affords a numerical method for solving Fundamental Problem I. (Plots of computational results for an example problem appear in figures 1 through 3).

Unit Tangent Load Problem

An essentially similar pattern of solution is employed for the Fundamental Problem II, the unit tangent load problem. First, consistent with problem symmetry, if the tangent unit load is considered to lie in the meridianal plane $\theta = 0$, displacement functions $\bar{u}$, $\bar{v}$ and $\bar{w}$ may be taken as:

$$\bar{u} = u \sin\theta, \quad \bar{v} = v \cos\theta, \quad \bar{w} = w \cos\theta$$

where u, v and w are functions of ϕ only.

Secondly, it should be noted that the function w is known from the solution of Problem I since by Betti's reciprocal theorem the displacement function v of Problem I is equal to -w of Problem II.

Substitution of expressions for $\bar{u}$, $\bar{v}$ and $\bar{w}$ into the elastic law equations (6) and these in turn into equilibrium equations (5a,b) leads to:

$$
\begin{aligned}
(1+\alpha) \Big\{ & \frac{1-\nu}{2} (u^{\circ\circ}\sin\phi + u^{\circ}\cos\phi) - u\Big[\frac{1}{\sin\phi} - \frac{1-\nu}{2}(1-\cot^2\phi)\sin\phi \Big] \\
& - \frac{1+\nu}{2} v^{\circ} - \frac{3-\nu}{2} v \cot\phi + (1+\nu)w \Big\} \qquad (19a) \\
& - \alpha \Big\{ w^{\circ\circ} + w^{\circ}\cot\phi + w\Big(2 - \frac{1}{\sin^2\phi}\Big) \Big\} = \beta u \sin\phi
\end{aligned}
$$

and

$$
\begin{aligned}
(1+\alpha) \Big\{ & \frac{1+\nu}{2} u^{\circ} - \frac{3-\nu}{2} u \cot\phi + v^{\circ\circ}\sin\phi + v^{\circ}\cos\phi \\
& - \frac{v}{\sin\phi}\Big[\cos^2\phi + \nu \sin^2\phi + \frac{1-\nu}{2} \Big] - (1+\nu)\, w^{\circ}\sin\phi \Big\} \qquad (19b) \\
& + \alpha \Big\{ w^{\circ\circ\circ}\sin\phi + w^{\circ\circ}\cos\phi + w^{\circ}\Big(2 - \frac{2}{\sin^2\phi}\Big)\sin\phi + 2w\frac{\cos\phi}{\sin^2\phi} \Big\} = \beta v \sin\phi
\end{aligned}
$$

If equation (19a) is differentiated, multiplied by $\sin\phi$ and added to (19b), we succeed in eliminating w and obtain upon reduction:

$$\tau^{\circ\circ} - \tau^{\circ}\cot\phi - \rho\tau = 0 \qquad (20)$$

where

$$\tau = \hat{u}^{\circ} + v$$

$$\hat{u} = u \sin\phi$$

$$\rho = \frac{2\beta}{(1-\nu)(1+\alpha)} - 2$$

A second useful differential equation is obtained through the elimination of v in equation (19a) by introducing τ. The resulting equation reduces to:

$$J(\hat{u}-\bar{\alpha}w) = \frac{1-\nu}{2}\rho(\hat{u}-\bar{\alpha}w) + \frac{1+\nu}{2}\tau^{\circ} + \frac{3-\nu}{2}\tau\cot\phi \qquad (21)$$
$$- \left[(1+\nu) - \bar{\alpha}\left(\frac{1-\nu}{2}\rho + 2\right)\right] w$$

where

$$J(\ ) = L(\ ) - (\ )$$
$$\bar{\alpha} = \frac{\alpha}{1+\alpha}$$

Introducing the symbols,

$$Z_1 = \frac{\tau^{\circ}}{\sin\phi} \qquad Z_2 = \frac{\tau}{\sin\phi}$$
$$Z_3 = Z_4^{\circ} + Z_4\cot\phi \qquad Z_4 = \hat{u} - \bar{\alpha}w$$

equations (20) and (21) are expressible as four first order equations:

$$Z_1^{\circ} = \rho Z_2$$
$$Z_2^{\circ} = Z_1 - Z_2\cot\phi \qquad (22)$$
$$Z_3^{\circ} = \frac{1+\nu}{2}\sin\phi\, Z_1 + \frac{3-\nu}{2}\cos\phi\, Z_2 + \frac{1-\nu}{2}\rho Z_4 - \left[(1+\nu) - \bar{\alpha}\left(\frac{1-\nu}{2}\rho + 2\right)\right] w$$
$$Z_4^{\circ} = Z_3 - Z_4\cot\phi$$

Unlike equations (15) for Fundamental Problem I, the above equations (22) do not contain the coefficient $1/\alpha$, and hence lend themselves to a stable numerical integration.

A solution of equations (22) for the complete sphere is then achieved through several steps as outlined below:

(1) Obtain a "particular" solution for the hemispherical shell consistent with the load singularity.

(2) Obtain two linearly independent "complimentary" solutions for the hemispherical shell consistent with the no unit load condition at the origin.

(3) Combine the three solutions employing symmetry principles to

to arrive at a solution in superposition for the complete spherical shell. We consider the steps further detail:

Step 1

A starting solution is first required. Near the load singularity, with the variable ϕ suitably small, only the first terms in the Maclaurin's series for $\sin\phi$ and $\cos\phi$ are retained. Consequently equation (20) becomes:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - (1+\frac{1}{x^2})y = 0 \qquad (23)$$

where

$$y = Z_2 \qquad \text{and} \qquad x = \sqrt{\rho}\,\phi$$

Retaining the Bessel function $K_1(x)$ as the solution consistent with the load singularity:

$$y = \frac{\tau}{\sin\phi} \doteq \frac{\tau}{\phi} \qquad \text{or} \qquad \tau = -AxK_1(x)$$

Ignoring the function w for small ϕ (the validity of which is evident upon examining resulting solutions) and introducing the solution for τ in equation (21) yields:

$$\frac{d^2\hat{u}}{ds^2} + \frac{1}{s}\frac{d\hat{u}}{ds} - (1+\frac{1}{s^2})\hat{u} = \left[\frac{1+\nu}{1-\nu}\,xK_0(x) - \frac{3-\nu}{1-\nu}\,K_1(x)\right]\frac{A}{\sqrt{\rho}} \qquad (24)$$

where

$$s = \sqrt{\frac{1-\nu}{2}}\,\rho\,\phi$$

Combining a particular and a complimentary solution of equation (24) one may obtain:

$$\hat{u} = \frac{A}{\rho}\,\frac{3-\nu}{1-\nu}\sqrt{\frac{1-\nu}{2}}\,\rho\,\left[\frac{1}{s} - K_1(s)\right]$$

which contains only the logarithmic singularity. In the resulting expression for u the dominant term (for small ϕ) is:

$$u = -A\frac{(3-\nu)}{4}\,\text{Log}\,(s)$$

and hence for v (from the definition of τ) we have:

$$v = A\left[\frac{3-\nu}{4}\,\text{Log }(s) - \frac{1+\nu}{4}\right]$$

In the neighborhood of the unit load the equilibrium boundary condition reduces to:

$$1 = \int_0^{2\pi} a\,(N_{\phi\theta}\sin\theta - N_\phi\cos\theta + Q_\phi\sin\phi\,\cos\theta)\,\sin\phi\,d\theta$$

Substitution of the elastic law for the membrane stresses together with the solutions for u and v permits the constant A to be evaluated:

$$A = -\frac{1}{\pi(1-\nu)}\cdot\frac{a}{K}$$

With initial values for Z_1, Z_2, Z_3 and Z_4 for an appropriately small value of ϕ taken directly from the starting solution, the Kutta-Merson procedure is then employed to extend the solution to the "equator", $\phi = \frac{\pi}{2}$. The resulting solution consitutes a "particular" solution for the hemispherical shell, that is, a solution consistent with the unit load singularity.

Step 2

Two additional independent solutions for the hemispherical shell are obtained from two separate sets of initial conditions.

In one case assuming a finite initial value for N_ϕ, a consistent solution for small ϕ is (considering low order terms):

$$Z_1 = 2$$

$$Z_2 = \phi$$

$$Z_3 = -2\,\frac{5+\nu}{1-\nu}\,\frac{1}{\rho}$$

$$Z_4 = Z_3\,\frac{\phi}{2}$$

A second solution is obtained for zero initial N_ϕ:

$Z_1 = 0$

$Z_2 = 0$

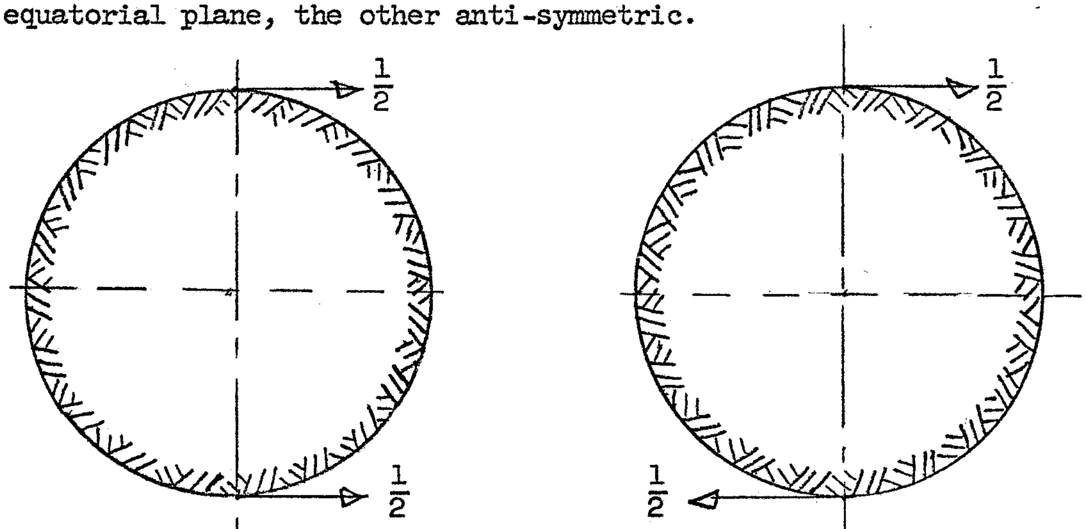
$Z_3 = 2$

$Z_4 = \phi$

Using these starting solutions, the Kutta-Merson integration provides two independent complimentary solutions for the hemispherical shell.

Step 3

The complete shell problem under the action of a unit tangential load may be resolved into two component problems, one symmetric with respect to the equatorial plane, the other anti-symmetric.



Symmetric Anti-symmetric

The boundary conditions at the equator for the symmetric problem are:

$v = 0$ and $u^\circ = 0$.

For the anti-symmetric case the equatorial boundary conditions are:

$v^\circ = 0$ and $u = 0$.

Each of the component problems may now be solved independently by superposition of solutions from Steps (1) and (2) to match the appropriate equatorial boundary conditions. Then finally, the two component problems are superimposed to obtain the solution to Fundamental Problem II. Computed results for a sample problem are plotted in figures 4 through 6 .

Unit Force-Couple Problem

The Fundamental Problem III entails a unit force couple at the origin acting on the complete spherical shell on the elastic foundation. To permit the meridianal plane, $\theta = 0$, to be a plane of symmetry, the unit force-couple vector is directed (in tangential plane) in the direction $\theta = -\pi/2$ radians. As with Problem II the displacement functions $\bar{u}$, $\bar{v}$ and $\bar{w}$ are then taken as $\bar{u} = u \sin\theta$, $\bar{v} = v \cos\theta$, $\bar{w} = w \cos\theta$.

The Betti reciprocal theorem may then be employed together with solutions from Fundamental Problems I and II to provide directly the solutions to Problem III.

We denote the displacement functions of the three Fundamental Problems as follows:

Problem I	v_I, w_I
Problem II	u_{II}, v_{II}, w_{II}
Problem III	u_{III}, v_{III}, w_{III}

Then observing from Betti's Theorem that the work done by a unit force couple rotated through the angle,

$$\frac{1}{r}\bar{w}'_{II} + \bar{u}_{II} = -\frac{w_{II}}{a \sin\phi} \sin\theta + u_{II} \sin\theta$$

is equal to the work of a unit tangent load displaced through the tangential displacement $\bar{u}_{III}$, we have:

$$u_{III} = -\frac{w_{II}}{a \sin\phi} + u_{II}$$

Similar applications of the reciprocal theorem lead to:

$$v_{III} = aX_{II} = w^{\circ}_{II} + v_{II}$$

$$w_{III} = -aX_{I} = -w^{\circ}_{I} - v_{I}$$

$$w_{II} = -v_{I}$$

Combining the four relationships above, Problem III displacement functions may be expressed as follows:

$$u_{III} = \frac{v_I}{a \sin\phi} + u_{II}$$

$$v_{III} = - v^{\circ}_{I} + v_{II} \qquad (25)$$

$$w_{III} = - w^{\circ}_{I} - v_{I}$$

<u>Summary of Required Functions</u>

The matrix influence functions A_{ij}, B_{Kj}, C_{KM}, D_{iM}, are expressed in terms of specific response functions associated with each of the three <u>fundamental problems</u>. These include:

a) the displacement functions,

u, v, w, w°,

b) the stress resultants and moment components,

$N_{\phi}, N_{\theta}, N_{\phi\theta}, Q_{\phi}, Q_{\theta}, M_{\phi}, M_{\theta}, M_{\phi\theta}$ and

c) the partial derivatives of moment components,

$M^{\circ}_{\phi}, M^{\circ}_{\theta}, M^{\circ}_{\phi\theta}, \frac{1}{r}M'_{\phi}, \frac{1}{r}M'_{\theta}, \frac{1}{r}M'_{\phi\theta}$

From the elastic law equations (6), the stress resultant, moments, and moment derivatives are expressible in terms strain components, curvature components and their derivatives:

$$\epsilon_\phi,\ \epsilon_\theta,\ \epsilon_{\phi\theta},\ \kappa_\phi,\ \kappa_\theta,\ \kappa_{\phi\theta},\ \kappa_\phi^\circ,\ \kappa_\theta^\circ,\ \kappa_{\phi\theta}^\circ\ .$$

The normal shear stress resultants, Q_ϕ and Q_θ may be obtained directly from equilibrium equations (5d,e) in terms moment components; (however, in Problem I, the shear Q_ϕ is most readily obtained from the dependent variable Y_4).

Assuming at this point that the set of equations (15) and the set of equations (22) are solved numerically for the functions Y_1 through Y_6 and Z_1 through Z_4, respectively, we now proceed to express the required response functions in terms of these solutions.

Without rewriting the elastic law equations (6) here, we may observe that:

$$\frac{M'_\phi}{r} = -\frac{M_\phi}{a\sin\phi}\ ,\ \frac{M'_\theta}{r} = -\frac{M_\theta}{a\sin\phi}\ ,\ \frac{M'_{\phi\theta}}{r} = \frac{M_{\phi\theta}}{a\sin\phi} \tag{26}$$

and also that M_ϕ°, M_θ°, and $M_{\phi\theta}^\circ$ require the curvature derivatives:

$$\kappa_\phi^\circ = \frac{1}{a^2}\left(v^{\circ\circ} - \mathring{w}^{\circ\circ}\right) \tag{27}$$

$$\kappa_\theta^\circ = \frac{1}{a^2}\left[\frac{u^\circ}{\sin\phi} - \frac{u\cos\phi}{\sin^2\phi} + (v^\circ + w^{\circ\circ})\cot\phi - (v + 2w^\circ)\frac{1}{\sin^2\phi} + \frac{2w\cos\phi}{\sin^3\phi}\right]$$

$$\kappa_{\phi\theta}^\circ = \frac{1}{a^2}\left[u^{\circ\circ} - u\cot\phi + \frac{u}{\sin^2\phi} - \frac{\dot{v}^\circ}{\sin\phi} + \frac{v\cos}{\sin^2\phi} - \frac{2w^{\circ\circ}}{\sin\phi}\right.$$

$$\left. + \frac{4w^\circ\cos\phi}{\sin^2\phi} - \frac{2w(1+\cos^2\phi)}{\sin^3\phi}\right]$$

It is seen from expressions (26) and (27) together with the elastic law equations (6) that the following displacements and their derivatives are required for each of three fundamental problems:

$$u,\ u^\circ,\ u^{\circ\circ},\ v,\ v^\circ,\ v^{\circ\circ};\ w,\ w^\circ,\ w^{\circ\circ},\ \mathring{w}^{\circ\circ}$$

Additionally, from the reciprocal relations for Problem III, the displacements $v_I^{\circ\circ\circ}$ and $w_I^{\circ\circ\circ\circ}$ are also required (for Problem I).

With aid of the definitions of the dependent variables Y_i and Z_i together with the governing differential equations (12) (15) and (22) the following expressions are developed for the required displacement quantities terms of Y_i and Z_i).

Problem I (non-zero functions only)

$$
\begin{aligned}
v_I &= Y_2 \\
v_I^{\circ} &= Y_2^{\circ} = Y_1 - Y_2 \cot\phi \\
v_I^{\circ\circ} &= v_I^{\circ} \cot\phi + \left(\beta - 2 + \frac{1}{\sin^2\phi}\right) Y_2 + Y_4 + (1+\nu)\, Y_6 \\
v_I^{\circ\circ\circ} &= -\, v_I^{\circ\circ} \cot\phi + v_I^{\circ} \left(\beta - 2 + \frac{1}{\sin^2\phi}\right) - \frac{2 \cos\phi}{\sin^3\phi}\, Y_2 \\
&\quad + Y_3 - Y_4 \cot\phi + (1+\nu)(Y_5 - Y_6 \cot\phi) \\
w_I &= \frac{1}{2 + \frac{\beta}{1+\nu}} \left(Y_1 + \frac{1}{1+\nu}\, Y_3\right) \\
w_I^{\circ} &= Y_6 - Y_2 \\
w_I^{\circ\circ} &= -\, v_I^{\circ} + Y_5 - Y_6 \cot\phi \\
w_I^{\circ\circ\circ} &= -\, v_I^{\circ\circ} - (w_I^{\circ\circ} + v_I^{\circ}) \cot\phi - \frac{1}{\alpha}\, Y_4 + \left[\frac{1}{\sin^2\phi} - (1-\nu)\right] Y_6 \\
w_I^{\circ\circ\circ\circ} &= -\, v_I^{\circ\circ\circ} - (w_I^{\circ\circ\circ} + v_I^{\circ\circ}) \cot\phi + (w_I^{\circ\circ} + v_I^{\circ}) \left[\frac{2}{\sin^2\phi} - (1-\nu)\right] \\
&\quad - \frac{1}{\alpha} (Y_3 - Y_4 \cot\phi) - \frac{2 \cos\phi}{\sin^3\phi}\, Y_6
\end{aligned}
\tag{28}
$$

Problem II

For convenience we first consider the quanity, $\hat{u} = u_{II} \sin \phi$, and express its derivatives as follows:

$$\hat{u} = Z_4 - \bar{\alpha}\, v_I$$

$$\hat{u}^{\circ} = Z_4^{\circ} - \bar{\alpha}\, v_I^{\circ}$$

where

$$Z_4^{\circ} = Z_3 - Z_4 \cot \phi$$

$$\hat{u}^{\circ\circ} = Z_4^{\circ\circ} - \bar{\alpha}\, v_I^{\circ\circ}$$

where

$$Z_4^{\circ\circ} = Z_3^{\circ} - Z_4^{\circ} \cot\phi + Z_4 \frac{1}{\sin^2\phi} \tag{29}$$

$$Z_3^{\circ} = \frac{1+\nu}{2} \sin\phi\, Z_1 + \frac{3-\nu}{2} \cos\phi\, Z_2 + \frac{1-\nu}{2}\rho\, Z_4 + \left[(1+\nu) - \bar{\alpha}\left(\frac{1-\nu}{2}\rho + 2\right)\right] v_I$$

Z_4° given above

$$\hat{u}^{\circ\circ\circ} = Z_4^{\circ\circ\circ} - \bar{\alpha}\, v_I^{\circ\circ\circ}$$

where

$$Z_4^{\circ\circ\circ} = Z_3^{\circ\circ} - Z_4^{\circ\circ} \cot\phi + Z_4^{\circ} \frac{2}{\sin^2\phi} - Z_4 \frac{2\cos\phi}{\sin^3\phi}$$

$$Z_3^{\circ\circ} = \frac{1+\nu}{2}\left(\sin\phi\, Z_1^{\circ} + \cos\phi\, Z_1\right) + \frac{3-\nu}{2}\left(\cos\phi\, Z_2^{\circ} - \sin\phi\, Z_2\right) \frac{1-\nu}{2}\rho\, Z_4^{\circ}$$

$$+ \left[(1+\nu) - \bar{\alpha}\left(\frac{1-\nu}{2}\rho + 2\right)\right] v_I^{\circ}$$

$$Z_1^{\circ} = \rho Z_2$$

$$Z_2^{\circ} = Z_1 - Z_2 \cot\phi$$

Z_4°, $Z_4^{\circ\circ}$ given above

Then with $\hat{u}$ and its derivatives known,

$$u_{II}\sin\phi = \hat{u}$$

$$u_{II}^{\circ}\sin\phi = \hat{u}^{\circ} - u_{II}\cos\phi$$

$$u_{II}^{\circ\circ}\sin\phi = \hat{u}^{\circ\circ} - 2\,u_{II}^{\circ}\cos\phi + u_{II}\sin\phi$$

$$v_{II} = Z_2\sin\phi - \hat{u}^{\circ}$$

$$v_{II}^{\circ} = Z_1\sin\phi - \hat{u}^{\circ\circ} \qquad (29)$$

$$v_{II}^{\circ\circ} = (\rho\sin\phi+\cos\phi)Z_2 - \hat{u}^{\circ\circ\circ}$$

$$w_{II} = - v_I$$

$$w_{II}^{\circ} = - v_I^{\circ}$$

$$w_{II}^{\circ\circ} = - v_I^{\circ\circ}$$

$$w_{II}^{\circ\circ\circ} = - v_I^{\circ\circ\circ}$$

Problem III

The required displacement functions for Problem III are expressed in terms of Problem I functions (28) as follows: (using a unit sphere)

$$u_{III} = v_I\,\frac{1}{\sin\phi} + u_{II}$$

$$u_{III}^{\circ} = v_I^{\circ}\,\frac{1}{\sin\phi} - v_I\,\frac{\cos\phi}{\sin^2\phi} + u_{II}^{\circ} \qquad (30)$$

$$u_{III}^{\circ\circ} = v_I^{\circ\circ}\,\frac{1}{\sin\phi} - 2v_I^{\circ}\,\frac{\cos\phi}{\sin^2\phi} + v_I\,\frac{1+\cos^2\phi}{\sin^3\phi} + u_{II}^{\circ\circ}$$

$$v_{III} = - v_I^{\circ} + v_{II}$$

$$
\begin{aligned}
v^{\circ}_{III} &= - v^{\circ\circ}_{I} + v^{\circ}_{II} \\
v^{\circ\circ}_{III} &= - v^{\circ\circ\circ}_{I} + v^{\circ\circ}_{II} \\
w_{III} &= - w^{\circ}_{I} - v_{I} \\
w^{\circ}_{III} &= - w^{\circ\circ}_{I} - v^{\circ}_{I} \\
w^{\circ\circ}_{III} &= - w^{\circ\circ\circ}_{I} - v^{\circ\circ}_{I} \\
w^{\circ\circ\circ}_{III} &= - w^{\circ\circ\circ\circ}_{I} - v^{\circ\circ\circ}_{I}
\end{aligned}
\tag{30}
$$

GREEN'S FUNCTION MATRIX

The Green's function approach to the vibration analysis in a specific problem entails a number of computational operations preliminary to the solution of eigenvalue problem symbolized by equation (4). In the discussion of these operations it is convenient to consider separately: (A) the mechanics of the reduction of the set of integral equations (1) and (2) to single matrix equation defining the eigenvalue problem numerically and (B) the computational operations to obtain the elements of the required matrices that combine to form the Green's function matrix.

Reduction of Integral Equations

The governing integral equations (1) and (2) are first replaced by finite difference approximations to the equations. The integral operators become rectangular matrices and the functions u_i, and q_i and L_M become column matrices (or "vectors").

Let the surface S (enclosed by the contour C) be subdivided into NS elements, the nth element denoted by $\Delta\sigma_n$.

Also let the bounding contour contour C be subdivided into NC segments, with Δs_m denoting the mth segment.

The integrals of equation (1) may then be approximated by mechanical guadratures; for example,

$$\iint_S B_{Kj} q_j d\sigma \doteq \sum_{n=1}^{N} B_{Kj}(m,n) q_j(n) \Delta\sigma_n$$

where $B_{Kj}(m,n)$ is B_{Kj} evaluated for the central points of the m^{th} segment on the contour $\underline{C}$ and the n^{th} surface element, and $q_j(n)$ is the central value of q_j for the surface element $\Delta\sigma_n$.

Next let the symbol [B] denote a $4\underline{NC}$ by $3\underline{NS}$ rectangular matrix whose element in row $[4(m-1) + K]$ and column $[3(n-1) + j]$ is $B_{Kj}(m,n)\Delta\sigma_n$. Also let $\vec{q}$ be a column matrix whose element in row $[3(n-1) + j]$ is $q_j(n)$. Then:

$$\sum_{n=1}^{N} B_{Kj}(m,n) q_j(n) \Delta\sigma_n = [B]\, \vec{q}$$

We define other rectangular and column matrices correspondingly to represent the other integral operators and functions appearing in equations (1) and (2):

Matrix	Element	Row	Column
[C]	$C_{KM}(m,p)\Delta s_p$	$4(m-1) + K$	$4(p-1) + M$
[A]	$A_{ij}(r,n)\Delta\sigma_n$	$3(r-1) + i$	$3(n-1) + j$
[D]	$D_{iM}(r,p)\Delta s_p$	$3(r-1) + i$	$4(p-1) + M$
$\vec{L}$	$L_M(p)$	$4(p-1) + M$	
$\vec{u}$	$u_i(r)$	$3(r-1) + i$	

The mechanical quadrature approximation to equations (1) and (2) is then:

$$[B]\, \vec{q} + [C]\, \vec{L} = 0 \qquad (1b)$$

$$[A]\, \vec{q} + [D]\, \vec{L} = \vec{u} \qquad (2b)$$

If $[C^{-1}]$ denotes the inverse of matrix $[C]$, then from equation (1b):

$$\vec{L} = - [C^{-1}]\,[B]\,\vec{q}$$

which substituted into (2b) leads to:

$$\Big\{[A] - [D][C^{-1}][B]\Big\}\,\vec{q} = \vec{u} \qquad (3b)$$

Finally, if $\vec{q} = \frac{1}{\lambda}\vec{u}$, we may write:

$$[G]\,\vec{u} = \lambda\,\vec{u} \qquad (4b)$$

where

$$[G] = \Big\{[A] - [D][C^{-1}][B]\Big\}$$

Equation (4b) is then the matrix eigenvalue equation corresponding to the integral equation (4) that defined the eigenvalue problem. Once the elements of matrix [G] are computed, it remains only to find the eigenvalues and associated eigenvectors.

Computation for Influence Matrices

The Green's function matrix is expressed in terms of influence matrices [A], [B], [C] and [D], whose elements must first be computed.

To facilitate the present discussion as well as actual computational steps, two spherical polar coordinate systems are introduced, one fixed in space (called the "global" system) to serve as a reference, the other oriented with respect to two selected points on the sphere and referred to as the "relative" system. The global coordinates (ϕ,θ) serve to locate sample points on the sperical surface $\overline{S}$ and on the enclosing contour $\overline{C}$. The relative coordinate system, with variables $(\hat{\phi},\hat{\theta})$ is oriented such that the pole, $\hat{\phi} = 0$, is located at a "stimulus" point, defined as a sample at which a unit force or unit couple is considered to act. Then the relative system meridianal plane, $\hat{\theta} = 0$, is oriented to contain a "response" point, defined as a sample point at which a displacement or stress resultant quantity is sought.

We have earlier observed that the influence functions (or for present purposes, the elements of the influence matrices) are the responses at one point on the sphere due to a unit stimulus at a second point, with both location of points and surface vector (or tensor) components expressed in the fixed global system. The solutions to the fundamental problems, on the other hand, are necessarily obtained in the relative system (as the stimulus is always at the pole). Consequently, tensor transformations are required at both the stimulus points and at the response points, in order to transform the fundamental problems from the relative coordinate system to the global system for each pair of points involved. Surface tensors which exist in a two dimensional space are resolved into rectangular cartesian components in the plane tangential to the spherical surface at the given point. The details of surface tensor transformations together with the required spherical geometric relationships are presented in Appendix III.

The response functions (matrices) relate response tensors components at one point to stimuli tensor components acting at the second point on the sphere. The response tensor quantities include the following: (Greek subscripts range over two dimensions)

displacement responses:

w scalar normal displacement

u_α surface vector with components $\bar{u}$ and $\bar{v}$ respectively.

response tensors for boundary conditions:

$w,_\alpha \equiv \frac{\partial w}{\partial x_\alpha}$ gradient vector of normal displacement w, where x_α represents are length along geodesic surface coordinates with origin at the point.

Q_α shear resultant on edges normal respectively to the x_1 and x_2 coordinate directions.

$N_{\alpha\beta}$ membrane stress resultant tensor (with components in the global system as N_ϕ, $N_{\phi\theta}$, N_θ)

$M_{\alpha\beta}$ moment tensor (with components in the global system M_ϕ, $M_{\phi\theta}$, M_θ)

$M_{\alpha\beta,\gamma} \equiv \frac{\partial M_{\alpha\beta}}{\partial x_\gamma}$ gradient of the moment tensor (with components in the global system, for example:

$$M_{11,1} = \frac{1}{a} M^\circ_\phi \quad , \quad M_{12,2} = \frac{1}{a} \frac{M'_{\phi\theta}}{\sin\phi}$$

The response points that lie on the contour $\underline{C}$, require as response functions for the influence matrices [B] and [C], the residuals, R_K, introduced earlier. For definition of these residuals at a given point on the contour $\underline{C}$ we introduce the surface unit vectors n_α and t_α. The unit vector n_α lies in the tangential plane but normal to the curve $\underline{C}$; while the vector t_α is tangential to the curve $\underline{C}$.

Depending on the boundary conditions to be specified, the required residuals, R_K, would be selected appropriately from the eight scalar invariants listed below (which make use of the Kirchoff formulation for edge reaction resultants):

$$
\begin{aligned}
&w \\
&u_n = n_\alpha u_\alpha \\
&u_t = t_\alpha u_\alpha \\
&\frac{\partial w}{\partial n} = n_\alpha w,_\alpha \\
&V_n = n_\alpha (Q_\alpha + t_\beta t_\gamma M_{\alpha\beta,\gamma}) \\
&N_n = n_\alpha n_\beta N_{\alpha\beta} \\
&S_{nt} = n_\alpha t_\beta (N_{\alpha\beta} - \frac{1}{a} M_{\alpha\beta}) \\
&M_n = n_\alpha n_\beta M_{\alpha\beta}
\end{aligned}
\tag{31}
$$

To evaluate these scalar quantities it is convenient to use the relative coordinate system as this entails a minimum of transformations. Only vectors n_α and t_α which are specified in the global coordinate system need be transformed; the response tensor quantities are already in the relative system. (The transformation of the vectors n_α and t_α is discussed in Appendix III).

The required tensor stimuli are defined in the following manner:

p unit normal load stimulus, a scalar.

$p_{1\alpha}$ unit tangential load vector directed in the increasing θ direction (in the global system).

$p_{2\alpha}$ unit tangent load vector directed in the increasing ϕ direction.

$F_{1\alpha}$ unit load vector directed tangentially to the contour curve C.

$F_{2\alpha}$ unit load vector in the direction of the surface vector n_α for a given point on curve C.

C_α unit force-couple vector directed tangentially to the curve C.

We have now defined the necessary symbols and developed the relationships to outline a step by step procedure for the computation of the elements of the influence matrices [A], [B], [C] and [D], and hence the Green's function matrix [G].

1) The spherical segment surface is sub-divided into NS surface elements with sample points for numerical computation located at the elemental geometric centers. Similarly, the enclosing curve C is sub-divided into NC elemental segments again with central points as sample points. Each sample point (whether on the surface or on the curve C) is considered in its turn to be a stimulus point and also in its turn is taken as response point, all combinations considered.

2) For a given stimulus point and a given response point the relative coordinate system for the two points is established and the required transformation matrices are computed. (Appendix III relationships employed).

3) Displacement functions of Fundamental Problems I, II and III are evaluated (using the relative coordinate $\hat{\phi}$, that locates the response point) by means of expressions (28), (29) and (30). These functions are then displacement responses (in the relative system) due to unit stimuli in the relative system.

4) If the response point lies on the curve C, the appropriate scalar invariants, from expression (31), are the residuals for that point. These residuals are then computed for each of the three fundamental problems, making use of the elastic law equations (6) and displacement functions of step 3.

5) Global unit stimuli, as defined earlier, are transformed from the global system into the relative system. The response functions as computed in steps 3 and 4 are then multiplied by the appropriate relative system components of each separate unit global stimulus. There results from this, displacement responses (in the relative system) and scalar residual responses due to separate unit global stimuli.

6) If the response point is a surface point, the displacement response functions of step 5 are transformed from the relative system to the global reference system, thus providing all responses in the global system due to unit global stimuli.

7) As the influence matrix [A] consists of displacement responses (in the global system) to unit load stimuli (also in the global system) for surface sample points, then should both the stimulus and the response points be surface points, the results of step 6 are elements in the matrix [A].

8) If the response point lies on the curve C. the residuals as computed in step 5 constitute elements of matrix [B] or matrix [C] (according to whether the stimulus point is a surface point or a boundary point).

9) If the stimulus point lies on the curve C while the response is sought at surface point, the results of step 6 constitute elements in the matrix [D].

10) Upon completion of the computation of all elements of matrices [A], [B], [C] and [D], the Green's function matrix [G] is then found by the matrix operations indicated in the definition of [G] (equation 4b).

A more complete and detailed algorithm for the computation of the Green's function matrix appears in Appendix IV.

With the construction of the Green's function matrix, the problem reduces to finding the eigenvalues and eigenvectors of [G]. Standard techniques exist to accomplish this operation. Finally, the natural angular frequencies for the vibration problem are found from the eigenvalues, λ_n, recalling:

$$\mu\, \omega_n^2 = \left(\frac{1}{\lambda_n} - k\right)$$

Also the associated mode shapes (for the original vibration problem) are defined by the eigenvectors which are the displacement components at the sample points in the static equivalent problem.

EXAMPLE PROBLEM

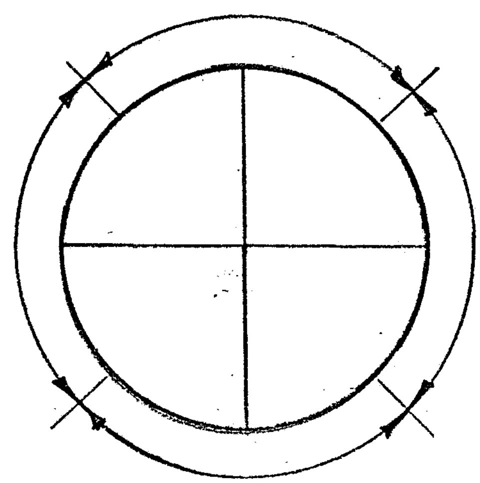
To demonstrate the procedures of the previous sections, natural frequencies have been obtained for a shallow spherical shell with clamped edges. Specifically, the particular properties

$$a/h = 10, \quad \nu = 0.3, \quad \phi_o = 30^\circ$$

were chosen where ϕ_o is the meridianal angle to the boundary edge.

As explained in the detailed algorithm of Appendix IV, the first step consists of numerically solving the three fundamental problems. After studying the behavior of the solutions in this particular example, the integration increment $\Delta\phi = 0.01$ radians was chosen as a compromise between high numerical accuracy and reasonable computer time. Typical results from these solutions are shown in Figures 1 through 6. Following the complete solution of the fundamental problems, each of the functions on page 22 was fitted with a polynomial in the Chebeychey sense.

For the next step the shell was subdivided into discrete surface elements and boundary arcs. In this particular instance the shell was coarsely sectioned into four surface elements and four boundary arcs as shown below. Following this subdivision the matrices A, B, C, and D were constructed in the manner described in Appendix IV from which the matrix C was then determined in accordance with equation (4b) on page 29.



NS = 4

NB = 4

The final step was to calculate the eigenvalues of the matrix G after which the natural frequencies were found by the equation on page

33. When non-dimensionalized, this equation takes the more convenient form

$$\Omega^2 = \left(\frac{a^2}{\lambda K} - \beta\right) \frac{1}{1 - \nu^2} \tag{32}$$

where

$$\Omega^2 = \mu\omega^2 \frac{a^2}{Eh}$$

For simplicity only, the fundamental (lowest) eigenvalue was determined using the Stodola-Vianello method. The resulting fundamental frequency Ω_1 is plotted in Figure 7 for various values of β. The dependence of the solution on β has been investigated in reference 40. In short, it is seen from equation 32 that for larger values of β, the eigenvalue λ must be determined with greater numerical accuracy in order to produce accurate approximations of Ω. That is, for large values of β, the value of Ω^2 depends on the difference of two large depends on the difference of two large numbers on the right hand side of equation 37. Hence, the degeneration of accuracy is to be expected for increasing β.

From Figure 7 it is noted that the asymptotic value of Ω_1 is 1.73. Upon comparing this value with the results obtainable by the methods of other authors such as 1.87 by Reissner's (27), 1.85 by Kalnins' (38), and 1.80 by Kalnins' (39), we see the present solution differs by approximately 4%. This difference seems quite acceptable in view of the coarseness of the present subdivision.

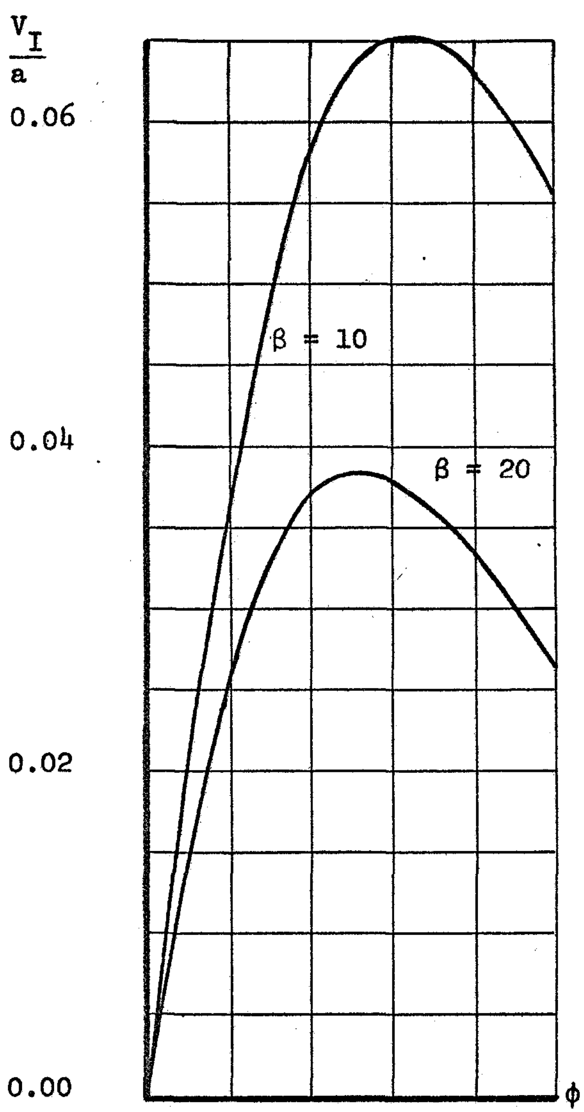


Figure 1

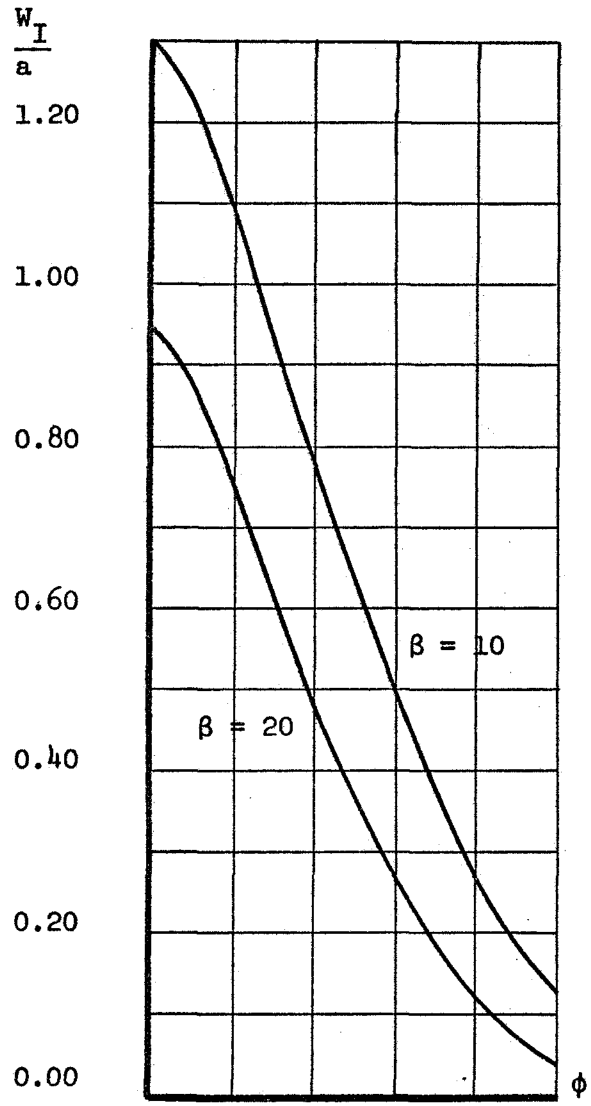


Figure 2

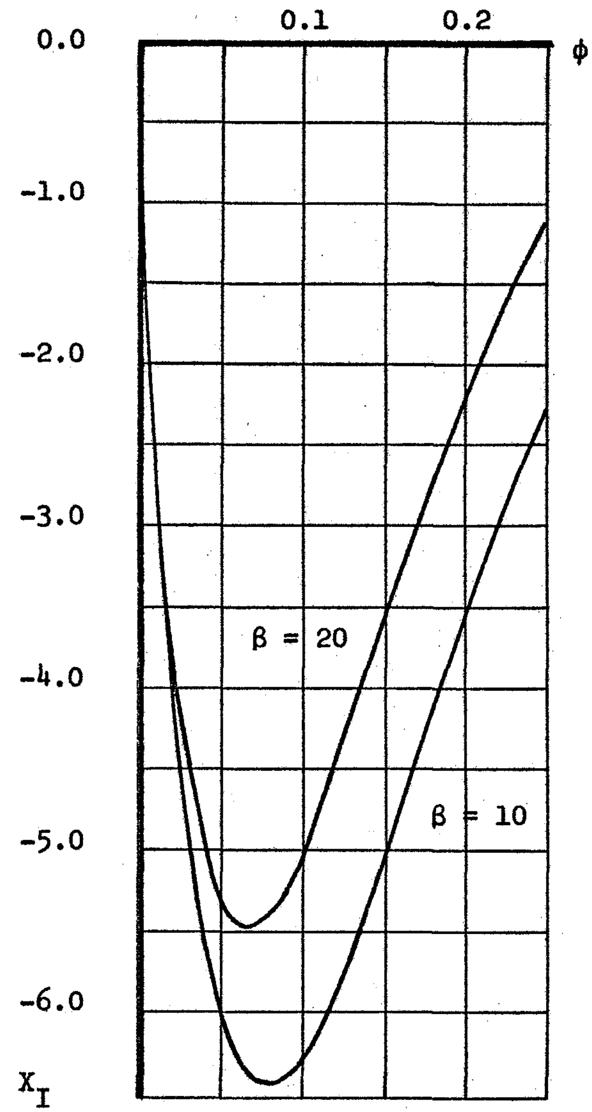


Figure 3

PLOTS OF TYPICAL FUNCTIONS FOR FUNDAMENTAL PROBLEM I

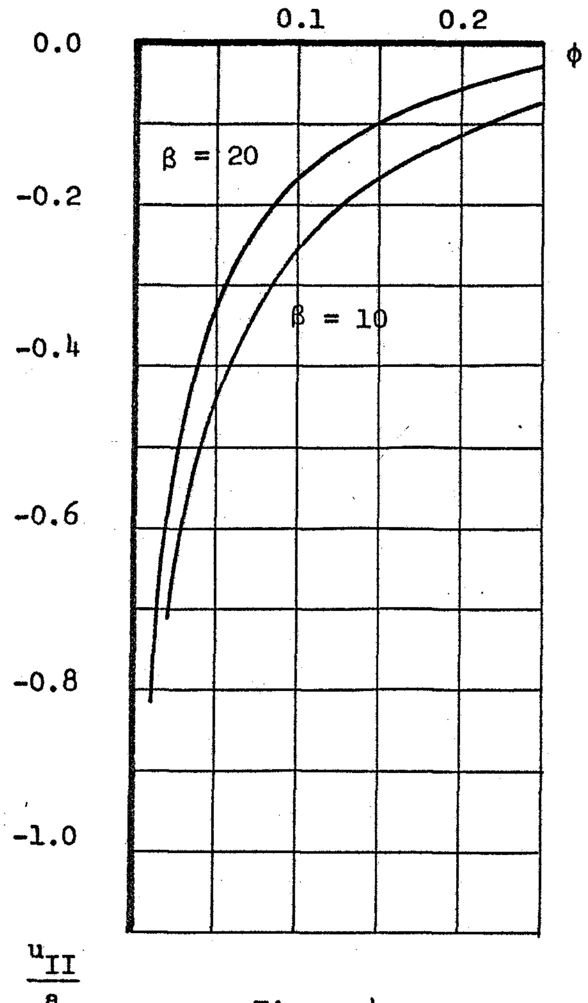


Figure 4

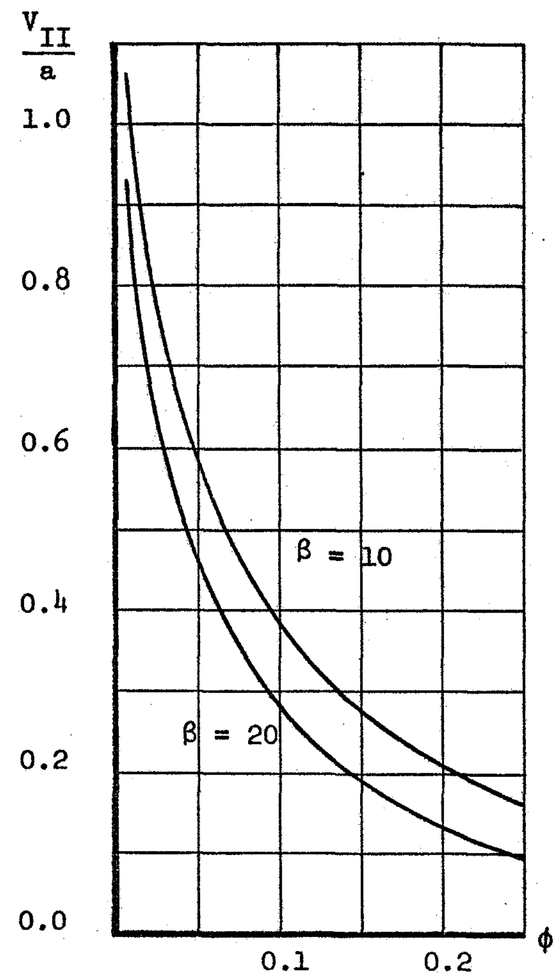


Figure 5

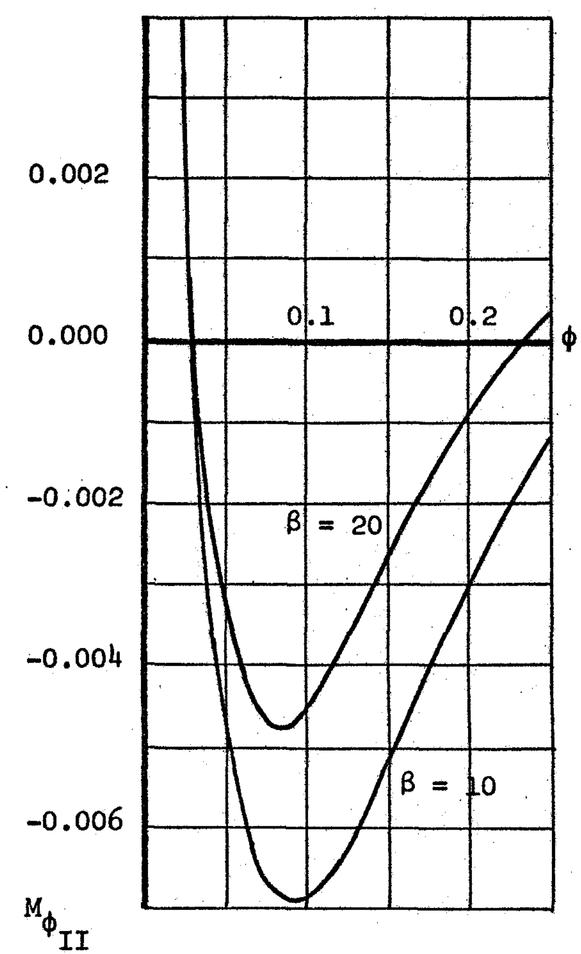


Figure 6

PLOTS OF TYPICAL FUNCTIONS FOR FUNDAMENTAL PROBLEM II

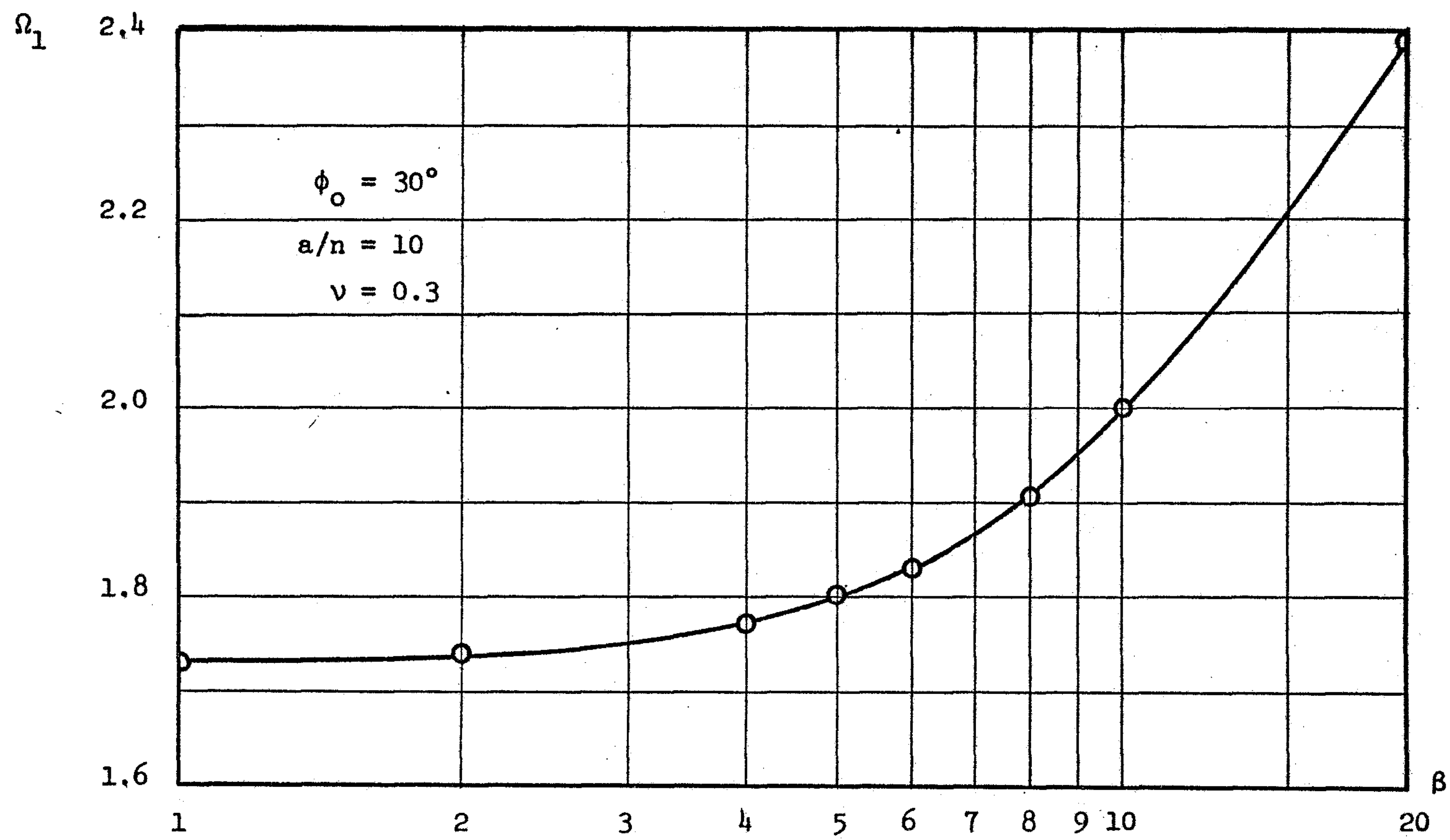


Figure 7 FUNDAMENTAL FREQUENCY FOR CLAMPED SHALLOW SHELL

APPENDIX I

Table of Symbols

A	constant
$[A]$	influence matrix
A_{ij}	i^{th} - j^{th} element of matrix function
a	radius of spherical shell
$[B]$	influence matrix
B_{Kj}	K^{th} - j^{th} element of matrix function
$\vec{b}_1, \vec{b}_2$	unit base surface vectors
$b_{\alpha\beta}$	surface tensor transformation matrix
$[C]$	influence matrix
C_{KM}	K^{th} - M^{th} element of matrix function
$\vec{c}_1, \vec{c}_2$	unit base surface vectors
$c_{\alpha\beta}$	surface tensor transformation matrix
D	shell flexural rigidity
$[D]$	influence matrix
D_{iM}	i^{th} - M^{th} element of matrix function
$d_{\alpha\beta}$	surface tensor transformation matrix
E	Young's modulus
$[G]$	Green's function matrix
G_{ij}	i^{th} - j^{th} element of matrix Green's function
K	shell elastic constant
k	foundation modulus
K_0, K_1	modified Bessel's functions of the second kind

L_J	J^{th} component of line load system
M_n	bending moment at boundary
M_{nt}	twisting moment at boundary
M_ϕ, M_θ	bending moment components
$M_{\phi\theta}$	twisting moment
$M_{\alpha\beta}$	bending moment tensor (surface)
N_n, N_{nt}	membrane stress resultants at boundary
N_ϕ, N_θ, $N_{\phi\theta}$	membrane stress resultant components
$N_{\alpha\beta}$	membrane stress resultant tensor (surface)
$\vec{n}$, n_α	surface vector, normal to curve $\underline{C}$
p	normal (to surface) unit force vector
$p_{1\alpha}$, $p_{2\alpha}$	surface unit force vectors
p_i	force vector with three unit magnitude components
Q_n	shear stress resultant at boundary
Q_ϕ, Q_θ	shear stress resultant components
Q_α	shear stress resultant tensor (surface)
$\vec{q}$, q_i	applied load vector (per unit area)
$\vec{q}$	column matrix for applied load
R_J	J^{th} boundary condition residual
r	cylindrical coordinate for point on spherical surface
$\vec{r}$	position vector of response point
S_{nt}	static equivalent edge membrane shear reaction
s	independent variable
$\vec{s}$	position vector of stimulus point
S_α	typical stimulus surface vector

$\vec{t}$, t_α	unit tangent vector (to curve $\underline{C}$)
$\vec{u}$, u_i	displacement vector (three dimensional)
$\vec{u}$	column matrix for displacements
u_n, u_t	components of surface vector displacement, normal and tangential to curve $\underline{C}$
u_α	surface vector displacement
$\bar{u}$, $\bar{v}$, $\bar{w}$	displacements of point on shell
u, v, w	displacement functions of ϕ alone
v_I, w_I	displacement functions for Problem I
u_{II}, v_{II}, w_{II}	displacement functions for Problem II
u_{III}, v_{III}, w_{III}	displacement functions for Problem III
V_n	static equivalent to normal edge reaction
x	independent variable
Y_i, Z_i	dependent functions in sets of differential equations
α	geometric parameter of shell
β	elastic foundation parameter
ϵ_ϕ, ϵ_θ, $\epsilon_{\phi\theta}$	strain components
θ	polar coordinate (for spherical surface)
$\varkappa$	dimensionless constant
$\varkappa_\phi$, $\varkappa_\theta$, $\varkappa_{\phi\theta}$	curvature change components
λ	eigenvalue
μ	shell mass per unit surface area
ν	Poisson's ratio
ρ	dimensionless constant
τ	dependent variable
ϕ	polar coordinate (for spherical surface)
χ	rotation of element

χ	dependent variable
ω	natural angular frequency

Subscripts and Operators

$\alpha, \beta, \gamma \ldots$	Greek indices denote components of two dimensional surface tensors
n,t	subscripts associated with directions of surface unit vectors $\vec{n}$ and $\vec{t}$, respectively
i,j,k...	lower case Latin indices denote components of three dimensional space vectors (or range over three dimensions in matrix functions)
J,K,M...	upper case Latin indices range over four dimensions in pseudo-vectors or in matrix functions
$(\ )_{,\alpha}$	denotes gradient of ( ) in two dimensional surface space
$(\ )^{\circ}$	denotes differentiation with respect to the variable ϕ
$(\ )'$	denotes differentiation with respect to the variable θ
L( )	denotes the operator $(\ )^{\circ\circ} + (\ )^{\circ} \cot\phi - (\ ) \cot^2\phi$
J( )	denotes the operator L( ) - ( )

APPENDIX II

Segmental Numerical Integration

In the numerical integration of equations (15) an instability condition arises from the large magnitude of the coefficient $1/\alpha$. If extended over a moderately large range in ϕ, the integration leads to a serious loss in significant figures (in the superposition process). To circumvent this difficulty, a segmental integration technique has been devised.*

Briefly, the technique entails the solutions of a succession of problems each of which is similar to the previous problem except for the addition to shell of a small segment associated with the angle $\Delta\phi$. The method is best explained through a consideration of its application to the problem at hand.

Suppose that for a spherical cap, $\phi \leq \phi_1$, we have four valid solutions of equations (15); one of which (we call the "particluar" solution) is consistent with the unit load singularity, the other three ("complementary" solutions) are linearly independent solutions that satisfy the no load boundary conditions at the origin. A general cap solution is then made up of the particular solution plus a linear combination of the three complementary solutions (or for no load, we omit the particular solution).

Consider next the adjacent spherical segment, $\phi_1 \leq \phi \leq \phi_2$. Six linearly independent solutions for the segment may be obtained from equations (15) by letting each of the functions Y_1 through Y_6 be non-zero independently as the initial conditions (at $\phi = \phi_1$). A linear combination of these six solutions then provides the general solution for the segment.

Nine boundary conditions are now required to determine the <u>three</u> coefficients of the complimentary solutions for the cap plus the <u>six</u> coefficients of the segmental solutions. Six of the required relationships arise

---

*The technique is similar in some respects to one developed by Kalnins. See reference 37.

from matching the functions Y_1 through Y_6 at the juncture between the cap and the segment. The remaining three conditions may be selected in any of several ways. In case (1) we let the functions Y_2, Y_4 and Y_6 be zero on the remote boundary ($\phi = \phi_2$) of the segment. In cases (2), (3) and (4) we let each of Y_2, Y_4, and Y_6 be non-zero independently at $\phi = \phi_2$. Moreover, in the latter three cases we omit the cap particular solution in the superposition. There results from these four cases four solutions to a new enlarged spherical cap which consists of the old cap with the added segment. The first of the resulting solutions is a new particular solution for the enlarged cap; the other three are new complementary solutions (linearly independent).

The process may then be repeated introducing a new segment, $\phi_2 \leq \phi \leq \phi_3$. A successive introduction of new segments extends the spherical cap to a suitably remote edge $\phi = \phi_n$. The succession of problem solutions would continue until either of two conditions are met:

(1) If ϕ reaches the value π radians thus defining a complete sphere, the process should terminate. It should be noted that the boundary conditions at $\phi = \pi$, are precisely that Y_2, Y_4 and Y_6 should equal zero; hence, the final "particular" solution is the solution to Fundamental Problem I.

(2) Since the functions Y_i are known to decay in the physical problem, if for any particular solution in the segmental process, the computed values of Y_1, Y_3 and Y_5 are found to be negligibly small (where Y_2, Y_4 and Y_6 are set equal to zero at $\phi = \phi_n$) the process is terminated. This condition is equivalent to having found the significant portion of the complete spherical shell solution for Problem I.

APPENDIX III

Spherical Geometry and Tensor Transformations

In the discussion of the computational steps to obtain the Green's function matrix, use is made of two surface polar coordinate systems, the global system which is fixed on the spherical surface, and the relative system which is oriented with respect to a given stimulus point and a given response point. To develop the required geometric relationships between these two coordinate systems and transformation relationships for surface tensors, a third coordinate system is introduced: A rectangular cartesian coordinate system with its origin at the center of the sphere, the y-axis along the global polar axis and the x-axis in the fixed meridianal plane $\theta = 0$.

The fundamental problems (I, II and III) which form the basis for the influence matrices, are developed in the relative coordinate system and in particular in terms of the relative polar angle $\hat{\phi}$ which locates the response point with respect to the stimulus point. Consequently, an expression is required for $\hat{\phi}$.

Let point S be the stimulus point and have coordinates, $\bar{\phi}$, $\bar{\theta}$ in the global system. Also let point R be the response point having coordinates ϕ, θ in the global system. In the relative system S is at the origin and $\hat{\phi}$ is the polar angle of R ($\hat{\theta} = 0$). For convenience we assume a unit sphere. Then position vectors $\vec{s}$ and $\vec{r}$ respectively of S and R may be expressed in the spacial cartesian coordinate system as:

$$\begin{aligned} \vec{s} &= \sin\bar{\phi}\,\cos\bar{\theta}\,\vec{i} + \cos\bar{\phi}\,\vec{j} + \sin\bar{\phi}\,\sin\bar{\theta}\,\vec{k} \\ \vec{r} &= \sin\phi\,\cos\theta\,\vec{i} + \cos\phi\,\vec{j} + \sin\phi\,\sin\theta\,\vec{k} \end{aligned} \qquad \text{(a)}$$

where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the usual unit base vectors.

The angle between $\vec{s}$ and $\vec{r}$ is the relative polar angle $\hat{\phi}$. Hence,

$$\cos\hat{\phi} = \vec{s}\cdot\vec{r} \quad \text{and} \quad \sin\hat{\phi} = \sqrt{1-\cos^2\hat{\phi}} \qquad \text{(b)}$$

where $\hat{\phi}$ is taken in the branch from zero to π radians.

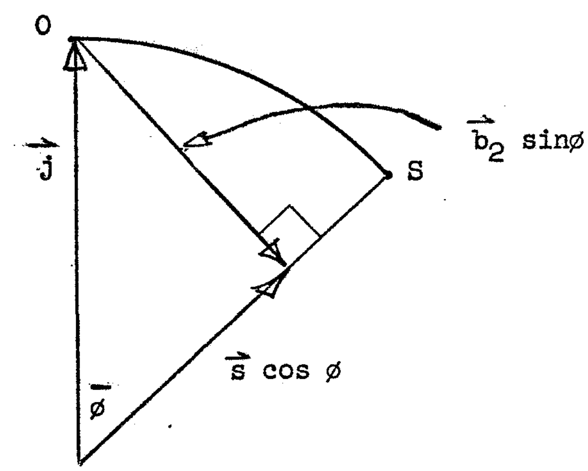
From equations (a) and (b), $\hat{\phi}$ and its functions may now be found.

At point $\underline{S}$, either scalar or vector stimuli may act. The vector unit stimuli include $p_{1\alpha}$, $p_{2\alpha}$, $F_{1\alpha}$, $F_{2\alpha}$, C_α as defined earlier. These are unit vectors tangential to the spherical surface and may be resolved into components either in the global system or the relative system. As the influences matrices [A], [B], [C] and [D] deal with unit stimuli defined in the global system but the fundamental problems require unit stimuli in the relative system, a transformation of vector components is required. To accomplish this it is helpful to define base unit vectors for each system.

For the global system we introduce the base unit tangent vectors $\vec{b}_1$ and $\vec{b}_2$ respectively, in the increasing $\bar{\theta}$ direction and in the increasing $\bar{\phi}$ direction. The unit vector $\vec{b}_2$ is given by:

$$\vec{b}_2 = (\vec{s} \cos\bar{\phi} - \vec{j}) \csc \bar{\phi} \qquad \text{(c)}$$

as may be verified from vector addition and the sketch below:



Then, as vector $\vec{b}_1$ is perpendicular both to $\vec{b}_2$ and $\vec{s}$, we have:

$$\vec{b}_1 = \vec{b}_2 \times \vec{s} \qquad \text{(d)}$$

For the relative coordinate system we introduce base unit tangent vectors $\vec{b}_1'$ and $\vec{b}_2'$: respectively, tangential to the great circle from S to R and in the appropriate perpendicular direction. Expressions analogous to (c) and (d) are obtained in a similar manner.

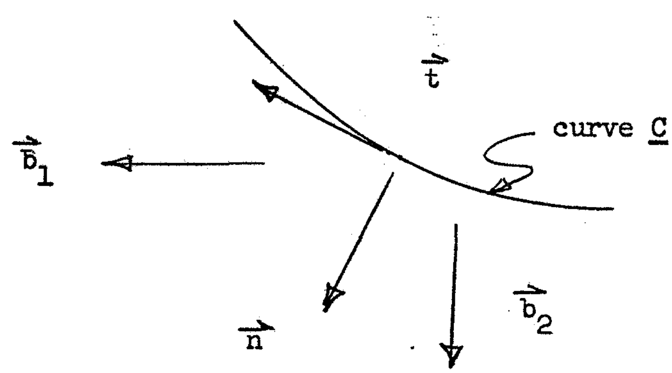
$$\begin{aligned} \vec{b}_1' &= (\vec{r} - \vec{s}\cos\hat{\phi}) \csc \hat{\phi} \\ \vec{b}_2' &= \vec{s} \times \vec{b}_1' \end{aligned} \qquad \text{(e)}$$

We recall that for any point on the contour curve C we have given the unit surface vectors $\vec{n}$ and $\vec{t}$, as defined earlier. If in the global system n has components n_α, given by:

$$\vec{n} = n_1\vec{b}_1 + n_2\vec{b}_2,$$

then from the sketch below we find the vector $\vec{t}$ must be:

$$\vec{t} = n_2\vec{b}_1 - n_1\vec{b}_2$$



From the definitions of unit vector stimuli we observe the components of these stimuli in the global system to be the following:

	Coefficient of $\vec{b}_1$	Coefficient of $\vec{b}_2$	
$P_{1\alpha}$	1	0	
$P_{2\alpha}$	0	1	
$F_{1\alpha}$	n_2	$-n_1$	(f)
$F_{2\alpha}$	n_1	n_2	
C_α	n_2	$-n_1$	

The transformation relations of these vectors to the relative system is treated in the general case if we let $\vec{S}$ represent a typical stimulus vector. If then S_α and S'_α are respectively the components of $\vec{S}$ in the global and in the relative system, we may resolve $\vec{S}$ into components within each system and equate vector expressions:

$$S'_1\vec{b}'_1 + S'_2\vec{b}'_2 = S_1\vec{b}_1 + S_2\vec{b}_2$$

From this it is seen that:

$$S'_1 = \vec{b}'_1 \cdot \vec{b}_1 S_1 + \vec{b}'_1 \cdot \vec{b}_2 S_2$$

$$S'_2 = \vec{b}'_2 \cdot \vec{b}_1 S_1 + \vec{b}'_2 \cdot \vec{b}_2 S_2$$

or in more compact form (with summation convection)

$$S'_\alpha = b_{\alpha\beta} S_\beta \tag{g}$$

where

$$b_{\alpha\beta} = \vec{b}'_\alpha \cdot \vec{b}_\beta .$$

At the response point R, response functions are first obtained in relative coordinate system (from Fundamental Problems I, II, and III). For the influence matrices [A] and [D], the displacement response vector, $\vec{u}$, requires transformation back to the global system. However, for the influence matrices [B] and [C], the scalar invariant residuals, expressions (31), only require the transformation of the contour unit vectors $\vec{n}$ and $\vec{t}$ from the global to the relative system (as the other functions within the scalar residual expressions are already in the relative system). For these transformations we must again define appropriate unit base vectors.

In the global system at a response point we define the base unit tangent vectors $\vec{c}_1$ and $\vec{c}_2$ (analogous to $\vec{b}_1$ and $\vec{b}_2$ for a stimulus point):

$$\begin{aligned} \vec{c}_2 &= (\vec{r} \cos\phi - \vec{J}) \csc \phi \\ \vec{c}_1 &= \vec{c}_2 \times \vec{r} \end{aligned} \tag{h}$$

For the relative system at a response point we introduce the base unit tangent vectors $\vec{c}_1'$ and $\vec{c}_2'$, where $\vec{c}_2'$ is tangential to the great circle that includes the stimulus point, but directed in the increasing ϕ direction; $\vec{c}_1'$ is perpendicular to $\vec{c}_2'$, in the increasing θ direction. Hence,

$$\begin{aligned} \vec{c}_2' &= (\vec{r} \cos\hat{\phi} - \vec{s}) \csc \hat{\phi} \\ \vec{c}_1' &= \vec{c}_2' \times \vec{r} \end{aligned} \tag{i}$$

Following a development similar to that for the stimulus point, we obtain the transformation of vector $\vec{u}$ from the relative system (using primes to denote the relative system components) to the global system:

$$u_\alpha = c_{\alpha\beta} u_\beta'$$

where

$$c_{\alpha\beta} = \vec{c}_\alpha \cdot \vec{c}_\beta' \tag{j}$$

For the transformation of the unit vectors $\vec{n}$ and $\vec{t}$ (at contour points) from the global system to the relative system (as required for the scalar invariant expressions 31) we obtain:

$$n'_\alpha = d_{\alpha\beta} n_\beta$$

$$t'_\alpha = d_{\alpha\beta} t_\beta$$

where the primed components are in the relative system, and

$$d_{\alpha\beta} = \vec{c}'_\alpha \cdot \vec{c}_\beta \qquad (k)$$

APPENDIX IV

Algorithm for Computation of the Green's Function Matrix

The numerical computation of the Green's function matrix may be divided into two separate phases. First, the Fundamental Problems I, II and III must be solved and the relevant response functions represented in suitably accurate yet numerically efficient approximate forms. Secondly, with the fundamental problems solved and the response functions available, the elements of the influence matrices may be computed,which matrices are then combined to provide the Green's function matrix. We now consider each phase separately:

Fundamental Problem Solution

1. The set of first order differential equations (15) are solved numerically by the Kutta-Merson technique (using also the segmental integration procedure outlined in Appendix II), providing a set values Y_i for a suitably closely spaced net of points.

2. At each point the displacement functions v_I, v_I°, $v_I^{\circ\circ}$, $v_I^{\circ\circ\circ}$, w_I, w_I°, $w_I^{\circ\circ}$, $w_I^{\circ\circ\circ}$, $w_I^{\circ\circ\circ\circ}$, are computed from the equations (28). ($u_I = 0$ from symmetry)

3. Employing expressions (27) and the elastic law equations (6) the stress-resultant functions N_ϕ, N_θ, M_ϕ, M_θ, M_ϕ° and M_θ° are evaluated (also at each point). $N_{\phi\theta}$, $M_{\phi\theta}$, and Q_θ are zero.

4. From the expression for Y_4, equations (14), Q_ϕ is evaluated.

5. Each of the displacement functions of step (2) and of the stress-resultant functions of steps (3) and (4) is represented by a Chebeychev polynomial fit (minimax fit) in each of separate ranges of the dependent variable ϕ. This provides the necessary response functions both for Problem I (and for Problem III, by virtue of equations 30 which arise from the reciprocal theorem.)

6. For the response function evaluation in the immediate neighborhood or directly under the unit load, a special approach is required to avoid difficulties arising from singularities. In this case the concentrated load is replaced by a statically equivalent distributed load acting over a line or surface element of suitable size. (The size of the element should correspond to the size of the finite element subsequently selected for the influence matrix computations.) The response at the center of the element is then obtained by numerical integration of response due to the distributed load.

7. The set of first order differential equations (22) are solved for the functions Z_i, by the Kutta- Merson technique. (The polynomial for v_I as obtained from step (5) is used for $-\underline{w}$ in equations $\underline{22}$).

8. The displacement functions u_{II}, $\overset{\circ}{u}_{II}$, $\overset{\circ\circ}{u}_{II}$, v_{II}, $\overset{\circ}{v}_{II}$, $\overset{\circ\circ}{v}_{II}$, w_{II}, $\overset{\circ}{w}_{II}$, $\overset{\circ\circ}{w}_{II}$, $\overset{\circ\circ\circ}{w}_{II}$ are computed for each point of the difference net by means of equations (29).

9. The elastic law equations (6) together with expressions (27) are used to compute the stress-resultant functions (except for Q_ϕ and Q_θ) at each point.

10. The equilibrium equations (5d,e) are then employed to evaluate the function Q_ϕ and Q_θ at each point.

11. The values for response functions directly under the unit load are evaluated in a manner similar to step (6).

12. A Chebeychev minimax fit is used to represent both displacement and stress resultant functions in each of the separate ranges of the variable ϕ, providing the required response functions for Fundamental Problem II.

13. From relationships (30), the Chebeychev polynomial representation displacement functions are obtained directly for Fundamental Problem III using the results of steps (5) and (6).

14. The combination of displacement functions (in polynomial form, from step 13) is used to obtain the stress-resultant functions in polynomial form. The elastic law equations (6) and equations (27) are used for this.

The forgoing fourteen steps then provide us with the necessary response functions for the three fundamental problems in an efficient polynomial form.

Matrix Element Computation

1. The surface of the spherical segment is sub-divided into NS surface elements with coordinates of element central points specified. Indentifying numbers are assigned each element. Similarly, the bounding contour is sub-divided into NC elemental arcs (also identified numerically) with coordinates of element central points specified. Additionally, the surface vector $\vec{n}$, with components n_α (in the global system), is specified for each contour point.

2. The position vector $\vec{r}$ and base vectors $\vec{b}_1$, $\vec{b}_2$ (and hence, vectors $\vec{s}$ and base vectors $\vec{c}_1$, $\vec{c}_2$) are evaluated for each central point (on the surface and on the contour) employing the equations (a), (c) and (d) of Appendix II.

3. Starting with the first surface point and continuing through all surface points, each point is taken in turn to be a response point. Then for any given surface response point, each surface point is also considered in turn to be a stimulus point. For each combination (say, response point r and stimulus point n) a contribution to influence matrix [A] is computed. If we consider matrix [A] to be a partitioned matrix composed of sub-matrices, one for each combination of response point and stimulus point, then the sub-matrix in the r^{th} row and n^{th} column of [A] is the 3 by 3 matrix of global displacement components for each of three unit global stimuli. The following sequence of operations is carried out to obtain the sub-matrix for row r and column n of [A]:

i) The relative polar angle $\hat{\phi}$ and the transformation coefficients $b_{\alpha\beta}$ and $c_{\alpha\beta}$ are calculated employing the equations (b), (e), (g), (i) and (j) of Appendix II.

ii) The response functions v_I, w_I, u_{II}, v_{II}, w_{II} are evaluated at $\hat{\phi}$.

iii) The global response-stimulus matrix for point r and point m is then obtained from the relationship:

$$\begin{bmatrix}\text{Global System}\\ \text{Response-Stimulus}\\ \text{Matrix}\end{bmatrix} = \begin{bmatrix}\text{Transformation}\\ \text{Matrix from Relative}\\ \text{to Global (reponse pt.)}\end{bmatrix}\begin{bmatrix}\text{Relative System}\\ \text{Response-Stimulus}\\ \text{Matrix}\end{bmatrix}\begin{bmatrix}\text{Transformation}\\ \text{Matrix from Global}\\ \text{to Relative (stimulus pt.)}\end{bmatrix}$$

or symbolically:

$$[RS] = [T_R][R'S'][T_S]$$

where

$$[T_R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{11} & c_{12} \\ 0 & c_{21} & c_{22} \end{bmatrix}$$

$$[R'S'] = \begin{bmatrix} w_I & w_{II} & 0 \\ 0 & 0 & u_{II} \\ v_I & v_{II} & 0 \end{bmatrix}$$

$$[T_S] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & b_{11} & b_{12} \\ 0 & b_{21} & b_{22} \end{bmatrix}$$

iv) The resulting response-stimulus matrix is contributed to influence matrix [A] as the sub-matrix in row r and column n.

4. For each surface response point (as described in step 3), each contour point is taken in turn as a stimulus point, to develop contributions to the influence matrix [D]. As with matrix [A], we consider matrix [D] to be a partitioned matrix composed of three by four sub-matrices, one for each combination of a surface response point with a contour stimulus point (say, surface point r and contour point p). The following sequence of steps are followed for each combination of points:

i) The polar angle $\hat{\phi}$ and transformation coefficients $b_{\alpha\beta}$ and $c_{\alpha\beta}$ are computed as in step (3).

ii) Response functions v_I, w_I, u_{II}, v_{II}, w_{II}, u_{III}, v_{III}, w_{III} are evaluated at ϕ.

iii) The global response-stimulus matrix for point r and point p is computed from the relationship:

$$[RS] = [T_R][R'S'][T_S]$$

where:

$[T_R]$ is the same as step (3)

$$[R'S'] = \begin{bmatrix} w_I & w_{II} & 0 & 0 & w_{III} \\ 0 & 0 & u_{II} & -u_{III} & 0 \\ v_I & v_{II} & 0 & 0 & v_{III} \end{bmatrix}$$

$$[T_S] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (b_{11}n_2 - b_{12}n_1) & (b_{11}n_1 + b_{12}n_2) & 0 \\ 0 & (b_{21}n_2 - b_{22}n_1) & (b_{21}n_1 + b_{22}n_2) & 0 \\ 0 & 0 & 0 & (b_{11}n_2 - b_{12}n_1) \\ 0 & 0 & 0 & (b_{21}n_2 - b_{22}n_1) \end{bmatrix}.$$

iv) The resulting response-stimulus matrix is contributed to influence matrix [D], as the sub-matrix in row r and column p.

5. Starting with the first contour point and continuing through all contour points, each point is taken in turn to be a response point. Then for any given contour response point each surface point in turn is considered to be a stimulus point. For each combination a contribution to influence matrix [B] is computed. The contribution is in the form of a sub-matrix where [B] is considered to be partitioned matrix. The following sequences of operations are performed to compute the sub-matrix associated with contour point m and surface point n:

i) The polar angle $\hat{\phi}$ and transformation coefficients $b_{\alpha\beta}$, $c_{\alpha\beta}$ and $d_{\alpha\beta}$ are computed as in step (3).

ii) Response displacement functions v_I, w_I, u_{II}, v_{II}, w_{II} as well as stress-resultant functions N_ϕ, N_θ, $N_{\phi\theta}$, M_ϕ, M_θ, $M_{\phi\theta}$, M°_ϕ, M°_θ, $M^\circ_{\phi\theta}$, Q_ϕ, Q_θ (for Fundamental Problems I and II) are evaluated at ϕ.

We recall that Problems II and III lack polar symmetry and the functions u, v, w are defined as functions of $\hat{\phi}$ in the expressions for displacement:

$$\bar{u} = u \sin\theta \qquad \bar{v} = v \sin\theta \qquad \bar{w} = w \sin\theta$$

It then follows from the elastic law equations (6) that the stress resultant functions (and components of grad $\bar{w}$) are expressible as functions of $\hat{\phi}$ each multiplied by either $\sin\theta$ or $\cos\theta$. Consequently, we may refer to the symmetric functions, those with the $\cos\theta$ coefficient, and to the anti-symmetric functions, those with $\sin\theta$ as the coefficient. A separation into symmetric and anti-symmetric response functions appears below:

<u>Symmetric</u>	<u>Anti-symmetric</u>
$\bar{v}$	$\bar{u}$
$\bar{w}$	$\bar{w}'$
$\bar{w}^\circ$	$N_{\phi\theta}$
N_ϕ	M'_ϕ
N_θ	M'_θ
M_ϕ	$M_{\phi\theta}$
M_θ	$M^\circ_{\phi\theta}$
M°_ϕ	Q_θ
M°_θ	
$M'_{\phi\theta}$	
Q_ϕ	

In fundamental Problem II the non-zero response functions due to the stimulus component (unit tangent load) in the direction of <u>base</u> <u>vector</u> $\vec{b}_1'$ includes only the symmetric; the anti-symmetric functions are due to the stimulus component in the $\vec{b}_2'$ direction. For Fundamental Problem III, the unit stimulus component <u>in the $\vec{b}_2'$ direction</u> produces only symmetric functions (at the response point) while for a unit stimulus component in the $\vec{b}_1'$ direction, the negative anti-symmetric response functions of $\hat{\phi}$ result (as $\sin\theta$ is evaluated at $\theta = -\pi/2$).

iii) The response-stimulus sub-matrix for response point m (on the contour) and stimulus point n (on the surface) is observed to be residuals, R_J, at point m due to global force stimuli at point n. This sub-matrix is then calculated from the relationship:

$[RS] = [R'S'][T_S]$

where:

$[T_S]$ is the same as the corresponding matrix in step (3)

$[R'S']$ is composed of the following columns:

column (1) contains the selected residuals, R_J', (for the specified boundary conditions) from Fundamental Problem I.

column (2) contains the residuals, R_J', associated with the symmetric functions of Problem II.

column (3) contains the residuals, R_J', associated with the anti-symmetric functions of Problem II.

R_J' are the selected four scalar invariants (from expressions 31) evaluated in the relative system, hence, containing n_α' and t_α'.

n_α', t_α' are components of unit vectors $\vec{n}$, $\vec{t}$ and are obtained from the transformation relations:

$$n_\alpha' = d_{\alpha\beta} n_\beta \ , \quad t_\alpha' = d_{\alpha\beta} t_\beta$$

iv) The sub-matrix computed in step (iii) is contributed to the partitioned influence matrix [B] in row m and column n.

6. With each contour response point (as introduced in step 5), a contour stimulus point (taking all points in turn) is associated to develop

contributions to the influence matrix [C]. As with other influence matrices, [C] is a partitioned matrix. It is composed of four by four sub-matrices, one for each combination of response point with a stimulus point (say, contour points m and p, respectively). The following sequence of operations are performed to obtain the required sub-matrix.

i) The quantities $\hat{\phi}$, $b_{\alpha\beta}$, $c_{\alpha\beta}$, $d_{\alpha\beta}$ are computed as in step (3).

ii) Response displacement functions v_I, w_I, u_{II}, v_{II}, w_{II}, u_{III}, v_{III}, w_{III} as well as stress-resultant functions are evaluated at $\hat{\phi}$ (for each fundamental problem). These functions are separated into a symmetric, anti-symmetric dichotomy for Problems II and III.

iii) The response-stimulus sub-matrix is calculated from the relationship:

$$[RS] = [R'S'][T_S]$$

where:

$[T_S]$ is the same as the corresponding matrix in step (4)

[R'S'] contains the following columns of elements:

columns (1) through (3) are identical with those of matrix [R'S'] in step (5).

column (4) contains the residuals, R'_J, associated with the negative anti-symmetric response functions of Problem III (as computed in operation (ii) above).

column (5) contains the residuals, R'_J, associated with the symmetric response functions of Problem III.

R'_J are the selected four scalar invariants described in step (5).

iv) The sub-matrix as computed above is contributed to influence matrix [C] in row m, column p of sub-matrices.

7. With influence matrices [A][B][C] and [D] computed, the Green's function matrix [G] is obtained through the following matrix operations:

i) $[C^{-1}]$ is obtained by inverting [C]

ii) The matrix [G] is then $[G] = [A] - [D][C^{-1}][B]$.

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