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NONZERO-SUM DIFFERENTIAL GAMES: CONCEPTS AND MODELS

By

Alan W. Starr

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ABSTRACT

A general class of differential games, where the N players try to minimize different cost criteria by controlling inputs to a single dynamic system, is investigated as an extension of optimal control theory. Dropping the usual zero-sum assumption makes it possible to model a more realistic class of competitive situations where mutual interest is important.

The nonzero-sum formulation has several interesting analytic and conceptual features not found in zero-sum differential games. It is no longer obvious what should be demanded of a "solution," and three types of solution concepts are discussed: Nash equilibrium, minimax, and noninferior (or Pareto optimal) strategies. For one special case, the "linear-quadratic" differential game, all of these solutions can be computed exactly by solving sets of coupled ordinary matrix differential equations.

Another feature not found in optimal control problems or in two-person, zero-sum differential games is the difference between "open loop" and "closed loop" equilibria. The "principle of optimality" of optimal control theory does not generalize in an obvious way to the nonzero-sum differential game. Some simple examples are given to illustrate this. It is shown that the various efficient algorithms of optimal control theory (such as "differential dynamic programming") do not readily extend to the computation of Nash equilibrium controls. However, approximate Nash solutions can be obtained in certain special cases.

Some simple examples are solved, and series of more difficult but more realistic nonzero-sum differential game situations are presented (but not solved) for models of economic oligopoly, advertising policy, labor-management negotiations, and international trade.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

1.1 Informal statement of the problem

A differential game is a mathematical model of a competitive situation which evolves over time. The structure of a general differential game is illustrated in Fig. 1.1. There are N "players," each continuously controlling a different set of inputs to a single dynamic system and each trying to minimize his own particular cost criterion. Associated with the dynamic system is an n-dimensional "state vector" which, at any time, contains all the information needed to predict the future behavior of the system if the future inputs are known. The state vector is governed by an n^{th} order differential equation in which the various inputs appear as driving terms.

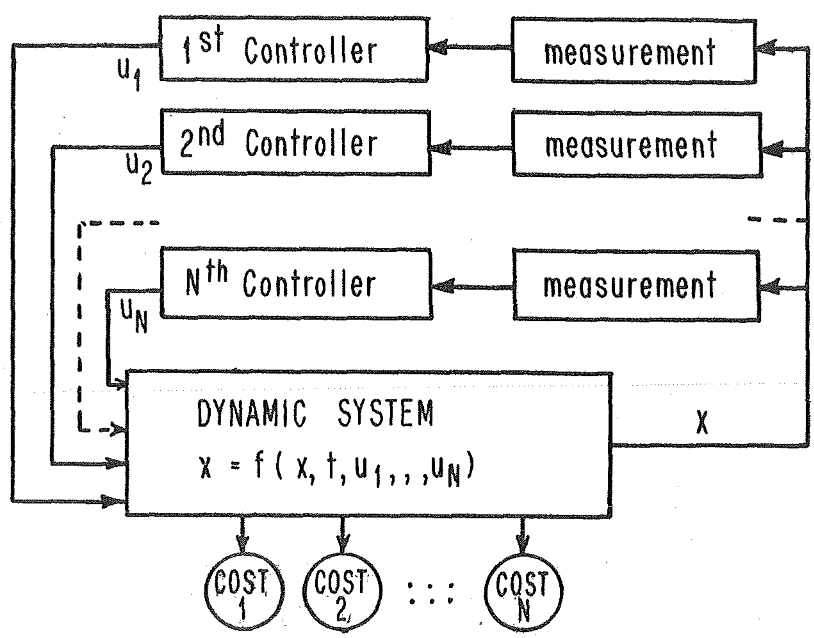

Fig. 1.1. Structure of a general differential game

A peculiar feature of this problem is that it is not generally clear what is meant by a "solution." In fact, there is a variety of interesting "solution" concepts, depending on the information available to the players during the course of the game and on the "rationales" used by the players. The remaining chapters will be concerned with defining, characterizing and computing several of the most interesting types of solutions.

1.2 History of the problem

Since the study of differential games was initiated by Isaacs[1] in 1954, many papers on the subject have appeared, mostly dealing with problems of the pursuit-evasion type. These papers have considered only two-player differential games with the "zero-sum" property, i.e., there is a single performance index which one player tries to minimize while the other player tries to maximize. A zero-sum game is a model of "total conflict" which excludes the possibility of mutual interest between the players. Since mutual interest plays an important role in realistic competitive situations (especially in those arising in economic contexts, but even in some military applications), a theory built on the zero-sum hypothesis is severely limited in its range of possible applications. On the other hand, zero-sum games are much simpler, both conceptually and analytically, than nonzero-sum games.

Apparently no journal articles have yet appeared on nonzero-sum differential games, except for two based on the present work.[2, 3] However, in a technical report, Case[4] extended some of Isaacs results to the N-player, nonzero-sum differential game. Case did not explore the implications of dropping the zero-sum hypothesis. Although several examples of the pursuit-evasion type were presented in that report, no examples of the application of the model to realistic competitive situations were discussed. Further work on the subject by Case and others should appear in the literature in the near future.

The theory of nonzero-sum differential games in effect merges general game theory with optimal control theory, and the literature of these two subjects is much more useful than the zero-sum differential game literature as a source of ideas in studying nonzero-sum differential games.

1.3 Motivation and general approach

This work has been motivated by a desire to understand the role of competition in economic processes which have a dynamic structure. If the dynamic behavior of such a structure can be adequately approximated by a continuous, finite-dimensional dynamic system (described by linear or nonlinear differential equations) then it is appropriate to model the process by a differential game.

It is by no means a trivial matter to formulate a meaningful differential game model for a realistic competitive situation, especially one with a "nonphysical" (e.g., economic) context. Even when the model has been formulated, it is not always clear what questions to ask. It is therefore important to consider the broad, general features of differential games before becoming immersed in the details of computing a particular type of solution to some specific model. Thus we shall take just the opposite approach to that taken by Isaacs in his book[1] on zero-sum differential games. The few specific examples which we shall consider will be intended only as illustrations. We shall not rely (as Isaacs does) on specific examples as a means for discovering general principles.

Although we shall discuss differential games in rather general terms, we shall occasionally make restrictive assumptions. For example, we shall consider only models with fixed terminal time. This eliminates some pursuit-evasion models (such as several of the examples considered by Case[(4)]) but does not appear to eliminate any interesting economic models. With fixed terminal time, one is always certain that the game will terminate, regardless of how the players behave.

Our approach to differential games will always be from the viewpoint of optimal control theory.[(5)] We shall consider only problems where, if all but one controller were eliminated, the remaining optimization problem would be most appropriately solved by the methods of optimal control theory, rather than by some type of nonlinear programming. It will be assumed that the reader is familiar with the better-known results of optimal control theory, and some of these results will be used without proof (and sometimes without a detailed statement of the conditions under which they hold) in analysing nonzero-sum differential games.

The spirit of this work is thus to attempt to generalize optimal control theory to allow for several controllers with different objectives. Such an extension of an optimal control problem results in a game situation, and many of the ideas of general game theory become relevant. However, it is important to remember that our starting point is optimal control theory, not game theory. We do not approach our study of differential games as a limiting case of a succession of "static" games.

Since this work is viewed as an extension of optimal control theory, it is not assumed that the reader is familiar with game theory. All the game-theoretic concepts which will be used in the later chapters are presented in Chapter II. Many of the important ideas of general game theory are not discussed in this work. For example, mixed strategies are not considered because they do not seem relevant to the applications envisioned here (see Chapter 8). Very little is said about coalitions, not because they are unimportant, but because there are simpler unresolved difficulties in analysing differential games which should be settled before such complications are introduced. In general, we shall rely on game theory for solution concepts but not for analytic methods.

1.4 Guide to the remaining chapters

Depending on the background and interest of the reader, there are various ways to approach this work. The reader who is only casually interested and who wonders if there really could be any use for nonzero-sum differential games could start by reading the first two pages of Section 3.1. He should then look at some of the examples of differential game models in Chapter VIII. If his interest is aroused, he may then return to Chapter II and proceed from there, skipping some of the more detailed sections.

The reader with a background in optimal control theory and a general interest in differential games may read the chapters in order, perhaps skipping over the less interesting sections. It is essential, however, that the reader thoroughly understands everything in Chapter II (except the last two pages in Section 2.5, concerning directional convexity) before reading Chapters III through VII. Chapter III is also essential to the remaining chapters.

The reader may also prefer to read the summary and conclusions in Chapter IX before proceeding with Chapter II.

CHAPTER II

NONZERO-SUM GAMES

2.1 Introduction

The primary concern of this report is differential games, where the decision variables are functions of time and possibly of other independent variables as well. The situations at different times in such a game are related through a dynamic system described by a set of ordinary differential equations.

However, many of the concepts in differential games are also important in the much better known theory of "static" games, where no dynamics are involved. In a static game, each player chooses his strategy from a given set of allowable strategies. The cost for each player is known (in advance) as a function of the strategies selected by all players. One usually adopts the view that all players select their strategies simultaneously without knowledge of what strategies the rivals will choose. Even differential games can conceptually be viewed as static games where the set of admissible strategies is some region in a function space.

The purpose of this chapter is to present some concepts from the general theory of games which will be useful in understanding differential games. There are no new results in this chapter, but the ideas and language established here form the basis for the discussion in all the remaining chapters.

The most general game which we shall consider is defined as follows:

Definition: A game G contains the following objects:

1) A set of N "players," where the i^{th} player $(i = 1, \ldots, N)$ selects a strategy s_i from a given set U_i of admissible strategies.

2) A set of cost functions $J = [J_1 J_2 \ldots J_N]$

$$J : U_1 x \ldots x U_N \longrightarrow E^N$$

3) A set of N orderings $\prec_i$, $i = 1, \ldots, N$, of set of all admissible strategy N-tuples $s \in U_1 x \ldots x U_N$ such that (letting $U = U_1 x \ldots x U_N$) for any u, $v \in U$,

$$u \prec_i v \quad \text{iff} \quad J_i(u) < J_i(v)$$

The symbol $\prec_i$ may be read "...is preferred by the i^{th} player to... "

Note that the orderings among the various strategy N-tuples would not be affected if, for $i = 1, \ldots, N$, the function $J_i(s)$ were replaced by

$$J_i'(s) = f_i(J_i(s)) \tag{2.1}$$

where f_i is any strictly monotone function. In many (but not all*) applications, these orderings are all that is needed to determine the outcome of the game, so that one is free to make transformations of the form (2.1).

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* One exception would be a stochastic game where players try to minimize the expected values of their cost functions.

Two special cases of the games defined above are especially well known:

1) If $N = 1$, then G is called a minimization problem, or an optimization problem.

2) If $N = 2$ and $J_1 + J_2 = 0$, then G is called a two-person zero-sum game.

Zero-sum games with more than two players may also be defined, but they are of little special interest.

In applications where only the orderings among the strategy N-tuples is important, any two-person game with the property that, for all $u, v \in U$

$$u \prec_1 v \quad \text{iff} \quad v \prec_2 u \tag{2.2}$$

can be converted via (2.1) to a zero-sum game. Whenever a game has property (2.2) and can be transformed to a zero-sum game, this should be done, since these games of perfect competition are easier to analyze than nonzero-sum games.

A third special case of some interest is:

3) If $J_1 = J_2 = \ldots = J_N$, then G is an identical goal game.

If only the orderings are needed to "solve" the game, then any game in which, for all $u, v \in U$

$$u \prec_i v \text{ for some } i \Longrightarrow u \prec_i v \text{ for all } i, \; i = 1, \ldots, N \tag{2.3}$$

can be converted via (2.1) to an identical-goal game. These "perfectly cooperative games" may be treated as minimization problems, assuming that the same information is available to all the players.

However, the nonzero-sum games which are of prime interest to us generally do not have any of the above special properties. Both mutual interest and conflict of interest are present in a general game.

2.2 Examples*

Some of the important features of nonzero-sum games can be illustrated by simple bimatrix games of the type presented in Luce and Raiffa(6) and in most other texts on game theory. Several such games will be presented in this section. They will be discussed further in the following two sections.

Game 1. Zero-sum

Player 1 \ Player 2	x	y
a	1, -1	0, 0
b	2, -2	-2, 2

Game 2. Zero-sum

Player 1 \ Player 2	x	y
a	-1, 1	0, 0
b	2, -2	-2, 2

In Game 1, Player 1 chooses between strategies a and b, while Player 2 simultaneously must choose x or y. The corresponding entries give the costs J_1, J_2 for the two players. For each strategy pair, $J_1 + J_2 = 0$, so the game is zero-sum. (In all games, each player wishes to minimize his own cost and is indifferent to the cost paid by the other player.) Player 2, if he is rational, will always play x, and Player 1, realizing this, will play a. This "saddle-point" solution is apparently the only reasonable one.

In Game 2, also zero-sum, no saddle-point solution exists. But Player 1 can minimize his maximum possible loss by choosing a,

*The material in Sections 2.2, 2.3, and 2.4 was presented in [2] but is repeated here for convenience. The reader may skip to Section 2.8, where these solution concepts are illustrated by a "continuous" static game.

on the assumption that Player 2 will ignore his own cost criterion and attempt to do maximum damage to Player 1's criterion. By the same reasoning, Player 2 would choose x. Thus (a, x) is a "minimax" solution, but it is not a saddle-point solution, i.e., it is not optimal for Player 2 against Player 1's strategy.

Game 3. "Dating game"

Player 2

		x	y
Player 1	a	0, 1	2, 2
	b	2, 2	1, 0

Game 4. "Prisoners' dilemma"

Player 2

		x	y
Player 1	a	2, 2	10, 1
	b	1, 10	5, 5

Game 3 is called the "dating game." Players 1 and 2 are "he" and "she" respectively, and decisions a and x represent "go to football game" while b and y represent "go to fashion show." The entries in the cost matrix indicate that, while "he" and "she" have different ideas as to which alternative is preferable, both prefer the other's company to going to either event alone. It is easily verified that neither (2.2) nor (2.3) hold, so Game 3 must be analysed as a nonzero-sum game.

Game 4, also nonzero-sum, is the celebrated "prisoners' dilemma." Two prisoners, held in separate cells, are charged with similar crimes. Each possesses information about the other's crime which, if divulged, would enable the state to obtain a conviction on a serious charge with a 10-year sentence. Without this information only a lesser conviction with a 2-year sentence could be obtained. The district attorney offers to halve the sentence of either or both prisoners if they divulge this information. Strategies a and x represent "do not talk," while b and y represent "talk." No communication is possible between the prisoners.

2.3 Nash equilibria

It should already be apparent from Games 3 and 4 that it is not obvious what is meant by a "solution" of a nonzero-sum game. In this section we define one type of solution which will be of central interest in differential games:

Definition: If $J_1(s_1, \ldots, s_N), \ldots, J_N(s_1, \ldots, s_N)$ are cost functions for players $1, \ldots, N$, then the strategy set $\{s_1^*, \ldots, s_N^*\}$ is a Nash equilibrium strategy set if, for $i = 1, \ldots, N$,

$$J_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_N^*) \geq J_i(s_1^*, \ldots, s_N^*) \quad (2.4)$$

where s_i is any admissible strategy for Player i.

In other words, the Nash equilibrium strategy is the optimal strategy for each of the players on the assumption that all of the other players are holding fast to their Nash strategies. In the two player, zero-sum case, the Nash solution is the familiar saddle-point solution.

The Nash solution is "secure" against unilateral attempts by any player to optimize. One must avoid the mistake of calling a Nash solution "optimal." In fact it is almost always possible for all players to achieve simultaneously lower costs than the Nash costs.

Note also that, for $N > 2$, the Nash solution is not secure against coalitions among a subset of the players.

The Nash solution for a continuous static game (not a bi-matrix game) will be illustrated in Section 8 below.

It can be seen by inspection that Games 1, 3 and 4 above have Nash equilibria, while Game 2 does not.

2.4 General properties of nonzero-sum games

This section considers some of the differences between nonzero-sum and zero-sum games.

It can easily be shown that in a zero-sum game

(i) All Nash equilibria are equivalent, i.e., have the same costs, and

(ii) If (s_1, s_2) and (s_1^*, s_2^*) are equilibrium pairs, then so are (s_1, s_2^*) and (s_1^*, s_2). (Interchangeability)

It is also clear that there can be no mutual interest in a zero-sum game; what is good for one player is harmful to the other. Nor can one player ever gain by disclosing his strategy in advance to his opponent.

Game 3, the "dating game," is nonzero-sum. It has two Nash equilibria, (a, x) and (b, y), with different costs. They are not interchangeable, since (a, y) and (b, x) are not equilibria. Notice what happens when both players seek to achieve their lowest possible costs. But if Player 1 announces in advance that he is committed to strategy a, then Player 2 has no choice but to play x! Thus it is advantageous in some (but not all) nonzero-sum games to disclose one's strategy in advance, i.e., to make the first "move."

Game 4 is the classical "prisoners' dilemma." The only equilibrium solution is (b, y), yet (a, x) gives a better result for both players. The solution (a, x) is vulnerable to "cheating" by one player, while (b, y) is not. This illustrates the non-optimality of the Nash equilibrium solution in the nonzero-sum game. There is mutual interest, since both players could gain if cooperation were possible.

These simple examples should convince the reader that there are important phenomena which can arise in nonzero-sum, but not in zero-sum, games, and that the Nash equilibrium is not the only interesting solution. The next three sections describe other types of solutions which may also be of interest.

2.5 Noninferior solutions

One may wish to know what could be gained by all players if a "negotiated" solution could be reached and enforced. Clearly such a solution should be selected from the following set of strategy N-tuples:

Definition: The strategy N-tuple $\theta = \{\theta_1, \ldots, \theta_N\}$ belongs to the non-inferior set if, for any other strategy N-tuple ϕ,

$$\{J_i(\phi) \leq J_i(\theta),\ i=1,\ldots,N\} \text{ only if } \{J_i(\phi) = J_i(\theta),\ i=1,\ldots,N\} \quad (2.5$$

Solution ϕ is said to dominate solution θ if

$$J_i(\phi) \leq J_i(\theta)\ , \quad i = 1,\ldots,N$$

with the inequality strict for at least one i. If the inequalities are strict for all i, we say that ϕ strictly dominates θ. The noninferior solutions are thus the only undominated solutions. They are sometimes called "efficient" or "Pareto-optimal" solutions. A solution is noninferior if any other solution which gives a better result for at least one player also gives a worse result for at least one player.

Solving for the set of noninferior controls is equivalent to solving a minimization problem with a vector cost criterion. There are several useful and well-known results for this class of problems[7]. They revolve around the question of whether or not the minimization problem with a <u>vector</u> cost criterion can be reduced to a family of minimization problems with <u>scalar</u> cost criteria. Such a process is called "scalarization," and is the subject of the remainder of this section.

Let M be the set of all N-vectors with strictly positive components, whose components add to unity:

$$M = \{\mu \mid \mu_j > 0 \text{ for all } j \text{ and } \sum_{j=1}^{N} \mu_j = 1\} \tag{2.7}$$

Let $\overline{M}$ be the closure of M (obtained by replacing > by ⩾).

Letting a strategy N-tuple $[u_1, \ldots, u_N]$ be denoted by u, we define the <u>scalar minimization problem P(μ)</u>:

$$\underset{u}{\text{minimize}} \quad J(\mu, u) = \sum_{i=1}^{N} \mu_i J_i(u) \quad \text{where} \quad \mu \in M. \tag{2.8}$$

Let the set of noninferior costs be denoted by Λ. Let Ω be defined as the set of cost vectors obtained by solving all problems of the form (2.8):

$$\Omega = \{J(\bar{u}) \mid \bar{u} \text{ solves } P(\mu) \text{ for some } \mu \in M\} \tag{2.9}$$

and let $\overline{\Omega}$ be the closure of Ω.

We can now state some of the relations between the noninferior set and the set of solutions to all scalar minimum problems of the type (2.8).

Theorem: $\Omega \subseteq \Lambda$.

Proof: Suppose $\bar{u} \in \Omega$ but $\bar{u}$ is not noninferior. Then there is a solution $v \in U$ such that $J_i(v) \leqslant J_i(\bar{u})$ for all i, with at least one inequality strict. But since all μ_i are positive, this means $\bar{u}$ does not minimize $J(\mu, u)$ and hence $J(\bar{u}) \notin \Omega$, a contradiction.

Thus we can always obtain at least some of the noninferior solutions by solving $P(\mu)$ for all $\mu \in M$.

Theorem: If for each $\mu \in \overline{M}$ either $P(\mu)$ has a unique solution or no solution, then $\overline{\Omega} \subseteq \Lambda$.

Proof: Suppose $u \in \overline{\Omega}$ for some $\mu \in \overline{M}$, and that $u \notin \Lambda$. Then $\exists v \in U$, $v \neq u$, such that v dominates u. But this implies, for any $\mu \in \overline{M}$, that $J(\mu, v) \leqslant J(\mu, u)$ and hence u is not the unique solution of $P(\mu)$.

DaGunha and Polak(7) prove that, if $J_i(u)$ is a convex function for all i and U is a convex set, then

$$\Lambda \subseteq \overline{\Omega} \tag{2.10}$$

The proof, which is long, is omitted here.

Note that solutions on the boundary of Ω correspond to weighting vectors μ with one or more zero components. Since such solutions completely ignore the interests of the players whose weightings are zero, they are of little interest to us. Thus for all practical purposes, the entire noninferior set can be found by solving scalar minimization problems whenever the conditions are such that (2.10) holds.

There are, in fact, less restrictive conditions under which (2.10) holds; they depend on the idea of directional convexity*:

Definition: Let P be a convex cone. A set A is said to be P-directionally convex, if for any $x, y \in A$ and any λ, $0 \leq \lambda \leq 1$,

$$\lambda x + (1 - \lambda)y + p = z, \quad \text{where } z \in A \text{ and } p \in P.$$

Let E^{N-} denote the negative orthant in the euclidean space E^N.

Theorem: If the set $Z = \{J \mid J = [J_1(u), \ldots, J_N(u)],\ u \in U\}$ (i.e., the set of all feasible cost vectors) is E^{N-}-directionally convex, then $\Lambda \subseteq \overline{\Omega}$.

Instead of proving this formally (see Ref. (7)), we illustrate the idea with some simple diagrams for the two-player game.

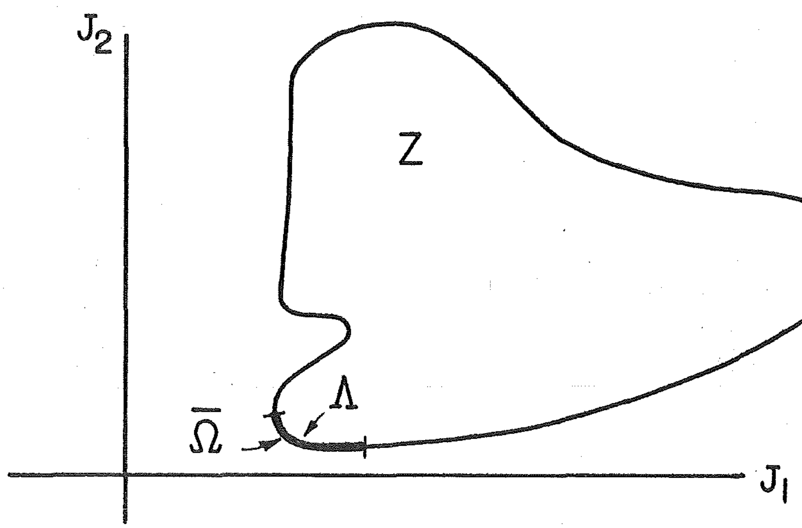

Fig. 2.1. A case where Z is not convex but $\Lambda \subset \overline{\Omega}$.

*This concept is not used in the remaining sections. The reader may skip to the top of page 2-13.

A case where Z is E^{N-}-directionally convex but not convex is shown in Fig. 2.1.

Note that $\overline{\Omega}$ only contains solutions which lie in the convex hull of Z. Thus (2.10) can hold only if all the noninferior solutions lie in the convex hull of Z. Fig. 2.2 illustrates a case where (2.10) does not hold. Note that Z is not E^{N-}-directionally convex.

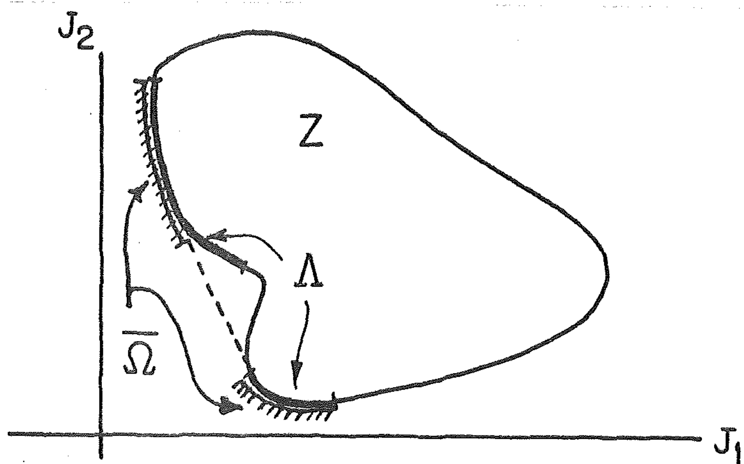

Fig. 2.2. A case where $\Lambda \not\subset \overline{\Omega}$.

Finally, Fig. 2.3 illustrates a case where $\overline{\Omega} \not\subset \Lambda$. Note that such a counterexample requires a μ which is in $\overline{M}$ but not in M, since $\Omega \subset \Lambda$ always. Note that, for $\mu = [1, 0]$, $P(\mu)$ does not have a unique solution.

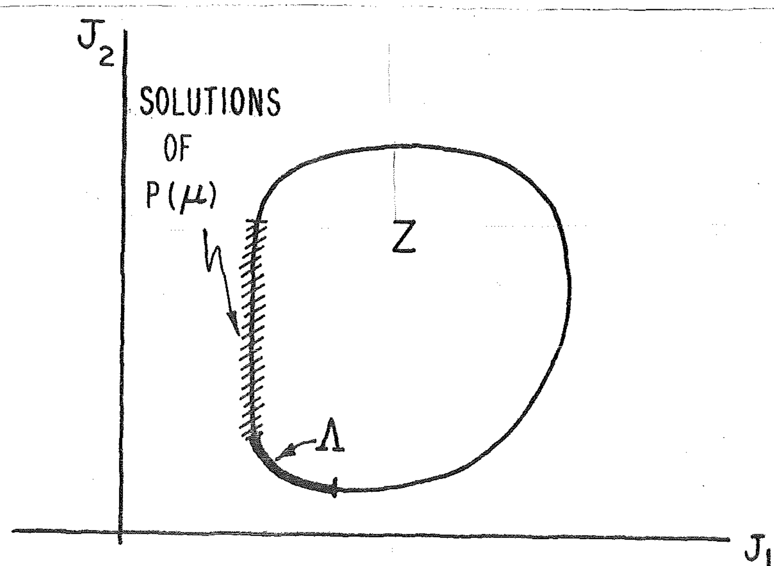

Fig. 2.3. A case where $\overline{\Omega} \not\subset \Lambda$.

When convexity conditions are such that 2.10 holds, all the noninferior solutions can be found by solving for $\overline{\Omega}$. However, solutions lying on the boundary of $\overline{\Omega}$ must still be checked to see if they are noninferior. Klinger[8] has given a counterexample to demonstrate this.

The members of the noninferior set Λ are not ordered by the vector criterion. The negotiating problem, equivalent (if the problem if scalarizable) to selecting a $\mu \in M$, can thus not be solved unless further rules are specified.

2.6 Minimax solutions

When a player believes that the other players will play Nash equilibrium strategies, he should also play the Nash strategies. But if he cannot be sure how his rivals will select their strategies, he may instead choose to minimize his cost against the worst possible set of strategies which they could choose.

Definition: A strategy $\overline{u}_i \in U_i$ is minimax for the i^{th} player if for all

$$[u_1, \ldots, u_N] \in U_1 x, \ldots, x U_N \quad , \tag{2.11}$$

$$\max_{u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_N} J_i(u_1,\ldots,u_{i-1},\overline{u}_i,u_{i+1},\ldots,u_N)$$

$$\leq \max_{u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_N} J_i(u_1,\ldots,u_i,\ldots,u_N)$$

Note that only the i^{th} player's cost function enters into the determination of his minimax strategy. This is equivalent to finding the equilibrium solution of a 2-player zero-sum game, where the opponent of Player i chooses all the strategies except the i^{th} and tries to maximize J_i. Player i can also calculate his minimax cost $\overline{J}_i$. If he plays $\bar{v}_i$, he will pay no more than $\bar{J}_i$. He will probably pay much less, since the other N - 1 players, each with his own cost to minimize, are unlikely to choose the combination of strategies which maximizes J_i (they may for example play their own minimax strategies). Since it fails to take account of the other players' cost criteria, and since it is excessively pessimistic, the minimax solution is somewhat unsatisfactory in the nonzero-sum game. In some reasonable, well-behaved games, $\bar{J}_i = \infty$. Of course, in the 2-player zero-sum game, the Nash solution, if it exists, is also minimax, but this is not true when $N > 2$, nor in a nonzero-sum game.

2.7 Coalitions

The most important new feature arising from the extension from 2 players to N players is the possibility of coalitions among groups of players. In Section 5 we have already considered one special coalition -- the one involving all the players, with no "side payments" allowed. There are many papers in the game theory literature dealing with various aspects of coalitions, for example, the existence of solutions which are stable against the formation or the disolving of certain types of coalitions. However, very little of practical importance can be said unless strict rules governing the formation and enforcement of coalitions are postulated.

In dealing with differential games, we shall (with the exception of noninferior solutions) leave the possibility of coalitions as a topic for further research.

2.8 Example: a function minimization game

In this section the ideas presented in this chapter are illustrated by a two-player, nonzero-sum game involving the minimization of continuous functions.

Consider the game

$$\text{Player 1 : } \min_{u_1} J_1(u_1, u_2)$$

$$\text{Player 2 : } \min_{u_2} J_2(u_1, u_2)$$

where u_1 and u_2 are scalars and where the functions J_1 and J_2 are assumed to be convex and twice differentiable with respect to both arguments.

Equicost contours in the space (u_1, u_2) are plotted for J_1 in Fig. 2.4a and for J_2 in Fig. 2.4b.

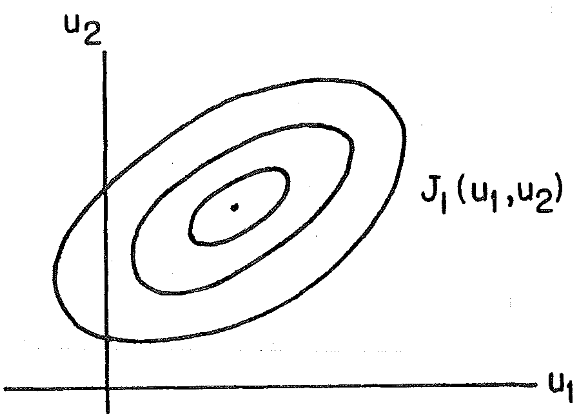

Fig. 2.4a

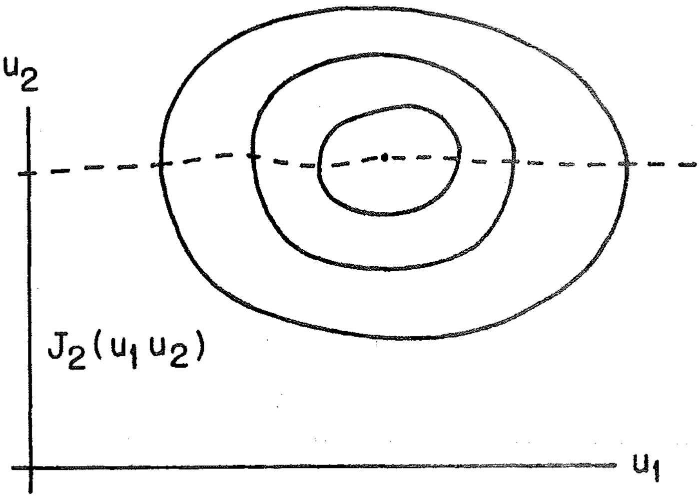

Fig. 2.4b

Suppose now that u_2, a variable over which Player 1 has no control, is fixed at level $\overline{u}_2$, as indicated in Fig. 2.4c. Then the best

Player 1 can do is to minimize J_1 along the line $u_2 = \bar{u}_2$. Clearly this is achieved by choosing u_1 such that the cost contour is tangent (externally) to the horizontal line $u_2 = \bar{u}_2$. The locus of such points for all possible $\bar{u}_2$ is given by the dashed curve in Fig. 2.4c. It is the locus of points where the cost contours of J_1 are horizontal.*

Similarly, if Player 2 must optimize against a given $\bar{u}_1$, the result is a u_2 such that the line $u_1 = \bar{u}_1$ is tangent externally to the cost contour of J_2. The locus of such points (where the cost contours of J_2 are vertical) is given by the dashed curve in Fig. 2.4b.

Superimposing Fig. 2.4b on Fig. 2.4c then shows us the Nash solution; it is the intersection of the two dashed curves in Fig. 2.4d.

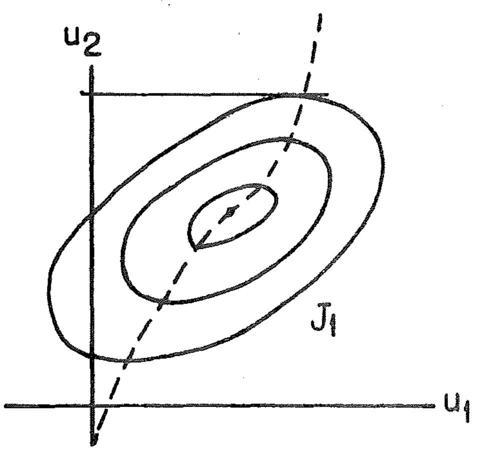

Fig. 2.4c

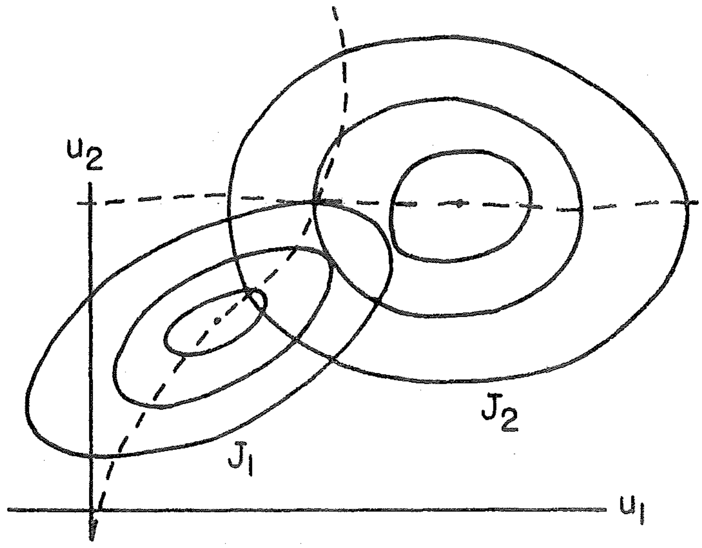

Fig. 2.4d

Our assumptions about J_1 and J_2 assure that if there is a Nash equilibrium, it is the solution of the coupled nonlinear equations:

$$\begin{aligned} J_{1u_1} &= 0 \\ J_{2u_2} &= 0 \end{aligned} \qquad (2.12)$$

*These points are the set of "rational solutions for Player 1."

A convenient procedure for solving (2.12) is to extend the familiar Newton-Raphson method: expanding (2.12) about the most recent iterative solution $u_1^{(k)}, u_2^{(k)}$ and evaluating at the Nash solution $\bar{u}_1, \bar{u}_2$ gives

$$J_{1u_1^{(k)}} + J_{1u_1^{(k)}u_1}(\bar{u}_1 - u_1^{(k)}) + J_{1u_1^{(k)}u_2}(\bar{u}_2 - u_2^{(k)}) = 0$$

$$J_{2u_2^{(k)}} + J_{2u_2^{(k)}u_1}(\bar{u}_1 - u_1^{(k)}) + J_{2u_2^{(k)}u_2}(\bar{u}_2 - u_2^{(k)}) = 0$$

(where higher order terms have been dropped). Solving these approximate equations for $\bar{u}_1, \bar{u}_2$ and letting the result be our next iterative solution $u_1^{(k+1)}, u_2^{(k+1)}$, we have the algorithm

$$\begin{bmatrix} u_1^{(k+1)} \\ u_2^{(k+1)} \end{bmatrix} = \begin{bmatrix} u_1^{(k)} \\ u_2^{(k)} \end{bmatrix} - \begin{bmatrix} J_{1u_1^{(k)}u_1} & J_{1u_1^{(k)}u_2} \\ J_{2u_2^{(k)}u_1} & J_{2u_2^{(k)}u_2} \end{bmatrix}^{-1} \begin{bmatrix} J_{1u_1^{(k)}} \\ J_{1u_2^{(k)}} \end{bmatrix} \qquad (2.13)$$

The generalization to more than two players is obvious.

It can be seen in Fig. 2.4d that both players could simultaneously achieve a better result than the Nash solution. The solutions which dominate the Nash solution are shown as a shaded region in Fig. 2.4e.

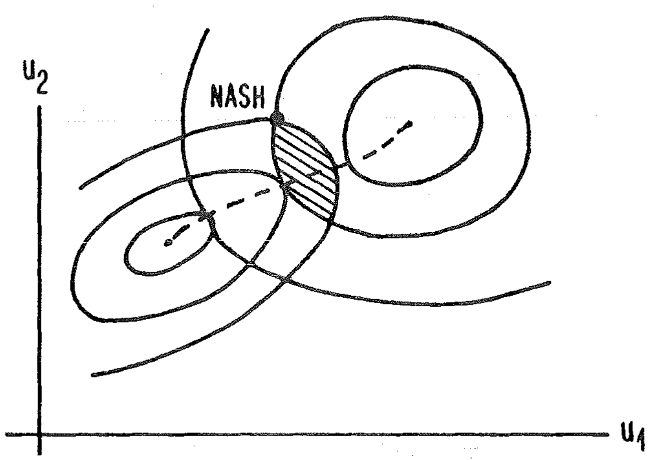

Fig. 2.4e

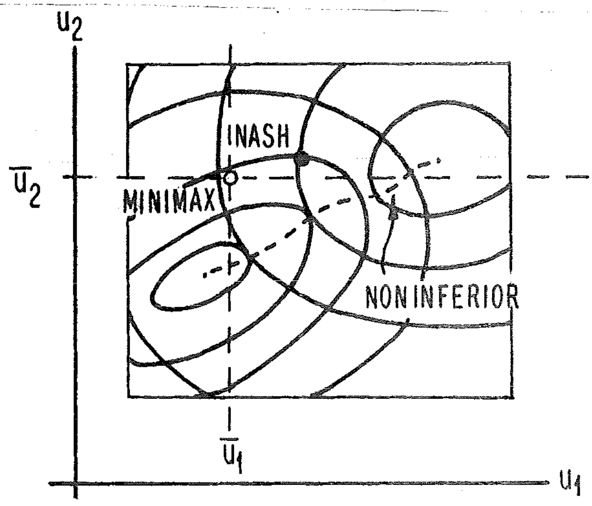

Fig. 2.4f

It should be clear that the noninferior solutions are those points where the cost contours of J_1 and J_2 are tangent externally. These are the only points which are undominated. The noninferior set is the dashed curve in Fig. 2.4e. As usual, it is a (N - 1)-parameter family of solutions when there are N players (in this case, N = 2).

To illustrate minimax solutions, we add the assumption that the feasible ranges for u_1 and u_2 lie between upper and lower bounds indicated in Fig. 2.4f. Player 1 then ignores J_2 and tries to optimize J_1 against the u_2 which would hurt him most. This leads to the pessimistic assumption that u_2 will be chosen at its upper bound, and consequently Player 1 choses $\overline{u}_1$. By similar reasoning, Player 2 chooses $\overline{u}_2$, assuming pessimistically that u_1 will be chosen at its lower bound. The resulting costs when $\overline{u}_1$ and $\overline{u}_2$ are played are, of course, a pleasant surprise to both players. In this case, the result happens to be worse than the Nash solution for Player 2 but better for Player 1.

Even when the functions J_1 and J_2 are convex and well-behaved as the ones we have considered, it may happen that there is no Nash solution. Such a situation is illustrated in Fig. 2.5.* Note that the noninferior solutions still exist. If constraints are added as in Fig. 2.4f, minimax solutions will also exist. Because the sets of rational solutions for Player 1 (upper dashed line) and for Player 2 (lower dashed line) do not intersect, there is no Nash solution.

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*It has been pointed out by K. Arrow that the nonexistence of the Nash solution in this example depends on the fact that the set of feasible controls is unbounded.

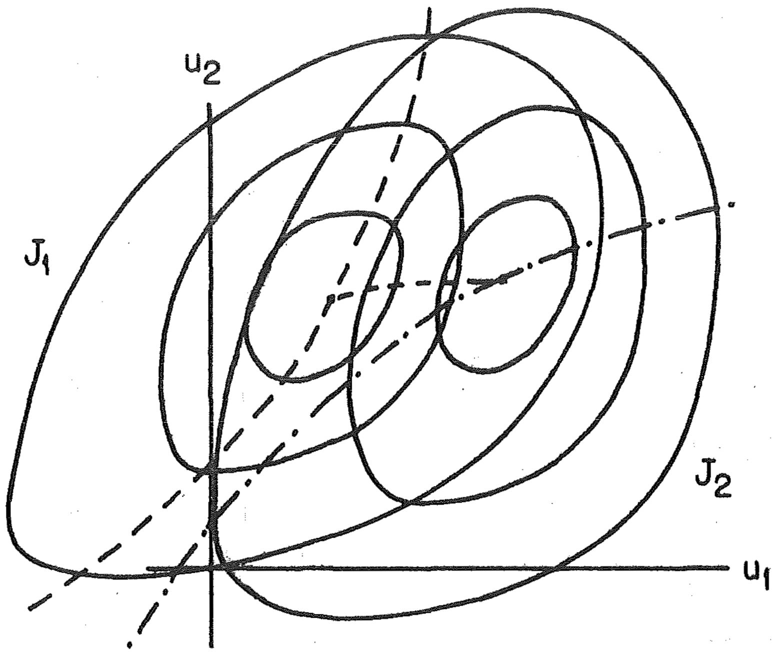

Fig. 2.5

2.9 Summary

Some of the basic features of general nonzero-sum games have been introduced as a background for the discussion of differential games to follow. There are N players, each trying to select a strategy to minimize his cost criterion. The interests of the players are not diametrically opposed, so there is an incentive to seek "cooperative" solutions, if such agreements can be enforced.

There is no single satisfactory definition of a "solution" to a nonzero-sum game. One type of solution, the Nash equilibrium, is secure against unilateral attempts by any player to optimize. If agreements can be reached and enforced, it is usually possible for all players to simultaneously achieve better results than the Nash solution. The set of noninferior solutions includes all solutions which are undominated. Finding this set, from which a "negotiated" solution would be chosen, is sometimes equivalent to solving an (N - 1)-parameter family of optimization problems. The information needed to select a particular member of the noninferior set is often not included in the formulation of the game.

If a player is unsure of what rationale his rivals are using, he might choose to minimize his worst possible cost by choosing his minimax strategy, which can be found by solving a two-person, zero-sum game.

All of these ideas were illustrated by a two-person game designed to give the kind of graphical insight which may be useful in studying differential games.

CHAPTER III

DIFFERENTIAL GAMES

3.1 Introduction

In the general N-player differential game, the i^{th} player ($i = 1, \ldots, N$) wishes to choose his "control" u_i at each time t in the interval $[t_o, t_f]$ to minimize

$$J_i = K_i(x(t_f), t_f) + \int_{t_o}^{t_f} L_i(x, t, u_1, \ldots, u_N)dt \tag{3.1}$$

subject to the constraint (common to all players)

$$\dot{x} = f(x, t, u_1, \ldots, u_N) \quad , \qquad x(t_o) = x_o \tag{3.2}$$

where x is a "state vector" of dimension n. There may also be inequality constraints on the control and/or state variables, but at this stage of the discussion such constraints need not be defined formally. The terminal time t_f may be fixed or variable; we shall generally consider it fixed.

When $N = 1$, the differential game is an optimal control problem. One naturally expects that extensions of various well-known results in optimal control theory will be useful in studying differential games. In fact, it would be foolish to even attempt to analyze an N-player differential game if the "corresponding" optimal control problem can not be solved by known methods.

With more than one player, the differential game still resembles an optimal control problem, but now game-theoretic considerations, such as those discussed in the previous chapter, complicate the situation. Recall that a general nonzero-sum game cannot be "solved" until one specifies what properties the "solution" should have. Similarly in a differential game, one must demand that the solution have some attribute such as minimax, Nash equilibrium, noninferiority, stability against coalition formation, etc.

One must also specify what information is available to each player during the course of the game. We shall always assume that each player knows the various parameters of the problem, including his rivals' cost criteria.

We shall generally make one of the following two additional assumptions: either (i) each player has continuous perfect measurements of the state vector x throughout the course of play, or (ii) no such measurements are available to any player. These assumptions will be called "closed loop" and "open loop," respectively.

Of course, many other assumptions about the type of information available to the players might be considered. It is perhaps worthwhile to list some of these possibilities, even though none of them will be pursued further in the remaining chapters.

1) The players may continually receive noisy measurements of the state vector. Some very special stochastic zero-sum differential games have been analyzed(9, 10, 11) but the general nonzero-sum stochastic differential game seems beyond the present state-of-the art.

2) Some players operate "open loop" while others operate "closed loop." Or perhaps a choice is offered, with a "measurement cost" assessed to those players who choose "closed loop" operation.

3) Although the controls are applied continuously, the players receive perfect measurements only at discrete times ("sampled data feedback controls").

4) The players do not have complete knowledge of all the parameters of the game (such as the exact cost criteria of the rivals) and must deduce them by measurements of the state vector during the course of the game. Such problems might loosely be termed "adaptive differential games."

5) Each player receives perfect measurements of some (not all) components of the state vector.

6) The players receive perfect measurements but they are not sure what <u>rationale</u> (i.e., minimax, Nash, coalition, etc.) their rivals are using.

While all of these possibilities arise in practical applications, it is obviously hopeless to try to analyze them until we have made some progress in the simpler open loop and closed loop "deterministic" differential games. Thus the five suggestions above are left as topics for future research.

3.2 Nash equilibrium solutions

When each player uses a strategy which is optimal against his rivals' strategies, the result is the Nash solution, defined formally in eq. (2.4). We have already seen that Nash solutions are non-optimal in the sense that it is usually possible for all players simultaneously to obtain lower costs, but this better result can only be achieved if the players can be trusted not to try to minimize their individual costs. In situations where " cooperative" arrangements cannot be made or enforced, our interest centers on the inefficient but "secure"* Nash solution. Of course, in some games Nash solutions may not exist, while other games may have more than one Nash solution.

If the i^{th} player in a differential game knew the Nash strategies of his rivals, he could find his own Nash strategy by solving an optimal control problem. Our approach to the problem of finding a set of Nash controls for all N players is then to use methods known in optimal control theory to solve the optimization problem for each player in terms of the other players' (still unknown) controls. Then by demanding consistency among these N solutions, we can hopefully solve for the N Nash strategies.

The usual procedure for solving optimal control problems is to obtain a set of <u>necessary conditions</u> with the following properties:

1) An algorithm can be devised for finding (numerically) all controls which satisfy these conditions.

2) The conditions are strong enough so that only a few controls (hopefully only one) satisfy them.

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*The Nash solution is secure only against <u>unilateral</u> attempts to optimize, not against coalitions involving two or more rivals.

3) A test is available for verifying whether or not a particular control actually is optimal.

We shall restrict our attention to differential games where the "corresponding" one-player game can be solved by this approach.

It may be objected that the discussion below does not treat differential games in their most general form. For example, problems with inequality constraints and singular problems are not given the attention they deserve. However, even the relatively simple unconstrained, nonsingular differential game provides an adequate means for illustrating the general features of differential games. We shall see that even with these restrictions, it is not easy to compute Nash solutions.

In Section 1 we discussed the types of information which might be available to the players and we decided to restrict our attention to two cases: no measurements (open loop) or perfect measurements (closed loop) of the state vector by all the players. We shall see that a differential game may have both an open loop and a closed loop Nash solution, and that these solutions give different paths and costs. This result may surprise the reader who is familiar with the theories of optimal control and two-person zero-sum differential games.

3.2a Open loop Nash solutions

Consider a general differential game of the kind described by (3.1), (3.2) where there are no inequality constraints on the state or control variables and no restrictions on the terminal state. Let the terminal time t_f be fixed. If the i^{th} player is given the open loop Nash control functions $u_j^*(t)$, $j \neq i$, for all his rivals, then he can obtain a set of necessary conditions for his own open loop Nash control $u_i^*(t)$ by variational methods which are well-known in optimal control theory.

Define the "Hamiltonian"† for the i^{th} player:

$$H_i(x, t, u_1, \ldots, u_N, \lambda_i) = L_i(x, t, u_1, \ldots, u_N) + \lambda_i^T f(x, t, u_1, \ldots, u_N) \tag{3.3}$$

where λ_i is a vector of dimension n. Then since $u_i^*(t)$ must minimize J_i when the other players use their Nash controls, the following first order necessary conditions must hold:

$$\dot{x} = f(x, t, u_1^*, (t), \ldots, u_N^*(t)) \tag{3.4}$$

$$\dot{\lambda}_i^T = -H_{ix}(x, t, u_1^*, (t), \ldots, u_N^*(t), \lambda_i) \quad , \quad \lambda_i^T(t_f) = K_{ix}(x(t_f)) \tag{3.5}$$

$$0 = H_{iu_i}(x, t, u_1^*, (t), \ldots, u_N^*(t), \lambda_i) \tag{3.6}$$

A second order necessary condition is that, along the "Nash path,"

$$H_{iu_iu_i} \quad \text{is positive semidefinite} \tag{3.7}$$

while a sufficient condition for u_i^* to be at least locally minimizing is that, along the Nash path,

$$H_{iu_iu_i} \quad \text{is positive definite} \tag{3.8}$$

The problem is said to be <u>nonsingular</u> if (3.8) holds for almost all $t \in [t_o, t_f]$ and for all possible control functions $u_i(t)$.

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*More precisely, the "Nash open loop Hamiltonian."

Eqs. (3.4), (3.5) and (3.6), for $i = 1, \ldots, N$, provide a set of necessary conditions for the entire N-tuple of Nash open loop control strategies. In some examples, the N coupled (vector) algebraic equations (3.6) can be solved, either numerically or analytically, to obtain a unique control N-tuple $\{u_1, \ldots, u_N\}$ as a function of $x, t, \lambda_1, \ldots, \lambda_N$. Often it can also be that (3.8) holds for this N-tuple for any $x, t, \lambda_1, \ldots, \lambda_N$. When this is the case, the controls can be eliminated from (3.4) and (3.5), leaving a nonlinear two-point boundary value problem with n differential equations with given initial conditions and nN with terminal values specified as functions of the terminal state. Various iterative algorithms are available[12, 13, 14] for solving problems of this type when $N = 1$. One would hope that some of these methods could be extended to the differential game ($N > 1$).

It should be noted that the i^{th} player's Hamiltonian is extremized only with respect to his own control; generally

$$H_{iu_j} \neq 0 \quad \text{for} \quad i \neq j \tag{3.9}$$

This fact is the source of considerable difficulty when one attempts to compute Nash equilibria.

3.2b Closed loop Nash solutions

If the i^{th} player were given the closed loop Nash strategies $\Psi_j(x, t)$, $j \neq i$, for all his rivals, he could find his own closed loop Nash strategy $\Psi_i(x, t)$, by solving optimal control problems starting from each point in the state-time space (i.e., by "filling the space with optimal trajectories").

One way to do this is to use variational necessary conditions similar to those used for the open loop control above. However, now the dependence of the rivals' controls on x must be included in considering variations of the i^{th} player's control. Defining the i^{th} Hamiltonian as in eq. (3.3), the closed loop necessary conditions are, for i = 1, . . . , N

$$\dot{x} = f(x, u_1, \ldots, u_N, t) \quad , \quad x(t_o) = x_o \tag{3.11}$$

$$\dot{\lambda}_i^T = -\frac{\partial}{\partial x} H_i(x; t; u_1, \ldots, u_N; \lambda_i) - \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\partial H_i}{\partial u_i} \frac{\partial \Psi_j}{\partial x}(x, t) \tag{3.12}$$

$$\lambda_i^T(t_f) = \frac{\partial}{\partial x(t_f)} K_i(x(t_f), t_f) \tag{3.13}$$

$$u_i = \Psi_i(x, t) \text{ minimizes } H_i(x; t; \Psi_1, \ldots, \Psi_{i-1}, u_i, \Psi_{i+1}, \ldots, \Psi_N; \lambda_i) \tag{3.14}$$

Notice that when N = 1 (optimal control problem) the second term in (3.12) is absent. The optimal "closed loop" control u(x, t) can then be obtained by solving for the "open loop" optimal control u(t) for every initial point (x, t). This method is not valid in the N-player game, however, due to the summation term in (3.12). In the optimal control problem, these necessary conditions are a set of ordinary differential equations, but in the N-player nonzero-sum game, they are a set of partial differential equations, generally very difficult to solve.

The presence of the summation term in (3.12) makes the necessary conditions (3.11)-3.14) virtually useless for deriving computational algorithms. Note that this troublesome term is absent in the optimal control problem (because N = 1), in the two-person zero-sum game (because $H_1 = -H_2$ so $\frac{\partial H_1}{\partial u_2} = -\frac{\partial H_2}{\partial u_2} = 0$), and in the open-loop nonzero-sum problem (because $\frac{\partial \Psi_j}{\partial x} = 0$). One certainly expects the open and closed loop solutions to be different whenever this term is nonzero.

Using reasoning familiar from optimal control theory, one may interpret (3.12) as follows: λ_i is the "influence function" for the i^{th} player, i.e., the sensitivity of his cost to a perturbation in the state vector. If the other players are using feedback strategies, any perturbation δx of the state vector will cause them to change their controls by an amount $\frac{\partial \Psi_j}{\partial x} \delta x$. If the i^{th} Hamiltonian were already extremized with respect to the control u_j, $j \neq i$, this would not affect the i^{th} player's cost, but since generally $\frac{\partial H_i}{\partial u_j} \neq 0$ for $i \neq j$, the reactions of the other players to the perturbation will influence the i^{th} player's cost, and the i^{th} player must account for this effect in considering variations of the trajectory.

A more satisfactory procedure for dealing with closed loop Nash controls is the value function approach. Let $\phi(x, t) = \{\phi_1(x, t), \ldots, \phi_N(x, t)\}$ be any set* of control strategies for the N players (resulting in piecewise

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*Strictly speaking, the set $\phi(x, t)$ must be defined so that the trajectory $x_\phi(t)$ satisfying (3.11) could be continued from any initial point (x_o, t_o).

continuous $u_i(t)$), and let $x_\phi(t)$ denote the trajectory through $x(t_o)$ resulting when these controls are used. Then the value function associated with this strategy set is piecewise continuously differentiable and defined as

$$V_i(x_o, t_o, \phi) = K_i(x(t_f), t_f) + \int_{t_o}^{t_f} L_i(x_\phi, \phi, t)dt \tag{3.15}$$

When $\phi(x, t)$ is a Nash strategy N-tuple, the functions defined in (3.15) will be called Nash value functions. Since these are the only type to be discussed here, the argument ϕ of V_i will be suppressed. One should remember, however, that some differential games have more than one closed loop Nash solution; a different value function then exists for each Nash strategy N-tuple.

By applying the definition of the Nash property (2.4), the usual "principle of optimality" argument can be extended in an obvious way to show that the value functions $V_i(x, t)$, $i = 1, \ldots, N$, are solutions of the partial differential equations

$$\frac{\partial V_i}{\partial t} = -\min_{u_i} H_i(x; t; \Psi_1, \ldots, \Psi_{i-1}, u_i, \Psi_{i+1}, \ldots, \Psi_N; \frac{\partial V_i}{\partial x}) \tag{3.16}$$

where

$$V_i(x(t_f), t_f) = K_i(x(t_f), t_f) \quad .$$

This is the generalized Hamilton-Jacobi-Bellman equation. The equilibrium strategy $\Psi_i(x, t)$ is the control u_i which achieves the minimum in (3.16). To integrate (3.16) backward from the terminal manifold, we must be able at each (x, t) to find the "Nash saddle-point" of the vector Hamiltonian $H = [H_1, \ldots, H_N]$, i.e., to solve an ordinary continuous nonzero-sum N-player game (not a differential game) at every instant t. This is not always possible, but it is possible in an important class of games. A differential game is said to be <u>normal</u> if (i) it is possible to find a unique Nash equilibrium point $\Psi^*(x, t, \lambda_1, \ldots, \lambda_N)$ for the vector H for all x, λ and t, and (ii) when the equations

$$\frac{\partial V_i}{\partial t} = -H_i[x; t; \Psi^*(x, t, \frac{\partial V_1}{\partial x}, \ldots, \frac{\partial V_N}{\partial x}); \frac{\partial V_i}{\partial x}] \tag{3.17}$$

$$\dot{x} = f(x, u_1, \ldots, u_N, t) \tag{3.18}$$

$$u_i = \Psi_i^*(x, t, \frac{\partial V_1}{\partial x}, \ldots, \frac{\partial V_N}{\partial x}) \tag{3.19}$$

are integrated backward from all the points on the terminal surface feasible trajectories are obtained.† The next chapter considers a class of games which are normal.

3.3 Noninferior solutions

If it is possible for all N players in a differential game to agree, prior to the starting time t_o, to coordinate their strategies, then the resulting set of controls should be chosen from the noninferior set of solutions, defined for the general game in eq. (2.5). We have already seen in Section 2.5 that finding the set of noninferior solutions to a "static" game is equivalent to solving a minimization problem with a vector cost function. Similarly, finding the noninferior strategy sets for an N-player differential game is equivalent to solving an optimal

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†Note that condition (i) requires that a unique set of controls giving a Nash point of H(x, t, u, λ) can be found as an explicit function of x, t, and λ for any λ. This is a <u>sufficient</u> (but not necessary) condition for the existence of a Nash trajectory. It is relatively easy to determine if (i) is satisfied, since no differential equation need be solved.

control problem with a vector cost criterion. All the statements made in Section 2.5 still apply, since they depend only on properties of the set of feasible results in the cost space E^N, not on how these results are obtained. However, in an optimal control problem it is impractical to try to generate the set Z of feasible results. Unless the functions involved in the cost criteria $J_1, \ldots, J_N$ are very special (e.g., convex in all the controls) there appears to be no practical way to answer the appropriate questions about the convexity properties of Z.

Whether or not these questions about Z can be resolved, we can in any case obtain <u>some</u> of the noninferior solutions by finding the set Ω, the set of solutions of all problems $P(\mu)$, where $\mu \in M$ (see definitions in eq. (2.7)). For the differential game, the problem $P(\mu)$ is an optimal control problem with a scalar cost criterion:

$$P(\mu): \quad \underset{u_1, \ldots, u_N}{\text{minimize}} \quad J(\mu) = \sum_{i=1}^{N} \mu_i [K_i(x(t_f)) + \int_{t_o}^{t_f} L_i(x, t, u_1, \ldots, u_N) d \tag{3.20}$$

subject to

$$\dot{x} = f(x, t, u_1, \ldots, u_N) \quad , \qquad x(t_o) = x_o \tag{3.21}$$

where

$$M = \{\mu \in E^N \,|\, \mu_i > 0, \; \sum_{i=1}^{N} \mu_i = 1\}$$

The components of μ are interpreted as the relative weights placed on the interests of the players entering the agreement. Since it is hard to see why any player should accept zero weighting on his cost criterion, the question of whether or not all the solutions in $\overline{\Omega}$ are noninferior is probably not important as far as negotiations are concerned. However, the situation could be as in Fig. 2.2, where an apparently "reasonable" portion of the efficient set Λ is not included in $\overline{\Omega}$. In practical applications, it might be possible to tell whether or not Ω contains all the "negotiable" solutions. Even if this can not be determined, it is certainly useful to compute the N - 1 parameter of solutions which generate Ω, since they are always noninferior and they can be used to eliminate large portions of the cost space from consideration.

DaCunha and Polak[15] have extended the Pontryagin maximum principle to obtain a set of necessary conditions for a solution of an optimal control problem with a vector cost criterion. Letting

$$L = [L_1, \ldots, L_N] \quad , \qquad u = [u_1, \ldots, u_N] \quad , \qquad \text{etc.}$$

we can state these conditions in our notation:

The vector optimal control problem is to find a control $\hat{u}(t)$, $t_o \leqslant t \leqslant t_f$, and a corresponding trajectory $\hat{x}(t)$ determined by (3.2), such that

(i) $\hat{u}(t)$ is a measurable, essentially bounded function whose range is contained in an arbitrary but fixed subset U of E^m.

(ii) For every control u(t) and corresponding trajectory x(t) satisfying (i), the relation

$$K(x(t_f), t_f) + \int_{t_o}^{t_f} L(x, t, u)dt \leqslant K(\hat{x}(t_f), t_f) + \int_{t_o}^{t_f} L(\hat{x}, t, \hat{u})dt$$

(componentwise) implies

$$K(x(t_f)t_f) + \int_{t_o}^{t_f} L(x,t,u)dt = K(\hat{x}(t_f),t_f) + \int_{t_o}^{t_f} L(\hat{x},t,\hat{u})dt$$

Let $\hat{u}(t)$ be a control which solves the vector optimal control problem. Assume $f(x,t,u)$ and $L(x,t,u)$ are continuous in u and x and are continuously differentiable in x. Then there exists a vector $\mu \in E^N$, $\mu \geq 0$, and a vector function $\lambda(t) \in E^n$, with $[\mu, \lambda] \not\equiv 0$, such that

(i) $$\dot{\lambda}^T(t) = -\mu^T \frac{\partial L(\hat{x},t,\hat{u})}{\partial x} - \lambda^T(t) \frac{\partial f(\hat{x},t,\hat{u})}{\partial x}$$

(ii) $$\lambda^T(t_f) = \mu^T \frac{\partial K(\hat{x}(t_f),t_f)}{\partial x}$$

(iii) for every $v \in U$ and almost all $t \in [t_o, t_f]$,

$$\mu^T L(\hat{x},t,\hat{u}) + \lambda^T(t) f(\hat{x},t,\hat{u}) \leq \mu^T L(x,t,v) + \lambda^T f(x,t,v) \quad .$$

Note that if $\mu \not\equiv 0$, then we can scale the problem so that $\sum_{i=1}^{N} \mu_i = 1$.

We then recognize the necessary conditions for problems of the type $P(\mu)$, where $\mu \in \overline{M}$. In fact, $\overline{\Omega}$ contains all solutions which satisfy the above conditions except those for which $\mu \equiv 0$ (all components). But when $\mu \equiv 0$, the solution to (i), (ii) above is $\lambda(t) = 0$, which is not allowed. Thus there appears to be no difference between the necessary conditions for the vector optimal control problem and for solutions in $\overline{\Omega}$. Thus the necessary conditions are not strong enough to distinguish between the sets $\overline{\Omega}$ and Λ.

We conclude that we can find at least some (and perhaps all) of the set of noninferior solutions to a differential game by solving an

(N - 1)-parameter family of optimal control problems, where the parameters are the relative weights on the various players' costs. Even if this set Ω of solutions (or its closure) does not contain all the efficient solutions, it may be satisfactory as a negotiation set. For example, it may contain all the efficient solutions which dominate the "threat point" (the result if no agreement can be reached). The problem of how to decide which member of the noninferior set to implement (the bargaining problem) cannot be solved unless further rules are specified.

3.4 Minimax solutions

If the i^{th} player has no idea of what rationale his rivals are using (perhaps he does not even know their cost functions) then he could make the pessimistic assumption that all the other players will join forces to try to maximize his cost. Player i thus envisions himself as playing in a two-person zero-sum differential game:

$$\underset{u_i}{\text{minimize}} \quad \underset{u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_N}{\text{maximize}} \quad J_i = K_i(x(t_f), t_f) + \int_{t_o}^{t_f} L_i(x,t,u)dt \tag{3.22}$$

subject to

$$\dot{x} = f(x,t,u_1,\ldots,u_N) \quad , \qquad x(t_o) = x_o \tag{3.23}$$

If no measurements are available to Player i, then the function $\bar{u}_i(t)$ which minimizes the worst damage which the other players can inflict on him is his open loop minimax strategy. Of course, the controls of the other players which the i^{th} player obtains from his calculation are not actually their minimax controls. To obtain <u>all</u> the minimax controls

$\bar{u}_j$, $j = 1, \ldots, N$, we must solve N separate two-person zero-sum differential games. When these controls are implemented, the resulting trajectory and costs will not be those predicted by any of the players (unless the game really is two-person zero-sum). Generally all players will achieve better than their minimax costs.

If the i^{th} player has continuous perfect measurements of the state vector, then when he implements his minimax control he will notice, after a very short time, that the trajectory x(t) does not correspond to his expectation; this is of course due to the other players not trying to maximize the i^{th} player's cost. But player i may assume that, although they have not done so so far, they will at all future times try to maximize his cost. He must then calculate a revised minimax strategy at each (x, t). Such a strategy $\bar{u}_i(x, t)$ is a closed loop minimax strategy. It has the same disadvantage as the open loop minimax solution -- it is excessively pessimistic.*

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*Ref. [3] discusses some interesting new phenomena which arise in nonzero-sum differential games, such as the important difference between open loop and closed loop solutions and the relationship between Nash solutions, noninferior solutions, and the "principle of optimality." These phenomena are illustrated by some simple multistage discrete (bimatrix) games.

CHAPTER IV

LINEAR-QUADRATIC DIFFERENTIAL GAMES

4.1 Introduction*

This chapter considers a special class of differential games where the system is linear and the cost functions are quadratic functions of the state vector and controls. Like its counterpart in optimal control theory, the linear-quadratic differential game (LQDG) is analytically tractable and of some practical interest. It is useful in modelling a situation where each player is trying to regulate an output of the common linear system, i.e., each player tries to make his particular output (a linear function of the state vector) follow as closely as possible some prescribed program, without expending too much control effort.

The LQDG is probably the only non-trivial class of differential games in which the Nash solutions, both open and closed loop, as well as solutions based on other rationales, can be obtained exactly without difficulty.

4.2 Definition

In a linear-quadratic differential game (LQDG) with N players, the i^{th} player chooses u_i trying to minimize

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*The material in Sections 4.1-4.5 was presented (in less detail) in Ref. [2].

$$J_i = \int_{t_o}^{t_f} \{ \frac{1}{2} x^T Q_i x + \frac{1}{2} \sum_{j=1}^{N} (x^T G_{ij} u_j + \frac{1}{2} \sum_{k=1}^{N} u_j^T R_{ijk} u_k)$$

$$+ \alpha_i^T x + \sum_{j=1}^{N} c_j^T u_j \} dt + \frac{1}{2} x(t_f)^T S_{if} x(t_f) + \xi_{if}^T x(t_f) \qquad (4.1)$$

subject to

$$\dot{x} = Ax + \sum_{j=1}^{N} B_j u_j + w \ , \qquad x(t_o) = x_o \qquad (4.2)$$

where Q_i, G_{ij}, R_{ijk}, α_i, c_i, A, B_i, and w are functions of time known to all the players. Depending on which type of solution is sought, the controls u_i may be functions of time only, or of the state vector and time.

This is the most general form of the LQDG.* However, all of the interesting features of this problem can be exhibited by considering a simpler, less cumbersome version where the cross terms, linear terms, and inhomogeneous terms are omitted:

$$J_i = \frac{1}{2} \int_{t_o}^{t_f} (x^T Q_i x + \sum_{j=1}^{N} u_j R_{ij} u_j) dt + \frac{1}{2} x_f^T S_{if} x_f \qquad (4.3)$$

subject to

$$\dot{x} = Ax + \sum_{j=1}^{N} B_j u_j \ , \qquad x(t_o) = x_o \ , \quad x_f = x(t_f) \qquad (4.4)$$

*The addition of inhomogeneous terms to the cost criteria would not affect the solutions.

In the remainder of this chapter, only the problem (4.3), (4.4) will be considered. The corresponding equations for the general LQDG are presented for reference purposes in Appendix A.

Since infinitely negative costs can be achieved if R_{ii} has any negative eigenvalues, we shall always assume that R_{ii} is positive semidefinite for all i. If R_{ii} is positive definite for all i, the problem is nonsingular. Unless otherwise stated, we shall always assume in this chapter that the game is nonsingular.

4.3 Nash equilibrium solutions

The Nash solutions, either open loop or closed loop, can be obtained by applying the results of Chapter III.

4.3a Open loop Nash solutions

In the open loop case, from the point of view of the i^{th} player, the controls of the other players must be considered functions of time only. Either the variational necessary conditions (3.4)-(3.6) or the value function approach (3.16) can be used, but if the latter is used the partial differential equation (3.16) must first be converted (via separation of variables) to a set of ordinary differential equations before any relation can be assumed between the other players' controls and the state vector. Consequently, it is easier to solve the open loop problem by applying the variational necessary conditions.

When all controls except the i^{th} player's are treated as given functions of time $\Psi_j(t)$, $j \neq i$, the variational Hamiltonian for the i^{th} player is

$$H_i = \frac{1}{2}x^TQ_ix + \frac{1}{2}u_i^TR_{ii}u_i + \frac{1}{2}\sum_{j\neq i}^{N}\Psi_j(t)^TR_{ij}\Psi_j(t)$$

$$+ \lambda_i^T(Ax + B_iu_i + \sum_{j\neq i}^{N}B_j\Psi_j(t)) \tag{4.5}$$

The necessary conditions for the i^{th} player are then

$$\dot{\lambda}_i = -Q_ix - A^T\lambda_i \quad , \quad \lambda_i(t_f) = S_{if}x_f \tag{4.6}$$

$$u_i = -R_{ii}^{-1}B_i^T\lambda_i \tag{4.7}$$

and the state equation becomes

$$\dot{x} = Ax - \sum_{j=1}^{N}B_jR_{jj}^{-1}B_j^T\lambda_j \quad , \quad x(t_o) = x_o \tag{4.8}$$

Eq. (4.6) (for $i = 1, \ldots, N$) and eq. (4.8) are a linear two-point boundary value problem consisting of $N + 1$ coupled vector differential equations, each of the same dimension as x. To solve it, we define the square matrix $S_i(t)$ by

$$\lambda_i(t) = S_i(t)x(t) \quad , \quad i = 1, \ldots, N \tag{4.9}$$

Conditions (4.6) and (4.8) are then satisfied when $S_i(t)$ is a solution of

$$\dot{S}_i = -A^TS_i - S_iA - Q_i + \sum_{j=1}^{N}S_iB_jR_{jj}^{-1}B_j^TS_j \tag{4.10}$$

$$S_i(t_f) = S_{if}$$

Note that (4.10) is a set of N coupled quadratic matrix differential equations. However, S_i is in general not symmetric. (One may verify that the asymmetric part of (4.10) has a nonzero driving term.)

When the Nash open loop strategies

$$u_j(t) = -R_{jj}^{-1}B_j^T S_j(t)x(t) \tag{4.11}$$

are played by each player, then the cost paid by the i^{th} player is

$$J_i(x_o, t_o) = \frac{1}{2}x_o^T P_i(t_o)x_o \tag{4.12}$$

where $P_i(t)$ is the solution of the linear differential equation:

$$\dot{P}_i = -A^T P_i - P_i A - Q_i + \sum_{j=1}^{N} (P_i B_j R_{jj}^{-1} B_j^T S_j + S_j^T B_j R_{jj}^{-1} B_j^T P_i$$
$$-S_j^T B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T S_j)$$

$$P_i(t_f) = S_{if} \tag{4.13}$$

This result is easily obtained by assuming the form $\frac{1}{2}x(t)^T P_i(t)x(t)$ for the remaining part of the cost starting from t. Equating this with the cost function defined in (4.3) and differentiating then yields (4.13). Note that the N matrix equations in (4.13) are uncoupled.*

The reader who is surprised by the asymmetry of (4.10) may find it instructive to follow the alternate derivation of these results, which is presented in Appendix B. It will be seen there that the

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*Note also that the S_j come from the open loop "Riccati-like" equation (4.10) and are merely parameters in the linear equations (4.13). The reader should not confuse (4.13) with (4.17) in the next section; they have different solutions because the S_j are different.

multiplier λ_i^T may be interpreted as $V_{ix}(x(t), t)$, where $V_i(x, t)$ is the i^{th} optimal return function based on the assumption that the other players do not change their control functions $u_j(t)$, $j \neq i$.

4.3b Closed loop Nash solutions

When the players all have measurements of the state vector, the Nash equilibrium strategies can be found either by using the closed loop variational necessary conditions (3.11)-(3.14) or by the value function approach (3.16). We choose the latter because it is conceptually clearer. Note that x must be treated as an independent variable.

For the LQDG the value function equation (3.16) is

$$-V_{it}(x, t) = \min_{u_i} [\tfrac{1}{2}x^T Q_i x + \tfrac{1}{2}u_i^T R_{ii} u_i + \tfrac{1}{2}\sum_{j \neq i}^{N} \Psi_j^T(x, t) R_{ij} \Psi_j(x, t)$$

$$+ V_{ix}(x, t)\{Ax + B_i u_i + \sum_{j \neq i}^{N} B_j \Psi_j(x, t)\}]$$

$$V_i(x_f, t_f) = \tfrac{1}{2}x_f^T S_{if} x_f \tag{4.14}$$

The minimizing controls are, for $i = 1, \ldots, N$

$$u_i = -R_{ii}^{-1} B_i^T V_{ix}(x, t)^T \tag{4.15}$$

Substituting (4.15) into (4.14) and guessing the following separation of variables:

$$V_i(x, t) = \tfrac{1}{2}x^T S_i(t) x \tag{4.16}$$

one immediately verifies that (4.16) is the solution to (4.14), where $S_i(t)$ is the solution of

$$\dot{S}_i = -A^T S_i - S_i A - Q_i + \sum_{j=1}^{N} (S_i B_j R_{jj}^{-1} B_j^T S_j + S_j B_j R_{jj}^{-1} B_j^T S_i - S_j B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T S_j)$$

$$S_i(t_f) = S_{if} \tag{4.17}$$

This set of N coupled symmetric quadratic matrix differential equations will be called "generalized Riccati equations," since it reduces to the familiar Riccati equation of optimal control theory when $N = 1$. The closed loop Nash costs are then given by (4.16) evaluated at (x_o, t_o).

It is also straightforward to obtain these same results using the closed loop variational necessary conditions (3.11)-(3.14).

It is evident from the fact that R_{ij} $(j \neq i)$ appears in (4.17) but not in (4.10) that the open loop and closed loop Nash controls will be different. Even when $R_{ij} = 0$, the equations are not the same. However, in the optimal control problem $(N = 1)$ both (4.17) and its open loop counterpart (4.10) reduce to the well-known Riccati equation

$$\dot{S} = -A^T S - SA - Q + SBR^{-1}B^T S \tag{4.18}$$

Similarly, in the two-person, zero-sum LQDG, which is obtained by setting

$$R_{12} = -R_{22} \quad , \quad R_{21} = -R_{11}$$

$$Q_2 = -Q_1 \quad , \quad S_{2f} = -S_{1f}$$

and trying a solution of the form $S_2 = -S_1 \triangleq -S$, both (4.17) and (4.10) reduce to

$$\dot{S} = -A^TS - SA - Q_1 + S(B_1R_{11}^{-1}B_1^T - B_2R_{22}^{-1}B_2^T)S$$

$$S(t_f) = S_{1f} \qquad (4.19)$$

in agreement with the results of Ho, Bryson and Baron[16]. But excepting these two special cases, the open loop and closed loop solutions generally do not coincide.

4.3c Steady state Nash solutions

If both the system and the cost parameters are time-invariant, one often is interested only in the steady state feedback solutions. These can be obtained by letting $t_f \longrightarrow \infty$. The resulting "Nash feedback laws" are then, for $i = 1, \ldots, N$

$$u_i(x) = -R_{ii}^{-1}B_i^T\tilde{S}_ix \qquad (4.20)$$

where the constant matrices $\tilde{S}_j$ are the solutions of the algebraic equations obtained by equating the right side of (4.17) to zero.

4.4 Noninferior solutions

In Section 3.3 it was seen that at least part of Λ, the noninferior set of solutions could be found by solving a (N - 1)-parameter family of optimal control problems. Whether or not this set Ω (or its closure $\overline{\Omega}$) includes all of the noninferior solutions depends on the convexity properties of Z, the set of cost vectors generated by all feasible control N-tuples. Specifically, if Z is E^{N}_{-}-directionally convex, that is, if any convex combination of any two points in Z is the sum of a vector in Z and a vector in the positive orthant, then $\overline{\Omega}$ contains Λ.

Unfortunately, even in the "simple" linear-quadratic case, it is not easy to apply this test. A sufficient condition for $\overline{\Omega}$ to contain Λ is (assuming as usual that R_{ii} is positive definite for all i) that Q_i and R_{ij} be positive semidefinite for all i, j. But this condition is very strong and excludes most of the interesting cases. For example, in the two-person zero-sum LQDG, Z is the straight line $J_1 = -J_2$, a convex set, yet $R_{12} = -R_{22}$.

In any case, whether or not we can obtain all the noninferior solutions this way, it is still worthwhile to find the set Ω. It may turn out that the only "negotiable" solutions belong to some subset of Ω.

For any given weighting vector μ, the corresponding member of Ω is easily found by solving a linear-quadratic optimal control problem. The controls corresponding to this solution are, for i = 1, . . . , N

$$\hat{u}_i(\mu) = -\left[\sum_{j=1}^{N} \mu_j R_{ji}\right]^{-1} B_i^T \hat{S}(\mu) x \tag{4.21}$$

where

$$\dot{\hat{S}}(\mu) = -\hat{S}A - A^T\hat{S} - \sum_{j=1}^{N} Q_j + \hat{S}\sum_{i=1}^{N} B_i \left[\sum_{j=1}^{N} \mu_j R_{ji}\right]^{-1} B_i^T \hat{S}$$

$$\hat{S}(\mu, t_f) = \sum_{i=1}^{N} \mu_i S_{if} \tag{4.22}$$

where $\mu_i \geq 0$, i = 1, . . . , N, and $\sum_{i=1}^{N} \mu_i = 1$. Of course, (4.21) and (4.22) hold only if μ is such that the matrix to be inverted is positive definite. If some of the R_{ij} have negative roots, then there will be some positive vectors μ for which no solution exists, i.e., the scalar cost resulting from such a weighting of the players' interests can be driven to $-\infty$. But such a solution would probably give at least one player an unacceptably high cost and so would not be "negotiable."

One might wish to compute the costs incurred when the players use arbitrary linear feedback controls of the form

$$u_i = -K_i(t)x \tag{4.23}$$

Starting at (x, t), these costs are

$$J_i = \frac{1}{2}x^T P_i x \tag{4.24}$$

where $P_i(t)$, $i = 1, \ldots, N$, satisfy the N uncoupled linear matrix differential equations

$$\dot{P}_i = -P_iA - A^TP_i - Q_i - \sum_{j=1}^{N} (K_j^T R_{ij} K_j - P_i B_j K_j - K_j^T B_j^T P_i)$$

$$P_i(t_f) = S_{if} \tag{4.25}$$

This formula can be used to compute the costs for the individual players when the noninferior controls in (4.21) are implemented.

4.5 Minimax solutions

Finding the minimax control for the i^{th} player is equivalent to solving a 2-player zero-sum differential game, where the opponent of the i^{th} player chooses all but the i^{th} control and tries to maximize

J_i. Applying the results of Section 3.5, the minimax control for the i^{th} player is

$$\bar{u}_i = -R_{ii}^{-1}B_i^T\bar{S}_i x \qquad (4.26)$$

where

$$\dot{\bar{S}}_i = -\bar{S}_i A - A^T\bar{S}_i - Q_i + \bar{S}_i \sum_{j=1}^{N} B_j R_{ij}^{-1} B_j^T \bar{S}_i$$

$$\bar{S}_i(t_f) = S_{if} \qquad (4.27)$$

provided that

$$R_{ii} > 0 \quad ; \qquad R_{ij} < 0 \;, \qquad j \neq i \;, \qquad j = 1, \ldots, N \qquad (4.28)$$

If conditions (4.28) are not satisfied, the i^{th} minimax control may fail to exist, so the i^{th} minimax cost is infinite. Note that a minimax control might exist for some players and fail to exist for others. A case of interest is $R_{ij} = 0$ for all $i, j \neq i$. In this special case, the minimax control is either identically zero or does not exist.

If Player i believes the minimax assumption, his minimax cost, or "security level" is $\frac{1}{2}x_o\bar{S}_i(t_o)x_o$. The actual costs obtained when all players use their minimax controls in feedback form (4.26) can be computed using (4.25).

4.6 Applications

As an example of a nonzero-sum LQDG, Ref. [2] considered a generalization of a simple pursuit-evasion model, one of the best-known models in the zero-sum differential game theory literature.

The extension to the nonzero-sum model allowed the pursuer and evader to have cost criteria which were not entirely in conflict.

The example presented here is concerned with heating duplex apartments at the lowest cost. While the example may seem frivolous, it illustrates a case where the coupling between two physical systems, separately controlled to achieve seemingly nonconflicting goals, produces a differential game situation.

Example: Heating an apartment

A nonzero-sum differential game situation can occur when several physical systems, separately controlled to achieve seemingly nonconflicting goals, interact through their common environment. When each controller is trying to regulate his system to keep certain variables close to a prescribed program, then the resulting interaction can sometimes be adequately modeled as a linear-quadratic differential game. To illustrate this idea, we shall consider a simple example involving the heating of several apartments in a single building.

Each apartment in our model has its own heating source (a gas heater) controlled by the tenant. The i^{th} tenant can instantaneously control the heat flow, u_i, (where $u_i \geq 0$) to maintain the temperature of his apartment, x_i, at a comfortable level. The cost, per unit time, of operating the heater is $c_i u_i + \frac{1}{2} p_i u_i^2$, where the quadratic term represents the damage done to the furnace by operating it at excessive levels. (Whatever the actual form of the limitation on the capacity of the furnace, the quadratic term may be considered an approximate "penalty function" model of this limitation.)

The reference level of the temperature is chosen to be the desired temperature (say 70°F). To simplify the model somewhat, we assume that all tenants have the same preferred temperature. It is then reasonable to model the i^{th} tenant's "discomfort cost" by a quadratic term in x_i. Thus, the cost criterion in our model is

$$J_i = \int_{t_o}^{t_f} (\tfrac{1}{2} q_i x_i^2 + c_i u_i + \tfrac{1}{2} p_i u_i^2) dt + \tfrac{1}{2} s_{if} x_i(t_f)^2 \qquad (4.29)$$

Note that the various tenants' cost criteria are not coupled.

The source of <u>conflict</u> in this model is heat flow through walls shared by adjoining apartments. We let the state variables in our model be the temperatures, assumed uniform in any apartment. Let $x_o(t)$ be the outdoor temperature (relative to the desired indoor temperature), assumed to be sufficiently negative so that all heaters will be operated at positive levels. (This assumption allows us to ignore the constraint $u_i \geq 0$.) The temperatures of the N apartments are then governed by the following set of N coupled linear differential equations:

$$\dot{x}_i = \frac{1}{V_i} [-\sigma_{io}(x_i - x_o) - \sum_{j \neq i} \sigma_{ij}(x_i - x_j) + u_i] \qquad (4.30)$$

where

V_i = heat capacity (per degree) of the i^{th} apartment

σ_{io} = heat conduction through exterior walls

$\sigma_{ij} = \sigma_{ji}$ = heat conduction through wall shared by apartments i and j.

We may consider the heat capacity V_i to be a measure of the size of the i^{th} apartment.

Finally, we must specify the information available to the tenants. We assume that each tenant has continuous knowledge of the temperatures of all N apartments, so that closed loop controls can be used.*

We then seek closed loop Nash solutions, and, for comparison, the set of noninferior solutions. Since there are linear terms in the cost functions and driving terms (involving x_o, the outside temperature, which is assumed known as a function of time) in the state equations, we use the Nash solutions for the more general form of the problem, which are given in Appendix A.** Letting

$$x = \begin{bmatrix} x_1 \\ \cdots \\ x_N \end{bmatrix} \quad , \quad w = \begin{bmatrix} \sigma_{1o}/V_1 \\ \cdots\cdots \\ \sigma_{No}/V_N \end{bmatrix} x_o \quad , \quad B_i = \frac{1}{V_i} e_i$$

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*An interesting and more realistic alternate assumption is that each tenant can measure only his own temperature. Questions would then arise concerning the observability of the other temperatures through measurements of the ith temperature.

**The reader may skip from this point to the paragraph following equation (4.65).

$$A = \begin{bmatrix} -(\sigma_{1o} + \sum_{j\neq 1} \sigma_{1j})/V_1 & \cdots & \sigma_{1N}/V_1 \\ \cdots & \cdots & \\ \sigma_{N1}/V_N & \cdots & -(\sigma_{No} + \sum_{j\neq N} \sigma_{Nj})/V_N \end{bmatrix}$$

where e_i is the i^{th} unit base vector, and using (A. 3)-(A. 7), the closed loop Nash cost functions are

$$V_i(x,t) = \frac{1}{2} x^T S_i(t) x + \xi_i(t)^T x + \eta_i(t) \tag{4.31}$$

and the closed loop Nash controls are

$$\Psi_i(x,t) = -(B_i^T S_i x + B_i^T \xi_i + c_i)/V_i \tag{4.32}$$

where the N x N symmetric matrix S_i, the N-vector ξ_i, and the scalar η_i are solutions of

$$\dot{S}_i = -S_i A - A^T S_i - q_i e_i e_i^T + S_i B_i B_i^T S_i / p_i$$

$$+ \sum_{j\neq i} (S_i B_j B_j^T S_j + S_j B_j B_j^T S_i)/p_j$$

$$S_i(t_f) = s_{if} e_i e_i^T \tag{4.33}$$

$$\dot{\xi}_i = -(A^T - \sum_j \frac{1}{p_j} S_j B_j B_j^T)\xi_i - S_i w + \sum_{j\neq i} S_i B_j B_j^T \xi_j / p_j$$

$$+ S_i(B_i c_i / p_i + B_j c_j / p_j)$$

$$\xi_i(t_f) = 0 \tag{4.34}$$

$$\dot{\eta}_i = (c_i + \xi_i^T B_i)^2/2p_i + \sum_{j \neq i} \xi_i^T B_j (B_j^T \xi_j + c_j)/p_j - \xi_i^T w$$

$$\eta_i(t_f) = 0 \qquad (4.35)$$

The set of noninferior solutions to this problem can be obtained by solving a set of scalar optimal control problems of the form: Choose $u_1, \ldots, u_N$ to minimize

$$J(\mu) = \int_{t_o}^{t_f} [\tfrac{1}{2} x^T Q(\mu) x + c(\mu)^T u + \tfrac{1}{2} u^T R(\mu) u] dt + x_f^T S_f(\mu) x_f \qquad (4.36)$$

subject to

$$\dot{x} = Ax + Bu + w \qquad (4.37)$$

where μ is the usual positive weighting vector and

$$B = [B_1, \ldots, B_N] \ , \qquad u^T = [u_1, \ldots, u_N]$$

$Q(\mu)$ = diagonal matrix whose diagonal elements are $\mu_1 q_1, \ldots, \mu_N$

$R(\mu)$ = diagonal matrix whose diagonal elements are $\mu_1 p_1, \ldots, \mu_N$

$S_f(\mu)$ = diagonal matrix whose diagonal elements are

$\mu_1 s_{1f}, \ldots, \mu_N s_{Nf}$

$$c(\mu)^T = [\mu_1 c_1, \ldots, \mu_N c_N]$$

Since any optimal control problem is a special case of a differential game, the solution to (4.36), (4.37) can also be obtained from Appendix A

$$\hat{u}(\mu) = -R(\mu)^{-1} [B^T \hat{S}(\mu) x + B^T \hat{\xi}(\mu) + c(\mu)] \qquad (4.38)$$

where

$$\dot{\hat{S}}(\mu) = -SA - A^T S - Q(\mu) + SBR(\mu)^{-1} B^T S \ , \quad S(\mu, t_f) = S_f(\mu) \qquad (4.39)$$

$$\dot{\hat{\xi}}(\mu) = -A^T\hat{\xi} + S(\mu)BR(\mu)^{-1}[B^T\hat{\xi} + c(\mu)] - \hat{S}w \quad ,$$

$$\hat{\xi}(\mu, t_f) = 0 \tag{4.40}$$

The noninferior control for the i^{th} tenant is, from (4. 58)

$$\hat{u}_i(\mu) = -(\hat{\xi}_i + \hat{S}_{i.}x)/\mu_i p_i V_i - c_i/p_i \tag{4.41}$$

(where $\hat{S}_{i.}$ denotes the i^{th} row of $\hat{S}$). The costs for the individual tenants corresponding to these noninferior solutions can be computed by applying (A. 12)-(A. 15). After some manipulation, the cost for the i^{th} tenant is found to be

$$\hat{J}_i(\mu, x_o, t_o) = \frac{1}{2}x_o^T\hat{P}_i(\mu, t_o)x_o + \hat{q}_i(\mu, t_o)^T x_o + \hat{r}_i(\mu, t_o) \tag{4.42}$$

where

$$\dot{\hat{P}}_i = -A^t\hat{P}_i - \hat{P}_iA - q_iI + \sum_j (\hat{S}_jB_jB_j^T\hat{P}_i + \hat{P}_iB_jB_j\hat{S}_j)/\mu_jp_j$$

$$- \hat{S}_iB_iB_i^T\hat{S}_i/\mu_i^2$$

$$\hat{P}_i(\mu, t_f) = S_i(t_f) \tag{4.43}$$

$$\dot{\hat{q}} = -(A^T - \sum_j \hat{S}_jB_jB_j^T/\mu_jp_j)\hat{q}_i + \hat{S}_iB_ic_i/\mu_ip_i - \hat{P}_iw$$

$$+ \sum_j \hat{P}_iB_j(c_j + B_j^T\hat{\xi}_j/\mu_j)/p_j$$

$$\hat{q}_i(\mu, t_f) = 0 \tag{4.44}$$

$$\dot{\hat{r}}_i = -\hat{q}_i^T w + \sum_j \hat{q}_i^T B_j(c_j + B_j^T\hat{\xi}_j/\mu_j)/p_j - \frac{1}{2}(c_i + B_i^T\hat{\xi}_i/\mu_i)^2/p_i$$

$$\hat{r}_i(\mu, t_f) = 0 \tag{4.45}$$

The set of formulas (4.32)-(4.45) will give the controls and costs for the closed loop Nash solutions and for the set of all noninferior solutions for any configuration of apartments. However, they are far too cumbersome to have any intuitive appeal. We shall demonstrate their use by computing the solutions to a specific model involving two apartments of unequal size, with different heat conduction to the outside and different "comfort criteria" for the tenants.

Specific example:

When there are two apartments, our model reduces to

$$J_1 = \int_{t_o}^{t_f} (\frac{1}{2}q_1x_1^2 + u_1 + \frac{1}{2}p_1u_1^2)dt$$

$$J_2 = \int_{t_o}^{t_f} (\frac{1}{2}q_2x_2^2 + u_2 + \frac{1}{2}p_2u_2^2)dt$$

$$\dot{x}_1 = -\frac{\sigma_{1o}}{V_1}(x_1 - x_o) - \frac{\sigma}{V_1}(x_1 - x_2) - u_1/V_1$$

$$\dot{x}_2 = -\frac{\sigma_{2o}}{V_2}(x_2 - x_o) - \frac{\sigma}{V_2}(x_2 - x_1) - u_2/V_2$$

where $\sigma = \sigma_{12} = \sigma_{21}$, and where the costs are measured in units such that the "fuel prices" c_1 and c_2 are unity. We have omitted the terminal cost terms because we intend to take t_f sufficiently large that the terminal costs would have a negligible effect on the solutions during most of the interval of play. In other words, we shall seek steady-state (but not necessarily constant) solutions.

The inhomogeneous terms in our state equation represent the outdoor temperature $x_o(t)$, measured with respect to the desired indoor temperature. A reasonable choice for the outdoor temperature might be

$$x_o(t) = -y - z\cos(2\pi t)$$

where $x > z \geq 0$, and where time is in days, measured from the time of day when the temperature is lowest. A general program has been written to compute the closed loop Nash solutions and the noninferior solutions for any set of parameters, by integrating eqs. (4.33)-(4.35) and (4.38)-(4.45). However, it has been observed that no interesting effects arise from the presence of the periodic driving term, and the results are much more easily presented when the outdoor temperature is constant.

Since there are nine parameters even when the outdoor temperature is constant, it would obviously be impractical to illustrate the dependence of the solutions on each of the parameters. Instead, we shall present solutions for two values of one of the parameters, the first tenant's "comfort parameter" q_1, holding the remaining parameters at the following values:

$$x_o = -40^\circ$$
$$\sigma_{1o}/V_1 = \sigma_{2o}/V_2 = 4$$
$$\sigma/V_1 = \sigma/V_2 = 10$$
$$p_1 = p_2 = .1$$
$$q_2 = 3$$

For the completely symmetric case, $q_1 = 3$, the equicost contours for the two tenants are plotted in the control space in Fig. 4.1. The dashed lines give the loci of rational solutions for the two players; their intersection is the Nash solution. The dotted curve gives part of the set of noninferior solutions (it extends to the centers of the sets of elliptical cost contours, which lie at (-10, 398) and (398, -10) for Players 1 and 2, respectively). At various points on this curve, the weighting μ on Player 1 is indicated. Note that, as usual, something must be sacrificed to obtain the security of the Nash solution. The shaded region indicates those solutions which dominate the Nash solution. They correspond to weightings $.4589 < \mu < .5411$. In particular, if each player plays the cooperative solution corresponding to equal weighting, each player's cost is 7% less than his Nash cost.

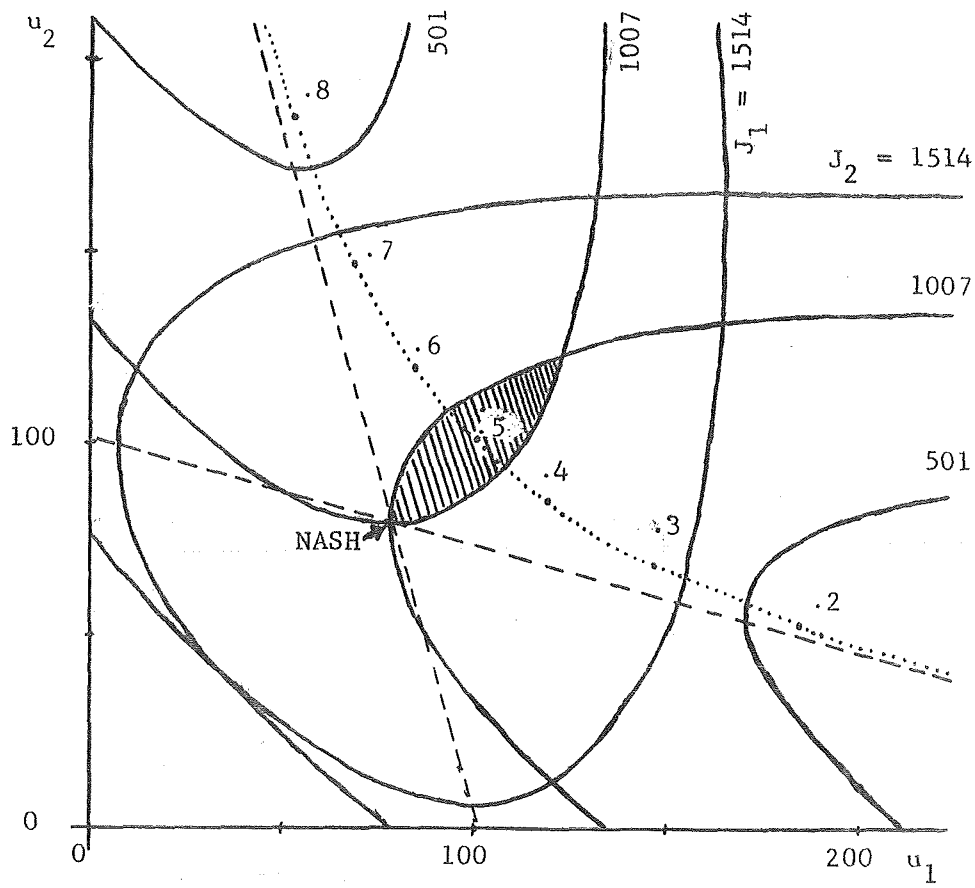

Fig. 4.1. Cost contours for the symmetric apartment heating problem

If we now lower q_1 from 3 to 1, we should expect Player 1 to use less heat, since he is now willing to tolerate a lower temperature. Player 2 will then lose more heat through the dividing wall, and he will partially compensate this by operating his heater at a higher level. The resulting situation is shown in Fig. 4.2. Again, there are solutions (shaded region) which dominate the Nash solution, but the noninferior solution with equal weights on the two players is not one of them. The noninferior solutions which dominate the Nash solution have weightings in the range $.6391 < \mu < .7116$. The noninferior solution with weighting .672 gives each player a cost which is 6% below his Nash cost.

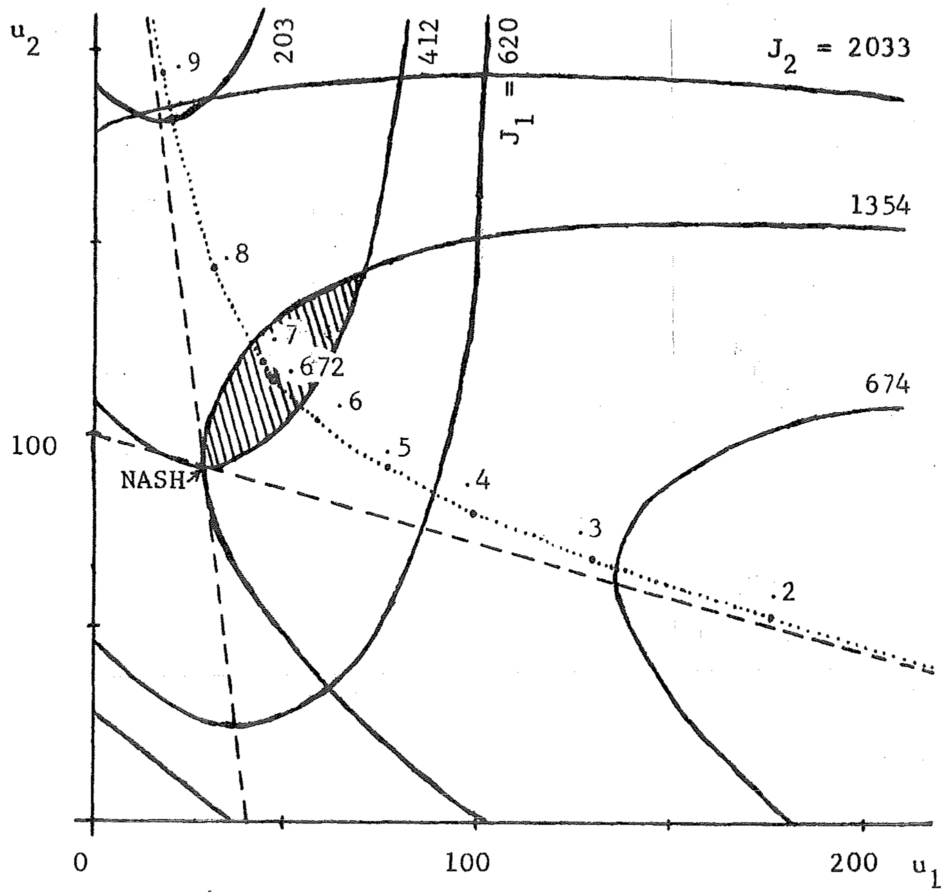

Fig. 4.2. Cost contours when first tenant places a lower penalty on discomfort.

It is also easy to find the risk involved in playing a noninferior solution, as well as the advantage to be gained by "cheating." Suppose, for example, that Player 2 cheats while Player 1 uses a noninferior control. Since Player 2 will try to optimize, the resulting solution will lie on the dashed line giving Player 2's rational solutions, at a point directly below the "nominal" noninferior point. The resulting costs (which can be computed by interpolation) can then be compared with the Nash costs. Note that if, in Fig. 4.1, a noninferior solution with equal weighting is agreed upon, and then both players cheat (each believing the other will not cheat) the result is worse for both players than the Nash solution.

CHAPTER V

COMPUTATION OF NASH EQUILIBRIA

5.1 Introduction

This chapter considers the problem of computing Nash equilibrium solutions, both open loop and closed loop, for general nonlinear differential games.* In Chapter IV we saw that when the system is linear, the cost criteria are all quadratic functions of the control and state variables, and there are no inequality constraints, then both kinds of Nash solutions can easily be obtained by integrating ordinary differential equations. Unfortunately, as can be seen from the examples given in Chapter VIII, most of the interesting applications of differential games cannot even be approximately modeled by the LQDG. Before differential games can be used to analyse realistic situations of "imperfect competition," numerical methods are needed for computing various kinds of solutions for a more general class of problems.

In Chapter III we have seen that the set of noninferior solutions, or at least part of it, can be obtained by solving a family of optimal control problems. The minimax solutions can be found by solving N two-person zero-sum differential games. Solving a deterministic two-person zero-sum differential game is very similar to solving an optimal control problem. Of the solutions we have considered in the previous chapters, only the Nash solution cannot be obtained by solving one or

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*The only "result" in this chapter is contained in Section 5.4. The general procedures described briefly in Sections 5.2 and 5.3 have not been implemented. They are included because they are conceptually useful as a background for considering more efficient computational methods.

more optimal control problems. This is the reason for giving it special attention in this chapter.

There are many optimal control problems which are too difficult to be solved in a reasonable time by any known numerical methods. For each of these problems, a differential game with a similar mathematical structure could be constructed, but we would not expect much success in analyzing such a game. Hence, we restrict our attention to differential games where the optimal control problem with the same structure can be solved by some practical method. This means that we shall always be considering problems where the part of the noninferior set which can be obtained by "scalarization," as well as the minimax solutions (if they exist), can be found by available numerical techniques.

In devising algorithms to solve optimal control problems, one need not distinguish between the open loop and closed loop cases, since (in deterministic problems) both assumptions lead to the same trajectories. But we have seen in Chapter III that in the nonzero-sum differential game entirely different solutions follow from the two alternative assumptions. Thus, in designing an algorithm for computing open loop Nash solutions, one must be careful to avoid using "closed loop ideas." Each player's control must optimize his cost, considering the other players' controls as fixed functions of time only. Conversely, the computation of each player's closed loop Nash control must correctly take account of the state dependence of the other players' controls.

The reader should be warned that the results in this chapter are discouraging. It appears that computational methods which have been successful in optimal control theory cannot readily be extended to find Nash solutions of differential games. The difficulties arise because one player's cost is neither minimized nor maximized with respect to another player's control when the game is nonzero-sum.

The next section describes a naive iterative method for computing either kind of Nash solution. If the procedure converges, the solutions have the desired Nash property, but there is no guarantee, and little reason to hope, that the method converges.

Section 5.3 describes the extension of the well-known dynamic programming idea. While it provides a conceptual method for finding closed loop Nash solutions without iteration, the same difficulties which make it impractical for most optimal control problems arise -- plus one additional difficulty.

Section 5.4 describes the extension of one of the most successful "second order" methods of optimal control theory, and shows that this approach -- surprisingly -- fails in the nonzero-sum differential game.

5.2 The "cycling" method

The definition of the Nash solution requires that each player's control function shall be optimized against the other players' control functions. This suggests the following simple iterative procedure:

1. Start by guessing control functions for players $2, \ldots, N$, and set $i = 1$.

2. Using an available optimal control algorithm, compute the optimal control for the i^{th} player, using the most recently computed control functions for the other players.

3. If the most recent computation for each player produced no change in his control function, stop. Otherwise, replace i by i(mod N) + 1, and return to step 2.

If this "cycling" procedure converges, the resulting solution will satisfy the definition of the Nash equilibrium. In principle, it could be used for either open loop or closed loop controls, but it seems rather impractical to compute and store a complete closed loop control function at each step in the algorithm. Thus, the method appears better suited for computing open loop controls.

The result of one "cycle" of this algorithm can be viewed as a mapping from the set of functions $[u_2^{(k)}, \ldots, u_N^{(k)}]$ to the set of functions $[u_2^{(k+1)}, \ldots, u_N^{(k+1)}]$ (where k denotes the k^{th} cycle). The algorithm will converge if this mapping is a contraction on the space of feasible strategy N-tuples. The Nash equilibrium N-tuple is the fixed point of this contraction. However, there is generally no simple test for determining if the mapping is a contraction, since the function giving the mapping is not available in explicit form. (One evaluation of this function requires solving N optimal control problems.)

One expects that the cycling method will succeed on some problems and fail on others. In some differential games, the success of the procedure may depend on the ordering of the players in the cycle. Convergence, if it occurs, may be slow, since near the Nash equilibrium each player's cost is insensitive to his own control but sensitive to the other players' controls.

5.3 Closed loop Nash controls via dynamic programming

This section describes briefly the extension of the well-known idea of dynamic programming to the problem of computing Nash solutions to differential games.

Let "DG" denote the usual differential game, where player i ($i = 1, \ldots, N$) chooses a closed loop control function $u_i(x, t)$ to minimize

$$J_i = \int_{t_o}^{t_f} L_i(x, t, u_1(x, t), \ldots, u_N(x, t))dt + K_i(x(t_f)) \tag{5.1}$$

subject to

$$\dot{x} = f(x, t, u_1(x, t), \ldots, u_N(x, t)) \quad , \qquad x(t_o) = x_o \tag{5.2}$$

$$u_i(x, t) \in U_i(x, t) \quad \text{for all} \quad t \in [t_o, t_f] \tag{5.3}$$

where the terminal time t_f is fixed.

Corresponding to DG, let "MG(K)" denote the multistage game obtained by dividing the interval $[t_o, t_f]$ into K equal parts* and requiring that the players use piecewise constant controls, with discontinuities occurring only at times $\frac{k}{K}(t_f - t_o)$, $k = 0, \ldots, (K - 1)$. In MG(K), player i chooses $u_i(x, k)$ to minimize

$$J_i^K = \sum_{k=0}^{K-1} L_i^K(x, k, u_1(x, k), \ldots, u_N(x, k)) + K_i(x_f) \tag{5.4}$$

subject to

*The reader familiar with dynamic programming may skip to the middle of page 5-7.

$$x(k+1) = x(k) + F^K(x(k), k, u_1(x(k), k), \ldots, u_N(x(k), k))$$

$$k = 0, \ldots, K-1$$

$$x(0) = x_o \quad , \qquad u_i(x, k) \in U_i^K(x, k) \tag{5.5}$$

where the functions L_i^K and F^K are obtained from L_i and f in (5. 1) and (5. 2) by

$$L_i^K = \int_{\frac{k}{K}(t_f - t_o)}^{\frac{k+1}{K}(t_f - t_o)} L_i[y(t), t, u_1(x(k), k), \ldots, u_N(x(k), k)]dt \tag{5.6}$$

$$F^K = \int_{\frac{k}{K}(t_f - t_o)}^{\frac{k+1}{K}(t_f - t_o)} f[y(t), t, u_1(x(k), k), \ldots, u_N(x(k), k)]dt \tag{5.7}$$

where y(t) is the solution of

$$\dot{y} = f[y, t, u_1(x(k), k), \ldots, u_N(x(k), k)]$$

$$y(\tfrac{k}{K}(t_f - t_o)) = x(k) \tag{5.8}$$

For any positive integer K, the idea of closed loop Nash controls for the multistage game MG(K) is well-defined. The value functions $V_i^K(x(k), k)$, $i = 1, \ldots, N$, are the remaining part of the sum in (5. 4) starting at stage k, when all players use closed loop Nash controls. They satisfy the following difference equation:

$$V_i^K(x, k) = \min_{u_i} \{ L_i^K[x + F^K(x, k, u_1^*(x, k), \ldots, u_i, \ldots, u_N^*(x, k)),$$

$$k, u_1^*(x, k), \ldots, u_i, \ldots, u_N^*(x, k)]$$

$$+ V_i^K[x + F^K(x, k, u_1^*(x, k), \ldots, u_i, \ldots, u_N^*(x, k)), k+1$$

$$V_i^K(x, K) = K_i(x) \tag{5.9}$$

where $u_i^*(x,k)$, $i = 1, \ldots, N$, is the control which achieves the minimum in (5.9). Since the N value functions are given for all x at the terminal stage K, the N coupled nonlinear difference equations (5.9) can be solved for all x, for $k = 1, \ldots, K$, provided that the "static Nash point" of the function in the brackets in (5.9) can be found.

To construct an algorithm based on (5.9), one requires that the function $V_i^K(x,k)$ be represented approximately by some finite set of numbers. The usual procedure in dynamic programming is to compute $V_i^K(x,k)$ only at points on a grid in x-space. In computing $V_i^K(x, k-1)$ on the same grid, one either only considers controls at stage $k - 1$ which lead to an exact grid point at stage k, or one obtains $V_i^K(x,k)$ at nongrid points by linear interpolation.

Even in optimal control problems $(N = 1)$ it is very difficult to obtain acceptable results by dynamic programming, especially if the dimension of x is greater than 2. The "curse of dimensionality" is even worse in the game MG(K), because (i) N value functions must be stored instead of one, and (ii) finding the Nash point of a vector function is more complicated than finding the minimum of a scalar function. In fact, if one represents a scalar function by a set of values given at points of a finite grid, then no matter how coarse the grid, this set of values will always have a minimum. But if an N-dimensional function of N control variables is represented by a finite set of values, it is possible that this array will have no Nash point even when the function itself does have a Nash point. Too coarse an approximation of the value functions may thus not only cause inaccuracy but may actually lead us to the erroneous conclusion that no Nash solution exists for our problem.

Even if an accurate solution to MG(K) could be obtained, it may be hard to prove that the solution to MG(K) approaches the solution to DG as $K \rightarrow \infty$. However, in practical applications, decisions are not really made continuously, and a multistage model may be more appropriate than a continuous differential game model.

5.4 A second order approach to the computation of closed loop Nash controls

In this section, an attempt is made to construct a computationally efficient method for finding closed loop Nash controls by generalizing a method which has succeeded in solving a wide class of nonlinear optimal control problems. Unfortunately, the conclusion of this section will be that this type of approach cannot be used to compute Nash solutions of nonzero-sum differential games. Although this negative result is disappointing, the reason for the failure of the method gives some insight about the nature of general differential games.

The dynamic programming approach described in the previous section is general enough to be applicable to almost any problem, but it is not practical to use this method except for very low-dimensional problems because of the enormous amount of computation involved. There are two features shared by many practical optimal control problems (as well as by most differential game models) which the dynamic programming method fails to exploit:

(i) The functions appearing in the system equations and the cost criteria are continuous and differentiable in all their arguments.

(ii) One does not need a solution for all points in the state-time space, but only along a trajectory through a specified initial point. (Plus, possibly, linear "correction" terms valid in a neighborhood of this trajectory.)

In optimal control theory, efficient iterative methods have been developed which exploit these two features by expanding the optimal cost surfaces around a nominal trajectory through the given initial point. A linear correction to the control is then computed to minimize the dominant terms in this expansion, and a new nominal trajectory through the same initial point is computed. An algorithm of this type is called "second order" if all the second order terms are included in the expansion of the function which is minimized by the control corrections.

One of the most successful of the second order techniques of optimal control theory is the "differential dynamic programming" (DDP) method of Jacobson and Mayne.[14, 17] The DDP approach is especially appropriate for our purpose because it is based on an expansion of the Hamilton-Jacobi equation and is thus a purely "closed loop" method. We have already seen in Chapter III that in nonzero-sum differential games (unlike optimal control problems) one cannot use closed loop and open loop arguments interchangeably.

The DDP method (and its extension to differential games) is most easily described if we restrict our attention to nonsingular problems (defined below) with no inequality constraints. Jacobson also extended the method to cover optimal control problems with control variable inequality constraints[14], and, more recently, extended it to singular problems[18], "bang-bang" problems[19], and problems with inequality constraints involving the state variables[20]. These extensions would be important in analyzing differential games, but we need not consider them here. The simple unconstrained, nonsingular case is enough to demonstrate the idea of the method as well as the reason for its future.

5.5 Description of the extended DDP algorithm

We are seeking an iterative procedure for finding the closed loop Nash controls to the differential game "DG" described by (5.1) and (5.2), with no constraints of the form (5.3). We are mainly interested in the Nash trajectory passing through the initial point (x_o, t_o), although if possible we would also like to compute the local expansion of the closed loop Nash controls about this trajectory.

Let the exact closed loop Nash value function for the i^{th} player be denoted $V_i^e(x, t)$. (We use the superscript to distinguish the exact function from approximate functions based on expansions.) Let the closed loop Nash controls be $\Psi_j^e(x, t)$, $i = 1, \ldots, N$. Then the value functions must satisfy the generalized Hamilton-Jacobi-Bellman equation: for $i = 1, \ldots, N$

$$-\frac{\partial V_i^e}{\partial t} = \min_{u_i} [L_i(x, t, \Psi_1^e(x, t), \ldots, u_i, \ldots, \Psi_N^e(x, t)) + V_{ix}^e(x, t) f(x, t, \Psi_1^e(x, t), \ldots, u_i, \ldots, \Psi_N^e(x, t))]$$

$$V_i^e(x, t_f) = K_i(x, t_f) \tag{5.10}$$

where $\Psi_i^e(x, t)$ is the u_i which achieves the global minimum. Define the vector Hamiltonian H as the N-vector whose i^{th} component is the function

$$H_i(x, t, u_1, \ldots, u_N, V^e_{ix}) = L_i(x, t, u_1, \ldots, u_N) + V^e_{ix} f(x, t, u_1, \ldots, u_N) \qquad (5.11)$$

Letting $V^e_x = [V^e_{1x}, \ldots, V^e_{Nx}]$, we write H as the vector function $H(x, t, u_1, \ldots, u_N, V^e_x)$, and the operation in (5.10) then is to find the (static) Nash point of H.

In order to concentrate on the simplest case, we now make the following two assumptions: given any arbitrary x, t, and V^e_x,

(i) If $(u^*_1, \ldots, u^*_N)$ is a Nash point of $H(x, t, u_1, \ldots, u_N, V^e_x)$, then $H_{iu_iu_i}(x, t, u_1, \ldots, u_N, V^e_{ix})$ is positive definite for $i = 1, \ldots, N$. This is the <u>nonsingularity</u> condition.

(ii) There is not more than one Nash point of H.

Note that (since the minimization in (5.10) is global) condition (ii) would be implied by condition (i) in an optimal control problem. But a nonzero-sum game can have two distinct Nash points, with different costs, even though all the minimizations are global.

Now let $V_x(x, t) = \{V_{1x}(x, t), \ldots, V_{Nx}(x, t)\}$ be any N-tuple of n-dimensional vector functions (our algorithm will furnish an approximation to V^e_x, but at this point V_x is considered arbitrary). Let

$$u^*(x, t, V_x) = [u^*_1(x, t, V_x), \ldots, u^*_N(x, t, V_x)] \qquad (5.12)$$

be the N-tuple of control vectors which gives the Nash point of $H(x, t, u_1, \ldots, u_N, V_x)$ whereever the Nash point exists. (By assumption (ii), the Nash point is unique if it exists.) Since there are no control inequality constraints,

$$\frac{\partial H_i}{\partial u_i}(x, t, u_1^*, \ldots, u_N^*, V_{ix}) = 0 \quad \text{for} \quad i = 1, \ldots, N \text{ and for all } x, t \tag{5.13}$$

Let $\overline{x}(t)$ be any arbitrary given "nominal trajectory."† We now expand the unknown Nash cost functions $V_i^e(x, t)$ around this nominal path:

$$V_i^e(\overline{x} + \delta x, t) = V_i^e(\overline{x}, t) + V_{ix}^e(\overline{x}, t)\delta x + \frac{1}{2}\delta x^T V_{ixx}^e(\overline{x}, t)\delta x + \ldots \tag{5.14}$$

It is also convenient (although not necessary) at this point to define $a_i^e(x, t)$ as the difference between the true Nash cost for the i^{th} player and the "nominal cost," starting from (x, t):

$$a_i^e(x, t) \equiv V_i^e(x, t) - V_i(x, t)$$

Henceforth, H_i, f, and all their partial derivatives in the expansions to follow will be evaluated at $\overline{x}(t), t, u_1^*(t), \ldots, u_N^*(t), V_x(\overline{x}, t)$ unless other arguments are given explicitly.

Using approximations a_i, V_{ix}, and V_{ixx} (to be furnished along the nominal path by the algorithm) for $a_i^e(\overline{x}(t), t)$, $V_{ix}^e(\overline{x}(t), t)$, and $V_{ixx}^e(\overline{x}(t), t)$, respectively, and substituting (5.13) into (5.10) gives:

$$-V_{it} - a_{it} - V_{ixt}\delta x - \frac{1}{2}\delta x^T V_{ixxt}\delta x$$

$$= \min_{\delta u_i}[L_i(\overline{x} + \delta x, t, u_1^* + \delta u_1, \ldots, u_N^* + \delta u_N)$$

$$+ (V_{ix} + \delta x^T V_{ixx} + \frac{1}{2}\delta x^T V_{ixxx}^{\delta x}) f(\overline{x} + \delta x, t, u_1^* + \delta u_1, \ldots, u_N^*$$

$$+ \delta u_N)] \tag{5.15}$$

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†Later, we shall denote by $\overline{u}_1, \ldots, \overline{u}_N$ the set of controls which generates this "nominal" path.

where $\delta x^T V_{ixxx}^{\delta x}$ denotes the row vector whose j^{th} component is

$$\sum_k \sum_m \delta x^k \delta x^m \frac{\partial^3 V_i}{\partial x^j \partial x^k \partial x^m} \quad , \quad \text{etc.}$$

Note that, to retain all second order terms in the expansion, we must expand V_i to third order.

We now expand the right side of (5.15) around $\bar{x}, t, u_1^*, \ldots, u_N^*, V_{ix}$ to second order:

$$\min_{\delta u_i} [H_i + \sum_j H_{iu_j} \delta u_j + H_{ix} \delta x + f^T V_{ixx} \delta x + \sum_j \delta u_j^T (H_{iu_j x} + f_{u_j}^T V_{ixx}) \delta x$$

$$+ \frac{1}{2} \delta x^T (H_{ixx} + f_x^T V_{ixx} + V_{ixx} f_x + V_{ixxx}^f) \delta x$$

$$+ \text{higher order terms}] \qquad (5.16)$$

where, to simplify the algebra somewhat, we have made the assumption that

$$H_{iu_j u_k} = 0 \quad \text{for} \quad j \neq k$$

Note again our notation: V_{ixxx}^f is the n x n matrix $\sum_j \frac{\partial V_{ixx}}{\partial x^j} f^j$.

We now assume an approximate relation between δu_i and δx of the form

$$\delta u_i = \beta_i \delta x + \frac{1}{2} \Gamma_i^{\delta x} \delta x \qquad (5.18)$$

where β_i is a m_i x n matrix and Γ_i is a m_i x n x n tensor whose meaning should by now be clear.

Differentiating (5.16) by δu_i to obtain the minimum, the first order terms are

$$\delta u_i = -H^{-1}_{iu_iu_i}(H_{iu_ix} + f^T_{u_i} V_{ixx})\delta x + o(\delta x^2) \tag{5.19}$$

so we must have

$$\beta_i = -H^{-1}_{iu_iu_i}(H_{iu_ix} + f^T_{u_i} V_{ixx}) \tag{5.20}$$

If one carries out the expansion (5.16) through all third order terms, then differentiates and substitutes for δu_i using (5.18) and (5.20), one finds by collecting all the second order terms that

$$\frac{1}{2}\Gamma_i = -H^{-1}_{iu_iu_i}(H_{iu_iu_ix}\beta_i + \frac{1}{2}H_{iu_ixx} + \frac{1}{2}H^{\beta_i}_{iu_iu_iu_i}\beta_i + \frac{1}{2}f^T_{u_i}V_{ixxx}$$

$$+ \frac{1}{2}f^T_{u_ix}V_{ixx} + f^{T\beta_i}_{u_iu_i}V_{ixx}) \tag{5.21}$$

Using these results for β_i and Γ_i, and collecting like-order terms in δx in (5.15), one obtains, for $i = 1, \ldots, N$,

$$-\overline{V}_{it} - a_{it} = H_i \tag{5.22}$$

$$-V_{ixt} = \sum_{j \neq i} H_{iu_j}\beta_j + H_{ix} + f^T V_{ixx} \tag{5.23}$$

$$-V_{ixxt} = \sum_{j \neq i} H_{iu_j}\Gamma_j + 2\sum_j \beta_j^T(H_{iu_jx} + f^T_{u_j}V_{ixx}) + \sum_j \beta_j^T H_{iu_ju_j}\beta_j$$

$$+ H_{ixx} + f^T_x V_{ixx} + V_{ixx}f + V^f_{ixxx} \tag{5.24}$$

Now since $\overline{V}_i$, a_i, V_{ix}, and V_{ixx} are functions of x and t, their total time derivatives along the nominal trajectory are

$$\frac{d}{dt}(\overline{V}_i + a_i) = \overline{V}_{it} + a_{it} + V_{ix}f(\overline{x}, t, \overline{u}_1, \ldots, \overline{u}_N) \tag{5.25}$$

$$\frac{d}{dt} V_{ix} = V_{ixt} + V_{ixx} f(\bar{x}, t, \bar{u}_1, \ldots, \bar{u}_N) \tag{5.26}$$

$$\frac{d}{dt} V_{ixx} = V_{ixxt} + V_{ixxx} f(\bar{x}, t, \bar{u}_1, \ldots, \bar{u}_N) \tag{5.27}$$

Taking the symmetric part of (5.24) (since V_{ixx} must be symmetric), eqs. (5.22)-(5.27) give (using $L_i = -\frac{d}{dt}\bar{V}_i$)

$$-\dot{a}_i = H_i - H_i(\bar{x}, t, \bar{u}_1, \ldots, \bar{u}_N, V_{ix})$$

$$a_i(x_f, t_f) = 0 \tag{5.28}$$

$$-\dot{V}_{ix} = H_{ix} + (f^T - f^T(\bar{x}, t, \bar{u}_1, \ldots, \bar{u}_N))V_{ixx}$$

$$- \sum_{j \neq i} H_{iu_j} H^{-1}_{ju_ju_j} (H_{ju_jx} + f^T_{u_j} V_{jxx})$$

$$V_{ix}(\bar{x}_f, t_f) = K_{ix}(\bar{x}_f) \tag{5.29}$$

$$-\dot{V}_{ixx} = H_{ixx} + V_{ixx} f_x + f^T_x V_{ixx}$$

$$+ \sum_j (H_{ju_jx} + f^T_{u_j} V_{jxx})^T H^{-1}_{ju_ju_j} H_{iu_ju_j} H^{-1}_{ju_ju_j} (H_{ju_jx} + f^T_{u_j} V_{jxx})$$

$$- \sum_j [(H_{ju_jx} + f^T_{u_j} V_{jxx})^T H^{-1}_{ju_ju_j} (H_{iu_jx} + f^T_{u_j} V_{ixx})$$

$$+ (H_{iu_jx} + f^T_{u_j} V_{ixx})^T H^{-1}_{ju_ju_j} (H_{ju_jx} + f^T_{u_j} V_{jxx})]$$

$$+ V_{ixxx}(f - f(\bar{x}, t, \bar{u}_1, \ldots, \bar{u}_N))$$

$$+ \text{symmetric part of} \sum_{j \neq i} H_{iu_j} \Gamma_j$$

$$V_{ixx}(x_f, t_f) = K_{ixx}(x_f) \qquad (5.30)$$

where Γ_i is given by (5.21).

Note that the next-to-last term in (5.30), which contains V_{ixxx}, would vanish if the nominal trajectory were the actual Nash trajectory, since then $(\bar{u}_1, \ldots, \bar{u}_N)$ would equal $(u_1^*, \ldots, u_N^*)$. Thus ignoring this term would not invalidate the result, as long as the algorithm converges. Unfortunately, V_{ixxx} also contributes to Γ_i (see (5.21)) which appears in the last term in (5.30), and in this case the contribution of V_{ixxx} does not vanish even if the nominal trajectory is the Nash trajectory. Ignoring the last term (or the part of it involving V_{ixxx}) would thus completely invalidate (5.30). Since the V_{jxx} for all $j \neq i$ do not vanish from (5.29) on the Nash trajectory, (5.29) will not yield the correct V_{ix}, and the controls $u_1^*, \ldots, u_N^*$ computed via (5.14) will not minimize the correct Nash Hamiltonian.

This difficulty could be eliminated if we had some way to compute V_{ixxx} for all i, by some approximation which became exact on the true Nash path. If V_{ixxx} were available, then the process described above could be repeated, using as the new nominal trajectory the path obtained by integrating the state equation from (x_o, t_o), using the controls

$$u_i' = u_i^* + \beta_i \delta x \qquad (5.31)$$

where $\delta x = x' - \bar{x}$ is found by integrating from (x_o, t_o)

$$\dot{x}' = f(x', t, u_1', \ldots, u_N')$$

$$\delta x(t_o) = 0 \qquad (5.32)$$

It is straightforward to verify that, if this iterative procedure converges (still assuming the correct V_{ixxx} is somehow included in (5.30)), then the solution will be the closed loop Nash solution satisfying (5.14).

Unfortunately, to obtain an equation for V_{ixxx} one must expand (5.15) to include all third order terms. The resulting differential equation for V_{ixxx} will then contain nonnegligible terms involving V_{ixxxx}, and so forth. Unless the problem has the special property that the k^{th} partial derivative of $V_i^e(x, t)$ vanishes, for all x, t, and i, for some finite k, the method will fail to produce the correct trajectory no matter how high the order of our expansions. Thus the DDP method apparently cannot be extended to find Nash closed loop solutions, and clearly the same difficulty would occur with any other iterative method based on "local" expansions of the Hamilton-Jacobi equation or of the variational necessary conditions. This is a remarkable property of the nonzero-sum differential game, and is a consequence of the fact that, for $i \neq j$, $\frac{\partial H_i}{\partial u_j} \neq 0$ at the Nash point.

There are several special cases where the difficulty described above does not occur, so that the algorithm succeeds:

(1) The optimal control problem (N = 1). The summation in the last term in (5.30) has N - 1 terms and thus is absent. The method then reduces exactly to Jacobson's DDP.

(2) The two-person, zero-sum differential game. Here it is easy to show that $H_1 = -H_2$, so that, since $H_{1u_1} = 0$, we also have $H_{1u_2} = 0$. Thus the last term in (5.30) vanishes, and the

algorithm succeeds. In fact, since $V_1^e = -V_2^e$, eqs. (5.28)-(5.30) can be simplified considerably, so that they very closely resemble the corresponding equations for the DDP method.

(3) The linear-quadratic problem (see Section 4.3b). This is a case where V_{ixxx} is identically zero, and (5.30) then reduces to (4.17). While this is reassuring, it is of little help since we already know how to solve the linear-quadratic differential game.

(4) The identical goal game, where all players have the same cost criteria. Since $H_1 = \ldots = H_N$, $H_{iu_i} = 0$ implies $H_{iu_j} = 0$ for all j, and the last term in (5.30) vanishes. This problem can be reduced to an optimal control problem and solved via DDP.

(5) The differential game which can be decomposed into N subproblems which are not coupled in the state equations nor in the cost criteria. This is not really a game but only N separate optimal control problems.

(6) The "ordered" differential game, where the players can be numbered so that the i^{th} player is concerned with the controls used by players 1 through (i - 1) but not with the controls of players (i + 1) through N. This can be solved as an ordered sequence of optimal control problems.

It is important to note that simply using the algorithm described above, omitting the terms involving the unavailable V_{ixxx}, will not in any acceptable sense produce an "approximate" Nash solution for the general nonlinear case.

It also appears that "bootstrap" procedures for generating an approximation either to V_{ixxx} or to Γ_i, based on computing several neighboring trajectories and numerically differentiating, are doomed to failure. Careful examination of such proposals reveals that they are based on using the same V_{ixx} on the set of neighboring paths, which is equivalent to assuming $V_{ixxx} = 0$.*

Thus the prospects for developing an efficient computational algorithm for Nash solutions appear rather poor. In the next chapter, some special cases (not quite so special as the six listed above) are presented for which approximate Nash solutions can be obtained, using ideas introduced in this section.

*This fault was pointed out by D. Jacobson (private communication).

CHAPTER VI

APPROXIMATE NASH SOLUTIONS FOR SOME SPECIAL CASES

6.1 Introduction

In the previous chapter, we concluded that there apparently was no computationally efficient method for computing closed loop Nash equilibrium trajectories through a given initial point for a general nonzero-sum differential game. However, at the end of Section 5.4, six special degenerate cases were given in which the closed loop Nash solutions could be computed efficiently by known methods such as Jacobson's differential dynamic programming (DDP)[14]. The first case, the optimal control problem, does not interest us, since we are concerned with games with two or more players. The remaining five special cases were:

(i) The two-person zero-sum differential game, where the cost criteria J_1 and J_2 have the property that there exists some strictly monotone transformation g such that $g(J_2) = -J_1$ for all feasible strategy pairs.

(ii) The identical goal differential game (the "perfectly cooperative" case) where (after appropriate monotonically increasing transformations) the cost criteria of the N players are identical.

(iii) The uncoupled differential game, which can be decomposed into N separate and unrelated optimal control problems.

(iv) The "ordered" differential game, where the players can be ordered so that each player is influenced by the players ahead of him in the ordering, but not by the players behind him. (This includes (iv) as a special case.) This can be solved as a sequence of control problems, using the controls for the first (i - 1) players as inputs to the i^{th} player's system.

(v) The linear-quadratic differential game, which is easily solved without iteration by the results of Section 4. 3b.

The purpose of this chapter is to consider differential game models where the formulation involves a "small" parameter ϵ, such that, if $\epsilon = 0$, the model reduces to one of the five special cases listed above. The "nominal" solution (for $\epsilon = 0$) will be assumed to have been found for some fixed initial point. Certain other quantities, such as various partial derivatives of the Nash cost functions and of the Hamiltonians will be assumed known along the nominal path. (These would be provided as byproducts of a DDP computation of the nominal path.) We shall then develop the closed loop Nash solution as Taylor series expansions in ϵ, and an algorithm will be derived which will compute exactly the first order terms in these expansions. By dropping the higher order terms (which are not provided by the algorithm) we will then obtain an approximate solution with an error proportional to ϵ^2. The actual range for which this approximation is acceptable will depend on the details of the example under consideration. However, as a general indication of the magnitude of the error, the second order term in the expansion of the cost functions can be obtained, without too much additional computation, at the initial time.

Since the basic idea of this type of approximation is the same for games which "almost" satisfy any of the five conditions listed above, a detailed derivation of the method will only be given for the "almost zero-sum" differential game. For the other four cases, only the problem statements and the results will be given.

It will be assumed in the following sections that there are no inequality constraints of any kind, and that all the partial derivatives required in the method exist. The idea can probably be extended to problems with certain simple types of inequality constraints (such as control constraints) but considerable complexity would be added, since the expansion in ϵ would not be valid at "entry" or "exit" points (times at which the number of active constraints changes).

6.2 The "almost zero-sum" differential game

If J_1 and J_2 are the cost functions for a 2-player differential game, then they can always be written in the form

$$J_1 = J^- + \epsilon J^+ \qquad\qquad J^+ = \frac{1}{2\epsilon}(J_1 + J_2)$$
$$\text{i.e.,} \tag{6.1}$$
$$J_2 = -J^- + \epsilon J^+ \qquad\qquad J^- = \frac{1}{2}(J_1 - J_2)$$

If nearly all the costs come from the J^- term, we shall say the game is "almost zero-sum." We may then consider J^+ and J^- to be of the same order of magnitude, so that the constant parameter ϵ is "small."

We then seek an approximate method for finding the closed loop Nash solutions to the problem:*

*It is not actually necessary that the coefficients of ϵ in J_1 and J_2 be the same (we could have L_1^+ and L_2^+) but we forgo this slightly more general formulation in favor of less cumbersome notation.

Choose $u_1(x,t)$ to minimize

$$J_1 = \int_{t_o}^{t_f} [L^-(x,t,u_1,u_2) + \epsilon L^+(x,t,u_1,u_2)]dt + K^-(x_f) + \epsilon K^+(x_f) \tag{6.2}$$

and choose $u_2(x,t)$ to minimize

$$J_2 = \int_{t_o}^{t_f} [-L^-(x,t,u_1,u_2) + \epsilon L^+(x,t,u_1,u_2)]dt - K^-(x_f) + \epsilon K^+(x_f) \tag{6.3}$$

both subject to

$$\dot{x} = f(x,t,u_1,u_2) \quad , \qquad x(t_o) = x_o \tag{6.4}$$

Let $\Psi_1^*(x,t,\epsilon)$ and $\Psi_2^*(x,t,\epsilon)$ denote the exact Nash closed loop control fields, and let $V_1(x,t,\epsilon)$ and $V_2(x,t,\epsilon)$ denote the corresponding "Nash remaining cost" functions. Since the problem reduces to a zero-sum game when $\epsilon = 0$, it follows that

$$V_1(x,t,0) = -V_2(x,t,0) = V(x,t) \tag{6.5}$$

Suppose the nominal problem ($\epsilon = 0$) had been solved for all x, t. Then we could find an approximate solution for small ϵ by expanding the generalized Bellman equations

$$-V_{1t}(x,t,\epsilon) = \min_{u_1} [L^-(x,t,u_1,\Psi_2^*) + \epsilon L^+(x,t,u_1,\Psi_2^*) + V_{1x}(x,t,\epsilon) f(x,t,u_1,\Psi_2^*)]$$

$$-V_{2t}(x,t,\epsilon) = \min_{u_2} [-L^-(x,t,\Psi_1^*,u_2) + \epsilon L^+(x,t,\Psi_1^*,u_2) + V_{2x}(x,t,\epsilon) f(x,t,\Psi_1^*,u_2)]$$

$$V_1(x_f,t_f,\epsilon) = K^-(x_f) + \epsilon K^+(x_f)$$

$$V_2(x_f,t_f,\epsilon) = -K^-(x_f) + \epsilon K^+(x_f) \tag{6.6}$$

(where $\Psi_1^*(x, t, \epsilon)$ and $\Psi_2^*(x, t, \epsilon)$ are the controls u_1 and u_2 which achieve the required minima) in a series in ϵ around the nominal field, treating both x and t as independent variables. The practical difficulty with this conceptually clear approach is that the nominal solution would ordinarily be available only along a single nominal trajectory (the one through x_o, t_o) and not in the entire x, t-space.

To avoid having to fill the entire x, t-space with nominal trajectories, we proceed by expanding all functions around the nominal trajectory $\bar{x}(t)$ through the given initial point x_o, t_o, as follows:

$$x(t, \epsilon) = \bar{x}(t) + \epsilon\xi + \frac{1}{2}\epsilon^2\theta + \dots \tag{6.7}$$

$$\begin{aligned} V_1(x, t, \epsilon) = {} & V(\bar{x}(t), t) + \epsilon(V_{1\epsilon}(\bar{x}, t) + V_x\xi) \\ & + \epsilon^2(\tfrac{1}{2}V_{1\epsilon\epsilon} + \tfrac{1}{2}V_x\theta + V_{1\epsilon x}\xi + \tfrac{1}{2}\xi^T V_{xx}\xi) \\ & + \dots \end{aligned} \tag{6.8}$$

$$\begin{aligned} V_2(x, t, \epsilon) = {} & -V + \epsilon(V_{2\epsilon} - V_x\xi) + \epsilon^2(\tfrac{1}{2}V_{2\epsilon\epsilon} - \tfrac{1}{2}V_x\theta + V_{2\epsilon x}\xi \\ & - \tfrac{1}{2}\xi^T V_{xx}\xi) + \dots \end{aligned} \tag{6.9}$$

$$\left.\begin{aligned} u_1 &= \Psi_1(\bar{x}(t), t) + \epsilon(w_1 + \Psi_{1x}\xi) = \Psi_1 + \epsilon v_1 \\ u_2 &= \Psi_2(\bar{x}(t), t) + \epsilon(w_2 + \Psi_{2x}\xi) = \Psi_2 + \epsilon v_2 \end{aligned}\right\} \tag{6.10}$$

$$\begin{aligned} V_{1x}(x, t, \epsilon) = {} & V_x + \epsilon(V_{1\epsilon x} + V_{xx}\xi) + \epsilon^2(\tfrac{1}{2}V_{1\epsilon\epsilon x} + \tfrac{1}{2}V_{xx}\theta \\ & + V_{1\epsilon xx}\xi + \tfrac{1}{2}\xi^T V_{xxx}\xi) + \dots \end{aligned} \tag{6.11}$$

etc.

where all functions, unless otherwise specified, are evaluated on the nominal trajectory. The first order terms in the expansion of (6.6) are then

$$-V_{1\epsilon t} - V_{tx}\xi = \min_{v_1}[L_x^- + L_{u_1}^- v_1 + L_{u_2}^- v_2 + L^+ + V_x(f_x\xi + f_{u_1}v_1 + f_{u_2}v_2) + (V_{1\epsilon x} + \xi^T V_{xx})f]$$

$$-V_{2\epsilon t} + V_{tx}\xi = \min_{v_2}[-L_x^- - L_{u_1}^- v_1 - L_{u_2}^- v_2 + L^+ - V_x(f_x\xi + f_{u_1}v_1 + f_{u_2}v_2) + (V_{2\epsilon x} - \xi^T V_{xx})f]$$

(6.12)

Let us define, as usual, the Hamiltonian on the nominal trajectory:

$$H(x, t, u_1, u_2, V_x) = L^-(x, t, u_1, u_2) + V_x f(x, t, u_1, u_2) \qquad (6.13)$$

Then on this nominal path,

$$H_{u_1} = 0 = H_{u_2} \qquad (6.14)$$

$$-\frac{d}{dt}V_x = -V_{tx} - f^T V_{xx} = H_x = L_x^- + V_x f_x \qquad (6.15)$$

so (6.12) reduces to

$$-V_{1\epsilon t} - V_{1\epsilon x}f = L^+ , \qquad V_{1\epsilon}(\bar{x}_f, t_f) = K^+(\bar{x}_f)$$

$$-V_{2\epsilon t} - V_{2\epsilon x}f = L^+ , \qquad V_{2\epsilon}(\bar{x}_f, t_f) = K^+(\bar{x}_f) \qquad (6.16)$$

These equations have identical solutions. Defining $V_{1\epsilon} = V_{2\epsilon} \triangleq V_\epsilon$,

$$-\frac{d}{dt}V_\epsilon = L^+ , \qquad V_\epsilon(\bar{x}_f, t_f) = K^+(\bar{x}_f) \qquad (6.17)$$

where the total derivative is taken along the nominal path.

It is convenient at this point to obtain a differential equation for $V_{\epsilon x}(\bar{x}(t), t)$. It is easy to verify that

$$-\frac{d}{dt} V_{\epsilon x} = -V_{\epsilon tx} - V_{\epsilon xx} f = L_x^+ + L_{u_1}^+ \Psi_{1x} + L_{u_2}^+ \Psi_{2x}$$

$$+ V_{\epsilon x}(f_x + f_{u_1}\Psi_{1x} + f_{u_2}\Psi_{2x})$$

$$V_{\epsilon x}(t_f) = K_x^+(\bar{x}(t_f)) \tag{6.18}$$

where, from the nominal solution*,

$$\Psi_{1x} = -H_{u_1u_1}^{-1}(H_{u_1x} + f_{u_1}^T V_{xx})$$

$$\Psi_{2x} = -H_{u_2u_2}^{-1}(H_{u_2x} - f_{u_2}^T V_{xx}) \tag{6.19}$$

Since v_1 and v_2 do not appear in (6.16), the minimizations required in (6.6) must be performed on the second order terms in the expansion. Collecting these terms, using the fact that, from the nominal solution,

$$-\dot{V}_{xx} = -V_{xxt} - V_{xxx} f = H_{xx} + V_{xx} f_x + f_x^T V_{xx}$$

$$- (H_{xu_1} + V_{xx} f_{u_1}) H_{u_1u_1}^{-1} (H_{u_1x} + f_{u_1}^T V_{xx})$$

$$- (H_{xu_2} + V_{xx} f_{u_2}) H_{u_2u_2}^{-1} (H_{u_2x} + f_{u_2}^T V_{xx})$$

$$V_{xx}(t_f) = K_{xx}^-(\bar{x}(t_f)) \tag{6.20}$$

and performing the required minimization yields

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*For the nominal solutions to exist, we must have $H_{u_1u_1} > 0$ and $H_{u_2u_2} < 0$.

$$v_1 = -H^{-1}_{u_1u_1}[(f^T_{u_1} V_{xx} + H_{u_1x})\xi + L^{+T}_{u_1} + f^T_{u_1} V^T_{\epsilon x}]$$

$$v_2 = -H^{-1}_{u_2u_2}[(f^T_{u_2} V_{xx} + H_{u_2x})\xi - L^{+T}_{u_2} - f^T_{u_2} V^T_{\epsilon x}] \quad (6.21)$$

or, alternatively, using (6.19) and the definitions in (6.10),

$$w_1 = -H^{-1}_{u_1u_1}(L^{+T}_{u_1} + f^T_{u_1} V^T_{\epsilon x})$$

$$w_2 = H^{-1}_{u_2u_2}(L^{+T}_{u_2} + f^T_{u_2} V^T_{\epsilon x}) \quad (6.21')$$

Using (6.21) to substitute for v_1 and v_2 in the second order terms of the expansion of (6.6), and using (6.18) and (6.20) to eliminate many terms, the following equations are finally obtained for $V_{1\epsilon\epsilon}$ and $V_{2\epsilon\epsilon}$:

$$-\dot{V}_{1\epsilon\epsilon} = -2V_{\epsilon x} f_{u_1} H^{-1}_{u_1u_1} L^{+T}_{u_1} + 6V_{\epsilon x} f_{u_2} H^{-1}_{u_2u_2} L^{+T}_{u_2} - L^{+}_{u_1} H^{-1}_{u_1u_1} L^{+T}_{u_1}$$

$$+ 3L^{+}_{u_2} H^{-1}_{u_2u_2} L^{+T}_{u_2} - V_{\epsilon x}(f_{u_1} H^{-1}_{u_1u_1} f^T_{u_1} - 3f_{u_2} H^{-1}_{u_2u_2} f^T_{u_2})V^T_{\epsilon x}$$

$$-\dot{V}_{2\epsilon\epsilon} = -6V_{\epsilon x} f_{u_1} H^{-1}_{u_1u_1} L^{+T}_{u_1} + 2V_{\epsilon x} f_{u_2} H^{-1}_{u_2u_2} L^{+T}_{u_2} - 3L^{+}_{u_1} H^{-1}_{u_1u_1} L^{+T}_{u_1}$$

$$+ L^{+}_{u_2} H^{-1}_{u_2u_2} L^{+T}_{u_2} - V_{\epsilon x}(3f_{u_1} H^{-1}_{u_1u_1} f^T_{u_1} - f_{u_2} H^{-1}_{u_2u_2} f^T_{u_2})V^T_{\epsilon x}$$

$$V_{1\epsilon\epsilon}(t_f) = V_{2\epsilon\epsilon}(t_f) = 0 \quad (6.22)$$

Finally, an equation is needed for ξ. This is obtained by expanding the state equation (6.4) and collecting the linear terms in ϵ:

$$\dot{\xi} = (f_x + f_{u_1}\Psi_{1x} + f_{u_2}\Psi_{2x})\xi - f_{u_1} H^{-1}_{u_1u_1}(L^{+T}_{u_1} + f^T_{u_1} V^T_{\epsilon x})$$

$$+ f_{u_2} H^{-1}_{u_2u_2}(L^{+T}_{u_2} + f^T_{u_2} V^T_{\epsilon x})$$

$$\xi(t_o) = 0 \quad (6.23)$$

where (6.21) was used to eliminate v_1 and v_2. By integrating (6.17) and (6.18) backward from the terminal time t_f, V_ϵ and $V_{\epsilon x}$ are obtained along the nominal trajectory. The first order correction to the trajectory can then be computed by integrating (6.23) forward from t_o. By integrating (6.22) backwards from t_f, one can obtain $V_{1\epsilon\epsilon}$ and $V_{2\epsilon\epsilon}$ at the initial time t_o, and these can then be used in the expansions of the cost functions (6.7), (6.8) to get the second order corrections to the Nash costs at time t_o. Note that these second order corrections can only be obtained at t_o, since at other times $\theta(t)$, the second order correction to the trajectory which appears in (6.7), (6.8), is not zero and is not provided by our method. However, knowing the second order terms in the costs at t_o might be useful in determining the range of ϵ over which the approximation is acceptable.

The procedure just described, although messy in appearance, is computationally simple. No iteration is involved, once the nominal solution has been found. However, there is no way to improve the accuracy of the approximation except by going to higher order expansions.

6.3 The "almost identical goal" differential game

In an "identical goal" game, all the players try to minimize the same cost criterion. In this special case, the Nash solution (which is also the only noninferior solution) is just the optimal solution any single player would achieve if he could choose all the controls. If, in a general differential game, the cost criteria involve a parameter ϵ such that, if $\epsilon = 0$, the game reduces to an identical goal game, then the Nash closed loop solutions for sufficiently small values of ϵ may be obtained approximately by expanding the solution in ϵ around the "nominal"

($\epsilon = 0$) trajectory. One might qualitatively describe such a game as one in which conflict plays a relatively minor role, compared to common interest.

Since the idea behind this approximate solution is the same used in the previous section for the "almost zero-sum" game, only the problem statement, the form of the expansions, and the resulting approximate solutions will be presented.

The problem is for the i^{th} player ($i = 1, \ldots, N$) to choose u_i to minimize

$$J_i = \int_{t_o}^{t_f} [L^+(x, t, u_1, \ldots, u_N) + \epsilon L_i(x, t, u_1, \ldots, u_N)]dt + K^+(x_f) + \epsilon K_i(x_f) \tag{6.24}$$

subject to

$$\dot{x} = f(x, t, u_1, \ldots, u_N) \quad , \qquad x(t_o) = x_o \tag{6.25}$$

For $i = 1, \ldots, N$, let $\Psi_i(x, t, \epsilon)$ be the field of Nash closed loop equilibrium controls, and let $V_i(x, t, \epsilon)$ be the corresponding (exact) value functions. Let $\bar{x}(t)$ be the exact nominal trajectory ($\epsilon = 0$) through the initial point x_o, t_o. Let the trajectory, controls, and cost functions be expanded in ϵ along the nominal trajectory as follows:

$$x(t, \epsilon) = \bar{x}(t) + \epsilon\xi + \frac{1}{2}\epsilon^2\theta + \ldots \tag{6.26}$$

$$V_i(x, t, \epsilon) = V(\bar{x}, t) + \epsilon(V_{i\epsilon}(\bar{x}, t) + V_x\xi) + \epsilon^2(\frac{1}{2}V_{i\epsilon\epsilon} + \frac{1}{2}V_x\theta + V_{i\epsilon x}\xi + \frac{1}{2}\xi^T V_{xx}\xi) + \ldots \tag{6.27}$$

$$u_i = \Psi_i(\bar{x}, t) + \epsilon(w_i + \Psi_{ix}\xi) = \Psi_i + \epsilon v_i \tag{6.28}$$

By expanding the value function equation as in the previous section, the following exact equations are obtained:*

$$v_i = -H^{-1}_{u_iu_i}[(f^T_{u_i}V_{xx} + H_{u_ix})\xi + L^T_{iu_i} + f_{u_i}V^T_{i\epsilon x}] \tag{6.29}$$

$$-\mathring{V}_{i\epsilon} = L_i \ , \qquad V_{i\epsilon}(t_f) = K_i(\bar{x}_f) \tag{6.30}$$

$$-\mathring{V}_{i\epsilon x} = L_{ix} + V_{i\epsilon x}(f_x + \sum_{j=1}^{N} f_{u_j}\Psi_{jx}) + \sum_{j=1}^{N} L_{iu_j}\Psi_{jx}$$

$$V_{i\epsilon x}(t_f) = K_{ix}(\bar{x}_f) \tag{6.31}$$

$$\mathring{\xi} = (f_x + \sum_{j=1}^{N} f_{u_j}\Psi_{jx})\xi - \sum_{j=1}^{N} f^T_{u_j}H^{-1}_{u_ju_j}(L^T_{ju_j} + f^T_{u_j}V^T_{j\epsilon x})$$

$$\xi(t_o) = 0 \tag{6.32}$$

where, from the nominal solution,

$$\Psi_{ix} = -H^{-1}_{u_iu_i}(H_{u_ix} + f^T_{u_i}V_{xx}) \tag{6.33}$$

and where all functions, including the various partial derivatives of the nominal Hamiltonian H, are evaluated on the nominal path. By integrating

$$-\mathring{V}_{i\epsilon\epsilon} = \sum_{j=1}^{N} [(L_{ju_j} + V_{j\epsilon x}f_{u_j}) - 2(L_{iu_j} + V_{i\epsilon x}f_{u_j})]H^{-1}_{u_ju_j}[L^T_{ju_j} + f^T_{u_j}V^T_{j\epsilon x}]$$

$$V_{i\epsilon\epsilon}(t_f) = 0 \tag{6.34}$$

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*These quantities may be interpreted as the sensitivities of the Nash controls, costs, cost gradients, and trajectory to the parameter ϵ.

and using the result in the expansion (6.27), the second order terms in the cost functions can be found at the initial time. (They cannot be found for $t > t_o$ because θ, the second order term in the expansion (6.26) of the trajectory, appears in (6.27). Nevertheless, it may be worthwhile to evaluate $V_{i\epsilon\epsilon}(t_o)$ via (6.34) in order to get an indication of the range of ϵ over which the "almost identical goal" approximation is valid.

6.4 Competitive interaction among weakly coupled systems

In this section we consider a set of N systems operated by separate controllers with different, unrelated objectives. If the systems operate completely independently of each other, the optimal controls for the set of systems can be found by solving N separate, unrelated optimal control problems. But suppose the systems interact (due to coupling between the N differential equations describing them). Then each controller must consider the actions of the other (N - 1) controllers, and the situation has the structure of a differential game.

We shall say that the N systems are weakly coupled if there is some scalar parameter ϵ such that, when the state equations are written in the form (for i = 1, . . . , N)

$$\dot{x}_i = f_i(x_i, t, u_i) + \epsilon g_i(x_1, \ldots, x_N, t, u_1, \ldots, u_N)$$

$$x_i(t_o) = x_{io} \qquad (6.35)$$

then ϵ is small enough so that it is reasonable to expand all solutions in Taylor series in ϵ. Here x_i is an n_i-dimensional vector which in the absence of coupling would be the state vector for the i^{th} system.

The i^{th} player wishes to choose u_i to minimize

$$J_i = \int_{t_o}^{t_f} L_i(x_i, t, u_i)dt + K_i(x_i(t_f)) \tag{6.36}$$

Lacking a means for reaching "cooperative" solutions, it is reasonable to assume that each player wants his strategy to be optimal against whatever strategies the other players are using. If each player also realizes that all the other players are thinking this way, the result will be the Nash equilibrium solution. Assuming each player has continuous knowledge of x_j for all j, we seek a first order (in ϵ) approximation to the closed loop Nash solution, following the same procedure used in the previous two sections.

When $\epsilon = 0$, the problem splits into N separate control problems which can presumably be solved by DDP or some other appropriate algorithm. The solution to these problems starting from x_o, t_o (the nominal solution) will be assumed known. Denoting the exact Nash controls and corresponding cost function for the i^{th} player by $\Psi_i(x_1, \ldots, x_N, t, \epsilon)$ and $V_i(x_1, \ldots, x_N, t, \epsilon)$ respectively, we use the following expansions around the nominal path:

$$x_i(t, \epsilon) = \bar{x}_i(t) + \epsilon\xi_i + \frac{1}{2}\epsilon^2\theta_i + \ldots \tag{6.37}$$

$$V_i(x_1, \ldots, x_N, t, \epsilon) = V_i(\bar{x}, t) + \epsilon(V_{i\epsilon} + V_{ix_i}\xi_i) + \frac{1}{2}\epsilon^2(V_{i\epsilon\epsilon} + \xi_i^T V_{ix_ix_i}\xi_i + V_{ix_i}\theta_i + 2\sum_{j=1}^{N} V_{i\epsilon x_j}\xi_j) + \ldots \tag{6.38}$$

$$\Psi_i(x_1, \ldots, x_N, t, \epsilon) = \Psi_i(\bar{x}_i, t) + \epsilon v_i + \ldots \tag{6.39}$$

By expanding the value function equations and following the procedure used in Section 6.2, the following exact equations can be obtained: for i = 1, ..., N

$$v_i = -H^{-1}_{iu_iu_i}[(H_{iu_ix_i} + f^T_{iu_i}V_{ix_ix_i})\xi_i + g^T_{iu_i}V^T_{ix_i} + f^T_{iu_i}V^T_{i\epsilon x_i}] \tag{6.40}$$

$$-\dot{V}_{i\epsilon} = V_{ix_i}g_i \quad , \qquad V_{i\epsilon}(t_f) = 0 \tag{6.41}$$

$$-\dot{V}_{i\epsilon x_j} = V_{i\epsilon x_j}(f_{jx_j} + f_{ju_j}\Psi_{jx_j}) + g_i^T V_{ix_ix_j} + V_{ix_i}(g_{ix_j} + g_{iu_j}\Psi_{jx_j})$$

$$V_{i\epsilon x_j}(t_f) = 0 \tag{6.42}$$

$$\dot{\xi}_i = (f_{ix_i} + f_{iu_i}\Psi_{ix_i})\xi_i - f_{iu_i}H^{-1}_{iu_iu_i}(g^T_{iu_i}V^T_{ix_i} + f^T_{iu_i}V^T_{i\epsilon x_i}) + g_i$$

$$\xi_i(t_o) = 0 \tag{6.43}$$

where, from the nominal solutions,

$$\Psi_{ix_i} = -H^{-1}_{iu_iu_i}(H_{iu_ix_i} + f^T_{iu_i}V_{ix_ix_i}) \tag{6.44}$$

All functions are, of course, evaluated on the nominal path. By integrating

$$-\dot{V}_{i\epsilon\epsilon} = (V_{i\epsilon x_i}f_{iu_i} + V_{ix_i}g_{iu_i})H^{-1}_{iu_iu_i}(f^T_{iu_i}V^T_{i\epsilon x_i} + g^T_{iu_i}V^T_{ix_i})$$

$$+ 2\sum_{j=1}^{N} V_{i\epsilon x_j}g_j - 2\sum_{j=1}^{N}(V_{ix_i}g_{iu_j} + V_{i\epsilon x_j}f_{ju_j})H^{-1}_{ju_ju_j}$$

$$\cdot (f^T_{ju_j}V^T_{j\epsilon x_j} + g^T_{ju_j}V^T_{jx_j})$$

$$V_{i\epsilon\epsilon}(t_f) = 0 \tag{6.45}$$

the exact second order term in the expansion (6.38) of the costs can be obtained at the initial time (not at later times because θ_i is unknown).

Thus, once the set of N unrelated nominal problems has been solved, by some control algorithm such as DDP, then the first order corrections to the closed loop Nash solution when $\epsilon \neq 0$ can be found by integrating the linear differential equation (6.42) backward from the terminal time, then integrating the linear "state sensitivity equation" (6.43) forward from the initial time.

It may be objected that, although we have found an approximation for the closed loop Nash solution, the results are not given in closed loop form. However, for small perturbations from the first order approximation to the Nash trajectory, corrections can be made by using the nominal linear feedback law (6.44). A more accurate feedback law could be obtained by further expansions, but this seems rather pointless since our solution is in any case only an approximation (with error proportional to ϵ^2).

6.5 The "almost ordered" differential game

In the "ordered" differential game, the players can be numbered in such a way that the i^{th} player need not be concerned with the actions of the last (N - i) players in the order, but only with the actions of the first (i - 1) players. This is not really a game, but only an ordered sequence of optimal control problems. The controls of the first (i - 1) players can be considered known inputs to the i^{th} player's system, and the i^{th} control obtained by solving a usual optimal control problem independent of the actions of players i + 1, . . . , N.

Even more generally, there might be some partial ordering of the players forming a heirarchy (see Fig. 6.1). Each player need only be concerned with the actions of the players above him in the heirarchy.

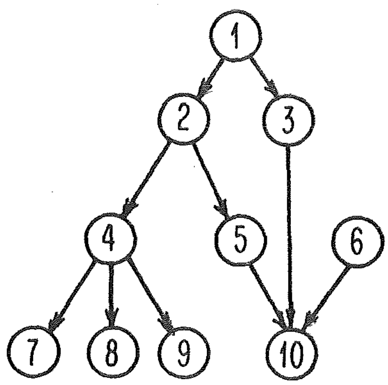

Fig. 6.1. A partially ordered 10-player game.

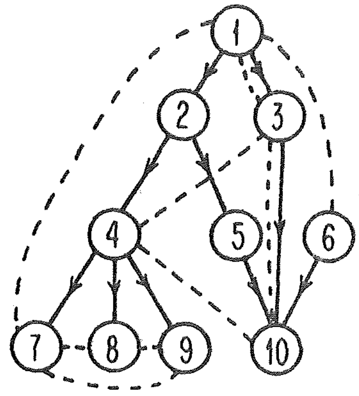

Fig. 6.2. An "almost partially ordered" 10-player ga[me]. Dashed lines indicate weak coupling.

For example, in Fig. 6.1, Player 5 need only consider the actions of Players 1 and 2. Again such a game can be solved by solving a heirarchy of optimal control problems.

In this section we wish to consider differential games where there exists strong unidirectional coupling as in Fig. 6.1, plus "weak" two-way coupling, so that all players must take account of the actions of all the other players. Such a situation is indicated in Fig. 6.2. The assumption that the "upward" coupling is weak makes it possible to obtain approximate Nash closed loop solutions by using the same approach used in the previous three sections. In fact, the weakly coupled game of Section 6.4 is a special case of the game considered in this section.

Because the approximation method proposed here is based on exactly the same idea used in the previous sections, only the statement of the problem will be given. It is straightforward (but tedious) to derive equations for $V_{i\epsilon}$, $V_{i\epsilon x_j}$, and $V_{i\epsilon\epsilon}$, and thus to obtain the first order terms in the power series expansion in ϵ of the closed loop Nash controls and costs.

To avoid excessively cumbersome notation, we state the problem only for a total ordering of the players (rather than the more general partial ordering illustrated in Fig. 6.2). The extension to the more general case is straightforward.

The problem is then for the i^{th} player ($i = 1, \ldots, N$) to choose a closed loop control function $u_i(x, t)$ to minimize

$$J_i = \int_{t_o}^{t_f} [L_i(x_1, \ldots, x_i, t, u_1, \ldots, u_i) + \epsilon M_i(x_1, \ldots, x_N, t, u_1, \ldots, u_N)]dt$$

$$+ K_i(x_{1f}, \ldots, x_{if}) + \epsilon P_i(x_{1f}, \ldots, x_{Nf}) \qquad (6.46)$$

where

$$\dot{x}_i = f_i(x_1, \ldots, x_i, t, u_1, \ldots, u_i) + \epsilon g(x_1, \ldots, x_N, t, u_1, \ldots, u_N)$$

$$x_i(t_o) = x_{io} \qquad (6.47)$$

When $\epsilon = 0$, the resulting "nominal" game can be solved as a sequence of optimal control problems, all of which are assumed to be solvable by some known algorithm such as DDP. An approximate closed loop Nash trajectory is then obtained by expanding in ϵ around this nominal path, using the same procedure followed in Section 6.2.

6.6 The "almost linear-quadratic" differential game

As the last in our collection of special differential games for which approximate closed loop Nash solutions can be obtained easily, we consider the case where the game is almost linear-quadratic.

The problem is for the i^{th} player ($i = 1, \ldots, N$) to choose control function $u_i(x, t)$ to minimize

$$J_i = \int_{t_o}^{t_f} [\frac{1}{2}x^T Q_i x + \sum_{j=1}^{N} \frac{1}{2}u_j^T R_{ij} u_j + \epsilon L_i(x, t, u_1, \ldots, u_N)]dt$$

$$+ \frac{1}{2}x_f^T S_{if} x_f + \epsilon K_i(x_f) \tag{6.48}$$

where

$$\dot{x} = Ax + \sum_{j=1}^{N} B_j u_j + \epsilon f(x, t, u_1, \ldots, u_N) \quad , \qquad x(t_o) = x_o \tag{6.49}$$

When $\epsilon = 0$, this problem reduces to the special form of the linear-quadratic differential game considered in Chapter IV. The most general linear-quadratic differential game, for which Nash solutions are presented in Appendix A, could also be extended to the "almost linear-quadratic" model.

If we follow the procedure used in the previous sections, we can obtain the first order terms in ϵ of the power series expansions of the closed loop Nash controls and costs. We would then find that these first order corrections depended on L_i and f only through f, L_i, f_x, f_{u_i}, L_{ix} and L_{iu_j} evaluated on the nominal path. It would be a simple matter to compute the first order approximate solutions this way. However, there would be an error (of order ϵ^2) even if L_i were a quadratic function and f were a linear function. This is somewhat disturbing, since we know that we could obtain the exact solution to this problem as easily as we can obtain the nominal solution!

If we only want a solution for a particular value of ϵ, we can obtain a more accurate approximation by the following iterative procedure:

(i) Compute the nominal solution with $\epsilon = 0$.

(ii) Expand L_i and f in power series in x and $u_1, \ldots, u_N$ around the nominal path.

(iii) Redefine the linear and quadratic terms in the nominal problem to include the corresponding terms in the expansions of L_i and f. (For example, replace Q_i by $Q_i + \epsilon L_{ixx}$, with L_{ixx} evaluated on the nominal path, and subtract $\frac{1}{2}x^T L_{ixx} x$ from L_i.)

(iv) With these new parameters, again set $\epsilon = 0$ and compute a new nominal path. Return to step (ii).

(v) When this process has converged, the problem will have been recast in a form such that, along the nominal path,

$$0 = L_{ixx} = L_{iu_ju_k} = L_{ixu_j} = L_{iu_j} = L_{ix} = f_x = f_{u_i} = K_{ixx} = K_{ix}$$

$$= K_i = L_i = f \tag{6.50}$$

If we then expand the Bellman equation for this recast version of the problem in a Taylor series in ϵ (pretending that Q_i, R_{ij}, etc. are not dependent on ϵ), we will see that the second order terms are already minimized by the nominal controls. Thus we have an approximate set of Nash controls, with error of order ϵ^2. (Higher accuracy could be obtained by expanding the Bellman equation to higher order in ϵ, although this is somewhat cumbersome.) In any case, the iterative procedure described above (which resembles the extended DDP algorithm described in Section 5.3) produces a correction to the control with error proportional to ϵ^2 and produces the exact Nash solution if the original problem (including the ϵ-terms) is linear-quadratic.

CHAPTER VII

CONSTRAINED LINEAR DIFFERENTIAL GAMES

7.1 Introduction

Very little has been said in the previous chapters about the possibility of having inequality constraints on the control or state variables. Our approach has been to try to extend known results of optimal control theory to the more general nonzero-sum differential game. Inequality constraints, especially those involving only the state variables, greatly complicate the analysis of optimal control problems. The known algorithms for handling inequality constraints on the state variables are rather unsatisfactory, especially if several such constraints are present. Thus one is naturally reluctant to tackle state variable inequality constraints in the more difficult differential game.

However, in realistic situations which have differential game structures (most of which arise in economic contexts) inequality constraints abound. The analysis of economic competition by differential game models will certainly require methods for handling at least a few inequality constraints. In this chapter we shall discuss what is probably the simplest constrained differential game of economic interest. The constraints, system equations, and cost criteria are all linear functions of the state and control variables.

The discussion of the constrained linear differential game (CLDG) in this chapter is preliminary in nature. No important results have been obtained for this problem. In fact, the model has apparently never been formulated before. The special linear structure of the CLDG and its resemblence to linear programming seem to offer the hope that practical computational algorithms may eventually be developed (if not for the continuous CLDG, then perhaps for some multistage version). The availability of an algorithm for solving CLDG with several constraints (at least four) would make the CLDG a potentially powerful tool analysing imperfect economic competition. It could in fact become the most useful of all differential game models.

In the following section, the CLDG is stated in a form which is an extension of the constrained linear optimal control problem. Some restrictions on the form of the constraints are necessary in order that the problem make sense as a game. These are discussed, and a general economic interpretation of the model is given.

In Section 7.3, the CLDG is presented in a more general form which is an extension of the "continuous linear programming" problem[21] and the model is interpreted economically in terms of a set of coupled "bottleneck" problems.

Some of the difficulties one would encounter in attempting to compute Nash solutions are discussed briefly in Section 7.4.

7.2 Extension of the linear optimal control problem

In the general version of the constrained linear differential game obtained as an extension of the corresponding optimal control problem, the i^{th} player wishes to choose u_i to maximize

$$J_i = \int_{t_o}^{t_f} (q_i^T x + \sum_{j=1}^{N} r_{ij}^T u_j) dt + k_i^T x_f \qquad (7.1)$$

subject to

$$\dot{x} = Ax + \sum_j B_j u_j + w \;, \qquad x(t_o) = x_o \qquad (7.2)$$

$$G_i u_i \leqslant D_i x + d_i \qquad (7.3)$$

$$u_i \geqslant 0 \qquad (7.4)$$

where q_i, r_{ij}, A, B_j, w, G_i, D_i, and d_i are time-dependent vectors and matrices, known by all the players.

It is important to realize that Player i is restricted in his choice of controls only by (7.3) and (7.4), not by the corresponding equations for the j^{th} player. In other words, the i^{th} player is not responsible for assuring that the j^{th} player will always have a feasible control.*

Various assumptions may be made about the state vector information available to the players during the course of play. However, it is essential that the model be constructed in such a way that at each instant:

(i) There is always a feasible control for each player, regardless of the past or present actions of all the other players.

(ii) Each player knows how to choose a feasible control.

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*In a strictly mathematical sense, certain types of "solutions" may exist even when it is possible for a player to "become infeasible." An example would be an open loop Nash solution where each player was required to choose his controls subject to (7.3), (7.4) for all i. But this is not the model we have defined.

We shall assume that the parameters are such that (i) is satisfied. If this were not the case, it would be possible, by some sequence of feasible controls, to reach a state where one player has no feasible control. The game could then not be continued, and we have defined no payoff to the N players associated with this result. A game is meaningless unless payoffs are defined for every feasible outcome. Hence one should reject any game model where (i) is not satisfied.

If condition (ii) does not hold, then some subset of the players could consider the following strategy: Work Player i into a position such that (even though, by (i), he has a feasible control) he does not know which controls are feasible. The game cannot proceed beyond this point if Player i makes a "wrong guess," and no payoffs are assigned for this result. Thus, open loop controls usually do not make sense when there are inequality constraints, and the least one can assume is that each player knows the right side of the constraints (7.3) associated with his own control. (Otherwise he might choose an infeasible control.) Alternatively, one might assume that each player has instantaneous knowledge of the entire state vector x, so that he can compute the constraints faced by all the players. His controls would then be closed loop.

Note also that the formulation does not allow a single constraint to involve two players' controls, for then cooperation would be required to see that the constraints are satisfied.

Many situations of economic competition can be represented by the differential game (7.1)-(7.4).* The following interpretation illustrates the idea. The N players might represent a collection of large corporations. The control vector u_i represents the set of levels at which the i^{th} firm carries out its activities. For example, the components of u_i could be the rates of production, labor training, capital investment, and dividend payment. The state vector x represents the various scarce resources needed to carry out the activities of the firm, e.g., machines, trained labor, market demand, etc. Some of these resources may be associated with individual firms, while others (for example, a lake for water supply and sewage disposal) may be shared by several firms. The term $q_i^T x$ in the payoff could represent the desire of the management to preside over a large organization (said to be the real objective of most modern managers) while the terms in J_i involving the controls could represent dividends paid to shareholders (the more traditional goal of profit maximization). The terms in J_i involving the rivals' controls could represent the nuisance value of certain activities of the other firms (most models would probably omit these terms). The terminal term $k_i^T x_f$ represents the estimated value of the final state, taking account of possible operations beyond the time horizon t_f (often the hardest part of the model to formulate).

The n x n matrix A represents the rate at which the various resources diminish or grow independent of the decisions of the managers (e.g., depreciation, retirement, growing affluence of consumers, etc.).

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*The reader who is familiar with linear economic models can skip the rest of this section.

An inhomogeneous term w covers external inputs to the resource supplies and also allows the lower bound on the activity levels to be 0 in (7.4) without loss of generality. The rates at which resources are increased or depleted by the various activities are given by the matrices B_j.

Eq. (7.3) says that various individual activities or linear combinations of activities of a single firm are limited by the available resources. For example, one component of u_i might be the rate of production. It would then be limited by the production function, assumed in this model to be a linear function of equipment, labor, and other scarce goods. A row of D_i which is all zeroes (with the corresponding component of d_i positive, represents an "institutional" constraint, independent of the resource levels (i.e., a fixed upper bound on the corresponding component of u_i).

Generally the model would be formulated so that the components of x are always nonnegative. The various matrices and vectors in the problem may be quite sparse, as long as there is enough coupling so that each firm must be concerned with decisions of all its rivals.

7.3 Extension of the continuous linear programming problem

A somewhat more general form of the CLDG may be obtained as an extension of the continuous linear programming problem[21] (CLP). In this version of the game, the i^{th} player wishes to choose u_i to

maximize

$$J_i = \int_{t_o}^{t_f} \sum_{j=1}^{N} a_{ij}^T(t)u_j(t)dt \tag{7.5}$$

subject to the constraints

$$B_i(t)u_i(t) \leq c_i(t) + \int_{t_o}^{t} \sum_{j=1}^{N} K_{ij}(t, s)u_j(s)ds \tag{7.6}$$

$$u_i(t) \geq 0 \tag{7.7}$$

where the functions a_{ij}, B_i, c_i, and K_{ij} are all bounded and measureable.

By using the transition matrices associated with the linear system, the version of the game in the previous section can always be put in the form (7.5)-(7.7). (This is not always easy computationally.) But the converse is not true, since not every $K(t, s)$ is the transition matrix of a finite-dimensional linear system. For example, the case where

$$K_{ij}(t, s) = \begin{cases} 0 & \text{if } t - s < \alpha \\ 1 & \text{if } t - s \geq \alpha \end{cases}$$

cannot be represented by a model of the form (7.1)-(7.4).

The economic interpretation of (7.5)-(7.7) is similar to that given in the previous section for (7.1)-(7.4). Note that at time t the i^{th} player does not merely choose the u_i satisfying (7.6) which gives him the largest value of $a_{ii}^T u_i$. He must also consider choosing u_i to increase the right side of (7.6), enabling him to choose larger controls at later times. In addition, his choice of u_i will affect the constraints of the j^{th} player. Since the values of u_j chosen by the j^{th} player at later times will affect both the costs and the constraints of Player i, the latter must also consider the future effect of his choice of $u_i(t)$ on the actions of his rivals. Moreover, he knows that the other players are thinking the same way.

For the case of a single player ($N = 1$), (7.5)-(7.7) is sometimes called a "bottleneck" problem; the name is suggested by the form of (7.6). For $N > 1$, the game might be called a set of coupled bottleneck problems.

7.4 Solutions

As in all nonzero-sum differential games, there are a variety of "solutions" to the CLDG which may be of interest. The set of noninferior solutions can be found by solving a ($N - 1$)-parameter family of continuous linear programming problems. Maximin solutions for each player can be found by solving a set of N zero-sum games, each about as difficult computationally as a CLP.

In most CLDG models, the solution of greatest interest will be the Nash solution. We have seen that the Nash solutions are different for different assumptions about the information available to the players. The most tractable model is obtained by assuming that all players know the values of the right side of (7.6) for all i (i.e., the controls are closed loop). Naturally, one would hope that a practical algorithm for computing closed loop Nash solutions could be obtained by extending a known algorithm for the linear optimal control problem or for the CLP. Unfortunately, although many theoretical results for the CLP have been given by Grinold[21] involving existence of solutions to primal and dual problems, a really satisfactory algorithm for solving CLP is still lacking. Correspondingly, optimal control theory has not produced efficient methods for solving constrained linear optimal control problems, especially when there are more than a few constraints.

One approach to the solution of CLP is to discretize time and solve a linear programming problem (LP) of the form

$$\underset{z}{\text{Maximize}}\ a^T z \qquad \text{subject to} \quad Bz \leqslant c \ , \qquad z \geqslant 0 \tag{7.8}$$

where z is the large dimensional vector formed by adjoining the control vectors at each of the discrete times. However, some difficulties arise when we attempt to extend this idea to the nonzero-sum CLDG. In (7.8) the controls are approximated by a piecewise constant function of time only. But our closed loop assumption requires that the controls be considered as functions of the state (i.e., the values of the constraints) as well as time. To represent such a control function as a static vector, we would have to discretize the levels of the constraints. Besides introducing further inaccuracies, this would enormously increase the number of constraints in (7.8). This problem does not arise in the single player case because then the closed loop and open loop assumptions lead to the same optimal trajectories.

CHAPTER VIII

EXAMPLES OF DIFFERENTIAL GAME MODELS

8.1 Introduction

Almost all the published work on differential games so far has dealt exclusively with two-person, zero-sum differential games. The examples have mostly been "pursuit-evasion" situations, motivated by such military applications as anti-missile defense, submarine warfare, deployment of ground forces, aerial combat, etc. Those who have constructed such models have, in effect, ruled out the possibility of mutual interest between the conflicting parties. The extension of the theory to the nonzero-sum case makes it possible to consider dynamic situations where both mutual interest and competition are important; a much broader and more interesting range of problems can then be plausibly formulated as differential game models. It would appear that most of the new applications are in economics, although it is clear that even in military situations there is always some mutual interest.

The purpose of this chapter is to illustrate a variety of possible applications of nonzero-sum differential games to the analysis of imperfectly competitive situations. Each of the remaining sections considers a different area of application and presents a model which, although simplified, hopefully at least approximates the real situation well enough to be interesting. No attempt is made to "solve" these examples in any of the senses discussed in Chapter III. Even the one-player versions of these models make difficult optimal control problems.

From the discussions in Chapters V, VI, and VII of the additional computational difficulties which arise when optimal control problems are extended to differential games, it should be clear that the analysis of the games presented below must await the development of better computational methods.

In addition to the computational difficulties involved in finding "solutions" to realistic differential game models, it may be very hard even to formulate the model in a plausible way, if the system is not "physical." A differential game model requires an underlying dynamic system, describable by a finite-dimensional vector differential equation. Generally not enough is known about economic, social, or psychological "systems" to permit one to have much confidence in such a description. For example, what set of variables would serve as an adequate "state vector" for representing the attitudes of consumers about a set of products? Moreover, even if one were reasonably sure of the form of the dynamic model, it might be an unreasonable task to determine empirically the many parameters involved in the model. Of course, these same difficulties arise in the application of optimal control theory (or even mathematical programming) to "nonphysical" problems.

8.2 Dividend policies of imperfectly competitive firms

Consider the dividend policies for N firms, each manufacturing a single product. The products are substitutable but not identical. This means that an increase in the price of the i^{th} product results in lower (but not zero) sales of the i^{th} product and increased sales for all other products. In this model, the amount produced by each firm is a function only of the firm's capital, and everything produced is sold at whatever price the market will offer. These "market clearing prices" are in turn determined by the amounts of all N products currently offered for sale. A firm can generate new capital only from its own profits (no borrowing allowed). Given the appropriate production, demand, and production cost functions, one can obtain a vector function giving the net profit flow for each firm as a function of the vector of capital levels of all the firms.

The task of the management of the i^{th} firm is to choose the (continuous) dividend rate u_i to maximize the "shareholder's utility function"

$$J_i = \int_{t_o}^{t_f} u_i e^{-\alpha(t-t_o)} dt + x_i(t_f) e^{-\alpha(t_f-t_o)} \tag{8.1}$$

subject to

$$\dot{x}_i = f_i(x_1, \ldots, x_N) - u_i \tag{8.2}$$

$$u_i \geqslant 0 \;, \qquad x_i \geqslant \bar{x}_i \tag{8.3}$$

where

x_i = capital level of i^{th} firm

f_i = net profit function

$\quad = F_i(x_i) P_i(F_1(x_1), \ldots, F_N(x_N)) - C_i(x_i)$

F_i = production function

P_i = market-clearing price function

C_i = production cost function

α = interest rate

This is clearly a nonlinear nonzero-sum differential game. Even with very simple F_i and P_i, the inequality constraints make it difficult to analyze. The form of the terminal cost function implies that the entire capital assets could be instantaneously liquidated (i.e., paid out as dividends) at the end of the planning period.

8.3 Competition among firms through advertising

In some industries, the firms do not compete through their prices, nor through the amount of their products they offer in the market. Instead, the prices are fixed by tradition, and any firm can easily supply any amount of its product to the market. The firms compete entirely through promotional campaigns, and the cost of this promotion is the only important cost to be considered by the decision-makers. All the other costs in the model to be considered here will be included in a fixed overhead cost (which can be ignored since it cannot vary) plus a cost which is a linear function of the amount produced. The (constant) marginal cost is assumed less than the (fixed) market price, so that it always pays to produce as much as the market demands. Many consumer industries resemble this model; typical examples are cigarettes and cosmetics.*

The decision variable for the i^{th} firm is the rate at which it spends money on advertising (it is assumed that each firm knows the optimal way to spend any given sum of money on advertising). It is important to recognize at this point that the "dynamic system" involved here is not the set of firms, but the market itself. The "state variables"

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*Krishnan and Gupta[23] consider a static model of duopoly where the control variables are price and promotional effort.

in the model must be some collection of quantities which represent the attitudes of the public toward the various "brands." These attitudes are, or course, influenced by advertising. We would expect that these aggregate measures of consumer attitude would change slowly and smoothly in response to advertising "inputs," so that the dynamic behavior of the market demand could be approximated by some finite-state model. We shall make the simple assumption that the demands themselves adequately represent the state of market. (A formidable amount of research in market psychology would be required to formulate a more accurate state-variable model.) The model is then formulated as follows:

The manager of the i^{th} firm ($i = 1, \ldots, N$) wants to choose his rate of advertising u_i to maximize profits, discounted to the present:

$$J_i = \int_{t_o}^{t_f} (c_i x_i - u_i) e^{\alpha(t_o - t)} dt + \text{possible terminal value} \tag{8.4}$$

subject to

$$\dot{x} = f(x, u_1, \ldots, u_N, t) \tag{8.5}$$

$$u_i \geq 0 \tag{8.6}$$

where

x_i = rate of demand (in $ per day) of i^{th} firm's product

= gross revenue of i^{th} firm, per day

c_i = fraction of revenue left after marginal costs ($0 < c_i \leq 1$)

α = interest rate

f_i = rate at which rate of demand for i^{th} product changes where f would have the following general properties:

(i) $f_{iu_i} > 0$ (positive marginal return on advertising)

(ii) $f_{iu_iu_i} < 0$ (saturation effect)

(iii) For any set of constant positive $u_1, \ldots, u_N$, and any initial x, (8.5) is stable.

Various other assumptions about f depend on the nature of the market. For example, if the market were highly competitive in the sense that one firm increases its sales mainly at the expense of the other firms, then we might assume a form of f such that $f_{iu_j} < 0$ for $j \neq i$.

8.4 Control of inventory through pricing

The previous two sections presented simple models for oligopolistic competition among firms producing related products. Each firm was large enough to influence the behavior of the market. In the model in Section 8.2, the prices offered by the market were assumed known as a function of the quantities of the products of all the firms offered for sale. An instantaneous change in the amount of goods offered would cause an immediate change in the "market-clearing" prices. The decision variables were not the amounts of goods offered but the rate at which production capacity was increased. The "lag" or "inertia" in the model came from the process of increasing production capacity by investment of profits.

In the model in Section 8.3, it was assumed that the rate of production could be changed instantaneously to fulfill exactly a varying demand. Prices in this model were fixed by tradition. The amount of each product demanded by the market was assumed to depend on the advertising done by all firms at all times up to the present. The "inertia" in this model came not from the production process but from the gradual changes in consumer attitudes due to continual exposure to advertising.

It should be clear by now that the essential feature of a differential game model is the "dynamic system"--the process which provides the "inertia" needed to link past, present and future decisions.

In this section, oligopoly is modeled by still another type of dynamic system. In this model, the production for each firm is given exogenously as a function of time. The demand for each product is a known function of the prices of the goods offered by all N firms. Each firm may instantaneously control its price, but the excess production which is not being sold must be stored, at a cost dependent on the amount stored. In addition to the storage costs, some of the inventories are lost through spoilage or depreciation. The state variables are then the inventories of each firm. A firm with no goods in stock can, of course, sell nothing. Each firm wishes to maximize profits. If the operation is "seasonal," the inventories left at the terminal time would be worthless. Otherwise, some terminal value might be assigned to inventories at the end of the planning period. Each firm tries to control its inventory by judicious choice of prices.

The operator of the i^{th} firm chooses his price u_i to maximize

$$J_i = \int_{t_o}^{t_f} [u_i r_i(x_i, u_1, \ldots, u_N, t) - c_i(x_i, t)]dt + \text{terminal value} \tag{8.7}$$

subject to

$$\dot{x}_i = F_i(t) - r_i(x_i, u_1, \ldots, u_N, t) - s_i(x_i, t) \tag{8.8}$$

$$u_i \geqslant 0 \tag{8.9}$$

$$r_i(x_i, u_1, \ldots, u_N, t) = \begin{cases} 0 & \text{if } x = 0 \\ D_i(u_1, \ldots, u_N, t) & \text{if } x > 0 \end{cases} \tag{8.10}$$

where

x_i = inventory of i^{th} firm

$F_i(t)$ = production rate (exogenous) ($F_i \geqslant 0$)

r_i = rate at which i^{th} product is sold

D_i = market rate of demand for i^{th} product

s_i = depreciation or spoilage rate for i^{th} firm's inventory ($s_i \geqslant 0$ and $s_i(0,t) = 0$)

c_i = storage cost rate for i^{th} firm ($c_i \geqslant 0$)

Many other "coupled" inventory problems could be modeled as nonzero-sum differential games. For example, prices might be fixed and inventory might be controlled through production, investment, or advertising. More complex models of oligopoly could be constructed where the inventories are only part of the state vector.

8.5 Negotiations between labor and management

A simple model for continuous negotiations between labor and management is presented in this section. The model has the structure of a two-person nonzero-sum differential game. This model might be appropriate for such problems because:

(i) The "system" with which both parties are involved is a dynamic one--present decisions affect future possibilities.

(ii) The interests of the two parties are not identical, but there is a considerable degree of mutual interest.

To make the model simple, we assume here that one single state variable--the firm's capital assets--adequately represents the "state" of the system. The decision variables are the fraction of the labor force employed and the fraction of the profits invested in new capital equipment, both chosen by Management, and the wage level, chosen by Labor. Management tries to maximize return to shareholders, while labor tries to maximize consumption.

Define:

x = capital level of firm

u = fraction of total labor force employed

w = total wage rate if entire labor force works

$f(x, u)$ = total production, in units so that price = 1

L = total labor force, including unemployed

s = fraction of profits invested in capital equipment

$(1 - s)f - w$ = rate of payment of dividends to shareholders

δ = capital depreciation rate

γ = worker's discount on future consumption

α = interest rate

$c(u, w)$ = utility function for Labor employed at wage w

$K_L(x_f)$ = present value to Labor of capital at terminal time

$K_M(x_f)$ = present value to Management of capital at terminal time

w_o = wage which can be earned at alternative work by total labor force

The problem is then for Labor to choose the wage level w to maximize

$$J_L = \int_{t_o}^{t_f} c(u, w)e^{\gamma(t_o - t)} dt + K_L(x_f) \tag{8.11}$$

while Management chooses the fraction u of the labor force employed and the fraction s of net profits reinvested to maximize

$$J_M = \int_{t_o}^{t_f} [(1 - s)f(x, u) - w]e^{\alpha(t_o - t)} dt + K_M(x_f) \tag{8.12}$$

both subject to

$$\begin{aligned} &\dot{x} = sf(x, u) - w - \delta x \quad , \qquad x_o \text{ given} \\ &0 \leq u \leq 1 \\ &0 \leq w \quad , \qquad w = 0 \text{ if } x = 0 \\ &0 \leq s \leq 1 \end{aligned} \tag{8.13}$$

It is assumed that $f(x, 0) = f(0, u) = 0$, i.e., both capital and labor are necessary for production. The utility function for Labor, $c(u, w)$, is probably the hardest part of such a model to formulate.

Since this model has only one state variable, it might be reasonable to compute closed loop Nash solutions by the "dynamic programming" approach (Section 5.4). The continuous problem would first be converted to a multistage game. In fact, since decisions in labor-management negotiations, as well as in most other economic processes, are made at discrete times rather than continuously, the multistage game might be a better model than the differential game.

8.6 A model for international economic competition

This section presents a very simple differential game model for the interaction of the economic planning policies of several countries.

The model is far too naive to be of practical importance, but it illustrates the idea that economic competition between nations has the mathematical structure of a nonzero-sum differential game.

Our approach will be to consider first a model for economic growth which could be used by the planners of each country if the effects of the other countries could be ignored. By adding coupling terms, the N separate economies are linked together, and the result is a differential game.

As the basic "uncoupled" growth model for each country, we use the one-state model due to Solow[22], probably the best-known model in economic growth theory. Letting

k = capital/labor ratio

$f(k)$ = production function

s = fraction of production saved (reinvested)

β = population growth constant ($P = P_o e^{\beta(t-t_o)}$)

$c = (1 - s)f(k)$ = consumption per capita

$U(c)$ = utility of consumption

γ = discount on future consumption (optional)

$\Phi(k(t_f))$ = present value of capital at terminal time

the problem is to choose the saving ratio s to maximize

$$J = \int_{t_o}^{t_f} U(c(t))e^{-\gamma(t-t_o)}dt + \Phi(k(t_f)) \qquad (8.14)$$

subject to

$$\dot{k} = sf(k) - \beta k \ , \qquad k(t_o) = k_o$$

$$0 \leq s \leq 1 \qquad (8.15)$$

For certain simple production and utility functions [e.g., $f(k) = ak^{\frac{1}{2}}$, $U(c) = c$], this optimal growth problem can be solved analytically.

We shall assume that the coupling between any two economies is some function of the difference in capital levels. Since the coupling effect would presumably be small compared to the whole economy, it seems reasonable to let this function be linear. Using the definitions above, and adding subscripts to denote countries, the problem becomes:

For $i = 1, \ldots, N$, choose s_i to maximize

$$J_i = \int_{t_o}^{t_f} U_i(c_i)e^{-\gamma_i(t-t_o)}dt + \Phi_i(k_{1f}, \ldots, k_{Nf}) \tag{8.16}$$

subject to

$$\dot{k}_i = s_i f_i(k_i) - \sum_{j=1}^{N} b_{ij}(k_j - k_i) - \beta_i k_i$$

$$0 \leq s_i \leq 1$$

$$c_i = (1 - s_i)f_i(k_i) \tag{8.17}$$

The coupling coefficients b_{ij} will generally be positive, reflecting the idea that the country with a higher capital/labor ratio enjoys a competitive advantage, and they will depend on the populations of both countries. Of course, the b_{ij} must be sufficiently small so that k_i will never be driven negative.

In studying optimal growth models, one is often interested in steady-state solutions, where the economy grows at the same rate as the population, so that the capital/labor ratio remains constant. One then chooses the constant saving ratio to maximize the constant per capita consumption. Let us do this for our simple model, taking as our production function $f_i(k_i) = a_i k_i^{\frac{1}{2}}$ and assuming linear utility. We also put an ϵ in front of the coupling terms, to emphasize their smallness and to allow us to determine the first order effect of coupling on the steady-state Nash solutions by using an expansion technique. Define

$$y_i = k_i^{\frac{1}{2}}$$

and expand y_i and s_i in ϵ as follows

$$y_i = \bar{y}_i + \epsilon x_i + \frac{1}{2}\epsilon^2 z_i + \dots$$

$$s_i = \bar{s}_i + v_i + \dots \tag{8.18}$$

Then by inserting these expansions in (8.17), with $\dot{k} = 0$, solving for x_i and z_i, and choosing s_i and v_i to maximize consumption, we obtain after routine manipulation the first terms in the expansion of the steady-state Nash saving rates:

$$s_i = \frac{1}{2}[1 - \epsilon \sum_{j \neq i} b_{ij}/\beta_i] + \dots \tag{8.19}$$

The corresponding expansions for the steady-state consumptions are

$$c_i = \frac{a_i^2}{4\beta_i}[1 + 2\epsilon \sum_{j \neq i} b_{ij}/\beta_i] + \dots \tag{8.20}$$

This somewhat surprising result says that, in the steady-state, a small amount of coupling is beneficial to all countries. However, it must be borne in mind that steady-state solutions are independent of initial values. In fact, the per capita steady-state production for this model is

$$f_i = \frac{a_i^2}{2\beta_i}[1 + \epsilon \sum_{j \neq i} b_{ij}/\beta_i] + \ldots \qquad (8.21)$$

Note also the presence of the population growth rate β_i in the denominator. While the steady-state solutions do not seem to favor the large or advanced countries (in fact, all countries are "developed" when the steady-state is reached) nothing is said about the transient solution. During this transition, which may take a very long time, it is possible that the nations with high capital/labor ratios will be exploiting the less advanced countries, to the detriment of the latter.

CHAPTER IX

SUMMARY AND CONCLUSIONS

9.1 Summary

A general class of differential games, where the N players try to minimize different criteria by deciding inputs to a single dynamic system, was introduced as an extension of optimal control theory. All of the work on differential games which has yet appeared in the literature has been based on a zero-sum formulation, which rules out the possibility of mutual interest between the players. The nonzero-sum formulation considered here made it possible to model a far richer and more realistic class of competitive situations (many of which arise in economic contexts) where mutual interest is important.

The nonzero-sum differential game not only has a wider range of applications than the zero-sum model, but it is mathematically much richer as well. The word "optimal" becomes meaningless, and one must consider a variety of "solution" concepts. Several "solutions" with different features were discussed, all from the viewpoint of optimal control theory. For one solution, the Nash equilibrium, which is secure against unilateral deviations by any one player, the appropriate controls could be obtained by solving a set of coupled partial differential equations, provided that a unique "Nash saddle point" of a vector "Hamiltonian" could be found. It was seen that the Nash solutions depend on what information is available to the players during the course of play;

for example, the "closed loop" and "open loop" assumptions lead to entirely different costs and controls. (In two special cases -- the optimal control problem and the two-person, zero-sum differential game* -- these two Nash solutions coincide.)

The minimax solution, where each player minimizes his maximum possible cost, could be found by solving a set of N two-person, zero-sum differential games. The minimax solution is rather unsatisfactory when there is a significant degree of mutual interest, because it is excessively pessimistic (in fact, it might be called the "paranoid solution").

Finding the set of noninferior (or pareto-optimal) solutions, from which any negotiated solution would be chosen, involved solving a (N - 1)-parameter family of optimal control problems.

In one special case, where the system is linear and the costs are quadratic, all of these solution types can be computed exactly with relative ease. The Nash solutions, both open loop and closed loop, were presented for a very general form of the linear-quadratic nonzero-sum differential game. Noninferior solutions were also presented.

The computation of Nash equilibria for more general nonzero-sum differential games is much more difficult than finding optimal solutions to optimal control problems or two-person, zero-sum differential games. In spite of the apparent similarity of the problems, it was seen that a successful efficient "second order" algorithm for solving

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*As long as open loop and closed loop Nash solutions both exist, and provided that neither player ever deviates from the Nash rationale. See ref. 5, Ch. 9.

optimal control problems could not be extended to obtain an iterative procedure for computing closed loop Nash solutions to nonzero-sum differential games. The efficient computation of Nash solutions, especially closed loop, remains an outstanding problem. For the general case, even approximate methods are lacking.

Certain differential games can be formulated in terms of a small parameter such that, when this parameter is zero, the model reduces to a special case where the exact closed loop Nash solution can be computed. In such games, approximate Nash solutions can be obtained by expansion techniques.

One other special type of nonzero-sum differential game was presented -- the constrained linear differential game. It was presented because of its great potential usefulness as a tool in analysing economic competition, even though no results have been obtained for this problem, either here or elsewhere. Its one-player version, the continuous linear program, can be solved approximately by linear programming, but it is not known whether this approach can be generalized to compute approximate Nash closed loop solutions.

9.2 Conclusions and final comments

The nonzero-sum differential game model offers a promising new framework for the analysis of the evolution of competitive processes in the economy. Competition is thought to play an important role in our economic system, yet it is not well understood. A differential game model (or an equivalent multistage model) is appropriate for analysing the type of competition which "evolves" slowly over time, that is, where the effects of decisions are felt not immediately, but only gradually after time has elapsed.

Obviously, just the <u>formulation</u> of a realistic differential game model for describing competition in some industry would be a formidable project, even if the analytic difficulties of "solving" the model are disregarded. However, simply attempting to construct such a model, even if one falls somewhat short of this goal, could be quite instructive. In seeking a differential game description, one might be led to ask many new and interesting questions.

To the mathematical analyst, the nonzero-sum differential game offers a great variety of challenging problems. It is to be hoped that the research effort in the near future will be devoted to making differential games a more powerful tool for analysing the types of competition which really exist. This has not been the case in most of the work which has been done on zero-sum differential games. Hopefully, workers in nonzero-sum differential games will not lavish too much attention on the tractable but economically uninteresting linear-quadratic case. It would be far more useful to make progress on the constrained linear differential game considered in Chapter VII.

APPENDIX A

THE MORE GENERAL LINEAR-QUADRATIC DIFFERENTIAL GAME

In Chapter IV, the N-player linear-quadratic differential game was stated in its most general form (4.1)-(4.2). However, all the results presented in the chapter were based on a simplified version (4.3)-(4.4), where the cross terms, linear terms, and inhomogeneous terms were omitted. Nothing really interesting is sacrificed by considering only this simpler model, since it has all of the "important" features of the more general problem. However, the terms omitted in the simpler version sometimes occur when one tries to model realistic situations by LQDG. For convenience in solving such problems, the Nash solutions, both open loop and closed loop, are presented below.

<u>Problem</u>: For $i = 1, \ldots, N$, choose u_i to minimize

$$J_i = \int_{t_o}^{t_f} [\tfrac{1}{2} x^T Q_i x + a_i^T x + \sum_{j=1}^{N} (x^T G_{ij} u_j + \tfrac{1}{2} u_j^T R_{ij} u_j + c_{ij}^T u_j) + z] dt$$

$$+ \tfrac{1}{2} x_f^T S_{if} x_f + \xi_{if}^T x_f \qquad (A.1)$$

all subject to

$$\dot{x} = Ax + \sum_{j=1}^{N} B_j u_j + w \;, \qquad x(t_o) = x_o \qquad (A.2)$$

<u>Note</u>: Comparison of (A.1) with (4.1) reveals that the above model is not quite as general as it could be, since cross terms in the controls of different players are not present in the cost integrals. However,

there seem to be no reasonable models which contain these terms, and since they make the analysis considerably messier, they have been omitted here.

<u>Closed loop Nash solutions</u> are obtained by solving the generalized Bellman equation (3.16) by separation of variables. The Nash cost functions are

$$V_i(x,t) = \frac{1}{2}x^T S_i(t)x + \xi_i(t)^T x + \eta_i(t) \tag{A.3}$$

and the closed loop Nash controls are

$$\Psi_i(x,t) = -R_{ii}^{-1}(B_i^T S_i x + G_{ii}^T x + B_i^T \xi_i + c_{ii}) \tag{A.4}$$

where

$$\dot{S}_i = -S_i A - A^T S_i - Q_i + \sum_j (G_{ij}R_{jj}^{-1}G_{jj}^T + G_{jj}R_{jj}^{-1}G_{ij}^T)$$

$$+ \sum_j [S_i B_j R_{jj}^{-1}(B_j^T S_j + G_{jj}^T) + (G_{jj} + S_j B_j)R_{jj}^{-1}B_j^T S_i]$$

$$- \sum_j (S_j B_j + G_{jj})R_{jj}^{-1}R_{ij}R_{jj}^{-1}(G_{jj}^T + B_j^T S_j)$$

$$S_i(t_f) = S_{if} \tag{A.5}$$

$$\dot{\xi}_i = -[A^T - \sum_j (G_{jj} + S_j B_j)R_{jj}^{-1}B_j^T]\xi_i - \alpha_i - S_i w + \sum_j (G_{jj} + S_j B_j R_{jj}^{-1}$$

$$- \sum_{j \neq i} [(G_{jj} + S_j B_j)R_{jj}^{-1}R_{ij} - S_i B_j - G_{ij}]R_{jj}^{-1}[B_j^T \xi_j + c_{jj}]$$

$$\xi_i(t_f) = \xi_{if} \tag{A.6}$$

$$\dot{\eta}_i = -\xi_i^T w - z + \sum_j [\xi_i^T B_j + c_{ij} - \frac{1}{2}(\xi_j^T B_j + c_{jj})R_{jj}^{-1}R_{ij}]R_{jj}^{-1}(B_j^T \xi_j + c_{jj})$$

$$\eta_i(t_f) = 0 \qquad (A.7)$$

The set (A. 6) of N coupled linear vector differential equations can be solved once the S_j $(j = 1, \ldots, N)$ have been found by solving (A. 5). It is then a simple matter to evaluate the quadratures (A. 7).

Open loop Nash solutions are most readily obtained from the variational necessary conditions (3. 11)-(3. 14). By guessing that the solution has the form

$$\lambda_i = S_i(t)x(t) + \xi_i(t) \qquad (A.8)$$

and substituting this into the necessary conditions, one obtains the open loop Nash controls

$$u_i(t) = -R_{ii}^{-1}[(B_i^T S_i + G_{ii}^T)x + c_{ii} + B_i^T \xi_i] \qquad (A.9)$$

where the asymmetric matrix $S_i(t)$ is the solution of

$$\dot{S}_i = -S_i A - A^T S_i - Q_i + S_i \sum_j B_j R_{jj}^{-1}(B_j^T S_j + G_{jj}^T)$$

$$+ \sum_j G_{ij} R_{jj}^{-1}(B_j^T S_j + G_{jj}^T)$$

$$S_i(t_f) = S_{if} \qquad (A.10)$$

and the vector $\xi_i(t)$ is the solution of

$$\dot{\xi}_i = -A^T \xi_i - \alpha_i + \sum_j (S_i B_j + G_{ij})R_{jj}^{-1}(c_{ij} + B_j^T \xi_j)$$

$$\xi_i(t_f) = \xi_{if} \qquad (A.11)$$

Note carefully that S_i and ξ_i which solve (A. 10) and (A. 11) are not the same as S_i and ξ_i which solve the closed loop equations (A. 5) and (A. 6), except in certain special cases.

The costs associated with the open loop Nash solutions cannot be obtained directly from the solutions to (A. 9)-(A. 11), but once these equations have been solved, the costs can be obtained by integrating a set of linear equations which are a generalization of (4. 13). Omitting the straightforward derivation, we merely state the result: When all players use open loop Nash controls, starting from the initial point (x_o, t_o), their costs are

$$J_i = \frac{1}{2} x_o^T P_i(t_o) x_o + q_i^T(t_o) x_o + r_i(t_o) \qquad \text{(A. 12)}$$

where the symmetric matrix $P_i(t)$ and the vector $q_i(t)$ are solutions of

$$\dot{P}_i = -A^T P_i - P_i A - Q_i + \sum_{j=1}^{N} \{(G_{jj} + S_j^T B_j) R_{jj}^{-1} B_j^T P_i$$

$$+ P_i B_j R_{jj}^{-1} (B_j^T S_j + G_{jj}^T) - (G_{jj} + S_j^T B_j) R_{jj}^{-j} R_{ij} R_{jj}^{-1} (B_j^T S_j + G_{jj}^T)$$

$$+ G_{ij} R_{jj}^{-1} (B_j^T S_j + G_{jj}^T) + (G_{jj} + S_j^T B_j) R_{jj}^{-1} G_{ij}^T\}$$

$$P_i(t_f) = S_{if} \qquad \text{(A. 13)}$$

$$\dot{q}_i = -A^T q_i - \alpha_i + \sum_j (G_{jj} + S_j^T B_j) R_{jj}^{-1} (B_j^T q_i + c_{ij})$$

$$+ \sum_j [G_{ij} + P_i B_j - (G_{jj} + S_j^T B_j) R_{jj}^{-1} R_{ij}] R_{jj}^{-1} (c_{jj} + B_j^T \xi_j)$$

$$q_i(t_f) = \xi_{if} \qquad \text{(A. 14)}$$

and

$$r_i(t_o) = \int_{t_o}^{t_f} \{-z + \sum_j [q_i^T B_j + c_{ij}^T - \frac{1}{2}(\xi_j^T B_j + c_{jj}^T) R_{jj}^{-1} R_{ij}] R_{jj}^{-1}$$

$$\cdot \; (c_{jj} + B_j^T \xi_j)\} dt \qquad (A.15)$$

APPENDIX B

ALTERNATE DERIVATION OF THE OPEN LOOP NASH SOLUTIONS

In Chapter IV, the open loop Nash solutions for the linear-quadratic differential game were obtained from the variational necessary conditions. The open loop controls were given in terms of the solution to an asymmetric "Riccati-like" matrix differential equation (4.10). The reader who is familiar with the solutions to linear-quadratic optimal control problems or two-person, zero-sum differential games may have been surprised that in the more general case the multiplier λ_i associated with the i^{th} player was related to the state vector by an asymmetric matrix S_i, and that the costs could not be obtained directly from this matrix.

To aid in understanding this result, and to provide an interpretation for the multipliers $\lambda_1, \ldots, \lambda_N$, an alternate derivation of the open loop solutions, based on the value function approach (i.e., the "dynamic programming" approach) is presented here.

The derivation is based on the following idea: if the i^{th} player knows the open loop control functions used by the other players, he can calculate his own open loop control by solving an optimal control problem with the other players' controls considered as known forcing terms in the state equation and cost functional. Either an open loop or a closed loop method may be used, since they give the same solution in any deterministic optimal control problem. Thus player i can use the Bellman equation to compute his open loop Nash control explicitly as a function of the arbitrary control functions used by the other players. The resulting partial differential equation can then (by the usual educated guess) be transformed to a set of ordinary differential equations with boundary conditions at the terminal time. The still unknown controls of the other players will appear in these equations. This is done for all the players, and the entire set of controls can then be found by demanding that, for each i, the Nash control found by solving the i^{th} player's optimal control problem be the same as the arbitrary control assumed for the i^{th} player by all the other players. Because the optimal controls found for each player will be given in terms of the state vector, this last step will require solving a two-point boundary value problem. This is quickly accomplished by applying another old familiar trick.

We are now ready for the details of the derivation. From the point of view of the i^{th} player, let the (unknown) controls for the other players be denoted $\phi_j(t)$, $j \neq i$. The Bellman equation for the i^{th} player (for the problem stated in (4.3), (4.4)) is

$$-V_{it} = \min_{u_i}\Big[\frac{1}{2}x^T Q_i x + \frac{1}{2}u_i^T R_{ii} u_i + \frac{1}{2}\sum_j \phi_j^T R_{ij}\phi_j$$

$$+ V_{ix}(Ax + B_i u_i + \sum_j B_j \phi_j)\Big]$$

$$V_i(x_f, t_f) = \frac{1}{2}x_f^T S_{if} x_f \qquad \text{(B.1)}$$

We then guess the following separation of variables:

$$V_i(x,t) = \frac{1}{2}x^T\tilde{S}_i(t)x + \xi_i^T(t)x + \eta_i(t) \tag{B.2}$$

Substituting this into (B.1) and collecting like powers in x, one then easily verifies that (B.2) is the solution, where

$$\dot{\tilde{S}}_i = -\tilde{S}_iA - A^T\tilde{S}_i - Q_i + \tilde{S}_iB_iR_{ii}^{-1}B_i^T\tilde{S}_i$$

$$\tilde{S}_i(t_f) = S_{if} \tag{B.3}$$

$$\dot{\xi}_i = -(A^T - \tilde{S}_iB_iR_{ii}^{-1}B_i^T)\xi_i - \tilde{S}_i\sum_{j\neq i} B_j\phi_j$$

$$\xi_i(t_f) = 0 \tag{B.4}$$

$$\dot{\eta}_i = \frac{1}{2}\xi_i^TB_iR_{ii}^{-1}B_i^T\xi_i - \sum_{j\neq i}(\xi_i^TB_j + \frac{1}{2}\phi_j^TR_{ij})\phi_j$$

$$\eta_i(t_f) = 0 \tag{B.5}$$

and the Nash control for the i^{th} player is

$$\phi_i(t) = -R_{ii}^{-1}B_i^T[S_ix(t) + \xi_i] \tag{B.6}$$

where x(t) is the trajectory resulting from integrating the state equation from (x_o, t_o), using the controls $\phi_1, \ldots, \phi_N$:

$$\dot{x} = Ax + \sum_j B_j\phi_j = Ax - \sum_j B_jR_{jj}^{-1}B_j^T(\tilde{S}_jx + \xi_j) \tag{B.7}$$

When (B.6) is substituted into (B.3) and (B.4), the resulting equations together with (B.7) form a two-point boundary value problem. Note carefully that we never made any "closed loop assumptions" about the other players' controls in reducing the i^{th} player's Bellman equation to a set of ordinary differential equations.

All that remains is to solve the two-point boundary value problem. The only independent variable in these equations is t. Let us define the n x n matrix $M_i(t)$ by

$$\xi_i(t) = M_i(t)x(t) \qquad (\text{assuming } x \neq 0) \tag{B.8}$$

Then

$$\phi_i = -R_{ii}^{-1}B_i^T(\tilde{S}_i + M_i)x \qquad \text{for } i = 1, \ldots, N \tag{B.9}$$

Substituting this into (B.4) then gives a differential equation for M_i:

$$\dot{M}_i = -M_iA - A^TM_i + (\tilde{S}_i + M_i)\sum_j B_jR_{jj}^{-1}B_j^T(\tilde{S}_j + M_j)$$

$$- \tilde{S}_iB_iR_{ii}^{-1}B_i^T\tilde{S}_i$$

$$M_i(t_f) = 0 \tag{B.10}$$

It is interesting to add this equation to (B.3). The result is

$$\frac{d}{dt}(\tilde{S}_i + M_i) = -(\tilde{S}_i + M_i)A - A^T(\tilde{S}_i + M_i) - Q_i$$

$$+ (\tilde{S}_i + M_i)\sum_j B_jR_{jj}^{-1}B_j^T(\tilde{S}_j + M_j)$$

$$\tilde{S}_i(t_f) + M_i(t_f) = S_{if} \tag{B.11}$$

If one defines $S_i = \tilde{S}_i + M_i$, one sees that (B.11), which was obtained from the Bellman equation, is the same as (4.10), obtained from the variational necessary conditions.

We now seek an interpretation of the multipliers λ_i, $i = 1, \ldots, N$, which appear in the necessary conditions for the open loop controls (4.5)-(4.8). Taking the gradient of (B.2),

$$V_{ix}(x,t) = x^T\tilde{S}_i(t) + \xi_i^T(t) \tag{B.12}$$

Evaluating V_{ix} on the Nash path by using (B.8),

$$V_{ix}(x(t),t) = x(t)^T[\tilde{S}_i(t) + M_i^T(t)] = x(t)^T S_i(t)^T = \lambda_i(t)^T \tag{B.13}$$

so λ_i can be interpreted as the influence of a perturbation of x on the i^{th} player's cost, when the i^{th} player is allowed to adjust (optimally) his control while the other players are forced to use their "nominal" Nash open loop controls.

The derivation presented here also gives an alternate way to compute the open loop Nash costs. Substituting for ϕ_i and ξ_i in (B.5),

$$\dot{\eta}_i = x^T\{\tfrac{1}{2}M_i^T B_i R_{ii}^{-1} B_i^T M_i - \sum_{j\neq i}[\tfrac{1}{2}(\tilde{S}_j^T + M_j^T(B_j R_{jj}^{-1} R_{ij}$$

$$- M_i^T B_j] R_{jj}^{-1} B_j^T(\tilde{S}_j + M_j)\}x \tag{B.14}$$

If we define the symmetric matrix $Z_i(t)$ by

$$\eta_i = \tfrac{1}{2}x(t)^T Z_i(t) x(t) \tag{B.15}$$

we find from (B.14) that Z_i satisfies

$$\dot{Z}_i = -[A^T - \sum_j(\tilde{S}_j + M_j^T(B_j R_{jj}^{-1} B_j^T]Z - Z[A - \sum_j B_j R_{jj}^{-1} B_j^T(\tilde{S}_j + M_j)]$$

$$+ M_i^T B_i R_{ii}^{-1} B_i^T M_i - \sum_{j\neq i}(\tilde{S}_j + M_j^T)B_j R_{jj}^{-1} B_j^T(\tilde{S}_j + M_j)$$

$$+ \sum_{j\neq i}\{M_i^T B_j R_{jj}^{-1} B_j^T(\tilde{S}_j + M_j) + (\tilde{S}_j + M_j^T)B_j R_{jj}^{-1} B_j^T M_i\}$$

$$Z_i(t_f) = 0 \tag{B.16}$$

From (B.2), (B.8) and (B.16), the Nash cost for the i^{th} player starting at (x_o, t_o) is

$$V_i(x_o, t_o) = \frac{1}{2}x_o^T[S_i(t_o) + 2M_i^T(t_o) + Z_i(t_o)]x_o$$

$$= \frac{1}{2}x_o^T[S_i(t_o) + M_i(t_o)^T + M_i(t_o) + Z_i(t_o)]x_o$$

$$= \frac{1}{2}x_o^T P_i(t_o)x_o \qquad \text{(B.1}$$

where

$$P_i(t) = S_i(t) + M_i^T(t) + M_i(t) + Z_i(t)$$

By adding (B.3), (B.10), the transpose of (B.10), and (B.16), a differential equation for P_i can be obtained. In fact, as one would expect, it is exactly the same as (4.13).

Thus the open loop solutions can be obtained by the value function approach, provided that one is careful to treat all the controls of the rivals of player i as functions of time only, until after the partial differential equation for $V_i(x, t)$ has been converted to a set of ordinary differential equations. The results for the controls and costs are then identical to the results obtained (somewhat more easily) by starting from the variational necessary conditions.

REFERENCES

1. Isaacs, R., "Differential Games," Wiley, N. Y., 1965.

*2. Starr, A. W., and Y. C. Ho, "Nonzero-sum differential games," J. Optimization Theory and Applications, 3, No. 3, March 1969.

*3. Starr, A. W., and Y. C. Ho, "Further properties of nonzero-sum differential games," JOTA, 3, No. 4, April 1969.

4. Case, J. H., "Equilibrium points of N-person differential games," Univ. of Michigan, Dept. of Ind. Eng. Tech. Rept. 1967-1.

5. Bryson, A. E., and Y. C. Ho, "Applied Optimal Control," Blaisdell, 1968.

6. Luce, R. D. and H. Raiffa, "Games and Decisions," Wiley, N. Y., 1957.

7. daCunha, N. O., and E. Polak, "Constrained minimization under vector-valued criteria in finite-dimensional spaces," Memorandum ERL-M188, Oct. 1966, Univ. of Calif., Berkeley.

8. Klinger, A., "Vector-valued performance criteria," IEEE Tr-AC, Jan. 1964 (corresp.).

9. Behn, R. D., and Y. C. Ho, "On a class of linear stochastic differential games," IEEE Tr-AC 13, No. 3, June 1968.

10. Rhodes, I. B., and D. G. Luenberger, "Differential games with imperfect state information," IEEE Tr-AC, 1969.

11. Willman, W. W., "Formal solutions for a class of stochastic pursuit-evasion games," IEEE Tr-AC, Oct. 1969.

12. McReynolds, S. R., and A. E. Bryson, Jr., "A successive sweep method for solving optimal programming problems," J.A.C.C. 6, 1965, p. 551.

13. Kelley, H. J., R. E. Kopp, and H. G. Moyer, "A trajectory optimization technique based upon the theory of the second variation," A.I.A.A. Astrodynamics Conference, New Haven, Conn.

14. Jacobson, D. H., "New second-order and first-order algorithms for determining optimal control: a differential dynamic programming approach," JOTA 2, No. 6, p. 411, Nov. 1968.

---

*These papers also appeared as Harvard Tech. Report No. 564, May 1968 and Harvard Tech. Report No. 577, November 1968, respectively.

15. daCunha, N. O., and E. Polak, "Constrained minimization under vector-valued criteria in linear topological spaces," Memorandum No. ERL-M191, Nov. 1966, Univ. of Calif., Berkeley.

16. Ho, Y. C., A. E. Bryson, and S. Baron, "Differential games and optimal pursuit-evasion strategies," IEEE Tr-AC 10, No. 4, Oct. 1965.

17. Jacobson, D. H., "A note on error analysis in differential dynamic programming," IEEE Tr-AC, April 1969.

18. Jacobson, D. H., S. B. Gershwin, and M. M. Lele, "Computation of optimal singular control," Harvard Univ. Tech. Rept. 580, Jan. 1969 (to appear in JOTA, 1969).

19. Jacobson, D. H., "Differential dynamic programming methods for solving bang-bang control problems," IEEE Tr-AC 13, No. 6, Dec. 1968, pp. 661-675.

20. Jacobson, D. H., and M. M. Lele, "A transformation technique for optimal control problems with a state-variable inequality constraint," IEEE Tr-AC, 1969 (to appear).

21. Grinold, R. C., "Continuous programming," Op. Res. Ctr. Tech. Rept. ORC 68-14, June 1968, U. of Calif., Berkeley.

22. Solow, R. M., "A contribution to the theory of economic growth," Qu. J. of Economics 70, pp. 65-94, Feb. 1956.

23. Krishnan, K. S. and S. K. Gupta, "Mathematical model for a duopolistic market," Management Science 13, No. 7, p. 568, March 1967.

24. Owen, G., "Game Theory," Saunders, Philadelphia, 1968.

25. Ruff, L. E., "Optimal growth and technological progress in a Cournot economy," Tech. Rept. 11, Inst. for Mathematical Studies in the Social Sciences, Stanford Univ., Feb. 1968.

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13. ABSTRACT

A general class of differential games, where the N players try to minimize different cost criteria by controlling inputs to a single dynamic system, is investigated as an extension of optimal control theory. Dropping the usual zero-sum assumption makes it possible to model a more realistic class of competitive situations where mutual interest is important.

The nonzero-sum formulation has several interesting analytic and conceptual features not found in zero-sum differential games. It is no longer obvious what should be demanded of a "solution," and three types of solution concepts are discussed: Nash equilibrium, minimax, and noninferior (or Pareto optimal) strategies. For one special case, the "linear-quadratic" differential game, all of these solutions can be computed exactly by solving sets of coupled ordinary matrix differential equations.

Another feature not found in optimal control problems or in two-person, zero-sum differential games is the difference between "open loop" and "closed loop" equilibria. The "principle of optimality" of optimal control theory does not generalize in an obvious way to the nonzero-sum differential game. Some simple examples are given to illustrate this. It is shown that the various efficient algorithms of optimal control theory (such as "differential dynamic programming") do not readily extend to the computation of Nash equilibrium controls. However, approximate Nash solutions can be obtained in certain special cases.

Some simple examples are solved, and series of more difficult but more realistic nonzero-sum differential game situations are presented (but not solved) for models of economic oligopoly, advertising policy, labor-management netotiations, and international trade.

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