Quantum-Limited Detection of an Incoherent Object with a Lorentz Spectrum

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ABSTRACT

When the appropriate quantum threshold detectors are used, an object emitting incoherent light with a Lorentz spectrum has a lower probability of detection in the presence of thermal background light than an object emitting light with a rectangular spectrum of the same bandwidth.

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The quantum threshold detector for an object emitting incoherent light that is received with thermal background light was worked out in a recent paper.[1] It measures a quantum-mechanical operator U that is a quadratic functional of the field at the aperture of the observing instrument. The outcome u of the measurement is compared with a decision level u_0; if $u > u_0$, the decision that the object is present is made.

As a step toward calculating the probability Q_d of detection, the moment-generating function (m. g. f.)

$$\mu_u(s;\beta) = \mathrm{Tr}\,[\rho_1(\beta)\, e^{Us}] \qquad (1)$$

of the distribution of u was derived. Here β is the strength of the object actually present relative to that of the object for which the detector was designed, and $\rho_1(\beta)$ is the density operator of the total field made up of object light and background. It was assumed that the product of the observation interval T and the bandwidth W of the object light is large, $TW >> 1$, as is normally the case.

Here we compare the detectabilities of two incoherent objects with a rectangular and a Lorentz spectrum, respectively. For simplicity we assume that the light from both has first-order spatial coherence at the aperture; the spatial factor[2] is $\mathcal{F} = 1$. It was shown[1] that when the temporal spectrum is rectangular, the outcome u, suitably normalized, is a random variable with a Poisson distribution, permitting the probability $Q_d(\beta)$ of detection to be easily calculated as a function of β for a fixed false-alarm probability Q_o.

When the object emits light with a Lorentz spectrum,

$$X(\omega) = 2w/(\omega^2 + w^2), \tag{2}$$

the m. g. f. of the suitably normalized statistic u is[3]

$$\mu_u(s;\beta) = \exp\{ N_0\, s e^s [I_0(s) - I_1(s)] + N_s [e^s I_0(s) - 1] \} \tag{3}$$

where N_s is the average total number of photons received from the object during the observation interval, and $N_0 = \mathcal{N} TW$, with

$$\mathcal{N} = [e^{\hbar \Omega / K\mathcal{T}} - 1]^{-1}, \tag{4}$$

$\hbar$ = Planck's constant $h/2\pi$, Ω = the central angular frequency of the object spectrum, to which ω in Eq. (2) is referred, K = Boltzmann's constant, and $\mathcal{T}$ = the effective absolute temperature of the background. It is assumed that $\mathcal{N} << 1$, but with TW >> 1, N_0 is of the order of 1.

The m. g. f. in Eq. (3) cannot be inverted analytically to obtain the probability density function (p. d. f.) $p(u;\beta)$ of the outcome u. Numerical calculations of the probability $Q_d(\beta)$ of detection were therefore undertaken. The results are shown in Fig. 1, where the probability Q_d is plotted versus the signal-to-noise ratio $D = N_s/N_0^{1/2}$ for $N_0 = 15$ and two values of the false-alarm probability Q_0. The equivalent bandwidths of the two spectra were taken equal, $W = w$. The object with the Lorentz spectrum has the lower probability of detection. Calculations for $N_0 = 5$ revealed the same relationship.

The detection probabilities for the rectangular spectrum were calculated as described previously.[4] For the Lorentz spectrum two methods were used. At low values of the signal-to-noise ratio the Laplace transform $\mu_u(-s;\beta)/s$ of the cumulative distribution of u was inverted by the method of steepest descents[5] to obtain the approximate formula

$$q(u;\beta) = \int_u^\infty p(u';\beta)\,du' \cong$$

$$[2\pi\Phi''(t)]^{-1/2} \exp[\Phi(t)]\,\{1+\frac{1}{8}\Phi''''(t)\,[\Phi''(t)]^{-2}\} \tag{5}$$

where

$$\Phi(s) = \ln \mu_u(s;\beta) - su - \ln s \tag{6}$$

and t is the root of the equation $\Phi'(t) = 0$, primes denoting differentiation. Here β is proportional to N_s. The equation $q(u_0; 0) = Q_0$ was solved by Newton's method to obtain the decision level u_0. The probability of detection is then $Q_d = q(u_0;\beta)$.

At large values of the signal-to-noise ratio the probability $Q_d(\beta)$ was calculated by writing it first as

$$Q_d(\beta) = q(u_0;\beta) = 1 - \int_0^\infty R(u'/u_0)\,p(u';\beta)\,du' \tag{7}$$

where

$$R(x) = 1, \ 0 < x < 1; \ R(x) = 0, \ x > 1. \tag{8}$$

is the rectangular function. R(x) was expanded in a series of Laguerre functions, which was substituted into Eq. (7) and integrated term by term to obtain a series whose terms involve the m. g. f. $\mu_u(s;\beta)$ and its derivatives evaluated at a certain value of s. Twenty terms of the Laguerre series were used. Details of the method are given elsewhere.[6] For $Q_d \lesssim 0.3$, the results of the two methods agreed closely.

Footnotes

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1. C. W. Helstrom, "Detection of Incoherent Objects by a Quantum-Limited Optical System," submitted to J. Opt. Soc. Am. The notation of that paper is used here.

2. C. W. Helstrom, J. Opt. Soc. Am. **59**, (1969); see Eq. (3.8).

3. See reference 1, Eq. (5.22).

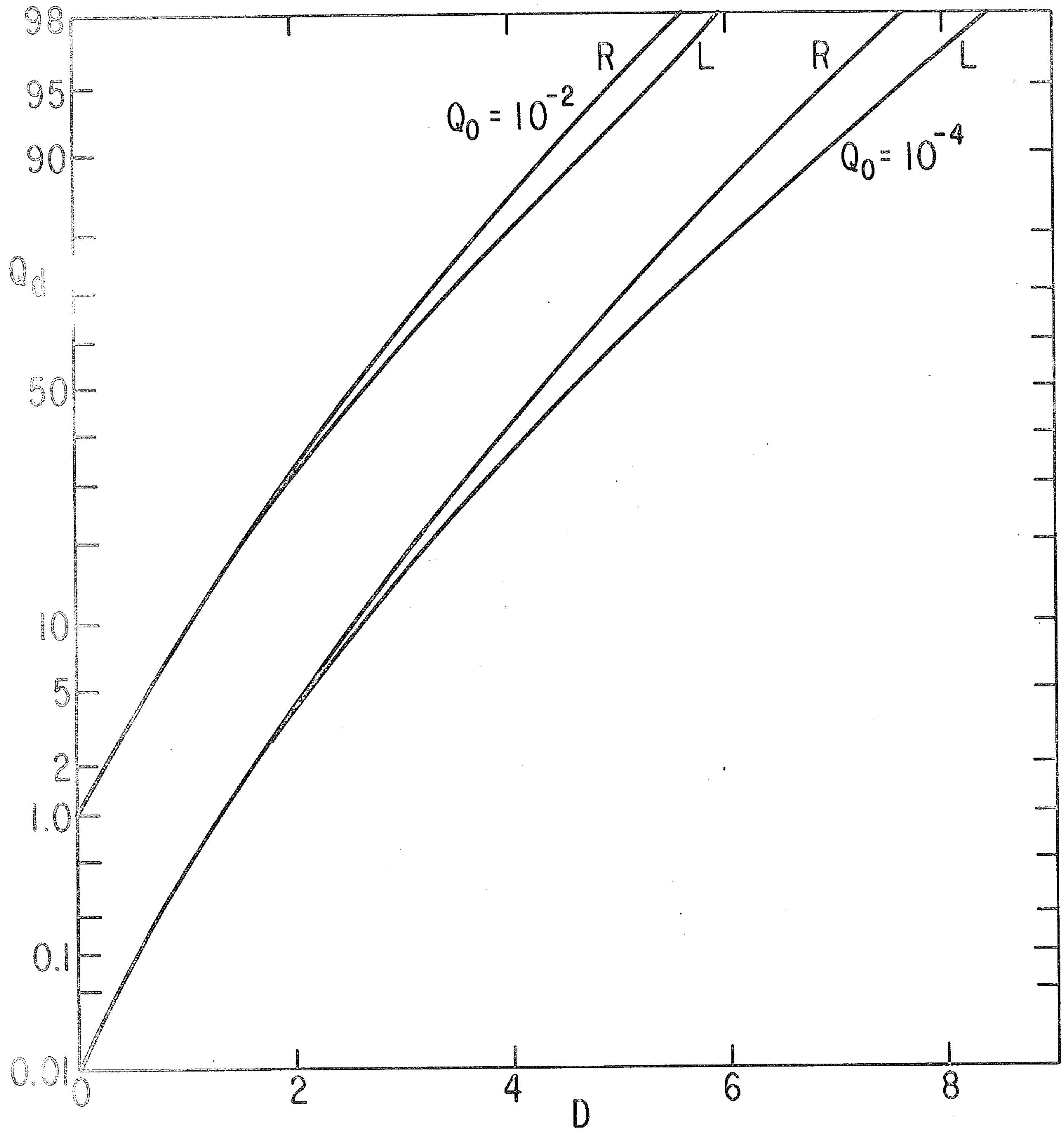
4. See reference 1, Section V.

5. G. Doetsch, Handbuch der Laplace-Transformation (Birkhäuser Verlag, Basel and Stuttgart, 1955), vol. 2, ch. 3, § 5, pp. 83-88.

6. C. W. Helstrom, Proc. IEEE **57**, (1969).

Figure Caption

FIGURE 1: Probability of detection Q_d vs. signal-to-noise ratio $D = N_s/N_0^{1/2}$ for false-alarm probabilities $Q_0 = 10^{-2}$ and 10^{-4}. R: rectangular spectrum; L: Lorentz spectrum.



98
95
90
50
10
5
2
1.0
0.1
0.01
Q_d
0
2
4
6
8
D
R
L
R
L
$Q_0 = 10^{-2}$
$Q_0 = 10^{-4}$