The Application of Prolate Spheroidal Wave Functions to the Detection and Estimation of Bandlimited Signals

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ABSTRACT

A solution to the detection Fredholm equation for bandlimited noise and signal is presented that makes use of the dual orthogonality and completeness of the prolate spheroidal wave functions. Although data are available only on a finite interval, the detector in principle achieves the same optimum performance as if the interval were infinite. This apparent anomaly is resolved in an example.

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The Fredholm integral equation

$$g(t) = \int_{-T/2}^{T/2} R(t - s)\, f(s)\, ds, \quad -T/2 < t < T/2 \tag{1}$$

and certain of its variants occur frequently in signal detection and estimation. The kernel $R(t - s)$ is symmetric positive definite, and the solution can be formally represented as an infinite summation of the eigenfunctions of $R(t - s)$[1]. These eigenfunctions have been computed only for a few simple kernels, and we normally assume $R(t)$ has a rational spectrum. Then $f(t)$ can be determined as the solution to a certain linear differential equation with constant coefficients.[2]

We will show that for $g(t)$ and $R(t)$ bandlimited to $|\omega| \leq W$, the solution to (1) can be expressed as an infinite summation of prolate spheroidal wave functions (PSWF's).

I. The Solution of (1) for Bandlimited g(t) and R(t)

With the definition

$$f_T(t) = \begin{cases} f(t) & |t| \leq T/2 \\ 0 & |t| > 0 \end{cases} \tag{2}$$

the integral equation (1) can be rewritten

$$\int_{-\infty}^{\infty} R(t - s)\, f_T(s)\, ds = g(t). \quad -T/2 < t < T/2. \tag{3}$$

Let $\hat{f}_T(\omega)$, $\hat{g}(\omega)$ and $\varphi(\omega)$ be the respective Fourier transforms of $f_T(t)$, $g(t)$, and $R(t)$. The left hand side of (3) is an entire function of t since it is a linear superposition of bandlimited, and therefore entire, functions $R(t - s)$. Now $g(t)$ for $|t| < T/2$ is a piece of an entire function. Hence the left and right hand sides of (3) are entire functions that coincide over a finite interval. We conclude that they are equal for all t, and we can Fourier transform both sides of (3) to obtain $\varphi(\omega)\ \hat{f}_T(\omega) = \hat{g}(\omega)$. Now both $\varphi(\omega)$ and $\hat{g}(\omega)$ are zero for $|\omega| > W$, so that this yields

$$\hat{f}_T(\omega) = \frac{\hat{g}(\omega)}{\varphi(\omega)} \quad . \tag{4}$$

only for $|\omega| < W$. But if we assume that $\hat{f}_T(\omega)$ is square integrable, we can then use the PSWF method[3] to find a time limited function that is defined for all ω and equal to $\hat{f}_T(\omega)$ almost everywhere in $(-W, W)$.

The definitions of $\psi_n(t)$, c, and $\lambda_n(c)$ are those of Slepian and Pollack. The PSWF expansion

$$\hat{f}_T(\omega) = \sum_{n=0}^{\infty} a_n\ \psi_n(T\omega/2W) \tag{5}$$

is defined for all ω, where

$$a_n = \frac{T}{2W\lambda_n(c)} \int_{-W}^{W} \hat{f}_T(\omega)\ \psi_n(T\omega/2W)\ d\omega \tag{6}$$

Moreover, by taking the inverse Fourier transform of (5) we have the desired result

$$f_T(t) = \begin{cases} \left(\frac{W}{\pi T}\right) \displaystyle\sum_{n=0}^{\infty} i^n\ a_n [\lambda_n(c)]^{1/2}\, \psi_n(t) & |t| \le T/2 \\ 0 & |t| > T/2\ . \end{cases} \tag{7}$$

An application of the general form of Parseval's theorem to (6) shows that

$$a_n = i^n \left(\frac{W}{\pi T}\right)^{-1/2} [\lambda_n(c)]^{-1/2} \int_{-\infty}^{\infty} h(t)\ \psi_n(t)\ dt \tag{8}$$

where

$$h(t) = \int_{-W}^{W} \frac{\hat{g}(\omega)}{\varphi(\omega)}\, e^{i\omega t}\ d\omega/2\pi . \tag{9}$$

Hence (7) can be rewritten

$$f_T(t) = \begin{cases} \sum_{n=0}^{\infty} h_n \, \psi_n(t)/\lambda_n & |t| \le T/2 \\ 0 & |t| > T/2, \end{cases} \tag{10}$$

where h_n is the n-th coefficient in the PSWF expansion of h(t). Equations (9) and (10), our main result, give the solution to (1).

II. An Example from Detection Theory

A sufficient statistic G for the detection of a known bandlimited signal s(t) in additive zero-mean, non-white gaussian noise n(t) is

$$G = \int_0^T q(t) \, \nu(t) \, dt \tag{11}$$

where

$$\nu(t) = \begin{cases} n(t) \text{ under hypothesis } H_0 \\ s(t) + n(t) \text{ under hypothesis } H_1. \end{cases} \tag{12}$$

q(t) is the solution to the integral equation

$$s(t) = \int_0^T R(t - s) \, q(s) \, ds \,, \quad 0 < t < T \tag{13}$$

where R(t) is the covariance of n(t).[4] With an appropriate change of variables (12) can be put in the form of (1), and we have the solution (10) with q(t + T/2) = $f_T(t)$ and

$$h(t) = \int_{-W}^{W} \frac{\hat{s}(\omega) e^{i\omega T/2}}{\varphi(\omega)} \, e^{i\omega t} \, d\omega/2\pi. \tag{14}$$

To evaluate the performance of this detection scheme we need only compute the mean and variance of G under the null hypothesis H_0 and the alternative hypothesis H_1.[5] We have from (11) and (10) that $E[G|H_0] = 0$ and

$$E[G|H_1] = \int_0^T q(t)\ s(t)\ dt = \sum_{n=0}^{\infty} h_n \int_{-T/2}^{T/2} s(t + T/2)\ \psi_n(t)\ dt/\lambda_n(c)$$

$$= \int_{-W}^{W} \frac{|\hat{s}(\omega)|^2}{\varphi(\omega)} \frac{d\omega}{2\pi} \triangleq d^2. \tag{15}$$

(15) follows from (9) and an application of Parseval's theorem. A similar computation will show that

$$\text{var } G = d^2 \tag{16}$$

under both hypotheses.

Note that (15) is independent of T. Thus for any T we achieve the same false-alarm and detection probabilities as for an infinite observation interval. In reality, because of computational difficulties, we could approach this theoretical limit only if nearly all the signal were contained in our observation interval T. This will be verified by an example.

Now consider the signal $s(t) = \text{sinc}\ \Omega(t - t_0)$ with $2\pi/\Omega << T$. We will take the noise spectral density to be the constant N_0 over the interval $|\omega| \leq W$. We have

$$h_n = \frac{1}{\lambda_n(c)N} \int_{-T/2}^{T/2} \text{sinc}\ \Omega[(\tfrac{1}{2}T - t_0) - t]\ \psi_n(t)\ dt. \tag{17}$$

If we take the PSWF's with $c = \Omega T/2$, then $h_n = \frac{1}{N_0}\ \psi_n\ (\frac{1}{2}T - t_0)$. If $|\frac{1}{2}T - t_0|$ $<< T/2$, i.e. s(t) is contained in T, h_n is nearly zero for $n > n_{crit} = [2\ \frac{\Omega}{2\pi}\ T]$. If $|\frac{1}{2}T - t_0| \geq T/2$, i.e. t_0 lies outside the interval (0, T), terms of higher order will be necessary to compute a detection statistic. However, for $n > n_{crit}$ the terms

$$\nu_n = \int_{-T/2}^{T/2} \nu(t + T/2)\ \psi_n(t)\ dt/\lambda_n \tag{18}$$

are extremely sensitive to error since $\psi_n(t) \approx 0$ in the interval $|t| \leq T/2$, and we cannot hope to improve the detection. Now the statistic

$$G = \sum_{n=0}^{\infty} h_n\ \nu_n \tag{19}$$

computed with $c = \Omega T/2$ is not optimum, since from (9) we should use $c = WT/2$. However,

$$\hat{f}_T(\omega) \cong \sum_{n=0}^{[\frac{2c}{\pi}]} a_n\ \psi_n(T\omega/2\Omega) \tag{21}$$

is nearly zero for $|\omega| > \Omega$, and we expect the improvement with $c = WT/2$ to be negligible, at least for $t_0 << T$.

We conclude that the PSWF's provide a convenient analytical tool for solving (1). That we attain theoretically maximum detectability ($T = \infty$) for only a finite T reflects the fact that all the information in a bandlimited signal is contained in a finite interval. With careful interpretation of the results, however, we see that we cannot in reality attain this limiting detectability. An example from estimation theory which makes use of an alternative approach has been presented.[6]

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FOOTNOTES

1. R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. I, Interscience Publishers, Inc., New York, N. Y., 1966, p. 122.
2. C. W. Helstrom, "Solution of the Detection Integral Equation for Stationary Filtered White Noise", IEEE Trans. on Information Theory, IT-11, July 1965, pp. 335-339.
3. D. Slepian and H. O. Pollak, "Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty--I", BSTJ, 40, January 1961, pp. 43-63.
4. C. W. Helstrom, Statistical Theory of Signal Detection, Pergamon Press, New York, N. Y., 1968, p. 116.
5. Ibid, p. 118.
6. C. L. Rino, "Bandlimited Image Restoration by Linear Mean-Square Estimation", J. Opt. Soc. Am., 59, May 1969, pp. 547-553.