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CONTROLLABILITY OF NONLINEAR SYSTEMS WITH LINEARLY OCCURRING CONTROLS

by George W. Haynes

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FOREWORD

This report contains the results of the research per-
formed during the past twelve months on Controllability and
the Singular Problem under NASA Contract NAS2-4898, Ames
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I. INTRODUCTION

In 1961, R. Hermann in a remarkable paper [1] , that is obscure
and difficult to read,developed algebraic techniques to solve the
accessibility or controllability problem of control theory, i.e. the
ability to transfer the state from some initial conditions to some
final conditions by suitable choice of the control vector. Hermann's
method is based on the work of Chow [2] which, in turn, is a generaliza-
tion to a system of pfaffians, of a result proven first by Carathéodory
[3] for a single pfaffian describing a thermodynamic process.

Carathéodory showed for a single pfaffian that if there
are some points or states that are inaccessible by trajectories satis-
fying the pfaffian then the pfaffian is integrable; that is, it can be
reduced to a perfect differential by a suitable integrating factor.
The contrapositive of this theorem yields the result that is useful to
the controllability problem; namely, if there are no integral manifolds
to the system then all points are accessible. In fact, Hermann's work
is based on the proposition that all points are accessible, that are
not obviously inaccessible. It is better to consider the integrability
conditions in terms of the dual formulation of a distribution of
vector fields being involutive, rather than use the standard Frobenius
integration theorem [8,9] for the pfaffian system. This is a more
natural formulation of the problem since the vector fields are simply
those defined by the differential equations describing the control system,
furthermore, the integral curves of the vector fields are, in fact,

the trajectories of the control system. If the system or distribution of vector fields form an involution, then all the integral curves lie on an integral manifold; and for the controllability problem this is equivalent to the existence of integrals (i.e. integral manifolds) to the control system independant of the choice of the controls, which is obviously contrary to the notion of controllability.

Demonstrating that a given distribution of vector fields is involutive is a relatively simple algebraic problem involving the Lie Bracket of vector fields. The Lie Bracket of two vector fields is another vector field and geometrically it is the vector field formed by traversing a rectangle of integral curves to the two vector fields. When the distribution is not involutive then at each point of the state space this process generates new directions or in other words the Lie Bracket generates an independent vector field.

Using these techniques, Hermann derived an algebraic criterion for the controllability of linear time varying systems. Subsequent to this Kalman et. al. [4] derived a less useful integral criterion for the complete controllability of linear time varying systems. However, it was Kalman [5,6] who really popularized the concept of controllability by showing that it provided the rationale for many assumptions invoked in the theory of control. The equivalence between the two criteria can be established by the linear dependence of the input/ output functions on the real line, and the various degrees of independence has generated various definitions of controllability.

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Demonstrating that there exists one interval on which these functions are linearly independent is equivalent to Kalman's integral criterion for complete controllability at t_o. If the functions are linearly independent on every interval then the system is said to be completely controllable or totally controllable; if the functions are linearly independent at every point (Hermann's criterion) then the system is uniformly controllable. These results were antedated by the earlier work of LaSalle [7] on the time optimal problem for linear systems. LaSalle not only derived the integral criterion for complete controllability; he also showed how the linear independence of the input/output functions on the real line yielded nontrivial applications of the maximum principle. This is a result of some significance, as pointed out by Hermes [11] , when dealing with linearized versions of nonlinear systems about the totally singular arc where the maximum principle does not yield information about any component of the control.

Hermann's differential geometric approach to controllability has obvious application to nonlinear control system, however, demonstrating that there are no integral manifolds to the control system does not imply that all points or states are accessible. There is a fallacy in Hermann's proposition, and this fallacy is manifest in the fact that the coordinates are not equal. For most physical problems the coordinate system is endowed with one special coordinate, namely time, which has to evolve or if parametrized has to be strictly monotone increasing. This monotonicity invalidates the geometric

interpretation of the Lie Bracket, we can no longer run trajectories "backwards", and it can be demonstrated by means of obvious counter-examples that the nonexistence of integral manifolds does not imply full neighborhoods of attainable points but rather it yields only "one-sided" sets of attainable points.

After the introduction the report is organized into five sections; the second section develops the mathematical concepts of the geometry of manifolds required for the differential geometric approach to controllability as expounded in section three. Section four deals with a special class of nonlinear control systems where the time coordinate is an ignorable coordinate; and global conditions for controllability in terms of noninvolutive systems are derived. In addition, the uniform approximation of trajectories by a given control system to the trajectories associated with the completed control system formed by augmenting the control vectors to include the vector fields generated by the Lie Bracket, is proven. This generalizes some recent work of Kučera. Section five deals with linear systems and developes some new algebraic equivalences with Kalman's integral criterion. Since linear systems are basically involutive, i.e., time is no longer an ignorable coordinate, it is shown that the skew symmetry of the vector fields about the origin implies full neighborhoods of attainable points, rather than one-sided sets of attainable points and this is why the algebraic tests for controllability, despite the monotonicity of the time coordinate, are valid.

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Section six deals with involutive nonlinear systems, and develops some meaningful equivalences between the control actuator vectors defining an involutive distribution and the existence of totally singular vector controls. Since time is no longer an ignorable coordinate the phenomena of one-sided sets of attainability can no longer be ruled out. The singular paradox, or the integrability of the linearized approximating system about the totally singular arc is reviewed together with the need to retain higher order approximations to establish controllability. Finally some nonautonomous versions of the techniques employed in section four are used to determine controllability criteria.

II   MATHEMATICAL PRELIMINARIES AND NOTATION

We shall briefly review some of the concepts and symbolism associated with the geometry of manifolds [9] that is pertinent to the geometric differential approach to controllability. For convenience all manifolds, vector fields, curves, maps, etc., will be assumed to be smooth, that is, differentiable as often as we please. Any exception to this rule will be stated in the text.

All sets, manifolds and spaces will be denoted by upper case letters, and vectors by lower case letters. To avoid the cumbersome problem of suffices, matrix notation will be used throughout and all vectors (including vector operators) will be considered as column vectors. The transpose of a vector or matrix $(\cdot)$ will be denoted by $(\cdot)^T$.

Composition of mappings will be denoted either by $\phi \cdot \psi$ when brevity of notation is required or by the obvious notation $\phi(\psi(\sigma))$. Throughout, we shall assume all manifolds M to be open sets of Euclidean n space E^n, having a fixed coordinate structure $(x_1, x_2, \ldots, x_n)$. If M is a manifold, then at a point $m\epsilon M$ we shall denote by $F(M, m)$ the set of smooth functions with domain a neighborhood $N(m)$ of m.

Let $\gamma(\sigma)$ be a parametrized curve in M, then the directional derivative of a function $f\epsilon F(M, m)$ at a given point m in the direction of the given curve γ gives rise to the notion of a <u>tangent vector</u>.

The curve $\gamma(\sigma)$ generates a tangent vector $\gamma_*(\sigma)$ which maps $F(M, m)$ into the reals E^1 as follows; if $f \epsilon F(M, m)$, $m = \gamma(\sigma)$ then

$$\gamma_*(\sigma)(f) = \frac{d}{d\sigma}(f \cdot \gamma)(\sigma) = \frac{\partial f}{\partial \gamma}(\gamma(\sigma))\frac{d\gamma(\omega)}{d\sigma}$$

which is the directional derivative of f in the direction γ at the point m. Therefore the notion of a tangent vector is simply an association of Euclidean vectors with directional differentiations. If $t^T = \{t_1, \ldots, t_n\}$ is a vector defined at a point $m \epsilon M$, then we may identify t with the operator

$$\left\{t_1\frac{\partial}{\partial x_1} + t_2\frac{\partial}{\partial x_2} + \ldots + t_n\frac{\partial}{\partial x_n}\right\}\Big|_m$$

which does the usual things to sums and products of functions as follows. If t is a tangent vector to M at a point m, then for any functions f, $g \epsilon F(M, m)$ and constants a, $b \epsilon E^1$ we have

i) $t(af + bg) = at(f) + bt(g)$

ii) $t(fg) \quad = t(f)g(m) + f(m)t(g)$

The totality of tangent vectors to M at a point m form a linear space denoted by M_m and is called the _tangent space_ to M at the point m. The dimension of M_m is n, the dimension of M; in fact, if $(x_1, x_2, \ldots, x_n)$ is a coordinate system in a neighborhood N(m) of a point m∈M, then $(D_{x_1}, D_{x_2}, \ldots, D_{x_n})$ is a basis for the tangent space M_m, where D_{x_1} means partial differentiation with respect to the x_i coordinate. Since we are dealing with a fixed coordinate system throughout, we can assume that $(D_{x_1}, D_{x_2}, \ldots, D_{x_n})$ is the basis for the tangent space M_m. Once the basis has been established the tangent vectors can be characterized by the following.

<u>Theorem 2.1</u>   If $(x_1, x_2, \ldots, x_n)$ is a coordinate system at m∈M, t a tangent vector at m, then $t = \sum t_i(x) D_{x_i}(m)$. The notation (m) following a given function or operator implies that the arguments are evaluated at the point m. Using matrix notation this representation of the tangent vector can be written as $t = t(x)^T D_x(m)$ where $t(m)$ is the vector to M at the point m and D_x is the gradient operator expressed as a column vector. Note, $t(x)^T D_x$ is a scalar operator.

A <u>vector field</u>, F, is a function defined on a manifold M which assigns at each point m∈M an element f(m) of the tangent space M_m. Let $(x_1, x_2, \ldots, x_n)$ be the coordinate system for the manifold M, then if F is a vector field defined on the manifold M we may write $F = \sum f_i(x) D_{x_i} = f^T(x) D_x$, where all the components $f_i(x)$ of

the vector function f(x) are real valued. Note once more that $f^T(x)D_x$ is a scalar operator defined for all points of the manifold M; for any smooth scalar function g defined on the manifold M, X(g) is another smooth function defined on the manifold M. With each vector field we can associate an <u>integral curve</u> $\gamma(\sigma)$ so that the vector field at each point on the integral curve $\gamma(\sigma)$ is equivalent to the tangent vector generated by the curve $\gamma(\sigma)$. In other words

$$\gamma_*(\sigma) = X(\gamma(\sigma))$$

so that the integral curves are solutions to the system of ordinary equations

$$\frac{d\gamma(\sigma)}{d\sigma} = f(\gamma(\sigma))$$

which are the characteristic equations to the partial differential equation

$$X(g) = f^T(x)D_x g(x) = 0.$$

If X and Y are smooth vector fields then we can define a smooth

vector field $[X, Y]$ called the <u>Lie Bracket</u> of X and Y. If X and Y have the representation

$$X = f^T(x)D_x; \quad Y = g^T(x)D_x$$

on the manifold M with a coordinate system $(x_1, x_2, \ldots, x_n)$, then the Lie Bracket is defined as

$$[X, Y] = (f^T(x)D_x g^T(x) - g^T(x)D_x f^T(x))D_x.$$

From the above form it is evident that the Lie Bracket is bi-linear with respect to real coefficients and is also skew-symmetric; that is $[X, X] = 0$, or equivalently $[X, Y] = -[Y, X]$.

Let X_1 be system of p $(p \leq dim(M))$ vector fields defined on the manifold M and assume that the system of vector fields is of maximal rank p for all points of the manifold M_i, i.e., rank $[f_1(m), \ldots f_p(m)] = p$ for all $m \epsilon M$. This system of vector fields so defined, can be regarded as a function θ defined on M which assigns to each $m \epsilon M$ a p dimensional linear subspace $\theta(m)$ of the tangent space M_m. The function θ is referred to as a <u>distribution</u>, and we say that a vector field X belongs to the distribution expressed as $X \epsilon \theta$ if for every point m of the manifold M, $X(m) \epsilon \theta(m)$. A distribution θ is involutive if for all vector fields X, Y which

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belong to θ, the Lie Bracket also belongs to the distribution θ, $[X, Y] \epsilon \theta$.

We define the differential of a smooth function $f \epsilon F(M, m)$ at the point $m \epsilon M$ by $df = \sum D_{x_i} f(m) dx_i = (D_x f(m))^T dx$. Each differential of a smooth function can be viewed as a linear function which maps the tangent space M_m into the reals as follows. If t is any tangent vector from the tangent space M_m and is defined by $t = \sum t_i(x) D_{x_i}(m) = t^T(x) D_x(m)$, then $t(f)(m) = t^T(m) D_x f(m)$ is the inner product between the components of the tangent vector t and the components of the differential of the function f, and is a real number. Therefore, the differential of any smooth function f can be regarded as an element of the <u>cotangent space</u> M_m^* which is dual to the tangent space M_m.

If $(x_1, x_2, \ldots, x_n)$ is a coordinate system for the manifold M at m, then the dual space M_m^*, or cotangent space, will have $(dx_1, dx_2, \ldots, dx_n)$ as a basis which is dual to the basis $(D_{x_1}, D_{x_2}, \ldots, D_{x_n})$ for the tangent space M_m.

A <u>differential one form</u> (or pfaffian) ω at a point $m \epsilon M$ is an expression of the form $\omega(m) = \sum a_i(m) dx_i = a^T(m) dx$, and from the previous discussion is a linear function on the tangent space into the reals, and are therefore elements of the cotangent space M_m^*.

It should be noted that every differential one form is not necessarily the differential of a smooth function.

In an analogous manner to the distributions of vector fields, we can speak of _codistributions_ of differential one forms. Let $\omega_i(x)$ be a system of p (p < dimM) differential one forms defined on the manifold M and assume that the system of differential one forms has maximal rank p for all points of the manifold M, i.e., rank $\left[a_1(m), a_2(m), \ldots, a_p(m) \right]$ = p for all mϵM. The codistribution of differential one forms on M can be regarded as a function π defined on M which assigns to each mϵM a p dimensional linear subspace $\pi(m)$ of the cotangent space M_m^*.

For a given codistribution π defined on the manifold M, certain elements of the linear subspace $\pi(m)$ of the cotangent space may be generated from differentials of smooth functions. In this case we can associate _integral manifolds_ with the codistribution as follows. Let N be a submanifold of M defined by $f_i(x) = 0$, (i = 1 . . . s $\leq$ p); then N is an integral manifold of the codistribution π if df$\epsilon\pi$.

III DIFFERENTIAL GEOMETRIC APPROACH TO CONTROLLABILITY

The basis of Hermann's [1] differential geometric approach to controllability is the use of Chow's theorem [2] which relates the accessibility of points to integral curves of a pfaffian system. Chow's theorem in turn, is a generalization to a system of pfaffians the important theorem due to Caratheodory [3] for a single pfaffian. We cite the following contrapositive form of Caratheodory's theorem since it appeals directly to the physical notion of controllability.

<u>3.1 Theorem</u> If the differential one form $\omega(x) = \sum a_i(x)dx_i$, defined on the manifold M with coordinate structure $(x_1, x_2, \ldots, x_n)$, is not integrable then there exists some neighborhood $N(x_o)$ of a given point $x_o \epsilon M$ in which all points are accessible by integral curves $\gamma(\sigma)$ satisfying $\omega(\gamma) = 0$.

This is a significant result since there are well defined procedures to determine whether or not a pfaffian system is integrable; and as previously mentioned theorem 3.1 was extended by Chow to systems of pfaffians or differential one forms. In the application of Chow's theorem to the controllability problem, Hermann's approach is based on the proposition that every point is accessible that is not obviously inaccessible. To prevent some points in $N(x_o)$ from being obviously inaccessible it is evident that we must negate the existence of any integral manifolds to the system of differential one forms. The existence of integral manifolds can be determined

by using Frobenius' integration theorem [9]. Equivalently, the
integrability conditions has a dual formulation in terms of dis-
tribution of vector fields being involutive, and this approach
has more direct appeal since the vector fields can be related
directly to the differential equations describing the control
system.

Let θ be a p-dimensional distribution of vector fields on a
manifold M ($p \leq \dim M$), then the following theorem is standard
for the existence of integral manifolds for involutive distribution.

3.2 Theorem   An involutive distribution θ on M is integrable.
Furthermore, through every mϵM there passes a unique maximal
connected integral manifold of θ and every other connected integral
manifold containing m is an open submanifold of this maximal one.

Hermann applied these results to the controllability problem
as follows. Consider the control system

$$\dot{x}_i = f_i(t, x, u)$$

where the state x is an n vector, the control u is an s vector
($s \leq n$) and the functions f are assumed to be smooth. In the
(n + s + 1) dimensional (t, x, u) space we can associate with
the control system a codistribution of one forms defined by

$$dx_i - f_i(t, x, u)dt = 0$$

The dual space of vector fields is spanned by

$$X = D_t + \sum f_i(t, x, u)D_{x_i}$$

$$Y = D_u$$

It is now a routine matter to demonstrate whether or not this distribution of vector fields is involutive under the Lie Bracket operation and thus determine the existence or nonexistence of an integral manifold. If an integral manifold does exist then it will constitute an integral to the system of differential equations independent of the choice of the controls. This is obviously contrary to the notion of controllability since the only accessible points will be those points on the integral manifold.

The relation of Hermann's proposition regarding the avoidance of obviously inaccessible points as well as the Chow-Caratheodory theorems on integrability and inaccessible points to the converse problem of accessible points follows from the geometrical interpretation of the Lie Bracket of vector fields. The tangent vectors associated with the integral curves of the vector fields do not span the tangent space M_m. However, if the distribution is not involutive

then the tangent vectors associated with the derived system of vector fields under the Lie Bracket operation do span the tangent space M_m. If the tangent vectors span M_m, then all points in some neighborhood $N(x_o)$ can be attained by integral curves of the vector fields, provided we can identify the integral curves with those vector fields that are generated by the Lie Bracket. The following theorem resolves this problem [9].

3.3 Theorem   Let X and Y be smooth vector fields both defined at m∈M. If $\gamma(\sigma)$ denotes the final point obtained by traversing in sequence the integral curves to the vector fields X, Y, -X, and -Y for a fixed parameter σ and an initial point m, then γ has [X, Y] (m) as the limit of its tangents.
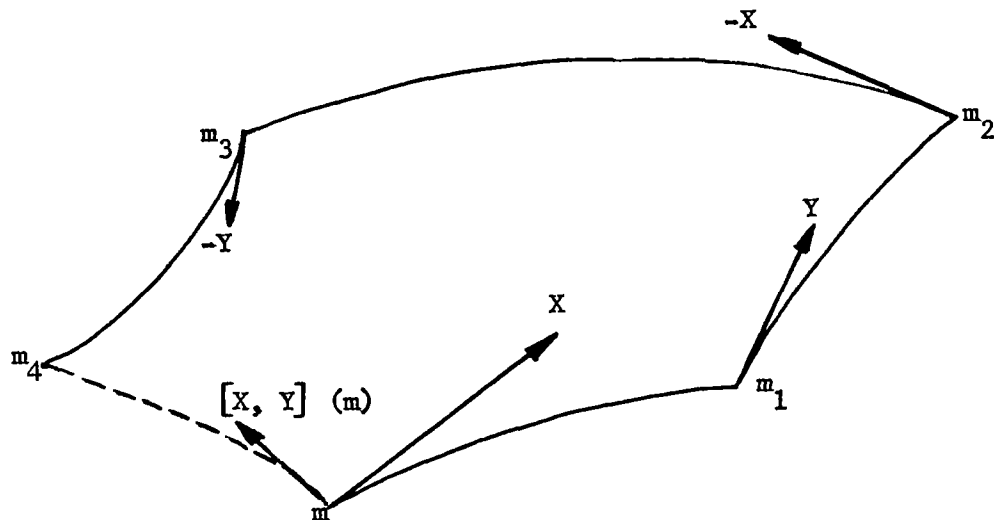

Figure 3.1

<u>Proof</u> Let the vector fields X and Y have the representations

$X = \sum f_i(x)D_{x_i}$, $Y = \sum g_i(x)D_{x_i}$. If $\phi(\sigma)$ and $\psi(\sigma)$ are the integral

curves to the vector fields X and Y respectively so that $\phi_*(\sigma) = X(\phi(\sigma))$,

and $\psi_*(\sigma) = Y(\psi(\sigma))$, then ϕ and ψ satisfy the differential equations

$$\frac{d\phi(\sigma)}{d\sigma} = f(\phi(\sigma)); \frac{d\psi(\sigma)}{d\sigma} = g(\psi(\sigma)).$$

As we traverse a rectangle of integral curves (Figure 3.1) we obtain
the following relations

$$m_1 = \phi(\sigma, m)$$

$$m_2 = \psi(\sigma, m_1)$$

$$m_3 = \phi(-\sigma, m_2)$$

$$m_4 = \psi(-\sigma, m_3)$$

Since we shall compare the point m_4 to the point m_1 for small σ,
we have on expanding the integral curves in a Taylor's series in σ

$$m_1 = m + f(m)\sigma + X(m) \ f(m)\frac{\sigma^2}{2} + 0(\sigma^3)$$

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$$m_2 = m_1 + g(m_1)\sigma + Y(m_1)g(m_1)\frac{\sigma^2}{2} + O(\sigma^3)$$

$$m_3 = m_2 - f(m_2)\sigma + X(m_2)f(m_2)\frac{\sigma^2}{2} + O(\sigma^3)$$

$$m_4 = m_3 - g(m_3)\sigma + Y(m_3)g(m_3)\frac{\sigma^2}{2} + O(\sigma^3)$$

Expanding these terms about the point m and only retaining terms in σ^2 or lower, yields

$$m_1 = m + f(m)\sigma + X(m)f(m)\frac{\sigma^2}{2}$$

$$m_2 = m + (f(m) + g(m))\sigma + (X(m)f(m) + Y(m)g(m) + 2X(m)g(m))\frac{\sigma^2}{2}$$

$$m_3 = m + g(m)\sigma + (2X(m)g(m) + Y(m)g(m) - 2Y(m)f(m))\frac{\sigma^2}{2}$$

$$m_4 = m + (2X(m)g(m) - 2Y(m)f(m))\frac{\sigma^2}{2}$$

Therefore the curve generated by the rectangle of integral curves of the vector fields X and Y is, for small σ, given by

$$\gamma(\sigma) = m_4 - m = (X(m)g(m) - Y(m)f(m))\sigma^2$$

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The relation between the integral curve γ and the Lie Bracket is obvious since

$$[X, Y] = \sum_i \{X(x)g_i(x) - Y(x)f_i(x)\} D_{x_i}$$

The Lie Bracket creates a second order tangent rather than a first order tangent since

$$\frac{d\gamma(o)}{d\sigma} = 0$$

Therefore for any function h we have

$$2[X, Y]h(m) = \frac{d^2}{d\sigma^2} (h \cdot \gamma)(o)$$

This geometrical interpretation of the Lie Bracket gives insight into the local attainability of points. Traversing a one parameter family of rectangles whose sides are tangent to a distribution might yield locally a curve whose tangent is not in the distribution. When the distribution is not involutive, we can generate an independent set of tangent vectors which span the manifold and implies local attainability of points by integral curves to the distribution.

It would appear at this stage that the controllability problem is completely solved; however, there is a fallacy in Hermann's proposition that all points are accessible that are not obviously inaccessible. Consider the following example,

$$\dot{x}_1 = x_2^2$$

$$\dot{x}_2 = u$$

The distribution of vector fields are given by

$$X = D_t + x_2^2 D_{x_1} + u D_{x_2}$$

$$Y = D_u$$

Application of the Lie Bracket yields

$$[X, Y] = -D_{x_2}$$

$$[X, [X, Y]] = 2x_2 D_{x_1}$$

Therefore provided $x_2 \neq 0$, these four tangent vectors span the

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(t, x_1, x_2, u) space. Hence, there are no integral manifolds to
the control system, and this would indicate that all points are
accessible. Despite this fact, it is apparent that the control
system is not controllable since the solution $x_1(t)$ for any control
u has to be monotone increasing. This example illustrates the
fallacy, which is that we deal with a coordinate structure that is
endowed with one special coordinate, namely time, which has to evolve
or if parametrized it must be monotone increasing. In the above
example if the time could be reversed then the control system
would be controllable. Therefore, due to the monotonicity of
time we can only speak of "one-side" sets of attainable points
in state space rather than full neighborhoods of attainable points.
Obviously, the use of the Chow-Caratheodory theorem is necessary
to the controllability problem to establish the nonexistence of
integral manifolds, so that the dimension of the "one-sided"
attainable sets is equal to the dimension of the state space.

There is a class of control systems however where time is
an ignorable coordinate and the Chow-Caratheodory Theorem can
be applied to yield global conditions for controllability.

IV. NONINVOLUTIVE CONTROL SYSTEMS

In this section we shall consider control systems of the form

$$\dot{x}(t) = B(x(t))u(t) \qquad\qquad 4.1$$

where x is an n vector and $B(x)$ an n x r matrix with columns denoted by $b_1(x), \ldots, b_r(x)$. We shall assume that the components of $B(\cdot)$ are smooth functions. The control vector u will always be assumed Lebesgue measurable; of particular interest will be the case where its values lie in a bounded set of Euclidean r dimension space E^r. Following Chow, we shall say the system (4.1) has <u>rank r at m</u> if the matrix B has rank r in every neighborhood of the point m. A point m is <u>regular</u> for the system (4.1), if the system has rank r at m and rank $B(m) = r$.

We now adjoin to B any linearly independent vectors formed by applying the Lie Bracket operation to the column vectors of B. Continuing this procedure we obtain the <u>derived</u> or <u>completed</u> system $\widetilde{B}$ associated with B. The columns of $\widetilde{B}$ will also be denoted by $b_1(x)$, $\ldots, b_s(x)$ where the rank, s, of the completed system at the point m satisfies $r \leq s \leq n$ if r is the rank of B at x. The integer s - r is called the <u>index</u> of B.

With $\widetilde{B}$ we associate the completed system of differential equations

$$\dot{x} = \widetilde{B}(x)\widetilde{u} \qquad\qquad 4.2$$

where $\tilde{u}$ is now an s dimensional control vector. For this system Chow's results give sufficient conditions that the set of points attainable by solutions of (4.2), starting from initial data $x(0) = x_o$, form an s dimensional manifold. It also follows that all points on this manifold can be attained by solutions of (4.1), starting at x_o and having controls with values at time t in the set of the first r co-ordinate vectors. This generalizes some recent wotk of Kučera [10].

The system of partial differential equations associated with the vector fields described by B are $B^T(x)D_x f = 0$; the i<u>th</u> equation has the form $b_i^T(x)D_x f = 0$, with $\dot{x} = b_i(x)$ as its characteristic equation. One should note that the i<u>th</u> characteristic equation of $B^T(x)D_x f = 0$ may be obtained from the control system (4.1) by placing the i<u>th</u> compo-nent of the control vector u equal to 1 and all other components equal to zero.

Now the results of Chow [2] pertain to points attainable by "piecing together" the characteristic solutions or integral curves of the vector fields as elucidated in Theorem 3.3 for the geometrical interpretation of the Lie Bracket. It is of fundamental importance to note that the Chow formulation allows the solutions to the charac-teristic equations to be considered with decreasing time, as well as in-creasing time. Thus if ϕ is a piecing together of characteristic solutions such that in some time interval I_i, ϕ is a solution of the i<u>th</u> characteristic equation, we only know that $\dot{\phi}(t) = \pm b_i(\phi(t))$, $t \in I_i$. However, this presents no problem for the control system (4.1),

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since the minus sign may be obtained by merely taking a control with -1 as its _ith_ component and all other components zero.

For $1 \leq i \leq r$, let $e_i \in E^r$ (real r dimensional Euclidean space) have a one in its _ith_ component and all other components zero. Define

$$V = \left\{ \pm e_1, \ldots, \pm e_r \right\} \subset E^r$$

and $\quad U = \left\{ u \text{ measurable; } u(\tau) \in V, \ \tau \geq 0 \right\}$.

Then a solution ϕ of the control system (4.1) corresponding to a control $u \in U$ is a piecing together of characteristic solutions in the sense of Chow. With the above in mind, we may combine theorems B and C of Chow [2] (See also Herman[1]) as follows.

Theorem 4.1   Let x_o be a regular point for the control system (4.1) and its completed system (4.2), and assume that the rank of B is r and the rank of $\widetilde{B}$ is s at x_o. Then there exists an s dimensional manifold M^s through x_o such that all points on this manifold are attainable by solutions of (4.1) with initial data $x(0) = x_o$ and control $u \in U$. Furthermore, given a sufficiently small neighborhood of x_o, the only points attainable by such solution of (4.1), which remain in the neighborhood, are points of M^s.

The following example illustrates the need for both control systems to be regular at the point x_o.

Example 4.1   Consider the three dimensional system $\dot{x} = B(x)u$ where

$$B(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & x_1 x_3 \end{pmatrix}$$

The point $x_o^T = (0,0,0)$ is regular for the system (4.1) since rank $B(x_o) = 2$. However, the completed system (4.2) is

$$[b_1, b_2] = \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix}, \qquad \widetilde{B}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x_1 x_3 & x_3 \end{pmatrix}$$

and we see that the completed system has rank three at x_o, (i.e., $\widetilde{B}(x)$ has rank three in every neighborhood of x_o) but x_o is not regular for the completed system since rank $\widetilde{B}(x_o) = 2$. As a result, all solution of the completed system $\dot{x} = \widetilde{B}(x)\widetilde{u}$ starting from x_o cannot leave the plane $x_3 = 0$; in other words the manifold of attainability from x_o has dimension two.

The second example we shall consider illustrates that if one does not restrict the solutions to lie in a small neighborhood of x_o, the last statement of theorem 4.1 need not be valid.

Example 4.2   Once again we shall consider a three dimensional system $\dot{x} = B(x)u$ where $B(x)$ is a 3 x 2 matrix with elements

$$b_{11} = \begin{cases} 1 - e^{(1 - |x|^2)} & \text{if } |x| \geq 1 \\ \\ 0 & \text{if } |x| \leq 1 \end{cases} \quad , \ b_{12} = 0 \ , \quad b_{13} = 1$$

and $\quad b_{21} = 0, \ b_{22} = 1, \ b_{23} = 0,$

where $|\cdot|$ denotes the Euclidean length of a vector. All points x are regular for the control system since rank $B(x) = 2$. If $|x| < 1$, $[b_1(x), b_2(x)] = 0$. If $|x| > 1$, $[b_1(x), b_2(x)]^T = (2x_2 e^{(1-|x|^2)}, 0, 0)$ if $|x| > 1$ and $x_2 \neq 0$ then x is regular for the completed control system.

Consider $x_2(0) \neq 0$ and $|x_o| < 1$, then $\widetilde{B}(x) = B(x)$ and x_o is regular for both control systems. In this case the integral manifold, M^2, of theorem 4.1 is the intersection of the unit ball (origin centered) with the plane $x_1 = x_1(0)$. If we choose a neighborhood of x_o, contained in the unit ball, the only points attainable by trajectories of the original system which remain in this neighborhood, are points on this plane. However, without this restriction, all points in some neighborhood of x_o may be attained by trajectories of the system with controls $u \epsilon U$. This occurs even though the unit ball is foliated by leaves $\{x : x_1 = \text{constant}\}$ since we may exit the ball on the leaf $x_1 = x_1(0)$, then move on an arbitrary path in the half space $x_2 > 0$ and re-enter the ball on a different leaf to reach point near x_o.

Motivated by this example, we introduce another concept of controllability for a general control system

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{4.3}$$

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Definition 4.1   The system (4.3) is _locally-locally controllable at_

x_o if given any $\epsilon > 0$ there exists a $\delta > 0$ such that all points of the

δ neighborhood of x_o are accessible by trajectories emanating from x_o

which do not leave the ϵ neighborhood.   Obviously $\delta \leq \epsilon$.

Definition 4.2   The system (4.3) is _globally-locally controllable_

_at x_o_ if all points in some neighborhood of x_o are accessible by

trajectories emanating from x_o.

In terms of these definitions we note that if in example 4.2,

$|x_o| < 1$, the system is not locally-locally controllable at x_o. However,

with $|x_o| < 1$ and $x_2(0) \neq 0$, the system is globally locally controllable

at x_o.   It is interesting to compare these two notions with that of

complete controllability, i.e., any two points can be joined by a

solution.   For example, the "Bushaw problem"

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + u$$

is completely controllable, but if $x_o \neq 0$ then the system is not

locally-locally controllable at x_o.   On the other hand, complete con-

trollability certainly implies global-local controllability.

Suppose that the system (4.1) has rank r at x_o, and the completed

system (4.2) has rank s at x_o and furthermore, x_o is regular for both

systems. Then it is a consequence of theorem 4.1 that a necessary condition for the system (4.1) to be locally-locally controllable at x_o is that $s = n$. To show that it is a sufficient condition also, requires that we can approximate the trajectories of the completed system (4.2) by trajectories of the original system (4.1).

To prove this result we shall assume the rank and regularity condition hold throughout. The tangent space to the manifold M^s of points accessible from x_o is spanned by $b_1(x), \ldots, b_s(x)$ for all x in a neighborhood of x_o. Thus if ψ is a smooth function satisfying $\psi(0) = x_o$, $\dot{\psi}(t)\epsilon$ span $\{b_1(\psi(t)), \ldots, b_s(\psi(t))\}$ for all $t \geq 0$, then $\psi(t)$ describes a curve on M^s. Let

$$\widetilde{U} = \{\widetilde{u} \text{ measurable: } \widetilde{u}(t)\epsilon E^s, \ |\widetilde{u}(t)| \leq 1, \ t \geq 0\}.$$

Then clearly a solution of the completed control system (4.2), with control $\widetilde{u}\epsilon\widetilde{U}$ and initial conditions $x(0) = x_o$, describes a curve on M^s. We now have to show that such a solution may be uniformly approximated (on a compact time interval) by a solution of the original control system (4.1), however the magnitude of the control required to do this may be very large.

Theorem 4.1 shows that all points on M^s are attainable by solution of the system (4.1) even with controls $u\epsilon U$. Therefore, it is natural to attempt to approximate a solution ψ of the completed system (4.2) on a compact interval $[0,T]$ by finding a solution ϕ of

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the system (4.1) which agrees with ψ at many points, i.e., say
$\psi(kT/m) = \phi(kT/m)$ for m a large integer and k = 0, 1, . . . , m.

The major difficulty that occurs in doing this is to show that the time it takes to reach an arbitrary point on M^S near x_o by a solution of (4.1) tends to zero as the distance of the point from x_o tends to zero. This will be the purpose of the next three lemmas.

Lemma 4.1   If the index of B is q, the highest number, p, of bracket operations needed to obtain any vector of the completed system $\tilde{B}$ is the qth term of the sequence, a_o, a_1, . . . , where $a_o = 0$, $a_1 = 1$, $a_k = 1 + a_{k-1} + a_{k-2}$.

Proof   If the index is zero, then clearly p = 0. If the index is one then p = 1. If the index is two, then p = 2. For an index of three, we may, in the worst case, have to form the bracket of the element in the complete system of rank r + 2 involving two brackets with the element of the incomplete system of rank r + 1 involving one bracket. From this it follows that for index 3, p = 2 + 1 + 1 = 4; and inductively the kth term of the sequence a_o, a_1, . . . , is $a_k = a_{k-1} + a_{k-2} + 1$.

Lemma 4.2   Let $\zeta_i(\cdot)$ denote a solution of $\dot{x} = b_i(x)$, $x(0) = x_o$ where b_i is obtained from b_1, . . . , b_r by p bracket operations. Then there exists a control $u \epsilon U$ such that the corresponding solution

$\phi(\cdot, u)$ of the system (4.1) satisfies

(i) $\phi(4^p\tau; u) - x_o = \tau^{p+1}b_i(x_o) + 0(\tau^{p+2})$ as $\tau \longrightarrow 0$

(ii) $\phi(4^p t^{1/1+p}; u) - \zeta_i(t) = 0(t^{p+2/p+1})$ as $t \longrightarrow 0.$

<u>Proof</u>   If b_i is one of the set $b_1, \ldots, b_r$ then $p = 0$ and (i) merely states that there exists a $u \epsilon U$ (in this case $u = e_i$) such that $\phi(\tau, e_i) - x_o = \tau b_i(x_o) + 0(\tau^2)$ as $\tau \longrightarrow 0$ which is obvious. Also (ii) merely reduces to $\phi(\cdot, e_i) = \zeta_i(\cdot)$. The proof proceeds by induction, however, the general step is similar to the case $p = 1$ thus for clarity of presentation and simplicity of notation we will present only this argument.

Suppose b_i is obtained by the use of one bracket operation, i.e., $b_i = [b_j, b_k]$. Let $T_j(t)y$ denote the solution at time t, of $\dot{x} = B(x)e_j = b_j(x)$ with $x(0) = y$, and $1 \leq j \leq r$. From the geometric interpretation of the Lie bracket (Theorem 3.3),

$$T_k(-t)T_j(-t)T_k(t)T_j(t)x_o - x_o = t^2[b_j(x_o), b_k(x_o)] + 0(t^3) \text{ as } t \longrightarrow 0$$

Let u be defined by

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$$u(\ ) = \begin{cases} e_j & \text{if } \tau \epsilon\ [0,t] \\ e_k & \text{if } \tau \epsilon\ [t,2t] \\ -e_j & \text{if } \tau \epsilon\ [2t,3t] \\ -e_k & \text{if } \tau \epsilon\ [3t,4t] \end{cases}$$

Then $\phi(4t;\ u) = T_k(-t)T_j(-t)T_k(t)T_j(t)x_o$. Let $\int_i(\cdot)$ denote the

solution of $\overset{\cdot}{x} = b_i(x)$, $x(0) = x_o$; then $\int_i(t) - x_o = t[b_j(x_o),\ b_k(x_o)] +$

$0(t^2)$. Since $\phi(4t;\ u) - x_o = t^2[b_j(x_o),\ b_k(x_o)] + 0(t^3)$ as $t \longrightarrow 0$,

equation (i) and (ii) follow easily for the case $p = 1$. The results

for arbitrary p, follow in this manner by induction.

<u>Lemma 4.3</u>   Let $\psi(\cdot)$ be a solution of (4.2) corresponding to a control

$\widetilde{u}\epsilon\widetilde{U}$ and initial data $x(0) = x_o$. Then there exists a solution $\phi(\cdot,\ u)$

of (4.1) with control $u\epsilon U$ and a $\tau = \tau(t_o)$ such that $\phi(\tau,\ u) = \psi(t_o)$ and

$\tau(t_o) \longrightarrow 0$ as $t_o \longrightarrow 0$.

<u>Proof</u>   Let $T_i(\cdot)y$ denote a solution of $\overset{\cdot}{x} = b_i(x)$, $x(0) = y, 1 \leq i \leq s$.

Since $b_1(x_o)$, . . . , $b_s(x_o)$ span the tangent space to the attainable

manifold M^s, at x_o, we may view the curves $T_i(\cdot)x_o$ as defining

local coordinates on M^s in a neighborhood of x_o.

Since x_o is a regular point for both systems (4.1) and (4.2) then

x is also regular for both systems if x is in a sufficiently small

neighborhood of x_o. Let t_o be sufficiently small so that $\psi(t_o)$ is

in this neighborhood. Then we may write

$$\psi(t_o) = T_s(t_s)T_{s-1}(t_{s-1}) \cdot \cdot \cdot T_1(t_1)x_o$$

and each $t_i \longrightarrow 0$ as $t_o \longrightarrow 0$.

From lemma 4.2, for each $1 \leq i \leq s$ we may find a control $u_i \epsilon U$ and a time $\tau(t_i)$ such that the corresponding solution $\phi(\cdot, u_i)$ through an initial point y satisfies $\phi(\tau(t_i), u_i) - T_i(t_i)y = 0(t_i^{p+2/p+1})$ as $t_i \longrightarrow 0$. To simplify notation we shall denote $\phi(\tau(t_i), u_i)$ by $S_i(\tau_i)y$.

Using lemma 4.2 successively s times we may "follow the coordinate rectangle" to $\psi(t_o)$ arbitrarily closely by trajectories of system (4.1), i.e.

$$S_s(\tau_s)S_{s-1}(\tau_{s-1}) \cdot \cdot \cdot S_1(\tau_1)x_o - T_s(t_s)T_{s-1}(t_{s-1}) \cdot \cdot \cdot T_1(t_1)x^o =$$

$$S_s(\tau_s) \cdot \cdot \cdot S_1(\tau_1)x_o - \psi(t_o) = 0(t_o) \text{ as } t_o \longrightarrow 0 \text{ where}$$

each $\tau_i = \tau_i(t_i), \tau_i(t_i) - t_i = 0(t_i)$ and $t_i \longrightarrow 0$ as $t_o \longrightarrow 0$.

Consider the map $h:E^s \longrightarrow M^s$ defined by $h(\tau_1, \ldots, \tau_s) = S_s(\tau_s) \ldots S_1(\tau_1)x_o$. Since the complete system of vectors $\dot{\zeta}_i(0) = b_i(x_o)$ $i = 1, 2, \ldots, s$ span the tangent space to M^s at x_o, using (ii) of lemma 4.2, it follows that the Jacobian of the map is non-singular. Thus, h is locally onto and the implicit function theorem applies to show that by slightly varying the times $\tau_1, \ldots \tau_s$ to

times $\tau_1^{'}, \ldots, \tau_s^{'}$ one may assume that $S_s(\tau_s^{'}) \ldots S_1(\tau_1^{'})x_o = \psi(t_o)$.

Also, each $\tau_i^{'} \longrightarrow 0$ as $t_o \longrightarrow 0$. We may now "piece together" a control $u \epsilon U$ in the obvious way such that its corresponding solution through x_o gives $\phi(\sum_1^s \tau_i^{'} ; u) = S_s(\tau_s^{'}) \ldots S_1(\tau_1^{'})x_o = \psi(t_o)$.

Let $\tau = \tau(t_o) = \sum_1^s \tau_i^{'}$. Then $\phi(\tau; u) = \psi(t_o)$ and $\tau(t_o) \longrightarrow 0$ as $t_o \longrightarrow 0$ as required.

<u>Theorem 4.2</u>   (Uniform approximation of a solution of (4.2) by a solution of (4.1))   Let ψ be any solution of the completed control system (4.2) with initial data $x(0) = x_o$ and $\widetilde{u} \epsilon \widetilde{U}$, on an interval $[0,T]$. Then given any $\epsilon > 0$ there exists a solution ϕ of (4.1) corresponding to initial data $x(0) = x_o$ and some bounded measurable control u, such that $\max_{0 \leq t \leq T} |\phi(t) - \psi(t)| < \epsilon$.

<u>Verification</u>   We first note that if $\phi(\cdot, u)$ denotes a solution of (4.1) with control u, then for any real α, $\phi(\alpha t, u) = \phi(t, \alpha u)$ for all t.

Let $N(\epsilon, \psi)$ denote a compact ϵ neighborhood of $\{\psi(t): 0 \leq t \leq T\}$ and let $\beta = \max_{1 \leq i \leq s} \{|b_i(x)| : x \epsilon N(\epsilon, \psi)\}$. Note that with $|u| \leq 1$, if $\phi(\cdot, u)$ is a solution of system (4.1) then $|\phi(t, u) - \psi(t)| \leq \epsilon$ on $[0, \mu]$ if $2\mu\beta < \epsilon$. The factor two is needed since $\overset{\bullet}{\psi}$ and $\overset{\bullet}{\phi}$ may have opposite directions.

For any integer k, consider $\psi(T/k)$. By the previous lemma there is a control $u \epsilon U$ and a τ_1 such that the corresponding solution ϕ of

the system (4.1) satisfies $\phi(\tau_1, u) = \psi(T/k)$ and we may choose k large

enough so that $\tau_1 < \mu$ (i.e. here we need $\tau_1 \longrightarrow 0$ as $T/k \longrightarrow 0$). Then

there exists an $\alpha > 0$ such that $\alpha T/k = t_1$, hence $\phi(t_1, u) = \psi(T/k) =$

$\phi(\alpha T/k, u) = \phi(T/k, \alpha u)$. Since $t_1 < \mu$, $|\phi(t, \alpha u) - \psi(t)| < \epsilon$

for $0 \leq t \leq T/k$. Now the solutions ϕ, ψ agree at T/k; we may repeat the

procedure with x_o replaced by $\psi(T/k)$ and obtain the result for $[0, 2T/k]$,

etc. The approximation procedure is best illustrated by the following

example; however, we must first cite one important consequence of

theorem (4.2).

<u>Corollary 4.1</u>   If each point $x \epsilon E^n$ is regular for both systems

(i.e, (4.1) and (4.2)) and rank $\widetilde{B}(x) = n$, then the system (4.1) is

completely controllable and locally-locally controllable at every point.

Furthermore, if ψ is any continuously differentiable map, $\psi: [0,1] \longrightarrow E^n$

and $\epsilon > 0$, there exists a bounded measurable control u such that the

corresponding solution $\phi(\cdot, u)$ of (4.1) satisfies $\max_{0 \leq t \leq 1} |\psi(t) - \phi(t)| < \epsilon$.

<u>Proof</u>   Clearly it suffices to prove the last statement. Let ψ be a

continuously differentiable map $\psi: [0, 1] \longrightarrow E^n$. Since rank $\widetilde{B}(x) = n$

for all x, define $v(t) = \widetilde{B}^{-1}(\psi(t)) \dot{\psi}(t)$. Then $\psi(t)$ satisfies

$\dot{\psi}(t) = \widetilde{B}(\psi(t))v(t)$ and the desired result follows as in theorem 4.2.

It should be noted that rank $\widetilde{B}(x) = n$ implies that the elements

of B(x) do not generate an involutive distribution of vector fields,

and since in this case systems of the form (4.1) are globally controll-

able, we have appropriately called them noninvolutive control systems.

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The following example illustrates the uniform approximation of a trajectory of the completed system (4.2) by a trajectory of the original system (4.1).

<u>Example 4.3</u>   Let $B(x)$ be a 3 x 2 matrix with columns $b_1(x)$ and $b_2(x)$ defined by

$$b_1(x)^T = (0 \, , \, 1 \, , \, \frac{1}{1 + (1 + x_1)^2})$$

$$b_2(x)^T = (1 \, , \, 0 \, , \, 0)$$

The completed system has an additional vector $b_3(x)$ generated by the Lie Bracket as

$$b_3(x)^T = [b_1(x), \, b_2(x)]^T = (0 \, , \, 0 \, , \, \frac{-2(1 + x_1)}{(2 + 2x_1 + x_1^2)^2})$$

The solution ψ of the completed system which we will approximate will be for $\tilde{u}^T = (0, 0, -1)$ and initial conditions $x_o^T = (0, 0, 0)$ thus $\psi(t)^T = (0, 0, t/2)$. If $T_i(t)y$ denotes the solution of $\dot{x} = b_i(x)$, $x(0) = y$, $i = 1, 2$; we know from the interpretation of the Lie Bracket that we should expect to approximate $\psi(t)$ by $T_2(-t)T_1(-t)T_2(t)T_1(t)x_o$.
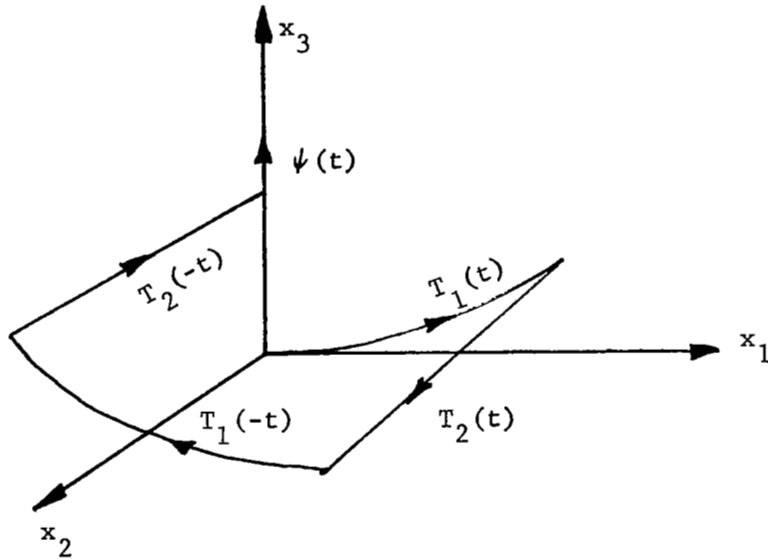
Figure 4.1

One may note that by varying the magnitude of the control vector u, we may vary the speed of traversing a solution of $\dot{x} = b_i(x)$. Let $u_1(\alpha)^T = (\alpha, 0)$ and $u_2(\alpha)^T = (0,\alpha)$ and define

$$u(t) = \begin{cases} u_1(\alpha) & \text{if } 0 \leq t \leq \gamma \\ u_2(\alpha) & \text{if } \gamma < t \leq 2\gamma \\ -u_1(\alpha) & \text{if } 2\gamma < t \leq 3\gamma \\ -u_2(\alpha) & \text{if } 3\gamma < t \leq 4\gamma \end{cases}$$

for $\alpha, \gamma > 0$. Let $\phi(\cdot, u)$ denote the solution of $\dot{x} = B(x)u$ for the choice of $u(t)$ and initial data $x_o^T = (0, 0, 0)$. Note that $\phi(\gamma, u) = T_1(\alpha\gamma)x_o$

$\phi(2\gamma, u) = T_2(\alpha\gamma)T_1(\alpha\gamma)x_o$, $\phi(3\gamma, u) = T_1(-\alpha\gamma)T_2(\alpha\gamma)T_1(\alpha\gamma)x_o$ and

$\phi(4\gamma, u) = T_2(-\alpha\gamma)T_1(-\alpha\gamma)T_2(\alpha\gamma)T_1(\alpha\gamma)x_o$.

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Calculating the actual solution yields

$$\phi(\gamma, u) = (0, \alpha\gamma, \frac{\alpha\gamma}{2})$$

$$\phi(2\gamma, u) = (\alpha\gamma, \alpha\gamma, \frac{\alpha\gamma}{2})$$

$$\phi(3\gamma, u) = (\alpha\gamma, 0, \frac{2(\alpha\gamma)^2 + (\alpha\gamma)^3}{4 + 4\alpha\gamma + 2(\alpha\gamma)^2})$$

$$\phi(4\gamma, u) = (0, 0, \frac{2(\alpha\gamma)^2 + (\alpha\gamma)^3}{4 + 4\alpha\gamma + 2(\alpha\gamma)^2})$$

as illustrated on figure 4.1. Now suppose we wish an $\epsilon > 0$ uniform approximation to $\psi(\cdot)$ where we take for $y \epsilon E^3$, $||y|| = \sum_1^3 |y_i|$ and $0 < \epsilon < 1$. Our object is to choose α and γ so that $\phi(4k\gamma, u) = \psi(4k\gamma)$ for $k = 0, 1, \ldots$ and $||\phi(t, u) - \psi(t)|| \leq \epsilon$ for all t.

Let $\alpha\gamma = \epsilon$. Then $|\phi_1(t_1 u) - \psi_1(t)| \leq \epsilon$, $|\phi_2(t, u) - \psi_2(t)| \leq \epsilon$ for $0 \leq t \leq 4\gamma$, and $|\psi_3(4\gamma) - \phi_3(4\gamma, u)| = |2\gamma - \frac{\epsilon^2(2+\epsilon)}{4 + 4\epsilon + 2\epsilon^2}|$.

Choose $\gamma = \frac{\epsilon^2[2+\epsilon]}{8 + 8\epsilon + 4\epsilon^2}$ so that $|\psi_3(4\gamma) - \phi_3(4\gamma, u)| = 0$ and obviously $|\psi_3(t) - \phi_3(t, u)| \leq \epsilon$ for $0 \leq t \leq 4\gamma$. This choice of γ gives $\gamma = \frac{8 + 8\epsilon + 4\epsilon^2}{\epsilon[2 + \epsilon]}$; since α determines the "speed" with which we move along the solution ϕ, we see that for small ϵ, γ is small (many switches)

and α is large.

The above choices of α and γ, therefore, yield $\psi(4k\gamma) - \phi(4k\alpha;u)=0$, $k = 0, 1, 2, \ldots$ and $||\psi(t) - \phi(t; u)|| \leq \epsilon$ for all t.

V   LINEAR SYSTEMS

The previous section has shown that global controllability
conditions can be derived for nonlinear systems with the control
appearing linearly, provided that the control actuator vectors do
not form an involutive distribution of vector fields. Therefore
in the remaining sections of this report we shall confine our
attention to those cases where the control actuator vectors when
viewed as vector fields are involutive and the exceptional "time"
coordinate is no longer an ignorable coordinate. The most obvious
case which we shall treat first are linear systems which can be
characterized as being involutive and for which a large body of
theory exists.

As an example of the differential geometric approach to con-
trollability, Hermann derived the following algebraic test for
the controllability of the linear system.

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad\qquad 5.1$$

<u>Theorem 5.1</u>   If the rank of $B(t)$, $\left[B(t), \ldots, \Gamma^{n-1}B(t)\right]$,
where $\Gamma = A(t) - D_t$, is n for each t then every point of the x-space
is accessible from the origin on paths that are solutions of the
linear system (5.1) for some choice of the control u(t).

The proof of this theorem is fairly trivial and proceeds as follows. With the vector fields defined by

$$X = D_t + (A(t)x + B(t)u)^T D_x$$

$$Y = D_u$$

successive application of the Lie Bracket yields

$$[Y, X] = B(t)^T D_x$$

$$[X, [Y, X]] = (\Gamma B(t))^T D_x$$

$$[X, [X, [Y, X]]] = (\Gamma^2 B(t))^T D_x \quad \text{etc.}$$

The distribution is not involutive if rank $[B(t), \Gamma B(t), \ldots, \Gamma^{n-1}B(t)]$ is n, which completes the proof.

Subsequent to this result, Kalman et. al. [4], derived the following integral test for controllability.

Theorem 5.2   The linear system (5.1) is <u>completely controllable</u> at t_o if and only if there exists a $t_1 > t_o$ such that $W(t_o, t_1)$

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is nonsingular, where $W(t_o, t_1)$ is the n x n matrix defined by

$$W(t_o, t_1) = \int_{t_o}^{t_1} \phi(t_o, t)B(t)B^T(t)\phi^T(t_o, t)dt$$

and $\phi(t, t_o)$ is the fundamental solution to the homogeneous differential equation.

Central to the proof of this theorem is the demonstration that there exists one interval $[t_o, t_1]$ on which the functions $\phi(t_o, t)B(t)$ are linearly independent. The reason for this is obvious. If the functions $\phi(t_o, t)B(t)$ are not independent on any interval, then this implies the existence of a constant vector c such that

$$c^T\phi(t_o, t)B(t) = 0$$

for all t. This, in turn, implies that the control system

$$\dot{y} = \phi(t_o, t)B(t)u$$

derived from (5.1) by the nonsingular transformation ϕ is not controllable since the integral manifolds would be given by $c^T y$.

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The equivalence between Hermann's algebraic test and Kalman's integral test follows from the demonstration of the linear independence of the functions $\phi(t_o, t)B(t)$. If we assume that $A(t) \epsilon C^{n-2}$ and $B(t) \epsilon C^{n-1}$, then by formally differentiating the expression $c^T \phi(t_o, t)B(t)$ $(n-1)$ times, we obtain Hermann's algebraic test on negating the existence of the constant vector c. In fact, there is an equivalence between this method and the differential geometric method of showing that the distribution is not involutive. However, the algebraic test implies that the functions $\phi(t_o, t)B(t)$ are linearly independent for all points of the interval $[t_o t_1]$. Kalman's integral test on the other hand requires that we find <u>one</u> interval where this is true for the linear system to be completely controllable at t_o. Having found one such interval $[t_o, t_1]$ on which the functions are linearly independent, there can exist subintervals of this interval on which the functions are not independent and integral manifolds exist. Since the integral manifolds are different on each subinterval, otherwise the functions would not be independent, it follows that the integral manifolds must span the manifold M for points to be accessible over the interval $[t_o, t_1]$. The algebraic test, therefore, constitutes only a sufficient condition for complete controllability of linear systems. The various ways in which the linear dependence of real functions on specified

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intervals can be defined has generated a whole hierarchy of controllability definitions. To cite a few of the more pertinent definitions, we have following Silverman and Meadows [15]. If $Q(t) = [B(t), \Gamma B(t), \ldots, \Gamma^{n-1} B(t)]$, then the system (5.1) is <u>totally controllable</u> on the interval $[t_o, t_1]$ if $Q(t)$ has rank n on every subinterval of $[t_o, t_1]$. This is equivalent to <u>complete controllability</u>. The system (5.1) is said to be <u>uniformly controllable</u> on the interval $[t_o, t_1]$ if $Q(t)$ has rank n for all $t \epsilon [t_o, t_1]$. Some equivalences are possible; for example, Chang [16] has proven that the algebraic test for <u>complete controllability at t_o</u> is fully equivalent to the integral criterion provided the matrices $A(t)$ and $B(t)$ are analytic. The equivalence is established in this case by showing that we cannot piece together integral manifolds to span the manifold since if any row or combination of rows of the matrix functions $\Phi(t_o, t) B(t)$ are zero on some interval, then they are zero everywhere by the analyticity condition.

The following algebraic tests are fully equivalent to Kalman's integral test for controllability.

<u>Theorem 5.3</u> Consider the linear system $\dot{y} = H(t)u$ where $H(t)$ is an n x r matrix composed of C^{n-1} elements. This system is completely controllable at t_o if and only if there exists n times $t_1, \ldots, t_n \geq t_o$ such that

rank $\left[H(t_1), \dot{H}(t_2), \ddot{H}(t_3), \ldots H^{(n-1)}(t_n) \right]$ is n.

Proof. To show sufficiency we prove the contrapositive. If the linear system is not completely controllable at t_o, this implies

rank $\left[H(t_1), \ldots, H^{(n-1)}(t_n) \right] < n$

for any set $t_1, \ldots, t_n \geq t_o$. In fact, if the linear system is not completely controllable at t_o, this implies there exists a non-zero vector c such that $c^T H(t) = 0$ for all $t \geq t_o$. This in turn implies that

$$c^T \dot{H}(t), \ldots, c^T H^{(n-1)}(t),$$

are also zero for all $t \geq t_o$. Hence, for any set $t_1, \ldots, t_n \geq 0$

rank $\left[H(t_1), \dot{H}(t_2), \ldots, H^{(n-1)}(t_n) \right] < n$.

For necessity we shall assume that the linear system is completely controllable at t_o and demonstrate the existence of a set $t_1, \ldots, t_n \geq t_o$ such that rank $\left[H(t_1), \dot{H}(t_2), \ldots, H^{(n-1)}(t_n) \right]$ is n.

This is equivalent to showing that for any nonzero vector c the nr dimensional vector

$$c^T \left[H(t_1), \ldots, H^{(n-1)}(t_n) \right] \neq 0.$$

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Let e_1 be any nonzero vector. Since the system is assumed to be controllable and $H(t)$ is continuous then there exists a $t_1 \geq t_o$ such that $e_1^T H(t_1) \neq 0$. If rank $H(t_1)$ is n then the proof is finished. If not, there exists a nonzero vector e_2 such that $e_2^T H(t_1) = 0$, so that e_2 and e_1 are linearly independent. Now there exists a $t_2 \geq t_o$ such that $e_2^T \dot{H}(t_2) \neq 0$, if not, then $e_2^T \dot{H}(t) = 0$ for all $t \geq t_o$. This implies

$$\int_{t_1}^{t} e_2^T \dot{H}(t) dt = e_2^T H(t) = 0$$

for all $t \geq t_o$ and contradicts the assumption that the system is completely controllable. Next consider $[H(t_1), \dot{H}(t_2)]$, if the rank of this matrix is n the proof is finished. If not, then there exists a nonzero vector e_3 such that

$$e_3^T [H(t_1), \dot{H}(t_2)] = 0; \text{ and a } t_3 > t_o \text{ such that } e_3^T \dot{H}(t_3) \neq 0.$$

Clearly e_1, e_2 and e_3 are linearly independent. Continuing inductively, either for some

$$j < n, \text{ rank } [H(t_1), \dot{H}(t_2), \ldots, H^{(j-1)}(t_j)] \text{ is n}$$

or else we generate n linearly independent vectors $e_1, \ldots, e_n$ such that

$$e_{j+1}^T \left[H(t_1), \dot{H}(t_2), H^{(j-1)}(t_j) \right] = 0$$

In the first instance, we are finished. In the second, any non-zero vector c can be expressed as $c = \sum \gamma_i e_i$ with not all the γ_i zero. From the property that the e_i satisfies, it follows that

$$c^T \left[H(t_1), \dot{H}(t_2), \ldots, H^{(n-1)}(t_n) \right] \neq 0$$

This completes the proof.

Corollary 5.1   Consider the linear system

$$\dot{x} = A(t)x + B(t)u,$$

where $A(t)$ is an n x n matrix of C^{n-2} elements, and $B(t)$ is an n x r matrix of C^{n-1} elements. This system is completely controllable at t_o if and only if there exists n times $t_1, \ldots, t_n \geq t_o$ such that rank

$$\left[B(t_1), \Phi(t_1, t_2)\Gamma B(t_2), \ldots, \Phi(t_1, t_n)\Gamma^{n-1}B(t_n) \right] = n$$

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where $\Gamma = A(t) - D_t$ and Φ is the fundamental solution to the homogeneous equation.

By means of the following lemma we can derive an even simpler algebraic test for complete controllability.

Lemma 5.1 Let $H(t)$ be an n x r matrix valued function. Suppose rank $\left[H(t_1), \ldots, H(t_n)\right] < n$ for all sets $\{t_1, t_2, \ldots, t_n\}$ with $t_i \geq t_o$. Then there exists a nontrivial constant vector c such that $c^T H(t) = 0$ for all $t \geq t_o$.

Proof. Let $t_1 \geq t_o$ be chosen so that rank $H(t_1)$ is maximal, and call this rank r_1. Select t_2 so that rank $\left[H(t_1), H(t_2)\right]$ is maximal and call this rank r_2. We continue this process to the choice of t_{n-1} such that rank $\left[H(t_1), \ldots, H(t_{n-1})\right] = r_{n-1} < n$ is maximal. Now either $r_j = r_{j+1}$ for some $j = 1, 2, \ldots, (n-1)$ or $r_{n-1} = n-1$. In the first case, let j be the smallest integer such that $r_j = r_{j+1}$. The columns of $H(t)$, therefore, must lie in the r_j dimensional subspace spanned by the columns of $H(t_1)$, $\ldots$, $H(t_j)$ for all t. Hence, if c is a nontrivial vector orthogonal to this subspace then $c^T H(t) = 0$ for all $t \geq t_o$. In the second case, since by hypothesis we cannot increase the rank of $\left[H(t_1), \ldots, H(t_{n-1})\right]$ by adjoining $H(t)$, the columns of $H(t)$ lie in the (n-1) dimensional subspace spanned by the columns of $H(t_1), \ldots, H(t_{n-1})$ and a nontrivial vector c orthogonal to

this subspace satisfies $c^T H(t) = 0$ for all $t \geq t_o$. We are now in a position to prove the following theorem.

<u>Theorem 5.4</u>   The system $\dot{y} = H(t)u$ is completely controllable at t_o if and only if there exists n times $t_1, \ldots, t_n \geq t_o$ such that rank $[H(t_1), \ldots, H(t_n)]$ is n.

Proof.   The system $\dot{y} = H(t)u$ is completely controllable at t_o if and only if $c^T H(t) = 0$ for $t \geq t_o$ implies $c = 0$. Suppose the system is not completely controllable at t_o. Then there exists a nontrivial vector c such that $c^T H(t) = 0$, $t \geq t_o$, which implies $c^T [H(t_1), \ldots, H(t_n)] = 0$ for all sets $\{t_1, \ldots, t_n\}$. The contrapositive of this shows, if the rank $[H(t_1), \ldots, H(t_n)]$ is n for some sets $\{t_1, \ldots, t_n\}$ then the system is completely controllable at t_o.

Next suppose that rank $[H(t_1), \ldots, H(t_n)] < n$ for all sets $\{t_1, \ldots, t_n\}$. Then by lemma 5.1 there exists a non-trivial vector c such that $c^T H(t) = 0$ for all $t \geq t_o$ which shows that the system is not completely controllable at t_o.

The concepts of complete controllability, the integral criterion for complete controllability and the linear independence of functions relating to the controllability problem, date back to the paper by LaSalle [7] on the time optimal problem for linear systems. It was subsequently popularized by Kalman who showed how the concept of complete controllability provided

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the rationale for many assumptions invoked in the general theory

of control systems. LaSalle defined the system (5.1) to be

proper if $\eta^T Y(t) = 0$ on each measurable interval implied $\eta = 0$.

It should be noted that in LaSalle's notation $Y(t) = \Phi(t_o, t)B(t)$

so that effectively the system (5.1) has been transformed to

$\dot{y} = \Phi(t_o, t)B(t)u$ under the nonsingular transformation Φ. Since

the vector $\Phi(t_o, t)^T \eta$ is the solution to the costate equations

$\dot{p} = -A^T(t)p$ for the time optimal problem with initial conditions

$p(t_o) = \eta$, then a proper system implies a nontrivial solution

to the maximum principle for the time optimal problem for some

components of the control vector. Those cases where the maximum

principle does not yield any information regarding the choice

of the optimal control are referred to as singular. Therefore,

a proper control system cannot be totally singular, that is,

all components of the control vector are singular. One central

theorem of LaSalle paper was the following.

Theorem 5.5   A proper control system is completely controllable.

As will be seen in a later section, this is a result of some

significance to linear systems which are derived as approximations

to nonlinear systems about a totally singular time optimal tra-

jectory.

One might question why the algebraic method of determining

the linear independence of functions on the real line, which is

basically equivalent to Hermann's method for linear systems, overcomes the problem of monotonicity associated with the exceptional time coordinate, so that full neighborhoods of attainable points can be achieved. This question does not arise in the case of the integral criterion for controllability (Theorem 5.2) since the sufficiency part of the proof provides a method whereby the control vector can be determined to achieve any desired state. The following lemma resolves this question.

Lemma 5.2 Completely controllable linear systems are locally-locally controllable at the origin.

Proof Since by assumption the linear system is completely controllable there are no integral manifolds to the system. If $\phi(t)$ is a solution of $\dot{x} = A(t)x + B(t)u$ corresponding to some $u(t)$, that starts from the origin, then $-\phi(t)$ corresponding to a control $-u(t)$ is also a solution that starts from the origin; hence, this skew-symmetry property eliminates one-sided sets of attainable points and the origin is locally-locally controllable.

In general, linear systems are locally-locally controllable at the origin and globally locally controllable elsewhere; the only exception to this rule is the case where rank $B(t)$ is n for all $t \in [t_o, t_1]$. In this instance all points of E^n are locally-locally controllable which is an expected result since the

vectors $B(t)$ for fixed t form a noninvolutive distribution of vector fields, and time would be an ignorable coordinate. For the majority of cases, however, rank $B(t)$ is strictly less than n for all $t \in [t_o, t_1]$ and the vectors $B(t)$ for fixed t form an involutive distribution of vector fields as intimated in the opening remarks.

Finally the notion of uniformly controllability or instanteous transfer of state only makes physical sense when applied to noninvolutive systems, where time is an ignorable coordinate so that the transfer can be achieved instantaneously. More important however is that noninvolutive systems are locally-locally controllable so that instantaneous transfers of the state can be achieved by trajectories that are contained within compact regions of the state space.

VI   TOTALLY SINGULAR CONTROL SYSTEMS

In this section we shall be concerned with nonlinear control systems in which the control appears linearly as defined by

$$\dot{x} = a(x) + B(x)u \qquad\qquad 6.1$$

where $a(x)$ is an n vector of smooth functions; $B(x) = \left[b_1(x), \ldots, b_r(x)\right]$ is an n x r matrix of smooth functions, and is assumed to be regular for all x in some neighborhood $N(x_o)$ of a given point $x_o \epsilon M$. We shall refer to B as the distribution of vector fields which are generated by the columns of $B(x)$; furthermore we shall assume that the distribution is involutive and the elements $b_1(x)$ form a basis for the distribution B, i.e. $\left[b_i(x), b_j(x)\right] = 0$ for all b_i, $b_j \epsilon B$. For a specified control system of the form (6.1) the dimension or number of column vectors in $B(x)$ will not, in general, be equal to the dimension of the distribution B, so that the augmentation procedure described in section 4. will have to be used to complete the system. However, since $B(x)$ now generates an involutive distribution of vector fields, the results on the uniform approximation of trajectories of the completed control system by trajectories of the original control system do not necessarily apply.

For example, consider the control system (6.1) with

$$a(x)^T = (0, \ 0, \ 0, \ x_1^2 + x_2^2)$$

and control actuator vectors

$$b_1(x)^T = (1, \ 0, \ 0, \ 0)$$

$$b_2(x)^T = (0, \ 1, \ x_1, \ 0)$$

Since $B(x)$ defines an involutive distribution of dimension 3, we can augment the actuator matrix $B(x)$ to $\widetilde{B}(x)$ by including

$$b_3(x)^T = \left[b_1(x), \ b_2(x)\right]^T = (0, \ 0, \ 1, \ 0).$$

By choosing the augmented control vector to be $\widetilde{u}^T = (u_1, \ u_2, \ u_3) = (0, \ 0, \ 1)$ the state of the augmented control system is transferred from $(0, \ 0, \ 0, \ 0)$ to $(0, \ 0, \ 1, \ 0)$ in the time interval $0 \le t \le 1$. If for the original control system we let $|u_1| < \epsilon_1$ and $|u_2| < \epsilon_2$, then it is easy to verify that the solution ϕ of the original control system cannot be made to uniformly approximate the solution ψ of the completed control systems since

$$\max_{0 \le t \le 1} |\phi(t) - \psi(t)|^2 < \epsilon_1^2 + \epsilon_2^2 + (1 - \epsilon_1\epsilon_2)^2 + (\epsilon_1^2 + \epsilon_2^2)^2$$

The minimum of the right hand side of the inequality is one, therefore we cannot choose any ϵ such that

$$|\phi(t) - \psi(t)| < \epsilon$$

which is required for the uniform approximation of trajectories.

For the involutive control system we have the following decomposition theorem.

Theorem 6.1   If the control actuator vectors $B(x)$ define a basis for an involutive distribution of vector fields of dimension r, then there exists a coordinate transformation which decomposes the control system (6.1) into

$$\dot{y} = f(y, z)$$

$$\dot{z} = g(y, z) + u \qquad\qquad 6.2$$

where y and f are (n - r) vectors and $f(\cdot)$ is smooth; and z and g are r vectors and $g(\cdot)$ is smooth.

Proof   The proof follows from the representation of involutive distributions as given by a theorem of Frobenius [8] which says that for each point $x \epsilon M$ we can find a coordinate system $(z_1, \ldots, z_n)$ such that the vector fields $(D_{z_1}, \ldots, D_{z_r})$

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generates the distribution B on the manifold M.

Let $(y_1, \ldots, y_{n-r})$ and $(z_1, \ldots, z_r)$, denoted by (y, z), represent a partitioned coordinate system of M'. If ϕ is a mapping of M' into M as given by $x = \phi(y, z)$, then the tangent vectors transform by

$$D_z = (D_z \phi^T) D_x \qquad 6.3$$

The theorem of Frobenius states that

$$b_i(\phi) = D_{z_i} \phi \qquad 6.4$$

The validity of this statement can be established by deriving the integrability conditions for the system of n x r nonhomogeneous partial differential equations (6.4).

Equating second partials yields

$$D_{z_j}(D_{z_i} \phi) = (D_{z_j} \phi^T) D_x b_i(\phi) = b_j(\phi)^T D_x b_i(\phi)$$

$$D_{z_i}(D_{z_j} \phi) = (D_{z_i} \phi^T) D_x b_j(\phi) = b_i(\phi)^T D_x b_j(\phi)$$

Hence $D_{z_j}(D_{z_i} \phi) = D_{z_i}(D_{z_j} \phi)$, the integrability conditions for the

system of nonhomogeneous partial differential equations (6.4) implies

$$b_j(\phi)^T D_x b_i(\phi) - b_i(\phi)^T D_x b_j(\phi) = [b_j, b_i] = 0.$$

which is true by assumption.

Differentiating the transformation ϕ with respect to time yields

$$\dot{x} = \dot{y}^T D_y \phi + \dot{z}^T D_z \phi = a(\phi) + B(\phi)u.$$

By virtue of equation (6.4) this reduces to

$$\dot{y}^T D_y \phi + (\dot{z} - u)^T D_z \phi = a(\phi)$$

Since the Jacobian of the transformation ϕ is assumed to be different from zero on the manifold M then the decomposition (6.2) follows. The vectors f and g are determined by inverting the matrix equation

$$[(D_y \phi^T)^T : \quad (D_z \phi^T)^T] \begin{pmatrix} f \\ \cdots \\ g \end{pmatrix} = a(\phi).$$

One obvious significance of the decomposition theorem is that it isolates out those transformed states that are locally-locally

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controllable. In fact there is no loss of generality if we choose

$$u = v - g(y, z).$$

so that the control system now assumes the form

$$\dot{y} = f(y, z)$$

$$\dot{z} = v$$

and the controllability problem viewed as determining the attainability of the states $y(t)$ for given inputs $z(t)$.

It should be noted that the pfaffian system associated with the control system (6.1) has been transformed to normal form.

$$dy - f(y, z)dt = 0$$

under the transformation φ.

We shall now develop some equivalences between "involutive" control systems and "totally singular" control system. In many optimal control problems a singular problem can arise that is characterized by the fact that the maximum principle fails to yield any information regarding the choice of the optimal controls. To

distinguish this condition, the controls and trajectories are termed singular. The controllability problem is intimately connected with the time optimal problem and this connection will be shown for the nonlinear system (6.1). The time optimal problem consists of finding the optimal controls $u(t)$ that transfer the state from some specified initial condition x_o to some specified final condition x_f in minimum time.

Following the Kalman-Caratheodory approach [17], we define the system Hamiltonian as,

$$H(t, x, p, u) = 1 + p^T a(x) + p^T B(x)u,$$

where p is an n vector describing the costate. For each x and p, the Hamiltonian is minimized with respect to the controls over the set of admissible controls Ω. If, for example, the set of admissible controls are constrained to an r-dimensional hypercube described by

$$\Omega = \left\{ u: \ |u_i| \leq 1, \ i = 1, 2, \ldots, r \right\}$$

the maximum principle yields

$$u = -\text{sgn}(B^T(x)p) \qquad\qquad 6.5$$

for the choice of the optimal controls.

Any solution of the nonlinear system (6.1) with controls (6.5) that pass through the desired terminal points x_o and x_f, with the costate satisfying

$$\dot{p} = -D_x a^T(x)p - u^T D_x B^T(x)p \qquad\qquad 6.6$$

would be regarded as a minimizing trajectory. However, singular controls $u_s(t)$ that are not necessarily bang-bang can exist so that the corresponding solutions to (6.1) and (6.6) $x = \phi(t)$ and $p = \psi(t)$ make some or all of the r components of the vector $B^T(\phi)\psi$ vanish over some measurable time interval. It is immediately obvious that this situation would invalidate the maximum principle for the singular components of the control vector. The singular components of the control vector, if they exist, are obtained by repeatedly differentiating the appropriate components of the singular condition $B^T(\phi)\psi = 0$ with respect to time. The problem is said to be totally singular if all components of $B^T(\phi)\psi$ vanish.

We shall now develop the equivalence between "involutive" and "totally singular". The first equivalence to be established is trivial but yields some geometric insight into the nature of totally singular arcs.

Lemma 6.1   All trajectories on an integral manifold are totally singular.

Proof   If $V(t, x)$ defines the integral manifold, which is a submanifold of E^n; then V satisfies

$$(D_t + a^T(x)D_x)V(t, x) = 0 \qquad\qquad 6.7$$

$$B(x)^T D_x V(t, x) = 0 \qquad\qquad 6.8$$

Since by assumption an integral manifold to the system (6.1) exists, then if we define the costate by $p(t) = D_x V(t, x(t))$ where $x(t)$ satisfies (6.1) for some $u(t)$; then the totally singular condition is automatically satisfied by (6.8). We now have to establish that this choice for the costate does indeed satisfy the Euler-Lagrange equations.

Formally differentiating $p(t)$ yields

$$\dot{p}(t) = D_t D_x V(t, x(t)) + \{a(x) + B(x)u\}^T D_x D_x V(t, x(t)).$$

The following identities follow trivially from (6.7) and (6.8)

$$D_x \left\{ (D_t + a(x)^T D_x) V(t,\ x) \right\} = (D_x a(x)^T) D_x V(t,\ x) + (D_t + a(x)^T D_x) D_x V(t,\ x) = 0$$

and for any choice of control $u(t)$,

$$D_x (B(x)u(t))^T D_x V(t,\ x) = (D_x (B(x)u(t))^T) D_x V(t,\ x) + (B(x)u(t))^T D_x D_x V(t,\ x) = 0$$

From these two identities we obtain

$$\dot{p}(t) = -(D_x a(x)^T) D_x V(t,\ x) - (D_x (B(x)u(t))^T) D_x V(t,\ x)$$

$$= -D_x \left\{ a(x(t)) + B(x(t))u(t) \right\}^T p(t)$$

which completes the proof.

The integral manifold is in fact generated by the totally singular solutions. However, the existence of totally singular controls does not necessarily imply the existence of an integral manifold. In fact, when an integral manifold does not exist, then the totally singular arcs define points along which the control system (6.1) appears to be integrable, i.e. has an integrable manifold. This phenomena results in the linearized approximating systems about totally singular arcs being integrable, and will be dealt with in detail later.

Before we establish necessary conditions for the existence of totally singular arcs we shall prove the following useful lemma which helps to avoid some of the algebraic tedium.

Lemma 6.2   If $p^T C(x) = 0$ along any time optimal extremals then this implies that $p^T [(a(x) + B(x)u), C(x)] = 0$ along the time optimal extremal.

Proof   Since the costate p satisfies the Euler-Lagrange equations $\dot{p} = -D_x(a(x) + B(x)u)^T p$; then formally differentiating the expression $p^T C(x) = 0$ yields

$$-(D_x(a(x) + B(x)u)^T p)^T C(x) + p^T (a(x) + B(x)u)^T D_x C(x) = 0$$

Rearranging terms and recalling the definition of the Lie Bracket gives the desired result.   It is standard terminology to refer to the Lie Bracket $[X, Y]$ as the Lie derivative of Y with respect to X.

We now come to the main result concerning the equivalence between the totally singular controls and involutive systems, which is a generalization of the result first proven for $(n - 1)$ control components [18].

Lemma 6.3   A necessary condition for the existence of a totally singular vector control is that the control actuator vectors be involutive.

Proof   We shall prove the contrapositive form of this theorem,

B(x) not involutive implies no totally singular vector control.

If the totally singular vector control exists then $p^T B(x) = 0$

along the totally singular arc. Differentiating this expression

along the singular arc yields $p^T [(a(x) + B(x)u), B(x)] = 0$, and

subsequent differentiations gives: $p^T [(a(x) + B(x)\dot{u}), B(x)] +$

$p^T [(a(x) + B(x)u), [(a(x) + B(x)u), B(x)]] = 0$, etc. Continuing

in this fashion we can generate n linearly independent vectors

orthogonal to the costate vector for all x and hence for all x(t),

which implies that the totally singular vector control does not

exist.

An equivalent proof follows from the fact that if B(x) is not

an involutive system; then by the results of section 4 all states

can be transferred instantaneously so that any bang-bang control

would be better than the totally singular control and hence the

totally singular arcs would not be extremals.

The geometric equivalence between "involutive" and "totally

singular" control systems can be established as follows. For

simplicity we shall consider the decomposed control system.

$$\dot{y} = f(y, z); \quad \dot{z} = v \qquad\qquad 6.9$$

It is evident from the pfaffian system dy - f(y, z)dt, that the

integral manifolds y associated with the reduced pfaffian system stratify M, and motion on these integrals $y(x)$ = constant, can be achieved instantaneously. In other words the minimum time required to transfer from one state to some other state will depend strictly on the vector y. Let us denote this cost by $t(y)$. The rate of change of the cost along the solution path is given by.

$$\frac{dt(y)}{dt} = f^T(y, z)D_y t(y) \qquad 6.10$$

Geometrically, the optimization problem consists of finding those points $z(y)$ on the stratification which extremizes the cost derivative (6.10). Assuming that the extreme points of the cost derivate occur at interior points of the manifold M, then these points are determined by

$$D_z f^T(y, z)D_y t(y) = 0 \qquad 6.11$$

If $D_z D_z f^T(y, z)D_y t(y)$ is nonsingular, then (6.11) can be inverted by the implicit function theorem to yield $z(y)$ so that (6.9) can be integrated with appropriate initial conditions to yield $y = \phi(t)$. Along each integral curve ϕ, we require the cost $t(\phi(t)) \equiv t$ to be equivalent to the time so that (6.10) becomes

$$f^T(\phi, \; z(\phi))D_y t(\phi) = 1$$

Differentiating this identity with respect to time yields

$$f^T(\phi, \; z(\phi))D_y f^T(\phi, \; z(\phi))D_y t(\phi) + f^T(\phi, \; z(\phi))D_y (D_y t(\phi))^T f(\phi, \; z(\phi)) = 0 \qquad 6.12$$

If we define the costate in the usual fashion by $p(t) = D_y t(\phi(t))$; then equation (6.11) becomes the condition for the control vector to be totally singular $D_z f^T(\phi(t), \; z(\phi(t)))p(t) = 0$; furthermore (6.12) is satisfied by the costate equations since it reduces to

$$f^T(\phi, \; z(\phi)) \left\{ D_y f^T(\phi, \; z(\phi))p + \dot{p} \right\} = 0$$

With involutive systems, controllability cannot be established on the basis of the control actuator vectors alone; in fact, if $a(x) = 0$ then by theorem 4.1 all solutions would lie on an integral manifold. Therefore, controllability can only be established with a nontrivial $a(x)$. Following Hermann, to preclude an integral manifold to the system (6.1) we require that the set of vectors $(a(x), b_1(x), \; . \; . \; ., b_r(x))$ do not generate an involutive distribution. However, this does not circumvent the problem of one-sided sets of attainability caused by the monotonicity of the time coordinate.

One approach to this problem is to replace the nonlinear control system by a linear approximating system about some preferred nominal trajectory so that the complete theory of linear systems can be used to establish controllability. It would appear therefore that we can generate an infinity of algebraic criteria for controllability, each one depending on the particular choice of the nominal trajectory defining the linear approximating system. This raises the question of what nominal trajectory should be chosen. Does there exist, for example, a nominal trajectory defining a linear approximating system whose algebraic controllability criteria immediately determines the complete controllability of the original nonlinear control system. This question leads to a paradox that was first observed by Hermes [11], however the result was essentially contained in LaSalle's Theorem, and is summarized in the following.

Theorem 6.2   The linear approximating system describing motions in some neighborhood of the totally singular trajectories for the time optimal problem is not completely controllable.

Proof   Let $\phi(t)$ and $\psi(t)$ describe the singular state and costate respectively, corresponding to a totally singular control $u^s(t)$. The linearized approximating system is obtained by the first order expansion of (6.1) under the substitution $x \longrightarrow \phi + x$, $u \longrightarrow u^s + u$ and is

$$\dot{x} = (D_x(a(\phi) + B(\phi)u^s)^T)^T)^T x + B(\phi)u \qquad\qquad 6.12$$

Since by assumption the control is totally singular this implies that $\psi^T B(\phi) = 0$, along the singular arc. The pfaffian system associated with the linear system (6.12) is

$$\psi^T(t)dx - \psi^T(x)(D_x(a(\phi) + B(\phi)u^s)^T)^T x\ dt = 0$$

which is integrable since by hypothesis ψ satisfies the Euler-Lagrange equations

$$\dot{\psi} = -D_x(a(\phi) + B(\phi)u^s)^T \psi$$

and $\psi^T x$ defines the integral manifolds to the system.

Corollary 6.1   The distribution of linearized vector fields about the totally singular trajectories associated with the time optimal problem is involutive.

A remarkable facet of theorem 6.2 is that the result is independent of the optimality of the totally singular vector control. If the singular arc is truly a minimizing arc, one would expect it to persist as a natural boundary to the set of reachable points, since by definition it would be better than any bang-bang control

irrespective of the magnitude of the control bounds. On the other hand, if the control system is completely controllable, one would expect all motions to completely fill Euclidean n-space by virtue of the system being linear in the control vector.

Therefore, one is tempted to conjecture that if the totally singular arc for the time optimal problem is not a minimizing arc, then the control system is completely controllable. However, the conjecture is not true as shown by the following counterexample which possesses a non-optimal singular arc for the time optimal problem. Consider the system [11]

$$\dot{x}_1 = u$$

$$\dot{x}_2 = 1 + x_2 x_1^2 u$$

then it is trivial to verify by the Green's Theorem approach that the non-optimal singular arc for the time optimal problem is described by $x_1(t) = 0$. Contrary to the proposed conjecture, the lack of controllability for the above system can be demonstrated by the following transformation which is nonsingular for all finite regions of Euclidean two space. With the transformation

$$y_1 = x_1; \ y_2 = x_2 \ \exp \ \frac{-x_1^3}{3}$$

the control system becomes

$$\dot{y}_1 = u; \ \dot{y}_2 = \exp \ \frac{-y_1^3}{3}$$

The lack of global controllability is evident since $\dot{y}_2 > 0$ irrespective of the choice of the control.

Hence, the method of using linear approximating system to a nonlinear control system to determine controllability is a perfectly valid technique provided the nominal trajectory is not a totally singular time optimal arc. In this case the higher order approximations are crucial to the establishment of controllability criteria, and cannot be neglected.

Let us return to Theorem 5.3 and give a geometric proof thereof, since it has obvious applications to nonlinear system. We recall that the theorem stated, "The system $\dot{y} = H(t)u$ is completely controllable at t_o if and only if there exist n times $t_1, \ \ldots, t_n \geq t_o$ such that rank $[H(t_1), \ H(t_2), \ \ldots, H(t_n)]$ is n".

The conclusion of the theorem follows from the integrability of the reduced pfaffian system and provides a technique for generating conditions for the local local controllability of nonlinear systems. For

the linear system $\dot{y} = H(t)u$ we can associate the hyperplanes $\Psi^t(t)y$

where $\Psi(t)$ is an $n \times n-r$ matrix of vectors orthogonal to $H(t)$. Since

the instantaneous motions are confined to these hyperplanes the con-

trollability of the linear system can be defined in terms of the

hyperplanes admitting a coordinate structure in M. That is to say,

the hyperplanes span M by suitable choice of the essential constants

(time) of the hyperplanes by $z = \Phi^t(t)y$, then the hyperplanes will

span M if the normal vectors $\Phi^t(t)$ form a basis, or equivalently if

the tangent vectors $H(t)$ form a basis. This requires that rank

$\left[H(t_1), H(t_2), \ldots, H(t_n)\right]$ is n. Alternatively this condition can

be derived by considering a sequence of n delta functions having

measures ξ at the points $t_1, t_2, \ldots, t_n$. The rank condition

defines a one to one mapping between the state y and the measures ξ.

This theorem has obvious generalizations to nonlinear systems

of the form $\dot{y} = H(t, y)u$, and the control system (6.1) $\dot{x} = a(x) + B(x)u$

can be put into this form by the following transformation. Let $\phi(t, y)$

denote the solution to $\dot{x} = a(x)$ then by a variation of parameters we

obtain

$$\dot{y} = \left[(D_y\phi(t, y)^T)^T\right]^{-1}B(\phi(t, y))u = H(t, y)u \qquad 6.13$$

This form of the control system is similar to the system described by

equation (4.1); however, for fixed t, the column vectors of H(t, y) describe an involutive system, because the involution of B(x) is invariant under the map ϕ(t, y). The geometric proof of theorem 5.3 yields the following theorem for the nonlinear system (6.13).

<u>Theorem 6.3</u> A necessary condition for the system $\dot{y}$ = H(t, y)u to be completely controllable at t_o in some neighborhood N of y_o is that there exist n times t_1, . . ., $t_n \geq t_o$ such that rank $[H(t_1, y), H(t_2, y), . . ., H(t_n, y)]$ is n for almost all yϵN.

<u>Proof</u> Let C(t, y) define (n-r) vectors orthogonal to H(t, y). H(t, y) is an involutive system of order r, then it follows that the pfaffian system C^T(t, y)dy is integrable for fixed t. If we now assume that the rank $[H(t_1, y), . . ., H(t_n, y)]$ is less than n for all sets $\{t_1, . . ., t_n\}$ and all yϵN then this implies that there exists a non-trivial vector c(y) such that c^T(y)H(t, y) = 0. Since the pfaffian system C^T(t, y)dy is integrable for fixed t it follows that c^T(y)dy is integrable so that an integral manifold exists. Hence, the control system is not completely controllable. The contrapositive of this yields the result of the theorem.

There is a unique relation between the singular arc and the points of measure zero where $[H(t_1, y), . . ., H(t_n, y)]$ vanishes and is summarized in the following theorem.

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<u>Theorem 6.4</u>   On totally singular arcs $y(t)$, Rank $\left[H(t_1, y(t)), H(t_2, y(t)), \ldots, H(t_n, y(t)) \right]$ is less than n for all sets $\left\{ t_1, t_2, \ldots, t_n \right\}$.

<u>Proof</u>   Since rank $\left[H(t_1, y(t)), \ldots, H(t_n, y(t)) \right]$ is less than n for all sets $\left\{ t_1, \ldots, t_n \right\}$ then there exists a nontrivial vector $\Psi(t)$ such that

$$\Psi^t(t) H(\tau, y(t)) \overset{t, \tau}{\equiv} 0 \qquad\qquad 6.14$$

Since this is an identity in t and τ then differentiating with respect to t yields

$$\dot{\Psi}^t(t) H(\tau, y(t)) + \Psi^t(t) D_y H(\tau, y(t)) H(t, y(t)) u(t) = 0$$

The r columns of $H(t, y)$, for fixed t, define a complete set of tangent vectors of order r.   Therefore, since the Lie Bracket does not generate new tangent vectors, the order of differentiation with respect to y in the above equation can be changed, on substituting $\tau = t$, to give

$$\left[\dot{\Psi}^t(t) + \Psi^t(t) D_y H(t, y(t)) u(t) \right] H(t, y(t)) = 0$$

The result is now obvious since $\Psi(t)$ can be identified with the co-state, and equation (6.14) is the singular condition.

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