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# STABILITY THEORY OF NONLINEAR OPERATIONAL DIFFERENTIAL EQUATIONS IN HILBERT SPACES



by

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# FOREWORD

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## Abstract


STABILITY THEORY OF NONLINEAR OPERATIONAL DIFFERENTIAL EQUATIONS IN HILBERT SPACES

Chia-Ven Pao, Ph.D.

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The object of this dissertation is to establish some criteria for the existence, uniqueness, stability, asymptotic stability and stability region of a solution of the linear or nonlinear, time-invariant or time-varying operational differential equations (i.e., equations of evolution) of the form

$$\frac{dx(t)}{dt} = A(t)x(t) \qquad (t \geq 0)$$

in Banach spaces and in Hilbert spaces, from which criteria for the same results of a solution of the corresponding type of partial differential equations can be deduced. In the case of linear time-invariant equations of evolution, linear semi-group theory is used; and by the introduction of an equivalent semi-scalar product on a Banach space, necessary and sufficient conditions on the linear operator $A(t) \equiv A$ for the generation of a semi-group in a real Banach space are obtained. By using the semi-group property, the existence, uniqueness, stability or asymptotic stability of a strong solution can be ensured. In the case of nonlinear time-invariant equations, the concept of nonlinear semi-group is introduced. Based on some properties of a monotone operator (or, a dissipative operator in the terminology of this dissertation), necessary and sufficient conditions on the nonlinear operator $A(t) \equiv A$ for the generation of a nonlinear semi-group in a complex Hilbert space are established, from which the existence, uniqueness and stability or

asymptotic stability of a weak solution are guaranteed by the nonlinear semi-group property. The introduction of an equivalent inner product in a complex Hilbert space makes it possible to develop a stability theory in terms of a Lyapunov functional which is defined through a defining sesquilinear functional. It is shown that such a functional defines an equivalent inner product and that the existence and stability property of a weak solution are invariant under equivalent inner products. In case of a Banach space, the defining sesquilinear functional is replaced by an equivalent semi-scalar product. The investigation of the existence, uniqueness and stability of weak solutions is extended to nonlinear time-varying operational differential equations. Under some additional restrictions on the nonlinear operator A(t) which is time-dependent, criteria for the existence, uniqueness, stability or asymptotic stability of a weak solution for the general nonlinear time-varying equation of evolution in a complex Hilbert space are obtained. Several special types of nonlinear equations which are more suitable for a class of nonlinear partial differential equations are deduced with particular attention on the class of nonlinear nonstationary equations of the form

$$\frac{dx(t)}{dt} = Ax(t) + f(t,x(t)) \qquad (t \geq 0)$$

where A is a linear or nonlinear time-independent operator mapping part of a real Hilbert space H into itself and f is a given (in general nonlinear) function defined on $R^+ \times H$ into H. Applications are given to a class of second order n-dimensional parabolic-elliptic type of partial differential equations with a detailed description of the formulation of an abstract operator having the desired property from a partial differential operator.

# I. INTRODUCTION

In the year of 1892, A. M. Lyapunov [16]* published in Russian his famous memoire on the stability of motion which originally received very little attention. About forty years later, the work in Lyapunov stability theory was resumed by some Soviet mathematicians and since then the so called "second method" or "direct method" of Lyapunov has been widely used as a mathematical tool in the investigation of linear and nonlinear stability problems governed by ordinary differential equations. The "direct method" of Lyapunov consists of a means for answering the question of stability of differential equations from the given form of the equations, including the boundary conditions, without explicit knowledge of the solutions. The central problem of the direct method in the investigation of stability of ordinary differential equations is the construction of a "Lyapunov function" $v(x)$ having the property that $v(x) \geq 0$ for $x$ in a finite dimensional space and the derivative of $v(x)$ along solutions of the given equation is negative. The development of the Lyapunov method has been moved toward the investigation of partial differential equations in recent years. This advance seems to be natural since many physical problems can be best described or must be represented by partial differential equations. It is also natural that the idea of constructing a Lyapunov function in finite dimensional spaces is extended to the construction of a "Lyapunov functional" in infinite dimensional spaces. This extension leads to the use of function spaces on which a topology can be defined. A first step toward applying the Lyapunov direct method to partial differential equations was the study of a denumerably

*Numbers in brackets designate references at the end of this dissertation.

infinite system of ordinary differential equations (e.g., see Massera [17]). A general stability theory by using a scalar functional was established by Zubov [24] who considered equations of the form

$$\frac{\partial u(t,x)}{\partial t} = f(x, u, \frac{\partial u}{\partial x}). \tag{I-1}$$

However, the existence of solutions of (I-1) was proved only for the case when f is linear in $\partial u/\partial x$ and for the general form of (I-1), the existence of solutions was assumed. Moreover, the requirement that the system of partial differential equations define a dynamical system (i.e., the solutions possess the group property) excludes a large class of differential equations whose solutions possess only the semi-group property. Since the stability problem of partial differential equations occurs in many fields of science such as reactor physics, control process, fluid mechanics, chemical process, etc. the study of stability behavior of solutions to partial differential equations has been accelerated by engineers, physicists and mathematicians in recent years as can be seen from a literature survey made by Wang [22]. However, most of the work listed in [22] deal with a specific partial differential operator, and in some of them the existence of a solution is either assumed or not mentioned. On the other hand, there are many works in the area of partial differential equations and in particular those works on operational differential equations (i.e., equations of evolution) in which only the existence and uniqueness are discussed. It should be mentioned that in some Russian literature, the stability problem of semi-linear operational differential equations has been investigated. Some earlier literature by Khalilov and Domshlak are described in a survey book edited by Gamkrelidze [7] in which numerous references concerning operational differential equations are also given. In the study of periodic solutions of the semi-linear operational differential equations of the form

$$\frac{dx(t)}{dt} = Ax(t) + F(t, \mu, \lambda) \tag{I-2}$$

Taam [20] also investigated the stability properties of solutions to (I-2). He assumed A either as a bounded linear operator or as the infinitesimal generator of a semi-group and established criteria for the existence and the asymptotic stability of a periodic solution.

## A. Recent Developments on Linear Equations

The difficulty of the direct extension from ordinary differential equations into partial differential equations by the Lyapunov direct method lies in the fact that the existence of a solution to a given partial differential equation must first be established because to ensure the stability of a solution the derivative of the "Lyapunov functional" is taken along the solutions of the given equation. More recently, in the study of stability problem of a system of linear partial differential equations, Buis [3] applied the semi-group and group theory to operational differential equations of the form

$$\frac{dx(t)}{dt} = Ax(t) \tag{I-3}$$

where A, which may be considered as an extension of a partial differential operator, is a linear operator with domain and range both in a real Hilbert space. By using semi-group or group theory, the solutions of (I-3) can be represented by a semi-group or a group in the sense that if a solution of (I-3) with initial condition $x \in \mathcal{D}(A)$ (the domain of A) is denoted by $\phi(t,x)$, then under suitable conditions the operator A generates a semi-group $\{T_t; t \geq 0\}$ or a group $\{T_t; -\infty < t < \infty\}$ of bounded linear operators such that the solution of (I-3) exists and is given by

$$\phi(t,x) = T_t x \qquad (t \geq 0)$$

for any $x \varepsilon \mathcal{D}(A)$. Thus the stability property of solutions to (I-3) is related to the property of the semi-group or group generated by A. Based on the known properties of the semi-group or group, Buis established sufficient conditions for A to generate a negative semi-group (of class $C_o$) and necessary and sufficient conditions for A to generate a negative group (see definitions III-9 and III-10) so that a solution of (I-3) exists and is asymptotically stable. All these conditions refer to the existence of a Lyapunov functional which is defined through a symmetric bilinear form. Following the same idea as in [3], Vogt, Buis and Eisen [21] considered a closed linear operator from a Banach space into itself and established the necessary and sufficient conditions for A to generate a negative group by using a semi-scalar product. Their results are, in fact, an extension of [3] for the case of a group from a Hilbert space into a Banach space.

B. Nonlinear Operational Differential Equations

In recent years, most of the investigations of differential equations (both ordinary and partial) are centered on nonlinear equations. This is perhaps due to the fact that many physical problems must be formulated by nonlinear differential equations as well as that nonlinear equations possess many properties of theoretical interest. In the case of operational differential equations, many results on the existence and uniqueness of semi-linear equations of the form similar to (I-2) have been established (e.g., see Browder [1], Kato [9]). Just recently (1967), Kōmura [13] studied an equation of evolution of the form

$$\frac{dx(t)}{dt} = Ax(t) \qquad (t \geq 0) \qquad \text{(I-4)}$$

where A is, in general, a nonlinear operator with domain and range in a Hilbert space H and x(t) is a vector-valued function defined on $R^+ = [0, \infty)$ to H. In his work, a general theory for nonlinear semi-groups of contraction operators in a Hilbert space is developed. However, Kōmura considered A of (I-4) as a multi-valued operator which makes his theory rather complicated. Motivated by the work in [13], Kato [11] refined and extended considerably Kōmura's results by considering a single-valued operator A(t) with domain and range both in a Banach space X where the operator A of (I-4) is also extended to A(t) which depends on the variable t. Following [13] and [11], Browder [2] further extended (in some sense), among others, Kato's results by including an additional function f(t,x) on the right of (I-4) with the simplification that the underlying space X is a real Banach space. All the above works are mainly concerned with the existence and uniqueness of solutions.

C. Area for Extension and New Development

It is seen in [3] that necessary and sufficient conditions for the operator A in (I-3) to generate a negative group (of class $C_o$), and that sufficient conditions for A to generate a semi-group were established by assuming the existence of a Lyapunov functional. Conversely, if A generates an equibounded or negative semi-group, is it possible to construct a Lyapunov functional as in the case for a group? Since the extension in [21] to a real Banach space of the above mentioned results in [3] was accomplished only for the case of a group, the investigation for a similar extension for a semi-group is also necessary. On the other hand, the class of nonlinear differential equations, either time-invariant or time-varying, are more important from both the applications and the theoretical points of view. All of these need further investigation. The introduction of the concept of nonlinear semi-groups opens a new road to the problem of nonlinear operational differential equations. The importance of the study of the stability problem by using the semi-group or nonlinear semi-group theory lies in the fact that the important problem of establishing the existence of a solution is an intrinsic part of the theory developed.

## II. STATEMENT OF PROBLEM

Many systems of partial differential equations can be written in the form of

$$\frac{\partial u(t,X)}{\partial t} = Lu(t,X) \qquad X\epsilon\Omega, \qquad t \geq 0 \tag{II-1}$$

where $u(t,X)$ is an m-vector function and L is a matrix whose elements are linear or nonlinear partial differential operators defined on a subset $\Omega$ of an n-dimensional Euclidean space $R^n$. In more general cases, the coefficients of the elements in L are both space and time dependent (linear or nonlinear). To specify solutions to the equation (II-1), a set of boundary conditions are given which can be put into the form

$$B\, u(t,X') = 0 \qquad X'\epsilon\partial\Omega, \tag{II-2}$$

where B is a matrix whose elements are linear or nonlinear partial differential operators and $\partial\Omega$ is the boundary of $\Omega$. In addition, an initial condition is given as

$$u(0,X) = u_o(X) \tag{II-3}$$

where $u_o(X)$ is a given space-dependent function. If all the elements of L and B are linear differential operators, (II-1) and (II-2) can be reduced to the form

$$\frac{dx(t)}{dt} = Ax(t) \tag{II-4}$$

where $x(t)$ is a vector-valued function (in the sense of a linear function space) defined on $R^+$ to a suitable Banach space or Hilbert space X and A is a (in general unbounded) linear operator from part of X to X; if one or more elements of L or B is nonlinear, then A is a nonlinear operator from part of X to X; in case one or more elements of L or B is space-time dependent, the systems (II-1) and (II-2) are reduced to the form

(II-4) with A replaced by A(t) which is a linear or nonlinear operator depending on t. In all cases, (II-1) and (II-2) can be considered as special cases of abstract operational differential equations which can be parabolic equations and certain hyperbolic equations, etc. The object of this research is to establish some stability criteria which intrinsically include the existence and uniqueness of solutions for the types of differential equations described above in an abstract setting, from which the behaviors of the corresponding type of partial differential equations can be deduced. The first two sections in the following introduce the types of operational differential equations (i.e., equations of evolution) to be investigated and the final section summarizes the results obtained in this investigation.

A. Linear Time-invariant Differential Equations

It has been seen in Chapter I that by using the semi-group or group theory, a Lyapunov stability theory for the linear operational differential equations of the form (II-4) in a real Hilbert space was established in [3]. There, a Lyapunov functional is defined through a symmetric bilinear functional. The main results concerning the equation of the form (II-4) is that if the domain of A is dense in H and the range of (I-A) is H (I is the identity operator) then A is the infinitesimal generator of a negative semi-group (of class $C_o$) if there exists a Lyapunov functional satisfying certain properties and it is the infinitesimal generator of a negative group (of class $C_o$) if and only if there exists a Lyapunov functional satisfying some additional properties. Unlike a group, however, a semi-group lacks the property of having a lower bound (in some sense) which makes the construction of a Lyapunov functional through a bilinear functional rather difficult.

Because of this difficulty the results given in [3] for the case of a semi-group do not parallel the case of a group, that is, the necessary condition for the existence of a Lyapunov functional having the desired property is not shown. To overcome this, an equivalent semi-scalar product is introduced. If the operator A in (II-4) is the infinitesimal generator of an equibounded or negative semi-group, a Lyapunov functional can be constructed through an equivalent semi-scalar product which gives the converse statement in [3] as described above. Moreover, by using the same idea in defining a Lyapunov functional, necessary and sufficient conditions for A to generate an equibounded or negative semi-group in the case of a real Banach space can also be established. This later extension to a Banach space is in analogy to the one in [21] for the case of a group. It is seen that with these additional extensions, the stability study of linear operational differential equation (II-4) by using semi-group or group theory would be, in a sense, completed (there is no difficulty in extending the above results to complex spaces).

B. Development of Nonlinear Operational Differential Equations

Owing to the importance of nonlinear differential equations in both pure theory and its applications, the investigation of the nonlinear operational differential equations is the main concern of this dissertation. The first stage in the development of nonlinear operational differential equations is to study the equations of evolution of the form

$$\frac{dx(t)}{dt} = Ax(t) \qquad (t \geqq 0) \qquad \text{(II-5)}$$

where x(t) is a vector-valued function defined on $R^+ = [0, \infty)$ to a Hilbert space H (in general, H is a complex Hilbert space) and A is a nonlinear operator (which is independent of t) with domain and range both in H.

Based on the results obtained by Kato in [11] in which the operator ($-A$) is assumed to be monotone (i.e., $A$ is dissipative in the terminology of this dissertation) and by using the nonlinear semi-group property, a stability theory as well as the extence and uniqueness theory for the equation (II-5) can be developed. Moreover, by introducing an equivalent inner product, the same results hold if the operator $A$ is dissipative with respect to this equivalent inner product. This fact motivates the construction of a Lyapunov functional through a sesquilinear functional which under some additional conditions defines an equivalent inner product. Thus a stability criteria can be established through the construction of a Lyapunov functional.

As a special case of (II-5), the semi-linear equations of evolution of the form

$$\frac{dx(t)}{dt} = A_o x(t) + f(x(t)) \qquad (t \geqq 0) \qquad \text{(II-6)}$$

is discussed to some extent where $A_o$ is an unbounded linear operator with domain and range both in a real Hilbert space $H$ and $f$ is a (nonlinear) function defined on $H$ into $H$. The purpose of doing this is that by utilizing the results established on the linear equation (II-4) (i.e., for $f(x) \equiv 0$ in (II-6)), the existence, uniqueness and stability or asymptotic stability of a solution to (II-6) can be ensured by imposing some additional conditions on the function $f$. Notice that (II-6) is a direct extension of the linear equation (II-4).

In case the elements of the partial differential operator $L$ in (II-1) or the elements of $B$ in the boundary conditions (II-2) possess time-dependent coefficients, equation (II-5) is not suitable as an abstract extension for this type of partial differential equation. The second stage in the development is to extend equation (II-5) to a more general type of operational differential equation of the form

$$\frac{dx(t)}{dt} = A(t)x(t) \qquad (t \geqq 0) \tag{II-7}$$

where $A(t)$ is, for each $t \geqq 0$, a nonlinear operator with domain and range both contained in a Hilbert space $H$. It is seen that this extension is a further advance in the generalization of nonlinear equations of evolution. In parallel to the case of the equation (II-4), criteria for the existence, uniqueness, stability and, in particular, asymptotic stability of a solution as well as the stability region are established. The concept of equivalent inner product is similarly introduced, and it is shown that stability property remains unchanged under equivalent inner product.

In the case of semi-linear equations of the form

$$\frac{dx(t)}{dt} = A_o(t)x(t) + f(t,x(t)) \qquad (t \geqq 0) \tag{II-8}$$

where $A_o(t)$ is, for each $t \geqq 0$, a linear unbounded operator with domain and range both in $H$ and $f$ is a (nonlinear) function defined on $R^+ x\ H$ into $H$, stability criteria are deduced from the results for the general equation (II-7). For the sake of applications as well as theoretical interest in certain partial differential equations which occur often in physical problems, some special equations of (II-7) are included. These equations can be written in the general form

$$\frac{dx(t)}{dt} = Ax(t) + f(t,x(t)) \qquad (t \geqq 0) \tag{II-9}$$

where $A$, which is independent of $t$, is a linear or nonlinear operator with domain and range both in a real Hilbert space $H$ and $f$ is a (nonlinear) function defined on $R^+ x\ H$ into $H$. The idea for considering equations of the form (II-9) is to transform and to simplify the conditions imposed on the general operator $A(t)$ into the conditions on $A$ and on $f$ so that the existence, uniqueness and stability or asymptotic stability of a solution as well as the stability region can be guaranteed. In case $A$ is linear and is the infinitesimal generator of a semi-group of class $C_0$ or is a self-adjoint operator, the results are particularly suitable for applications to certain partial differential equations. When $A$ is a bounded operator on $H$ into $H$, (II-9) can be put into the form

$$\frac{dx(t)}{dt} = f(t,x(t)) \qquad (t \geq 0) \tag{II-10}$$

which is, in fact, an ordinary differential equation. Criteria for the existence and stability of a solution are also given for this case.

C. Summary of Results and Contributions to the Problem

The object of this research is to establish a stability theory so that a solution of a given operational differential equation (i.e., equation of evolution) not only exists and is unique but also is stable or asymptotically stable. This given operational differential equation is, in general, an abstract generalization of a class of partial differential equations such as heat conduction equations and wave equations etc,. The contribution of this dissertation is the establishment of criteria for the existence, uniqueness, stability, asymptotic stability and stability region of a solution on several types of nonlinear (including linear) operational differential equations. This contribution can be stated as four stages which are discussed in chapters IV, V, VI and VII respectively. The results obtained in these chapters are summarized as follows:

(a) In chapter IV, the central idea is to show the existence of a Lyapunov functional and to show the necessary and sufficient conditions for the operator A to generate an equibounded or negative semi-group in a Banach space from which the existence and stability or asymptotic stability of a solution are ensured. This is done in theorems IV-7, IV-8, IV-11, IV-12 and IV-13.

(b) The central idea in chapter V is to establish a stability theory for nonlinear operational differential equations by extending the theory of linear semi-groups to nonlinear semi-groups with the hope that this theory can be applied to some nonlinear partial differential equations. Results on general nonlinear equations are given in theorems V-2 through V-9 and on semi-linear equations are given in theorems V-11, V-12, V-15, V-16 and V-17.

(c) The object in chapter VI is to extend the stability theory for time-invariant nonlinear equations in chapter V to time-varying nonlinear equations with the hope that this theory might be used for a larger class of non-stationary partial differential equations. Particular attention has been given to several special cases which are easier to apply for certain partial differential equations. Results on general nonlinear equations are given in theorems VI-2 through VI-5, those on nonlinear nonstationary equations are given in theorems VI-6 and VI-7 and those on semi-linear equations are given in theorems VI-8, VI-9, VI-13, VI-14 and VI-15.

(d) Finally, the applications of the results developed for operational differential equations to partial differential equations are given in chapter VII in which stability criteria for a class of parabolic-elliptic partial differential equations are established and are given in theorems VII-2, VII-4 and VII-6.

It is seen from this summary that the results of this dissertation cover several types of differential equations, and to the knowledge of this author, most of the above results on the part of stability theory have not been previously shown. It is thought that these results contribute to the stability theory of operational differential equations as well as of partial differential equations.

# III. A PRELIMINARY ON FUNCTIONAL ANALYSIS

Because of the importance of functional analysis in the study of operational differential equations (i.e., equations of evolution), it is desirable to give some of the basic definitions and properties that will be used in the stability analysis of operational differential equations. The following sections give an outline of some of the necessary topics. Proofs and further details may be found in most standard books on this subject (for example, references [5], [8], [10], [12] and [23]), in particular, most of the materials in this chapter can be found in [23].

## A. Banach and Hilbert Spaces

A set X is called a linear space over a field K if the following conditions are satisfied:

(i) X is an Abelian group (written additively);

(ii) A scalar multiplication is defined: to every element $x \in X$ and each $\alpha \in K$ there is associated an element of X, denoted by $\alpha x$, such that

$$\alpha(x+y) = \alpha x + \alpha y \qquad (\alpha \in K;\ x,y \in X),$$

$$(\alpha+\beta)x = \alpha x + \beta x \qquad (\alpha, \beta \in K;\ x \in X),$$

$$(\alpha\beta)x = \alpha(\beta x) \qquad (\alpha, \beta \in K;\ x \in X),$$

$$1 \cdot x = x \quad (1 \text{ is the unit element of the field } K).$$

Let X be a linear space over the field of real or complex numbers. If for every $x \in X$, there is associated a real number $||x||$, the norm of the vector x, such that for any $\alpha \in K$ and any $x,y \in X$

(i) $||x|| \geqq 0$, and $||x|| = 0$ if and only if $x = 0$,

(ii) $||x+y|| \leqq ||x|| + ||y||$ ,

(iii) $||\alpha x|| = |\alpha|\ ||x||$.

Then the linear space X together with the norm $||\cdot||$ is called a normed linear space and is denoted by $(X, ||\cdot||)$ or simply by X. A sequence $\{x_n\}$ in a normed linear space X is called a Cauchy sequence if for any $\varepsilon > 0$, there exists an integer $N=N(\varepsilon) > 0$ such that $||x_m - x_n|| < \varepsilon$ for all $m,n \geqq N$. If every Cauchy sequence in X converges to an element $x \in X$, the space is said to be a complete normed linear space or a Banach space (or simply a B-space). The convergence is said to be a strong convergence (or norm convergence) and is designated by $\lim x_n \to x$ as $n \to \infty$ or simply by $x_n \to x$. X is said to be a real or a complex Banach space according to whether the field K is the real or complex numbers. A complex linear space is called a complex inner product space (or a pre-Hilbert space) if there is defined on X x X a complex-valued function $(x,y)$, called the inner product of x and y, with the following properties:

(i) $(x+y,z) = (x,z) + (y,z)$

(ii) $(x,y) = \overline{(y,x)}$ (the bar denoting complex conjugate)

(iii) $(\alpha x,y) = \alpha (x,y)$

(iv) $(x,x) \geqq 0$, and $(x,x) = 0$ if and only if $x = 0$.

A real linear space is called a real inner product space if the properties (i)-(iv) are satisfied except that (ii) is replaced by $(x,y) = (y,x)$. By defining $||x|| = (x,x)^{1/2}$, an inner product space is a normed linear space and the norm is said to be induced by the inner product $(\cdot,\cdot)$. The converse is, in general, not true. However, if the norm in a normed linear space X (real or complex) satisfies the parallelogram law:

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2) \quad x,y \in X$$

then an inner product can be defined so that X is an inner product space. If an inner product space H (real or complex) is complete with respect to the norm induced by the inner product $(\cdot,\cdot)$, it is called a Hilbert space or an H-space and is denoted by $(H, (\cdot,\cdot))$ or simply by H. H is called a real or complex Hilbert space if K is the field of real or complex numbers respectively. A Hilbert space is a special Banach space. By the properties of (i), (ii), (iii) of an inner product, it is seen that an inner product is bilinear for a real Hilbert space and is sesquilinear for a complex Hilbert space. The sesquilinearity means that:

$$(\alpha_1 x + \alpha_2 y, z) = \alpha_1(x,z) + \alpha_2(y,z), \quad (\alpha_1,\alpha_2 \in K, \quad x,y,z \in H)$$
$$(x, \beta_1 y + \beta_2 z) = \bar{\beta}_1(x,y) + \bar{\beta}_2(x,z) \quad (\beta_1,\beta_2 \in K, \quad x,y,z \in H).$$

If $\bar{\beta}_1$ and $\bar{\beta}_2$ in the above equality are replaced by $\beta_1$ and $\beta_2$ respectively, the inner product is said to be bilinear.

Examples of Banach space and Hilbert space:

(1) $(\ell^p)$, $1 \leq p < \infty$: The set of all sequences $x = (x_1, x_2, \ldots)$ of complex numbers such that $\sum_{i=1}^{\infty} |x_i|^p < \infty$ constitutes a normed linear space $(\ell^p)$ by the norm $||x|| = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$. $(\ell^p)$ is a Banach space; in particular $\ell^2$ is a Hilbert space with the inner product defined by $(x,y) = \sum_{i=1}^{\infty} x_i \bar{y}_i$.

(2) $L^p(\Omega)$, $1 \leq p < \infty$: The set of all real valued (or complex-valued) measurable functions f(x) defined a.e. (almost everywhere) on $\Omega$, where $\Omega$ is an open subset of $R^n$, such that $|x(s)|^p$ is Lebesque integrable

over $\Omega$ constitutes a normed linear space $L^p(\Omega)$; it is a linear space by

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (\alpha f)(x) = \alpha f(x)$$

and the norm is defined by

$$||x|| = \left( \int_\Omega |f(x)|^p dx \right)^{1/p} \quad (dx = dx_1 dx_2 \cdots dx_n).$$

$L^p(\Omega)$ is a Banach space whose elements are the classes of equivalent $p^{th}$-power integrable functions. In particular, $L^2(\Omega)$ is a Hilbert space with the inner product defined by

$$(f, g) = \int_\Omega f(x)\, \overline{g(x)}\, dx.$$

Let X be a normed linear space. A point $x \in X$ is said to be a limit point of a set $D \subset X$ if there exists a sequence of distinct elements $\{x_n\} \subset D$ such that $x_n \to x$ as $n \to \infty$. The closure of a set D, denoted by $\bar{D}$, is the set comprised of D and all the limit points of D. A set D is said to be closed if $D = \bar{D}$ and is said to be dense in X if $\bar{D} = X$. Hence if D is closed and dense in X then $D = X$.

<u>Definition III-1</u>. Let $X_1 = (X, ||\cdot||_1)$, $X_2 = (X, ||\cdot||_2)$ where X is a linear space. The two norms $||\cdot||_1$ and $||\cdot||_2$ are said to be equivalent if there exist real numbers $\delta$, $\gamma$ with $0 < \delta \leq \gamma < \infty$ such that

$$\delta \, ||x||_2 \leq ||x||_1 \leq \gamma \, ||x||_2 \quad \text{for all } x \in X.$$

Thus, if $X_1$ is a Banach space so is $X_2$.

<u>Definition III-2</u>. A normed linear space is uniformly convex if for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $||x|| \leq 1$, $||y|| \leq 1$ and $||x-y|| \geq \varepsilon$ implies $||x+y|| \leq 2(1-\delta)$.

A Hilbert space is uniformly convex, for by the parallelogram law if $||x|| \leq 1$, $||y|| \leq 1$ and $||x-y|| \geq \varepsilon$ then

$$||x+y||^2 = 2||x||^2 + 2||y||^2 - ||x-y||^2 \leq 4-\varepsilon^2$$

which implies that $||x+y|| \leq 2(1-\delta)$ for some $\delta = \delta(\varepsilon) > 0$.

## B. Linear and Nonlinear Operators

Let X and Y be linear spaces on the same field of scalars K. Let A be an operator (or function or mapping) which maps part of X into Y. The domain of A, denoted by $\mathcal{D}(A)$, is the set of all $x \in X$ such that there exists a $y \in Y$ for which $Ax=y$. The range of A, denoted by $\mathcal{R}(A)$, is the set $\{Ax; x \in \mathcal{D}(A)\}$. The null space (or kernel) of A is $\mathcal{N}(A) = \{x; Ax = 0\}$. If $\mathcal{D}(A_1) \subset \mathcal{D}(A_2)$ and $A_1x = A_2x$ for all $x \in \mathcal{D}(A_1)$, then $A_2$ is called an extension of $A_1$ or $A_1$ is called a restriction of $A_2$ and this is denoted by $A_1 \subset A_2$. If $\mathcal{D}(A_1) = \mathcal{D}(A_2)$ and $A_1x = A_2x$ for all $x \in \mathcal{D}(A_1)$, then $A_1 = A_2$. The operator A is called one-to-one if distinct elements in $\mathcal{D}(A)$ are mapped into distinct elements of $\mathcal{R}(A)$ and in this case, A is said to have an inverse and is denoted by $A^{-1}$. An operator A with domain $\mathcal{D}(A)$ a linear subspace of X and range $\mathcal{R}(A)$ in Y is called linear if for all $x,y \in \mathcal{D}(A)$ and all $\alpha,\beta \in K$, $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$, and is called nonlinear if it is not linear. A linear operator A is one-to-one if and only if $\mathcal{N}(A) = \{0\}$.

If X and Y are normed linear spaces and T is a linear operator with $\mathcal{D}(T) \subset X$ and range $\mathcal{R}(T) \subset Y$, the following statements are equivalent: (a) T is continuous on $\mathcal{D}(T)$, (b) T is bounded, i.e., there exists a number $M > 0$ such that for all $x \in \mathcal{D}(T)$, $||Tx|| \leq M||x||$ (note that the two norms of the inequality are, in general, not the same). If T is bounded, the norm of T is defined by:

$$||T|| = \sup(||Tx||; ||x|| \leq 1, x \in \mathcal{D}(T)).$$

With this norm, the space of all bounded linear operators with domain X and range in Y denoted by $L(X,Y)$ is a normed linear space if we define addition of operators and multiplication of operators by scalars in the natural way, namely

$$(T+S)x = Tx+Sx. \quad (\alpha T)x = \alpha Tx \qquad T, S \in L(X,Y) \text{ and } x \in X.$$

If, in addition, Y is a Banach space, so is $L(X,Y)$.

Let X, Y be normed linear spaces on the same scalar field. Then the product space X x Y is a normed linear space of all ordered pairs $\{x,y\}$ $x \in X$, $y \in Y$ with addition and scalar multiplication defined by

$$\{x_1,y_1\} + \{x_2,y_2\} = \{x_1 + x_2, y_1 + y_2\}$$
$$\alpha\{x,y\} = \{\alpha x, \alpha y\}$$

and with norm given by

$$||\{x,y\}|| = (||x||^2 + ||y||^2)^{1/2}.$$

If X and Y are Banach spaces, so is X x Y. If T is a linear operator with $\mathcal{D}(T) \in X$ and $\mathcal{R}(T) \in Y$, the graph of T, $\mathcal{G}(T)$, is the set $(\{x,Tx\};$ $x \in \mathcal{D}(T))$. Since T is linear, $\mathcal{G}(T)$ is a subspace of X x Y. A linear operator T is said to be closed in X if the graph $\mathcal{G}(T)$ of T is closed in X x Y. A useful criterion to test whether a linear operator is closed is the following: A linear operator T is closed if and only if $x_n \in \mathcal{D}(T)$, $x_n \to x$, $Tx_n \to y$ imply $x \in \mathcal{D}(T)$ and $Tx = y$. The above criterion is sometimes used as the definition of a closed operator. If T is closed then the inverse $T^{-1}$, if it exists, is closed. It is to be noted that a continuous (or bounded) linear operator need not be closed and a closed operator may be unbounded. However, if T is continuous and Y is a Banach space, T has a unique extension $\bar{\bar{T}}$ to $\mathcal{D}(\bar{\bar{T}})$ such that $||\bar{\bar{T}}|| = ||T||$ and $\bar{\bar{T}}$ is closed; if in addition, $\mathcal{D}(T)$ is dense in a Banach space X, then $\bar{\bar{T}} \in L(X,Y)$. The following theorem is known as the Banach Closed Graph Theorem.

<u>Theorem III-1.</u> A closed linear operator T defined on a Banach space X into a Banach space Y is continuous.

A linear operator T is said to be closable if there exists a linear extension of T which is closed in X. When T is closable, there is a closed operator $\bar{T}$ with $G(\bar{T}) = \overline{G(T)}$; $\bar{T}$ is called the closure of T and is the smallest closed extension of T, in the sense that any closed extension of T is also an extension of $\bar{T}$. A linear operator T is closable if and only if $x_n \varepsilon \mathcal{D}(T)$, $x_n \to 0$ and $Tx_n \to y$ imply that $y=0$. In such cases, the closure $\bar{T}$ of T can be defined as follows: $x \varepsilon \mathcal{D}(\bar{T})$ if and only if there exists a sequence $\{x_n\} \subset \mathcal{D}(T)$ such that $x_n \to x$ and $\lim_{n\to\infty} Tx_n = y$ exists; and we define $\bar{T}x = y$. It can be shown that y is uniquely defined by x and $\bar{T}$ is closed. Let X and Y be normed linear spaces on the same scalar field and T be a one-to-one operator with $\mathcal{D}(T) \subset X$ and $\mathcal{R}(T) \subset Y$. The inverse of T is the map from $\mathcal{R}(T)$ into X given by $T^{-1}(Tx) = x$. If T is linear, then $T^{-1}$ is linear with domain $\mathcal{R}(T)$ and range $\mathcal{D}(T)$. $T^{-1}$ exists and is continuous if and only if there exists an $m > 0$ such that $||Tx|| \geqq m||x||$ for $x \varepsilon \mathcal{D}(T)$. If this is the case, $||T^{-1}|| \leqq m^{-1}$. $T^{-1}$ is closed if and only if T is closed.

Definition III-3. Let $H = (H, (\cdot,\cdot))$ be a Hilbert space and S be an operator with domain dense in H and range in H. The adjoint operator of S, denoted by S*, is defined as follows: $y \varepsilon H$ is in the domain of S* if and only if there exists a $y^* \varepsilon H$ such that

$$(Sx,y) = (x,y^*) \quad \text{for all } x \varepsilon \mathcal{D}(S)$$

and we define $S^*y = y^*$. S* exists if and only if $\mathcal{D}(S)$ is dense in H and in this case, S* is a closed linear operator. S is called symmetric if $S \subset S^*$, i.e., S* is an extension of S, and is called self-adjoint if $S=S^*$. Thus, a self-adjoint operator is closed. S is said to be positive definite if there exists a $\delta > 0$ such that

$$(Sx,x) \geqq \delta||x||^2 \quad \text{for all } x \varepsilon \mathcal{D}(S).$$

Let X and Y be normed linear spaces. Suppose T is a linear operator with domain X and range in Y. T is said to be completely continuous (or compact) if, for each bounded sequence $\{x_n\}$ in X, the sequence $\{Tx_n\}$ contains a subsequence converging to some limit in Y. Compact operators possess many interesting properties (see, e.g., [23]). Since these properties are not needed in the present discussion of stability analysis we shall not state them here.

C. Linear Functionals, Conjugate Spaces and Weak Convergence

A numerical function f(x) defined on a normed linear space X is called a functional. A functional is said to be linear if for any $x, y \in X$ and $\alpha, \beta \in K$ (real or complex number field)

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y);$$

and it is said to be continuous if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$||x-y|| < \delta \quad \text{implies} \quad |f(x)-f(y)| < \varepsilon.$$

f is said to be bounded if there exists a constant M such that

$$|f(x)| \leqq M||x|| \quad \text{for all } x \in X.$$

The following statements are equivalent: (a) f is continuous at any fixed element $x_o \in X$; (b) f is continuous on X; (c) f is uniformly continuous on X; (d) f is bounded on X.

Let X,Y be normed linear spaces on the same scalar field of real or complex numbers and let $L(X,Y)$ be the class of all bounded linear operators on X to Y. If Y is the real or complex number field topologized in the usual way (i.e., the absolute value $|\alpha|$ is taken as the norm of $\alpha$ in Y), $L(X,Y)$ is called the conjugate space (or dual space or adjoint space) of X and is denoted by $X^*$. Thus $X^*$ is the set of all continuous linear functionals on X. The pairing between any elements x of X and f of $X^*$ is denoted by $f(x)$ or by $\langle x, f\rangle$. If we define the norm of $f \in X^*$ by

$$||f|| = \sup_{||x||\leq 1} |f(x)|$$

then $X^*$ is a Banach space. Note that X is not necessarily a Banach space. For a given normed linear space X, the existence of a non-trivil continuous linear functional on X can be ensured by the Hahn-Banach extension theorem which is stated as follows for the case of a normed linear space.

Theorem III-2 (Hahn-Banach theorem). Let X be a normed linear space, M a linear subspace of X and f a continuous linear functional defined on M. Then there exists a continuous linear functional F defined on X such that F is an extension of f (i.e., $F(x) = f(x)$ for all $x \in M$) with $||F|| = ||f||$.

A direct consequence of the Hahn-Banach theorem is the following:

Theorem III-3. Let X be a normed linear space and $x_o \neq 0$ be any element of X. Then there exists a continuous linear functional f on X such that $f(x_o) = ||x_o||^2$ and $||f|| = ||x_o||$.

Corollary. If $f(x) = 0$ for every $f \in X^*$ then $x = 0$. In particular, if $f(x) = f(y)$ for every $f \in X^*$ then $x = y$.

In case X is a Hilbert space, $X^*$ can be identified with X as can be seen from the Riesz representation theorem.

Theorem III-4 (Riesz representation theorem). For any linear functional f on a Hilbert space $H = (H, (\cdot,\cdot))$, there exists an element

$y_f \in H$, uniquely determined by the functional $f$, such that

$$f(x) = (x, y_f) \quad \text{for every } x \in H.$$

Moreover, $||f|| = ||y_f||$.

<u>Corollary</u>. Let H be a Hilbert space. Then the totality of all bounded linear functionals $H^*$ on H constitutes also a Hilbert space, and there is a norm-preserving, one-to-one correspondence $f \leftrightarrow y_f$ between $H^*$ and H.

It should be remarked here that by the correspondence in the above corollary, $H^*$ may be identified with H as an abstract set; but it is not allowed to identify, by this correspondence, $H^*$ with H as linear spaces, since the correspondence $f \leftrightarrow y_f$ is conjugate linear:

$$(\alpha_1 f_1 + \alpha_2 f_2) \leftrightarrow (\bar{\alpha}_1 y_{f_1} + \bar{\alpha}_2 y_{f_2})$$

where $\alpha_1$, $\alpha_2$ are complex numbers. However if we define the space $H^*$ to be the set of all bounded semi-linear forms on H (i.e., by defining $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ and $(\alpha f)x = \bar{\alpha} f(x)$ for any $x \in H$, $f \in H^*$ and $\alpha \in K$, the complex field) then H can be identified with $H^*$ not only as an abstract set but also as a linear space.

Let X be a normed linear space and $X^*$ its conjugate space. The conjugate space of $X^*$, denote by $X^{**}$, is called the second conjugate (or second dual or bidual) of X. Obviously, $X^{**}$ is a Banach space. It can be shown that each $x_o \in X$ defines a continuous linear functional $f_o(x^*)$ on $X^*$ by $f_o(x^*) = \langle x_o, x^* \rangle$. The mapping

$$x_o \to f_o = Jx_o$$

of X into $X^{**}$ satisfies the conditions

$$J(x_1 + x_2) = Jx_1 + Jx_2, \quad J(\alpha x) = \alpha J(x), \quad \text{and } ||Jx|| = ||x||.$$

The mapping J is called the canonical mapping of X into $X^{**}$.

Definition III-4. A normed linear space X is said to be reflexive if X may be identified with its second dual X** by the correspondence $x \leftrightarrow Jx$ above.

In general, a Banach space X can be identified with only a subspace of its second dual space X**. However, under the condition of local compactness of X, it may be identified with X**. The following theorem is important in view of its applications.

Theorem III-5 (Eberlein-Shmulyan). A Banach space X is reflexive if and only if every strongly bounded sequence of X contains a subsequence which converges weakly to an element of X (i.e., locally sequentially compact).

For a proof of the above theorem see, e.g., [23].

Theorem III-6. A uniformly convex Banach space is reflexive. In particular, a Hilbert space is reflexive.

It is known that, for $1 < p < \infty$, the spaces $L^p$ and $\ell^p$ are uniformly convex (see Clarkson [4]) and thus are reflexive.

In the development of stability theory in Chapters V and VI, we have introduced the concept of equivalent inner product. The following theorem which was formulated by P. Lax and A. N. Milgram plays an important role in the construction of an equivalent inner product.

Theorem III-7 (Lax-Milgram). Let H be a Hilbert space. Let $V(x,y)$ be a complex-valued functional defined on the product space $H \times H$ which satisfies the conditions:

(i) Sesqui-linearity, i.e.,

$V(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 V(x_1, y) + \alpha_2 V(x_2, y)$ and

$V(x, \beta_1 y_1 + \beta_2 y_2) = \bar{\beta}_1 V(x, y_1) + \bar{\beta}_2 V(x, y_2)$.

(ii) Boundedness, i.e., there exists a positive constant $\gamma$ such that

$$|V(x,y)| \leq \gamma||x|| \; ||y||.$$

(iii) Positivity, i.e., there exists a positive constant $\delta$ such that

$$V(x,x) \geq \delta||x||^2.$$

Then there exists a uniquely determined bounded linear operator S with a bounded linear inverse $S^{-1}$ such that

$$V(x,y) = (x,Sy) \quad \text{whenever} \quad x,y \in H$$

and $||S|| \leq \gamma$, $||S^{-1}|| \leq \delta^{-1}$.

A proof of the above theorem can be found in [23].

<u>Definition III-5</u>. A sequence $\{x_n\}$ in a normed linear space X is said to converge weakly to an element $x \in X$ if $\lim_{n\to\infty} f(x_n) = f(x)$ for every $f \in X^*$. In this case, x is uniquely determined in virtue of Hahn-Banach theorem; we shall write $\underset{n\to\infty}{\text{w-lim}}\, x_n = x$ or simply $x_n \xrightarrow{w} x$ in the sense of weak convergence. It is to be recalled that $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ denotes convergence in the strong topology (i.e., norm topology).

<u>Theorem III-8</u>. Let $\{x_n\}$ be a sequence of elements in a normed linear space X. (a) If $x_n \to x$ then $x_n \xrightarrow{w} x$ but not conversely. (b) If $x_n \xrightarrow{w} x$ then $||x_n|| < \infty$ for all n and $||x|| \leq \underline{\lim}_{n\to\infty} ||x_n||$. (c) $x_n \xrightarrow{w} x$ if and only if (i) $\sup_{n\geq 1} ||x_n|| < \infty$, and (ii) $\lim_{n\to\infty} f(x_n) = f(x)$ for every $f \in D$ where D is a dense subset of X* (in the strong topology of X*).

As an example of a weakly convergent sequence which is not strongly convergent, we take the sequence of vectors

$$e_1 = (1,0,0,\ldots.), \quad e_2 = (0,1,0,\ldots), \quad \ldots$$

in the Hilbert space $(\ell^2)$. This sequence converges weakly to zero since by theorem III-4, given any $f \in (\ell^2)^*$ there exists an $x = (x_1, x_2, \dots)$ $\in (\ell^2)$ such that $f(e_n) = (e_n, x) = x_n \to 0$. However, $\{e_n\}$ does not converge strongly to zero since $||x_n|| = 1$ for every $n = 1, 2, \dots$.

In a Hilbert space H, if the sequence $\{x_n\}$ of H converges weakly to $x \in H$ and $\lim_{n\to\infty} ||x_n|| = ||x||$, then $\{x_n\}$ converges strongly to x. In the case of a finite dimensional space, weak convergence coincides with strong convergence. Weak convergence is related to the weak topology of X, as strong convergence is related to the strong topology. In the development of our results, there is no need of the deeper notion of weak topology; the use of the simple notion of weak convergence is sufficient for our purpose.

<u>Definition III-6</u>. A sequence $\{f_n\}$ in the conjugate space X* of a normed linear space X is said to converge weakly* to an element $f \in X^*$ if $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$. We shall write $w^*\text{-}\lim_{n\to\infty} f_n = f$ or simply $f_n \xrightarrow{w^*} f$.

<u>Theorem III-9</u>. Let $\{f_n\}$ be a sequence of elements in the conjugate space X* of a normed space X. (a) If $f_n \to f$ then $f_n \xrightarrow{w^*} f$ but not conversely. (b) If X is a Banach space and if $f_n \xrightarrow{w^*} f$ then $||f_n|| < \infty$ for every n and $||f|| \leq \underline{\lim}_{n\to\infty} ||f_n||$.

The weak continuity and weak differentiability are defined similarly.

<u>Definition III-7</u>. Let x(t) be a vector-valued function defined on $[0, \infty)$ to X. x(t) is said to be weakly continuous in t if $\langle x(t), f\rangle$ is continuous for each $f \in X^*$; it is said to be weakly differentiable in t if $\langle x(t), f\rangle$ is differentiable for each $f \in X^*$. If the derivative of $\langle x(t), f\rangle$ has the form $\langle y(t), f\rangle$ for each $f \in X^*$, y(t) is the weak derivative of x(t) and we write $dx(t)/dt = y(t)$ weakly. Similar terminology applies if x(t) is defined on $(-\infty, \infty)$.

Theorem III-10. For any interval $(a,b)$, if $x(t)$ is weakly differentiable for $t \varepsilon (a,b)$ with weak derivative identically zero, then by using the corollary of theorem III-3 $x(t)$ is constant.

D. Spectral Theory, Semi-groups and Groups

Let T be a linear operator with domain $\mathcal{D}(T)$ and range $\mathcal{R}(T)$ both contained in a normed linear space X. The distributions of values $\lambda$ for which the linear operator $(\lambda I-T)$ has an inverse and the properties of the inverse when it exists are called the spectral theory for the operator T.

Definition III-8. If $\lambda_o$ is such that $\mathcal{R}(\lambda_o I-T)$ is dense in X and $\lambda_o I-T$ has a continuous inverse $(\lambda_o I-T)^{-1}$, $\lambda_o$ is said to be in the resolvent set $\rho(T)$ of T; the inverse $(\lambda_o I-T)^{-1}$ is denoted by $R(\lambda_o;T)$ and is called the resolvent of T at $\lambda_o$. All complex numbers $\lambda$ not in $\rho(T)$ form a set $\sigma(T)$, called the spectrum of T.

Theorem III-11. Let X be a Banach space and T a closed linear operator with $\mathcal{D}(T)$ and $\mathcal{R}(T)$ both in X. Then for any $\lambda \varepsilon \rho(T)$, the resolvent $R(\lambda; T)$ is an everywhere defined continuous linear operator. The resolvent $\rho(T)$ of T is an open set of the complex plane.

The above theorem implies that for any $\lambda \varepsilon \rho(T)$, $\mathcal{R}(\lambda I-T) = = \mathcal{D}(R(\lambda; T)) = X$, and that the spectrum $\sigma(T)$ of T is a closed set of the complex plane. Further details on spectral theory can be found in [5] or [23].

In the study of stability of solutions to linear operational differential equations in the following chapter, we have used extensively the semi-group and group theory developed by Hille and Yosida. Much about this basic concept can be found in their respective books [8], [23]. However, we shall introduce some of the basic notions and theorems in the remainder of this section. The concept of nonlinear semi-groups, which is used in the study of nonlinear operational differential equations, will be introduced in a later chapter (see Chapter V). In the following, X is assumed to be a real Banach space.

<u>Definition III-9</u>. For each $t \in [0, \infty)$, let $T_t \in L(X,X)$. The family $\{T_t; t \geq 0\} \in L(X,X)$ is called a strongly continuous semi-group of class $C_o$ or simply a semi-group of class $C_o$ if the following conditions hold:

(i) $T_s T_t = T_{s+t}$ for $s,t \geq 0$.

(ii) $T_o = I$ (I is the identity operator).

(iii) $\lim_{t \to t_o} T_t x = T_{t_o} x$ for each $t_o \geq 0$ and each $x \in X$.

<u>Definition III-10</u>. The family $\{T_t; -\infty < t < \infty\} \in L(X,X)$ is called a strongly continuous group of class $C_o$ or simply a group of class $C_o$ if the following conditions hold:

(i) $T_s T_t = T_{s+t}$ for $-\infty < s,t < \infty$

(ii) $T_o = I$

(iii) $\lim_{t \to t_o} T_t x = T_{t_o} x$ for $-\infty < t_o < \infty$ and each $x \in X$.

It is clear that if $\{T_t; -\infty < t < \infty\}$ is a group, then both $\{T_t; t \geq 0\}$ and $\{T_t; t \leq 0\}$ are semi-groups. If $\{T_t; t \geq 0\}$ is a semi-group, its norm satisfies for some $M \geq 1$ and $\beta < \infty$

$$||T_t|| \leq M e^{\beta t} \qquad \text{for } t \geq 0.$$

If $\beta$ can be taken as $\beta = 0$, $\{T_t; t \geq 0\}$ is said to be an equibounded semi-group of class $C_o$; if in addition M=1, it is called a contraction semi-group of class $C_o$. If $\beta$ can be taken as $\beta < 0$, $\{T_t;\ t \geqq 0\}$ is said to be a negative semi-group of class $C_c$ and if, in addition, M=1, c is called a negative contraction semi-group of class $C_o$. If $\{T_t;\ -\infty < t < \infty\}$ is a group then the above inequality is replaced by

$$||T_t|| \leqq M\ e^{\beta|t|} \qquad \text{for } -\infty < t < \infty$$

Similar terminology applies for a group.

Definition III-11. The infinitesimal generator A of the semi-group $\{T_t;\ t \geqq 0\}$ is defined by

$$Ax = \lim_{h \downarrow 0} \frac{T_h x - x}{h}$$

for all $x \in X$ such that the limit exists.

For the infinitesimal generator A of a semi-group of class $C_o$, the following properties of A are known (e.g., see Yosida [23]).

Theorem III-12. Let A be the infinitesimal generator of a semi-group $\{T_t;\ t \geqq 0\}$. Then (a) A is a closed linear operator with domain $\mathcal{D}(A)$ dense in X and the zero vector $0 \in \mathcal{D}(A)$, (b) if $x \in \mathcal{D}(A)$ then $T_t x \in \mathcal{D}(A)$ for all $t \geqq 0$ and $d/dt\ (T_t x) = AT_t x = T_t Ax$, and (c) if $||T_t|| \leqq M\ e^{\beta t}$, then all $\lambda$ with $\mathrm{Re}(\lambda) > \beta$ is in the resolvent set $\rho(A)$ of A.

The following result is due to E. Hille and K. Yosida independently of each other around 1948 and is called the Hille-Yosida theorem. We state it w'. X as a Banach space rather than the more general locally convex linear topological space.

Theorem III-13 (Hille-Yosida theorem). Let A be a closed linear operator with domain $\mathcal{D}(A)$ dense in X and range $\mathcal{R}(A)$ in X. Then A is the infinitesimal generator of a semi-group $\{T_t;\ t \geqq 0\}$ satisfying

$||T_t|| \leq M e^{\beta t}$ with $M \geq 1$ and $\beta < \infty$ if and only if there exists real numbers $M$ and $\beta$ as above such that for every integer $n > \beta$, $n \in \rho(A)$ and

$$||R(n;A)^m|| \equiv ||(nI-A)^{-m}|| \leq M(n-\beta)^{-m} \quad (m=1,2,\cdots).$$

Notice that in the above theorem, $\beta$ can be positive as well as negative.

Definition III-12. Let A be a linear operator with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ both contained in a Hilbert space H. A is called dissipative with respect to the inner product $(\cdot,\cdot)$ of H if

$$\mathrm{Re}(Ax,x) \leq 0 \qquad \text{for } x \in \mathcal{D}(A)$$

and is called strictly dissipative if there exists a $\beta > 0$ such that

$$\mathrm{Re}(Ax,x) \leq -\beta(x,x) \qquad \text{for } x \in \mathcal{D}(A).$$

Theorem III-14. Let A be a linear operator with domain $\mathcal{D}(A)$ dense in H and range $\mathcal{R}(A)$ in H. Then A is the infinitesimal generator of a contraction semi-group of class $C_o$ in H if and only if A is dissipative and $\mathcal{R}(I-A) = H$; and A is the infinitesimal generator of a negative contraction semi-group of class $C_o$ in H if and only if A is strictly dissipative and $\mathcal{R}((1-\beta)I-A) = H$ where $\beta$ is the constant in definition III-12.

Corollary. Let A be a densely defined closed linear operator from a Hilbert space H into H. If A and its adjoint operator A* are both dissipative, then A is the infinitesimal generator of a contraction semi-group of class $C_o$.

## E. Distributions and Sobolev Spaces

In this section, we shall introduce some of the fundamental definitions and theorems on the theory of distributions and on the class of Sobolev spaces.

A real-valued function $q(x)$ defined on a linear space X is called a semi-norm on X, if the following conditions are satisfied:

(i) $q(x + y) \leqq q(x) + q(y)$

(ii) $q(\alpha x) = |\alpha|\, q(x)$.

It follows directly from the definition that $q(0) = 0$, $q(x-y) \geqq |q(x) - q(y)|$ and $q(x) \geqq 0$. Let $f(x)$ be a complex-valued (or real-valued) function defined in an open subset $\Omega$ of the Euclidean space $R^n$. The support of f, denoted by supp(f), means the smallest closed set containing the set $\{x \in \Omega;\ f(x) \neq 0\}$ (or equivalently, the smallest closed set of $\Omega$ outside of which f vanishes identically).

<u>Definition III-13</u>. By $C^m(\Omega)$, $0 \leqq m \leqq \infty$, we denote the set of all complex-valued (or real-valued) functions defined in $\Omega$ which have continuous partial derivatives of order up to and including m (of order $< \infty$ if $m = \infty$). By $C_o^m(\Omega)$, we denote the set of all functions of $C^m(\Omega)$ with compact supports, i.e., those functions of $C^m(\Omega)$ whose supports are compact subsets of $\Omega$. (A subset of $R^n$ is compact if and only if it is closed and bounded). In the case of $m = \infty$ the linear space $C_o^\infty(\Omega)$ defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \quad (\alpha f)(x) = \alpha f(x)$$

is of particular importance.

For any compact subset K of $\Omega$, let $D_K(\Omega)$ be the set of all functions $f \in C_o^\infty(\Omega)$ such that $\mathrm{supp}(f) \subseteq K$. Define a family of semi-norms on $D_K(\Omega)$ by

$$q_{K,p}(f) = \sup_{|\alpha| \leqq p,\, x \in K} |D^\alpha f(x)| \qquad (p < \infty)$$

where

$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \ \text{ with } \ \alpha_j \geqq 0 \qquad (j=1,2,\ldots,n),$$

$$D^{\alpha} = \frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}$$

$D_K(\Omega)$ is a locally covex linear topological space. The strict inductive limit of $D_K(\Omega)$'s, where K ranges over all compact subsets of $\Omega$, is a locally convex linear topological space. Topologized in this way, $C_o^{\infty}(\Omega)$ will be denoted by $D(\Omega)$. The convergence $\lim_{n\to\infty} f_n \to f$ in $D(\Omega)$ means that the following two conditions are satisfied: (i) there exists a compact subset K of $\Omega$ such that $\mathrm{supp}(f_n) \in K$ $(n=1,2,\ldots)$, and (ii) for any differential operator $D^{\alpha}$, the sequence $D^{\alpha} f_n(x)$ converges to $D^{\alpha} f(x)$ uniformly on K.

Definition III-14. A linear functional f defined and continuous on $D(\Omega)$ is called a distribution or a generalized function in $\Omega$; and the value $f(\phi)$ is called the value of the distribution f at the testing function $\phi \in D(\Omega)$. The set of all distributions in $\Omega$ is denoted by $D(\Omega)^*$ since it is the conjugate space (or dual space) of $D(\Omega)$. It is a linear space by

$$(f + g)(\phi) = f(\phi) + g(\phi), \quad (\alpha f)(\phi) = \alpha f(\phi).$$

Concerning the criteria for a linear functional to be a distribution, the following two theorems are useful.

Theorem III-15. A linear functional f defined on $D(\Omega)$ is a distribution in $\Omega$ if and only if f is bounded on every bounded set of $D(\Omega)$ (in the topology of $D(\Omega)$).

Theorem III-16. A linear functional f defined on $C_o^{\infty}(\Omega)$ is a distribution in $\Omega$ if and only if f satisfies the condition: To every compact subset K of $\Omega$, there correspond a positive constant C and a positive integer m such that $|f(\phi)| \leq C \sup_{|\alpha| \leq m,\, x \in K} |D^{\alpha}\phi(x)|$ whenever $\phi \in D_K(\Omega)$.

<u>Definition III-15</u>. The derivative of a distribution f is defined by

$$\frac{\partial}{\partial x_i} f(\phi) = -f\left( \frac{\partial \phi(x)}{\partial x_i} \right) \quad (i=1,2,\ldots,n), \quad \phi \in D(\Omega).$$

Thus, a distribution in $\Omega$ is infinitely differentiable and

$$(D^{\alpha}f)(\phi) = (-1)^{|\alpha|} f(D^{\alpha}\phi) \quad \alpha = (\alpha_1,\alpha_2, \ldots,\alpha_n), \quad |\alpha| = \sum_{j=1}^{n} \alpha_j.$$

<u>Sobolev Spaces $W^{m,p}(\Omega)$</u>. Let $\Omega$ be an open subset of the Euclidean space $R^n$, and m a positive integer. For $1 \leq p < \infty$, we denote by $W^{m,p}(\Omega)$ the set of all complex-valued (or real-valued) functions $f(x) = f(x_1,x_2,\ldots,x_n)$ defined in $\Omega$ such that f and its distributional derivatives $D^{\alpha}f$ of order $|\alpha| = \sum_{j=1}^{n} \alpha_j \leq m$ all belong to $L^p(\Omega)$. $W^{m,p}(\Omega)$ is a normed linear space by

$$(f + g)(x) = f(x) + g(x), \quad (\alpha f)(x) = \alpha f(x) \quad \text{and}$$

$$||f||_{m,p} = \Big( \sum_{|\alpha| \leq m} \int_{\Omega} |D^{\alpha}f(x)|^p \, dx\Big)^{1/p},$$

where $dx = dx_1 dx_2 \cdots dx_n$ is the Lebesgue measure in $R^n$, under the convention that two functions f and g are considered as the same vector of $W^{m,p}(\Omega)$ if f=g a.e. in $\Omega$. Thus $W^{m,p}(\Omega)$ is a subspace of $L^p(\Omega)$. It is easy to see that $W^{m,2}(\Omega)$ is an inner product space by the inner product

$$(f,g)_{m,2} = \sum_{|\alpha| \leq m} \int_{\Omega} D^{\alpha}f(x) \, \overline{D^{\alpha}g(x)} \, dx.$$

In fact, the space $W^{m,p}(\Omega)$ is a Banach space. In particular, $W^m(\Omega) \equiv W^{m,2}(\Omega)$ is a Hilbert space by the norm $||f||_m \equiv ||f||_{m,2}$ and the scalar product $(f,g)_m \equiv (f,g)m,2$.

<u>The spaces $H^m(\Omega)$ and $H_0^m(\Omega)$</u>. Let $\Omega$ be an open domain of $R^n$ and $0 \leq m < \infty$. Then the totality of functions $f \in C^m(\Omega)$ for which

the norm $||f||_m$ is given by the form as for $W^{m,2}(\Omega)$ constitutes an inner product space $\hat{H}^m(\Omega)$ by the inner product

$$(f,g)_m = \sum_{|\alpha| \leq m} \int_\Omega D^\alpha f(x) \quad \overline{D^\alpha g(x)}\ dx \qquad f,g \in C^m(\Omega).$$

The completion of $\hat{H}^m(\Omega)$ is a Hilbert space and is denoted by $H^m(\Omega)$. Similarly, the totality of functions $f \in C_o^m(\Omega)$ with the norm $||f||_m$ and the inner product $(f,g)_m$ defined as for $f \in C^m(\Omega)$ constitutes an inner product space $\hat{H}_o^m(\Omega)$ whose completion is a Hilbert space denoted by $H_o^m(\Omega)$.

The above definition implies that $C_o^\infty(\Omega)$ is dense in $H_o^m(\Omega)$. In fact, we have

Theorem III-17. The subset $C_o^\infty(\Omega)$ of $L^p(\Omega)$, $1 \leq p \leq \infty$, is dense in $L^p(\Omega)$.

# IV. STABILITY THEORY OF LINEAR DIFFERENTIAL EQUATIONS IN BANACH SPACES

This chapter is concerned with the stability as well as the existence and uniqueness of a solution of the operational differential equation

$$\frac{dx(t)}{dt} = Ax(t) \qquad (t \geqq 0) \tag{IV-1}$$

where the unknown function x(t) is a vector-valued function defined on $[0, \infty)$ to a real Banach space X and A is a given, in general unbounded, linear operator with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ both in X. It is well known that some linear systems of differential equations, both ordinary and partial, can be reduced to the form as in (IV-1) and in such cases A may be considered as an extension of a linear differential operator. In order to examine the stability of solutions to (IV-1), it is only necessary to characterize their properties without actually constructing the solutions. This is done by considering the properties of a semi-group because if A is the infinitesimal generator of a semi-group $\{T_t;\ t \geqq 0\}$ of bounded linear operators on a Banach space X then a solution to (IV-1) starting at $t_o \geqq 0$ from $x_o \in \mathcal{D}(A)$ is given by $x(t;\ x_o,\ t_o) = T_t x_o$ for all $t \geqq t_o$ with $x(t_o;\ x_o,\ t_o) = x_o$. Thus it is important to impose conditions on the operator A so that it is the infinitesimal generator of a semi-group from which the existence of a solution is ensured. Then, the stability criteria can be established from the semi-group properties.

## A. Background

It was seen in Chapter II that by using semi-group or group theory, a Lyapunov stability theory for the linear operational differential equation (IV-1) in a real Hilbert space was established in [3] and the extension to a real Banach space for the case of a group was accomplished in [21]. In order to describe these results and the further developments, it is convenient to state some fundamental definitions and known results.

Definition IV-1. A solution $x(t)$ of the equation (IV-1) with initial condition $x(0) = x \in \mathcal{D}(A)$ means:

(a) $x(t)$ is uniformly continuous in $t$ for each $t \geq 0$ with $x(0) = x$;

(b) $x(t) \in \mathcal{D}(A)$ for each $t \geq 0$ and $Ax(t)$ is continuous in $t$ for each $t \geq 0$;

(c) the derivative of $x(t)$ exists (in the strong topology) for all $t \geq 0$ and equals $Ax(t)$.

Definition IV-2. An equilibrium solution of (IV-1) is a solution $x(t)$ of (IV-1) such that

$$||x(t)-x(0)|| = 0 \quad \text{for all } t \geq 0,$$

and is denoted by $x(t) = x_e$.

Definition IV-3. An equilibrium solution $x_e$ of (IV-1) is said to be stable (with respect to initial perturbations) if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$||x-x_e|| < \delta \text{ implies } ||x(t)-x_e|| < \varepsilon \text{ for all } t \geq 0;$$

$x_e$ is said to be asymptotically stable if

(i) it is stable; and

(ii) $\lim_{t\to\infty} ||x(t)-x_e|| = 0$

where x(t) is any solution of (IV-1) with $x(0) = x \in \mathcal{D}(A)$. If there exists positive constants M and $\beta$ such that

(ii)' $||x(t)-x_e|| \leq M e^{-\beta t} ||x-x_e||$

then $x_e$ is called exponentially asymptotically stable.

It is clear from the above definition that if $0 \in \mathcal{D}(A)$ then x=0, the null solution, is an equilibrium solution of (IV-1). Suppose that an equilibrium solution $x_e$ exists. By letting $y(t) = x(t)-x_e$, equation (IV-1) becomes $dy(t)/dt = Ay(t)$ $(t \geq 0)$ which is the same form as the original equation with initial condition $y(0) = x(0)-x_e$. Since the domain of the operator A which we are concerned with contains the zero vector, it follows that the study of the stability problem of an equilibrium solution of a linear system is equivalent to the study of the stability property of the null solution. Throughout this chapter, the null solution is assumed as the underlying equilibrium solution which implies that definition IV-3 for stability or asymptotic stability of an equilibrium solution can be simplified by taking $x_e = 0$. It should be remarked that the stability theory developed in this and the following two chapters is not limited to equilibrium solutions; in fact, it is valid by starting from any initial element $x_o$ in $\mathcal{D}(A)$ with solution $x(t; x_o, t_o)$ which is not an equilibrium solution (such as a periodic solution or any unperturbed solution).

The following three theorems are from [3].

Theorem IV-1. Let $H_1 = (H, (\cdot,\cdot)_1)$ be a real Hilbert space. An inner product $(\cdot,\cdot)_2$ defined on the linear space H is equivalent to the inner product $(\cdot,\cdot)_1$ if and only if there exists a symmetric bounded positive definite linear operator $S \varepsilon L(H_1, H_1)$ such that

$$(x,y)_2 = (x,Sy)_1 \quad \text{for all } x,y \varepsilon H.$$

Remarks. (a) The above theorem is stated in a slightly different way from the original form for the sake of definiteness; proof of the above result remains the same. It is to be noted that if $S \varepsilon L(H_1,H_1)$, the terminologies of symmetry and self-adjointness of S are the same. (b) Theorem IV-1 has been extended in Chapter V to the case of a complex Hilbert space where the symmetricity condition is not explicitly needed.

A Lyapunov functional on a real Hilbert space $H_1$ is defined in [3] through the symmetric bilinear form

$$V(x,y) = (x,Sy)_1 = (y,Sx)_1 \qquad x,y \varepsilon H_1$$

where $S \varepsilon L(H_1,H_1)$ is a self-adjoint (symmetric) bounded positive definite linear operator. The Lyapunov functional is defined by

$$v(x) = V(x,x) \qquad x \varepsilon H_1.$$

It follows from the above definition and theorem IV-1 that $V(x,y)$ defines an equivalent inner product with respect to $(\cdot,\cdot)_1$ (see definition V-7).

Theorem IV-2. Let A be a linear operator with domain $\mathcal{D}(A)$ dense in $H_1$, range $R(A)$ in $H_1$ and $R(I-A) = H_1$. Then the null solution of (IV-1) is asymptotically stable if there exists a Lyapunov functional $v(x)$ such that

$$\mathring{v}(x) = 2V(x,Ax) \leqq -2\beta\, ||x||_1^2 \qquad x \varepsilon \mathcal{D}(A).$$

It has been shown in [3] that under the hypothesis of theorem IV-2, A generates a negative semi-group so that the null solution of IV-1 is asymptotically stable.

Theorem IV-3. Let A be a linear operator with domain $\mathcal{D}(A)$ dense in $H_1$ and range $\mathcal{R}(A)$ in $H_1$ such that $\mathcal{R}(\alpha I-A) = H_1$ for real $\alpha$ with $|\alpha|$ sufficiently large. Then A is the infinitesimal generator of a negative group (i.e., a group of exponential type) if and only if there exists a Lyapunov functional $v(x) = V(x,x)$ such that for some constant $\delta$, $\gamma$ with $0 < \delta \leqq \gamma < \infty$

$$-2\gamma V(x,x) \leqq \mathring{v}(x) = 2V(x,A\, x) \leqq -2\delta V(x,x) \quad x \in \mathcal{D}(A).$$

Remark. By the definition of a Lyapunov functional, $(x,y)_2 \equiv V(x,y)$ defines an equivalent inner product and thus the above inequality is the same as

$$-\gamma||x||_2^2 \leqq (x,Ax)_2 \leqq -\delta||x||_2^2$$

where $(\cdot,\cdot)_2$ is equivalent to $(\cdot,\cdot)_1$ (see definition V-7).

In order to extend theorems IV-2 and IV-3 to a Banach space, the notion of semi-scalar product, introduced by Lumer and Phillips [15] in the study of contraction semi-groups, is used. The following two theorems are from [15] and their proofs can also be found in [23].

Theorem IV-4 (Lumer). To each pair $\{x,y\}$ of a complex (or real) normed space X, we can associate a complex (or real) number $[x,y]$ such that

(i) $[x + y,z] = [x,z] + [y,z]$;

(ii) $[\alpha x,y] = \alpha[x,y]$;

(iii) $[x,x] = ||x||^2$;

(iv) $|[x,y]| \leqq ||x||\ ||y||$

$[x,y]$ is called a semi-scalar product of the vectors x and y.

Because the construction of a semi-scalar product is essential in our later development, we give a brief proof of this theorem.

According to the Hahn-Banach theorem (theorem III-3), given any $x_o \in X$ there exists at least one (let us choose exactly one) bounded linear functional $f_{x_o} \in X^*$, the dual space of X, such that $||f_{x_o}|| = ||x_o||$ and $f_{x_o}(x_o) = ||x_o||^2$. This is true for any $x_o \in X$. It is clear that

$$[x,y] = f_y(x)$$

defines a semi-scalar product.

Definition IV-4. Let a complex (or real) Banach space X be endowed with a semi-scalar product $[x,y]$. A linear operator A with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ both in X is called dissipative (with respect to $[\cdot,\cdot]$) if

$$\mathrm{Re}[Ax,x] \leqq 0 \qquad x \in \mathcal{D}(A);$$

and is called strictly dissipative (with respect to $[\cdot,\cdot]$) if there exists a real number $\beta > 0$ such that

$$\mathrm{Re}[Ax,x] \leqq -\beta[x,x] = -\beta||x||^2 \qquad x \in \mathcal{D}(A).$$

The supremum of all the positive numbers $\beta$ satisfying the above inequality is called the dissipative constant of A.

Theorem IV-5 (Phillips and Lumer). Let A be a linear operator with $\mathcal{D}(A)$ and $\mathcal{R}(A)$ both contained in a complex (or real) Banach space X such that $\mathcal{D}(A)$ is dense in X. Then A generates a contraction semi-group in X if and only if A is dissipative (with respect to any semi-scalar product) and $\mathcal{R}(I-A) = X$.

Corollary. Let A be a linear operator with $\mathcal{D}(A)$ and $\mathcal{R}(A)$ both contained in a real Banach space X such that $\mathcal{D}(A)$ is dense in X. Then A generates a negative contraction semi-group in X if and only if A is strictly dissipative with dissipative constant $\beta$ and $\mathcal{R}(I-(\beta I + A)) = X$.

The extension of theorem IV-3 from a real Hilbert space to a real Banach space has been accomplished in [21] where an important lemma which is also useful in the case of a semi-group is proved. Before stating these results, we introduce one more definition of equivalent semi-scalar product.

Definition IV-5. Let $[\cdot,\cdot]$ be a semi-scalar product on the Banach space $(X, ||\cdot||)$ with $[x,x] = ||x||^2$. Then the semi-scalar product $[\cdot,\cdot]_1$ with $[x,x]_1 = ||x||_1^2$ is said to be equivalent to $[\cdot,\cdot]$ on X if and only if $||\cdot||_1$ and $||\cdot||$ are equivalent on X.

Lemma IV-1. Let A be the infinitesimal generator of an equi-bounded (negative) semi-group $\{T_t;\ t \geqq 0\}$ in a real Banach space $(X, ||\cdot||)$. Then there exists an equivalent semi-scalar product $[\cdot,\cdot]$ inducing an equivalent norm $||\cdot||_1$ with respect to which A is dissipative (strictly dissipative).

This lemma implies that there exist constants $\beta$, $\gamma$, $\delta$ with $0 < \delta \leqq \gamma < \infty$ and $0 < \beta < \infty$ such that

$$\delta||x||^2 \leqq ||x||_1^2 \leqq \gamma||x||^2$$

and

$$[Ax,x] \leqq 0 \qquad ([Ax,x] \leqq -\beta||x||_1^2) \quad x \in \mathcal{D}(A).$$

Theorem IV-6. Let A be a linear operator with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ both contained in a real Banach space $(X, ||\cdot||)$ such that $\mathcal{D}(A)$ is dense in X. Then A generates a group $\{T_t;\ -\infty < t < \infty\}$ in X such that $\{T_t;\ t \geqq 0\}$ is a negative contraction semi-group with respect to an equivalent norm $||\cdot||_1$ if and only if

$$-\gamma_1||x||_1^2 \leqq [Ax,x] \leqq -\delta_1||x||_1^2 \qquad x \in \mathcal{D}(A),$$

where $0 < \delta_1 \leq \gamma_1 < \infty$ and $[\circ,\circ]$ is an equivalent semi-scalar product consistent with $||\circ||_1$, and

$$R(I(1-\delta_1)-A) = X, \qquad R(I(1+\gamma_1)+A) = X.$$

B. Construction of Lyapunov Functionals

In a real Hilbert space, a Lyapunov functional can be defined through a bilinear functional $V(x,y)$ on the product space $H \times H$ which satisfies the conditions of symmetry, boundedness and positive definiteness. In case of a general Banach space, it can be defined through an equivalent semi-scalar product which possesses most of the properties of the above bilinear functional. (e.g., bilinearity, boundedness and positive definiteness). We shall give a formal definition of a Lyapunov functional in this chapter.

<u>Definition IV-6</u>. Let $X = (X, ||\circ||)$ be a Banach space, and let $[\circ,\circ]$ be an equivalent semi-scalar product inducing an equivalent norm $||\circ||_1$ on X. The scalar functional $v(x)$ defined by

$$v(x) = [x,x] \quad \text{for all } x \in X$$

is called a Lyapunov functional.

It follows from the above definition that there exist constants $\delta$ and $\gamma$ with $0 < \delta \leq \gamma < \infty$ such that

$$\delta||x||^2 \leq v(x) \leq \gamma||x||^2 \quad \text{for all } x \in X$$

since $||\circ||$ and $||\circ||_1$ are equivalent.

In order to prove the main results, we show the following lemma which plays an essential role in the construction of a Lyapunov functional.

<u>Lemma IV-2</u>. Let A be the infinitesimal generator of a semi-group $\{T_t; t \geq 0\}$ in a Banach space X with norm $||\circ||$, and let $[\circ,\circ]$ be any

semi-scalar product on X. Then

$$2[AT_t x, T_t x] = \frac{d}{dt} ||T_t x||^2 \quad (t \geq 0, x \in \mathcal{D}(A)). \qquad \text{(IV-2)}$$

Proof. Let $t > 0$ be fixed. Choose h with $|h| < t$ so that $T_{t+h}x$ is defined for any $x \in \mathcal{D}(A)$. By the property of semi-scalar product, we have

$$[T_{t+h}\, x - T_t x, T_t x] = [T_{t+h}x, T_t x] - [T_t x, T_t x] \leq$$

$$\leq ||T_{t+h}x||\ ||T_t x|| - ||T_t x||^2 = ||T_t x||\ (||T_{t+h}x|| - ||T_t x||).$$

Hence for $h > 0$, the above inequality implies, on dividing both sides by h, that

$$[\frac{T_{t+h}x - T_t x}{h}, T_t x] \leq ||T_t x||\ (\frac{||T_{t+h}x|| - ||T_t x||}{h}).$$

As $h \downarrow 0$, this becomes

$$[AT_t x, T_t x] \leq ||T_t x|| \frac{d}{dt} ||T_t x|| = 1/2 \frac{d}{dt} ||T_t x||^2$$

since the differentiability of $T_t x$ implies the differentiability of $||T_t x||$. For the case of $h < 0$, we have on dividing both sides by h

$$[\frac{T_{t+h}x - T_t x}{h}, T_t x] \geq ||T_t x||\ (\frac{||T_{t+h}x|| - ||T_t x||}{h}).$$

Since $h^{-1}(T_{t+h}x - T_t x) = |h|^{-1}(T_t x - T_{t-|h|}x)$, i follows by taking $h \uparrow 0$ in the above inequality that

$$[AT_t x, T_t x] \geq ||T_t x|| \frac{d}{dt} ||T_t x|| = 1/2 \frac{d}{dt} ||T_t x||^2$$

Comparing the two inequalities involving the same term $1/2\, d||T_t x||^2/dt$ yields

$$2[AT_t x, T_t x] = \frac{d}{dt} ||T_t x||^2$$

which proves the lemma for $t > 0$. The validity of (IV-2) for $t = 0$

follows from a theorem which will be shown in a later section (see theorems IV-10 and IV-11) where the derivative of $||T_t x||^2$ at $t=0$ is taken as the right side derivative.

Remarks. (a) By following the same proof as above, it can be shown that if A is the infinitesimal generator of a group $\{T_t;\ -\infty < t < \infty\}$, then

$$2[AT_t x,\ T_t x] = \frac{d}{dt}\,||T_t x||^2 \qquad -\infty < t < \infty.$$

(b) The requirements in lemma IV-2 can be replaced by a weaker assumption: Let x(t) be a vector valued function defined on [a,b] to a Banach space X. Suppose that x(t) is strongly differentiable with respect to t (and so $||x(t)||$ is also differentiable in t), then for any semi-scalar product $[\cdot,\cdot]$

$$2[\frac{d}{dt}\,x(t),\ x(t)] = \frac{d}{dt}\,||x(t)||^2 \qquad a < t < b.$$

The proof is the same as in lemma IV-2 by replacing $T_t x$ by x(t).

The application of the "direct method" to stability problems consists of defining a Lyapunov functional with appropriate properties whose existence implies the desired type of stability. In this chapter, we are particularly interested in the stable and the exponentially asymptotically stable type. In case the operator A of (IV-1) is an infinitesimal generator of an equibounded or negative semi-group, then the existence of a Lyapunov functional having the desired property can be constructed as is seen in the following.

Theorem IV-7. If A is the infinitesimal generator of an equibounded semi-group $\{T_t;\ t \geq 0\}$ (of class $C_0$) in a real Banach space X, then there exists a Lyapunov functional v(x) such that

$$\dot{v}\,(x(t)) \leq 0 \qquad (t \geq 0)$$

where $x(t) = T_t x$ is an arbitrary solution of (IV-1) with $x \in \mathcal{D}(A)$.

Proof. By lemma IV-1, there exists an equivalent semi-scalar product $[\cdot,\cdot]$ inducing an equivalent norm $||\cdot||_1$ with respect to which A is dissipative. Define $v(x) = [x,x] = ||x||_1^2$, then by the equivalence relation of $||\cdot||$ and $||\cdot||_1$ there exists constants $\delta$, $\gamma$ with $0 < \delta \leqq \gamma < \infty$ such that

$$\delta||x||^2 \leqq v(x) = ||x||_1^2 \leqq \gamma||x||^2. \tag{IV-3}$$

Moreover, by lemma IV-2 and the dissipativity of A, for any $x \in \mathcal{D}(A)$

$$\mathring{v}(T_t x) = \lim_{h\to 0} h^{-1}(v(T_{t+h}x)-v(T_t x)) = \lim_{h\to 0} h^{-1}(||T_{t+h}x||_1^2 - ||T_t x||_1^2) =$$

$$= \frac{d}{dt}||T_t x||_1^2 = 2[AT_t x, T_t x] \leqq 0 \quad (t \geqq 0)$$

since $T_t x \in \mathcal{D}(A)$ for all $t \geqq 0$. Hence the theorem is proved.

In case A is the infinitesimal generator of a negative semi-group, we have an analogous theorem.

Theorem IV-8. If A is the infinitesimal generator of a negative semi-group $\{T_t: t \geqq 0\}$ (of class $C_o$) in a real Banach space X, then there exists a Lyapunov functional $v(x)$ such that for some $\beta > 0$

$$\mathring{v}(x(t)) \leqq -\beta\ ||x(t)||^2 \qquad (t \geqq 0)$$

where $x(t) = T_t x$ is an arbitrary solution of (IV-1) with $x \in \mathcal{D}(A)$.

Proof. By lemma IV-1, A is strictly dissipative with respect to an equivalent semi-scalar product $[\cdot,\cdot]$. By lemma IV-2 and the strict dissipativity of A we have, following the same reasoning as in the proof of theorem IV-7,

$$\mathring{v}(T_t x) = 2[AT_t x, T_t x] \leqq -2\ \beta_1||T_t x||_1^2$$

for some $\beta_1 > 0$ where $||\cdot||_1$ is induced by $[\cdot,\cdot]$. The equivalence between $||\circ||$ and $||\circ||_1$ implies by using (IV-3) that

$$\mathring{v}(T_t x) \leqq -2\ \beta_1 \delta ||T_t x||^2 = -\beta ||T_t x||^2 \quad (t \geqq 0)$$

where $\beta = 2\beta_1\delta > 0$. Thus the theorem is proved.

In case X is a Hilbert space with norm $||x|| = (x,x)^{1/2}$, the existence of a Lyapunov functional is still valid although the space X with the induced norm $||x||_1 = [x,x]^{1/2}$ is not necessarily a Hilbert space. However $(X, ||\circ||_1)$ is at least a Banach space since these two norms are equivalent and so the completeness of one space implies the completeness of the other.

The purpose of constructing a Lyapunov functional with the property as in theorems IV-7 and IV-8 can be seen from the following considerations: Suppose that a Lyapunov functional $v(x) = [x,x]$ satisfying

$$\mathring{v}(x(t)) \leqq -\beta ||x(t)||^2 \qquad (t \geqq 0)$$

for some $\beta \geqq 0$ can be constructed. Regarding $v(x(t)) \equiv v(t)$ as a function of t, we have

$$\mathring{v}(t) \leqq -\beta ||x(t)||^2 \leqq -\beta/\gamma\ ||x(t)||_1^2 = -\beta_1 v(t)$$

since $||x(t)||_1^2 = [x(t),\ x(t)] = v(x(t))$ where $\beta_1 = \beta/\gamma$. Upon integrating the above inequality yields

$$v(t) \leqq v(0)\ e^{-\beta_1 t} \qquad (t \geqq 0)$$

which implies that

$$\delta ||x(t)||^2 \leqq ||x(t)||_1^2 = v(x(t)) \leqq v(x(0))\ e^{-\beta_1 t}$$

$$= ||x(0)||_1^2\ \ e^{-\beta_1 t} \leqq \gamma ||x(0)||^2\ e^{-\beta_1 t}.$$

Thus

$$||x(t)|| \leqq (\gamma/\delta)^{1/2}\ e^{-1/2\ \beta_1 t}\ \ ||x(0)|| \quad (\beta_1 \geqq 0)$$

which shows that the null solution is stable for $\beta = 0$ and is exponentially asymptotically stable for $\beta > 0$.

It is to be noted that the construction of a Lyapunov functional having the desired property as in the above consideration is based on the assumption that solutions to (IV-1) exist. Thus the existence of a Lyapunov functional alone is not sufficient for solving the stability problem of a partial differential equation unless the existence of a solution is assured. The assurance of the existence of a solution requires further restriction.

C. Stability of Linear Operational Equations

As seen in the previous section the existence of a Lyapunov functional and the satisfaction of certain conditions by its derivative evaluated along solutions if they exist imply certain stability properties. Thus, to investigate the stability behavior of the solutions of (IV-1) by the Lyapunov's direct method, it is important to know that a Lyapunov functional exists. In this section, the necessary and sufficient conditions for the existence of a Lyapunov functional is established. This relation is valid for a Banach space as the underlying space as well as for a Hilbert space. Throughout this section, X denotes a real Banach space and H denotes a real Hilbert space. It has been seen that in the case of a real Hilbert space H, a Lyapunov functional can be defined through a symmetric bilinear form

$$V(x,y) = (x, Sy) \quad x,y \in H$$

where $S \in L(H,H)$ is a self-adjoint bounded positive definite linear operator. The boundedness of S implies that

$$|V(x,y)| = |(x,Sy)| \leq ||S||\ ||x||\ ||y|| \quad (x,y \in H)$$

which shows that $V(x,y)$ is continuous in both x and y; that is, for any sequences $\{x_n\}$ and $\{y_n\}$ in H such that $x_n \to x$ and $y_n \to y$ then

$$\lim_{n\to\infty} V(x_n,y_n) = V(x,y).$$

In the case of a real Banach space X, a Lyapunov functional is defined through an equivalent semi-scalar product by $V(x,y) = [x,y]$ which, as is seen in theorem IV-4, is defined through the choice of a continuous linear functional $f_y \in X^*$ for each fixed $y \in X$. This semi-scalar product has the property that $[x,y] = f_y(x)$ for each $x \in X$ and $||f_y|| = ||y||$. Although the linear functional $f_y(x)$ is continuous in x, it is not clear that $f_y(x)$ is also continuous in y since we know only that $||f_y|| = ||y||$. From the Lyapunov stability point of view it is desirable to know whether or not

$$\lim_{t\downarrow 0} [AT_t x,\ T_t x] = [Ax,x] \qquad x \in \mathcal{D}(A)$$

where A is the infinitesimal generator of the semi-group $\{T_t;\ t \geq 0\}$. If this last can be verified, then solutions need not be constructed. We shall show that the answer is affirmative by first establishing a series of lemmas which are essential in the proof of the above convergence relation. Before proving these lemmas, it is convenient to give the following notations: Let $x(t)$ be a vector-valued function defined on $[0,\infty)$ to a real Banach space X such that $x(t)$ is continuous in t wich $\lim_{t\downarrow 0} x(t) = x(0) \equiv x$ in the strong topology. For each fixed $t \geq 0$, let

$$M_t = \{m;\ m = \alpha x(t),\ \alpha \text{ real}\} \quad \text{and}$$

$$Y_t = \{y;\ y = m + \beta x_o,\ m \in M_t,\ \beta \text{ real}\}$$

where $x_o$ is a fixed element in X but not in $M_t$. It is clear that $M_t \subseteq Y_t$. With this notation, we have the following.

Lemma IV-3. (a) For any fixed $t \geq 0$, the functional $f_t$ on $M_t$ defined by

$$f_t(m) = \alpha||x(t)||^2 \quad \text{for } m = \alpha x(t) \in M_t$$

is a continuous linear functional on $M_t$ with $||f_t|| = ||x(t)||$.

(b) For the same t as in (a) and for any number $c_t$ the functional $F_t$ on $Y_t$ defined by

$$F_t(y) = f_t(m) + \beta c_t \quad \text{for } y = m + \beta x_o \in Y_t$$

is a continuous linear functional on $Y_t$.

Proof. Part (a) of the lemma is obvious, for if $m_1, m_2 \in M_t$, then $f_t(\gamma_1 m_1 + \gamma_2 m_2) = f_t((\gamma_1\alpha_1 + \gamma_2\alpha_2)\ x(t)) = (\gamma_1\alpha_1 + \gamma_2\alpha_2)||x(t)||^2 = \gamma_1 f_t(m_1) + \gamma_2 f_t(m_2)$ and $|f_t(m)| = |\alpha|\ ||x(t)||^2 = ||x(t)||\ ||m||$ for all $m \in M_t$ which implies that $||f_t||=||x(t)||$. To show that $F_t$ is a linear functional on $Y_t$, let $y_1, y_2 \in Y_t$ with $y_1 = m_1 + \beta_1 x_o$ and $y_2 = m_2 + \beta_2 x_o$, then

$$F_t(\gamma_1 y_1 + \gamma_2 y_2) = F_t((\gamma_1 m_1 + \gamma_2 m_2) + (\gamma_1\beta_1 + \gamma_2\beta_2)x_o) =$$

$$= f_t(\gamma_1 m_1 + \gamma_2 m_2) + (\gamma_1\beta_1 + \gamma_2\beta_2)c_t = \gamma_1 f_t(m_1) + \gamma_1\beta_1 c_t +$$

$$+ \gamma_2 f_t(m_2) + \gamma_2\beta_2\ c_t = \gamma_1 F_t(y_1) + \gamma_2 F_t(y_2).$$

This shows part (b) of the lemma.

Lemma IV-4. For the same fixed $t \geq 0$ as in lemma IV-3, there exists a number $c_t$ in defining the functional $F_t$ such that

$$||F_t|| = ||f_t|| = ||x(t)|| \quad (t \geq 0).$$

In particular, for $t = 0$ there exists an number c such that the functional $F_o$ on $Y_o$ defined by

$$F_o(y) = f_o(m_o) + \beta\ c \quad \text{for } y = m_o + \beta\ x_o \in Y_o \text{ with } m_o \in M_o$$

is a continuous linear functional on $Y_o$ with $||F_o|| = ||f_o|| = ||x||$.

<u>Proof.</u> It suffices to show that $||F_t|| \leqq ||f_t||$ since $F_t$ is an extension of $f_t$ which implies that $||f_t|| \leqq ||F_t||$. To accomplish this, we show that there exists a number $c_t$ in the definition of $F_t$ such that

$$|F_t(y)| \leqq ||f_t||\ ||y|| \quad \text{for all } y \in Y_t. \tag{IV-4}$$

Since $|F_t(y)| = |f_t(m) + \beta\, c_t|$ for $y = m + \beta\, x_o$, (IV-4) is equivalent to

$$-||f_t||\ ||m + \beta\, x_o|| - f_t(m) \leqq \beta\, c_t \leqq ||f_t||\ ||m + \beta\, x_o|| - f_t(m). \tag{IV-4)'$$

Now if $\beta = 0$, then $y = m \in M_t$ and $F_t(y) = f_t(m)$ which implies that (IV-4) is satisfied for arbitrary fixed t. We assume that $\beta \neq 0$. Hence for $\beta > 0$ (IV-4)' is equivalent to

$$-||f_t||\ ||\tfrac{m}{\beta} + x_o|| - f_t(\tfrac{m}{\beta}) \leqq c_t \leqq ||f_t||\ ||\tfrac{m}{\beta} + x_o|| - f_t(\tfrac{m}{\beta}) \tag{IV-4)''$$

and for $\beta < 0$ it is equivalent to

$$\frac{1}{\beta}||f_t||\ ||m+\beta\, x_o|| - \frac{1}{\beta}\, f_t(m) \leqq c_t \leqq -\frac{1}{\beta}||f_t||\ ||m+\beta\, x_o|| - \frac{1}{\beta}\, f_t(m)$$

which can immediately be reduced into the same form as in (IV-4)". Thus it is sufficient to choose $c_t$ satisfying

$$-||f_t||\ ||m'+x_o|| - f_t(m') \leqq c_t \leqq ||f_t||\ ||m'+x_o|| - f_t(m') \qquad m' \in M_t. \tag{IV-5}$$

The choice of $c_t$ is possible since for any $m', m'' \in M_t$

$$f_t(m') + f_t(m'') = f_t(m'+m'') \leqq ||f_t||\ ||m'+m''|| = ||f_t||\ ||m'+x_o+m''-x_o|| \leqq$$

$$\leqq ||f_t||\ ||m'+x_o|| + ||f_t||\ ||m''-x_o||$$

which implies that

$$-||f_t||\ ||m''-x_o|| + f_t(m'') \leqq ||f_t||\ ||m'+x_o|| - f_t(m').$$

The arbitrariness of m'' in $M_t$ implies

$$\sup_{m''\in M_t}[-||f_t||\ ||m''-x_o||+f_t(m'')] \leqq ||f_t||\ ||m'+x_o||-f_t(m') \qquad m' \in M_t;$$

and the arbitrariness of m' in $M_t$ yields

$$\sup_{m''\in M_t}[-||f_t||\ ||m''-x_o||+f_t(m'')] \leqq \inf_{m'\in M_t}[||f_t||\ ||m'+x_o||-f_t(m')]. \qquad \text{(IV-5)}'$$

In order to satisfy (IV-5), we need only to choose $c_t$ satisfying

$$\sup_{m''\in M_t}[-||f_t||\ ||m''-x_o||+f_t(m'')] \leqq c_t \leqq \inf_{m'\in M_t}[||f_t||\ ||m'+x_o||-f_t(m')]. \qquad \text{(IV-5)}''$$

It follows that (IV-5)'' reduced to the form (IV-5) by letting m'' = -m' for any $m' \in M_t$. With this choice of $c_t$, (IV-4) is satisfied and from which $||F_t|| \leqq ||f_t||$. Since $F_t$ is an extension of $f_t$, $||F_t|| \geqq ||f_t||$. Therefore, $||F_t|| = ||f_t||$. The above is true for each fixed $t \geqq 0$ and, in particular for $t = 0$, $F_0$ is a continuous functional on $Y_o$ where c corresponds to $c_o$.

In general, $c_t$ depends on t and there may be infinitely many of them for any t. The object in the following lemma is to select a number $c_t$ satisfying (IV-5) such that $c_t$ is a continuous function of t with $c_t \to c$ as $t \to 0$.

<u>Lemma IV-5</u>. The constant $c_t$ in lemma IV-4 can be choosen as a continuous real-valued function of t for $t \in [0, t_o]$ with $t_o$ a fixed positive number such that $c_t \to c$ as $t \downarrow 0$.

<u>Proof</u>. Since if $m \in M_t$, then $m = \alpha x(t)$ and $f_t(m) = \alpha||x(t)||^2$ for some real $\alpha$, it follows from $||f_t|| = ||x(t)||$ that (IV-5)'' becomes

$$\sup_{\alpha}[-||x(t)||\ ||\alpha x(t)-x_o|| + \alpha||x(t)||^2] \leqq c_t \leqq \inf_{\beta}[||x(t)||\ ||\beta x(t)+x_o||- \\ - \beta||x(t)||^2].$$

Since the continuity of $x(t)$ in $t$ in the strong topology implies the continuity of $||x(t)||$ in $t$, and since the product or the sum of two continuous functions is continuous, it follows that the real-valued scalar functions

$$f(\alpha,t) \equiv -||x(t)|| \; ||\alpha x(t)-x_o||+ \alpha||x(t)||^2 \quad \text{and}$$
$$g(\beta,t) \equiv ||x(t)|| \; ||\beta\, x(t)+x_o||- \beta||x(t)||^2$$

are continuous functions in $t$ and $\alpha$, and in $t$ and $\beta$ respectively. From $\sup_\alpha f(\alpha,t) \leqq \inf_\beta g(\beta,t)$, we can choose $c_t$ as a right continuous function of $t$ in the interval $[0,t_o]$ such that

$$\sup_\alpha f(\alpha,t) \leqq c_t \leqq \inf_\beta g(\beta,t) \text{ for } t \in [0,t_o].$$

It follows that

$$f(\alpha,t) \leqq c_t \leqq g(\beta,t) \quad \text{for all } \alpha,\ \beta.$$

The continuity of $c_t$ implies, as $t \downarrow 0$, that

$$f(\alpha,0) \leqq c_o \leqq g(\beta,0) \quad \text{for all } \alpha,\ \beta$$

which, by the same reasoning as in obtaining (IV-5)', yields

$$\sup_\alpha[-||x|| \; ||\alpha x-x_o|| + \alpha||x||^2] \leqq c_o \leqq \inf_\beta[||x|| \; ||\beta x+x_o||-\beta||x||^2].$$

By choosing $c=c_o$, the above inequality implies that for each $\beta$

$$-||x|| \; ||\beta x+x_o||-\beta||x||^2 \leqq c \leqq ||x|| \; ||\beta x+x_o||-\beta||x||^2$$

that is

$$-||f_o|| \; ||m_o+x_o||-f_o(m_o) \leqq c \leqq ||f_o|| \; ||m_o+x_o||-f_o(m_o) \text{ for all } m_o \in M_o.$$

Therefore, with this choice of c the functional $F_o$ defined by

$$F_o(y) = F_o(m_o + \beta x_o) = f_o(m_o) + \beta c$$

is a continuous linear functional on $M_o$ with $||F_o|| = ||f_o|| = ||x||$ such that $c_t \to c$ as $t \downarrow 0$ which proves the lemma.

As we have mentioned before, if there is a sequence $\{y_n\}$ in X such that $y_n \to y$ strongly, one can not draw a conclusion that

$[x,y_n] \to [x,y]$ since $[x,y_n] = f_{y_n}(x)$ where $||f_{y_n}|| = ||y_n||$ does not ensure that $\{f_{y_n}(x)\}$ converges to $f_y(x)$ for every $x \in X$. However, by using the above lemmas the following theorem can be shown

Theorem IV-9. Let A be the infinitesimal generator of an equi-bounded (negative) semi-group $\{T_t;\ t \geqq 0\}$ (of class $C_o$) in a real Banach space X. Then there exists a semi-scalar product such that

$$\lim_{t \downarrow 0} [Ax, T_t x] = [Ax, x] \qquad x \in \mathcal{D}(A).$$

Proof. By lemma VI-4, the functional $F_t$, with t fixed, is a continuous linear functional on $Y_t$ with $||F_t|| = ||f_t|| = ||x(t)||$. It follows from the Hahn-Banach theorem that there exists a continuous linear extension $G_t$ on X such that $||G_t|| = ||F_t|| = ||x(t)||$. Since $x(t) \in M_t$

$$|G_t(x(t))| = |f_t(x(t))| = ||x(t)||^2 .$$

It is clear that for arbitrary fixed $t \geqq 0$

$$G_t(y) = [y, x(t)]$$

defines a semi-scalar product (see theorem IV-4). In particular, when $t = 0$, then

$$G_o(y) = [y,x]$$

defines a semi-scalar product. For fixed $x \in \mathcal{D}(A)$, let $T_t x = x(t)$ and let $x_o = Ax - m_o$ where $m_o = \alpha_o T_t x \in M_t$ with $\alpha_o$ fixed. We choose this $x_o$ as the fixed element in the definition of $Y_t$ (if $x_o \in M_t$, we consider $f_t$ in place of $F_t$). Hence $Ax = m_o + x_o \in Y_t$, and

$$[Ax, T_t x] = G_t(Ax) = F_t(Ax) = F_t(m_o + x_o) = f_t(m_o) + c_t = \alpha_o ||T_t x||^2 + c_t$$

On the other hand,

$$[Ax,x] = G_o(Ax) = F_o(Ax) = f_o(m_o) + c = \alpha_o ||x||^2 + c .$$

Therefore, by lemma IV-5

$$\lim_{t \downarrow 0} |[Ax, T_t x] - [Ax,x]| \leqq \lim_{t \downarrow 0} |\alpha_o ||T_t x||^2 - \alpha_o ||x||^2| + \lim_{t \downarrow 0} |c_t - c| = 0,$$

and the theorem is proved.

<u>Corollary</u>. Let $x(t)$ be a vector-valued function defined on $[0, \infty)$ to $X$ such that $x(t)$ is continuous in $t$ in the strong topology, and let $A$ be a linear operator with $\mathcal{D}(A)$ and $\mathcal{R}(A)$ both contained in $X$ with $x(0) \equiv x \in \mathcal{D}(A)$. Then

$$\lim_{t \downarrow 0} [Ax, x(t)] = [Ax, x] \qquad x \in \mathcal{D}(A).$$

<u>Proof</u>. By the same argument as in the proof of the theorem, the result follows.

<u>Theorem IV-10</u>. Let $A$ be the infinitesimal generator of an equi-bounded (negative) semi-group $\{T_t;\ t \geq 0\}$ (of class $C_o$) in $X$, then

$$\lim_{t \downarrow 0} [AT_t x,\ T_t x] = [Ax, x] \qquad x \in \mathcal{D}(A).$$

<u>Proof</u>.

$$\begin{aligned} |[AT_t x, T_t x] - [Ax, x]| &= |[T_t Ax - Ax, T_t x] + [Ax, T_t x] - [Ax, x]| \leq \\ &\leq |[T_t Ax - Ax, T_t x]| + |[Ax, T_t x] - [Ax, x]| \leq ||T_t Ax - Ax||\ ||T_t x|| + \\ &\quad + |[Ax, T_t x] - [Ax, x]| \end{aligned}$$

since $AT_t x = T_t Ax$ for $x \in \mathcal{D}(A)$. Thus, by theorem IV-9

$$\lim_{t \downarrow 0} |[AT_t x, T_t x] - [Ax, x]| \leq \lim_{t \downarrow 0} ||T_t Ax - Ax||\ ||T_t x|| + \lim_{t \downarrow 0} |[Ax, T_t x] - [Ax, x]| = 0$$

which implies the desired result.

<u>Corollary</u>. Let $x(t)$ be a solution to (IV-1) with $x(0) = x$ where $x \in \mathcal{D}(A)$. Then

$$\lim_{t \downarrow 0} [Ax(t),\ x(t)] = [Ax, x].$$

<u>Proof</u>. Since $x(t)$ is a solution of (IV-1), it is differentiable in $t$ and satisfies

$$Ax(t) = \frac{d}{dt} x(t) \qquad (t \geq 0)$$

with $x(0) = x \in \mathcal{D}(A)$. Hence $Ax(t)$ is continuous in $t$ in the strong topology. By the corollary of theorem IV-9 and the continuity of $Ax(t)$ in $t$, we have

$$\lim_{t\downarrow 0}|[Ax(t), x(t)] - [Ax,x]| \leq$$

$$\lim_{t\downarrow 0}(|[Ax(t) - Ax,x(t)]| + |[Ax,x(t)] - [Ax,x]|) \leq$$

$$\lim_{t\downarrow 0} ||Ax(t)-Ax|| \; ||x(t)|| + \lim_{t\downarrow 0} |[Ax,x(t)]-[Ax,x]| = 0$$

and the result follows.

It is known [15] that the infinitesimal generator of a contraction semi-group is independent of the choice of semi-scalar product. It follows that an operator A with dense domain and $R(I-A) = X$ which is dissipative with respect to one semi-scalar product defined on a Banach space X, is also dissipative with respect to any other semi-scalar product compatible with the norm of X since under the given conditions A is the infinitesimal generator of a contraction semi-group. This fact enables us to choose any semi-scalar product on X consistent with the norm of X such as the one constructed in the proof of theorem IV-9 without affecting the dissipativity of A. The following two theorems give the necessary and sufficient conditions for A to generate equibounded and negative semi-groups respectively.

<u>Theorem IV-11</u>. Let A be a linear operator with domain $\mathcal{D}(A)$ dense in $X = (X, ||\circ||)$ and range $R(A)$ in X. Then A is the infinitesimal generator of an equibounded semi-group $\{T_t; t \geq 0\}$ if and only if there exists a Lyapunov functional $v(x) = [x,x]$ such that

$$\mathring{v}(x) = 2[Ax,x] \leq 0 \qquad x \in \mathcal{D}(A) \qquad \text{(IV-6)}$$

and $R(I-A) = X$ where $[\circ,\circ]$ is an equivalent semi-scalar product on X consistent with $||\circ||_1$.

<u>Proof</u>. Let A be the infinitesimal generator of an equibounded semi-group $\{T_t; t \geq 0\}$. By lemma IV-1, there exists an equivalent semi-scalar product $[\circ,\circ]$ inducing an equivalent norm $||\circ||_1$ such that

$[Ax,x] \leq 0$. Define $v(x) = [x,x]$, then by lemma IV-2 and theorem IV-10

$$\dot{v}(x) \equiv \lim_{t\downarrow 0} \frac{1}{t} (v(T_t x)-v(x)) = \lim_{t\downarrow 0} \frac{1}{t} (||T_t x||_1^2-||x||_1^2) =$$

$$= \frac{d}{dt}( ||T_t x||_1^2)_{t=0+} = 2 \lim_{t\downarrow 0} [AT_t x, T_t x] = 2[Ax,x] \leq 0.$$

By theorem III-12, moreover, for any $\lambda > 0$, $\lambda \in \rho(A)$ (the resolvent set of A), it follows by theorem III-11 that $R(I-A) = \mathcal{D}(R(1;A)) = X$. Conversely, if there exists a Lyapunov functional $v(x) = [x,x]$ satisfying (IV-6) where $[\circ,\circ]$ is an equivalent semi-scalar product inducing an equivalent norm $||\circ||_1$, then A is dissipative with respect to $[\circ,\circ]$. By the equivalence relation between the two norms $||\circ||$ and $||\circ||_1$, $\mathcal{D}(A)$ is dense in $X_1=(X,||\circ||_1)$ and $R(I-A) = X_1$ since $\mathcal{D}(A)$ is dense in $X = (X,||\circ||)$ and $R(I-A) = X$ by hypothesis. It follows by theorem IV-5 that A generates a contraction semi-group $\{T_t; t \geq 0\}$ in $X_1$ with $||T_t||_1 \leq 1$ since the dissipativity of A is independent of semi-scalar product on $X_1$. It is known that semi-group properties are invariant under equivalent norms and the equivalence between $||\circ||$ and $||\circ||_1$ implies that $||T_t|| \leq M$ for some $M > 0$, hence $\{T_t; t \geq 0\}$ is an equibounded semi-group in X. Therefore, the desired result is proved.

For the case of a negative semi-group, we have the following results.

<u>Theorem IV-12.</u> Let A be a linear operator with domain $\mathcal{D}(A)$ dense in X and range $R(A)$ in X. Then A is the infinitesimal generator of a negative semi-group $\{T_t; t \geq 0\}$ if and only if there exists a Lyapunov functional $v(x) = [x,x]$ such that

$$\dot{v}(x) = 2[Ax,x] \leq -2\beta||x||_1^2 \quad (x \in \mathcal{D}(A),\ \beta > 0)$$

and $R(I-(\beta I+A)) = X$ where $[\circ,\circ]$ is an equivalent semi-scalar product on X consistent with $||\circ||_1$.

Proof. The proof is essentially the same as for theorem IV-11. The "only if" part follows from lemma IV-1 with $\dot{v}(x) = 2[Ax,x] \leq -2\beta||x||_1^2$, and the "if" part follows from the corollary of theorem IV-5 with $||T_t||_1 \leq e^{-\beta t}$ for some $\beta > 0$ so that $||T_t|| \leq M e^{-\beta t}$ with $M > 0$ $(t \geq 0)$.

The above two theorems just proved can be applied to a Hilbert space H although the linear space H with the norm $||\circ||_1$ induced by the semi-scalar product $[\circ,\circ]$ may no longer be a Hilbert space. However if $[\circ,\circ]$ is an equivalent semi-scalar product on H, then the space $(H, ||\circ||_1)$ is at least a Banach space, and the semi-scalar product can still be used to define a Lyapunov functional.

Based on the results obtained in the above two theorems, we can define a pair of functionals v(x) and w(x) in X such that if certain conditions are satisfied by these two functionals the stability or asymptotic stability of the null solution are ensured. These two functionals, which in a sense are in parallel to those used by Zubov in [24], are defined by

$$v(x) = [x,x] \quad (x \in X) \quad \text{and} \quad w(x) = [Ax,x] \quad (x \in \mathcal{D}(A))$$

where $[\circ,\circ]$ is an equivalent semi-scalar product and A is the linear operator in (IV-1). Thus, v(x) is in fact a Lyapunov functional on X as defined in definition IV-6. The following theorem stated in terms of these two functionals is an immediate consequence of theorems IV-11 and IV-12.

Theorem IV-13. Let A be a linear operator with $\mathcal{D}(A)$ dense in X and $\mathcal{R}(I-(\beta I+A)) = X$ where $\beta \geq 0$ and X is a Banach space or a Hilbert space. If there exist two functionals v(x) and w(x) defined by

$$v(x) = [x,x] \qquad x \in X$$

$$w(x) = [Ax,x] \qquad x \in \mathcal{D}(A)$$

such that

(i) $\mathring{v}(x) = 2w(x)$; and

(ii) $w(x) \leq -\beta \; ||x||_1^2 \qquad x \in \mathcal{D}(A)$

where $[\cdot,\cdot]$ is an equivalent semi-scalar product on X. Then the null solution of (IV-1) is stable if $\beta = 0$ and is asymptotically stable if $\beta > 0$.

Proof. Under the assumption of (i) and (ii),

$$\mathring{v}(x) = 2[Ax,x] \leq -2\beta \; ||x||_1^2 \qquad x \in \mathcal{D}(A).$$

Thus by hypotheses all the conditions in theorems IV-11 and IV-12 are satisfied for $\beta = 0$ and $\beta > 0$, respectively. These imply that A generates an equi-bounded or negative semi-group depending on $\beta = 0$ or $\beta > 0$. The stability or asymptotic stability of the null solution follows from the equibounded or negative property of a semi-group respectively.

Remark. Under the assumptions of the above theorem, the condition $R(I-(\beta I+A)) = X$ in the theorem can be weakened by assuming that $R(\alpha I-A)=X$ for some $\alpha > 0$. This is due to the fact that the condition $R(I-(\beta I+A))=X$ can be replaced by $R(\lambda I - (\beta I+A))=X$ for sufficiently large $\lambda$ (e.g., see [23], p. 250) and thus for any $\beta \geq 0$ a number $\lambda_o > \beta$ can be chosen such that $R((\lambda_o-\beta)I-A) = X$. This will be satisfied if $R(\alpha I-A) = X$ for some $\alpha > 0$ since by lemma V-1 in the next chapter the condition $R(\alpha I-A) = X$ for some $\alpha > 0$ and the dissipativity of A imply that $R(\alpha I-A) = X$ for every $\alpha > 0$.

Thus in case of a Hilbert space, the Lyapunov functional $v(x)$ can be constructed from an equivalent semi-scalar product other than an equivalent inner product. The importance of theorems IV-11 and IV-12 lies in the fact that the existence of a Lyapunov functional alone does not necessarily ensure the existence of a solution to (IV-1), and in fact the proof of the existence of a solution to (IV-1) is, in general, rather complicated. However, under the additional conditions $\overline{\mathcal{D}(A)} = X$ and $\mathcal{R}(I-A) = X$ the existence of a solution with any initial element $x \in \mathcal{D}(A)$ is assured. This assurance makes the stability of a solution meaningful.

# V. STABILITY THEORY OF NONLINEAR TIME-INVARIANT DIFFERENTIAL EQUATIONS IN HILBERT SPACES

Many physical and engineering problems are formulated by differential equations, often, by nonlinear partial differential equations. Since the stability problem of solutions to partial differential equations occurs in many fields of science the study of the stability behavior of solutions to partial differential equations has been extensively investigated in recent years. However, most of this work is concerned with specific partial differential operators and sometimes the existence of a solution is assumed. In order to unify a theory for a class of partial differential equations and to develop a stability theory on this class, it is desirable to consider a general nonlinear operator from a function space into itself. In this chapter, Hilbert spaces are taken as the underlying spaces, and only in some special cases (section C), real Hilbert spaces are considered.

Consider the nonlinear operational differential equation

$$\frac{dx(t)}{dt} = Ax(t) \qquad (t \geq 0) \qquad \text{(V-1)}$$

where the unknown x(t) is a vector-valued function defined on $[0, \infty)$ to a Hilbert space H, and A is a given, in general, nonlinear operator with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ both contained in H. The object of this chapter is to develop criteria for the stability and the asymptotic stability as well as the existence and uniqueness of solutions to (V-1).

The stability and the asymptotic stability properties of the solutions of (V-1) are developed in terms of nonlinear contraction and negative contraction semi-groups. By the introduction of an equivalent inner product, these properties are related to the existence and the construction of a Lyapunov functional which is a direct extension of the linear case due to Buis [3]. Finally, the semi-linear differential equation

$$\frac{dx}{dt} = A_o x + f(x) \tag{V-2}$$

is discussed as a special case where $A_o$ is a linear closed operator and f is a nonlinear function defined on a real Hilbert space H. It turns out that if $A_o$ is a self-adjoint operator in H or in a topologically equivalent Hilbert space $H_1$, the conditions imposed on $A_o$ are particularly simple.

A. Nonlinear Semi-groups and Dissipative Operators

In order to describe the results in this and the following sections, it is necessary to give some basic definitions.

Definition V-1. Let H be a Hilbert space. The family $\{T_t; t \geq 0\}$ is called a continuous semi-group of nonlinear contraction operators on H or simply (nonlinear) contraction semi-group on H if and only if the following conditions hold:

(i) for any fixed $t \geq 0$, $T_t$ is a continuous (nonlinear) operator defined on H into H;

(ii) for any fixed $x \in H$, $T_t x$ is strongly continuous in t;

(iii) $T_s T_t = T_{s+t}$ for s, $t \geq 0$, and $T_o = I$ (the identity operator);

(iv) $||T_t x - T_t y|| \leq ||x-y||$ for all x, y, $\in$ H and all $t \geq 0$.

If (iv) is replaced by

(iv') $||T_t x - T_t y|| \leq e^{-\beta t}||x-y||$ $(\beta > 0)$ for all x, y $\in$ H and all $t \geq 0$, then $\{T_t;\ t \geq 0\}$ is called a (nonlinear) negative contraction semi-group on H. The supremum of all the numbers $\beta$ satisfying (iv') is called the contractive constant of $\{T_t;\ t \geq 0\}$. For a subset $\mathcal{D}$ of H, the family $\{T_t;\ t \geq 0\}$ is said to be a nonlinear contraction (negative contraction) semi-group on $\mathcal{D}$ if the properties (i)-(iv) ((i)-(iv')) are satisfied for all $x, y \in \mathcal{D}$.

Definition V-2. The infinitesimal generator A of the nonlinear semi-group $\{T_t;\ t \geq 0\}$ is defined by

$$Ax = \underset{h \downarrow 0}{\text{w-lim}} \ \frac{T_h x - x}{h}$$

for all $x \in H$ such that the limit on the right-side exists in the sense of weak convergence.

Definition V-3. An operator (nonlinear) A with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ both contained in a Hilbert space is said to be monotone [18] if

$$\operatorname{Re}(Ax - Ay,\ x-y) \geq 0 \qquad \text{for } x,\ y \in \mathcal{D}(A). \tag{V-3}$$

The operator A is called dissipative if $-A$ is monotone; and A is called strictly dissipative if there exists a real number $\beta > 0$ such that $-(A + \beta I)$ is monotone.

It follows from the above definition that

$$\operatorname{Re}(Ax - Ay,\ x-y) \leq 0 \quad \text{for } x,\ y \in \mathcal{D}(A) \tag{V-4}$$

if and only if A is dissipative; and

$$\operatorname{Re}(Ax-Ay,\ x-y) \leq -\beta\,(x-y,\ x-y),\ \beta > 0 \qquad x,y \in \mathcal{D}(A) \tag{V-4)'$$

if and only if A is strictly dissipative. The supremum of all the numbers $\beta$ such that (V-4)' holds is called the dissipative constant of A. Note that these conditions coincide with the usual definitions of dissipativity when A is a linear operator (see definition III-12).

The definition of a monotone operator has been extended to the case when A is an operator in a Banach space X. In this case, A is said to be monotone if

$$||x-y + \alpha\,(Ax - Ay)|| \geqq ||x-y|| \quad \text{for all } \alpha > 0 \text{ and } x,y \in \mathcal{D}(A). \quad (V-3)'$$

Let X* be the set of all bounded semi-linear forms on X; that is, the pairing between $x \in X$ and $f \in X^*$ denoted by $<x,f>$ is linear in x and semi-linear in f (If X is a Hilbert space, X* is identified with X and $<\cdot,\cdot>$ with the inner product in X). For any fixed $x \in X$, define

$$F(x) = \{f \in X^*;\ <x,f> = ||x||^2 = ||f||^2\}.$$

Then it can be shown that [11] (V-3)' is equivalent to

$$\mathrm{Re} <Ax - Ay,\ f> \geqq 0 \text{ for some } f \in F(x-y),\ x,y \in \mathcal{D}(A). \quad (V-3)''$$

Note that the inequality (V-3)" is not required to hold for every $f \in F(x-y)$. Hence if X is a Hilbert space, (V-3)" is reduced to (V-3), since in this case $F(x-y) = \{x-y\}$ consists of a single element and

$$\mathrm{Re} <Ax-Ay,\ f> = \mathrm{Re}(Ax-Ay,\ x-y).$$

The condition (V-3)' implies that $(I + \alpha A)^{-1}$ exists and is Lipschitz continuous for all $\alpha > 0$, where $I + \alpha A$ is an operator with domain $\mathcal{D}(A)$ which maps x into $x + \alpha Ax$. As to the domain of $(I + \alpha A)^{-1}$, we have the following lemma (see [11]) which was proved essentially by Kōmura [13] (see also [19]).

Lemma V-1. Let A be monotone. If the domain of $(I + \alpha A)^{-1}$ is the whole of X for some $\alpha > 0$, then the same is true for all $\alpha > 0$.

Hence for a monotone operator A, the operator $(I + \alpha A)^{-1}$ has domain X either for every $\alpha > 0$ or for no $\alpha > 0$.

Definition V-4. If A is a monotone operator such that $\mathcal{D}((I+\alpha A)^{-1}) = \mathcal{R}(I+\alpha A) = X$ for every $\alpha > 0$ (or for some $\alpha > 0$), then A is said to be m-monotone.

Because of the generality of the problem considered in [11], the theorems developed in that paper are somewhat complicated. However, in case the operator A in (V-1) is independent of t, as in this chapter, those theorems are relatively simple and can be stated in terms of non-linear contraction semi-groups. Now we restate the main theorems in [11] when A in (V-1) is independent of t.

Theorem V-1. Let X and X* be both uniformly convex spaces, and let -A be m-monotone. Then A is the infinitesimal generator of a non-linear contraction semi-group $\{T_t; t \geqq 0\}$ on $\mathcal{D}(A)$ such that for any $x \in \mathcal{D}(A)$, $T_t x$ is the unique solution of (V-1) with the initial condition $T_o x = x$. A solution x(t) of (V-1) satisfies: (i) For each $x(0) \in \mathcal{D}(A)$, $x(t) \in \mathcal{D}(A)$ for all $t \geqq 0$; (ii) x(t) is uniformly Lipschitz continuous in t; (iii) the weak derivative of x(t) exists for all $t \geqq 0$ and equals Ax(t); (iv) the strong derivative $dx(t)/dt = Ax(t)$ exists and is strongly continuous except at a countable number of values t.

Through out this chapter, conditions (i)-(iv) of the above theorem specify what is meant by a solution of the differential equation of the form (V-1). It should be remarked here that except for the assumption of m-monotonicity, the operator A is arbitrary. This is different from much of the work on nonlinear evolution equations in Hilbert spaces or in Banach spaces in which only semi-linear equations of the form (V-2) were considered (cf. Browder [1], Kato [9]). This latter type of equation will be discussed in a later section by applying the results for the general form (V-1).

It is clear from the above theorem that if A is dissipative in the sense of (V-4) and X and X* are uniformly convex, then an equilibrium solution (or a periodic solution) if it exists, would be stable by the contraction property of the semi-group. However, it is not trivial to relate exponentially asymptotic stability directly to such a property. If A is linear and is the infinitesimal generator of a contraction semi-group $\{T_t;\ t \geq 0\}$ of class $C_o$, then the family $\{e^{-\beta t}T_t;\ t \geq 0\}$ for some $\beta > 0$ is a negative contraction semi-group with the infinitesimal generator $A - \beta I$. But when A is nonlinear, the contraction semi-group $\{T_t;\ t \geq 0\}$ generated by A is nonlinear and so the family $\{e^{-\beta t}T_t;\ t \geq 0\}$ is not, in general, a semi-group since property (iii) in definition V-1 does not hold. However, with a slight modification, necessary and sufficient conditions for the exponentially asymptotic stability analogous to the linear case still holds. This can be achieved by using the negative contraction semi-group property. Before doing this, we show in this section some basic results which will be needed in the later sections. We leave the development of stability and asymptotic stability to section B of this chapter in which we introduce the concept of equivalent inner product.

Theorem V-2. Let A be a nonlinear operator with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ both contained in a Hilbert space H such that $\mathcal{R}(I-A)=H$. Then A is the infinitesimal generator of a nonlinear contraction semi-group $\{T_t;\ t \geq 0\}$ on $\mathcal{D}(A)$ if and only if A is dissipative (i.e. $-A$ is monotone).

Proof. Sufficiency: suppose A is dissipative, (i.e. $-A$ is monotone). Then $-A$ is m-monotone, for by hypothesis, $\mathcal{R}(I+(-A)) = \mathcal{R}(I-A) = H$. Since $H^*$ is identified with H, it is also a Hilbert space. Thus H and $H^*$ are both uniformly convex. The sufficiency follows from theorem V-1.

Necessity: Let A be the infinitesimal generator of a non-linear contraction semi-group $\{T_t;\ t \geqq 0\}$ on $\mathcal{D}(A)$. Then for any $x, y \in \mathcal{D}(A)$

$$\operatorname{Re}(h^{-1}(T_h x - x) - h^{-1}(T_h y - y),\ x-y) = h^{-1}\operatorname{Re}[(T_h x - T_h y,\ x-y) - (x-y,\ x-y)]$$
$$\leqq h^{-1}[\,||T_h x - T_h y||\ ||x-y|| - ||x-y||^2] = h^{-1}||x-y||[\,||T_h x - T_h y|| - ||x-y||\,] \leqq 0$$

for all $h > 0$ since $\{T_t,\ t \geqq 0\}$ is contractive. Letting $h\downarrow 0$ in the above inequality, we have, by the continuity of inner product and by definition V-2

$$\operatorname{Re}(Ax - Ay,\ x-y) \leqq 0 \qquad \text{for any } x, y \in \mathcal{D}(A).$$

Hence the theorem is proved.

It should be noted that in the above theorem, it is not assumed that the domain of A is dense in H. However, if A is a linear operator in a Hilbert space, the m-monotonicity of $-A$ implies that $\mathcal{D}(-A)$ is dense in H (cf. [11]), and the above theorem is reduced into the well-known results due to Lumer and Phillips [15]. But it is not known yet whether or not $\mathcal{D}(A)$ is dense in H if A is a m-monotone nonlinear operator. It will be shown that the nonlinear contraction semi-group $\{T_t;\ t \geqq 0\}$ can be extended by continuity to a nonlinear contraction semi-group on $\overline{\mathcal{D}(A)}$, the closure of $\mathcal{D}(A)$. Hence if $\mathcal{D}(A)$ is dense in H, $\{T_t;\ t \geqq 0\}$ can be extended to the whole space H which is a direct generalization of a strongly continuous semi-group of class $C_o$. The condition $\mathcal{R}(I-A) = H$ can also be weakened by assuming $\mathcal{R}(I-\alpha_o A) = H$ for some $\alpha_o > 0$ since the monotonicity of $-A$ implies: (i) the existence of $(I-\alpha A)^{-1}$ for all $\alpha > 0$, and (ii) if $\mathcal{D}((I-\alpha_o A)^{-1}) = H$ for some $\alpha_o > 0$, then $\mathcal{D}((I-\alpha A)^{-1}) = H$ for all $\alpha > 0$.

The nonlinear contraction semi-group $\{T_t;\ t \geqq 0\}$ generated by A in Theorem V-2 can be extended to the closure $\mathcal{D}(A)$ denoted by $\overline{\mathcal{D}(A)}$. In order to do this, we consider the approximate equation of the form

$$\frac{dx_n(t)}{dt} = A_n x_n(t) \qquad x_n(0) = x \in H,\ n = 1,2,\cdots \tag{V-5}$$

where $A_n = A(I-n^{-1}A)^{-1}$, and show the following lemma which is proved based on some of Kato's work in the construction of a solution to (V-1).

<u>Lemma V-2</u>. Let A be a dissipative operator, and let $\mathcal{R}(I-A)=H$. Then for any $x \in H$ there exists a unique solution $T_t^{(n)}x$ of (V-5) which is continuously differentiable in the strong topology such that $T_o^{(n)}x = x$ for each $n=1,2,\cdots$. Moreover, for any $x \in \overline{\mathcal{D}(A)}$, $T_t^{(n)}x$ converges uniformly in t as $n \to \infty$, and for $x_k \in \mathcal{D}(A)$ such that $x_k \to x$ as $k \to \infty$

$$\lim_{n\to\infty} T_t^{(n)}x = \lim_{n\to\infty}\lim_{k\to\infty} T_t^{(n)}x_k = \lim_{k\to\infty}\lim_{n\to\infty} T_t^{(n)}x_k. \tag{V-6}$$

<u>Proof</u>. The operator $A_n = A(I-n^{-1}A)^{-1}$ is defined everywhere on H for each n since $-A$ is monotone and by lemma V-1 $\mathcal{D}((I-A)^{-1}) = \mathcal{R}(I-A)=H$ implies $\mathcal{D}((I-n^{-1}A)^{-1}) = H$ for every n. $A_n$ is dissipative for each n and satisfies $||A_nx-A_ny|| \leq n||x-y||$ (cf. Kato [11]). Hence for each n, $A_n$ satisfies the following conditions:

(i) $A_n$ is continuous and carries bounded subsets of H into bounded subsets of H since $||A_nx|| \leq ||A_ny_o|| + n||x-y|| \leq ||A_ny_o|| + n||x||+n||y_o||$ where $y_o$ is a fixed element in H.

(ii) For each fixed n, $(A_nx-A_ny,x-y) \leq n||x-y||^2$ since $||A_nx-A_ny|| \leq n||x-y||$. The above conditions imply that for any $x \in H$ there exists a unique solution $T_t^{(n)}x$ which is continuously differentiable in the strong topology such that $T_o^{(n)}x = x$ for each n (cf. Browder [1] or Kato [9]). It can be shown by the dissipativity of $A_n$ that

$$||T_t^{(n)}x-T_t^{(n)}y|| \leq ||x-y|| \qquad x,y \in H \tag{V-7}$$

uniformly in t and n (see lemma V-5 with $T_t^nx = x(t)$). Since the solution

$T_t x$ of (V-1) is constructed as the limit of $T_t^{(n)}x$ as $n \to \infty$ and for $y \in \mathcal{D}(A)$ the strong limit $T_t y = \lim_{n\to\infty} T_t^{(n)}y$ converges uniformly in t (cf. [11]), it follows by (V-7) that $T_t^{(n)}x$ converges uniformly in t for $x \in \overline{\mathcal{D}(A)}$. Moreover by (V-7) for $x_k \in \mathcal{D}(A)$ and $x_k \to x$ as $k \to \infty$

$$\lim_{k\to\infty} ||T_t^{(n)}x - T_t^{(n)}x_k|| \leqq \lim_{k\to\infty} ||x-x_k|| = 0$$

uniformly in t which is the same as

$$T_t^{(n)}x = \lim_{k\to\infty} T_t^{(n)}x_k \quad \text{uniformly in } t.$$

This last equality relation and the fact that

$$\lim_{k\to\infty}\lim_{n\to\infty} ||T_t^{(n)}x - T_t^{(n)}x_k|| \leqq \lim_{k\to\infty}\lim_{n\to\infty} ||x-x_k|| = 0$$

imply that

$$\lim_{n\to\infty} T_t^{(n)}x = \lim_{n\to\infty}\lim_{k\to\infty} T_t^{(n)}x_k = \lim_{k\to\infty}\lim_{n\to\infty} T_t^{(n)}x_k.$$

Thus the lemma is proved.

Following the results of lemma V-2, it is natural to extend the nonlinear contraction semi-group $\{T_t;\ t \geqq 0\}$ to the closure of $\mathcal{D}(A)$ by the relation (V-6). More precisely, we have the following

Lemma V-3. Let $\{T_t;\ t \geqq 0\}$ be the nonlinear contraction (negative contraction) semi-group generated by A on $\mathcal{D}(A)$ in theorem V-1. Then it can be extended to a contraction (negative contraction) semi-group $\{\bar{T}_t;\ t \geqq 0\}$ on $\overline{\mathcal{D}(A)}$ by defining

$$\bar{T}_t x = \lim_{k\to\infty} T_t x_k \qquad \text{for } x \in \overline{\mathcal{D}(A)} \tag{V-8}$$

where $x_k \in \mathcal{D}(A)$ and $x_k \to x$ as $k \to \infty$.

Proof. The limit defined by (V-8) exists and is independent of the choice of $x_k$ in $\mathcal{D}(A)$. The first assertion follows from the fact that for fixed $t \geqq 0$

$$||T_t x_k - T_t x_j|| \leqq ||x_k - x_j|| \to 0 \quad \text{as} \quad k,j \to \infty$$

which shows that $\{T_t x_k\}$ is a Cauchy sequence and so it converges to an element in H. To see that (V-8) is unambiguously defined, let $y_k \in \mathcal{D}(A)$ such that $y_k \to x$. Then

$$\lim_{k\to\infty} ||T_t x_k - T_t y_k|| \leqq \lim_{k\to\infty} ||x_k - y_k|| = 0$$

which implies that $\bar{T}_t x = \lim_{k\to\infty} T_t x_k = \lim_{k\to\infty} T_t y_k$. Next we show that $\{\bar{T}_t;\ t \geqq 0\}$ is a nonlinear contraction semi-group from $\overline{\mathcal{D}(A)}$ into $\overline{\mathcal{D}(A)}$. For any fixed t and any pair $x, y \in \overline{\mathcal{D}(A)}$ with $x_k, y_k \in \mathcal{D}(A)$ and $x_k \to x$, $y_k \to y$, we have

$$||\bar{T}_t x - \bar{T}_t y|| = \lim_{k\to\infty} ||T_t x_k - T_t y_k|| \leqq \lim_{k\to\infty} ||x_k - y_k|| = ||x-y||.$$

$$(||\bar{T}_t x - \bar{T}_t y|| = \lim_{k\to\infty} ||T_t x_k - T_t y_k|| \leqq \lim_{k\to\infty} e^{-\beta t}||x_k - y_k|| = e^{-\beta t}||x-y||).$$

Thus $\bar{T}_t$ is, for each $t \geqq 0$, continuous and contractive (negative contractive) on $\overline{\mathcal{D}(A)}$. $\bar{T}_t x$ is continuous in t for any fixed $x \in \overline{\mathcal{D}(A)}$. To see this, let $x_k \in \mathcal{D}(A)$ and $x_k \to x$. Then

$$\bar{T}_t x = \lim_{k\to\infty} T_t x_k = \lim_{k\to\infty}\lim_{n\to\infty} T_t^{(n)} x_k = \lim_{n\to\infty} T_t^{(n)} x$$

by using lemma V-2. Since $T_t^{(n)}$ is continuous in t and converges uniformly in t in the strong topology, we have

$$\lim_{t\downarrow 0} \bar{T}_t x = \lim_{t\downarrow 0}\lim_{n\to\infty} T_t^{(n)} x = \lim_{n\to\infty}\lim_{t\downarrow 0} T_t^{(n)} x = x.$$

Hence for any $t \geqq 0$

$$\lim_{h\to 0} ||\bar{T}_{t+h} x - \bar{T}_t x|| = \lim_{h\to 0}\lim_{k\to\infty} ||T_{t+h} x_k - T_t x_k|| \leqq \lim_{h\to 0}\lim_{k\to\infty} ||T_h x_k - x_k||$$

$$= \lim_{h\to 0} ||\bar{T}_h x - x|| = 0$$

since $T_{t+h} x = T_t T_h x$ and $T_t$ is contractive on $\mathcal{D}(A)$. (Similarly for a negative contractive semi-group). The continuity of $\bar{T}_t x$ in t is proved. To show that $\bar{T}_s \bar{T}_t = \bar{T}_{s+t}$, we first show that $\bar{T}_t$ maps $\overline{\mathcal{D}(A)}$ into $\overline{\mathcal{D}(A)}$.

This follows directly from definition since for any $x \in \overline{\mathcal{D}(A)}$ with $x_k \in \mathcal{D}(A)$ and $x_k \to x$, then $T_t x_k \in \mathcal{D}(A)$ for all k which implies that $\bar{T}_t x = \lim_{k\to\infty} T_t x_k \in \overline{\mathcal{D}(A)}$. Now if $x \in \overline{\mathcal{D}(A)}$ then $\bar{T}_t x \in \overline{\mathcal{D}(A)}$ and so $\bar{T}_s \bar{T}_t x$ is defined. Moreover $\bar{T}_s(\bar{T}_t x) = \lim_{k\to\infty} T_s(T_t x_k) = \lim_{k\to\infty} T_{s+t} x_k = \bar{T}_{s+t} x$ since the limit is independent of the choice of any sequence which converges to $\bar{T}_t x$. Note that $T_t x_k \to \bar{T}_t x$. Furthermore, $\bar{T}_o x = \lim_{k\to\infty} T_o x_k = x$, that is $\bar{T}_o = I$ on $\overline{\mathcal{D}(A)}$. Therefore $\{\bar{T}_t;\ t \geq 0\}$ is a nonlinear contraction (negative contraction) semi-group, and the lemma is proved.

Owing to the importance of asymptotic stability in the study of the stability theory of differential equations, it should be desirable to extend theorem V-2 to the case where A is the infinitesimal generator of a nonlinear negative contraction semi-group. For this purpose, we first prove the following lemmas which will be used in the proof of the next theorem and which will play an important role in the construction of a Lyapunov functional.

<u>Lemma V-4</u>. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in H such that $x_n \overset{w}{\to} x$ and $y_n \to y$ as $n \to \infty$ where $\overset{w}{\to}$ denotes weak convergence. Then

$$\lim_{n\to\infty} (x_n, y_n) = (x, y) \qquad\qquad x, y \in H.$$

<u>Proof</u>. Since a weakly convergent sequence is strongly bounded i.e., $||x_n|| < \infty$ for all n (theorem III-8), it follows by the strong convergence of $\{y_n\}$ that

$$\lim_{n\to\infty} |(x_n,\ y_n - y)| \leq \lim_{n\to\infty} ||x_n||\ ||y_n - y|| = 0$$

which implies that

$$\lim_{n\to\infty} (x_n, y_n) = \lim_{n\to\infty} (x_n, y).$$

By the weak convergence of $x_n$, we have

$$\lim_{n\to\infty} (x_n, y_n) = \lim_{n\to\infty} (x_n, y) = (x, y).$$

Lemma V-5. Let $x(t)$, $y(t)$ be any two solutions of (V-1) (in the sense of theorem V-1). Then $||x(t)-y(t)||^2$ is differentiable in $t$ for each $t \geq 0$, and is given by

$$\frac{d}{dt}||x(t)-y(t)||^2 = \mathrm{Re}(Ax(t)-Ay(t),\ x(t)-y(t)) \quad \text{for each } t \geq 0. \quad \text{(V-9)}$$

Proof. For any fixed $t > 0$, let $h \neq 0$ and $|h| < t$ so that $x(t+h)$ and $y(t+h)$ are defined By hypothesis, $h^{-1}(x(t+h)-x(t)) \overset{w}{\to} Ax(t)$ and $h^{-1}(y(t+h)-y(t)) \overset{w}{\to} Ay(t)$ we have by the continuity of inner product and by lemma V-4 that

$$\lim_{h\to 0} h^{-1}[||x(t+h)-y(t+h)||^2 - ||x(t)-y(t)||^2] = \lim_{h\to 0} h^{-1}[(x(t+h)-y(t+h),x(t+h)-y(t+h))-(x(t)-y(t),x(t)-y(t))] = \lim_{h\to 0} h^{-1}[(x(t+h)-y(t+h)-(x(t)-y(t)),x(t+h)-y(t+h)) + (x(t)-y(t),(x(t+h)-y(t+h)) - (x(t)-y(t)))]$$

$$= \lim_{h\to 0} h^{-1}[(x(t+h)-x(t),x(t+h)-y(t+h))-(y(t+h)-y(t),x(t+h)-y(t+h)) + (x(t)-y(t),x(t+h)-x(t)) - (x(t)-y(t),y(t+h)-y(t))]$$

$$= (Ax(t),x(t)-y(t)) - (Ay(t),x(t)-y(t)) + (x(t)-y(t),Ax(t)) - (x(t)-y(t),Ay(t))$$

$$= (Ax(t)-Ay(t),x(t)-y(t)) + (x(t)-y(t),Ax(t)-Ay(t))$$

$$= 2\ \mathrm{Re}(Ax(t)-Ay(t),x(t)-y(t)).$$

Hence, $||x(t)-y(t)||^2$ is differentiable and (v-9) holds for $t > 0$. For $t = 0$, the above is still valid by taking $h > 0$ and $h \downarrow 0$ in place of $h \to 0$ and by defining $\frac{d}{dt}||x(t)-y(t)||^2$ at $t = 0$ as the right-side limit.

The following theorem is an immediate extension of theorem V-2 and is fundamental for the construction of a Lyapunov functional from which the asymptotic stability of solutions to (V-1) can be ensured.

Theorem V-3. Let $A$ be a nonlinear operator with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ both contained in a Hilbert space $H$ such that $\mathcal{R}(I-A) = H$. Then $A$ is the infinitesimal generator of a nonlinear negative contraction semi-group $\{T_t;\ t \geq 0\}$ with contractive constant $\beta$ on $\mathcal{D}(A)$, that is

$$||T_t x - T_t y|| \leq e^{-\beta t}\ ||x-y|| \qquad x,\ y \in \mathcal{D}(A) \tag{V-10}$$

if and only if A is strictly dissipative with dissipative constant $\beta$, that is

$$\mathrm{Re}(Ax-Ay,\ x-y) \leq -\ \beta(x-y,\ x-y) \qquad x,y \in \mathcal{D}(A). \tag{V-11}$$

Proof. Necessity: Let A be the infinitesimal generator of $\{T_t;\ t \geq 0\}$ such that (V-10) is valid. Then

$$||T_t x - T_t y||^2 \leq e^{-2\beta\ t}||x-y||^2 \quad \text{for all } t \geq 0 \tag{V-10)'}$$

since both side of (V-10) are positive. Subtracting $||x-y||^2$ and then dividing by $t > 0$ in the above inequality, (V-10)' becomes

$$t^{-1}(||T_t x - T_t y||^2 - ||x-y||^2) \leq t^{-1}(e^{-2\beta t}-1)\ ||x-y||^2 \qquad t > 0.$$

As $t \downarrow 0$, we obtain

$$\frac{d}{dt}\ ||T_t x - T_t y||^2_{t=0} \leq -2\beta||x-y||^2\ .$$

Since for any $x,\ y \in \mathcal{D}(A)$, $T_t x$, $T_t y$ are solutions of (V-1), it follows by lemma V-5 that

$$\mathrm{Re}(Ax-Ay,\ x-y) \leq -\beta(x-y,\ x-y) \qquad x,\ y \in \mathcal{D}(A).$$

Sufficiency: Let (V-11) holds. Then A is dissipative and by theorem V-2, it is the infinitesimal generator of a nonlinear contraction semi-group $\{T_t;\ t \geq 0\}$ on $\mathcal{D}(A)$. Moreover, by lemma V-5

$$\frac{d}{dt}\ ||T_t x - T_t y||^2 = 2\mathrm{Re}(AT_t x - AT_t y, T_t x - T_t y) \leq -2\beta||T_t x - T_t y||^2 \quad t \geq 0$$

since $T_t x$, $T_t y$ are solutions of (V-1). By integrating the above inequality, we have

$$||T_t x - T_t y||^2 \leq e^{-2\beta t}\ ||x-y||^2$$

and the result follows.

Theorem V-3 is a direct generalization of theorem 1' in [21] when X is a Hilbert space, for the strict dissipativity in theorem V-3 is a generalization of the strict dissipativity in the sense of [21]. Moreover, it can be shown (for instance, see [23]) that the condition $R((1-\beta)I-A) = H$ in theorem 1' of [21] can be replaced by $R((\lambda-\beta)I-A) = H$ for sufficiently large $\lambda > 0$. Hence for any $\beta > 0$, we can choose $\lambda$ such that $\lambda - \beta > 0$ which implies that the condition $R((1-\beta)I-A)= H$ can be replaced by $R(I-(\lambda-\beta)^{-1}A) = H$ for $\lambda-\beta > 0$. However, the latter condition is equivalent to $R(I-A) = H$ in virture of lemma V-1, since under the assumption of (V-10) or (V-11) in the theorem, $-A$ is monotone. The equivalence between $R(I-(\lambda-\beta)A)=H$ and $R(I-A) = H$ follows directly from lemma V-1.

B. Stability Theory of Nonlinear Time-invariant Equations

The object of this section is to develop some criteria in terms of the operator A so that the stability or the asymptotic stability as well as the existence and uniqueness of solutions to (V-1) is assured. In the particular case of partial differential operators, these criteria are in terms of the properties of the coefficients of the original system of differential equations and possibly include the given boundary conditions. The results obtained in the previous section serve as the basis for the development of a stability theory which can be applied to certain classes of nonlinear partial differential equations. Before showing these results, it would be appropriate to give some definitions of stability and asymptotic stability of an equilibrium solution.

<u>Definition V-5</u>. An equilibrium solution of (V-1) is an element $x_e$ in $\mathcal{D}(A)$ satisfying (V-1) (in the weak topology) such that for any solution x(t) of (V-1) with $x(0) = x_e$

$$||x(t)-x_e|| = 0 \qquad \text{for all } t \geqq 0.$$

It follows from the above definition that if x(t) is a solution to (V-1) with x(0) = x, then it is an equilibrium solution if and only if Ax(t) = 0 for all $t \geqq 0$. To show this, let Ax(t) = 0 where x(t) is a solution of (V-1). Then by theorem V-1 the strong derivative dx(t)/dt = Ax(t) = 0 exists and is strongly continuous except at a countable number of values t. This means $x(t) = x_o$ (a constant vector) except at a countable number of values t. But x(0) = x and since any solution of (V-1) is strongly continuous it follows that x(t) = x for all $t \geqq 0$ (see also theorem III-10). Conversely, let x(t) be an equilibrium solution of (V-1). Then

$$(Ax(t),z) = (dx(t)/dt, z) = \lim_{h\to 0} h^{-1}(x(t+h)-x(t),z) = \lim_{h\to 0} h^{-1}(0,z) = 0$$

for every z ε H and every $t \geqq 0$. Since x(t) is a solution of (V-1), x(t) ε $\mathcal{D}(A)$ and Ax(t) ε H for each $t \geqq 0$; thus the orthogonality of Ax(t) to every z in H implies that for each $t \geqq 0$, Ax(t) = 0. Hence the existence of an equilibrium solution is equivalent to the existence of a solution to (V-1) satisfying

$$Ax(t) = 0 \qquad \text{for every } t \geqq 0 .$$

Definitions of stability, asymptotic stability and exponentially asymptotic stability of an equilibrium solution are the same as given in definition IV-3. However, we introduce here one more definition of stability region.

<u>Definition V-6</u>. Let x(t) be a solution to (V-1) with x(0) = x. A subset $\mathcal{D}$ of H is said to be a stability region of the equilibrium

solution $x_e$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $x \in \mathcal{D}$ and $||x-x_e|| < \delta$ imply $||x(t)-x_e|| < \varepsilon$ for all $t \geq 0$

The dissipativity in theorems V-2 and V-3 are defined with respect to the original inner product of the space. Since the semi-group property is invariant under equivalent norms, the possibility occurs that by defining other inner products inducing equivalent norms, the semi-group could be made contractive and the infinitesimal generator dissipative. This follows from the fact that stability and asymptotic stability are invariant under equivalent norms and may be verified by the dissipativity of A with respect to an equivalent inner product.

Definition V-7. Two inner products $(\circ , \circ)$ and $(\circ , \circ)_1$ defined on the same vector space H are said to be equivalent if and only if the norms $||\circ||$ and $||\circ||_1$ induced by $(\circ , \circ)$ and $(\circ , \circ)_1$ respectively are equivalent, that is, there exists constants $\delta$, $\gamma$ with $0 < \delta \leq \gamma < \infty$ such that

$$\delta||x|| \leq ||x||_1 \leq \gamma||x|| \qquad \text{for all } x \in H. \tag{V-12}$$

The Hilbert space $H_1$ equipped with the inner product $(\circ , \circ)$ is said to be an equivalent Hilbert space of H and is denoted by $(H, (\circ , \circ)_1)$ or simply by $H_1$.

Under the equivalent inner product $(\circ , \circ)_1$, the vector space $(H, (\circ , \circ)_1)$ is a Hilbert space if and only if the original space $(H, (\circ , \circ))$ is, since the completeness of one space implies the completeness of the other. This fact enables us to weaken the dissipativity condition on the operator A in theorem V-2 and V-3.

Theorem V-4. Let A be a nonlinear operator with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ both contained in a Hilbert space $H = (H, (\circ , \circ))$ such that $R(I-A) = H$. Then A is the infinitesimal generator of a nonlinear contraction (negative contraction) semi-group $\{T_t;\ t \geqq 0\}$ on $\mathcal{D}(A)$ in an equivalent Hilbert space $(H, (\cdot\ ,\ \cdot)_1)$ if and only if A is dissipative (strictly dissipative) with respect to $(\cdot,\cdot)_1$. In this case the family $\{T_t;\ t \geqq 0\}$ is a nonlinear (nonlinear negative) semi-group $\{T_t;\ t \geqq 0\}$ on $\mathcal{D}(A)$ in H. (i.e. conditions (iv) and (iv') are replaced by $||T_t x - T_t y|| \leqq M||x-y||$ and $||T_t x - T_t y|| \leqq Me^{-\beta t}||x-y||$ respectively for some $M \geqq 1$).

<u>Proof</u>. Since the inner product $(\cdot\ ,\ \cdot)_1$ is equivalent to $(\cdot\ ,\ \cdot)$, the space $H_1 = (H, (\cdot\ ,\ \cdot)_1)$ is a Hilbert space and $R(I-A) = H_1$. Hence by considering $H_1$ as the underlying space, all the conditions in theorem V-2 (theorem V-3) are satisfied implying the first assertion is proved. To show the second part of the theorem, let A be the infinitesimal generator of a nonlinear contraction (negative contraction) semi-group $\{T_t;\ t \geqq 0\}$ on $\mathcal{D}(A)$ with respect to the norm $||\cdot||_1$, that is

$$||T_t x - T_t y||_1 \leqq ||x-y||_1 \qquad (||T_t x - T_t y||_1 \leqq e^{-\beta t}||x-y||_1) \qquad x,y \in \mathcal{D}(A).$$

By the equivalence relation (V-12), we have

$$||T_t x - T_t y|| \leqq \delta^{-1}||T_t x - T_t y||_1 \leqq \delta^{-1}||x-y||_1 \leqq \gamma\delta^{-1}||x-y||$$

$$(||T_t x - T_t y|| \leqq \gamma\delta^{-1}\, e^{-\beta t}\, ||x-y||) \qquad x,\ y \in \mathcal{D}(A).$$

Since the properties of a semi-group in definition V-1 remains unchanged under equivalent norms except for possibly the contraction prope y, it follows that $\{T_t;\ t \geqq 0\}$ is a nonlinear (nonlinear negative) semi-group on $\mathcal{D}(A)$ with respect to the origina norm (with $M = \gamma\delta^{-1}$).

The application of the "direct method" to the stability problem consists of defining a Lyapunov functional with appropriate properties whose existence implies the desired type of stability. In order to give the definition of a Lyapunov functional on a complex Hilbert space, we first introduce the following:

Definition V-8. Let H be a Hilbert space, and let $V(x,y)$ be a complex-valued sesquilinear functional defined on the product space $H \times H$ (i.e. $V(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 V(x_1, y) + \alpha_2 V(x_2, y)$ and $V(x, \beta_1 y_1 + \beta_2 y_2) = \bar{\beta}_1 V(x, y_1) + \bar{\beta}_2 V(x, y_2)$). Then $V(x,y)$ is called a defining sesquilinear functional if it satisfies the following conditions:

(i) $V(x,y) = \overline{V(y,x)}$ (symmetry)

(ii) $|V(x,y)| \leq \gamma ||x|| \, ||y||$ for some $\gamma > 0$ (boundedness)

(iii) $V(x,x) \geq \delta ||x||^2$ for some $\delta > 0$ (positive definiteness)

Note that condition (ii) implies that $V(x,y)$ is continuous both in x and in y.

Definition V-9. Let $V(x,y)$ be a defining sesquilinear functional. Then the scalar functional $v(x)$ defined by $v(x) = V(x,x)$ is called a Lyapunov functional.

By applying a theorem due to Lax and Milgram, we show the following.

Lemma V-6. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $H = (H, (\cdot,\cdot))$ such that $x_n \overset{w}{\to} x$ and $y_n \to y$ as $n \to \infty$. Then

$$\lim_{n\to\infty} V(x_n, y_n) = V(x,y) \qquad x, y \in H.$$

Proof. By definition of $V(x,y)$, all the conditions (i.e. sesquilinearity, boundedness and positivity) in the Lax-Milgram theorem (see theorem III-7) are satisfied. Thus, there exists a bounded linear operator S with a bounded linear inverse $S^{-1}$ such that

$$V(x,y) = (x, Sy) \qquad \text{for all } x, y \in H. \tag{V-13}$$

Since a weakly convergent sequence is strongly bounded so that $||x_n|| < \infty$ for all n, it follows by the sesquilinearity of $V(x,y)$ and by the relation (V-13) that

$$\lim_{n\to\infty} |(V(x_n,y_n) - V(x_n,y))| = \lim_{n\to\infty} |V(x_n,y_n-y)| = \lim_{n\to\infty} |(x_n,S(y_n-y))| \leqq$$

$$\leqq \lim_{n\to\infty} ||x_n||\ ||S||\ ||y_n-y|| = 0$$

which shows that

$$\lim_{n\to\infty} V(x_n, y_n) = \lim_{n\to\infty} V(x_n,y).$$

Again, by the relation (V-13) and by the weak convergence of $\{x_n\}$

$$\lim_{n\to\infty} V(x_n,y) = \lim_{n\to\infty} (x_n, Sy) = (x, Sy) = V(x, y).$$

Therefore, the lemma is proved by the above two equality relations.

It follows from the above definitions and lemma V-6 that the following results can easily be shown.

Lemma V-7. For any $x \in H$.

$$\delta_1||x||^2 \leqq v(x) \leqq \gamma_1||x||^2 \tag{V-14}$$

and for any pair of solutions $x(t)$, $y(t)$ of (V-1)

$$\mathring{v}(x(t) - y(t)) = 2\mathrm{Re}\ V(Ax(t) - Ay(t), x(t) - y(t)) \tag{V-15}$$

where $\mathring{v}(z(t))$ denotes the derivative of $v(z(t))$ with respect to t.

Proof. (V-14) follows from the definition of $V(x,y)$. To show (V-15), note that by the sesquilinearity of $V(x,y)$ it is easily seen that

$$V(x-y, x+y) + V(x+y , x-y) = 2(V(x,x) - V(y,y)) \quad \text{for any } x,y \in H,$$

and by the symmetry of $V(x,y)$, the above equality implies that

$$v(x) - v(y) = V(x,x) - V(y,y) = \frac{1}{2}(V(x-y,x+y) + \overline{V(x-y,x+y)}) = \mathrm{Re}\ V(x-y,x+y).$$

Hence for any fixed $t \geqq 0$ and for any number h

$$v(x(t+h)-y(t+h))-v(x(t)-y(t)) = \mathrm{Re}\ V(x(t+h)-x(t)-y(t+h)+y(t),x(t+h)+x(t) -$$

$$- y(t+h)-y(t)).$$

Dividing both sides by h in the above equality, and by the sesquilinearity of $V(x,y)$, this becomes

$$h^{-1}[v(x(t+h)-y(t+h))-v(x(t)-y(t))] = \text{Re } V(h^{-1}(x(t+h)-x(t))-h^{-1}(y(t+h)-y(t)), x(t+h)+x(t)-y(t+h)-y(t))$$

Since $h^{-1}(x(t+h)-x(t)) \xrightarrow{w} Ax(t)$ and $x(t+h) \to x(t)$ as $h \to 0$, (similarly these two limits hold by replacing x by y) we have by lemma V-6, as $h \to 0$

$$\frac{d}{dt} v(x(t)-y(t)) = \text{Re } V(Ax(t)-Ay(t), 2x(t)-2y(t)) = 2\text{Re } V(Ax(t)-Ay(t), x(t)-y(t)).$$

Thus (V-15) is proved for $t > 0$. For the case of $t = 0$, we take $h > 0$ and let $h \downarrow 0$. Therefore (V-15) holds for all $t \geq 0$ by defining $\mathring{v}(x(0)-y(0))$ as the right-side limit at $t = 0$.

It is easily seen from the above lemma that if we define $V(x,y) = (x,y)$ where $(\cdot,\cdot)$ is the inner product of the Hilbert space H, then $\mathring{v}(x(t)-y(t)) \leq 0$ along any two solutions $x(t)$ and $y(t)$ if A is dissipative. This follows from (V-15) that $\mathring{v}(x(t)-y(t)) = 2\text{Re}(Ax(t)-Ay(t), x(t)-y(t))$ for all $t \geq 0$ and $x(t), y(t) \in \mathcal{D}(A)$. Conversely, if $\mathring{v}(x(t)-y(t)) \leq 0$ and $\mathring{v}(x(0)-y(0))=2\text{Re}(Ax(0)-Ay(0), x(0)-y(0))$ where $x(0) \equiv x$, $y(0) \equiv y$ are any two elements in $\mathcal{D}(A)$, then A is dissipative. The above argument holds true for the strict dissipativity of A and the relation $\mathring{v}(x(t)-y(t)) \leq -2\beta||x(t)-y(t)||^2$ where $\beta$ is the dissipative constant of A. Hence we have the following theorem which is equivalent to theorem V-2 (theorem V-3).

<u>Theorem V-5</u>. Let A be a nonlinear operator with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ both contained in a Hilbert space H such that $\mathcal{R}(I-A)=H$. Then A is the infinitesimal generator of a nonlinear contraction

(negative contraction) semi-group $\{T_t;\ t \geq 0\}$ on $\mathcal{D}(A)$ if and only if the Lyapunov functional $v(x) = (x,x)$ satisfies

$$\mathring{v}(x-y) = 2\mathrm{Re}(Ax-Ay,x-y) \leq 0 \quad (\mathring{v}(x-y) = 2\mathrm{Re}(Ax-Ay,x-y) \leq -2\beta||x-y||^2) \tag{V-16}$$

where $x \equiv x(0)$, $y \equiv y(0)$ are any two elements of $\mathcal{D}(A)$.

Proof. Let A be the infinitesimal generator of $\{T_t;\ t \geq 0\}$, then for any $x \in \mathcal{D}(A)$ there exists a solution $T_t x$ of (V-1) with $T_0x=x$, and by theorem V-2 (theorem V-3) A is dissipative (strictly dissipative). Applying lemma V-7 for $t = 0$

$$\mathring{v}(x(0)-y(0)) = 2\mathrm{Re}(Ax(0)-Ay(0),\ x(0)-y(0)) \qquad (x(0)=x,\ y(0)=y),$$

and by the dissipativity (strict dissipativity) of A, it follows that

$$\mathring{v}(x-y) = 2\mathrm{Re}(Ax-Ay,\ x-y) \leq 0 \quad (\mathring{v}(x-y)=2\mathrm{Re}(Ax-Ay,x-y) \leq -2\beta||x-y||^2)$$

where $\beta$ is the dissipative constant of A. Conversely, let the Lyapunov functional $V(x) = (x,x)$ satisfy (V-16). Then A is dissipative (strictly dissipative) and theorem V-2 (theorem V-3) implies that A is the infinitesimal generator of a nonlinear contraction (negative contraction) semi-group.

Lemma V-8. Let $V(x,y)$ be a defining sesquilinear functional defined on the product space $H \times H$. Then

$$(x,y)_1 = V(x,y) \qquad x,y \in H$$

defines an inner product $(\cdot,\cdot)_1$ which is equivalent to $(\cdot,\cdot)$.

Proof. By the symmetry and the sesquilinearity properties of $V(x,y)$

$$(x,y)_1 = V(x,y) = \overline{V(y,x)} = \overline{(y,x)}_1 \qquad \text{for any } x,y \in H$$

and

$$(\alpha_1x_1+\alpha_2x_2,y)_1 = V(\alpha_1x_1+\alpha_2x_2,y) = \alpha_1V(x_1,y)+\alpha_2V(x_2,y) = \alpha_1(x_1,y)_1+\alpha_2(x_2,y)_1$$

for any $x_1, x_2, y \in H$; by the positivity of $V(x,y)$

$$(x,x)_1 = V(x,x) \geqq \delta||x||^2$$

so that $(x,x)_1 \neq 0$ if $x \neq 0$.

Hence $(\cdot,\cdot)_1$ is an inner product. The boundedness of $V(x,y)$ implies that

$$(x,x)_1 = V(x,x) \leqq \gamma||x||^2.$$

Therefore, $\delta||x||^2 \leqq ||x||_1^2 \leqq \gamma||x||^2$ which shows that $(\cdot,\cdot)_1$ is equivalent to $(\cdot,\cdot)$.

Lemma V-9. Let S be a bounded linear operator on a complex Hilbert space H. If $(Sx,x)$ is real for any $x \in H$, then S is self-adjoint. In particular, if S is positive definite (i.e. there exists a real number $\delta > 0$ such that $(Sx,x) \geqq \delta||x||^2$ $x \in H$), then S is self-adjoint.

Proof. Since S is a linear operator, it is easily seen that for any $x, y \in H$

$$(S(x+y), x+y) - (S(x-y),x-y) = 2((Sx,y) + (Sy,x)), \qquad \text{(V-17)}$$

and on replacing y by iy in (V-17) we have

$$(S(x+iy), x+iy) - (S(x-iy), x-iy) = -2i((Sx,y) - (Sy,x)). \qquad \text{(V-17)'}$$

By multiplying (V-17)' by i and adding to (V-17) yields

$$4(Sx,y) = [(S(x+y),x+y)-(S(x-y),x-y)] + i[(S(x+iy),x+iy)-(S(x-iy),x-iy)].$$

Since the above equality holds for arbitrary $x, y \in H$ and by hypothesis, the expressions in brackets are real, we have on interchanging x and y:

$$4(Sy,x)=[(S(y+x),y+x)-(S(y-x),y-x)] +i\ [(S(y+ix),y+ix)-(S(y-ix),y-ix)]$$

$$= [(S(x+y),x+y)-(S(x-y), x-y)] + i[(S(x-iy),x-iy)-(S(x+iy),x+iy)]$$

$$= 4\overline{(Sx,y)} = 4(y,Sx).$$

Thus $(x,Sy) = (Sx,y)$ which shows that S is self-adjoint. In particular, if S is positive definite then $(Sx,x)$ is real and so S is self-adjoint.

From the above two lemmas, the following theorem can easily be shown.

Theorem V-6. Let $H_1 = (H, (\cdot,\cdot)_1)$ be a complex Hilbert space. An inner product $(\cdot,\cdot)_2$ defined on the same complex vector space H is equivalent to the inner product $(\cdot,\cdot)_1$ if and only if there exists a positive definite operator $S \in L(H_1,H_1)$ such that

$$(x,y)_2 = (x, Sy)_1 \qquad \text{for all } x,y \in H. \tag{V-18}$$

Proof. Suppose that $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ are equivalent, then by definition there exists constants $\delta$ and $\gamma$ with $0 < \delta \leq \gamma < \infty$ such that

$$\delta||x||_1 \leq ||x||_2 \leq \gamma||x||_1 \qquad \text{for all } x \in H.$$

Define $V(x,y) = (x,y)_2$, then by definition of inner product, $V(x,y)$ is a sesquilinear functional defined on $H_1 \times H_1$ and that $V(x,y) = \overline{V(y,x)}$. Moreover, by the equivalence relation between $||\cdot||_1$ and $||\cdot||_2$

$$|V(x,y)| = |(x,y)_2| \leq ||x||_2\ ||y||_2 \leq \gamma^2||x||_1\ ||y||_1 \quad \text{and}$$

$$V(x,x) = (x,x)_2 \geq \delta^2||x||_1^2 .$$

Hence by the Lax-Milgram theorem there exists a bounded linear operator S on $H_1$ such that

$$(x,y)_2 = V(x,y) = (x,Sy)_1 \qquad \text{for all } x,y \in H.$$

The operator S is positive on $H_1$ since

$$(x,Sx)_1 = (x,x)_2 \geq \delta^2||x||_1^2 \qquad \text{for all } x \in H.$$

Conversely, let $S \in L(H_1,H_1)$ be a positive definite operator satisfying (V-18), then the functional $V(x,y)$ defined by $V(x,y) = (x,y)_2 = (x,Sy)_1$ is a sesquilinear functional on $H_1 \times H_1$ since S is linear. The positive definiteness of S implies that

$$V(x,x) = (x,Sx)_1 \geq \delta_1||x||_1^2 \qquad \text{for some } \delta_1 > 0$$

and that by applying lemma V-9

$$V(x,y) = (x,Sy)_1 = (Sx,y)_1 = \overline{(y,Sx)_1} = \overline{V(y,x)}.$$

Moreover, since S is a bounded operator we have

$$|V(x,y)| = |(x,Sy)_1| \leqq ||S||\ ||x||_1\ ||y||_1.$$

Hence $V(x,y)$ is a defining sesquilinear functional. By lemma V-8 $(x,y)_2 = V(x,y)$ defines an equivalent inner product $(\cdot,\cdot)_2$ of $(\cdot,\cdot)_1$ which proves the theorem.

Theorem V-6 is, in fact, an extension of theorem IV-1. It should be noted that the condition of self-adjointness of S is not required since the positive definiteness of S in a complex Hilbert space implies that it is self-adjoint.

Theorem V-7. Let A be a nonlinear operator with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ both contained in a Hilbert space $H = (H,(\cdot,\cdot))$ such that $\mathcal{R}(I-A) = H$. Then A is the infinitesimal generator of a nonlinear contraction semi-group $\{T_t;\ t \geqq 0\}$ on $\mathcal{D}(A)$ in an equivalent Hilbert space $H_1 = (H,(\cdot,\cdot)_1)$ if and only if there exists a Lyapunov functional $v(x) = V(x,x)$ such that

$$\dot{v}(x-y) = 2\mathrm{Re}\ V(Ax-Ay,\ x-y) \leqq 0 \qquad x,y \in \mathcal{D}(A) \tag{V-19}$$

where $V(x,y)$ is the defining sesquilinear functional of $v(x)$ on $H \times H$.

Proof. Let A be the infinitesimal generator in the Hilbert space $H_1$ as given in the theorem. Then by theorem V-4, A is dissipative with respect to $(\cdot,\cdot)_1$, that is

$$\mathrm{Re}(Ax-Ay,\ x-y)_1 \leqq 0 \qquad x,y \in \mathcal{D}(A).$$

Define $V(x,y) = (x,y)_1$. Then $V(x,y)$ is a defining sesquilinear functional defined on $H \times H$. To see this, note that $V(x,y)$ is sesquilinear, $V(x,y) = \overline{V(y,x)}$ and by the relation (V-12)

$$|V(x,y)| \leqq ||x||_1\ ||y||_1 \leqq \gamma^2||x||\ ||y|| \qquad \text{for all } x,y \in H$$

and

$$V(x,x) = ||x||_1^2 \geqq \delta^2||x||^2 \qquad \text{for all } x,y \in H.$$

Hence the scalar functional $v(x) = V(x,x) = (x,x)_1$ is a Lyapunov functional on the space H. By lemma V-7, for any x, y $\varepsilon$ $\mathcal{D}(A)$

$$\mathring{v}(T_tx-T_ty) = 2\mathrm{Re}V(AT_tx-AT_ty, T_tx-T_ty) \quad (t \geqq 0).$$

In particular, for t = 0

$$\mathring{v}(x-y) = 2\mathrm{Re}V(Ax-Ay, x-y) \qquad x,y \in \mathcal{D}(A).$$

Thus the dissipativity of A with respect to $(\cdot,\cdot)_1$ implies that

$$\mathring{v}(x-y) = 2\mathrm{Re}V(Ax-Ay, x-y) = (Ax-Ay, x-y)_1 \leqq 0.$$

Conversely, suppose that there exists a Lyapunov functional $v(x) = V(x,x)$ such that (V-19) holds, where $V(x,y)$ is a defining sesquilinear functional defined on H x H. By lemma V-8, the functional $(x,y)_1 = V(x,y)$ defines an equivalent inner product of $(\cdot,\cdot)$. Hence, by the hypothesis (V-19)

$$\mathrm{Re}(Ax-Ay,x-y)_1 = \mathrm{Re}V(Ax-Ay, x-y) \leqq 0 \qquad x,y \in \mathcal{D}(A)$$

which implies that A is dissipative with respect to $(\cdot,\cdot)_1$. The result follows by applying theorem V-4.

<u>Theorem V-8</u>. Let A be a nonlinear operator with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ both contained in a Hilbert space $H = (H,(\cdot,\cdot))$ such that $\mathcal{R}(I-A) = H$. Then A is the infinitesimal generator of a nonlinear negative contraction semi-group $\{T_t;\ t \geqq 0\}$ on $\mathcal{D}(A)$ in an equivalent Hilbert space $H_1 = (H, (\cdot,\cdot)_1)$ if and only if there exists a Lyapunov functional $v(x) = V(x,x)$ such that

$$\mathring{v}(x-y) = 2\mathrm{Re}V(Ax-Ay, x-y) \leqq -2\beta||x-y||^2 \qquad x,y \in \mathcal{D}(A) \qquad \text{(V-20)}$$

for some $\beta > 0$ where $V(x,y)$ is the defining sesquilinear functional of $v(x)$ on H x H.

<u>Proof</u>. The proof is essentially the same as for theorem V-7. To show the "only if" part, define $V(x,y) = (x,y)_1$ then $V(x,y)$ is a

defining sesquilinear functional defined on $H \times H$ as has been shown in theorem V-7. Since A generates a nonlinear negative contraction semi-group, it is strictly dissipative with respect to $(\cdot,\cdot)_1$ with the dissipative constant $\beta_1$ (theorem V-4). Thus by lemma V-7 and the equivalence relation between $||\cdot||$ and $||\cdot||_1$

$$\dot{v}(x-y) = 2\mathrm{Re}V(Ax-Ay,x-y) = (Ax-Ay,x-y)_1 \leqq -2\beta_1||x-y||_1^2 \leqq$$
$$\leqq -2\beta_1\delta^2||x-y||^2$$

for any $x$, $y \in \mathcal{D}(A)$ where we have used the relation (V-12). The result follows by letting $\beta=\beta_1\delta^2$. Conversely, let a Lyapunov functional $v(x) = V(x,x)$ exist and satisfy the relation (V-20), then by lemma V-8 the functional

$$(x,y)_1 = V(x,y) \qquad \text{for all } x,y \in H$$

defines an equivalent inner product $(\cdot,\cdot)_1$. Hence by (V-20) and the relation (V-12), we have for any $x,y \in \mathcal{D}(A)$

$$\mathrm{Re}(Ax-Ay,x-y)_1 = \mathrm{Re}V(Ax-Ay,x-y) \leqq -\beta||x-y||^2 \leqq$$
$$\leqq -\beta/\gamma^2 \ ||x-y||_1^2$$

which shows that A is strictly dissipative. Hence the result follows by applying theorem (V-4).

In theorem V-5 the Lyapunov functional $v(x)$ is defined by the original inner product and in theorem V-6 $v(x)$ is defined by an equivanent inner product $(\cdot,\cdot)_1$. If the defining sesquilinear functional $V(x,y)$ of $v(x)$ satisfies (V-16) and (V-19) respectively, then together with the assumption $\mathcal{R}(I-A) = H$, A is the infinitesimal generator of a contraction semi-group on $\mathcal{D}(A)$ in the respective space H and $H_1$. The contraction semi-group $\{T_t;\ t \geqq 0\}$ generated by A in the $H_1$-space satisfies for any $x \in \mathcal{D}(A)$ and $t \geqq 0$

$$\left(\frac{dT_t x}{dt}, z\right)_1 = (AT_t x, z)_1 \qquad \text{for every } z \in H_1.$$

However, if is not obvious that the same equality holds for the inner product $(\cdot,\cdot)$. In other words, if $T_t x$ is a solution of (V-1) in an equivalent $H_1$-space, does it imply that it is also a solution of (V-1) in the original H-space? The answer is affirmative as can be seen from the following.

Lemma V-10. Let A be the infinitesimal generator of a nonlinear contraction (negative contraction) semi-group $\{T_t;\ t \geqq 0\}$ on $\mathcal{D}(A)$ in an equivalent Hilbert space $H_1 = (H, (\cdot,\cdot)_1)$. Then A is the infinitesimal generator of a nonlinear (negative) semi-group $\{T_t;\ t \geqq 0\}$ on the same domain $\mathcal{D}(A)$ in the original Hilbert space $H = (H, (\cdot,\cdot))$.

Proof. By the equivalence relation between the two inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_1$, the sesquilinear functional $V(x,y) = (x,y)$ defined on the product space $H_1 \times H_1$ satisfies all the hypotheses in the Lax-Milgram theorem. Thus there exists a bounded linear operator S with a bounded inverse $S^{-1}$ defined on all of $H_1$ such that

$$(x,y) = V(x,y) = (x,Sy)_1 \qquad \text{for all } x,\ y \in H. \tag{V-21}$$

By hypothesis, A generates the semi-group $\{T_t;\ t \geqq 0\}$ in $H_1$ so that

$$\lim_{t \downarrow 0} t^{-1}(T_t x - x,\ z)_1 = (Ax,\ z)_1 \qquad \text{for every } z \in H. \tag{V-22}$$

It follows from (V-21) and (V-22) that for each $z \in H$

$$\lim_{t \downarrow 0} t^{-1}(T_t x - x,\ z) = \lim_{t \downarrow 0} t^{-1}(T_t x - x,\ Sz)_1 = (Ax, Sz)_1 = (Ax, z)$$

which shows that A is the infinitesimal generator of the semi-group $\{T_t;\ t \geqq 0\}$ on $\mathcal{D}(A)$ in the space H. The fact that $\{T_t;\ t \geqq 0\}$ remains as a semi-group in H is that semi-group property is invariant under equivalent norms except for possibly the contraction property. Since $\{T_t;\ t \geqq 0\}$ is a contraction semi-group in $H_1$ and $||\cdot||$ and $||\cdot||_1$ are equivalent, we have by the relation (V-12)

$$||T_t x - T_t y|| \leqq \gamma/\delta\ ||x-y|| \qquad x,\ y \in \mathcal{D}(A)$$

$$(||T_t x - T_t y|| \leqq \gamma/\delta\ e^{-\beta t}||x-y|| \qquad x,\ y \in \mathcal{D}(A))$$

and the lemma is proved.

Corollary. Let the operator A appearing in (V-1) be the infinitesimal generator of a nonlinear contraction (negative contraction) semi-group $\{T_t;\ t \geqq 0\}$ on $\mathcal{D}(A)$ in the space $H_1 = (H, (\cdot,\cdot)_1)$ so that for any $x \in \mathcal{D}(A)$, $T_t x$ is the unique solution of (V-1) with $T_o x = x$. Then $T_t x$ is also the unique solution of (V-1) with $T_o x = x$ in the space $H = (H, (\cdot,\cdot))$ where $(\cdot,\cdot)_1$ and $(\cdot,\cdot)$ are equivalent.

Proof. Since (V-21) and (V-22) in the proof of the above lemma hold for any $x, y \in H$, we have for any $x \in \mathcal{D}(A)$ and $t \geqq 0$

$$\lim_{h\to 0} h^{-1}(T_{t+h}x - T_t x, z) = \lim_{h\to 0} h^{-1}(T_h T_t x - T_t x,\ Sz)_1 = (AT_t x,\ Sz)_1$$

$$= (AT_t x,\ z) \qquad \text{for every } z \in H$$

which implies that $T_t x$ is a solution of (V-1) in the space $(H, (\cdot,\cdot))$ since all the other properties listed in theorem V-1 remain unchanged under equivalent norms.

Theorem V-9. Let the nonlinear operator A appearing in (V-1) be such that $R(I-A) = H$. If there exists a Lyapunov functional $v(x) = V(x,x)$, where $V(x,y)$ is a defining sesquilinear functional defined on $H \times H$, such that for any $x,\ y \in \mathcal{D}(A)$

(i) $\mathring{v}(x-y) = 2\mathrm{Re}V(Ax-Ay,\ x-y) \leqq 0$ or

(ii) $\mathring{v}(x-y) = 2\mathrm{Re}V(Ax-Ay,\ x-y) \leqq -2\beta||x-y||^2 \qquad (\beta > 0)$

Then, (a) for any $x \in \mathcal{D}(A)$ there exists a unique solution $x(t)$ of (V-1) with $x(0) = x$, (b) any equilibrium solution $x_e$ (or periodic solution), if it exists, is stable under the condition (i) and is asymptotically stable under the condition (ii), and (c) a stability region of $x_e$ is $D(A)$ which can be extended to $\overline{D(A)}$, the closure of $\mathcal{D}(A)$, in the sense of lemma V-3. If, in addition, $0 \in \mathcal{D}(A)$ and $A0 = 0$, then the zero vector is an equilibrium solution, called the null solution, of (V-1) which is stable or asymptotically stable according to (i) or (ii), respectively.

<u>Proof</u>. By hypothesis and applying theorem V-7, A is the infinitesimal generator of a nonlinear contraction semi-group on $\mathcal{D}(A)$ in an equivalent space $H_1 = (H, (\circ,\circ)_1)$ under the condition (i) and is the infinitesimal generator of a nonlinear negative contraction semi-group on $\mathcal{D}(A)$ in $H_1$ under the condition (ii), where the norm $||\circ||_1$ induced by $(\circ,\circ)_1$ satisfies

$$\delta||x|| \leqq ||x||_1 \leqq \gamma||x|| \qquad \text{for some} \quad 0 < \delta \leqq \gamma < \infty .$$

By lemma V-10, A is the infinitesimal generator of a nonlinear semi-group $\{T_t;\ t \geqq 0\}$ on $\mathcal{D}(A)$ in H such that under the condition (i)

$$||T_t x - T_t y|| \leqq \gamma\ \delta^{-1}||x-y|| \qquad x,y \in \mathcal{D}(A)$$

and under the condition (ii)

$$||T_t x - T_t y|| \leqq \gamma\ \delta^{-1}\ e^{-\beta t}\ ||x-y|| \qquad x,y \in \mathcal{D}(A) \qquad (t \geqq 0).$$

Since for any $x \in \mathcal{D}(A)$, $T_t x$ is the unique solution in $H_1$ with $T_o x = x$, it follows from the corollary of lemma V-10 that $T_t x$ is also the unique solution in H with $T_o x = x$. By the semi-group property of $\{T_t;\ t \geqq 0\}$ in H, we have under the conditions (i) or (ii)

$$||T_t x - x_e|| \leqq \gamma\ \delta^{-1}||x-x_e|| \qquad (t \geqq 0)$$

or

$$||T_t x - x_e|| \leqq \gamma\ \delta^{-1}e^{-\beta t}||x-x_e|| \qquad (t \geqq 0),$$

which shows that the equilibrium solution $x_e$, if it exists, is stabl and asymptotically stable, respectively. Note that $T_t x_e = x_e$ for all $t \geq 0$. Since by lemma V-3, the contraction semi-group $\{T_t; t \geq 0\}$ on $\mathcal{D}(A)$ in the space $H_1$ can be extended to $\overline{\mathcal{D}(A)}$ in the $||\cdot||_1$-topology, the same is true for the semi-group $\{T_t; t \geq 0\}$ on $\mathcal{D}(A)$ in the space H because the closure of $\mathcal{D}(A)$ in the $||\cdot||_1$-topology is the closure of $\mathcal{D}(A)$ in the $||\cdot||$-topology by the equivalence relation of these two norms. Hence the results of (a), (b) and (c) are proved. The stability property of the null solution follows from (b).

The purpose for the construction of a Lyapunov functional can be demonstrated as follows: Let $v(x) = V(x,x)$ be a Lyapunov functional such that for some $\alpha \geq 0$

$$\dot{v}(x(t)-y(t)) \leq -\alpha \; ||x(t)-y(t)||^2 \qquad (t \geq 0) \qquad \text{(V-23)}$$

for any two solutions $x(t)$, $y(t)$ of (V-1), where $V(x,y)$ is a defining sesquilinear functional. By lemma V-8, the functional

$$(x,y)_1 = V(x,y) \qquad x,\ y \in H$$

defines an equivalent inner product of $(\cdot,\cdot)$. Since

$$v(x) = V(x,x) = (x,x)_1 \leq \gamma||x||^2 \qquad \text{for all } x \in H,$$

it follows from (V-23) that

$$\dot{v}(x(t)-y(t)) \leq -\alpha/\gamma \;\; v(x(t)-y(t)) = -2\lambda v(x(t)-y(t)) \qquad (2\lambda \equiv \alpha/\gamma).$$

Integrating the above inequality with respect to t, we have

$$v(x(t)-y(t)) \leq v(x(0)-y(0))e^{-2\lambda t} \qquad (t \geq 0)$$

which is equivalent to

$$||x(t)-y(t)||_1^2 \leq ||x(0)-y(0)||_1^2 \;\; e^{-2\lambda t} \qquad (t \geq 0)$$

since $v(x) = (x,x)_1 = ||x||_1^2$ (for all $x \in H$). By the equivalence relation of $||\cdot||$ and $||\cdot||_1$, there exists constants $\delta$, $\gamma$ with $0<\delta\leq\gamma<\infty$

such that (V-12) holds. Thus the above inequality implies that

$$||x(t)-y(t)||^2 \leq 1/\delta^2 \; ||x(t)-y(t)||_1^2 \leq e^{-2\lambda t}/\delta^2 ||x(0)-y(0)||_1^2 \leq$$
$$(\gamma/\delta)^2 e^{-2\lambda t} \; ||x(0)-y(0)||^2$$

which is the same as

$$||x(t)-y(t)|| \leq \gamma/\delta \; e^{-\lambda t} \; ||x(0)-y(0)|| \qquad \text{for } t \geq 0.$$

Hence, if an equilibrium solution $x_e$ (or any unperturbed solution) exists, then by choosing $y(0) = x_e$ in the above inequality, we have

$$||x(t)-x_e|| \leq \gamma/\delta \; e^{-\lambda t} ||x(0)-x_e|| \qquad \text{for all } t \geq 0$$

which shows that the equilibrium solution $x_e$ is exponentially asymptotically stable if $\alpha > 0$, and is stable if $\alpha = 0$.

The importance of theorems V-5, V-7, V-8 and V-9 is the fact that the existence of a Lyapunov functional satisfying (V-16) or (V-20) alone does not guarantee the existence of a solution to (V-1) and in general, it is rather complicated to prove such solutions exist. However under the additional assumption that $\mathcal{R}(I-A) = H$ the existence of a solution with any initial element $x \in \mathcal{D}(A)$ is assured. This assurance makes the stability of solutions of (V-1) meaningful.

## C. Stability Theory of Semi-linear Stationary Equations

In this section, we consider the operational differential equations of the semi-linear form

$$\frac{dx}{dt} = A_o x + f(x) \qquad x \in \mathcal{D}(A_o) \qquad \text{(V-24)}$$

where $A_o$ is a linear operator with domain $\mathcal{D}(A_o)$ and range $\mathcal{R}(A_o)$ both contained in a real Hilbert space H, and f is a given function (in general, nonlinear in x) defined on H to H. By considering the operator $A_o + f(\cdot)$ as the nonlinear operator A in the previous sections, (V-24) becomes a special case of (V-1) and hence all the results developed in the previous sections are applicable to this case. In particular, if $A_o$ is the infinitesimal generator of a linear contraction semi-group of class $C_o$, it is natural to ask that under what conditions on f the operator $A_o + f(\cdot)$ is the infinitesimal generator of a non-linear contraction semi-group, or equivalently under what conditions on f a solution of (V-24) exists and is stable (or asymptotically stable). One simple answer to this question is that $(f(x)-f(y), x-y) \leq 0$ and $R(I-A_o-f(\cdot)) = H$ since under these assumptions $A=A_o + f(\cdot)$ is dissipative and the result follows by applying theorem V-2. However the requirement $R(I-A_o-f(\cdot)) = H$ by itself is not easy to verify since it is equivalent to the functional equation

$$x - A_o x - f(x) = z$$

having a solution for every $z \in H$. In order to eliminate this assumption and to refine some assumptions on the operator $A_o$, we shall make use of some results due to Browder [1], [2] for the case of a Hilbert space. The results obtained in this section include:

(a) The existence and the uniqueness of a solution of (V-24).

(b) The stability or asymptotic stability of an equilibrium solution as well as the stability region with respect to the equilibrium solution.

In order to show the following results, it is convenient to state a theorem due to Browder [2].

Theorem V-10 (Browder). Let X be an uniformly convex Banach space with its conjugate space X* also uniformly convex, and let T and $T_o$ be two accretive mappings with domain and range in X. Suppose that

(i) The range of T+I is all of X. $\mathcal{D}(T)$ is dense in X.

(ii) $T_o$ is defined and demicontinuous (i.e. continuous from X in the strong topology to the weak topology of X) on all of X and maps bounded subsets of X into bounded subsets of X.

(iii) The mapping $T+T_o$ defined with domain $\mathcal{D}(T)$ satisfies the condition that

$$||Tx+T_ox|| \to +\infty, \text{ as } ||x|| \to +\infty \qquad (x \in \mathcal{D}(T)).$$

Then, the range of $(T+T_o)$ is all of X, i.e., for each z in X, there exists an element x in $\mathcal{D}(T)$ such that

$$Tx + T_ox = z.$$

It is to be noted that in the case of a Hilbert space X, both X and X* are uniformly convex since X* is also a Hilbert space. Moreover, the definition of accretive operator coincides with monotone operator when X is a Hilbert space. Now we show the following:

<u>Theorem V-11</u>. Let $A_o$ be the infinitesimal generator of a (linear) contraction semi-group of class $C_o$. Assume that f satisfies the following conditions:

(i) f is defined on all of H into H such that it is continuous from H in the strong topology to the weak topology, and is bounded on every bounded subset of H.

(ii) $(f(x) - f(y), x-y) \leq 0 \qquad$ for all x, y $\varepsilon$ H.

Then,

(a) For any $x \in \mathcal{D}(A_o)$, there exists a unique solution of (V-24) (in the sense of theorem V-1) with $T_ox = x$ such that $T_tx$ is strongly continuous and is weakly differentiable with respect to t.

(b) Any equilibrium solution $x_e$ (or any unperturbed solution), if it exists, is stable.

(c) A stability region with respect to the equilibrium solution $x_e$ (or any unperturbed solution) is $\mathcal{D}(A_o)$ which can be extended to the whole space H in the sense of lemma V-3.

<u>Proof</u>. Let $A=A_o + f(\circ)$ with $\mathcal{D}(A) = \mathcal{D}(A_o)$. Since an infinitesimal generator of a contraction semi-group of class $C_o$ is densely defined, dissipative and $\mathcal{R}(I-A_o) = H$ (see theorems III-12 and III-14), it follows by the dissipativity of $A_o$ and by the assumption (ii) that

$$(Ax-Ay, x-y) = (A_o x-A_o y,\ x-y) + (f(x)-f(y),\ x-y) \leqq 0 \quad \text{for all } x, y \in \mathcal{D}(A)$$

which shows that A is dissipative. To show that $\mathcal{R}(I-A) = H$, we apply theorem V-10. Note that the operator $-A_o$ is monotone and the range of $-A_o + I$ is all of H with $\mathcal{D}(-A_o) = \mathcal{D}(A_o)$ dense in H. Thus the operator $T = -A_o$ is accretive (or monotone) and satisfies the condition (i) of theorem V-10. To show the conditions (ii) and (iii) of theorem V-10, let $T_o = I-f(\circ)$. Then from assumption (i) $T_o$ is defined on all of H and is continuous from H in the strong topology to the weak topology and maps bounded subsets of H into bounded subsets of H which shows (ii) of theorem V-10. $T_o$ is monotone, for

$$(T_o x-T_o y,\ x-y) = (x-y,\ x-y) - (f(x)-f(y),\ x-y) \geqq ||x-y||^2 \qquad x, y \in H$$

where we have used assumption (ii). Moreover, by letting y=0 in (ii) gives

$$(f(x), x) \leqq (f(0), x) \leqq ||f(0)||\ ||x|| \qquad \text{for all } x \in H. \qquad \text{(V-25)}$$

It follows by the dissipativity of $A_o$ and by (V-25) that

$$||-A_o x+T_o x|| \geqq (-A_o x+T_o x, x)/||x|| \geqq (T_o x, x)/||x|| = ((x, x)-(f(x), x))/||x||$$

$$\geqq ||x|| - ||f(0)|| \qquad \text{for all } x \in \mathcal{D}(A_o) \quad (x \neq 0)\ .$$

Thus $||Tx+T_o x|| \to +\infty$ as $||x|| \to \infty$, that is, condition (iii) of theorem V-10 is satisfied. Hence by applying that theorem we have $\mathcal{R}(I-A) = \mathcal{R}(T+T_o) = H$.

This later condition and the dissipativity of A imply that A is the infinitesimal generator of a nonlinear contraction semi-group $\{T_t;\ t \geq 0\}$ on $\mathcal{D}(A_o)$ by applying theorem V-2. Therefore, for any $x \in \mathcal{D}(A_o)$, $T_t x \in \mathcal{D}(A)$ and is the unique solution of (V-24) with $T_o x = x$ and such that $T_t x$ is strongly continuous and weakly once differentiable with respect to t. Since

$$||T_t x - T_t y|| \leq ||x-y|| \qquad \text{for all } t \geq 0 \qquad x, y \in \mathcal{D}(A_o)$$

it follows that by taking y as the equilibrium solution $x_e$, if it exists, then it is stable. Note that $T_t x_e = x_e$. The above inequality holds for any $x, y \in \mathcal{D}(A_o)$ which implies that a stability region is $\mathcal{D}(A_o)$, and by lemma V-3 this region can be extended to the whole space H since $\mathcal{D}(A_o)$ is dense in H. Therefore, the theorem is proved.

The above theorem can be extended to the asymptotic stability of an unperturbed solution. This can be achieved by making use of theorem V-3.

Theorem V-12. Let $A_o$ be the infinitesimal generator of a (linear) negative contraction semi-group of class $C_o$ with contractive constant $\beta$. Assume that f satisfies the following conditions:

(i) f is defined on all of H into H such that it is continuous from H in the strong topology to the weak topology and is bounded on every bounded subset of H,

(ii) $(f(x) - f(y), x-y) \leq k||x-y||^2$ with $k < \beta$ for all $x, y \in H$.

Then,

(a) For any $x \in \mathcal{D}(A_o)$, there exists a unique solution $T_t x$ to (V-24) with $T_o x = x$ such that $T_t x$ is strongly continuous and is weakly differentiable with respect to t.

(b) Any equilibrium solution (or any unperturbed solution), if it exists, is asymptotically stable.

(c) A stability region with respect to any unperturbed solution, including an equilibrium solution, is $\mathcal{D}(A_o)$ which can be extended to the whole space H in the sense of lemma V-3.

<u>Proof</u>. Let $A = A_o + f(\cdot)$. Since $A_o$ is the infinitesimal generator of a negative contraction semi-group, it is densely defined, dissipative and $\mathcal{R}(I-A_o) = H$. Applying theorem V-3 for the linear case, $A_o$ is strictly dissipative with dissipative constant $\beta$, that is

$$(A_o x, x) \leq -\beta ||x||^2 \qquad \text{for all } x \in \mathcal{D}(A_o).$$

Thus the operator A is strictly dissipative with dissipative constant $\beta-k$ for

$$(Ax-Ay,\ x-y) = (A_o x - A_o y,\ x-y) + (f(x)-f(y),\ x-y) \leq -(\beta-k)||x-y||^2$$

for all $x, y \in \mathcal{D}(A)$. To show that $\mathcal{R}(I-A) = H$, we prove $\mathcal{R}(I-\alpha A) = H$ for some $\alpha > 0$, since the monotonicity of $-A$ implies that $(I-\alpha A)^{-1}$ exists for every $\alpha > 0$, and by applying lemma V-1 if $\mathcal{R}(I-\alpha A) = H$ for some $\alpha > 0$ then $\mathcal{R}(I-A) = H$. The reason for doing this is that if the same argument as in the proof of theorem V-11 is used it will lead to the unnecessary requirement $k \leq 1$. Let $I-\alpha A=-\alpha A_o+(I-\alpha f(\cdot))=T+T_o$ where $T = -\alpha A_o$ and $T_o=I-\alpha f(\cdot)$. Since $-A_o$ is monotone and is densely defined so is $T=-\alpha A_o$, and since $A_o$ is the infinitesimal generator of a semi-group, $\alpha \in \rho(A_o)$ (the resolvent set of $A_o$) for all $\alpha > 0$ (theorem III-12) which implies that $\mathcal{R}(I+T) = \mathcal{R}(I-\alpha A_o) = H$. Thus the condition (i) of theorem V-10 is satisfied. The mapping $T_o=I-\alpha f(\cdot)$ is monotone for $\alpha \leq k^{-1}$ since by the assumption (ii)

$$(T_o x-T_o y, x-y) = (x-y, x-y) - \alpha(f(x)-f(y), x-y) \geq (1-\alpha k)||x-y||^2 \geq 0.$$

It is obvious by the assumption (i) that $T_o$ is continuous on H and is bounded on every bounded subset of H, which shows that $T_o$ satisfies the condition (ii) of theorem V-10. Finally, the relation $||Tx+T_ox|| \to \infty$ as $||x|| \to \infty$ is also satisfied. This is due to the fact that the dissipativity of $\alpha A_o$ and the relation (V-25) imply that

$$||Tx+T_ox|| = ||-\alpha A_ox+T_ox|| \geqq (-\alpha A_ox+T_ox,x) \;/||x|| \geqq (T_ox,x)/||x||=$$
$$((x,x)-\alpha(f(x),x))/||x|| \geqq (||x||^2-\alpha||f(0)|| \; ||x||)/||x||=||x||-\alpha||f(0)||$$

where $\alpha > 0$ is a fixed number. Hence by choosing $\alpha \leqq k^{-1}$, all the hypotheses in theorem V-10 are satisfied and the result $\mathcal{R}(I-\alpha A)=\mathcal{R}(T+T_o)=H$ follows. It should be noted that $k > 0$ so that $0 < \alpha \leqq k^{-1}$ exists. (if $k \leqq 0$, then $T_o$ is monotone by taking, for instance, $\alpha = 1$ and the other conditions remain unchanged). By theorem V-3, A is the infinitesimal generator of a nonlinear negative contraction semi-group $\{T_t;\ t \geqq 0\}$ on $\mathcal{D}(A_o)$ with the contractive constant $\beta$-k. Therefore the results listed in (a), (b) and (c) follow directly from the negative contraction property of the semi-group $\{T_t;\ t \geqq 0\}$ and by lemma V-3 for the extension of the stability region.

Remark. If $A_o$ is the infinitesimal generator of a contraction semi-group instead of a negative contraction semi-group, any unperturbed solution is still asymptotically stable provided that the constant k appearing in the condition (ii) is negative, since in this case, we may take $\beta=0$ and the operator $A=A_o+f(\circ)$ remains strictly dissipative with dissipative constant $-k$. The proof of $\mathcal{R}(I-A) = H$ remains the same.

Corollary 1. Under the hypothesis of theorem V-11 (theorem V-12) and in addition, if $f(0)=0$, then the null solution is stable (asymptotically stable) with the stability region the whole space H.

Proof. If $f(0) = 0$ then $x(t) \equiv 0$ is an equilibrium solution (called the null solution) of (V-24). Hence by theorem V-11 (theorem V-12), the null solution is stable (asymptotically stable) with the stability region extended to the whole space H.

Corollary 2. Let $A_o$ be the infinitesimal generator of a (linear) negative contraction semi-group of class $C_o$ with contractive constant $\beta$, and let f be Lipschitz continuous on H with Lipschitz constant $k < \beta$, that is

$$||f(x)-f(y)|| \leq k||x-y|| \quad \text{for all } x, y \in H. \tag{V-26}$$

Then for any $x \in \mathcal{D}(A_o)$ there exists a unique solution $T_t x$ to (V-24) with $T_o x = x$ such that any equilibrium solution $x_e$ to (V-24) is asymptotically stable. In particular, if $f(0)=0$ the null solution is asymptotically stable. Moreover, a stability region is $\mathcal{D}(A_o)$ which can be extended to the whole space H.

Proof. By the Lipschitz continuity of f on H, it follows that condition (i) in theorem V-12 is satisfied. This is due to the fact that strong continuity implies weak continuity, and by (V-26) with $x_o$ a fixed element in H

$$||f(x)|| \leq ||f(x_o)||+k\,||x-x_o|| \leq ||f(x_o)|| +k\,||x|| +k\,||x_o||$$

which is bounded whenever $||x||$ is bounded. Moreover, by (V-26)

$$(f(x)-f(y), x-y) \leq ||f(x)-f(y)||\,||x-y|| \leq k||x-y||^2$$

and so condition (ii) in theorem V-12 is satisfied. Hence, by theorem V-12 the existence and the uniqueness of a solution as well as the stability property of an equilibrium solution are proved. In particular, if $f(0) = 0$ then corollary 1 implies that the null solution is asymptotically stable.

Theorem V-13. Let the linear operator $A_o$ appearing in (V-24) be such that $0 \in \mathcal{D}(A_o)$ and that for some finite number $\beta$ (i.e., $|\beta|<\infty$),

$$(A_o x, x) \leq \beta(x,x) \qquad \text{for all } x \in \mathcal{D}(A_o).$$

Let f be defined on $\mathcal{D}(A_o)$ to H such that $f(0)=0$ and such that for some finite number k (i.e., $|k| < \infty$)

$$(f(x),x) \leq k||x||^2 \qquad \text{for all } x \in \mathcal{D}(A_o).$$

If $\beta > k$ then the null solution of (V-24) is the only equilibrium solution.

Proof. It is obvious that the zero vector is an equilibrium solution of (V-24). Let $x_e$ be any other equilibrium solution, then $x_e \in \mathcal{D}(A_o)$ and by the statement following definition V-5, $A_o x_e + f(x_e) = 0$. It follows that

$$0 = (A_o x_e + f(x_e), x_e) = (A_o x_e, x_e) + (f(x_e), x_e) \leq -(\beta-k)||x_e||^2$$

which implies that $x_e=0$ since by hypothesis $\beta-k > 0$. Hence the uniqueness of the equilibrium solution is proved.

Corollary. Under the conditions of theorem V-12 and in addition if $f(0) = 0$, then the null solution is the only equilibrium solution.

Proof. Since $A_o$ is the infinitesimal generator of a negative contraction semi-group with contractive constant $\beta$, it is strictly dissipative with dissipative constant $\beta$ and $0 \in \mathcal{D}(A_o)$. By the assumption (ii) of theorem V-12 we have, by letting y=0 in the condition (ii)

$$(f(x),x) \leq k||x||^2 \qquad \text{with } k < \beta, \qquad x \in H$$

since $f(0) = 0$. Hence the uniqueness of the equilibrium solution follows from the theorem.

Most of the theorems developed in this section up to now assumed that the linear part $A_o$ of (V-24) is the infinitesimal generator of a contraction semi-group of class $C_o$. A necessary and sufficient condition for $A_o$ having this property is that $A_o$ is dissipative, $\overline{\mathcal{D}(A_o)} = H$ and $\mathcal{R}(I-A_o) = H$ (see theorem III-14). Again the requirement $\mathcal{R}(I-A_o) = H$ means the existence of a solution of the functional equation

$$x - A_o x = z$$

for every $z \in H$ which by itself needs further justification. However in case $A_o$ is a self-adjoint operator which occurs often in physical applications, this requirement can be eliminated in these theorems. In order to show this, we first state a theorem from [1] by Browder and then we consider a densely defined closed operator and take a self-adjoint operator as a special case.

Theorem V-14 (Browder). Let X be a reflexive Banach space, T a mapping from the dense linear subset $\mathcal{D}(T)$ of X into X*. Suppose that T=L+G where L is a densely defined closed linear operator from X to X*, G a hemi-continuous mapping from X to X* with $\mathcal{D}(G) = X$ and G taking bounded subsets of X into bounded subsets of X*. Suppose that:

(i) There exists a completely continuous mapping C from X to X* such that T+C is monotone;

(ii) L* is the closure of its restriction to $\mathcal{D}(L) \cap \mathcal{D}(L^*)$;

(iii) There exists a real-valued function c(r) on $R^1$ with $c(r) \to \infty$ as $r \to \infty$ such that

$$(Tx,x) \geqq c(||x||)\ ||x|| \qquad \text{for all } x \in \mathcal{D}(T).$$

Then $\mathcal{R}(T)$, the range of T, is all of X*.

Remarks. (a) G is said to be hemi-continuous if G is continuous from every line segment in $\mathcal{D}(G)$ to the weak* topology of X*.
(b) A Hilbert space is reflexive.

Theorem V-15. Let $A_o$ be a densely defined closed operator from H into H. Suppose that:

(i) $A_o$ is strictly dissipative with dissipative constant $\beta$, that is

$$(A_o x, x) \leqq -\beta||x||^2 \qquad \text{for all } x \in \mathcal{D}(A_o);$$

(ii) $A_o^*$ is the closure of its restriction to $\mathcal{D}(A_o) \cap \mathcal{D}(A_o^*)$ where $A_o^*$ is the adjoint operator of $A_o$;

(iii) f is defined on all of H into H such that it is continuous from the strong topology to the weak topology and is bounded on every bounded subset of H;

(iv) $(f(x)-f(y), x-y) \leqq k||x-y||^2$ with $k < \beta$ for all $x,y \in H$.

Then

(a) For any $x \in \mathcal{D}(A_o)$ there exists a unique strongly continuous solution $T_t x$ to (V-24) with $T_o x = x$;

(b) An equilibrium solution $x_e$, if it exists, is asymptotically stable. In particular, if $f(0)=0$ the null solution exists and is asymptotically stable;

(c) The stability region can be extended to the whole space in the sense of lemma V-3.

Proof. Let $A = A_o + f(\cdot)$, then A is strictly dissipative, since by hypothesis

$$(Ax-Ay, x-y) = (A_o x - A_o y, x-y) + (f(x)-f(y), x-y) \leqq -(\beta-k)||x-y||^2$$

for all $x,y \in \mathcal{D}(A_o) = \mathcal{D}(A)$. To show that $\mathcal{R}(I-A) = H$, let $T = I-A = -A_o + (I-f(\cdot))$, then $\mathcal{D}(T) = \mathcal{D}(A_o)$ is densely defined. Since $-A_o$ is densely defined, $A_o^*$ exists and is closed, and by the assumption (ii) $-A_o^*$ is the closure of its restriction to $\mathcal{D}(-A_o) \cap \mathcal{D}(-A_o^*)$. By (iii) the operator $G = I-f(\cdot)$ is continuous from all of H to H in the strong topology to the weak topology which implies its hemi-continuity from H to H with $\mathcal{D}(G) = H$. The boundedness of G on bounded subsets of H also follows from (iii). Moreover

$$(Tx-Ty,x-y) = (x-y,x-y) - (Ax-Ay,x-y) \geqq (1+\beta-k)||x-y||^2 \qquad x,y \in \mathcal{D}(T)$$

so that T is monotone. In particular by letting $y=0$ ($0 \in \mathcal{D}(A_o)=\mathcal{D}(T)$) in the above inequality and since $T\circ 0 = 0-A\circ 0 = -f(0)$, it follows that

$$(Tx,x) \geqq (1+\beta-k)||x||^2-(f(0),x) \geqq ((1+\beta-k)||x||-||f(0)||)||x||,$$

$$\text{for all } x \in \mathcal{D}(T)$$

and since $\beta - k > 0$ the real valued function $c(||x||)$ defined by

$$c(||x||) = (1+\beta-k)||x||-||f(0)||$$

has the property that $c(||x||) \to \infty$ as $||x|| \to \infty$. Hence all the conditions in theorem V-14 are satisfied if we take, for instance, the completely continuous mapping $C=0$ (the zero operator which maps all $x \in H$ into the 0 vector in H). Therefore $\mathcal{R}(I-A) = \mathcal{R}(T) = H$. By applying theorem V-3, A is the infinitesimal generator of a non-linear negative contraction semi-group on $\mathcal{D}(A) = \mathcal{D}(A_o)$ with the contractive constant $\beta-k$. Thus, the stated results in the theorem follow directly from the negative contraction semi-group property as in the proof of theorem V-11.

Remarks. (a) The above theorem can also be proved with $\beta=k=0$, in which case the equilibrium solution is stable with a stability region $\mathcal{D}(A_o)$. The proof is exactly the same by letting $\beta=k=0$ and by applying theorem V-2. (b) If $A_o$ is dissipative (i.e. $\beta=0$ in (i)) and $k < 0$ in (iv), then the theorem is still valid. In this case, $A_o+f(\circ)$ is the infinitesimal generator of a nonlinear negative contraction semi-group with the contractive constant $-k$.

Since an unbounded self-adjoint operator $A_o$ is a densely defined closed operator having the property that $\mathcal{D}(A_o) = \mathcal{D}(A_o^*)$ (in fact $A_o=A_o^*$, see definition III-3) we have, with a stronger assumption on the function f, the following result which is stated as a theorem because of its usefulness in applications.

Theorem V-16. Let $A_o$ be an unbounded self-adjoint operator from H to H and assume that it is strictly dissipative with dissipative constant β, that is

$$(A_o x, x) \leqq -\beta(x,x) \qquad \text{for all } x \in \mathcal{D}(A_o).$$

Let f be Lipschitz continuous on H with Lipschitz constant $k < \beta$, that is

$$||f(x)-f(y)|| \leqq k||x-y|| \qquad \text{for all } x,y \in H.$$

Then for any $x \in \mathcal{D}(A_o)$ there exists a unique strongly continuous solution $T_t x$ to (V-24) with $T_o x=x$. Moreover any equilibrium solution $x_e$ of (V-24), if it exists, is asymptotically stable with $\mathcal{D}(A_o)$ a stability region, and this region can be extended to the whole space H. In particular, if $f(0) = 0$ then the null solution is asymptotically stable.

Proof. The self-adjointness of $A_o$ implies that $A_o$ is a densely defined closed operator and $\mathcal{D}(A_o^*) = \mathcal{D}(A_o)$. By the Lipschitz continuity of f, f is continuous in the strong topology and is bounded on every bounded subset of H. This assumption (Lipschitz continuity) also implies that

$$(f(x)-f(y),x-y) \leqq ||f(x)-f(y)||\ ||x-y|| \leqq k||x-y||^2 \quad \text{for all } x,y \in H.$$

Hence, all the conditions in theorem V-15 are satisfied, and the result follows by applying that theorem.

Remark. The Lipschitz continuity of f in the theorem can be weakened by using the conditions (iii) and (iv) in theorem V-15.

In section B, it has been shown that stability and asymptotic stability are invariant if the inner product (·,·) is replaced by an equivalent inner product $(\cdot,\cdot)_1$ with respect to which A is dissipative. In the special case of $A=A_o+f(\cdot)$, where $A_o$ and $f(\cdot)$ are defined as in (V-24), theorem V-11 (also theorem V-12) remains valid if $A_o$ is the infinitesimal generator of a contraction (negative contraction) semigroup of class $C_o$ in the Hilbert space $(H,(\cdot,\cdot)_1)$ and the inner product $(\cdot,\cdot)$ in condition (ii) is replaced by $(\cdot,\cdot)_1$ (in theorem V-12, $(\cdot,\cdot)$ and $||\cdot||$ in (ii) should be replaced by $(\cdot,\cdot)_1$ and $||\cdot||_1$ respectively). Because of its usefulness in applications (for instance, a non-self-adjoint operator in a Hilbert space $(H,(\cdot,\cdot))$ can sometimes be made self-adjoint in $(H,(\cdot,\cdot)_1)$ where $(\cdot,\cdot)_1$ is an equivalent inner product) we show one theorem, which is an extension of theorem V-16, as an illustration.

Theorem V-17. Let $A_o$ be a densely defined linear operator from $H=(H,(\cdot,\cdot))$ into H, and let f be defined from all of H into H such that it is continuous from the strong topology to the weak topology of H and is bounded on every bounded subset of H. If there exists an equivalent inner product $(\cdot,\cdot)_1$ such that $A_o$ is a self-adjoint operator in $H_1 = (H,(\cdot,\cdot)_1)$ satisfying

$$(A_o x,x)_1 \leqq -\beta||x||_1^2 \qquad x \in \mathcal{D}(A_o)$$

and if

$$(f(x)-f(y),x-y)_1 \leqq k||x-y||_1^2 \qquad \text{with } k < \beta, \qquad x,y \in H.$$

Then, all the results stated in theorem V-15 are valid.

Proof. Consider $A_o$ as an operator from the space $H_1 = (H,(\cdot,\cdot)_1)$ into $H_1$. Since $A_o$ is self-adjoint in the space $H_1$, it is a densely defined closed operator and $\mathcal{D}(A_o) = \mathcal{D}(A_o^*)$. The continuity and the boundedness of f with respect to the $||\cdot||$-norm topology implies the same property of f with respect to the $||\cdot||_1$-norm topology since these two norms are equivalent. By assumption, $A_o$ is strictly dissipative and the condition (iv) in theorem V-15 is satisfied with respect to $(\cdot,\cdot)_1$. Hence all the hypothesis in theorem V-15 are satisfied by considering $H_1$ as the underlying space which implies that the operator $A=A_o+f(\cdot)$ is the infinitesimal generator of a nonlinear negative contraction semi-group $\{T_t;\ t \geqq 0\}$ on $\mathcal{D}(A_o)$ with contractive constant $\beta-k$ in the space $H_1$. By lemma V-10, A is the infinitesimal generator of a nonlinear negative semi-group $\{T_t;\ t \geqq 0\}$ on $\mathcal{D}(A_o)$ in the original space H. Therefore all the results in theorem V-15 hold good in this case (The proof is the same as in the proof of theorem V-9).

# VI. STABILITY THEORY OF NONLINEAR TIME-VARYING DIFFERENTIAL EQUATIONS IN HILBERT SPACES

A large class of physical problems are described by a system of nonlinear partial differential equations which can be reduced to the form (V-1) but with either time-dependent coefficients of the partial differential operator or time-dependent boundary conditions. In a more general case both the coefficients of the differential operator and the boundary conditions are time-varying. In order to investigate this type of differential equation in the abstract setting, it is necessary to extend the operator A in the previous chapter to a more general type of operator A(t) which depends on the variable t. The object in this chapter is to extend the principle result in Chapter V for the case of nonlinear time-varying operational differential equations of the form

$$\frac{dx(t)}{dt} = A(t)x(t) \qquad (t \geqq 0) \qquad \text{(VI-1)}$$

where the unknown vector x(t) is a vector-valued function defined on $R^+ = [0, \infty)$ to a Hilbert space H and A(t) is, for each $t \geqq 0$, a given nonlinear operator with domain $\mathcal{D}(A(t))$ and range $\mathcal{R}(A(t))$ both contained in H. In the first section, we give a formal definition of a solution and state the main results from [11]. In section B, we present some results on the general operational differential equations of the form (VI-1), and in section C we consider, as a special case of (VI-1), operational differential equations of the form

$$\frac{dx(t)}{dt} = Ax(t) + f(t,x(t)) \qquad (t \geqq 0) \qquad \text{(VI-2)}$$

where A is a nonlinear operator as in Chapter V and f is a given function from $R^+ \times H$ into H. It is seen that equation (VI-2) is a direct extension of equation (V-1). In section D, we first discuss briefly another special case of (VI-1), the equations of the form

$$\frac{dx(t)}{dt} = A_o(t)x(t) + f(t,x(t)) \qquad (t \geqq 0) \qquad \text{(VI-3)}$$

where $A_o(t)$ is, for each $t \geqq 0$, a linear operator with domain $\mathcal{D}(A_o(t))$ and range $\mathcal{R}(A_o(t))$ both contained in a Hilbert space H and f is a given function from $R^+ \times H$ into H. The object of this section is to deduce a number of theorems from the results obtained in section C on a special form of (VI-3) where $A_o(t) = A_o$ which is independent of t. We discuss in more detail this type of equation which is a direct extension of equation (V-24) with $f(t,x(t)) = f(x(t))$. Finally, a few results on the ordinary differential equations of the form

$$\frac{dx(t)}{dt} = f(t,x(t)) \qquad (t \geqq 0) \qquad \text{(VI-4)}$$

with the same f as in (VI-3) are included in this section since it is a special form of (VI-3) with $A_o(t) \equiv 0$.

## A. Background

As in the case of Chapter V, the stability theory developed in this chapter is again based on the recent paper by Kato [11] in which the existence and uniqueness of a solution to (VI-1) are established. In order to state the results in [11], we give a formal definition of a solution of (VI-1) and according to some additional properties of the solutions, different terminology is used as given in the following:

Definition VI-1. By a solution x(t) of (VI-1) with initial condition $x(0) = x \in \mathcal{D}(A(0))$ in a Hilbert space H (real or complex), we mean the following:

(a) x(t) is uniformly Lipschitz continuous in t for each $t \geqq 0$ with $x(0) = x$.

(b) $x(t) \in \mathcal{D}(A(t))$ for eac.. $- \geq 0$ and $A(t)x(t)$ is weakly continuous in t.

(c) The weak derivative of $x(t)$ exists for all $t \geq 0$ and equals $A(t)x(t)$.

(d) The strong derivative of $dx(t)/dt = A(t)x(t)$ exists and is strongly continuous except at a countable number of values t.

(e) For any $x(t)$, $y(t)$ satisfying (a)-(c) with $x(0) = x$, $y(0) = y$ both in $\mathcal{D}(A(0))$, there exists a positive constant M such that

$$||x(t)-y(t)|| \leq M||x-y|| \qquad \text{for all } t \geq 0.$$

The above definition of a solution $x(t)$ is in the sense of a "weak solution" since $x(t)$ satisfies (VI-1) in the weak topology of H. However, by the condition (d), $x(t)$ is an almost everywhere strong solution in the sense that $x(t)$ satisfies (VI-1) for almost all values of $t \in R^+$ in the strong topology of H.

Definition VI-2. Let $x(t)$ be a solution of (VI-1) with $x(0)=x$ (in the sense of definition VI-1). If $M \leq 1$, where M is the positive constant appearing in (e), then $x(t)$ is called a contraction solution; if M is replaced by $Me^{-\beta t}$ or by $e^{-\beta t}$ for some $\beta > 0$, then $x(t)$ is called a negative solution and a negative contraction solution respectively.

It follows from the condition (e) that the solution $x(t)$ of (VI-1) with $x(0) = x \in \mathcal{D}(A(0))$ is unique, and if $y(t) \equiv x_e$ is an equilibrium solution of (VI-1) then the condition (e) implies that $x_e$ is stable.

On setting $x(t) = T_t x$ for any $x \in \mathcal{D}(A(0))$ where $x(t)$ is the contraction solution of (VI-1) with $x(0) = x$, it can easily be shown that the family $\{T_t;\ t \geq 0\}$ forms a nonlinear contraction semi-group on $\mathcal{D}(A(0))$. However, in this chapter, we do not follow the semi-group property as in Chapter V, but rather use directly the properties (a)-(e) of a solution given in definition VI-1. Yet, if we set $x(t) = T_t x$, then by lemma V-3 $\{T_t;\ t \geqq 0\}$ can be extended to the closure of $\mathcal{D}(A(0))$ which implies that the existence of a contraction solution can be extended for any initial element $x \in \overline{\mathcal{D}(A(0))}$. Hence we can state the following:

<u>Lemma VI-1</u>. If for any $x \in \mathcal{D}(A(0))$ there exists a contraction (negative contraction) solution $x(t)$ of (VI-1) with $x(0) = x$, then for any $x \in \overline{\mathcal{D}(A(0))}$, we can define a "solution" $x(t)$ of (VI-1) with $x(0) = x$ by

$$x(t) = \lim_{n\to\infty} x_n(t)$$

where $x_n(0) = x_n \in \mathcal{D}(A(0))$ for each n and $x_n \to x$ as $n \to \infty$. The "solution" $x(t)$ is also a contraction solution (negative contraction solution).

It has been shown in the proof of lemma V-3 that the limit defined above exists and is independent of the choice of any sequence $\{x_n\}$ (in $\mathcal{D}(A(0))$) which converges to x. Moreover, $x(t) \in \overline{\mathcal{D}(A(0))}$ for all $t \geqq 0$ and the condition (e) in definition VI-1 with M=1 (with M replaced by $e^{-\beta t}$ for a negative contraction solution), is satisfied for any "solution" $y(t)$ with $y(0) = y \in \overline{\mathcal{D}(A(0))}$.

For convenience, we introduce the following basic assumptions on the operator $A(t)$ and refer to them thereafter as the condition I or the conditions I, II etc. to mean that $A(t)$ satisfies the respective assumptions.

I. The domain $\mathcal{D}$ of $A(t)$ is independent of t.

II. For each $t \geqq 0$, there is a real number $\alpha(t) > 0$ such that $\mathcal{R}(I - \alpha(t)A(t)) = H$.

III. There exists a positive, nondecreasing function $L(r)$ of $r > 0$ such that for all $x \in \mathcal{D}$ and any $s,t \geqq 0$

$$||A(t)x-A(s)x|| \leqq L(||x||)\ |t-s|\ (1 + ||A(s)x||)$$

where the norm $||\cdot||$ is induced by the inner product $(\cdot,\cdot)$ of the Hilbert space $H=(H,(\cdot,\cdot))$.

In the development of the stability and the asymptotic stability properties of the solutions to (VI-1), we have used some of the results obtained in [11]. Because of their importance in the development of our stability theory, we state the main results from [11] as the following theorem where we take a Hilbert space as the underlying space.

Theorem VI-1. Let the nonlinear operator $A(t)$ appearing in (VI-1) satisfies the conditions I, II, III. Assume that for each $t \geqq 0$, $A(t)$ is dissipative (i.e. $-A(t)$ is monotone). Then for any $x \in \mathcal{D}$, there exists a unique contraction solution $x(t)$ (in the sense of definition VI-1) with $x(0) = x$.

It follows from definition V-4 that for each $t \geqq 0$, the dissipativity of $A(t)$ and the condition II imply that $-A(t)$ is m-monotone which is one of the hypotheses in the main theorems of [11]. It is to be noted that if the initial time is not at $t=0$ but at $t=t_o > 0$, then the result of the above theorem remains valid in the sense that for any $x \in \mathcal{D}(A(t_o)) = \mathcal{D}$ there exists a unique contraction solution starting at $x(t_o) = x$. Here definitions VI-1 and VI-2 of a contraction solution should be modified by replacing 0 by $t_o$ whenever it appears; and in the case of a negative solution or a negative contraction solution, $Me^{-\beta t}$ or $e^{-\beta t}$ should be replaced by $Me^{-\beta(t-t_o)}$ and $e^{-\beta(t-t_o)}$ respectively.

B. Stability Theory of General Nonlinear Equations

The contraction property of the solution of (VI-1), obtained in theorem VI-1 implies that any equilibrium solution $x_e$, if it exists, is stable. However, in many physical and engineering problems, it is important to know the asymptotic behavior of solutions of the differential equations describing these systems. In order to extend theorem VI-1 to show the asymptotic stability of solutions to (VI-1), we first show the following:

Lemma VI-2. For any pair of strongly continuous and weakly differentiable functions $x(t)$, $y(t)$ which satisfy (VI-1) in the weak sense, then the real-valued function $||x(t)-y(t)||^2$ is differentiable in t for each $t \geq 0$ and is given by

$$\frac{d}{dt}||x(t)-y(t)||^2 = 2\mathrm{Re}(A(t)x(t)-A(t)y(t),\ x(t)-y(t)) \qquad \text{(VI-5)}$$

where $d/dt\ ||x(t)-y(t)||^2$ at $t=0$ is defined as the right-side derivative.

Proof. For any fixed $t > 0$, let $h \neq 0$ be such that $|h| < t$. Then $t+h > 0$ so that $x(t+h)$ and $y(t+h)$ are defined. Following the same proof as for lemma V-5, we have

$$h^{-1}[||x(t+h)-y(t+h)||^2-||x(t)-y(t)||^2]=h^{-1}[(x(t+h)-x(t),x(t+h)-y(t+h)) - (y(t+h)-y(t),x(t+h)-y(t+h))+(x(t)-y(t),x(t+h)-x(t))-(x(t)-y(t), y(t+h)-y(t))].$$

By hypothesis $h^{-1}(x(t+h)-x(t)) \overset{w}{\to} A(t)x(t)$ and $x(t+h)-y(t+h) \to x(t)-y(t)$ as $h \to 0$ (Similarly, $h^{-1}(y(t+h)-y(t)) \overset{w}{\to} A(t)y(t)$), we have on applying lemma V-4 as $h \to 0$

$$\frac{d}{dt}||x(t)-y(t)||^2=(A(t)x(t),x(t)-y(t))-(A(t)y(t),x(t)-y(t)) + +(x(t)-y(t),A(t)x(t)) - (x(t)-y(t), A(t)y(t))=(A(t)x(t)- -A(t)y(t),x(t)-y(t))+(x(t)-y(t), A(t)x(t)-A(t)y(t)) = = 2\mathrm{Re}(A(t)x(t)-A(t)y(t),x(t)-y(t))$$

which shows that $||x(t)-y(t)||^2$ is differentiable and satisfies (VI-5) for $t > 0$. For $t=0$, (VI-5) is still valid by taking $h > 0$ and $h \downarrow 0$ in place of $h \to 0$, where we define $d/dt\ ||x(0)-y(0)||^2$ as the right-side derivative.

Theorem VI-2. Assume that the nonlinear operator $A(t)$ appearing in (VI-1) satisfies the conditions I, II, III and that there exists a positive real-valued continuous function $\beta(t)$ defined on $R^+$ such that for each $t \geq 0$, $A(t)$ is strictly dissipative with dissipative constant $\beta(t)$, i.e.,

$$\mathrm{Re}(A(t)x-A(t)y,x-y) \leq -\beta(t)(x-y,x-y) \quad \text{for all } x,y \in \mathcal{D}.$$

Then for any $x \in \mathcal{D}$, there exists a unique contraction solution $x(t)$ of (VI-1) with $x(0)=x$, and for any solution $y(t)$ with $y(0) = y \in \mathcal{D}$

$$||x(t)-y(t)|| \leq e^{-\int_0^t \beta(s)ds}||x-y|| \quad \text{for all } t \geq 0. \tag{VI-6}$$

In particular, if $\beta(t) = \beta$ which is independent of $t$ then $x(t)$ is a negative contraction solution.

Proof. For each fixed $t \geq 0$, the strict dissipativity of $A(t)$ implies the dissipativity of $A(t)$ (see definition V-3) and thus the existence and the uniqueness of the solution $x(t)$ with $x(0)=x \in \mathcal{D}$ follows from theorem VI-1. To show the inequality (VI-6), let $y(t)$ be any solution of (VI-1) with $y(0)=y \in \mathcal{D}$. Since by definition VI-1 $x(t)$ and $y(t)$ are strongly continuous, weakly differentiable and satisfy (VI-1), it follows by lemma VI-2 and by the strict dissipativity of $A(t)$ that

$$\frac{d}{dt}||x(t)-y(t)||^2 = 2\mathrm{Re}(A(t)x(t)-A(t)y(t),x(t)-y(t)) \leq$$

$$\leq -2\beta(t)||x(t)-y(t)||^2$$

for each $t \geq 0$. Note that the function $||x(t)-y(t)||^2$ is a positive real-valued function defined on $R^+ = [0, \infty)$. Writing the above inequality in the form

$$d(||x(t)-y(t)||^2)/(||x(t)-y(t)||^2 \leq -2\beta(t)dt$$

and integrating on both sides, we have

$$||x(t)-y(t)||^2 \leqq ||x(0)-y(0)||^2 \, e^{-2\int_0^t \beta(s)ds}$$

which is equivalent to

$$||x(t)-y(t)|| \leqq e^{-\int_0^t \beta(s)ds} ||x-y|| \qquad \text{for all } t \geqq 0.$$

In particular, if $\beta(t) = \beta$ then

$$||x(t)-y(t)|| \leqq e^{-\beta t} ||x-y|| \qquad \text{for all } t \geqq 0$$

and thus $x(t)$ is a negative contraction solution. Hence the theorem is proved.

Lemma VI-3. Let $H_1=(H,(\cdot,\cdot)_1)$ be an equivalent Hilbert space of the space $H=(H,(\cdot,\cdot))$. For any $x \in \mathcal{D}$, let $x(t)$ be the solution of (VI-1) with $x(0)=x$ in the equivalent space H. (i.e., the underlying space in definition VI-1 is $H_1$). Then $x(t)$ is also the solution of (VI-1) with $x(0)=x$ in the original space H.

Proof. The equivalence relation between $(\cdot,\cdot)$ and $(\cdot,\cdot)_1$ implies that there exist constants $\delta$, $\gamma$ with $0 < \delta \leqq \gamma < \infty$ such that

$$\delta||x|| \leqq ||x||_1 \leqq \gamma \, ||x|| \qquad \text{for all } x \in H \qquad \text{(VI-7)}$$

where $||\cdot|| = (\cdot,\cdot)^{1/2}$ and $||\cdot||_1 = (\cdot,\cdot)_1^{1/2}$. By hypothesis, $x(t)$ satisfies the conditions (a)-(e) of definition VI-1 in the $H_1$-space, we shall show that the same is true for $x(t)$ in the H-space. The conditions (a) and (d) are obviously satisfied with $x(t)$ in the H-space, for strong continuity in the norm topology is invariant under equivalent norms. By the relation (VI-7), the condition (e) is satisfied for some $N > 0$ since

$$||x(t)-y(t)|| \leqq \delta^{-1}||x(t)-y(t)||_1 \leqq \delta^{-1}M||x-y||_1 \leqq \gamma/\delta \; M||x-y|| \qquad \text{(VI-8)}$$

where $N = \gamma/\delta \; M$. To show that the conditions (b) and (c) are satisfied in H, define $V(x,y) = (x,y)$. Then $V(x,y)$ is a sesquilinear functional defined on the product space $H_1 \times H_1$ and satisfies the following conditions:

(i) Sesquilinearity: $V(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 V(x_1, y) + \alpha_2 V(x_2, y)$

$(x_1, x_2, y \in H_1)$

$V(x, \beta_1 y_1 + \beta_2 y_2) = \bar{\beta}_1 V(x, y_1) + \bar{\beta}_2 V(x, y_2)$

$(x, y_1, y_2 \in H_1)$

which follows from the definition of inner product defined on a complex vector space.

(ii) Boundedness: $|V(x,y)| = |(x,y)| \leq ||x||\ ||y|| \leq \delta^{-2}||x||_1||y||_1$

(iii) Positivity: $V(x,x) = (x,x) = ||x||^2 \geq \gamma^{-2}||x||_1^2.$

Hence by the Lax-Miligram theorem (III-7), there exists a bounded linear operator S with a bounded inverse $S^{-1}$ defined on all of $H_1$ such that

$$(x,y) = V(x,y) = (x, Sy)_1 \qquad \text{for all } x,y \in H. \qquad \text{(VI-9)}$$

Thus for each fixed $t > 0$, the relation (VI-9) and the weak differentiability of $x(t)$ with its derivative equals $A(t)x(t)$ in $H_1$ imply that

$$\lim_{h\to 0} h^{-1}(x(t+h)-x(t), z) = \lim_{h\to 0} h^{-1}(x(t+h)-x(t), Sz)_1 =$$

$$= (A(t)x(t), Sz)_1 = (A(t)x(t), z) \qquad \text{for every } z \in H \qquad \text{(VI-10)}$$

which shows that $x(t)$ is weakly differentiable for $t > 0$ and equals $A(t)x(t)$. For $t=0$, we take $h > 0$ with $h \downarrow 0$ in place of $h \to 0$ so that (VI-10) is valid by defining the weak derivative of $x(0)$ as the right side weak derivative. This proves condition (c) in the H-space. The condition (b) in the space H follows from (VI-9) and the weak continuity of $A(t)x(t)$ in $H_1$ since for each $t \geq 0$

$$\lim_{h\to 0}(A(t+h)x(t+h), z) = \lim_{h\to 0} (A(t+h)x(t+h), Sz)_1 = (A(t)x(t), Sz)_1 = (A(t)x(t), z)$$

for every $z \in H$

where for $t=0$ the limit in the above relation is taken as the right-side limit. Therefore, all the conditions of definition VI-1 are satisfied in the space H and thus the lemma is proved.

It should be noted that if the solution $x(t)$ of (VI-1) is contractive in $H_1$, it is not necessarily contractive in the space $H$ since the constant $N = \gamma/\delta\, M$ in the relation (VI-8) is, in general, not less than 1 even though $M \leq 1$.

Theorem VI-3. Let $(H, (\cdot,\cdot))$ be a Hilbert space and assume that the conditions I, II, III are satisfied in $H$. If there exists an equivalent inner product $(\cdot,\cdot)_1$ with respect to which $A(t)$ is dissipative for each $t \geq 0$, then for any $x \in \mathcal{D}$ there exists a unique solution $x(t)$ of (VI-1) in the space $(H, (\cdot,\cdot))$ with $x(0)=x$.

Proof. Consider $A(t)$ as an operator with domain $\mathcal{D}$ and range $\mathcal{R}(A(t))$ both contained in the equivalent Hilbert space $H_1=(H,(\cdot,\cdot)_1)$, we shall show that conditions I, II, III are satisfied with $H_1$ as the underlying space. The conditions I, II remain valid in $H_1$. To show that the condition III is satisfied with respect to $||\cdot||_1$, note that $L(||x_1||) \leq L(||x_2||)$ if $||x_1|| \leq ||x_2||$ since $L$ is nondecreasing. By hypothesis the condition III holds with respect to $||\cdot||$, we have on using the relation (VI-7)

$$||A(t)x-A(s)x||_1 \leq \gamma||A(t)x-A(s)x|| \leq \gamma L(||x||)\ |t-s|(1+||A(s)x||) \leq$$

$$\leq \gamma\ L(\delta^{-1}||x||_1)\ |t-s|(1+\delta^{-1}||A(s)x||_1) \leq \gamma\ \lambda\ L(\delta^{-1}||x||_1)\ |t-s|(1+||A(s)x||_1)$$

where $\lambda = \max(1,\delta^{-1})$. Let $L_1(||x||_1)=\gamma\lambda L(\delta^{-1}||x||_1)$, then $L_1(r)$ as a function of $r > 0$ is positive since $L(r)$ is; it is also nondecreasing, for given any pair of positive numbers $r_1, r_2$ with $r_1 < r_2$ which is equivalent to $\delta^{-1}r_1 < \delta^{-1}r_2$, then $L(\delta^{-1}r_1) \leq L(\delta^{-1}r_2)$ which shows that $L_1(||x||_1)$, is non-decreasing. Hence on replacing $L(||x||)$ by $L_1(||x||_1)$, the condition III is satisfied with respect to $||\cdot||_1$. By hypothesis $A(t)$ is dissipative with respect to $(\cdot,\cdot)_1$, it follows by theorem VI-1 that for any $x \in \mathcal{D}$ there exists a unique contraction solution $x(t)$ in $H_1$ with $x(0) = x$. Therefore by lemma VI-3, x(t) is also the solution of (VI-1) in the space H with x(0)=x (in general, x(t) is not contractive). Thus the theorem is proved.

Following the same proof of the above theorem and applying theorem VI-2, we can prove the following theorem for the existence of a negative solution.

Theorem VI-4. Let $H=(H,(\cdot,\cdot))$ be a Hilbert space and assume that the conditions I, II, III are satisfied in H. If there exists an equivalent inner product $(\cdot,\cdot)_1$ with respect to which A(t) is strictly dissipative with dissiaptive constant $\beta(t)$ for each $t \geq 0$ where $\beta(t)$ is a positive continuous function defined on $R^+$, then for any $x \in \mathcal{D}$ there exists a unique solution x(t) of (VI-1) in H with x(0) = x, and for any solution y(t) with $y(0)=y \in \mathcal{D}$ there is a finite number $M \geq 1$ such that

$$||x(t)-y(t)|| \leq Me^{-\int_0^t \beta(s)ds}||x-y|| \quad \text{for all } t \geq 0. \qquad \text{(VI-11)}$$

In particular, if $\beta(t) = \beta$ which is independent of t, x(t) is a negative solution.

Proof. Since all the hypotheses of theorem VI-3 are fulfilled, the existence of a unique solution follows. To show that the solution is negative, let x(t), y(t) be any two solutions with x(0)=x, y(0)=y both contained in $\mathcal{D}$. From the proof of theorem VI-3, A(t) satisfies the conditions I, II, III in $H_1$, and by hypothesis A(t) is strictly dissipative with dissipative constant $\beta(t)$ with respect to $(\cdot,\cdot)_1$. Hence by applying theorem VI-2

$$||x(t)-y(t)||_1 \leq e^{-\int_0^t \beta(s)ds}||x-y||_1 \qquad (t \geq 0).$$

It follows by the equivalence relation (VI-7) that

$$||x(t)-y(t)|| \leq \delta^{-1}||x(t)-y(t)||_1 \leq \delta^{-1}e^{-\int_0^t \beta(s)ds}||x-y||_1 \leq$$

$$\leq (\gamma/\delta)e^{-\int_0^t \beta(s)ds}||x-y||=Me^{-\int_0^t \beta(s)ds}||x-y|| \qquad (t \geq 0)$$

where $M=\gamma/\delta \geq 1$. If $\beta(t) = \beta$ which is independent of t, then

$$||x(t)-y(t)|| \leq Me^{-\beta t}||x-y|| \qquad \text{for all } t \geq 0$$

which shows that the solution is negative. This completes the proof.

An immediate consequence of the relation (VI-11) is that under the hypotheses of theorem VI-4, and if $\inf_{t\geq 0} \beta(t) > 0$, then an equilibrium solution $x_e$ (or a periodic solution) of (VI-1), if it exists, is asymptotically stable since $\int_0^t \beta(s)ds \to \infty$ as $t \to \infty$. In particular, if $\beta(t) = \beta$ then the equilibrium solution $x_e$ is exponentially asymptotically stable.

By an equilibrium solution $x_e$ of (VI-1), we mean the same thing as in definition V-5 except with the words "$x_e$ in $\mathcal{D}(A)$" replaced by "$x_e$ in $\mathcal{D}(A(t))$ for all $t \geq 0$". It can easily be shown that (see the proof following definition V-5) the existence of an equilibrium solution is equivalent to the existence of a solution to (VI-1) satisfying

$$A(t)x(t) = 0 \qquad \text{for all } t \geq 0 . \tag{VI-12}$$

Theorem VI-5. Assume that the conditions I, II, III are satisfied. If there exists a Lyapunov functional $v(x) = V(x,x)$ such that for each $t \geq 0$

$$\mathrm{Re}V(A(t)x-A(t)y,\ x-y) \leq 0 \qquad \text{for any } x,y \in \mathcal{D} \tag{VI-13}$$

where $V(x,y)$ is a defining sesquilinear functional defined on $H \times H$. Then:

(a) For any $x \in \mathcal{D}$, there exists a unique solution $x(t)$ of (VI-1) with $x(0)=x$;

(b) An equilibrium solution $x_e$ (or a periodic solution), if it exists, is stable;

(c) The stability region of $x_e$ is $\mathcal{D}$ which can be extended to $\bar{\mathcal{D}}$, the closure of $\mathcal{D}$, in the sense of lemma VI-1.

If the relation (VI-13) is replaced by

$$\mathrm{Re}\, V(A(t)x-A(t)y, x-y) \leq -\beta(t)||x-y||^2 \qquad \text{for any } x,y \in \mathcal{D} \quad \text{(VI-13)'}$$

where $\beta(t)$ is a positive continuous function on $R^+$ with $\inf_{t \geq 0} \beta(t) > 0$, then (b) can be replaced by:

(b)' An equilibrium solution $x_e$ (or a periodic solution), if it exists, is asymptotically stable.

<u>Proof.</u> Since $V(x,y)$ is a defining sesquilinear functional defined on $H \times H$, it follows by lemma V-8 that

$$(x,y)_1 = V(x,y) \qquad x, y \in H$$

defines an inner product $(\cdot,\cdot)_1$ which is equivalent to $(\cdot,\cdot)$. By the assumption (VI-13), for each $t \geq 0$

$$\mathrm{Re}\,(A(t)x-A(t)y, x-y)_1 = \mathrm{Re}V(A(t)x-A(t)y, x-y) \leq 0 \qquad x,y \in \mathcal{D}$$

which shows that $A(t)$ is dissipative with respect to $(\cdot,\cdot)_1$ for each $t \geq 0$. Hence, by applying theorem VI-3, for any $x \in \mathcal{D}$ there exists a unique solution $x(t)$ of (VI-1) in the original space $H$ with $x(0)=x$. By definition VI-1, for any solution $y(t)$ with $y(0)=y \in \mathcal{D}$

$$||x(t)-y(t)|| \leq M||x-y|| \qquad \text{for all } t \geq 0. \qquad \text{(VI-14)}$$

It follows by taking $y=x_e$ (if it exists) in the above inequality and noting that $y(t) \equiv x_e$

$$||x(t)-x_e|| \leq M||x-x_e|| \qquad \text{for all } t \geq 0 \qquad \text{(VI-14)'}$$

which shows that the equilibrium solution $x_e$ is stable. Since (VI-14) holds for any solution $x(t)$ with $x(0)=x \in \mathcal{D}$, the stability region is thus the whole domain $\mathcal{D}$. The extension of $\mathcal{D}$ into its closure $\bar{\mathcal{D}}$ follows from lemma VI-1. In case (VI-13) is replaced by (VI-13)', then

$$\mathrm{Re}(A(t)x-A(t)y, x-y)_1 \leq -\beta(t)||x-y||^2 \leq -\beta(t)/\gamma\ ||x-y||_1^2 \qquad (x,y \in \mathcal{D})$$

and so for each $t \geq 0$, $A(t)$ is strictly dissipative with dissipative constant $\beta(t)/\gamma$ with respect to $(\cdot,\cdot)_1$. Thus by applying theorem VI-4, for any $x \in \mathcal{D}$ there exists a unique solution $x(t)$ in the space $(H,(\cdot,\cdot))$ with $x(0)=x$. If an equilibrium solution $x_e$ exists, then by the relation (VI-11)

$$||x(t)-x_e|| \leq Me^{-\gamma^{-1}\int_0^t \beta(s)ds}||x-x_e|| \qquad \text{for all } t \geq 0.$$

Therefore the equilibrium solution $x_e$ is asymptotically stable since $\inf_{t\geq 0} \beta(t) > 0$ implies $\lim_{t\to\infty} \int_0^t \beta(s)ds = \infty$.

Corollary 1. Assume that the conditions I, II, III are satisfied and that (VI-13) is valid. Then for any two solutions $x(t)$ and $y(t)$ of (VI-1) with $x(0)=x$, $y(0)=y$ both in $\mathcal{D}$

$$\mathring{v}(x(t)-y(t)) \leq 0 \qquad \text{for all } t \geq 0.$$

If (VI-13)' is satisfied, then

$$\mathring{v}(x(t)-y(t)) \leq -2\beta(t)||x(t)-y(t)||^2 \qquad \text{for all } t \geq 0.$$

Proof. It can easily be shown by following the proof of lemma V-7 that for any two solutions $x(t),y(t)$

$$\mathring{v}(x(t)-y(t))=2\text{Re}V(A(t)x(t)-A(t)y(t),x(t)-y(t)).$$

The results follow directly from (VI-13) and (VI-13)' since $x(t),y(t) \in \mathcal{D}$ for all $t \geq 0$.

A direct consequence of theorem VI-5 is the following:

Corollary 2. Under the assumptions of theorem VI-5, and in addition if $0 \in \mathcal{D}$ and $A(t)\cdot 0=0$. Then the null solution is stable under the condition of (VI-13) and is asymptotically stable under the condition of (VI-13)'.

C. Nonlinear Nonstationary Equations

Based on the theorems developed in the previous section, we shall develop some results on the nonstationary differential equations of the form

$$\frac{dx(t)}{dt} = Ax(t) + f(t,x(t)) \qquad \text{(VI-15)}$$

where A, which is independent of t, is a nonlinear operator with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ both contained in a real Hilbert space H and f is a given (nonlinear) function on $R^+ \times H$ into H. On setting $A(t)=A+f(t,\cdot)$, the equation of the form (VI-15) becomes a special case of the general nonlinear equation (VI-1) and thus the results in section B can be applied to this type of equation. On the other hand, equations of the form (VI-15) are direct extensions of the nonlinear differential equations of the form (V-1) where f can be regarded as identically equal to zero. The purpose of this section is to modify the basic assumptions I, II, III of section A so that the existence, the uniqueness, the stability and the asymptotic stability of a solution can be investigated. For the sake of convenience in the statements of our results in this and in the remaining sections of this chapter, we state some basic assumptions on the function f. These basic assumptions are:

(i) f is defined on $R^+ \times H$ into H and for each $t \geq 0$, f is continuous from the strong topology to the weak topology of H and is bounded on every bounded subset of H;

(ii) For each $t \geq 0$,

$$(f(t,x)-f(t,y),\ x-y) \leq 0 \qquad \text{for all } x,y \in H;$$

(ii)' There exists a continuous real-valued function k(t) on $R^+$ such that $\sup_{t\geq 0} k(t) < \beta$ where $\beta$ is the dissipative constant of A, and such that for each $t \geq 0$

$$(f(t,x)-f(t,y),x-y) \leqq k(t)||x-y||^2 \qquad \text{for all } x,y \in H;$$

(iii) There exists a positive nondecreasing function $L(r)$ of $r > 0$ such that for all $x \in \mathcal{D}$ and any $s,t \geqq 0$

$$||f(t,x)-f(s,x)|| \leqq L(||x||)|t-s|(1+||Ax+f(s,x)||).$$

Theorem VI-6. Let the operator $A$ of (VI-15) be densely defined, dissipative and $\mathcal{R}(I-A)=H$. Assume that $f$ satisfies the conditions (i), (ii), (iii). Then

(a) For any $x \in \mathcal{D}(A)$, there exists a unique contraction solution of (VI-15) with $x(0)=x$;

(b) An equilibrium solution $x_e$ (or a periodic solution), if it exists, is stable;

(c) A stability region of the equilibrium solution $x_e$ is $\mathcal{D}(A)$ which can be extended to the whole space $H$.

Proof. Let $A(t)=A+f(t,\cdot)$. We shall show that $A(t)$ satisfies all the conditions in theorem VI-1. Since $A$ is independent of $t$ and $f$ is defined on all of $t \in R^+$, it follows that $\mathcal{D}(A(t))=\mathcal{D}(A)$ which is independent of $t$ and thus the condition I is satisfied. By the condition (iii), for each $x \in \mathcal{D}(A)$

$$||A(t)x-A(s)x||=||f(t,x)-f(s,x)|| \leqq L(||x||)|t-s|(1+||Ax+f(s,x)||)$$

which shows that the condition III is satisfied. To show the condition II, we shall apply theorem V-10 as in the proof of theorem V-11. Let $T=-A$ and for each $t \geqq 0$ let $T_t=I-f(t,\cdot)$. Then both $T$ and $T_t$ are monotone since the dissipativity of $A$ implies the monotonicity of $T$ and by the condition (ii), for any $x,y \in H$

$$(T_tx-T_ty,x-y) = (x-y,x-y) - (f(t,x)-f(t,y),x-y) \geqq ||x-y||^2$$

which implies that $T_t$ is monotone. By hypothesis, $\mathcal{R}(I+T)=\mathcal{R}(I-A)=H$ and $\mathcal{D}(T)=\mathcal{D}(A)$ is dense in $H$. For each $t \geqq 0$, $T_t$ is, by the condition (i), defined and demicontinuous (i.e., continuous from the strong topology to the weak topology of H) on H and is bounded on every bounded subset of H since the identity operator I also possesses this property. On setting y=0 in the condition (ii), we have

$$(f(t,x),x) \leqq (f(t,0),x) \leqq ||f(t,0)|| \, ||x||. \qquad \text{(VI-16)}$$

Hence the dissipativity of A and the relation (VI-16) imply that

$$||Tx+T_t x|| = ||-Ax+T_t x|| \geqq (-Ax+T_t x,x)/||x|| \geqq (T_t x,x)/||x|| =$$

$$= ((x,x) - (f(t,x),x))/||x|| \geqq ||x||-||f(t,0)||$$

which shows that

$$||Tx+T_t x|| \to +\infty \qquad \text{as } ||x|| \to +\infty .$$

Therefore, all the conditions in theorem V-10 are satisfied. It follows by applying that theorem that $\mathcal{R}(I-A(t))=\mathcal{R}(T+T_t)=H$ for each $t \geqq 0$ which shows condition II with $\alpha(t) \equiv 1$. Finally, the dissipativity of A and the condition (ii) imply that for each $t \geqq 0$

$$(A(t)x-A(t)y,x-y) = (Ax-Ay,x-y) + (f(t,x)-f(t,y),x-y) \leqq 0$$

for all $x,y \in \mathcal{D}(A)$. Thus A(t) is dissipative for each $t \geqq 0$ and so all the conditions in theorem VI-1 are satisfied. Hence for any $x \in \mathcal{D}(A)$ there exists a unique contraction solution of (VI-15) with x(0)=x. The contraction property of solutions of (VI-15) implies that if an equilibrium solution $x_e$ exists, then for any solution x(t) with $x(0)=x \in \mathcal{D}(A)$

$$||x(t)-x_e|| \leqq ||x-x_e|| \qquad \text{for all } t \geqq 0$$

which shows that the equilibrium solution is stable with a stability region $\mathcal{D}(A)$. Since $\mathcal{D}(A)$ is dense in H, the extension of the stability region to the whole space H follows from lemma VI-1. Hence the theorem is completely proved.

The above theorem has a counter part for the asymptotic stability of an unperturbed solution (e.g. equilibrium solution or periodic solution), we shall show this in the following.

Theorem VI-7. Let the operator A of (VI-15) be densely defined, strictly dissipative with dissipative constant $\beta$ and let $\mathcal{R}(I-A)=H$. Assume that f satisfies the conditions (i), (ii)', (iii). Then:

(a) For any $x \in \mathcal{D}(A)$ there exists a unique contraction solution of (VI-15) with $x(0)=x$ and for any solution $y(t)$ with $y(0)=y \in \mathcal{D}$

$$||x(t)-y(t)|| \leq e^{-\int_0^t(\beta-k(s)ds)}||x-y|| \qquad \text{for all } t \geq 0; \qquad \text{(VI-17)}$$

(b) An equilibrium solution $x_e$ (or a periodic solution), if it exists, is asymptotically stable;

(c) A stability region of the equilibrium solution $x_e$ is $\mathcal{D}(A)$ which can be extended to the whole space H.

Proof. Let $A(t)=A+f(t,\circ)$, we shall show that $A(t)$ satisfies all the conditions in theorem VI-2. As in the proof of theorem VI-6, the conditions I and III are satisfied. To show the condition II, note that the dissipativity of A and $\mathcal{R}(I-A)=H$ imply that $\mathcal{R}(I-\alpha A)=H$ for all $\alpha > 0$ (see lemma V-1). Let $T_t = I-\alpha(t)f(t,\circ)$. For each $t \geq 0$, choose a real number $\alpha(t)$ such that $0 < \alpha(t) \leq k(t)^{-1}$ (if $k(t) \leq 0$, choose, e.g., $\alpha(t)=1$) then $T_t$ is monotone, for by the condition (ii)'

$$(T_tx-T_ty,x-y)=(x-y,x-y)-\alpha(t)\ (f(t,x)-f(t,y),x-y) \geq (1-\alpha(t)k(t))||x-y||^2 \geq 0.$$

With $\alpha(t)$ so choosen for each $t \geq 0$, the operator $T=-\alpha(t)A$ is monotone with $\mathcal{R}(I+T)=\mathcal{R}(I-\alpha(t)A)=H$ and with $\overline{\mathcal{D}(T)} = \overline{\mathcal{D}(A)} = H$. By the condition (i), $T_t$ is defined and demicontinuous on all of H and is bounded on every bounded subset of H, and by the dissipativity of $\alpha(t)A$ and the relation (VI-16)

$$||Tx+T_tx||=||-\alpha(t)Ax+T_tx|| \geq (-\alpha(t)Ax+T_tx,x)/||x|| \geq (T_tx,x)/||x||$$
$$= (||x||^2-\alpha(t)\ (f(t,x),x))/||x|| \geq ||x||-\alpha(t)||f(t;0)||$$

which implies that $||Tx+T_t x|| \to +\infty$ as $||x|| \to +\infty$. It follows by applying theorem V-10 that for each $t \geqq 0$ we can choose an $\alpha(t) > 0$ such that $R(I-\alpha(t)A(t))=R(T+T_t)=H$ which shows the condition II. Moreover by the strict dissipativity of A and the condition (ii)', for any $x,y \in \mathcal{D}$

$$(A(t)x-A(t)y,x-y)=(Ax-Ay,x-y)+(f(t,x)-f(t,y),x-y) \leqq -(\beta-k(t))||x-y||^2$$

$$\text{for each } t \geqq 0$$

which shows that A(t) is strictly dissipative with dissipative constant $\beta-k(t)$ for each $t \geqq 0$. It follows by applying theorem VI-2 that (a) is proved and the relation (VI-6) holds with $\beta(s)$ replaced by $\beta-k(s)$. Hence if an equilibrium solution $x_e$ exists, then for any solution x(t) with $x(0)=$ $=x \in \mathcal{D}$

$$||x(t)-x_e|| \leqq e^{-\int_0^t(\beta-k(s)ds)}\, ||x-x_e|| \qquad \text{for all } t \geqq 0$$

which proves (b) since $\int_0^t(\beta-k(s))ds \geqq (\beta-\sup_{s\geqq 0}k(s))t$ for any $t \geqq 0$. Note that $\beta-\sup_{s\geqq 0} k(s) > 0$. It also proves that a stability region is $\mathcal{D}(A)$. The extension of $\mathcal{D}(A)$ into $\overline{\mathcal{D}(A)}=H$ follows from lemma VI-1 which completes the proof of part (c).

<u>Corollary</u>. Let the operator A of (VI-15) be densely defined, strictly dissipative with dissipative constant $\beta$ and let $R(I-A)=H$. Assume that f(t,x) is uniformly Lipschitz continuous in x with Lipschitz constant $k < \beta$, that is

$$||f(t,x)-f(t,y)|| \leqq k||x-y|| \qquad \text{for all } x,y \in H \qquad \text{(VI-18)}$$

and let there exist a positive nondecreasing function L(r) of $r > 0$ such that for all $x \in \mathcal{D}(A)$

$$||f(t,x)-f(s,x)|| \leqq L(||x||)\ |t-s| \qquad \text{for all } s,t \geqq 0.$$

Then the results (a), (b), (c) in theorem VI-7 are valid.

<u>Proof</u>. We shall show that $f(t,x)$ satisfies all the conditions (i), (ii)', (iii). For each $t \geq 0$, the condition (VI-18) implies that $f$ is continuous from the strong topology to the strong topology and that for any fixed $y_o \in H$

$$||f(t,x)|| \leq ||f(t,y_o)||+ k||x||+k||y_o|| \qquad \text{for all } x \in H$$

which is bounded whenever $||x||$ is bounded. Thus the condition (i) is satisfied. The condition (ii)' also follows from (VI-18) since for each $t \geq 0$

$$(f(t,x)-f(t,y),x-y) \leq ||f(t,x)-f(t,y)|| \; ||x-y|| \leq k||x-y||^2 \quad x,y \in H.$$

Finally, the condition (iii) follows by hypothesis. Hence all the hypotheses in theorem VI-7 are fullfilled and the result (a), (b), (c) follows immediately.

<u>Remarks</u>. (a) In theorem VI-6, theorem VI-7 and the Corollary of theorem VI-7, the condition $R(I-A)=H$ can be weakened by the condition $R(I-\alpha A)=H$ for some $\alpha > 0$ since by lemma V-1 $R(I-\alpha A)=H$ for some $\alpha > 0$ implies $R(I-A)=H$. (b) In theorem VI-7, if $A$ is dissipative rather than strictly dissipative and if the function $k(t)$ appearing in the condition (ii)' is such that $\sup_{t\geq 0} k(t) < 0$, the results still hold. (c) The continuity of the real-valued function $k(t)$ can be weakened to some extent, for example, $k(t)$ can be discontinuous at a finite number of points on $R^+$ with the values of $k(t)$ properly defined at these points of discontinunity (e.g., $k(t_o) = k(t_o + 0)$ or $k(t_o)=1/2\,(k(t_o + 0) + k(t_o - 0))$ where $t_o$ is a point of discontinuity of k( ).

D. Semi-linear Nonstationary Equations

Another application of the results obtained in section B is for the differential equations of the form

$$\frac{dx(t)}{dt} = A_o(t)x(t) + f(t,x(t)) \qquad \text{(VI-19)}$$

where $A_o(t)$ is, for each $t \geq 0$, a linear unbounded operator with $\mathcal{D}(A_o(t))$ and $\mathcal{R}(A_o(t))$ both contained in a real Hilbert space H and f is a given function from $R^+ \times H$ into H. Again, on setting $A(t)=A_o(t)+f(t,\cdot)$, the equation of the form (VI-19) becomes a special form of (VI-1). Differential equations of the semi-linear form (VI-19) have been investigated rather extensively (e.g., see Browder [1] or Kato [9]), and in [9] it gives a survey of the results obtained for this type of equation by using semi-group theory. The object in this section is not to prove any new theorems on the existence of a solution but rather to deduce some results from the general theorem developed in section B and to extend these results for the investigation of the asymptotic stability property of a solution. In part 1, we introduce some theorems based on the general results of section B, and in Parts 2 and 3, which are the main object of this section, we shall discuss some special equations of the form (VI-19). Because of the hypothesis in these special forms is relatively simple, it is expected that these results would be more convenient for applications on certain physical problems, that is, on some concrete partial or ordinary differential equations.

1. General Semi-linear Equations

Consider the operator differential equations of the form (VI-19), we first show the following:

Theorem VI-8. Assume that $A_o(t)$ satisfies the conditions I and II (given in section B) and that for each $t \geq 0$, $A_o(t)$ is dissipative with $\mathcal{D}(A_o(t))=\mathcal{D}$ dense in H. If the operator $A(t)=A_o(t)+f(t,\cdot)$ satisfies the condition III and f satisfies the conditions (i) and (ii) (given in section C). Then all the results (a), (b), (c) of theorem VI-6 hold.

Proof. Consider the operator $A(t)=A_o(t)+f(t,\cdot)$ as a nonlinear operator in the equation (VI-1), we shall show that all the hypotheses in theorem VI-1 are satisfied. Since $\mathcal{D}(A_o(t))=\mathcal{D}$ is independent of t and that f is defined on all of $R^+ \times H$, it follows that $\mathcal{D}(A(t)) = \mathcal{D}(A_o(t))=\mathcal{D}$ is independent of t and thus A(t) satisfies the condition I. By hypothesis for each $t \geq 0$, $A_o(t)$ is dissipative and by lemma V-1, the condition II implies that $R(I-A_o(t))=H$. It follows from the same proof as in theorem VI-6 that $R(I-A(t))=H$ since for each fixed $t \geq 0$ we may take $A_o(t)$ as the operator A in theorem VI-6. Note that all the hypotheses for the proof of $R(I-A)=H$ in that theorem are fullfilled if we replace A by $A_o(t)$ where t is fixed. Since this is true for each $t \geq 0$, the condition II is satisfied. The condition III is given by hypothesis. By the dissipativity of $A_o(t)$ and by the condition (ii), we have for each $t \geq 0$

$$(A(t)x-A(t)y,x-y) = (A_o(t)x-A_o(t)y,x-y) + (f(t,x)-f(t,y),x-y) \leq 0$$

for all $x,y \in \mathcal{D}$. Hence A(t) is dissipative for each $t \geq 0$. By applying theorem VI-1, the result (a) is proved. The proof of (b) and (c) is the same as in that of theorem VI-6.

Remark. The assumptions I and III in the above theorem can be replaced by $(I-A_o(t))^{-1}$ is strongly continuously differentiable in t and f is demicontinuous in t. For a direct proof of this theorem see [9]. It should be noted that the solution obtained in [9] is the so-called "mild solution" which is the solution of an integral equation reduced from the differential equation (VI-19).

Theorem VI-9. Assume that $A_o(t)$ satisfies the conditions I and II with $\mathcal{D}$ dense in H and for each $t \geq 0$, let $A_o(t)$ be strictly dissipative with dissipative constant $\beta(t)$ where $\beta(t)$ is a positive real-valued continuous function on $R^+$. If the operator $A(t)=A_o(t)+f(t,\cdot)$ satisfies the condition III and if f satisfies the conditions (i) and (ii)' with $k(t) < \beta(t)$ for each $t \geqq 0$ and $\int_o^t(\beta(s)-k(s))ds \to +\infty$ as $t \to \infty$, then all the results (a), (b), (c) of theorem VI-7 hold.

<u>Proof</u>. It suffices to show that the operator $A(t)=A_o(t)+f(t,\cdot)$ satisfies all the hypotheses in theorem VI-2. The condition I is obviously satisfied and by hypothesis the condition III is satisfied. The proof of the condition II follows the same argument as in the proof of theorem VI-7. Since for each fixed $t \geqq 0$, $A_o(t)$ is strictly dissipative with dissipative constant $\beta(t)$, and by hypothesis f satisfies the condition (ii)', it follows that for any $x,y \in \mathcal{D}$

$$(A(t)x-A(t)y,x-y) = (A_o(t)x-A_o(t)y,x-y)+(f(t,x)-f(t,y),x-y) \leqq$$
$$\leqq -(\beta(t)-k(t))||x-y||^2 \qquad \text{for all } t \geqq 0$$

which shows that for each $t \geqq 0$, $A(t)$ is strictly dissipative with dissipative constant $(\beta(t)-k(t))$. Note that $\beta(t)-k(t) > 0$ for all $t \geqq 0$. Hence by theorem VI-2, (a) and (c) are proved with the relation (VI-17) for $\beta-k(s)$ replaced by $\beta(s)-k(s)$. Since by hypothesis $\lim_{t\to\infty} \int_o^t (\beta(s)-k(s))ds = \infty$, it follows by the relation (VI-17) that if an equilibrium solution $x_e$ exists, it is asymptotically stable which proves (b).

2. Some Special Semi-linear Equations

The results developed in the preceeding sections of this chapter are not easy to apply for partial differential equations. However, a number of physical and engineering problems are fromulated by a system of partial differential equations which can be reduced to the simplier form

$$\frac{dx(t)}{dt} = A_o x(t) + f(t,x) \qquad \text{(VI-20)}$$

where $A_o$, which is independent of t, is a linear unbounded operator with domain $\mathcal{D}(A_o)$ and range $\mathcal{R}(A_o)$ both contained in a real Hilbert space H and f is a given function from $R^+ \times H$ into H. Since (VI-20) is a special form of (VI-15) with $A=A_o$ a linear operator, the results obtained in section C are directly applicable. Note that the equation (VI-20) is an extension of the equation (V-24) where $f(t,x)=f(x)$. The object in this section is to deduce some results similar to those in section V-C, which would be easier to apply for a certain class of non-stationary partial differential equations.

According to theorem III-14, if $A_o$ is the infinitesimal generator of a contraction semi-group of class $C_o$, then $A_o$ is densely defined, dissipative and $\mathcal{R}(I-A_o)=H$. Hence the following theorem is a direct consequence of theorem VI-6.

Theorem VI-10. Let $A_o$ be the infinitesimal generator of a (linear) contraction semi-group of class $C_o$. Assume that f satisfies the conditions (i), (ii), (iii). Then all the results (a), (b), (c) of theorem VI-6 hold.

As to the asymptotic stability of a solution of (VI-20), we have the following theorem which is a special case of theorem VI-7.

Theorem VI-11. Let $A_o$ be the infinitesimal generator of a (linear) negative contraction semi-group of class $C_o$ with the contractive constant $\beta$. Assume that f satisfies the conditions (i), (ii)', (iii). Then all the results (a), (b), (c) of theorem VI-7 hold.

Proof. Since $A_o$ is the infinitesimal generator of a negative contraction semi-group of class $C_o$, it is densely defined, dissipative and $\mathcal{R}(I-A_o)=H$. By applying theorem V-3 for $A=A_o$ as a special case, $A_o$ is strictly dissipative with dissipative constant $\beta$ since the dissipativity of $A_o$ in the sense of definition V-3 for a linear operator coincides with the dissipativity of $A_o$ in the ordinary sense. Hence all the results (a), (b), (c) follow from theorem VI-7.

Corollary. Let $A_o$ be the infinitesimal generator of a (linear) negative contraction semi-group of class $C_o$ with the contractive constant $\beta$, and let f be uniformly Lipschitz continuous on $R^+ \times H$ with $k < \beta$ where k is the Lipschitz constant with respect to x. Then all the results (a), (b), (c) of theorem VI-7 hold.

Proof. We show that all the hypothesis in the corollary of theorem VI-7 are fulfilled. As in the proof of theorem VI-11, $A_o$ is densely defined, strictly dissipative with dissipative constant $\beta$ and $R(I-A_o)=H$. The uniform Lipschitz continuity of f on $R^+ \times H$ implies that the relation (VI-18) holds (with $k < \beta$) and that there exists a positive real number L such that for any $x \in H$

$$||f(t,x)-f(s,x)|| \leq L|t-s| \qquad \text{for all } s, t \geq 0$$

which implies that the condition (iii) is satisfied. Hence by the corollary of theorem VI-7, all the results in theorem VI-7 hold.

So far in this section, we have assumed that $A_o$ is the infinitesimal generator of a contraction semi-group of class $C_o$ (The conditions imposed on $A_o(t)$ in theorems VI-8 and VI-9 imply that for each $t \geq 0$, $A_o(t)$ is the infinitesimal generator of a contraction semi-group of class $C_o$ (theorem III-14)). In the remainder of this section, we shall consider $A_o$ as an unbounded closed linear operator. (The infinitesimal generator of a semi-group is always closed). Before looking into this type of operator, let us make some observations about the equation (VI-20). Suppose that there exists an equilibrium solution $x_e$ of (VI-20). Let

$z(t) = x(t) - x_e$. On substituting $x(t)$ by $z(t)+x_e$ in (VI-20), we have

$$\frac{dz(t)}{dt} = A_o z(t) + F(t, z(t)) \qquad \text{for all } t \geq 0$$

where

$$F(t, z(t)) = A_o x_e + f(t, z(t) + x_e).$$

Since by (VI-12)

$$A_o x_e + f(t, x_e) = 0 \qquad \text{for all } t \geq 0$$

it follows that $F(t,0) = 0$. Moreover, if $f$ satisfies the conditions (i) (ii) (iii) (or (i), (ii)', (iii)), so does $F$ with possibly different $L(||x||)$ in the condition (iii). To show this, note that the translation mapping from $x$ to $x+x_e$ is a continuous one-to-one mapping from all of $H$ onto $H$ so that $F$ is defined on $R^+ \times H$ into $H$. For each $t \geq 0$ and any $z_1(t)$, $z_2(t) \in H$

$$(F(t, z_2(t)) - F(t, z_1(t)), u) = (f(t, z_2(t)+x_e) - f(t, z_1(t)+x_e), u)$$

for every $u \in H$ which implies that $F$ is continuous from the strong topology to the weak topology of $H$ and is bounded on every bounded subset of $H$ since $f$ has these properties. Note that $z_1(t) \to z_2(t)$ if and only if $z_1(t)+x_e \to z_2(t)+x_e$ and that $||z(t)||$ is bounded if and only if $||z(t)+x_e||$ is bounded where $x_e$ is a fixed element in $H$. Thus $F$ satisfies the condition (i). For any $x, y \in H$

$$(F(t,x) - F(t,y), x-y) = (f(t, x+x_e) - f(t, y+x_e), (x+x_e) - (y+x_e))$$

which shows that $F$ satisfies the condition (ii) if $f$ does. In case $f$ satisfies the condition (ii)', so does $F$ since the above equality implies that

$$(F(t,x) - F(t,y), x-y) \leq k(t)||(x+x_e) - (y+x_e)||^2 = k(t)||x-y||^2.$$

Finally, if $f$ satisfies the condition (iii), then by the definition of $F$ for any $z \in \mathcal{D}(A_o)$

$$||F(t,z)-F(s,z)|| = ||f(t,z+x_e)-f(s,z+x_e)|| \leq L(||z+x_e||)|t-s|.$$
$$(1+||A_o(z+x_e)+f(s,z+x_e)||) = L(||z+x_e||)|t-s|(1+||A_oz+F(s,z)||) \leq$$
$$\leq L(||x||+||x_e||)|t-s|(1+||A_oz+F(s,z)||)$$

since $L(||z+x_e||)$ is nondecreasing (which implies that $L(||z+x_e||) \leq \leq L(||z||+||x_e||)$). The function $L_1(||z||)=L(||z||+||x_e||)$ is a positive nondecreasing function of $||z|| > 0$, for if $||z_1|| \leq ||z_2||$ then $||z_1||+||x_e|| \leq ||z_2||+||x_e||$ which implies that

$$L(||z_1||+||x_e||) \leq L(||z_2||+||x_e||).$$

The positivity of $L_1$ follows directly from the positivity of L. This completes the proof.

It follows from the above observation that if an equilibrium solution of (VI-20) exists, we may assume that $f(t,0) = 0$ and thus the investigation of the stability property of an equilibrium solution is the same as that of the null solution.

Another observation about equilibrium solutions of (VI-20) is the following theorem.

Theorem VI-12. Let H be a real Hilbert space, and let $A_o$ be a strictly dissipative operator from H into H with the dissipative constant $\beta$, i.e.,

$$(A_ox,x) \leq -\beta||x||^2 \qquad \text{for all } x \in \mathcal{D}(A_o).$$

Assume that for any $x,y \in \mathcal{D}(A_o)$

$$(f(t,x)-f(t,y),x-y) \leq k(t)||x-y||^2 \qquad \text{for all } t \geq 0$$

where $k(t)$ is a real-valued function with $k(t_o) < \beta$ for some $t_o \geq 0$. Then an equilibrium solution $x_e$ of (VI-20), if it exists, is unique. In particular, if $f(t,0)=0$ for all $t \geq 0$, then the null solution is the only equilibrium solution.

<u>Proof</u>. Let $y_e$ be an equilibrium solution. By (VI-12)

$$A_o x_e + f(t,x_e) = 0 \quad \text{and} \quad A_o y_e + f(t,y_e) = 0 \quad \text{for all } t \geqq 0$$

which implies that

$$A_o(x_e - y_e) + f(t,x_e) - f(t,y_e) = 0.$$

Hence, for all $t \geqq 0$

$$0 = (A_o(x_e-y_e), x_e-y_e) + (f(t,x_e)-f(t,y_e), x_e-y_e) \leqq -(\beta - k(t))\,||x_e-y_e||^2.$$

By hypothesis $\beta - k(t_o) > 0$ for some $t_o \geqq 0$, it follows from the above inequality that $||x_e-y_e|| = 0$ which proves the uniqueness of $x_e$.

<u>Remark</u>. The above theorem remains true if $A_o$ is dissipative and the function $k(t)$ is negative for some $t_o \geqq 0$ since under this condition, we have $0 \leqq k(t)||x_e-y_e||$ for all $t \geqq 0$ which is a contradiction unless $||x_e-y_e||=0$ since $k(t_o) < 0$.

<u>Corollary</u>. Under the hypothesis in theorem VI-11 (or in theorem VI-7) if an equilibrium solution exists, it is unique.

The uniqueness of the equilibrium solution in theorem VI-11 (or in theorem VI-7) is also a direct consequence of the negative contraction property of the solution. For, if $x_e$ and $y_e$ are two equilibrium solutions then since $x(t)=x_e$ and $y(t)=y_e$ for all $t \geqq 0$

$$||x_e-y_e|| \leqq e^{-\beta t}||x_e-y_e|| \qquad \text{for all } t \geqq 0$$

which is impossible unless $x_e = y_e$.

Now we return to the equation (VI-20) where $A_o$ is an unbounded closed linear operator. In analogy to theorems V-15 to V-17, the following theorems may be regarded as their respective extension.

<u>Theorem VI-13</u>. Let $A_o$ be densely defined, closed and strictly dissipative with dissipative constant $\beta$. Assume that $A_o^*$ is the closure of its restriction to $\mathcal{D}(A_o) \cap \mathcal{D}(A_o^*)$ and that f satisfies the conditions (i), (ii)', (iii) where $A_o^*$ is the adjoint operator of $A_o$. Then all the results (a), (b), (c) in theorem VI-7 hold.

Proof. It suffices to show that $\mathcal{R}(I-A_c)=H$ since all the other conditions in theorem VI-7 are fulfilled by hypothesis. Note that (VI-20) is a special form of (VI-15) with $A=A_o$. But is has been shown in the proof of theorem V-15 that $\mathcal{R}(I-A_o)=H$. Hence the results follow.

Theorem VI-14. Let $A_o$ be an unbounded self-adjoint operator from part of H to H and let it be strictly dissipative with dissipative constant $\beta$. Assume that for each $t \geq 0$, f is uniformly Lipschitz continuous in x with Lipschitz constant k(t) where k(t) is a positive continuous function on $R^+$ satisfying $\sup_{t\geq 0} k(t) < \beta$ and assume that for each $x \in \mathcal{D}(A_o)$, f is uniformly Lipschitz continuous in t with Lipschitz constant $L(||x||)$ where $L(||x||)$ is a positive non-decreasing function of $||x||$. Then all the results (a), (b), (c) of theorem VI-7 hold.

Proof. Since the self-adjoint operator $A_o$ is densely defined, closed and equals its adjoint operator $A_o^*$ (in particular, $\mathcal{D}(A_o)=\mathcal{D}(A_o^*)$), it follows that $A_o$ satisfies the requirements in theorem VI-13. By hypothesis, for each $t \geq 0$

$$||f(t,x)-f(t,y)|| \leq k(t)||x-y|| \qquad \text{for all } x,y \in H \tag{VI-21}$$

which implies that f satisfies the conditions (i) and (ii)'. This is due to the fact that for each $t \geq 0$, (VI-21) implies that f is a continuous in x (from the strong topology to the strong topology of H) and that for a given $y_o \in H$

$$||f(t,x)|| \leq ||f(t,y_o)||+k(t)||x||+k(t)||y_o||.$$

Hence for each $t \geq 0$, $||f(t,x)||$ is bounded whenever $||x||$ is bounded since $k(t) < \beta$ and $||f(t,y_o)||$ is bounded for each t (see (VI-22) below). This proves the condition (i). Condition (ii)' follows also from (VI-21) since for any x, $y \in H$

$$|(f(t,x)-f(t,y),x-y)| \leq ||f(t,x)-f(t,y)|| \, ||x-y|| \leq k(t)||x-y||^2$$

for all $t \geq 0$. By the assumption of uniform continuity of f in t, for each $x \in \mathcal{D}(A_o)$

$$||f(t,x)-f(s,x)|| \leq L(||x||)|t-s| \qquad \text{for all } s,t \geq 0 \quad \text{(VI-22)}$$

which shows that f satisfies the condition (iii). Hence the theorem is proved by applying theorem VI-13.

Remark. It is obvious that the assumptions on f can be weakened by assuming that f satisfies the conditions (i), (ii)', (iii). On the other hand, a stronger assumption is that f is uniformly Lipschitz continuous on $R^+ \times H$, that is, $k(t) = k < \beta$ and $L(||x||)=L>0$.

It can happen that the given linear operator $A_o$ of (VI-20) is not self-adjoint in the original space $H=(H,(\cdot,\cdot))$ but it is possible to find an equivalent inner product $(\cdot,\cdot)_1$ such that $A_o$ is self-adjoint in the space $H_1=(H,(\cdot,\cdot)_1)$. In such a case, we have the following theorem which is an extension of theorem VI-14.

Theorem VI-15. Let $A_o$ be a densely defined linear operator from $H = (H,(\cdot,\cdot))$ into H, and let f satisfy the conditions (i) (iii) in H. If there is an equivalent inner product $(\cdot,\cdot)_1$ such that $A_o$ is self-adjoint and is strictly dissiaptive with the dissipative constant $\beta$ with respect to $(\cdot,\cdot)_1$, and such that for any $x,y \in H$

$$(f(t,x)-f(t,y),x-y)_1 \leq k(t)||x-y||_1^2 \qquad \text{for all } t \geq 0 \qquad \text{(VI-23)}$$

where k(t) is a continuous real-valued function on $R^+$ such that $\sup_{t \geq 0} k(t)<\beta$. Then, (a) For any $x \in \mathcal{D}(A_o)$, there exists a unique solution x(t) of (VI-20) with $x(0)=x$. (b) If an equilibrium solution $x_e$ exists, it is asymptotically stable. (c) A stability region of $x_e$ is $\mathcal{D}(A_o)$ which can be extended to the whole space H in the sense of lemma VI-1.

Proof. Consider $A_o$ as an operator from the space $H_1=(H,(\cdot,\cdot)_1)$ into $H_1$. Since $A_o$ is self-adjoint in $H_1$, it is densely defined, closed and $\mathcal{D}(A_o)=\mathcal{D}(A_o^*)$ in $H_1$. It follows by hypothesis that $A_e$ satisfies the conditions in theorem VI-13 where the underlying space is $H_1$. The continuity of f being invariant under equivalent norms together with the relation (VI-9) imply that if f is demicontinuous in H, it is demicontinuous in $H_1$. Thus f satisfies the condition (i) in the $H_1$-space since by hypothesis, f possesses this property in the H-space. Note that the boundedness of f is also invariant under equivalent norms. Moreover, by the condition (iii) in H and the equivalence relation (VI-7)

$$||f(t,x)-f(s,x)||_1 \leq \gamma||f(t,x)-f(s,x)|| \leq \gamma L(||x||)|t-s|(1+||A_ox+f(s,x)||)$$
$$\leq \gamma\, L(\delta^{-1}||x||_1)|t-s|(1+\delta^{-1}||A_ox+f(s,x)||_1)$$

since $||x|| \leq \delta^{-1}||x||_1$ and $L(||x||)$ is nondecreasing. Let $\lambda = \max(1,\delta^{-1})$ and set $L_1(||x||_1)=\gamma\lambda L(\delta^{-1}||x||_1)$, then $L_1$ is a positive nondecreasing function since L is. Hence

$$||f(t,x)-f(s,x)||_1 \leq L_1(||x||_1)|t-s|(1+||A_ox+f(s,x)||_1)$$

which shows that the condition (iii) is satisfied with respect to $||\cdot||_1$. By applying theorem VI-13, all the results (a), (b), (c) of theorem VI-7 hold in the space $H_1$. Since for any $x \in \mathcal{D}(A_o)$, there exists a unique contraction solution x(t) of (VI-20) with x(0)=x in $H_1$, it follows by lemma VI-3 that x(t) is also the unique solution with x(0)=x in H. Thus (a) is proved. Since the relation (VI-17) holds in $H_1$, and by lemma VI-3 if $x_e$ is an equilibrium solution in $H_1$ it is also an equilibrium solution in H. It follows that for any solution x(t) in H with $x(0)=x \in \mathcal{D}(A_o)$

$$||x(t)-x_e|| \leq \delta^{-1}||x(t)-x_e||_1 \leq \delta^{-1}e^{-\int_0^t(\beta-k(s))ds}||x-x_e||_1 \leq$$
$$\leq(\gamma/\delta)e^{-\int_0^t(\beta-k(s))ds}||x-x_e|| \qquad \text{for all } t \geq 0$$

which shows that the equilibrium solution $x_e$ is asymptotically stable since $\sup_{t\geq 0} k(t)<\beta$ implies $\lim_{t\to\infty}\int_0^t(\beta-k(s))ds= +\infty$. The above inequality is true for any $x \in \mathcal{D}(A_o)$ showing that a stability region is $\mathcal{D}(A_o)$. By lemma VI-1, this region can be extended to the whole space since $\mathcal{D}(A_o)$ is dense in H. Hence the theorem is completely proved.

It is clear that theorems VI-13 to VI-15 are particularly useful for the class of partial differential equations which can be formulated in the form of (VI-20) where $A_o$ is a concrete partial differential operator defined in a suitable Hilbert space H into H and f is a (non-linear) function defined on $R^+ \times H$ into H. It happens often that the operator $A_o$ reduced from a partial differential operator is a densely defined closed operator or its extension is a closed operator (i.e., $A_o$ is closable). Theorem VI-14 and VI-15 suggest that if $A_o$ is self-adjoint in H or if an equivalent inner product can be found such that $A_o$ is self-adjoint in the equivalent Hilbert space $H_1$, then the strict dissipativity imposed on $A_o$ in these theorems is likel to give some stability criteria for the coefficients of the partial differential operator and possibly including the parameters involved in the boundary conditions. On the other hand, in certain design or control processes, the function f itself or the parameters involved in this function can be varied so that the conditions imposed on f such as (VI-22) and (VI-23) are also likely to yield some criteria among this class of functions or among the parameters involved in the given function. In practical problems, these criteria are often in terms of physical properties, dimensional parameters, control functions, etc. which are originated from the derivation of the differential équations describing this system. Thus they are not only important for the design or control prupose but also gives some interpretation of the physical meaning about the system.

3. Ordinary Differential Equations

In case the operator $A_o$ in the equation (VI-20) is a bounded linear operator on H to H, we can write (VI-20) as an ordinary differential equation of the form

$$\frac{dx(t)}{dt} = f(t,x(t)) \qquad \text{(VI-24)}$$

where $f(t,x)$ is a function from $R^+ \times H$ into H. Since the equation (VI-24) is also a special form of (VI-15) with $A \equiv 0$ which is densely defined, dissipative, and $R(I-0)=H$, we have immediately the following theorems.

Theorem VI-16. Let f satisfies the conditions (i), (ii), (iii) (given in section C). Then, (a) For any $x \in H$, there exists a unique contraction solution of (VI-24) with $x(0)=x$. (b) If an equilibrium solution $x_e$ exists, it is stable. (c) The stability region is H.

Theorem VI-17. If f satisfies the conditions (i), (ii)', (iii) with $\beta=0$ (i.e., $\sup_{t\geq 0} k(t)<0$), then the results (a), (c) of theorem VI-16 hold, and in addition: (b)' For any solution $y(t)$ with $y(0)=y \in H$

$$||x(t)-y(t)|| \leq e^{\int_0^t k(s)ds}||x-y|| \qquad \text{for all } t \geq 0.$$

Thus, if an equilibrium solution $x_e$ exists, it is asymptotically stable.

The above two theorems can be proved directly by considering the operator $A(t)$ of (VI-1) as $f(t,\circ)$ and show that the conditions in theorem VI-1 and theorem VI-2 are satisfied respectively. To see this, we first note that $A(t)=f(t,\circ)$ satisfies the conditions I and III by the assumption (i) and (iii) respectively. To show that $A(t)$ satisfies the condition II, let $T=I$ and $T_t=-f(t,\circ)$. By following the proof of theorem VI-6, it can easily be shown that all the conditions in theorem (V-10) are satisfied which implies that for each $t \geq 0$, $R(I-A(t))=R(I-f(t,\circ))=H$. The dissipativity of $A(t)$ follows from the assumption (ii). Hence all the results of theorem VI-16 follow by applying theorem VI-1. A direct proof for theorem VI-17 can similarly be shown.

It should be noted that the existence and the uniqueness of a solution of (VI-24) do not require that $k(t)$ be negative (c.f. [1], [9]). However under this condition, the asymptotic stability property of a solution can not be ensured.

Theorem VI-16 and VI-17 remain true if an equivalent inner product $(\cdot,\cdot)_1$ can be found such that f satisfies respectively the conditions (ii) and (ii)' with respect to $(\cdot,\cdot)_1$. In fact, we have the following theorem whose proof follows that of theorem VI-15.

Theorem VI-18. Assume that f satisfies the conditions (i), (iii) in the Hilbert space $H=(H,(\cdot,\cdot))$. If there exists an equivalent inner product $(\cdot,\cdot)_1$ such that

$$(f(t,x)-f(t,y),x-y)_1 \leqq k(t)||x-y||_1^2 \qquad \text{for all } t \geqq 0$$

where $k(t)$ is a continuous real-valued function defined on $R^+$ with $\sup_{t\geqq 0} k(t) < 0$, then the results (a), (b)', (c) of theorem VI-17 hold except the contraction property of the solutions. If $k(t)=0$, (b)' should be replaced by (b) in theorem VI-16.

In theorems VI-17 and VI-18, if an equilibrium solution $x_e$ exists, it is unique. A weaker condition for the uniqueness of an equilibrium solution can be obtained by applying theorem VI-12. We show this in the following.

Theorem VI-19. Assume that for any $x,y \in H$

$$(f(t,x)-f(t,y),x-y) \leqq k(t)||x-y||^2 \qquad \text{for all } t \geqq 0$$

where $k(t)$ is a real-valued function with $k(t_o) < 0$ for some $t_o \geqq 0$.

Then an equilibrium solution $x_e$, if it exists, is unique. In particular, if $f(t,0)=0$ for all $t \geq 0$, then the null solution is the only equilibrium solution.

<u>Proof</u>. Let $y_e$ be any equilibrium solution. By (VI-12)

$$f(t,x_e) = 0 \quad \text{and} \quad f(t,y_e) = 0 \qquad \text{for all } t \geq 0$$

which implies that

$$0 = (f(t,x_e)-f(t,y_e),x_e-y_e) \leq k(t)||x_e-y_e||^2 \quad \text{for all } t \geq 0.$$

But $k(t_o) < 0$, the above inequality is impossible unless $||x_e-y_e||=0$. Thus the uniqueness of $x_e$ is proved.

# VII. APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS

The stability and existence thoery of the operational differential equations developed in Chapters IV, V, VI deals with unbounded and nonlinear cperators which are extensions of certain concrete linear and nonli.ear partial differential operators respectively. Thus the solutions of the operational differential equations are closely related to the concept of generalized solutions (distribution solutions, weak solutions, etc.) of boundary-value problems for partial differential equations. By a suitable choice of a function space (such as $L^2(\Omega)$, $H^m(\Omega)$), the results obtained in the previous mentioned chapters are directly applicable. In this chapter, we do not intend to solve general nonlinear partial differential equations but rather to apply some of the results obtained in Chapters IV, V, VI to certain semi-linear partial differential equations (which occurs often in physical problems) in order to illustrate some steps in applying the theorems developed for operational differential equations.

## A. Elliptic Formal Partial Differential Operators

It is known that a linear partial differential operator can be, under suitable conditions, formulated as a linear operator in a function space such as Banach space or Hilbert space. In this section, we shall formulate an elliptic partial differential operator as an unbounded linear operator in the real Hilbert space $L^2(\Omega)$. Before giving a formal definition of an elliptic partial differential operator, it is convenient to use the following conventional notations:

$x=(x_1,x_2,\ldots,x_n)$ and $\xi=(\xi_1,\xi_2,\ldots,\xi_n)$ denote variable points in $R^n$; $|\alpha|=\sum_{j=1}^{n}\alpha_j$ where $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ whose components are non-negative integers; $D^\alpha=D_1^{\alpha_1}D_2^{\alpha_2}\ldots D_n^{\alpha_n}$ where $D_j=\frac{\partial}{\partial x_j}$ for $j=1,2,\ldots,n$; if $|\alpha|=0$ the operator $D^\alpha$ is defined to be the identity operator; $\xi^\alpha$ denotes the expression $\xi_{\alpha_1}\ \xi_{\alpha_2}\ \ldots\ \xi_{\alpha_n}$ and $a_\alpha(x)$ denotes the expression $a_{\alpha_1\alpha_2\ldots\alpha_n}(x)$. With these notations, we first give the following definition of a formal partial differential operator.

Definition VII-1. Let the operator

$$L=\sum_{|\alpha|\leq p} a_\alpha(x)\ D^\alpha,$$

where p is a positive integer and the coefficients $a_\alpha(x)$ are infinitely differentiable functions in an open set $\Omega\subseteq R^n$. Then L is called a formal partial differential operator. The differential operator

$$L^*(\cdot)=\sum_{|\alpha|\leq p}(-1)^{|\alpha|}D^\alpha(a_\alpha(x)(\cdot))$$

which is also a formal partial differential operator is called the (real) formal adjoint of L. If $L=L^*$, then L is said to be formally self-adjoint.

Now we give a formal definition of an elliptic differential operator.

Definition VII-2. Let

$$L=\sum_{|\alpha|\leq p} a_\alpha(x)\ D^\alpha$$

be a formal partial differential operator of order p defined in a domain $\Omega$ of the Euclidean space $R^n$. If for each non-zero vector $\xi$ in $R^n$

$$\sum_{|\alpha|=p} a_\alpha(x)\ \xi^\alpha \neq 0 \qquad x\in\Omega,$$

then the operator L is said to be elliptic. Thus, the requirement of ellipticity for a partial differential operator is the analogue of the condition that the leading coefficient should be non-vanishing.

For the case of second order elliptic partial differential operator (i.e., p=2), the operator L can be written in the form

$$L = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c(x)$$

with the requirement that for any non-zero vector $\xi$ in $R^n$

$$\sum_{i,j=1}^{n} a_{ij}(x)\ \xi_i \xi_j \neq 0 \qquad\qquad x \in \Omega.$$

The elliptic partial differential operator L can be formulated as an operator in $L^2(\Omega)$ in different ways. For example, we may define the operator T to be the restriction of L with domain $\mathcal{D}(T)=C_0^\infty(\Omega)$, the set of all infinitely differentiable functions with compact support in $\Omega$. T is a densely defined linear operator from $L^2(\Omega)$ into $L^2(\Omega)$ since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$ (see theorem III-17). The domain of T is narrower than necessary; in the above definition we could replace $C_0^\infty(\Omega)$ by $C_0^p(\Omega)$ since we need only p-th order derivatives in constructing L, there by obtaining an extension of T. We can also define a larger extension $T_1$ of T by admitting in its domain all functions $u \in L^2(\Omega)$ such that $u \in C^p(\Omega)$ and $Lu \in L^2(\Omega)$ (here u need not have compact support). Since T is densely defined and $T \subset T_1$, it follows that $T_1$ is densely defined and so both $T^*$ and $T_1^*$ exist. The question may arise that if the formal partial differential operator L is self-adjoint, that is, $L=L^*$, whether or not $T^*$ (or $T_1^*$) is also self-adjoint. To answer this question for the case of the operator T, we state the following theorem whose proof can be found in the book by Dunford and Schwartz [6].

<u>Theorem VII-1</u>. Let L be an elliptic formal partial differential operator of even order 2p defined in a domain $\Omega_0$ in $R^n$. Suppose that L

is of the form

$$L = \sum_{|\alpha| \leq 2p} a_\alpha(x)\, D^\alpha \tag{VII-1}$$

and that

$$(-1)^p \sum_{|\alpha|=2p} a_\alpha(x)\, \xi^\alpha > 0, \quad x \in \Omega_o,\ \xi \in R^n,\ \xi \neq 0. \tag{VII-2}$$

Let $\Omega$ be a bounded subdomain whose closure is contained in $\Omega_o$. Suppose that the boundary of $\Omega$ is a smooth surface $\partial\Omega$, and that no point in $\partial\Omega$ is interior to the closure of $\Omega$. Let T and $\hat{T}$ be the operators in the Hilbert space $L^2(\Omega)$ defined by the equation

$$\mathcal{D}(T)=\mathcal{D}(\hat{T}) = \{u \in C_o^\infty(\bar{\Omega});\ u(x) = \partial_\nu u(x)=\dots \partial_\nu^{p-1} u(x)=0,\ x \in \partial\Omega\}$$

$$Tu = Lu,\ \hat{T}u = L^*u, \quad u \in \mathcal{D}(T) = \mathcal{D}(\hat{T})$$

where $\partial_\nu^k$ denotes the k-th normal derivatives on $\partial\Omega$. Let A and $\hat{A}$ be the closure of T and $\hat{T}$, respectively. Then (i) $A^* = \hat{A}$ and $(\hat{A})^* = A$. (ii) $\sigma(A)$, the spectrum of A, is a countable discrete set of points with no finite limit point. (iii) If $\lambda \notin \sigma(A)$, $(\lambda I-A)^{-1}$ is a compact operator.

Corollary. Under the hypotheses of theorem VII-1 and, in addition, L is formally self-adjoint so that $L=L^*$. Then (i) the operator A is self-adjoint, $A=A^*$; (ii) The spectrum $\sigma(A)$ is a sequence of points $\{\lambda_n\}$ tending to $\infty$, and for $\lambda \notin \sigma(A)$; $R(\lambda;A)$ is a compact operator.

Remark. Suppose that the condition (VII-2) in theorem VII-1 is replaced by the condition

$$(-1)^p \sum_{|\alpha|=2p} a_\alpha(x)\, \xi^\alpha < 0, \quad x \in \Omega_o,\ \xi \in R^n,\ \xi \neq 0 \tag{VII-2)'$$

then $-L$ satisfies the hypothesis in the above theorem in which $-T$, $-\hat{T}$, $-A$ and $-\hat{A}$ would be the operators associated with $-L$ where T, $\hat{T}$, A and $\hat{A}$ are the operators defined in the theorem for the operator L. Thus if L is formally self-adjoint so is $-L$ and by applying the above corollary

$-A=(-A)^*$ which implies $A=A^*$. Hence theorem VII-1 and its corollary, on the part of self-adjointness of A, remains valid if the condition (VII-2) is replaced by the condition (VII-2)'.

It follows from the above theorem that under suitable conditions on the leading coefficients of L and a smooth boundary condition on $\Omega$, the elliptic partial differential operator L can be formulated as a linear operator T in $L^2(\Omega)$ such that if L is formally self-adjoint then the closure of T is also self-adjoint. This formulation enables us to apply some of the results developed in Chapters V and VI for certain semi-linear partial differential equations.

It is known that [6] under the conditions of the above theorem and if $\Omega$ is a bounded open set contained in $\Omega_o$ then the Garding's Inequality holds, that is there exists constant $K < \infty$ and $k > 0$ such that

$$(Lu,u) + K(u,u) \geq k||u||_p^2 \qquad u \in C_o^\infty(\Omega)$$

where $||\circ||_p$ is the norm of the Hilbert space $H_o^p(\Omega)$.

B. Semi-linear Partial Differential Equations

The formulation of a formal linear partial differential operator as a linear operator in $L^2(\Omega)$ in the previous section enables us to establish some existence and stability criteria among the coefficients of the formal differential operator for a certain class of stationary and non-stationary partial differential equations. In this section, we give some applications of the results obtained in Chapters IV, V and VI to a class of linear and semi-linear partial differential equations which can be served as an illustration of some steps in applying the theorems developed for operational differential equations. In the following, the first simple example of a linear partial differential equation gives a fairly detailed description of the application from which some more general equations or non-zero boundary conditions can easily be obtained.

Example VII-1. Consider the simple case of the linear partial differential equation

$$\frac{\partial u}{\partial t} = a(x)\,\frac{\partial^2 u}{\partial x^2} + b(x)\,\frac{\partial u}{\partial x} + c(x)u \qquad x \in (0,1) \qquad \text{(VII-3)}$$

with the boundary conditions

$$u(t,0) = u(t,1) = 0 \qquad (t \geq 0). \qquad \text{(VII-4)}$$

Assume that the coefficient $a(x)$ is positive (or negative) on $[0,1]$ and that $a(x)$, $b(x)$, $c(x)$ are all infinitely differentiable functions in an open interval $I_o$ containing $[0,1]$. Then the linear operator

$$L = a(x)\,\frac{\partial^2}{\partial x^2} + b(x)\,\frac{\partial}{\partial x} + c(x)$$

is a formal partial differential operator defined in $I_o$. Moreover, by the assumption $a(x) > 0$ for all $x \in [0,1]$ we have

$$-a(x)\xi^2 < 0 \quad \text{for all } \xi \in R^1 \quad \text{with } \xi \neq 0 \text{ and } \ x \in [0,1].$$

It follows that $-L$ is an elliptic partial differential operator. The formal adjoint operator of $L$ is given as

$$L^*(\cdot) = \frac{\partial^2}{\partial x^2}(a(x)(\cdot)) - \frac{\partial}{\partial x}\,(b(x)(\cdot)) + c(x)(\cdot)$$

which is also an elliptic partial differential operator. It is easily shown by a simple calculation that equation (VII-3) can be reduced to the form

$$\frac{\partial u}{\partial t} = \frac{1}{q(x)} \quad \frac{\partial}{\partial x}\,(P(x)\,\frac{\partial u}{\partial x}) + c(x)u \qquad \text{(VIII-3)'}$$

where

$$q(x) = (a(x))^{-1}\, e^{\int_{x_o}^{x} (b(\xi)/a(\xi))d\xi} \qquad (x_o \in [0,x] \text{ fixed})$$

$$P(x) = e^{\int_{x_o}^{x} (b(\xi)/a(\xi))d\xi} = a(x)q(x). \qquad \text{(VII-5)}$$

Let us seek a solution in the real Hilbert space $L^2(0,1)$ in which the inner product between any pair of elements $u, v \in L^2(0,1)$ is defined by

$$(u, v) = \int_0^1 u(x)\ v(x)\ dx. \qquad \text{(VII-6)}$$

Define the operator T in $L^2(0,1)$ as the restriction of L on $C_0^\infty(0,1)$ and $\hat{T}$ the restriction of L* on $C_0^\infty(0,1)$, that is

$$\mathcal{D}(T) = \mathcal{D}(\hat{T}) = C_0^\infty(0,1); \quad Tu=Lu \quad \text{and} \quad \hat{T}u=L^*u, \qquad u \in \mathcal{D}(T).$$

Let A and $\hat{A}$ denote the closure of T and $\hat{T}$ respectively (T and $\hat{T}$ are closable). Then $\mathcal{D}(A)$ is dense in $L^2(0,1)$ since $\mathcal{D}(A) \supset \mathcal{D}(T) = C_0^\infty(0,1)$ which is dense in $L^2(0,1)$. Thus A* and $(\hat{A})^*$ both exist. In general, T is not self-adjoint with respect to the inner product defined in (VII-6) as can be seen by "integration by parts" of the integral

$$(u,Tv) = \int_0^1 u(x)\ T\, v\ (x)\ dx \qquad u, v \in \mathcal{D}(T)$$

which, in general, is not equal to (v, Tu) for all $u,v \in \mathcal{D}(T)$. However, by defining the scalar functional V(u,v) by

$$V(u,v) = (u, qv) = \int_0^1 u(x)\ q(x)\ v(x)\ dx \qquad \text{(VII-6)'}$$

where the function q(x) is the known function given in (VII-5) then V(u,v) defines an equilvaent inner product $(\cdot,\cdot)_1$ such that

$$(Tu,v)_1 = (u, Tv)_1 \qquad \text{for all } u, v \in \mathcal{D}(T).$$

To see this, define

$$(u,v)_1 = V(u,v)$$

then it is obvious that $(\cdot,\cdot)_1$ possesses all the properties of an inner product. Since $(u,u)_1 = (u,qu) = \int_0^1 qu^2 dx$, it follows that

$$\left(\min_{0\leq x\leq 1} q(x)\right) ||u||^2 \leq ||u||_1^2 \leq \left(\max_{0\leq x\leq 1} q(x)\right) ||u||^2$$

which implies that $(\cdot,\cdot)_1$ and $(\cdot,\cdot)$ are equivalent. Notice that $q(x)>0$

and is continuous over the closed interval [0,1] so that it actually attains its maximum and minimum values bounded away from zero and $\infty$. For any u, v $\varepsilon$ $\mathcal{D}(T)$, on integrating by parts and taking notice that the boundary conditions are satisfied for any u $\varepsilon$ $\mathcal{D}(T)$ we have

$$(u,Tv)_1 = (u,qTv) = \int_0^1 uq[q^{-1} \frac{\partial}{\partial x} (P \frac{\partial v}{\partial x}) + cv]dx$$

$$= \int_0^1 (-P \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + c\ q\ u\ v)dx = \int_0^1 [v \frac{\partial}{\partial x} (P \frac{\partial u}{\partial x}) + c\ q\ u\ v]dx = (Tu,v)_1$$

which shows that $T=\hat{T}$. It follows by applying theorem VII-1 and the remark following that theorem that $A=(\hat{A})^*=A^*$ which shows that A is self-adjoint in the equivalent Hilbert space $L_1^2$ (0,1) equipped with the inner product $(\cdot,\cdot)_1$. Moreover, the above equality implies that for any u $\varepsilon$ $\mathcal{D}(T)$

$$(u,Tu)_1 = -\int_0^1 [P(\frac{\partial u}{\partial x})^2 - c\ q\ u^2]dx = -\int_0^1 [a\ q(\frac{\partial u}{\partial x})^2 - c\ q\ u^2]dx\ .$$

On setting $u_1 = q^{1/2}$ u then $||u_1||=||u||_1$ and by an elementary calculation we have

$$aq(\frac{\partial u}{\partial x})^2 = a(\frac{\partial u_1}{\partial x})^2 - \frac{1}{2}(b-a')\frac{\partial u_1^2}{\partial x} + \frac{1}{4}\frac{(b-a')^2}{a} u_1^2 \tag{VII-7}$$

where $a' \equiv \frac{d}{dx} a(x)$. Hence, integrating by parts and using the well known inequality

$$\int_0^1 (\frac{du}{dx})^2\ dx \geq \pi^2 \int_0^1 u^2\ dx \tag{VII-8}$$

which is valid for any u(x) satisfying the condition (VII-4), we have

$$(u,Tu)_1 = -\int_0^1 [a(\frac{\partial u_1}{\partial x})^2 + (\frac{1}{2}(b'-a'') + \frac{1}{4}\frac{(b-a')^2}{a} - c)\ u_1^2]dx$$

$$\leq -\int_0^1 [\pi^2\ a_{min} + \frac{1}{2}(b'-a'') + \frac{1}{4}\frac{(b-a')^2}{a} - c]u_1^2\ dx \leq -\beta||u||_1^2$$

where

$$a_{min} = \min_{0\leq x\leq 1} a(x)$$

$$\beta = \min_{0\leq x\leq 1} [\pi^2\ a_{min} + \frac{1}{2}(b'(x)-a''(x)) + \frac{1}{4}\frac{(b(x)-a'(x))^2}{a(x)} - c(x)\ ].$$

It follows that if $\beta=0$ or $\beta>0$ then T is dissipative or strictly dissipative, respectively, with respect to $(\cdot,\cdot)_1$. The dissipativity or strict dissipativity of T implies the dissipativity or strict dissipativity, respectively, of A. To see this, let $u \in \mathcal{D}(A)$ then by the construction of the closure of a closable operator there exists a sequence $\{u_n\} \subset \mathcal{D}(T)$ such that $u_n \to u$ and $\lim_{n\to\infty} Tu_n$ exists and equals Au (see the definition of closable operator following theorem III-1). Hence by the continuity of inner product, we have

$$(Au,u)_1 = \lim_{n\to\infty} (Tu_n, u_n)_1 \leq \lim_{n\to\infty} (-\beta||u_n||_1^2) = -\beta||u||_1^2$$

which shows the dissipativity and strict dissipativity of A. Therefore, by applying theorems V-17 and V-13 with $f \equiv 0$ we have the following results.

<u>Theorem VII-2</u>. Assume that the coefficients a(x), b(x) and c(x) of (VII-3) are infinitely defferentiable over any open interval $I_o$ containing [0,1] and that a(x) is positive on [0,1]. If the condition

$$\beta = \min_{0\leq x\leq 1} [\pi^2 a_{min} + \frac{1}{2}(b'(x)-a''(x)) + \frac{1}{4}\frac{(b(x)-a'(x))^2}{a(x)} - c(x)] \geq 0 \tag{VII-9}$$

is satisfied where $a_{min} = \min_{0\leq x\leq 1} a(x)$ and $a'(x) = \frac{d}{dx}a(x)$, $a''(x) = \frac{d^2}{dx^2}a(x)$, then for any initial element $u_o(x) \in \mathcal{D}(A)$ there exists a unique solution u(t,x) in the sense of definition VI-1 with $u(0,x)=u_o(x)$. Moreover, the null solution of (VII-1) is stable if $\beta=0$ and is asymptotically stable if $\beta > 0$ and in the later case the null solution is the only equilibrium solution.

As an example of the above theorem, take $a(x) = \frac{1}{R}$, $b(x) = \frac{2}{\sqrt{R}}x$, $c(x) = (x^2 + \frac{2}{\sqrt{R}})$ where R is a positive constant to be determined, then

$$\beta = \min_{0\leq x\leq 1} [\frac{\pi^2}{R} + \frac{1}{\sqrt{R}} + \frac{1}{4}R(\frac{2}{\sqrt{R}}x)^2 - (x^2 + \frac{2}{\sqrt{R}})] = \frac{\pi^2}{R} - \frac{1}{\sqrt{R}}.$$

Hence $\beta > 0$ if $0 < R < \pi^4$ which shows the same result as given in [3].

Remark. The solution $u(t,x)$ in theorem VII-2 is in fact a solution of (VII-3) in the strong sense i.e., $\frac{du(t,x)}{dt} = Au(t,x)$ in the norm topology as can be seen by applying the corollary of theorem III-14. However, in the case of semi-linear equations which are to be discussed in the following, theorem III-14 and its corollary do not apply. Thus, we shall assume that any solution in the following discussion is in the sense of definition VI-1.

In case $a(x)$ is negative instead of positive, then L is an elliptic partial differential operator satisfying (VII-2). By defining $(u,v)_1 = (u,-qv)$ and note that $-q(x) > 0$ for all $x \in [0,1]$, $(\circ,\circ)_1$ is equivalent to $(\circ,\circ)$ and that A remains to be a self-adjoint operator with respect to $(\circ,\circ)_1$. Moreover,

$$(u,Tu)_1 = (u,-q\ Tu) = -\int_0^1 uq\ [\frac{1}{q}\frac{\partial}{\partial x}(P\frac{\partial u}{\partial x}) + c\ q\ u^2]dx$$

$$= -\int_0^1 [-P(\frac{\partial u}{\partial x})^2 + c\ q\ u^2]dx = -\int_0^1[-aq(\frac{\partial u}{\partial x})^2 + c\ q\ u^2]dx.$$

On setting $u_1=(-q)^{1/2}\ u$, then $||u_1||=||u||_1$, and using the identity (VII-7) and the relation (VII-8) we have

$$(u,Tu)_1 = -\int_0^1[-a(\frac{\partial u_1}{\partial x})^2-(\frac{1}{2}(b'-a'')-\frac{1}{4}\frac{(b-a')^2}{a}-c)u_1^2]dx$$

$$\leqq -\int_0^1[\pi^2(-a)_{min} - \frac{1}{2}(b'-a'')-\frac{1}{4}\frac{(b-a')^2}{a}-c]u_1^2\ dx \leqq -\beta||u||_1^2$$

where

$$\beta = \min_{0\leqq x\leqq 1}\ [\pi^2(-a)_{min} - \frac{1}{2}(b'(x)-a''(x))-\frac{1}{4}\frac{(b(x)-a'(x))^2}{a(x)} - c(x)]. \tag{VII-10}$$

Hence we have the following results.

Theorem VII-3. Under the hypotheses of theorem VII-2 with the assumption $a(x)$ positive replaced by $a(x)$ negative and with (VII-9) replaced by (VII-10), all the results in theorem VII-2 hold.

The results obtained in theorems VII-2 and VII-3 can be applied to the case of semi-linear equations with the same linear part as in example VII-1, i.e., a nonlinear function f is included in equation (VII-3). With some additional restrictions on f, the existence and stability of a solution can be ensured. These conditions are given in the following.

Example VII-2. Consider the partial differential equation

$$\frac{\partial u}{\partial t} = a(x)\,\frac{\partial^2 u}{\partial x^2} + b(x)\,\frac{\partial u}{\partial x} + c(x)u + f(u) \tag{VII-11}$$

with the boundary conditions $u(t,0)=u(t,1)=0$ where $a(x)$, $b(x)$, $c(x)$ are the same as in theorem VII-2 and f is a nonlinear function defined on $L^2(0,1)$ to $L^2(0,1)$. According to theorem V-17, if f is continuous on $L^2(0,1)$ and is bounded on bounded subsets of $L^2(0,1)$ such that

$$(f(u)-f(v),u-v)_1 \leq k_1||u-v||_1^2 \quad \text{with } k_1 < \beta, \quad u,v \in L^2(0,1)$$

where $(\cdot,\cdot)_1$ is the equivalent inner product defined in (VII-6)' and $\beta$ is given by (VII-9), then all the results in theorem VII-2 with respect to an equilibrium solution, if it exists, remain valid. In particular if $f(0)=0$, the null solution is exponentially asymptotically stable.

To illustrate the above statement take, for example, the function

$$f(u) = k\,\frac{u^2}{\lambda^2+u^2} \qquad (\lambda^2 > 0).$$

It is obvious that f is continuous on $L^2(0,1)$ (in the strong topology) and is bounded on $L^2(0,1)$. By the definition of $(\cdot,\cdot)_1$ in (VII-6)'

$$(f(u)-f(v),u-v)_1 = \int_0^1 k\left(\frac{u^2}{\lambda^2+u^2} - \frac{v^2}{\lambda^2+v^2}\right)q(u-v)\,dx$$

$$= k\,\lambda^2 \int_0^1 \frac{u+v}{(\lambda^2+u^2)(\lambda^2+v^2)}\,q(u-v)^2\,dx \leq$$

$$\leq \lambda^2 \max_{0\leq x\leq 1} \frac{|k(u(x)+v(x))|}{(\lambda^2+u^2(x))(\lambda^2+v^2(x))}\,||u-v||_1^2\,.$$

It is easily shown that for any real number $u, v$

$$\frac{|u+v|}{(\lambda^2+u^2)(\lambda^2+v^2)} < \frac{1}{|\lambda^3|} \qquad \text{(VII-12)}$$

which implies that

$$(f(u)-f(v), u-v)_1 < \left|\frac{k}{\lambda}\right| \; ||u-v||_1^2 \text{ .}$$

It follows that if $|\frac{k}{\lambda}| \leqq \beta$ then the existence and uniqueness of a solution for any initial element $u_o(X) \in \mathcal{D}(A)$ are ensured. Moreover the null solution is exponentially asymptotically stable with stability region $\mathcal{D}(A)$.

The above example gives general conditions on the coefficients of the partial differential operator L and on the nonlinear function f which depends on u. In case f is a function of both t and u, additional restriction on f is necessary. These conditions are given as an example.

Example VII-3. Consider the non-stationary semi-linear equation

$$\frac{\partial u}{\partial t} = a(X)\frac{\partial^2 u}{\partial X^2} + b(X)\frac{\partial u}{\partial X} + c(X)u + f(t,u) \qquad \text{(VII-13)}$$

with the same boundary conditions $u(t,0)=u(t,1)=0$ where $a(X)$, $b(X)$ and $c(X)$ remain the same as in example VII-1. According to theorem VI-15, if f satisfies the conditions (i) and (iii) given in section C of Chapter VI and if there exists a continuous real-valued function $k(t)$ on $R^+=[0,\infty)$ with $\sup_{t\geqq 0} k(t) < \beta$ where $\beta$ is given by (VII-9) such that for any $u, v \in L^2(0,1)$

$$(f(t,u)-f(t,v), u-v)_1 \leqq k(t) \; ||u-v||_1^2 \qquad (t \geqq 0) \qquad \text{(VII-14)}$$

then for any initial element $u_o(X) \in \mathcal{D}(A)$ there exists a unique solution $u(t,X)$ with $u(0,X)=u_o(X)$, and if an equilibrium solution exists, it is unique and is asymptotically stable.

Take, for instance, the function

$$f(t,u) = \frac{ku^2}{(\lambda^2+u^2)(c_1+c_2t)} \qquad (c_1, c_2 > 0).$$

It is obvious that f is defined on $R^+ \times L^2(0,1)$ into $L^2(0,1)$ and is such that for each $t \geq 0$ it is continuous on $L^2(0,1)$ (in the strong topology) and is bounded uniformly which implies that f satisfies the condition (i) in theorem VI-15. For any $u(x) \in \mathcal{D}(A_o)$ and any $s,t \geq 0$

$$||f(t,u)-f(s,u)|| = ||\frac{ku^2}{\lambda^2+u^2}\,\frac{c_2(s-t)}{(c_1+c_2t)(c_1+c_2s)}|| \leq$$

$$\leq \frac{|c_2k|}{c_1^2}\,||\frac{u^2}{\lambda^2+u^2}||\ |s-t| \leq \frac{|c_2k|}{c_1^2}\ |s-t|$$

which shows that f satisfies the condition (iii). Finally, by using (VII-12) for any $u,v \in L^2(0,1)$

$$(f(t,u)-f(t,v),u-v)_1 = \frac{k}{c_1+c_2t}\int_o^1 (\frac{u^2}{\lambda^2+u^2} - \frac{v^2}{\lambda^2+v^2})q(u-v)dx <$$

$$< |\frac{k}{\lambda}|\,\frac{1}{c_1+c_2t}\ ||u-v||_1^2 = k(t)||u-v||_1^2$$

where $k(t) = |\frac{k}{\lambda}|\,\frac{1}{c_1+c_2t}$ is a continuous function on $R^+$ with $\sup_{t\geq 0} k(t) = \frac{|k|}{c_1\,|\lambda|}$. It follows by applying theorem VI-15 that if $\frac{|k|}{c_1|\lambda|} \leq \beta$ then all the results stated above are valid. Since in this particular case, $f(t,0) = 0$, which implies that the null solution is asymptotically stable.

In the examples above, we assumed that the boundary conditions were $u(t,0)=u(t,1)=0$. In the case of non-zero boundary conditions, a suitable transformation of the unknown function can reduce these conditions into zero boundary conditions without affecting the existence or stability of the original system. The following example gives such an illustration.

Example VII-4. Consider the same problem as in example VII-3 except with the boundary conditions replaced by

$$u(t,0)=h_o(t) \quad \text{and} \quad u(t,1)=h_1(t) \qquad (t \geq 0) \tag{VII-15}$$

where $h_o$ and $h_1$ are two given continuously differentiable functions of $t \geq 0$. On setting

$$v(t,x) = u(t,x) - (1-x)h_o(t) - xh_1(t) \qquad (t \geq 0) \tag{VII-16}$$

equation (VII-13) is reduced to

$$\frac{\partial v}{\partial t} = a(x)\frac{\partial^2 v}{\partial x^2} + b(x)\frac{\partial v}{\partial x} + c(x)v + f_1(t,v) \tag{VII-13'}$$

with the boundary conditions $v(t,0)=v(t,1)=0$ where

$$f_1(t,v) = f(t,v_1) - (1-x)h_o'(t) - xh_1'(t) + b(x)(h_1(t)-h_o(t)) +$$
$$+ c(x)(xh_1(x) + (1-x)h_o(t)) \tag{VII-17}$$

with $v_1(t,x)=v(t,x)+(1-x)h_o(t)+xh_1(t)$. Suppose that $f_1$ satisfies all the conditions in theorem VI-15, then for any two initial elements $v_1(0,x)$ and $v_2(0,x) \in \mathcal{D}(A)$ theorem VI-15 implies that there exists two solutions $v_1(t,x)$ and $v_2(t,x)$, respectively, such that

$$||v_1(t,x)-v_2(t,x)|| \leq M\, e^{-\int_0^t (\beta-k(s))ds}\, ||v_1(0,x)-v_2(0,x)||$$

where $M \geq 1$, $\beta$ is given in (VII-9) and $k(t)$ is given in (VII-14) with $f$ replaced by $f_1$. By the relation (VII-16)

$$u_1(t,x)-u_2(t,x)=v_1(t,x)-v_2(t,x) \qquad (t \geq 0,\ x \in [0,1]),$$

it follows that

$$||u_1(t,x)-u_2(t,x)|| \leq M\, e^{-\int_0^t (\beta-k(s))ds}\, ||u_1(0,x)-u_2(0,x)||$$

which shows that the existence, uniqueness and stability of a solution of the transformed system with homogeneous boundary condtions implies the same property of a solution of the original system with non-homogeneous boundary conditions. Hence the investigation of the equation (VII-13) with the non-homogeneous boundary conditions (VII-15) is reduced to the one with homogeneous boundary conditions by taking the transformed function $f_1$ as the given nonlinear function.

It is to be noted that if an equilibrium solution $v_e$ exists for the transformed equation, it does not, in general, imply the existence of an equilibrium solution $u_e$ of the original equation. In fact, if $h_o(t)$ and $h_1(t)$ are not both constant no equilibrium solution of the original system can exist. (In physical problems, this type of boundary conditions often generates periodic solutions).

The above examples are given in the one-dimensional space which serve as an illustration of some needed technique in formulating linear operators in a Hilbert space from formal partial differential operators and which give an application of some of the results developed for operational differential equations to partial differential equations. Following the same idea as in the one-dimensional case, the extension of the above results to more general n-dimensional space-dependent partial differential operators bears no difficulty. For the sake of simplicity, we limit our discussion to second order partial differential equations which occur often in physical problems.

Example VII-5. Consider the second order linear differential equations of the form

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j}\right) + c(x)u \qquad x \in \Omega \tag{VII-18}$$

with the boundary conditions

$$u(t,x')=0 \qquad x' \in \partial\Omega \qquad t \geq 0 \tag{VII-19}$$

where $x=(x_1,x_2,\ldots,x_n)$, $\Omega$ is a bounded open subset of the Euclidean space $R^n$ with boundary $\partial\Omega$ which is a smooth surface and no point in $\partial\Omega$ is interior to $\bar{\Omega}$, the closure of $\Omega$. Assume that $a_{ij}(x) = a_{ji}(x)$ $(i,j=1,2,\ldots,n)$ and together with $c(x)$ are infinitely differentiable real-valued functions in a domain $\Omega_o$ which contains $\bar{\Omega}$ and that there exists a positive constant $\alpha$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqq \alpha \sum_{i=1}^{n} \xi_i^2 \qquad x \in \Omega_o,\ \xi \in R^n. \tag{VII-20}$$

By definition VII-2, the operator

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + c(x)$$

is an elliptic partial differential operator in $\Omega_o$ since under the assumption (VII-20)

$$(-1)\sum_{i,j=1}^{n} a_{ij}(x)\ \xi_i\xi_j \neq 0, \qquad x \in \Omega_o,\ \xi \in R^n,\ \xi \neq 0.$$

In fact, if the operator L satisfies the condition (VII-20), it is said to be strongly elliptic. It is easily seen by definition that the operator L is self-adjoint i.e., $L=L^*$. Let T be the operator in $L^2(\Omega)$ defined by

$$\mathcal{D}(T) = \{u \in C^\infty(\bar{\Omega});u(x')=0,\ x' \in \partial\,\Omega\}$$

$$Tu=Lu \qquad\qquad u \in \mathcal{D}(T),$$

and let A be the closure of T. By the corollary of theorem VII-1, A is self-adjoint. For any $u \in \mathcal{D}(T)$, integration by parts yields

$$(u,Tu) = \int_\Omega uTu\,dx = \int_\Omega \left[\sum_{i,j=1}^{n} u\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + c(x)u^2\right]dx$$

$$= -\int_\Omega \left[\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} - c(x)\ u^2\right]dx$$

where $dx = dx_1dx_2\cdots dx_n$. By the assumption (VII-20) and using the well known inequality [24]

$$\int_\Omega \sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2 dx \geqq \gamma \int_\Omega u^2\ dx \tag{VII-21}$$

where $\gamma$ is a positive real number, we obtain

$$(u,Tu) \leqq -\int_\Omega [\alpha \sum_{i=1}^{n} (\frac{\partial u}{\partial x_i})^2 - c(x)\ u^2]dx \leqq -\int_\Omega (\alpha\gamma - c(x))u^2\ dx$$

$$\leqq -(\alpha\gamma - c_m)||u||^2 = -\beta||u||^2$$

where $c_m = \max_{x\epsilon\bar{\Omega}} c(x)$ and $\beta = \alpha\gamma - c_m$. Hence, T is dissipative if $\beta=0$ and is strictly dissipative if $\beta>0$. The dissipativity and strict dissipativity of A follow from the dissipativity and strict dissipativity, respectively, of T as has been shown in example VII-1 since A is the closure of T. Therefore, A satisfies all the hypotheses in theorem V-16. To summarize, we can state the following theorem by applying theorem V-16 with $f \equiv 0$.

<u>Theorem VII-4</u>. Assume that all the real-valued functions $a_{ij}(x)=a_{ji}(x)$ $(i,j=1,2,\ldots,n)$ and $c(x)$ in equation (VII-18) are infinitely differentiable in a domain $\Omega_o$ containing $\bar{\Omega}$, the closure of $\Omega$, where $\Omega$ is a bounded open set in $R^n$ whose boundary $\partial\Omega$ is a smooth surface and no point of $\partial\Omega$ is interior to $\bar{\Omega}$. If the condition (VII-20) is satisfied and if

$$\beta = \alpha\gamma - \max_{x\epsilon\bar{\Omega}} c(x) \geqq 0 \qquad \text{(VII-22)}$$

where $\alpha$ is given in (VII-20) and $\gamma$ is given in (VII-21), then for any $u_o(x) \ \epsilon\ \mathcal{D}(A)$ there exists a unique solution $u(t,x)$ to (VII-18) strongly continuous in t with respect to the $L^2(\Omega)$ norm with $u(0,x)=u_o(x)$. Moreover, the null solution is stable for $\beta=0$ and is asmyptotically stable if $\beta > 0$ and in the later case the null solution is the only equilibrium solution. The stability region is $\mathcal{D}(A)$ which, in some sense, can be extended to the whole space $L_2(\Omega)$.

It is seen from the above theorem that the major conditions imposed on the coefficients of the operator L are conditions (VII-20) and (VII-22). Notice that if $c(x)$ is a non-positive function, then (VII-22) is automatically satisfied. As a special form of (VII-18) we consider the equation

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(a_i(x) \frac{\partial u}{\partial x_i}\right) + c(x)\, u \qquad x \in \Omega \qquad \text{(VII-18)}'$$

with the boundary conditions (VII-19). The following theorem is an immediate consequence of theorem VII-14.

Theorem VII-5. Assume that the real-valued functions $a_i(x)$ $(i=1,2,\ldots,n)$ and $c(x)$ in equation (VII-18)' are infinitely differentiable in a domain $\Omega_o$ containing $\bar{\Omega}$ where $\Omega$ is a bounded open set in $R^n$ whose boundary $\partial\Omega$ is sufficiently smooth. If, in addition, $a_i(x)$ is positive for each i and $c(x)$ is non-positive then all the results in theorem VII-4 hold.

Proof. Consider (VII-18)' as a special form of (VII-18) with $a_{ij}(x)=a_i(x)$ for $i=j$ and $a_{ij}(x)=0$ for $i\neq j$. Then the condition (VII-20) is satisfied since by hypothesis $\alpha = \min_{1\leq i\leq n} (\min_{x\in\bar{\Omega}} a_i(x))>0$ which implies

$$\sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i\xi_j = \sum_{i=1}^{n} a_i(x)\xi_i^2 \geq \alpha \sum_{i=1}^{n} \xi_i^2 .$$

The condition (VII-22) follows from the non-positivity of $c(x)$. Hence the results follow by applying theorem VII-4.

As an example of the above theorem, consider the equation

$$\frac{du}{dt} = \Delta\, u - c^2 u \qquad \text{(c real)}$$

where $\Delta$ is the Laplacian operator in $\Omega \subseteq R^3$ with $\partial\Omega$ sufficiently smooth. Then all the conditions in the above theorem are fulfilled since in this case $a_i(x) = 1$ for each i and $c(x) = -c^2$.

Just as in the case of one-dimensional space case, semi-linear equations of the form

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j}\right) + c(x)u + f(t,u) \qquad x \in \Omega \qquad \text{(VII-23)}$$

with the boundary conditions

$$u(t,x) \Big|_{\partial\Omega} = h(t,x') \qquad x' \in \partial\Omega \qquad \text{(VII-24)}$$

can similarly be treated where f is a function on $R^+ \times L^2(\Omega)$ to $L^2(\Omega)$. For the sake of application, we state a theorem which is a direct consequence of theorems VI-14 and VII-4.

Theorem VII-6. Suppose that the semi-linear equation (VII-23) with the boundary conditions

$$u(t,x') = 0 \qquad x' \in \partial\Omega \qquad \text{(VII-24)'}$$

possesses the same linear part as given in theorem VII-4. If for each $t \geqq 0$, f is uniformly Lipschitz continuous in u with Lipschitz constant k(t) where k(t) is a positive continuous function on $R^+$ satisfying $\sup_{t\geqq 0} k(t) < \beta$ with $\beta$ given by (VII-22); and if for each $u \in \mathcal{D}(A)$, f is uniformly Lipschitz continuous in t with Lipschitz constant $g(||u||)$ where g is a positive non-drecreasing function on $R^+$. Then

(a) For any $u_o(x) \in \mathcal{D}(A)$ there exists a unique solution of (VII-23) with $u(0,x)=u_o(x)$.

(b) An equilibrium solution (or a periodic solution), if it exists, is stable if $\sup_{t\geqq 0} k(t)=\beta$; and is asymptotically stable if $\sup_{t\geqq 0} k(t)<\beta$.

(c) A stability region of the equilibrium solution is $\mathcal{D}(A)$ which can be extended, in some sense, to the whole space $L^2(\Omega)$.

Remarks. (a) The conditions of uniform Lipschitz continuity imposed on f can be weakened by assuming that f satisfies the conditions (i), (ii) (or (ii)') and (iii) listed in section C of Chapter VI. (b) The continuity condition on k(t) can be weakened to allow discontinuous at a finite number of points on $R^+$ with k(t) properly defined at the points of discontinuity (see the remarks following theorem VI-7).

Example VII-6. As an example of the above theorem, consi[der] the partial differential equation

$$\frac{\partial u}{\partial t} = \Delta u - c^2 u + \frac{ku^2}{(\lambda^2+u^2)(c_1+c_2 t)} \qquad (c^2_, \lambda^2_, c_1, c_2 > 0) \qquad \text{(VII-25)}$$

with the boundary conditions

$$u(t, x') = 0 \qquad x' \varepsilon \, \partial \, \Omega$$

where $\Delta$ is the Laplacian operator in a bounded open set $\Omega$ in $R^3$ and $u=u(t,x)$ with $x=(x_1,x_2,x_3)$. The coefficients of $\Delta$ are $a_{ij}(x)=\delta_{i,j}$, the Kronecker delta, which implies that the condition (VII-20) is satisfied with $\alpha = 1$ since

$$\sum_{i,j=1}^{3} a_{ij}(x) \, \xi_i \xi_j = \sum_{i=1}^{3} \xi_i^2 \, .$$

Since $c(x) = -c^2 < 0$, the condition (VII-22) is satisfied. Hence all the hypotheses in theorem VII-4 are fullfilled with $\beta=\gamma+c^2$. It is easily shown that for any $u \, \varepsilon \, \mathcal{D}(A)$ and $s,t \geq 0$ (see example VII-3)

$$||f(t,u)-f(s,u)|| \leq \frac{|c_2 k|}{c_1^2} |s-t|$$

which shows that f is uniformly Lipschitz continuous in t with $g(||u||)= \frac{|c_2 k|}{c_1^2}$. By using the relation (VII-12), for each $t \geq 0$

$$||f(t,u)-f(t,v)|| = \left|\frac{k}{c_1+c_2 t}\right| \, \left|\left|\frac{\lambda^2(u^2-v^2)}{(\lambda^2+u^2)(\lambda^2+v^2)}\right|\right| =$$

$$= \left|\frac{k \, \lambda^2}{c_1+c_2 t}\right| \, \left( \int_\Omega \frac{(u+v)^2}{(\lambda^2+u^2)^2(\lambda^2+v^2)^2} (u-v)^2 \, dx\right)^{1/2} <$$

$$< \left|\frac{k\lambda^2}{c_1+c_2 t}\right| \frac{1}{|\lambda^3|} \left( \int_\Omega (u-v)^2 dx)\right)^{1/2} = \left|\frac{k}{\lambda(c_1+c_2 t)}\right| \, ||u-v||$$

which implies that f is uniformly Lipschitz continuous with Lipschitz constant

$$k(t) = \left|\frac{k}{\lambda(c_1+c_2t)}\right|.$$

Hence if $\sup_{t\geq 0} k(t) = |k/\lambda\ c_1| \leq \beta$, all the results in theorem VII-6 follow. In this particular case, $f(t,0)=0$ it follows that the null solution is the only equilibrium solution and is asymptotically stable.

In case the boundary conditions are given by (VII-24) where the function $h(t,x')$ is a continuously differentiable function of t on $R^+$ and twice continuously differentiable in x on all the (n-1)-dimensional subspace of $\bar{\Omega}$. On setting

$$v(t,x) = u(t,x) - h(t,x') \qquad x \in \bar{\Omega},\ x' \in \partial\,\Omega,$$

equation (VII-23) reduced to

$$\frac{\partial v}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial v}{\partial x_j}\right) + c(x)\,v + f_1(t,v) \quad (x \in \Omega) \qquad \text{(VII-23)'}$$

with the boundary conditions $v(t,x') = 0 \quad (x' \in \partial\,\Omega)$ where

$$f_1(t,v) = f(t,v+h) + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial h}{\partial x_j}\right) + c(x)h - \frac{\partial h}{\partial t} \qquad \text{(VII-26)}$$

which is a known function since both f and h are given functions. It follows that the nonhomogeneous boundary conditions can be reduced to the homogeneous boundary conditions as for the one-dimensional case from which theorem VII-6 may be used for the existence and stability of a solution. Knowing the property of the solution $v(t,x)$ in (VII-23)', the property of $u(t,x)$ of (VII-23) with boundary conditions (VII-24) can be deduced.

# VIII. CONCLUSIONS

## A. The Objective of the Research

The objective of this dissertation is to establish some criteria for the stability and the existence and uniqueness of solutions for some linear or nonlinear, time-invariant or time-varying operational differential equations (i.e., equations of evolution) from which stability criteria for the corresponding type of partial differential equations can be deduced. In the case of linear time-invariant differential equation, a Lyapunov stability theory for this type of equations in a real Banach space is established. By using the linear semi-group theory and by the introduction of semi-scalar product, the existence of a Lyapunov functional is shown. In addition, necessary and sufficient conditions for the generation of an equibounded or negative semi-group are obtained from which the existence and stability of a solution can be ensured.

In parallel to the linear semi-group theory, the introduction of nonlinear semi-group theory enables the extension of linear differential equations to nonlinear operational differential equations. A stability theory as well as the existence and uniqueness theory for nonlinear differential equations in a complex Hilbert space are established. Moreover, by introducing an equivalent inner product, the same results hold in an equivalent Hilbert space. This fact makes possible the construction of a Lyapunov functional through a sesquilinear functional which under suitable conditions defines an equivalent inner product and from which a stability criteria is obtained. In the special case of semi-linear differential equations, the known results on the linear part simplifies the criteria on a general nonlinear operator. Upon imposing some additional conditions on the nonlinear part which is an everywhere defined function, stability and existence of a solution are guaranteed. This type of equation is particulerly useful for some physical problems.

The development of the nonlinear time-invariant differential equation is further extended to a more general type of nonlinear time-varying operational differential equation. Criteria for the existence, uniqueness, stability and in particular, asymptotic stability of a solution, including the stability region, are obtained. The invariance of the existence and stability property of this type of equation in two equivalent Hilbert spaces is also proved. Particular attention has been paid to the nonlinear non-stationary operational differential equation. Some special cases of this type of equation possess many possibilities for applications to partial differential equations.

In order to apply the results obtained for the above mentioned type of operational differential equations to partial differential equations, some second order stationary and nonstationary equations in one-dimensional and in n-dimensional spaces are considered. These applications not only yield results on the type of partial differential equations under consideration but also illustrate some steps in the formulation of a linear operator in a Hilbert space from a formal partial differential operator. These steps may be needed in solving more general partial differential equations. In the following section, a brief description of the main results in this research are given.

## B. The Main Results

### 1. The Existence of a Lyapunov Functional

The linear time-invariant operational differential equations are investigated in Chapter IV. Through the use of an equivalent semi-scalar product, the existence of a Lyapunov functional in a Banach space is proved in theorems IV-7 and IV-8; and in terms of this Lyapunov functional, necessary and sufficient conditions on A to generate an equibounded and negative semi-group are established in theorems IV-11 and IV-12 respectively. With these additional results, the stability study of the linear time-invariant equations by using semi-group or group theory in a Banach space or a Hilbert space is (in a sense) completed. In addition to the above results, some interesting properties of semi-scalar product in terms of a semi-group are given in theorems IV-9 and IV-10, the proofs of which are based on an useful lemma (lemma IV-5) which is proved through the construction of a continuous linear functional.

### 2. Nonlinear Time-Invariant Operational Differential Equations

Linear time-invariant differential equations have been extended in Chapter V to nonlinear differential equations with the underlying space a complex Hilbert space. By introducing the concept of nonlinear semi-groups, stability criteria in terms of the infinitesimal generator of a nonlinear contraction semi-group are given in theorem V-2 and is extended to theorem V-3 for asymptotic stability. The proof of theorem V-3 is based on a very useful lemma which is shown as lemma V-5. These two theorems are fundamental for the development of stability theory. Moreover, the semi-group on $\mathcal{D}(A)$ generated by A in theorems V-2 and V-3 are extended into the closure of $\mathcal{D}(A)$ as is shown in lemma V-3. The inner product with respect to which the nonlinear operator A is dissipative required in theorem V-2 can be replaced by an equivalent inner product which is shown in theorem V-4. In this case, the semi-group generated by A is not necessarily contractive in the original space. However, from the stability point of view, there is no loss whatsoever of the stability property. This fact enables one to define a Lyapunov functional through a sesquilinear functional so that stability property can be determined by the construction of a Lyapunov functional. These results are obtained in theorems V-7 to V-9. In addition to the above results which are directly related to stability theory, lemma V-6, lemma V-10 and its corollary all have their own values. Moreover, theorem V-6 gives the necessary and sufficient conditions for the existence of an inner product equivalent to the given inner product of a complex Hilbert space. It should be remarked that theorem V-5 is an alternative form of theorems V-2 and V-3.

As a special case, the semi-linear equation is discussed with the underlying space a real Hilbert space. If the linear part is the infinitesimal generator of a semi-group of class $C_o$, then the existence, uniqueness, stability or asymptotic stability of a solution are established in theorems V-11, V-12 and their corollaries. Moreover, under some weaker conditions than those required in theorem V-12, the uniqueness of an equilibrium solution is established in theorem V-13 and a special case of the null solution is given in its corollary. This theorem is contributed in a large part by Dr. Vogt during the discussion between him and the author. In case the linear part is a closed operator, a general theorem for the existence, uniqueness and stability property is established in theorem V-15, and in the special case of a self-adjoint operator the results are given in theorem V-16. Finally, theorem V-17 shows that theorem V-16 remains true if the inner product of H is replaced by an equivalent inner product.

3. Nonlinear Time-Varying Operational Differential Equations

The nonlinear time-invariant differential equations are further extended in Chapter VI to the nonlinear time-varying differential equations. In parallel to the development of Chapter V, a stability criterion for the general equations of evolution is established in theorem VI-2. Through the use of lemma VI-3, theorem VI-2 is extended to an equivalent Hilbert space as is shown in theorems VI-3 and VI-4 for the stability and asymptotic stability resepctively. By defining a Lyapunov functional through a sesquilinear functional, theorems VI-3 and VI-4 are, in fact, equivalent to theorem VI-5. Additional properties are stated as corollaries 1 and 2.

An important special form of nonlinear time-varying equations is the nonlinear nonstationary differential equation which is also an extension of the nonlinear equation discussed in Chapter V. Theorems VI-6 and VI-7, which are very useful to the applications of concrete nonlinear partial differential equations, have established general criteria for the stability and asymptotic stability, respectively, of a solution.

Another special form of the nonlinear time-varying equations is the semi-linear equations. In the general case where the linear part is a time-varying unbounded operator, criteria for the stability and asymptotic stability of a solution are given in theorems VI-8 and VI-9 respectively. In case the linear part is time-invariant and if it is the infinitesimal generator of a semi-group of class $C_o$, theorems VI-10 and VI-11 give conditions for the existence, uniqueness and stability or asymptotic stability, respectively, of a solution. Theorem VI-12 shows the uniqueness of an equilibrium solution; if it is a closed unbounded linear operator, a general theorem is shown in theorem VI-13; when it is a self-adjoint operator either in the original Hilbert space H or in an equivalent Hilbert space $H_1$, conditions imposed on it turn out to be particularly simple, and these results are shown in theorems VI-14 and VI-15 which are very useful for the application of a class of partial differential equations. Finally, if the linear part is a bounded operator on H, the semi-linear equations is reduced to an ordinary differential equation. Results on this type of equations are given in theorems VI-16 to VI-19 which are direct consequences of the semi-linear equation.

4. Applications

Applications of the results developed for operational differential equations to partial differential equations are given in Chapter VII in which stability criteria for a class of second order partial differential equations are established and are given in theorems VII-2 through VII-6. These applications and special examples also illustrate some steps for solving the stability problem of certain partial differential equations through the use of the results for operational differential equations.

## C. Some Suggested Further Research

The stability theory developed in this research can be extended in two broader directions, namely; theoretical extensions to some more general function spaces such as Banach space on the one hand, and applications to the class of nonlinear partial differential equations which can be reduced to the form of operational differential equations on the other. As it has been shown in Chapter IV, that the stability criteria of linear time-invariant operational differential equations in Hilbert spaces can be extended to Banach spaces by the introduction of semi-scalar product. This suggests that through the use of semi-scalar product it might be possible to extend the stability and existence theory for nonlinear operational differential equations from Hilbert spaces to Banach spaces. It is believed that this extension is possible for some class of Banach spaces which are not Hilbert spaces. On the other hand, the results obtained for the operational differential equations can be used for a large class of nonlinear partial differential equations which are not limited to semi-linear equations. The formulation of a nonlinear operator in a suitable Hilbert space from a given nonlinear partial differential operator and the associated abstract operator possessing the desired property both need further investigation. One of the immediate extensions along this line is the formulation of a nonlinear partial differential operator of elliptic type as a nonlinear operator in some suitable function spaces such that this nonlinear operator has the required property to ensure the stability of a solution of the parabolic-elliptic partial differential equations. Moreover, applications to nonlinear wave equations and to Schrodinger equations also need additional attention.

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