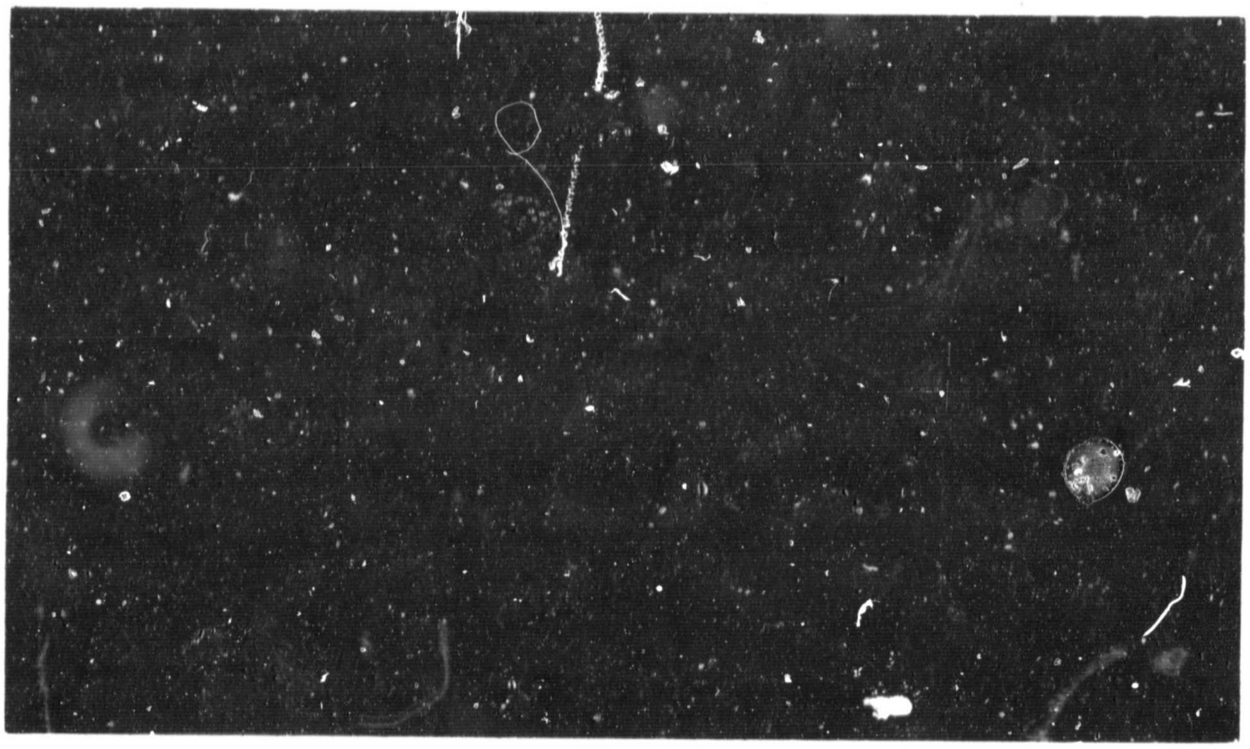
AD

PLASMA PHYSICS GROUP



33

Theory of Stability of Large Amplitude Periodic (BGK) Waves in Collisionless Plasmas

Martin V. Goldman

Physics Department, University of California at Los Angeles

June, 1969 R-52

Plasma Physics Group
Department of Physics
University of California
Los Angeles, California 90024

This paper was prepared for submittal to Physics of Fluids

Theory of Stability of Large Amplitude Periodic (BGK) Waves in Collisionless Plasmas

Martin V. Goldman

Department of Physics, University of California, Los Angeles, California 90024

Abstract

A method is developed for examining the stability of a large amplitude periodic (BGK) wave, E_o, in a collisionless plasma. Vlasov's equation is integrated by the method of characteristics to yield a polarization charge density response, ρ_1, linear in a small-amplitude field E_1, but nonlinear in E_o. The susceptibility linking ρ_1 and E_1 is expressed in terms of the exact orbits of trapped and untrapped particles in the field E_o, distributed in energy according to the assumed BGK distribution function, f_o. These susceptibilities couple the Fourier components of E_1 in the usual mode-coupling fashion, but trapping effects are now included. For fields E_o which are not too large, the mode-coupling problem reduces to finding the zeroes of a 2×2 or 3×3 determinant. Distribution functions which are localized at the bottom of the potential energy troughs of E_o give identically the growing side-band instability of Dawson, Kruer, and Sudan.(3)

I. INTRODUCTION

In 1957, Bernstein, Greene, and Kruskal[1] showed that for a given large amplitude stationary traveling wave in a collisionless plasma it is possible to construct many distribution functions which are consistent solutions to Vlasov's equation. They raised the question: which, if any, of these solutions are stable? This paper describes a procedure for testing (in principle) the stability of a given BGK distribution function and periodic wave. The amplitude of the wave can be large enough to trap particles, but not so large as to drive modes at frequencies very far from the linear resonances of the plasma.

Our approach was motivated by recent experimental and theoretical work[2-4] on large amplitude sinusoidal waves. Wharton, Malmberg, and O'Neil[2] measured the spatial decay of a large amplitude wave driven by R.F. voltage on a probe. As the wave decayed spatially, growing upper and lower side-band waves were observed, displaced from the original wave by roughly a bounce frequency $(eE_o k_o/m)^{1/2}$ for electrons trapped in the potential energy troughs of the original wave. Dawson, Kruer, and Sudan[3] proposed a simple theory to account for the side bands. They computed the charge density produced by a "bunched beam" of trapped oscillating electrons in a simple way, ignoring the distribution in number and bounce frequency of trapped electrons as a function of their total energy. This charge density was then shown to couple the side bands and make them unstable. The mechanism has some similarity to a two stream instability.

The present theory develops, in terms of the Vlasov equation, the idea that the charge density of trapped particles or particles with perturbed orbits can couple waves and lead to instability. We present a linear stability theory of small perturbations about any assumed BGK distribution function, f_{o}, and the periodic stationary traveling wave E_o consistent with it. In this way the stability of various BGK distribution functions can (in principle) be studied.

There are two main ingredients of the theory. The method of characteristics is used to calculate from Vlasov's equation the charge density in terms of the exact orbits of particles in the field of E_o, distributed in energy according to the assumed BGK distribution function, f_o. This charge density is proportional, through electrostatic susceptibilities, to a sum of Fourier components of the perturbing field E_1 (differing by multiples of the frequency and wavenumber of E_o). The susceptibilities which connect the charge density to the field contain important orbital effects such as trapping due to E_o. Poisson's equation then produces an infinite matrix of such susceptibilities coupling the Fourier components of E_1. If the field E_o is not too large, only the components of E_1 which have frequencies near the linear modes of the plasma in the absence of E_o need be retained. This is the customary truncation procedure of mode-coupling theories[6,12] and leads to a 2×2 or 3×3 determinant whose zeroes determine the stability of the system. The entries in this determinant contain the orbital modifications, such as trapping, modified Landau damping, etc.

The mode-coupling determinants simplify for trapped particle distribution functions which are localized at the bottom of the potential energy troughs of E_o. In this limit the growing sideband instability of Dawson, Kruer, and Sudan[3] is obtained. The sideband instability has the character of a parametric instability, in which modulation of the susceptibility at $2\omega_o$ drives the sidebands (whose frequencies add up to $2\omega_o$).

II. GENERAL PERTURBATIONS ABOUT "INHOMOGENEOUS EQUILIBRIA"

Consider a one-dimensional steady state in which there is an arbitrary (not necessarily periodic) stationary traveling electrostatic wave $E_o(x,t)$. In the wave frame electrons and ions obey the Vlasov equation and Poisson's equation:

$$v \frac{\partial f_{os}}{\partial x} + \frac{q_s E_o(x)}{m_s} \frac{\partial f_{os}}{\partial v} = 0$$

$$\frac{\partial E_o}{\partial x} = 4\pi \int_{-\infty}^{+\infty} dv \sum_s q_s f_{os}(x,v) \tag{1}$$

The distribution functions f_{os} are functions only[1] of the sign of v and of the energy $W_s = \frac{1}{2} m_s v^2 + q_s \phi_o(x)$, where $-\nabla\phi_o = E_o$:

$$f_{os}(W, \mathrm{sgn}(v)) = \Theta(v)\, f_{os}^{>}(W_s) + \Theta(-v)\, f_{os}^{<}(W_s) \tag{2}$$

where $\theta(v) = +1$ if $v > 0$, and $= 0$ if $v < 0$. Bernstein, Greene, and Kruskal[1] showed that for any given $E_o(x)$ a variety of self-consistent distribution functions f_{os} may be constructed (in principle).[5] Assume a pair of functions f_{oe}, f_{oi} have been so constructed. This is our (inhomogeneous) "equilibrium" and the object is to study the linear stability of small test waves $E_1(x,t)$ and perturbations $f_{1s}(x,v,t)$ about E_o and f_{os} . Thus,

$$\frac{\partial f_{1s}}{\partial t} + v\frac{\partial f_{1s}}{\partial x} + \frac{q_s E_o(x)}{m_s}\frac{\partial f_{1s}}{\partial v} = -\frac{q_s}{m_s}E_1(x,t)\frac{\partial f_{os}}{\partial v} \tag{3}$$

$$\frac{\partial}{\partial x}E_1(x,t) = 4\pi\left[\rho_{1e}(x,t) + \rho_{1i}(x,t)\right], \tag{4}$$

where ρ_{1s} is the perturbed charge density of either species:

$$\rho_{1s}(x,t) = q_s\int_{-\infty}^{+\infty} dv\, f_{1s}(x,v,t) \tag{5}$$

We view the right hand side of Eq. (3) as an arbitrary source [i.e., the constraint of Eqs. (4) and (5) on the class of possible fields E_1 is deferred until later--see Eq. (16)]. Equation (3) is then solved by the following causal Green's function:

$$G_s(x,x';v,v';t-t') =$$

$$\Theta(t-t')\,\delta(x-x^s_{orb}(t-t';x',v'))\,\delta(v-v^s_{orb}(t-t';x',v')) \tag{6}$$

where x^s_{orb} and v^s_{orb} are the phase space coordinates of a particle orbit due to the force $q_s E_o(x_{orb})$, with initial conditions (x',v') at time $t-t' = 0$:

$$\begin{aligned} \ddot{x}^s_{orb} &= \frac{q_s}{m_s}E_o(x^s_{orb}), \qquad & v^s_{orb} &= \frac{\partial x_{orb}}{\partial t} \\ x^s_{orb}(0;x',v') &= x', \qquad & v^s_{orb}(0;x',v') &= v' \end{aligned} \tag{7}$$

It is easy to verify by direct differentiation that

$$\frac{\partial G}{\partial t} + v\frac{\partial G}{\partial x} + \frac{q}{m}E_0(x)\frac{\partial G}{\partial v} = \delta(t-t')\,\delta(x-x')\,\delta(v-v'), \tag{8}$$

so that

$$f_1(x,v,t) = -\frac{q}{m}\int dt'\,dv'\,dx'\; G(x,x';v,v';t-t')\,\frac{\partial f_0(x',v')}{\partial v'}\,E_1(x',t')$$

(from now on we omit the species subscripts s).

The use of this Green's function is closely related to the method of characteristics. Upon integration over velocities v , we have for the charge density of either species,

$$4\pi\rho_1(x,t) = -\int dx'\,dt'\,\frac{\partial \chi}{\partial x}(x,x';t-t')\,E_1(x',t'), \tag{9}$$

where we have defined an electrostatic susceptibility χ for each species by

$$-\frac{\partial \chi}{\partial x}(x,x';t-t') = \Theta(t-t')\frac{4\pi q^2}{m}\int_{-\infty}^{+\infty} dv'\,\frac{\partial f_0}{\partial v'}(x',v')\,\delta(x-x_0(t-t';x'v')) \tag{10}$$

Equation (9) may be viewed as a linear constitutive relation for the polarization charge density ρ_1 in terms of the total field E_1. All of the nonlinear effects of $E_o(x)$ are contained in the electrostatic susceptibility χ, which therefore characterizes the steady state or "inhomogeneous equilibrium" about which we are perturbing. All fields and charge densities are in the wave frame. In particular, $E_o(x)$ is time-independent in this frame. This has allowed us to assume that x_{orb} and χ will depend only on the time difference, $t-t'$.

Fourier analyzing Eq. (9) in time:

$$4\pi \rho_1(x,\omega) = -\int_{-\infty}^{+\infty} dx' \frac{\partial}{\partial x} \chi(x,x';\omega) E_1(x',\omega) \qquad (11)$$

The stability properties of the perturbation field E_1 are studied (in principle) by inserting (11) into the time Fourier transform of Poisson's equation (4). The resulting integral equation for $E_1(x,\omega)$ is probably a useful starting point for investigating the stability of non-periodic BGK solutions such as solitary waves, against small perturbations. In the next section we specialize to the case of periodic BGK solutions.

III. PERTURBATIONS ABOUT SPATIALLY PERIODIC EQUILIBRIA

Let $E_o(x)$ be an arbitrary spatially periodic BGK field in the wave frame with period λ_o. An example might be $E_o(x) = E_o \sin k_o x$, where $k_o = 2\pi/\lambda_o$, but the analysis below is perfectly general. Since the "equilibrium" is no longer translationally invariant, the susceptibility $\chi(x,x';\omega)$ is not a function of the combination $x-x'$ alone. We shall now prove that it is a function of the two independent variables $x-x'$ and x' and as a function of the latter it is periodic with period λ_o. The right hand side of Eq. (10) is proportional to

$$\int_{-\infty}^{+\infty} dv' \frac{\partial f_o}{\partial v'}(W, \operatorname{sgn} v')\, \delta\left([x-x'] - (x_{orb}(t-t';x'v') - x')\right),$$

where $W = \frac{1}{2} m(v')^2 + q\phi_o(x')$. The dependence of the above function on $x-x'$ is in the square bracket. The remaining dependence is on x'. The x'-dependence contained in f_o is obviously periodic since $\phi_o(x')$ is periodic. The x'-dependence contained in $x_{orb}(t-t';x',v') - x'$ is also periodic, as we see by inspection of Eq. (7) for x_{orb} $[x_{orb}(t-t';x'+n\lambda_o,v') = x_{orb}(t-t';x',v') + n\lambda_o]$. This suggests a Fourier series expansion,

$$\chi(x-x';x';\omega) = \sum_{n=-\infty}^{+\infty} \chi_n(x-x';\omega) e^{-ink_o x'} \tag{12}$$

$$\chi_n(x-x';\omega) = \int_{-\frac{\lambda_0}{2}}^{+\frac{\lambda_0}{2}} \frac{dz}{\lambda_0} e^{ik_0 nz} \chi(x-x';z;\omega) \tag{13}$$

followed by a Fourier integral transformation of Eq. (11):

$$\rho_1(k,\omega) = \int_{-\infty}^{+\infty} dx\, e^{-ikx} \rho_1(x,\omega)$$

$$4\pi\rho_1(k,\omega) = -ik \sum_{n=-\infty}^{+\infty} \chi_n(k,\omega) E_1(k+nk_0,\omega) \tag{14}$$

The components $E_1(k+nk_0,\omega)$ may be understood as the components of a Bloch function $E_1(x) = e^{ikx} \sum_n E_1(k+nk_0)\, e^{ink_0x}$. The steps leading to Eq. (14) essentially demonstrate Bloch's theorem. From Eqs. (13) and (10) and the preceding discussion, the susceptibilities $\chi_n(k\omega)$ are given by

$$\chi_n(k,\omega) = \frac{-4\pi e^2 i}{mk} \int_{-\infty}^{+\infty} dv' \int_{-\frac{\lambda_0}{2}}^{+\frac{\lambda_0}{2}} \frac{dx'}{\lambda_0} \int_0^{\infty} dt\, e^{i\omega t} e^{ix'(nk_0+k)} e^{-ikx_{ow}(t,x'v')} \frac{\partial f_0(x';v')}{\partial v'}, \quad (\mathrm{Im}\,\omega > 0) \tag{15}$$

This expression is valid for either species, provided the correct mass and distribution function for that species is inserted.

Equation (14) may be understood in simple physical terms. As a consequence of the spatial periodicity in the wave frame, the susceptibility χ is periodically modulated, allowing all components $E_1(k+nk_0)$ to produce polarization charge density which can then act through Poisson's equation as a source for $E_1(k)$. In this way all the Fourier components $E_1(k+nk_0)$ are coupled together. Such effects are well known in solids[7] and have been studied (without trapping effects) in plasmas[8] with simple models for susceptibilities χ_n representing non-propagating periodic stationary states. Equation (14) is even more illuminating in the laboratory frame. If x_w is position in the wave frame, laboratory and wave frame quantities are related by:

$$\rho_1^{LF}(x_w+v_0t\,;t) = \rho_1(x_w,t)$$

$$E_1^{LF}(x_w+v_0t\,;t) = E_1(x_w,t)$$

In Fourier space, therefore, the laboratory frame charge density, ρ_1^{LF} , and field, E_1^{LF} , are obtained by making the appropriate Doppler shifts:

$$\rho_1^{LF}(k,\omega+kv_0) = \rho_1(k,\omega)$$

$$E_1^{LF}(k,\omega+kv_0) = E_1(k,\omega)$$

Equation (14) in the laboratory frame reads:

$$4\pi\rho_1^{LF}(k,\omega) = -ik\sum_{n=-\infty}^{+\infty}\chi_n(k,\omega-kv_0)\,E_1^{LF}(k+nk_0,\omega+n\omega_0) \qquad (14a)$$

The Doppler shifts produce coupling between waves at different frequencies $\omega + n\omega_0$ and the associated wave numbers. As is well known, such coupling can produce parametric instability of the waves E_1 . D. F. DuBois and Goldman have previously computed[9-12] susceptibilities χ_n for a collisionless plasma in a sinusoidal field, neglecting contributions from trapped electrons.

The present treatment includes such effects which can be important for certain [illegible] distribution functions. Equation (14a) combined with the Fourier transform of Poisson's equation (4) yields:

$$E_1^{LF}(k,\omega) + \sum_{n=-\infty}^{+\infty} \chi_n^{TOTAL}(k,\omega-kv_0)\, E_1^{LF}(k+nk_0,\omega+n\omega_0) = 0, \tag{16}$$

where χ_n^{TOTAL} is the sum of electron and ion susceptibilities,

$$\chi_n^{TOTAL}(k,\omega-kv_0) = \chi_n^{ION}(k,\omega-kv_0) + \chi_n^{ELECTRON}(k,\omega-kv_0) \tag{17}$$

In Eq. (16) send (k,ω) into $(k+\ell k_o,\ \omega+\ell\omega_o)$ and change the sum over n into a sum over $s = n+\ell$. This generates a matrix equation for coupled normal modes E_1:

$$\sum_{s=-\infty}^{+\infty} \epsilon_{\ell s}(k,\omega)\, E_1^{LF}(k+sk_0,\ \omega+s\omega_0) = 0, \tag{18}$$

where

$$\epsilon_{\ell s}(k,\omega) = \delta_{\ell,s} + \chi_{\ell s}^{TOTAL}(k,\omega) \tag{19a}$$

$$\chi_{\ell,s}^{TOTAL}(k,\omega) = \chi_{s-\ell}^{TOTAL}(k+\ell k_0,\omega-kv_0) \tag{19b}$$

For one species, Eq. (15) yields

$$\chi_{\ell,s}(k,\omega) =$$

$$-\frac{4\pi e^2}{m[k+\ell k_0]}\int_{-\infty}^{+\infty}dv'\int_{-\lambda_0/2}^{+\lambda_0/2}\frac{dx'}{\lambda_0}\int_0^{\infty}dt\, e^{i(\omega-kv_0)t}\, e^{ix'(k+sk_0)}\, e^{-i(k+\ell k_0)x_{orb}(t;x'v')}\,\frac{\partial f_0}{\partial v'} \qquad (19c)$$

The time integral is well defined only for $\mathrm{Im}\,\omega > 0$.

The general stability properties of the field E_1 are contained in the zeroes of the infinite determinant $\varepsilon_{\ell s}$:

$$\det \varepsilon_{\ell s}(k,\omega) = 0 \qquad (20)$$

Of course, in the absence of the periodic field E_0 , the system is homogeneous, the susceptibility a function only of $x-x'$ and therefore

$$\lim_{E_0 \to 0} \chi^{TOTAL}_{s-\ell}(k+\ell k_0, \omega-kv_0) = \delta_{s,\ell}\, \chi^{T}_{HOM}(k,\omega)$$

where $\chi^{T}_{HOM}(k,\omega)$ is the usual total linear homogeneous suceptibility of the collisionless plasma. For either species

$$\chi_{HOM}(k,\omega) = \frac{4\pi e^2}{m k^2}\int_{-\infty}^{+\infty}dv\,\frac{k\,\partial f_0(v)/\partial v}{\omega - kv + i\epsilon} \qquad (21)$$

In order to make progress with the infinite determinant we must now specialize to the case of fields with energy small compared to particle thermal energies:

$$E_o \ll \sqrt{4\pi nkT} \quad .$$

In this limit the zeroes of the infinite determinant (20) cannot be far removed from the zeroes of the homogeneous system, given by

$$\epsilon(k,\omega) = 1 + \chi^T_{Hom}(k,\omega) = 0$$

This means the infinite system of Eq. (18) may be truncated, retaining only the components $E_1^{LF}(k+sk_o, \omega+s\omega_o)$ which are near the zeroes of ϵ_{HOM}. Thus, if we look for a root ω which is less than ω_o but near the plasma frequency, $\omega-\omega_o$ may be near the (negative) acoustic mode, $-V_A|k-k_o|$, and $\omega-2\omega_o$ will be near the (negative) plasma frequency, $-[\omega_p^2 + 3V_T^2(k-2k_o)^2]^{1/2}$. All other frequencies $\omega \pm s\omega_o$ correspond to multiples of the plasma frequency, where the linear system has no resonances.

The infinite matrix equation can then be replaced[6,12] by a 3 × 3 matrix equation of a familiar type from the theory of four-mode parametric interaction[12]:

$$\begin{pmatrix} 1+\chi^{TOT}_{-2,-2} & \chi^{TOT}_{-2,-1} & \chi^{TOT}_{-2,0} \\ \chi^{TOT}_{-1,-2} & 1+\chi^{TOT}_{-1,-1} & \chi^{TOT}_{-1,0} \\ \chi^{TOT}_{0,-2} & \chi^{TOT}_{0,-1} & 1+\chi^{TOT}_{0,0} \end{pmatrix} \begin{pmatrix} E_1(k-2k_o,\omega-2\omega_o) \\ E_1(k-k_o,\omega-\omega_o) \\ E_1(k,\omega) \end{pmatrix} = 0 \qquad (22)$$

The field E_o must be small enough to guarantee that all off-diagonal matrix elements are $\ll 1$, for this approximation to be valid. This 3 × 3 determinant differs from the usual mode-coupling determinant such as described in Ref. 10 in the following respect: The susceptibilities contain contributions to all orders in E_o, not just perturbative effects of order $(E_o^2/nkT)^{1/2}$ and E_o^2/nkT, which come from high energy orbits. In particular, they contain contributions from electrons trapped in the potential energy troughs of E_0. These contributions will be seen in Eq. (36) to be of the form of an integral and a sum over generalized oscillators similar to the type proposed by Dawson, Kruer, and Sudan[3] in their elementary treatment of the side bands observed in the experiments of Wharton, Malmberg, and O'Neil.[2] Figure 1 shows the coupled modes which generally must be included: a lower and upper side band and a low frequency mode. The Dawson-Kruer-Sudan theory omits the low frequency mode. This is valid only if the frequency difference between ω_0 and a side band frequency does not lie in the vicinity of acoustic mode frequencies.

We wish to turn now to the structure of the susceptibilities and express the contributions from the various orbits in a useful and revealing form.

IV. EVALUATION OF SUSCEPTIBILITIES

Equation (19c) may be understood by examination of Fig. 2, which shows the phase space trajectories of particles in the periodic field $E_0(x)$ (e.g., a sinusoidal function). The right hand side of (19c) is a Fourier transform over positive times of the orbit function $\exp[-i(k+\ell k_0)\, X_{orb}(t;x'v')]$ followed by a weighted average over initial phase space points (x',v') in the shaded region of phase space. Two kinds of orbits occur: Those which remain in the (darker) shaded region, corresponding to trapped particles, and those untrapped particles which may leave the (lighter) shaded region. The orbit is classified by the total energy,

$$W = \frac{1}{2} m(v')^2 + V(x') , \qquad V(x') = q\,\phi_0(x') \tag{23}$$

which is fixed by the initial phase space point, (x',v'). Trapped particles have energy

$$V_{min} \le W < V_{max} \tag{24}$$

while untrapped particles have energies

$$V_{max} < W \tag{25}$$

The susceptibility χ_n can therefore be broken up into a contribution from trapped and untrapped orbits:

$$\chi_{\ell s}(k\omega) = \chi_{\ell s}^{TRAPPED}(k\omega) + \chi_{\ell s}^{UNTRAPPED}(k\omega) \tag{26}$$

With some manipulation we may rewrite Eq. (19c) in terms of an energy integral:

$$\chi^{TRAPPED}_{\ell s}(k,\omega) = \frac{-4\pi e^2 i}{m[k+\ell k_0]} \int_{V_{min}}^{V_{max}} dW \int_{-x_W}^{+x_W} \frac{dx'}{\lambda_0} \int_0^{\infty} dt\, e^{i(\omega - k v_0)t} \,.$$

$$\cdot\, e^{i x'(k+sk_0)} \frac{\partial f_0(W)}{\partial W} \left[e^{-i(k+\ell k_0) x_{os}(t;x',W)} - e^{-i(k+\ell k_0) x_{os}(-t;x',W)} \right] \quad (27)$$

$$\chi^{UNTRAPPED}_{\ell s}(k,\omega) = \frac{-4\pi e^2 i}{m[k+\ell k_0]} \int_{V_{max}}^{\infty} dW \int_{-\lambda_0/2}^{+\lambda_0/2} \frac{dx'}{\lambda_0} \int_0^{\infty} dt\, e^{i(\omega - k v_0)t} e^{i x'(k+sk_0)} \,.$$

$$\cdot \left[\frac{\partial f_0^{>}(W)}{\partial W} e^{-i(k+\ell k_0) x_{os}(t;x',W)} - \frac{\partial f_0^{<}(W)}{\partial W} e^{-i(k+\ell k_0) x_{os}(-t;x',W)} \right] \quad (28)$$

$$(\mathrm{Im}\, \omega > 0)$$

This transformation from initial velocity to positive initial velocity to energy variable makes use of the following relations:

$$x_{os}(t; x', -v') = x_{os}(-t; x', v') \,, \quad (29)$$

$$\left[\begin{array}{c} f_0(x',v') = f_0(x',-v') \quad \text{or} \quad f_0^{>}(W) = f_0^{<}(W) = f_0(W) \\ \underline{\text{in the trapped region only.}} \end{array} \right] \quad (30)$$

Equation (29) says that reversal of a particle's initial velocity is equivalent to letting it run backwards in time. Equation (30) follows from the time independence of the distribution function in the trapped region and is discussed by Bernstein, Greene, and Kruskal.[1] The notation $f_0^{\pm}(W)$ is defined in Eq. (2). All the orbits in (27) and (28) have positive initial velocity. $\pm x_W$ are the turning points for an orbit of energy W.

We now make use of the periodic nature of the orbits. A solution for x_{orb} follows from the first integral of the equations of motion:

$$\int_{x'}^{x_{orb}(t;x'v')} dx \frac{1}{\left[\left(\frac{2}{m}\right)\left(W-V(x)\right)\right]^{1/2}} = t, \quad \text{for } v'>0, \text{ and } x_{orb} \text{ before a turning point.} \tag{31}$$

There are two characteristic times. Let $\tau_T(W)$ be the "bounce" time for the untrapped particle to complete one period of its motion, and $\omega_T(W)$ the corresponding bounce frequency:

$$\tau_T(W) \equiv \frac{2\pi}{\omega_T(W)} = 4\left(\frac{m}{2}\right)^{1/2}\int_0^{x_W} dx \frac{1}{\left[W-V(x)\right]^{1/2}}, \quad V(x_W) = W \tag{32}$$

For untrapped orbits we define a time τ_u and an average velocity $u(W)$

$$\tau_u(W) \equiv \frac{\lambda_0}{u(W)} = \left(\frac{m}{2}\right)^{1/2}\int_{-\lambda_0/2}^{+\lambda_0/2} dx \frac{1}{\left[W-V(x)\right]^{1/2}} \tag{33}$$

For untrapped orbits,

$$x_{orb}(t;x'v') - u(W)t$$

is a periodic function of time t with period τ_u. The periodicity in the two regions allows us to make Fourier series expansions in time. In the trapped region,

$$e^{-i(k+\ell k_o)x_{orb}(t)} = \sum_{n=-\infty}^{+\infty} A_n^{\ell} e^{-in\omega_T t}$$

$$A_n^{\ell} = \frac{1}{\tau_T}\int_0^{\tau_T} dt\, e^{-i(k+\ell k_o)x_{orb}(t)} e^{in\omega_T t} \tag{34}$$

In the untrapped region,

$$e^{-i(k+\ell k_o)x_{orb}(t)} = \sum_{n=-\infty}^{+\infty} B_n^{\ell} e^{-i[k+(\ell+n)k_o]u(W)t}$$

$$B_n^{\ell} = \frac{1}{\tau_u}\int_0^{\tau_u} dt\, e^{-i[k+\ell k_o]x_{orb}(t)} e^{i(k+[\ell+n]k_o)u(W)t} \tag{35}$$

The coefficients A_n and B_n are functions of W and x' since x_{orb} depends on these quantities. With these expansions, the time integrations in Eqs. (27) and (28) may be performed. We obtain expressions for the susceptibilities in the classic form of a sum over oscillators.

$$\chi_{\ell,s}^{TRAPPED}(k,\omega) = -\frac{4\pi e^2}{m}\sum_{n=1}^{\infty}\int_{V_{min}}^{V_{max}} dW \frac{\mathcal{F}_{\ell s,n}(W)}{(\omega-kv_o)^2 - n^2\omega_T(W)^2}, \quad \mathrm{Im}\,\omega > 0 \tag{36}$$

$$ {}^{o}\mathcal{F}_{\ell s,n}(W) = \frac{-2m\,\omega_n(W)}{k+\ell k_0}\,\frac{\partial f_0}{\partial W}\int_{-\lambda_W}^{+\lambda_W}\frac{dx'}{\lambda_0}\,e^{ix'(k+s k_0)}\left(\Lambda_n^{\ell}(W,x')\cdot\Lambda_n^{\ell}(W,x')\right) \tag{37} $$

$\mathcal{F}_{\ell s,n}(W)$ is an oscillator strength for the oscillator of frequency $n\omega_n(W)$

$$ X^{\text{UNTRAPPED}}_{\ell,s}(k,\omega) = -\frac{4\pi e^2}{m}\sum_{n=-\infty}^{+\infty}\int_{V_{max}}^{\infty} dW \cdot $$

$$ \cdot\left[\frac{F^{>}_{\ell s,n}(W)}{\omega-kv_0-(k+[\ell+n]k_0)u(W)} - \frac{F^{<}_{\ell s,n}(W)}{\omega-kv_0+(k+[\ell+n]k_0)u(W)}\right], \quad \mathrm{Im}\,\omega > 0 \tag{38} $$

$$ F^{\gtrless}_{\ell s,n}(W) = \frac{-1}{k+\ell k_0}\int_{-\lambda_0/2}^{+\lambda_0/2}\frac{dx'}{\lambda_0}\,e^{ix'(k+s k_0)}\,\frac{\partial f_0^{\gtrless}}{\partial W}\,B_n^{\ell}(W,x') \tag{39} $$

V. CONTRIBUTIONS FROM THE BOTTOM OF THE WELL

It is useful to completely evaluate the portion of $\chi^{TRAPPED}_{\ell,s}(k,\omega)$ which comes from energies near the potential well minimum. Suppose there is one minimum per period λ_o at $x = 0$ and we indicate the susceptibility arising from oscillators near this minimum by

$$\chi^{BOT}_{\ell,s}(k,\omega) = -\frac{4\pi e^2}{m}\sum_{n=1}^{\infty}\int_{V_{min}}^{V_c} dW \frac{F_{\ell s n}(W)}{(\omega - kv_0)^2 - n^2\omega_B^2(W)}, \quad \mathrm{Im}\,\omega > 0, \tag{40}$$

where V_c is an energy cut-off $\ll V_{max}$. We can evaluate the oscillator strengths for these low energies since turning points x_W are $\ll \lambda_o$, the potential is nearly parabolic, and the phase space orbits are nearly circular:

$$V(x) \cong V_{min} + \frac{1}{2}x^2 V''(0)$$

$$x_W \cong a\sin(\omega_B t + \theta), \qquad a^2 \equiv 2(W - V_{min})/V''(0)$$

$$\frac{2\pi}{\tau_B} \equiv \omega_B \equiv \left(\frac{V''(0)}{m}\right)^{1/2}, \qquad a\sin\theta = x' \tag{41}$$

The Fourier coefficient A_n may be evaluated with the aid of the Bessel function expansion $e^{-ib\sin\phi} = \sum_{n=-\infty}^{+\infty} J_n(b)\, e^{-i\phi n}$:

$$A_n^{\ell} = e^{-in\theta} J_n(|k+\ell k_0|a) \tag{42}$$

The oscillator strength integral may be evaluated by changing to θ as variable and employing the Bessel function identity $b[J_{n-1}(b) + J_{n+1}(b)] = 2nJ_n(b)$. We find

$$F_{\ell s,n}(W) = -(2\pi)^2 \omega_B k_0 \frac{\partial f_0}{\partial W} \frac{J_n(|k+\ell k_0|a)}{|k+\ell k_0|} \frac{J_n(|k+sk_0|a)}{|k+sk_0|} \tag{43}$$

The special case of most interest is when $(k+\ell k_0)a$ and $(k+sk_0)a$ are $\ll 1$. The Bessel function expansions then yield $F_{\ell s,n} \propto [(k+\ell k_0)a]^n \, [(k+sk_0)a]^n/(2^n \, n!)^2$, so that the $n = 2$ oscillator has a smaller oscillator strength than the $n = 1$ oscillator by a factor $\{(k+\ell k_0)a \, (k+sk_0)a\}/16$. Ignoring terms of this order, we find simply

$$F_{\ell s,n}(W) = -\delta_{n,1} \omega_B k_0 \frac{\partial f_0}{\partial W} a^2 = -2\delta_{n,1} \frac{\omega_B k_0}{V''(0)} (W - V_{min}) \frac{\partial f_0}{\partial W} \tag{44}$$

This is independent of the matrix indices ℓ and s and includes only the $n = 1$ oscillator. We can now integrate Eq. (40) to obtain

$$\chi^{MIN}_{\ell,s}(k,\omega) = -\frac{4\pi e^2 \mathcal{N}_T}{m} \frac{1}{(\omega - kV_o)^2 - \omega_B^2} ,$$

(45)

where the effective "number" of trapped oscillators

$$\mathcal{N}_T = \frac{-2\omega_B k_o}{V''(0)} \int_{V_{min}}^{V_c} dW \frac{\partial f_o}{\partial W} \cdot (W - V_{min})$$

$$= \frac{2\omega_B k_o}{V''(0)} \left[\int_{V_{min}}^{V_c} dW \cdot f_o(W) - f_o(V_c) \cdot (V_c - V_{min}) \right] \qquad (46)$$

The integration of Eq. (40) to obtain (45) assumes an integrand denominator which does not get too small. This condition is $|(\omega-kv_o)^2 - \omega_B^2| \gg \omega_B^2 (V_c - V_{min})/(V_{max} - V_{min})$. Also, the truncation procedure of the infinite determinant assumes a denominator large enough so that $|X^{MIN}_{\ell,s}| \ll 1$. Both assumptions must be verified a posteriori, and can be satisfied for the cases discussed below. The susceptibility of Eq. (45) is identical in form to the trapped particle susceptibility proposed by Dawson, Kruer, and Sudan[1] on the basis of a bunched beam approximation. It is here derived from the Vlasov equation. If the entire trapped particle distribution function is completely localized to vicinity of the potential energy minima of $E_o(x)$, V_c may be taken equal to V_{max} , and the above susceptibility gives the total contribution of trapped particles. $E_o(x)$ may be a sinusoidal, distorted sinusoidal, or other periodic wave. The stability of BGK modes with more general trapped particle distribution functions will be the subject of a forthcoming paper with

H. Berk. Preliminary investigations show that distributions localized at the minima are consistent with almost-sinusoidal BGK waves, $E_o(x)$. When the distribution function extends to the top of the well, many more oscillators participate, the energy dependence of their bounce frequency becomes important and the oscillator strengths must be evaluated using the true orbits. In the case of a sinusoidal E_o these orbits can be expressed in terms of elliptic functions.

Finally, there is the matter of the untrapped susceptibilities $\chi_{\ell,s}^{UNTRAPPED}(k,\omega)$ expressed in Eq. (28). If the orbits just above V_{max} are neglected (e.g., $f_o(W)$ or $\partial f_o/\partial W$ is zero or small for $W \gtrsim V_{max}$) then the off-diagonal $\chi_{\ell,s}^{UNTRAPPED}$ are proportional to powers of $\Lambda_o = (E_o^2/4\pi nkT)^{1/2}$, just as in the ordinary parametric mode-coupling theory.(12) Such terms are obtained by doing perturbation theory in E_o and E_1 about, say, a Maxwellian distribution of non-resonant electrons with wave-frame energy much greater than V_{max}. In terms of orbit theory, such contributions come from the slightly perturbed straight-line trajectories of electrons which see a "weak" periodic field E_o. (The perturbation theory breaks down for orbits having energy W close to V_{max}, and the present theory takes such orbits into account properly.) The diagonal untrapped susceptibilities contain terms of order E_o^2, etc., as well as the linear susceptibilities (zeroth order in E_o) given in Eq. (21). The latter, of course, come from unperturbed straight-line orbits.

The Dawson-Kruer-Sudan dispersion relation is obtained by neglecting coupling to the low frequency mode [second column and row in the determinant of Eq. (22)], including only single-oscillator contributions such as Eq. (45) to $\chi_{ij}^{TRAPPED}$, and working to zeroth order in E_o in all $\chi_{ij}^{UNTRAPPED}$. The resulting 2×2 determinant is

$$\begin{vmatrix} \epsilon(k,\omega) + \chi^T(k,\omega) & \chi^T(k,\omega) \\ \chi^T(k,\omega) & \epsilon(k-2k_o, \omega-2\omega_o) + \chi^T(k,\omega) \end{vmatrix} = 0$$

where $\chi^T = -\omega_{pT}^2/(\omega-kv_o)^2 - \omega_B^2$, $\omega_{pT}^2 \equiv 4\pi n_T e^2/m$, Dawson, Kruer, and Sudan show this leads to growing sidebands at frequencies for which $|\chi^T| << 1$. Their theory is rigorously correct only if there exists a BGK distribution function f_{oT} (compatible with a sine wave) which localizes trapped particles at the bottom of the well. Even in this case there will be corrections from nonresonant electrons with orbits slightly perturbed by E_o and possibly by (parametric) coupling to a low frequency mode.

The contribution to the susceptibility from untrapped electrons will be of order $(E_o^2/4\pi nkT)$. For the parameters of the Wharton-Malmberg-O'Neil experiment we estimate this to be several orders of magnitude smaller than $|\chi^T| \approx \omega_{pT}^2/\omega_B^2$. If we ignore such contributions but retain a low frequency mode, the resulting 3×3 determinant [from (22)] is

$$\begin{vmatrix} \epsilon(k,\omega) + \chi^T & \chi^T & \chi^T \\ \chi^T & \epsilon(k-k_o, \omega-\omega_o) + \chi^T & \chi^T \\ \chi^T & \chi^T & \epsilon(k-2k_o, \omega-2\omega_o) + \chi^T \end{vmatrix} = 0$$

where, again, $\chi^T = -\omega_{pT}^2/[(\omega-kv_o)^2 - \omega_B^2]$. This determinant reduces to

$$\frac{1}{\epsilon(k,\omega)} + \frac{1}{\epsilon(k-k_o,\omega-\omega_o)} + \frac{1}{\epsilon(k-2k_o,\omega-2\omega_o)} = \frac{1}{\chi^T(k,\omega)}$$

Dawson, Kruer, and Sudan use simple fluid dielectric functions for $\epsilon(k,\omega)$ and $\epsilon(k-2k_o, \omega-2\omega_o)$. A simple resonant form for $\epsilon(k-k_o, \omega-\omega_o)^{-1}$, corresponding, for example, to the ion acoustic mode in a two-temperature plasma should determine the effect of this mode on the sideband instability. However, in the Wharton-Malmberg-O'Neil experiment, ω_B is much greater than the ion plasma frequency, so the ion acoustic mode probably does not participate. On the other hand, finite size effects in their experiment give rise to low frequency modes which should be included through $\epsilon(\omega-\omega_o)$. (Kruer[4] has shown that the experimentally determined dispersion relation is mocked up by the simple relation,

$$\epsilon(k,\omega) = 1 - \omega_{p\ eff}^2/(\omega^2-3k^2V_T^2) ,$$

where

$$\omega_{p\ eff}^2 = \omega_p^2k^2R^2/1+k^2R^2 , \quad k_oR = 2 \quad .)$$

VI. CONCLUDING REMARKS

To sum up, the stability of a large amplitude plasma wave against decay into sidebands and (possibly) a low frequency mode is determined by the zeroes of a 2 × 2 or 3 × 3 determinant whose entries are susceptibilities containing the exact orbits of trapped and untrapped particles in the large wave. These particles are distributed in energy according to the BGK distribution function consistent with the large wave. For distributions localized at the bottom of the troughs of the large wave, the stability analysis reduces to the "bunched beam" approximation of Dawson, Kruer, and Sudan.[3]

Dawson, Kruer, and Sudan emphasize the similarity of the sideband instability to a two-stream instability. However, it was pointed out to the author by Dr. D. F. DuBois that the sideband instability can also be viewed as a parametric instability, even in the absence of an acoustic-like mode. Modulation of χ at frequency and wave number $(2\omega_o, 2k_o)$ couples the sidebands and drives them unstable. If there are no trapped particles and the untrapped particles are treated perturbatively, this is just a decay instability. The $2\omega_o$ component of E_o^2 decays into lower (ω) and upper $(2\omega_o-\omega)$ sidebands, $2\omega_o \to \omega + (2\omega_o-\omega)$. Such decay instabilities have been discussed by Jackson[14] for transverse E_o . The sideband displacement $|\omega-\omega_o|$ is found to be small. The effect of trapped particles is to enhance this instability when $|\omega-\omega_o|$ is in the vicinity of ω_B .

There is another interesting piece of information which falls out of the present calculation. According to Eq. (37), the oscillator strengths for $\chi^{TRAPPED}_{\ell,s}$ are proportional to $\partial f_o/\partial W$. This means a distribution function which is perfectly flat in the entire trapped region makes no contribution to $\chi^{TRAPPED}_{\ell,s}$, and hence is stable, at least in the "bunched beam" approximation. Such flat regions have been found by O'Neil(14) in his long-time coarse-grain average distribution function.

Recent computer experiments(4) which show the sideband instability in time suggest that the trapped particle distribution function may be localized near the top of the well. The stability theory would then require the computation of $\chi_{\ell,s}$ matrix elements from orbits near the top of the well. For a general distribution function the matrix elements and 2 × 2 or 3 × 3 determinants probably require computer analysis. Further analytic work is also necessary and will undoubtedly soon be forthcoming.

ACKNOWLEDGMENTS

We wish to thank Drs. D. DuBois, S. Liu, R. Pellat, and H. Berk for stimulating discussions and helpful suggestions.

This work was supported in part by the National Science Foundation, Grant # GP 6817; the Office of Naval Research, Grant # NONR-4756(01); the Atomic Energy Commission, Contract AT(11-1)-34, Project #157; and the National Aeronautics and Space Administration, Contract # NGR-05-007-116.

REFERENCES AND FOOTNOTES

1. I. B. Bernstein, J. M. Greene, and M. D. Kruskal, Phys. Rev. 108, 3 (1957).
2. C. B. Wharton, J. H. Malmberg, and T. M. O'Neil, Phys. Fluids 11, 1761 (1968).
3. J. Dawson, W. Kruer, and R. Sudan, Bull. Am. Phys. Soc. 11, 13, 1504; 2B4 (1968).
4. W. Kruer, Ph.D. Dissertation, Princeton University, April 1969.
5. H. Berk has shown it is sometimes more useful to invert this procedure, solving for $E_0(x)$ from assumed simple distribution functions (private communication).
6. See, for example, D. F. DuBois, 1968 Tokyo Summer Lectures in Theoretical Physics (to be published by Syokabō and W. A. Benjamin, 1969).
7. S. Adler, Phys. Rev. 126, 413 (1962).
8. G. Knorr, Phys. Fluids 11, 885 (1968).
9. D. F. DuBois and M. V. Goldman, Phys. Rev. Letters 14, 544 (1965).
10. M. V. Goldman, Ann. Phys. (N.Y.) 38, 95 (1966).
11. D. F. DuBois and M. V. Goldman, Phys. Rev. 164, 207 (1967).
12. D. F. DuBois and M. V. Goldman, Phys. Rev. Letters 19, 1105 (1967).
13. E. A. Jackson, Phys. Rev. 153, 235 (1967).
14. T. O'Neil, Phys. Fluids 8, 2255 (1965).

FIGURE CAPTIONS

Fig. 1. Coupling of a large amplitude wave at ω_o to sidebands at ω and $2\omega_o-\omega$ (both near the plasma frequency) and a possible low frequency wave at $\omega_o-\omega$.

Fig. 2. Phase-space trajectories of particles in the field of E_o . Orbit integrals are over trajectories of particles starting at initial phase-space points in the two shaded regions. Particles in the light shaded regions are untrapped. Those in the dark shaded region are trapped.

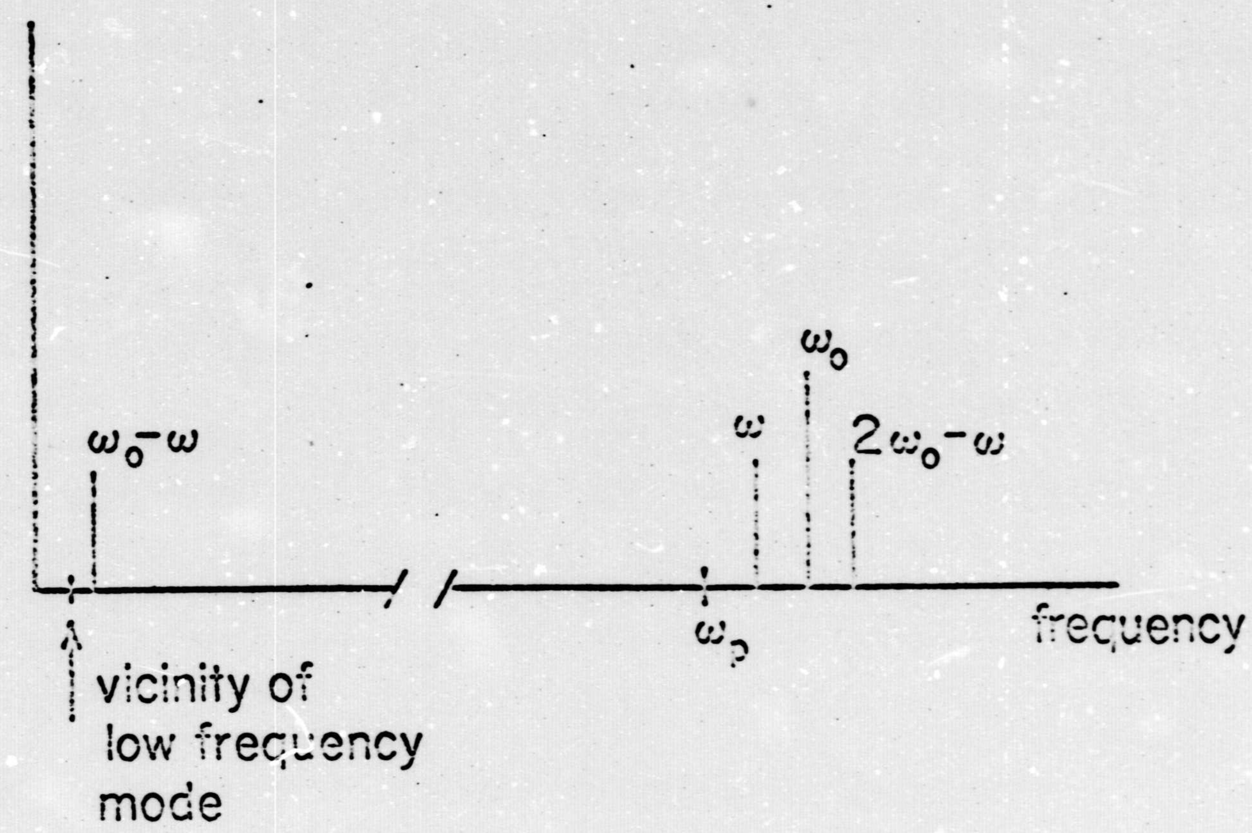


Fig. 1

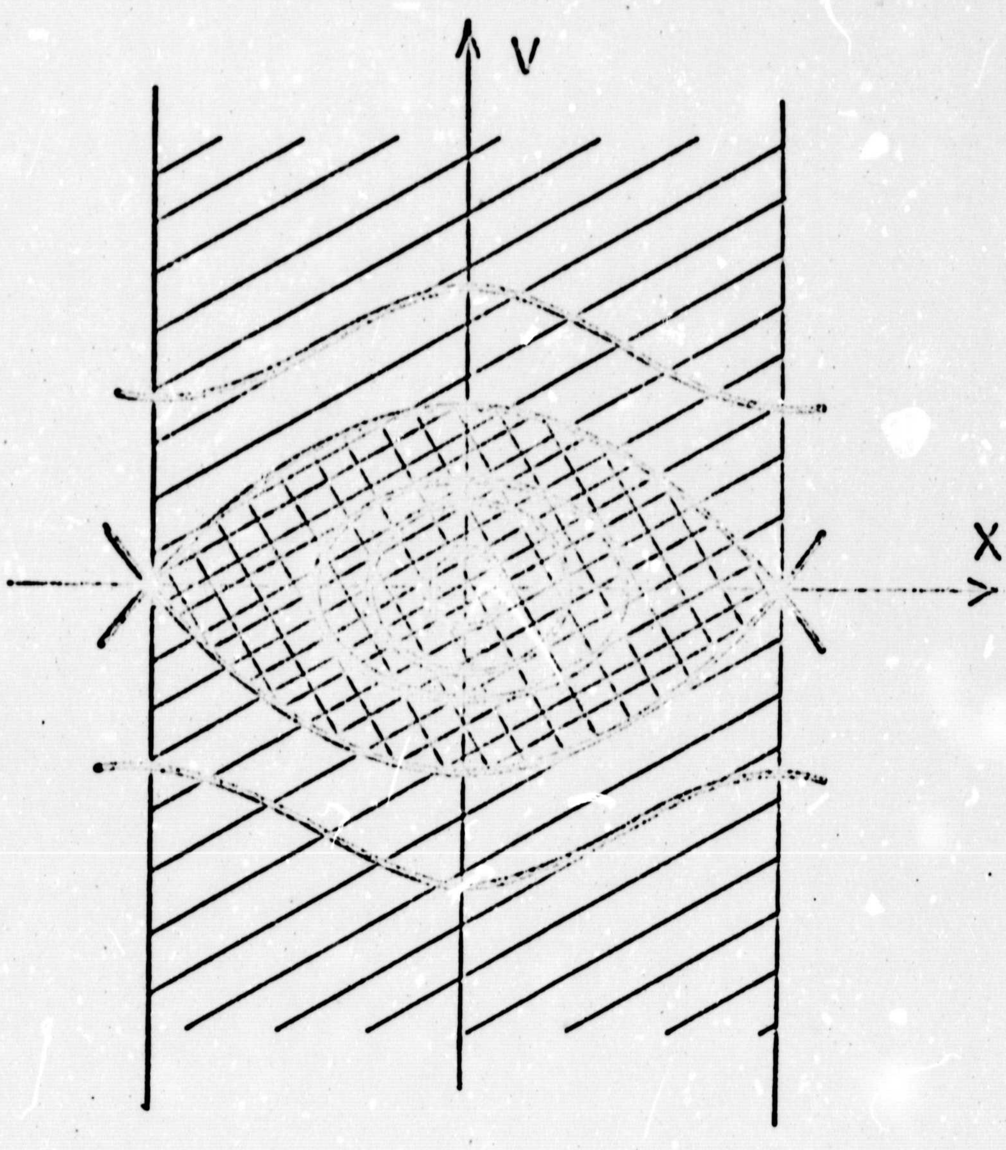


Fig. 2.