BELLCOMM, INC.
955 L'ENFANT PLAZA NORTH, S.W., WASHINGTON, D.C. 20024

COVER SHEET FOR TECHNICAL MEMORANDUM

TITLE- Response of Simply Supported Timoshenko Beams

TM- 69-1022-11

DATE- November 14, 1969

FILING CASE NO(S)- 620

AUTHOR(S)- R. J. Ravera

FILING SUBJECT(S) (ASSIGNED BY AUTHOR(S))- Structural Dynamics
Transform Techniques
Vibrations

ABSTRACT

A method to determine the response of simply supported, finite Timoshenko beams under any general time and space dependent loads is illustrated. The simply supported end conditions permit the use of finite Fourier transform methods which, when coupled with the Laplace transform technique, leads to a relatively straightforward solution. The general method is applied to the problem of a suddenly applied load at the beam mid-span.

SEE REVERSE SIDE FOR DISTRIBUTION LIST

BELLCOMM, INC.

TM-69-1022-11

DISTRIBUTION

COMPLETE MEMORANDUM TO

CORRESPONDENCE FILES:

OFFICIAL FILE COPY
plus one white copy for each
additional case referenced

TECHNICAL LIBRARY (4)

NASA Headquarters

H. Cohen/MLR
P. E. Culbertson/MLA
J. H. Disher/MLD
W. B. Evans/MLO
L. K. Fero/MLV
J. P. Field, Jr./MLP
C. C. Gay/MAT
T. A. Keegan/MA-2
C. H. King, Jr./MAT
(Acting Director/MAT)
R. Petrone/MA
J. F. Saunders, Jr./MAT
M. Savage/MLT
W. C. Schneider/ML

MSC

L. A. Bernardi/KS
C. T. Modlin/ES2
D. C. Wade/ES2

MSFC

E. E. Beam/S&E-ASTN-AA
J. H. Furman/S&E-ASTN-AA
J. Nichols/S&E-ASTN-ADS
N. Showers/S&E-ASTN-A
J. B. Sterett, Jr./S&E-ASTN-A

JPL

R. M. Banford

Martin Marietta Col, Denver, Colorado

G. Morosow
G. Pfaff

COMPLETE MEMORANDUM TO CONTINUED

A. P. Boysen
D. R. Hagner
W. G. Heffron
H. A. Helm
J. J. Hibbert
N. W. Hinners
B. T. Howard
D. B. James
R. K. McFarland
J. Z. Menard
J. M. Nervik
J. J. O'Connor
I. M. Ross
J. A. Saxton
R. V. Sperry
J. W. Timko
R. L. Wagner
M. P. Wilson
Department 1024 File
Department 2031
Division 102
Central File

BELLCOMM, INC.
955 L'ENFANT PLAZA NORTH, S.W. WASHINGTON, D. C. 20024

SUBJECT: Response of Simply Supported Timoshenko Beams - Case 620

DATE: November 14, 1969

FROM: R. J. Ravera

TM-69-1022-11

TECHNICAL MEMORANDUM

I. INTRODUCTION

Anderson[1] has developed a method to determine dynamic response of finite Timoshenko beams which involves the use of the Laplace transform for both space and time variables. For arbitrary boundary conditions, this technique is very attractive since, in general, the eigenfunctions of the beam are composed of sines, cosines, hyperbolic sines, and hyperbolic cosines, making the application of finite spatial transform techniques[2] inordinately difficult. However, for the case of simply supported, finite Timoshenko beams, the eigenfunctions are sines and cosines and finite Fourier transforms lead to a fairly straightforward solution for the most general kind of load inputs. It should be emphasized that the method to be shown is applicable only to simply supported Timoshenko beams.

II. STATEMENT OF PROBLEM

Consider a Timoshenko beam of length l, shown in Figure 1. The beam is acted on by a concentrated force $\psi(t)$. The force distribution per unit length, $q(z,t;\zeta)$ is therefore given by

$$q(z,t;\zeta) = \psi(t)\delta(z-\zeta) \tag{1}$$

$\psi(t)$ being any general time dependent function and $\delta(z-\zeta)$ being the Dirac delta function defined by

$$\delta(z-\zeta) = 0, \quad z \neq \zeta$$

and

$$\delta(z-\zeta) \to \infty, \quad z = \zeta$$

such that

$$\int_{-\infty}^{\infty} f(z)\delta(z-\zeta)dz = f(\zeta)$$

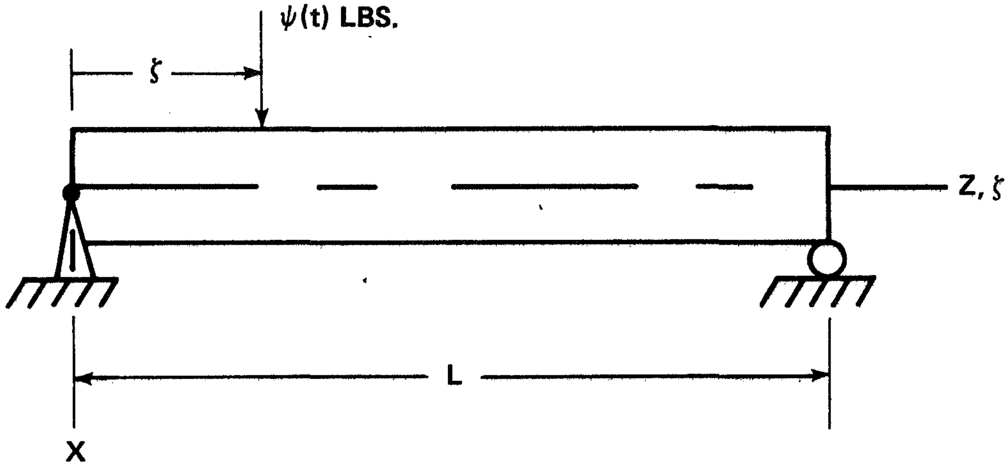

FIGURE 1 - BEAM CONFIGURATION

The magnitude of the concentrated force acting on the beam is recovered by integrating the distribution of force per unit length over the span of the beam. Thus, from (1) and the definition of the delta function,

$$F = \int_0^L q(z,t)dz = \psi(t) \tag{2}$$

The lateral (x) displacement of the beam for the problem with the concentrated load is denoted by $w(z,t;\zeta)$; the rotation of the cross section is represented by $\phi(z,t;\zeta)$. The variable z indicates the position at which the deflection is measured, while the variable ζ indicates the point of application of the concentrated load. The terms w and ϕ are consistent with the notation of Anderson[1] and Cowper[3]. For convenience the arguments $(z,t;\zeta)$ will be dropped except where it is felt to be necessary. The equations of motion for w and ϕ are, after Cowper[3]:

$$\frac{\partial^2 w}{\partial t^2} - a^2 \frac{\partial^2 w}{\partial z^2} - a^2 \frac{\partial \phi}{\partial z} = \frac{\psi(t)}{\rho A} \delta(z-\zeta) \tag{3}$$

and

$$\frac{\partial^2 \phi}{\partial t^2} - b^2 \frac{\partial^2 \phi}{\partial z^2} + c^2 \frac{\partial w}{\partial z} + c^2 \phi = 0 \tag{4}$$

where ρ is mass density of the beam, A is the cross section area and

$$\left.\begin{aligned} a^2 &= KG/\rho \quad , & \text{(a)} \\ b^2 &= E/\rho \quad , & \text{(b)} \\ c^2 &= KAG/\rho I \quad , & \text{(c)} \end{aligned}\right\} \tag{5}$$

with K being the shear coefficient, E, the modulus of elasticity, G, the shear modulus, and I, the second moment of area. The boundary conditions for a simply supported beam are:

$$w(0,t;\zeta) = w(L,t;\zeta) = 0 \tag{6}$$

and

$$\frac{\partial \phi}{\partial z}(0,t;\zeta) = \frac{\partial \phi}{\partial z}(L,t;\zeta) = 0 \tag{7}$$

Zero initial conditions are assumed.

The problem described in this section is concerned with the response of the beam to a concentrated load. However, a solution of Equations (3) and (4) with a delta function input can be used as a Green's function to generate a solution to a problem with the most general kind of spatially dependent input.*

III. METHOD OF SOLUTION

The Laplace transform - Finite Fourier transform method is not, in general, suitable for solving Equations (3) and (4) with arbitrary boundary conditions. However, for the simply supported case, the eigenfunctions are such that the Fourier transform method is easily executed.

The following transform and transform pairs are defined.

(i) Laplace transform:

$$f^*(p) = \int_0^\infty f(t) e^{-pt} dt \tag{8}$$

(ii) Finite Fourier Sine transform pair;

$$\bar{f}_s(n) = \int_0^L f(z) \sin \frac{n\pi z}{L} dz \tag{9a}$$

$$f(z) = \frac{2}{L} \sum_{n=1}^{\infty} \bar{f}_s(n) \sin\left(\frac{n\pi z}{L}\right) \tag{9b}$$

---

*See Appendix B.

(iii) Finite Fourier Cosine transform pair:

$$\bar{f}_c(n) = \int_0^L f(z)\cos\frac{n\pi z}{L}\,dz \tag{10a}$$

$$f(z) = \frac{1}{L}\bar{f}_c(0) + \frac{2}{L}\sum_{n=1}^{\infty}\bar{f}_c(n)\cos\frac{n\pi z}{L} \tag{10b}$$

It is generally known that the Laplace transform of a quantity $\frac{\partial^2 f}{\partial t^2}$ where f has zero initial conditions is

$$\int_0^{\infty}\frac{\partial^2 f}{\partial t^2}e^{-pt}dt = p^2 f^*(p) \tag{11}$$

The following relationships concerning finite Fourier transforms may also be proven[2]:

$$\int_0^L \frac{\partial f}{\partial z}\sin\frac{n\pi z}{L}\,dz = -\frac{n\pi}{L}\bar{f}_c \tag{12}$$

$$\int_0^L \frac{\partial^2 f}{\partial z^2}\sin\frac{n\pi}{L}\,zdz = -\frac{n\pi}{L}[f(L)(-1)^n - f(0)] - \frac{n^2\pi^2}{L^2}\bar{f}_s \tag{13}$$

$$\int_0^L \frac{\partial f}{\partial z} \cos \frac{n\pi}{L} z dz = [f(L)(-1)^n - f(0)] + \frac{n\pi}{L} \bar{f}_s \tag{14}$$

$$\int_0^L \frac{\partial^2 f}{\partial z^2} \cos \frac{n\pi}{L} z dz = [f'(L)(-1)^n - f'(0)] - \frac{n^2\pi^2}{L^2} \bar{f}_c \tag{15}$$

where the prime denotes derivative with respect to z. It should be noted that the expressions in the brackets of Equations (13) through (14) are simply the boundary conditions on the function f. Indeed, $w(0)$, $w(L)$, $\phi'(0)$, $\phi'(L)$ are precisely the boundary conditions known in Equations (6) and (7) and requested by Equations (13), (14) and (15).

IV. SOLUTION

For the purposes of clarity, the Laplace and Fourier transformations of (3) will be illustrated on a term by term basis. Only those terms containing time (t) will be affected when taking the Laplace transformation.

Term 1: The Laplace transform of the first term of (3) is from (11)

$$\int_0^\infty \frac{\partial^2 w}{\partial t^2} e^{-pt} dt = p^2 w^* \tag{i}$$

Term 2: Since a^2 and the operator $\frac{\partial^2}{\partial z^2}$ are independent of time, the Laplace transform of the second term of (3) is, from (8),

$$\int_0^\infty a^2 \frac{\partial^2}{\partial z^2} (w) e^{-pt} dt = a^2 \frac{\partial^2}{\partial z^2} \int_0^\infty w e^{-pt} dt = a^2 \frac{\partial^2 w^*}{\partial z^2} \tag{ii}$$

<u>Terms 3 and 4</u>: The remaining terms transform exactly as term 2; thus

$$\int_0^\infty a^2 \frac{\partial \phi}{\partial z} e^{-pt} dt = a^2 \frac{\partial \phi^*}{\partial z} \qquad \text{(iii)}$$

and

$$\int_0^\infty \frac{\psi(t)\,\delta(z-\zeta)}{\rho A} e^{-pt} dt = \frac{\psi^*}{\rho A}\,\delta(z-\zeta) \qquad \text{(iv)}$$

Substituting terms (i) through (iv) into the corresponding positions in (3) gives the Laplace transform of (3) in the form

$$p^2 w^* - a^2 \frac{d^2 w^*}{dz^2} - a^2 \frac{d\phi^*}{dz} = \frac{\psi^*}{\rho A}\,\delta(z-\zeta) \qquad \text{(v)*}$$

(Note that (v) contains ordinary derivatives. Since time has been transformed out of the equation, partial derivatives are unnecessary.) We now take the finite Fourier sine transform of (v) on a term by term basis. Only those terms dependent on z (w^*, ϕ^* and $\delta(z-\zeta)$) will be affected.

---

*The argument of w^*, for example, is now $(z,p;\zeta)$; the same is true for ϕ^*. ψ^*, being originally a time dependent function only, has the argument (p).

Term 1: The finite Fourier sine transform of the first term in (v) is from (9a)

$$\int_0^L p^2 w^* \sin \frac{n\pi}{L} z dz = p^2 \int_0^L w^* \sin \frac{n\pi}{L} z dz = p^2 \bar{w}_s^* \qquad \text{(vi)}$$

Term 2: From (13), the finite Fourier sine transform of the second term of (5) is

$$\int_0^L a^2 \frac{d^2 w^*}{dz^2} \sin \frac{n\pi z}{L} dz = - a^2 \frac{n\pi}{L} [w^*(L,p;\zeta)(-1)^n - w^*(0,p;\zeta)]$$

$$- a^2 \left(\frac{n\pi}{L}\right)^2 \bar{w}_s^*$$

or noting boundary conditions (6),

$$w^*(0,p;\zeta) = w^*(L,p;\zeta) = 0$$

it follows

$$\int_0^L a^2 \frac{d^2 w^*}{dz^2} \sin \frac{n\pi z}{L} dz = - a^2 \left(\frac{n\pi}{L}\right)^2 \bar{w}_s^* \qquad \text{(vii)}$$

Term 3: From (12), the third term of (v) is transformed according to

$$\int_0^L a^2 \frac{d\phi^*}{dz} \sin\frac{n\pi z}{L}\, dz = -\frac{a^2 n\pi}{L}\, \bar{f}^*_c \tag{viii}$$

Term 4: Finally, the finite Fourier transform of the fourth term of (v) is from the definition of the delta function

$$\int_0^L \frac{\psi^*}{\rho A}\, \delta(z-\zeta)\, \sin\frac{n\pi z}{L}\, dz = \frac{\psi^*}{\rho A} \sin\frac{n\pi\zeta}{L} \tag{ix}$$

Substituting terms (vi) through (ix) for the corresponding terms of (v) gives the doubly transformed equation (3) in the form

$$\left(p^2+a^2\left(\frac{n\pi}{L}\right)^2\right)\bar{w}^*_s + \frac{a^2 n\pi}{L}\,\bar{\phi}^*_s = \frac{\psi^*}{\rho A}\sin\frac{n\pi\zeta}{L} \tag{x}$$

By dividing through equation (x) by a^2, we obtain the desired form of the transformed equation (3) as shown in (16) below. Taking the Laplace transform and finite Fourier cosine transform of (4) and noting boundary conditions (7) yields in a similar fashion the doubly transformed equation (4) as shown in (17) below. The resulting set of transformed equations of motion are grouped for convenience.

$$\left((p/a)^2 + \left(\frac{n\pi}{L}\right)^2\right)\bar{w}^*_s + \frac{n\pi}{L}\,\bar{\phi}^*_c = \frac{\psi^*}{\rho A a^2}\sin\frac{n\pi\zeta}{L} \tag{16}$$

and

$$\left(\frac{c}{b}\right)^2 \frac{n\pi}{L}\,\bar{w}^*_s + \frac{1}{b^2}\left(p^2+c^2+\left(\frac{n\pi b}{L}\right)^2\right)\bar{\phi}^*_c = 0 \tag{17}$$

Equations (16) and (17) are now two linear algebraic equations in $\bar{w}_s^*$ and $\bar{\phi}_c^*$. The determinant of (16) and (17) is

$$D = \begin{vmatrix} \frac{p^2}{a^2} + \frac{n^2\pi^2}{L^2} & n\pi/L \\ \frac{c^2}{b^2}\frac{n\pi}{L} & \frac{1}{b^2}(p^2+c^2+\frac{n^2\pi^2b^2}{L^2}) \end{vmatrix}$$

$$= \frac{1}{a^2b^2}\left\{p^4+p^2[c^2+(a^2+b^2)\ \frac{n^2\pi^2}{L^2}] + \frac{n^4\pi^4}{L^4}\ a^2b^2\right\} \quad (18)$$

The solution for $\bar{w}_s^*$ is given by

$$\bar{w}_s^* = \frac{1}{D}\begin{vmatrix} \frac{\psi^*}{\rho A a^2}\sin\frac{n\pi\zeta}{L} & \frac{n\pi}{L} \\ 0 & \frac{1}{b^2}\left(p^2+c^2+\frac{n^2\pi^2b^2}{L^2}\right) \end{vmatrix}$$

or

$$\bar{w}_s^* = \frac{\psi^*}{\rho A}\sin\frac{n\pi\zeta}{L}\ \frac{(p^2+c^2+\frac{n^2\pi^2b^2}{L^2})}{\left\{p^4+[c^2+\frac{n^2\pi^2}{L^2}\ (a^2+b^2)]p^2 + \frac{n^4\pi^4}{L^4}\ a^2b^2\right\}} \quad (19)$$

Similarly,

$$\overline{\phi}^*_c = \frac{\psi^*}{\rho A} \sin \frac{n\pi\zeta}{L} \frac{\frac{c^2}{b^2}\frac{n\pi}{L}}{\left\{p^4+[c^2+\frac{n^2\pi^2}{L^2}(a^2+b^2)]p^2+\frac{n^4\pi^4}{L^4}a^2b^2\right\}} \tag{20}$$

Let

$$2T_1(n) = [c^2+\frac{n^2\pi^2}{L^2}(a^2+b^2)] \tag{21}$$

$$T_2(n) = \frac{n^4\pi^4a^2b^2}{L^4} \tag{22}$$

and

$$d(n) = [c^2+\frac{n^2\pi^2b^2}{L^2}] \tag{23}$$

The bracketed portion of the denominator in (19) and (20) may then be written

$$p^4 + 2T_1(n)\ p^2 + T_2(n)$$

or upon factoring

$$\{(p^2+\alpha^2(n))(p^2+\beta^2(n))\}$$

where

$$\alpha^2(n) = T_1(n) - \sqrt{T_1^{\,2}(n) - T_2(n)} \tag{24}$$

$$\beta^2(n) = T_1(n) + \sqrt{T_1^{\,2}(n) - T_2(n)} \tag{25}$$

Equations (19) and (20) may then be written

$$\bar{w}_s^* = \frac{\psi^*}{\rho A} \sin\frac{n\pi\zeta}{L}\left[\frac{p^2+d(n)}{(p^2+\alpha^2(n))(p^2+\beta^2(n))}\right] \tag{26}$$

and

$$\bar{\phi}_c^* = \frac{\psi^*}{\rho A} \sin\frac{n\pi\zeta}{L} \frac{\frac{n\pi}{L}\frac{c^2}{b^2}}{(p^2+\alpha^2(n))(p^2+\beta^2(n))} \tag{27}$$

In Appendix A it is shown that $\alpha^2(n)$ and $\beta^2(n)$ are real positive numbers. This guarantees real frequencies (see (33) and (34)).

Taking the inverse Laplace trasform, represented by L^{-1} ( ), of (26) and (27) gives

$$\bar{w}_s = \psi(t) * F_1(t,n) \frac{\sin\frac{n\pi\zeta}{L}}{\rho A} \tag{28}$$

and

$$\bar{\phi}_c = \psi(t) * F_2(t,n) \frac{\frac{n\pi}{L}\left(\frac{c}{b}\right)^2 \sin\frac{n\pi\zeta}{L}}{\rho A} \tag{29}$$

where $\psi(t) * F(t)$ is the notation for the convolution integral

$$\psi(t) * F(t) = \int_0^t \psi(t)F(t-\tau)d\tau \tag{30}$$

and

$$F_1(t,n) = L^{-1}\left[\frac{p^2 + d(n)}{(p^2+\alpha^2(n))(p^2+\beta^2(n))}\right] \tag{31}$$

$$F_2(t,n) = L^{-1}\left[\frac{1}{(p^2+\alpha^2(n))(p^2+\beta^2(n))}\right] \tag{32}$$

From tables,

$$F_1(t,n) = A_1(n)\sin[\alpha(n)t] + B_1(n)\sin[\beta(n)t] \tag{33}$$

$$F_2(t,n) = A_2(n)\sin[\alpha(n)t] + B_2(n)\sin[\beta(n)t] \tag{34}$$

where

$$A_1(n) = \frac{-d(n) - \alpha^2(n)}{\alpha(n)[\beta^2(n) - \alpha^2(n)]} \tag{35}$$

$$B_1(n) = \frac{-[d(n)-\beta^2(n)]}{\beta(n)[\beta^2(n) - \alpha^2(n)]} \tag{36}$$

$$A_2(n) = \frac{-1}{\alpha(n)[\alpha^2(n)-\beta^2(n)]} \tag{37}$$

and

$$B_2(n) = \frac{1}{\beta(n)[\alpha^2(n)-\beta^2(n)]} \tag{38}$$

Taking the inverse Fourier sine transform (Equation (9b)) of (28) and the inverse Fourier cosine transform (Equation (10b)) of (29) while noting that

$$m = \rho AL = \text{mass of beam} \tag{39}$$

yields

$$w(z,t;\zeta) = \frac{2}{m}\sum_{n=1}^{\infty} \psi(t)*F_1(t,n)\sin\frac{n\pi\zeta}{L}\sin\frac{n\pi z}{L} \tag{40}$$

and

$$\phi(z,t;\zeta) = \frac{2}{m}\frac{c^2}{b^2}\frac{\pi}{L}\sum_{n=1}^{\infty} n\psi(t)*F_2(t,n)\sin\frac{n\pi\zeta}{L}\cos\frac{n\pi z}{L} \qquad (41)$$

Equations (40) and (41) represent a formal solution to the problem of response to a concentrated load posed in Section II. To obtain a solution to the problem where the loading takes the more general spatial form

$$q(z) = \psi(t)g(z) \qquad (42)$$

$g(z)$ being any well behaved function, it is noted that (40) and (41) are Green's functions*. Thus, a solution to (3) and (4) where $g(z)$ replaces $\delta(z-\zeta)$, is given by

$$w(z,t) = \int_0^L w(z,t;\zeta)\ g(\zeta)d\zeta \ .$$

and

$$\phi(z,t) = \int_0^L \phi(z,t;\zeta)\ g(\zeta)d\zeta \ .$$

V. <u>EXAMPLE</u>

Consider a beam acted upon by a suddenly applied concentrated load at mid-span; that is, let

$$q(z,t) = PH(t)\delta(z-L/2) \quad , \qquad (43)$$

---

*See Appendix B.

where P is the magnitude of the force, and H(t) is the Heaviside unit step function defined by

$$H(t) = \begin{cases} 0, & t<0 \\ 1, & t\geq 0 \end{cases} \tag{44}$$

then, $\psi(t) = PH(t)$, and combining (44), (34), (33) with (30) gives

$$\begin{aligned} \psi * F_1(t,n) &= -P\int_0^t H(t-\tau)\{A_1(n)\sin[\alpha(n)t]+B_1(n)\sin[\beta(n)t]\}d\tau \\ &= -P\left\{\frac{A_1(n)}{\alpha(n)}[\cos[\alpha(n)t]-1] + \frac{B_1(n)}{\beta(n)}[\cos[\beta(n)t]-1]\right\} \end{aligned} \tag{45}$$

and

$$\psi * F_2(t,n) = -P\left\{\frac{A_2(n)}{\alpha(n)}[\cos[\alpha(n)t]-1] + \frac{B_2(n)}{\beta(n)}[\cos[\beta(n)t]-1]\right\} \tag{46}$$

Substituting $\zeta = L/2$ (from (43)) into (40) and (41) and noting

$$\sin\frac{n\pi}{2} = \begin{cases} 0, & n = 0, 2, 4, \ldots \\ (-1)^{\frac{n-1}{2}}, & n = 1, 3, 5, \ldots \end{cases} \tag{47}$$

gives

$$w(z,t) = -\frac{2P}{m} \sum_{n=1,3,5...}^{\infty} (-1)^{\frac{n-1}{2}} \left\{ \frac{A_1(n)}{\alpha(n)} [\cos[\alpha(n)t]-1] \right.$$

$$\left. + \frac{B_1(n)}{\beta(n)} [\cos[\beta(n)t]-1] \right\} \sin \frac{n\pi z}{L} \qquad (48)$$

$$\phi(x,t) = -\frac{2}{M} \left(\frac{c}{b}\right)^2 \frac{\pi}{L} P \sum_{n=1,3,5...}^{\infty} n\ (-1)^{\frac{n-1}{2}} \left\{ \frac{A_2(n)}{\alpha(n)} [\cos[\alpha(n)t]-1] \right.$$

$$\left. + \frac{B_2(n)}{\beta(n)} [\cos[\beta(n)t]-1] \right\} \cos \frac{n\pi z}{L} \qquad (49)$$

The summation over odd integers insures a symmetric w deflection pattern about mid-span; this is to be expected since the load is applied at mid-span. These results are in agreement with the results of R. A. Anderson.[4]

Another interesting check for this solution (and, indeed, for the entire formulation) may be accomplished by an application of the Abel-Tauber "final value" theorem. This theorem states that if $f^*(p)$ is the Laplace transform of $f(t)$, the the final value of $f(t)$ (steady state value) is

$$\lim_{t\to\infty} f(t) = \lim_{p\to 0} pf^*(p) \qquad (50)$$

For this example, the final value or steady state solution should reduce to the static solution* of a concentrated load at the beam mid-span. If it is noted from (1) and (43) that

$$\psi(t) = PH(t)$$

and, therefore,

$$\psi^*(p) = P/p \tag{51}$$

then (51) and (26) give, after setting $\zeta = L/2$

$$\bar{w}_s^* = \frac{P}{\rho A p} \sin\frac{n\pi}{2}\left[\frac{p^2 + d(n)}{(p^2+\alpha^2(n))(p^2+\beta^2(n))}\right] \tag{52}$$

From (52) and (50)

$$\lim_{t\to\infty} \bar{w}_s = \lim_{p\to 0} p\bar{w}_s^* = \frac{P}{\rho A} \sin\frac{n\pi}{2}\left[\frac{d(n)}{\alpha^2(n)\beta^2(n)}\right] \tag{53}$$

---

*See Appendix C.

Equations (25), (24) and (22) give

$$\alpha^2(n)\beta^2(n) = \frac{n^4\pi^4}{L^4} a^2b^2 \tag{54}$$

Substituting (54) and (23) into (53) yields

$$\lim_{t\to\infty} \overline{w}_s = \frac{P}{\rho A} \sin\frac{n\pi}{2}\left[\frac{L^4c^2}{n^4\pi^4a^2b^2} + \frac{L^2}{n^2\pi^2a^2}\right] \tag{55}$$

or from the definitions (5),

$$\lim_{t\to\infty} \overline{w}_s = P \sin\frac{n\pi}{2}\left[\frac{L^4}{n^4\pi^4EI} + \frac{L^2}{n^2\pi^2KGA}\right] \tag{56}$$

Taking the inverse Fourier sine transform (9b) of (56) and noting (47) yields for the deflection at mid-span

$$\lim_{t\to\infty} w(\tfrac{L}{2},t;\tfrac{L}{2}) = P\,\frac{2}{L} \sum_{n=1,3,5}^{\infty} (-1)^{n-1}\left[\frac{L^4}{n^4\pi^4EI} + \frac{L^2}{n^2\pi^2KGA}\right]$$

$$= P\,\frac{2}{L}\left\{\frac{L^4}{\pi^4EI}\sum_{n=1,3,5}^{\infty}(-1)^{n-1}\frac{1}{n^4} + \frac{L^2}{\pi^2KGA}\sum_{n=1,3,5}^{\infty}(-1)^{n-1}\frac{1}{n^2}\right\} \tag{57}$$

From Sommerfeld,[5]

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots$$

Then (57) becomes

$$\lim_{t\to\infty} w(\frac{L}{2},t;\frac{L}{2}) = \frac{PL^3}{48EI} + \frac{PL}{4KGA} \quad (58)$$

Equation (58) is the correct static solution for a simply supported beam loaded at mid-span by a concentrated force P.

ACKNOWLEDGEMENT

Interest in this problem was generated through discussions with G. M. Anderson concerning his work on the dynamic response of beams.

R. J. Ravera

1022-RJR-jf R. J. Ravera

Attachments
References
Appendix A, B and C

REFERENCES

1. Anderson, G. M., "Dynamic Distortion of a Gimbaled Telescope." Bellcomm TM-69-1022-3, May 21, 1969.

2. Sneddon, I. N., "Fourier Transforms," McGraw-Hill, 1951, p. 71.

3. Cowper, G. R., "The Shear Coefficient in Timoshenko's Beam Theory," J. of Applied Mechanics, V. 33, June 1966, pp. 335-340.

4. Anderson, R. A., "Flexural Vibrations in Uniform Beams According to the Timoshenko Theory," J. of Applied Mechanics, December 1953, pp. 504-510.

5. Sommerfeld, A., "Partial Differential Equations in Physics," Academic Press Inc., 1949, pp. 12-13.

APPENDIX A

The object of this section is to show that $\alpha^2(n)$ and $\beta^2(n)$, defined in (24) and (25), are real positive numbers for all n. From Equation (21)

$$T_1^2(n) = \frac{1}{4}\left[c^4+\frac{2n^2\pi^2}{L^2}c^2(a^2+b^2) + \frac{n^4\pi^4}{L^4}(a^4+2a^2b^2+b^4)\right] \tag{A-1}$$

The term $T_2(n)$ may be written, after noting (22),

$$T_2(n) = \frac{4n^4\pi^4a^2b^2}{4L^4} \tag{A-2}$$

Subtracting (A-2) from (A-1) gives

$$T_1^2(n) - T_2(n) = \frac{1}{4}\left[c^4+\frac{2n^2\pi^2c^2}{L^2}(a^2+b^2)+\frac{n^4\pi^4}{L^4}(a^4+b^4-2a^2b^2)\right] \tag{A-3}$$

Since

$$a^4 + b^4 - 2a^2b^2 = (a^2-b^2)^2 \quad ,$$

(A-3) may be written

$$T_1^2(n) - T_2(n) = \frac{1}{4}\left[c^4+\frac{2n^2\pi^2c^2}{L^2}(a^2+b^2) + \frac{n^4\pi^4}{L^4}(a^2-b^2)^2\right] \tag{A-4}$$

It is clear that every term on the right hand side of (A-4) is positive; thus

$$T_1^2(n) - T_2(n) > 0$$

and the radicals in (24) and (25) are real. Furthermore, it is clear from (21) and (22) that $T_1(n) > 0$ and $T_2(n) > 0$. Hence,

$$T_1(n) > \sqrt{T_1^2(n) - T_2(n)}$$

and the quantities $\alpha^2(n)$ and $\beta^2(n)$ are always real and positive.

APPENDIX B

The solution to any linear differential equation where the non-homogeneous term (input or forcing function) is a point source (Dirac delta function) is a Green's function. The Green's function may be used to generate a solution to more general types of input. The following heuristic argument will illustrate how this is accomplished.

Consider the beams shown in Figure B-1. The solution to the concentrated load problem for the beam of Figure B-1

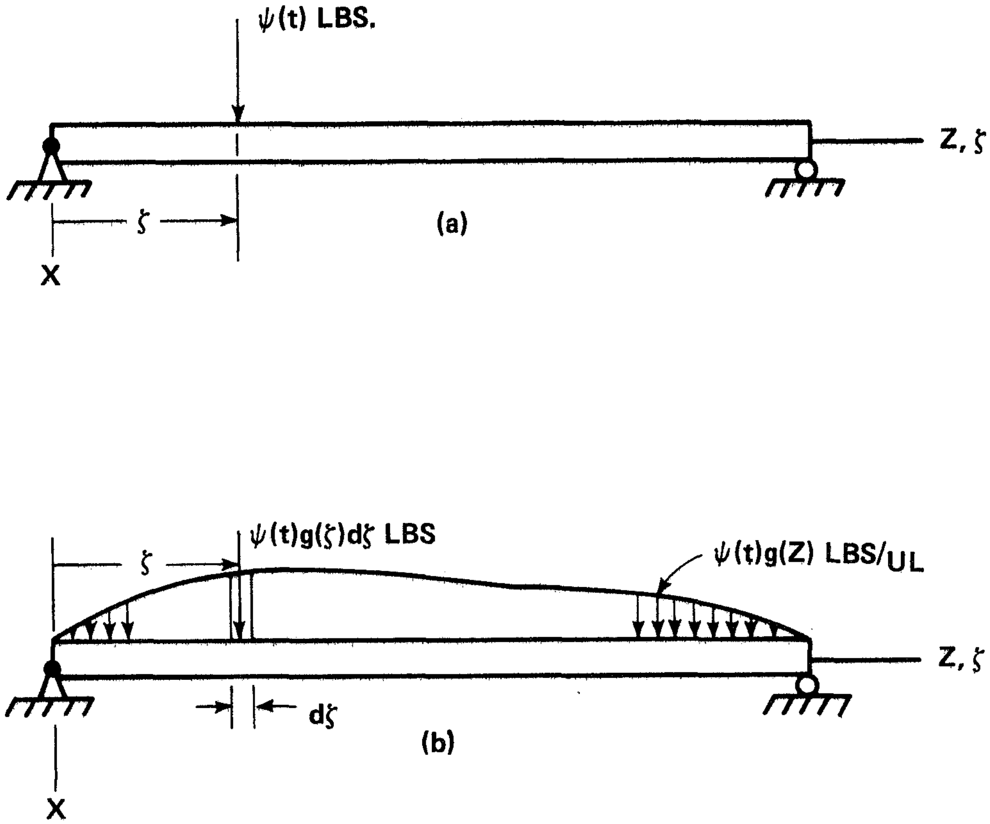

FIGURE B-1

(a) was obtained in Section IV; the solutions for displacement, $w(z,t;\zeta)$ and rotation, $\phi(z,t;\zeta)$, are shown in equations (40) and (41). Consider the beam of Figure B-1 (b) acted upon by a general distributed force per unit length $\psi(t)g(z)$. At any point ζ, the quantity $\psi(t)g(\zeta)d\zeta$, where $d\zeta$ is an increment of length, may be considered a concentrated force by making $d\zeta$ as small as we wish. Since the equations of motion are linear, the displacement and rotation due to a concentrated force $\psi(t)g(\zeta)d\zeta$ are $w(z,t;\zeta)g(\zeta)d\zeta$ and $\phi(z,t;\zeta)g(\zeta)d\zeta$. Summing up the effect of all the concentrated loads $\psi(t)g(\zeta)d\zeta$ gives the solution to the general load input per unit length, $\psi(t)g(z)$, in the form

$$w(z,t) = \int_0^L w(z,t;\zeta)g(\zeta)d\zeta$$

and

$$\phi(z,t) = \int_0^L \phi(z,t;\zeta)g(\zeta)d\zeta \quad .$$

APPENDIX C

The $f^*(p)$ of this example is $\bar{w}_s^*$, defined in (52), and since it contains poles on the imaginary axis it is actually incorrect to use the term "final value" in applying (50) to (52). Since all the poles of (52) are simple, a partial fraction expansion takes the form

$$\bar{w}_s^* = c_o/p + \sum \frac{C_k}{(p+i\omega_k)} - \sum \frac{C_k}{(p-i\omega_k)} \tag{C-1}$$

Application of (50) to (C-1) thus extracts the contribution of the pole at the origin only, and this is the static solution. Contributions from the remaining terms, representing undamped oscillations at beam natural frequencies are suppressed. Therefore, application of the final value theorem (50) in this case does not give us the complete final value but rather the static part only. By postulating even a small amount of damping, a more realistic supposition, the oscillatory terms will damp out and the true final value will be the static solution.