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ON PERIODIC SOLUTIONS OF AUTONOMOUS HAMILTONIAN SYSTEMS

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I. INTRODUCTION

Investigations of periodic solutions near an equilibrium point for an autonomous differential system (i.e., the differential equations do not explicitly contain time as an independent variable) are of special interest to the mathematician working in the field of dynamical astronomy. Lagrange, for example, established the existence of certain periodic motions about the equilibrium points (called the Lagrangian points) of the equilateral-triangle solution of the restricted three-body problem. The actual existence of periodic motions of this type was verified with the discovery that the Trojan asteriods have periodic orbits about the Lagrangian points of the Jupiter-Sun system.

In 1956, Siegel [4] wrote a tract on celestial mechanics in which he developed a criterion for establishing the existence of periodic solutions of autonomous Hamiltonian systems near an equilibrium point. He went on to describe a method for obtaining an approximate, periodic solution of the autonomous system through the use of a periodic solution of the associated linear system. The usefulness of his method was dramatically demonstrated by Pars [3] who applied it to Lagrange's equilateral - triangle solution of the three-body problem to establish new periodic solutions about the Lagrangian points.

An English translation of Siegel's work appeared in 1966, but its circulation is so limited by publication rights that individual copies are difficult to obtain. The National Aeronautics and Space Administration who prints the translation has so restricted its distribution list that it does not abstract or record it in its bi-monthly publication Scientific and Technical Abstract Reports.

Siegel's text is based upon a series of lectures on celestial mechanics which he gave at Göttingen in the winter of 1951-52. The emphasis in his lectures was on the elaboration of ideas and results, attendant to the three-body problem, which had arisen in the 70 years preceding his lecture series. Since his audience was largely composed of people who were quite knowledgeable in celestial mechanics, little motivational detail is included in the text. Thus, it is quite difficult for the nonspecialist in celestial mechanics to read with any degree of real understanding.

In Chapter II of this paper a systematic description of Siegel's criterion and method is given. It is intended for the nonspecialist in celestial mechanics who may be interested in the subject or concerned with orbital problems. The material in Siegel has been rearranged somewhat so that the continuity of thought can be more readily appreciated by a reader who may not be familiar with dynamical astronomy. Additional detail has been supplied whenever its inclusion would make for easier reading.

The method consists of expanding the Hamiltonian of a system of canonical equations in a Taylor series about an equilibrium solution of the system. The constant term of the expression can be ignored and all first order terms are identically zero as will be seen later. If the expansion is truncated after the quadratic terms, an associated linear canonical differential system can be formed from the quadratic Hamiltonian. The associated linear system is used as a basis for studying periodic solutions of the original differential system. If a periodic solution of the linear system exists then, Siegel establishes that in general, a periodic solution exists for the original canonical system. By applying a device suggested by the idea of the variation of parameters, Siegel also shows that a periodic solution of the linear system is either an actual or approximate periodic solution of the original canonical system.

Matrix notation is used throughout the study of the linear system. In this regard two interesting types of matrix arise in the development. One of the two matrices is referred to as a symplectic matrix and the other is referred to as a Hamiltonian matrix. Relatively little information about either of these two types of matrices is available and what little is found is scattered throughout the literature. In fact, Diliberto points out [3] that there does not appear to be any reasonably complete source giving an adequate account of these matrices. For this reason a treatment of some of of their more important properties is given in Chapter III.

In Chapter IV, three example problems are presented which illustrate the use of Siegel's method and some of the properties of symplectic and Hamiltonian matrices.

II. SIEGEL'S METHOD

We consider a system of m first-order ordinary differential equations

$$\dot{X} = F(X) \qquad (2.1)$$

where X represents a column matrix $\{x_1, x_2, \ldots, x_m\}$* whose elements x_i, ($i = 1,2,\ldots,m$) are the independent unknown variables. The symbol $\dot{X}$ represents a column matrix $\{\dot{x}_1, \dot{x}_2, \ldots \dot{x}_m\}$, where $\dot{x}_i$ ($i = 1,2,\ldots m$) is the time derivative of x_i. The symbol F(X) represents a column matrix $\{f_1(X), f_2(X), \ldots, f_m(X)\}$, whose elements $f_i(X)$ ($i = 1,2,\ldots,m$) are assumed to be autonomous (i.e., they do not explicitly contain time as an independent variable) functions of the independent variables $x_1, x_2, \ldots, x_m$.

We shall assume also that Eq. (2.1) has an equilibrium solution, $X(t) = X^{(0)}$ ($t \geq 0$), defined by

$$F(X^{(0)}) = \underline{0} \qquad (2.2)$$

where $\underline{0}$ represents the column matrix $\{0, 0, \ldots, 0\}$ of m zeros and each of the m functions, $f_i(X)$, is analytic in a neighborhood of the equilibrium solution. It will be sufficient here to consider only equilibrium solutions of the form $X^{(0)} = \underline{0}$ since other equilibrium points in state space

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*For economy of space, the elements of an r x 1 column matrix will be displayed as an ordered r-tuple enclosed in braces.

can be translated to the origin by simple coordinate transformations.

If we expand each $f_i(X)$ about an equilibrium point corresponding to an equilibrium solution of Eq. (2.1), we obtain expressions of the form

$$f_i(X) = f_i(\underline{0}) + \sum_{j=1}^{m} \frac{\partial f_i}{\partial x_j}\Big]_{(0,0,\ldots,0)} X_j + \cdots \quad (i=1,2,\cdots m). \tag{2.3}$$

Since the first term on the right-hand side of each equation in Eq. (2.3) is zero, the matrix F(X) can be written in the form

$$F(X) = AX + \cdots, \tag{2.4}$$

where A is an m x m matrix in which the entry a_{ij} in the i-th row and j-th column is the constant $\frac{\partial f_i}{\partial x_j}\Big]_{(0,0,\ldots,0)}$. If quadratic and higher degree terms in the elements of X are dropped, Eq. (2.1) reduces to the linear system

$$\dot{X} = AX. \tag{2.5}$$

We shall refer to Eq. (2.5) as the linear system associated with the original system of Eq. (2.1).

Siegel's approach to finding periodic solutions of Eq. (2.1) is to first find the periodic solutions of the associated linear system, if any exist. If we assume that Eq. (2.5) has a periodic solution which satisfies certain conditions, to be given later, then we can use this periodic solution of the associated linear system as a basis for constructing a periodic solution of Eq. (2.1).

For an arbitrary system of first-order ordinary differential equations satisfying the conditions stipulated for

Eq. (2.1), the fact that the associated linear system has a periodic solution does not quarantee that the original system has a periodic solution. But if Eq. (2.1) represents a Hamiltonian system, Siegel shows that it is generally possible to construct either an exact periodic solution or at least an approximate periodic solution of Eq. (2.1) if a periodic solution of Eq. (2.5) is known.

We assume that Eq. (2.1) represents a Hamiltonian system and we let m = 2n, where n is the number of degrees of freedom of the system. The canonical equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = \frac{-\partial H}{\partial q_i}, \quad (i = 1,2,\ldots,n) \tag{2.6}$$

where the Hamiltonian, $H = H(q_1,q_2,\ldots q_n,p_1,\ldots,p_n)$, is understood to be a function of the generalized coordinates, q_i, and the generalized momenta, p_i, alone. Time does not appear in H as an independent variable. Equations (2.6) can be combined and expressed as a single equation if we set $\dot{x}_i = \dot{q}_i$ and $\dot{x}_{i+n} = \dot{p}_i$, $(i = 1,2,\ldots,n)$, and write

$$\dot{X} = JH_x, \tag{2.7}$$

where $\dot{X}$ represents the column matrix $\left\{\dot{q}_1,\dot{q}_2,\ldots\dot{q}_n,\dot{p}_1,\ldots\dot{p}_n\right\}$ and H_x represents the column matrix $\left\{\frac{\partial H}{\partial q_1},\ldots,\frac{\partial H}{\partial q_n},\frac{\partial H}{\partial p_1},\ldots,\frac{\partial H}{\partial p_n}\right\}$.

If I_n and 0_n represent the n x n identity and the n x n zero matrices, respectively, then J is defined by the matrix equation

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}. \tag{2.8}$$

Since the matrix J will play an important role in the development to follow, we note, at this point, certain of its properties. It is readily verified that

$$J^T = -J, \qquad (2.9a)$$

$$J^T = J^{-1}, \qquad (2.9b)$$

$$J^2 = J \cdot J = -I_{2n}, \qquad (2.9c)$$

$$\det(J) = 1, \qquad (2.9d)$$

where the superscript notation T and -1 is used herein to designate the transpose and inverse, respectively, of a given matrix, I_{2n} is the 2n x 2n identity matrix and the operator det( ) represents the operation of forming the determinant of the matrix indicated inside the parenthesis.

To obtain the linear system associated with Eq. (2.7) expand the Hamiltonian in a Taylor series about an equilibrium solution and then form the column matrix, H_x, by carrying out the necessary partial derivatives of the expanded Hamiltonian. The Taylor series expansion has the form

$$H(x_1, x_2, \ldots, x_{2n}) = H(0,0,\ldots,0) + \sum_{i=1}^{2n} \left.\frac{\partial H}{\partial x_i}\right]_{(0,0,\ldots,0)} x_i + \frac{1}{2} \sum_{i=1}^{2n} \sum_{j=1}^{2n} \left.\frac{\partial^2 H}{\partial x_i\, \partial x_j}\right]_{(0,0,\ldots,0)} x_i x_j + \cdots. \qquad (2.10)$$

The first term on the right-hand side of Eq. (2.10) is a constant which may be set equal to zero since we are concerned only with forming partial derivatives of H. Furthermore, the coefficient of x_i in each term of the first sum appearing on the right-hand side of Eq. (2.10) is zero for each i since the point of expansion is an equilibrium point. In matrix notation, Eq. (2.10) can be written

$$H = \frac{1}{2} X^T S X + \ldots, \tag{2.11}$$

where S is a real symmetric matrix of constant coefficients $\sigma_{ij} = \left.\frac{\partial^2 H}{\partial x_i \, \partial x_j}\right]_{(0,0,\ldots,0)}$. If we neglect terms of third degree and higher, Eq. (2.11) may be replaced by

$$H = \frac{1}{2} X^T S X. \tag{2.12}$$

If Eq. (2.12) is used as a basis for forming the column matrix H_X we find that

$$H_X = SX. \tag{2.13}$$

Thus, the linear system associated with Eq. (2.7) may be written in the form

$$\dot{X} = UX, \tag{2.14}$$

where the matrix $U = (u_{ij}) = JS$, and u_{ij} is the entry in the i-th row and j-th column of U. The matrix U is called a Hamiltonian matrix (for an alternative definition, see Definition 3, Chapter III).

If U is a diagonal matrix the linear system can be integrated immediately to yield the solution

$$x_i = \alpha_i e^{\lambda_i t}, \quad (i = 1,2,\ldots,2n) \tag{2.15}$$

where $\lambda_i = u_{ii}$ and the α_i are constants of integration determined by the initial conditions.

If U is not a diagonal matrix, we seek a canonical linear transformation*

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*The reason that we demand a transformation which is both canonical as well as linear is simply that Siegel's method of finding periodic solutions of Eq. (2.7) requires that the new variables must satisfy canonical equations of motion.

$$X = CY \qquad (2.16)$$

where C is a constant matrix and Y is a column matrix $\{Q_1, Q_2, \ldots, Q_n, P_1, P_2, \ldots, P_n\}$, such that the equations of motion in the new variables, $Q_1, Q_2, \ldots, Q_n, P_1, P_2, \ldots, P_n$, have the form

$$\dot{Y} = DY, \qquad (2.17)$$

where D is a diagonal matrix. If the eigenvalues λ_i, $(i = 1, 2, \ldots, 2n)$, of U are distinct we can be sure that a matrix C exists which will diagonalize U. That is to say there exists a matrix C such that

$$C^{-1}UC = D, \qquad (2.18)$$

where D is a diagonal matrix whose diagonal elements are the eigenvalues of the Hamiltonian matrix U.

The eigenvalues of U are the 2n roots $\lambda_2, \ldots, \lambda_{2n}$ of the 2n-th degree characteristic equation

$$\det(\lambda I_{2n} - U) = 0. \qquad (2.19)$$

The matrix C is determined by the conditions

$$(\lambda_j I_{2n} - U)C^{(j)} = \underline{0}, \quad (j = 1, 2, \ldots, 2n) \qquad (2.20)$$

where $C^{(j)}$ denotes the j-th column of C. Since the eigenvalues of U are assumed to be distinct, the matrix

$$(\lambda_j I_{2n} - U) = C(\lambda_j I_{2n} - D)C^{-1}$$

is of rank 2n-1. It is possible, therefore, to express 2n-1 elements of $C^{(j)}$ in terms of any remaining nonzero element, ρ_j. For any arbitrary choice of the ρ_j's a matrix will be determined which will diagonalize the matrix U. Our task is to choose the ρ_j's such that the transformation in Eq. (2.16) is canonical.

In the next chapter we show that if the Jacobian matrix of a transformation, M, satisfies the equation

$$M^TJM = J, \qquad (2.21.)$$

the transformation is canonical. Any matrix which satisfies Eq. (2.21) is called a symplectic matrix. The constant matrix C in Eq. (2.16) is the Jacobian matrix of the transformation. If we now require the matrix C to be symplectic, we obtain n equations in the 2n unknowns, ρ_j , of the form

$$\rho_j\rho_{j+n} = c_j \quad (j = 1,2,\ldots,n), \qquad (2.22)$$

where the c_j are known constants. If the ρ_j ,($j = 1,2,\ldots,n$), are chosen arbitrarily, then the values of ρ_{j+n} are fixed by Eq. (2.22). The canonical linear transformation is thus determined.

If we apply the canonical linear transformation of Eq. (2.16) to the Hamiltonian system, Eq. (2.7), we obtain the new Hamiltonian system

$$\dot{Y} = JH^*_y. \qquad (2.23)$$

The symbol H^*_y is the column matrix $\left\{\frac{\partial H^*}{\partial y_1}, \frac{\partial H^*}{\partial y_2}, \ldots, \frac{\partial H^*}{\partial y_{2n}}\right\}$ and the new Hamiltonian, H*, is obtained from the old Hamiltonian by transforming to the new variables j,($j = 1,2,\ldots,2n$). Explicitly, we have

$$H^* = \frac{1}{2} Y^TC^TSCY+\cdots. \qquad (2.24)$$

If we apply Theorem 9, which is discussed in the next chapter, the first term on the right-hand side of Eq. (2.24) may be written in the form $\sum_{j=1}^{n}\lambda_j y_j y_{j+n}$, so that Eq. (2.24) can be written in the form

$$H^* = \sum_{j=1}^{n} \lambda_j y_j y_{j+n} + \cdots . \qquad (2.25)$$

If Eq. (2.25) is used as a basis for forming the column matrix H_y^*, Eq. (2.23) takes the form

$$\dot{y}_j = \lambda_j y_j + g_j \quad (j = 1,2,\cdots,2n), \qquad (2.26)$$

where for $1 \leq j \leq n$

$$g_j = \frac{\partial}{\partial y_{j+n}} (H^* - \sum_{k=1}^{n} \lambda_k y_k y_{k+n}) \qquad (2.27)$$

and for $n < j \leq 2n$

$$g_j = \frac{-\partial}{\partial y_{j-n}} (H^* - \sum_{k=1}^{n} \lambda_k y_k y_{k+n}) . \qquad (2.28)$$

To obtain periodic solutions of Eq. (2.26), we adopt a device suggested by the idea of variation of parameters. Let two of the eigenvalues, say λ_1 and λ_{n+1}, be pure imaginary numbers such that $\lambda_1 = -\lambda_{n+1}$. We assume that a solution exists in which each y_j can be represented in a multiple power series of two new variables ξ and η . The solutions are to have the form

$$y_1 = \xi + z_1, \qquad (2.29)$$

$$y_{n+1} = \eta + z_{n+1}, \qquad (2.30)$$

and

$$y_j = z_j \; (j = 2,3,\cdots,n,n+2,\cdots,2n), \qquad (2.31)$$

where

$$z_j = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} a_{rs}^{j} \xi^r \eta^s (j = 1,2,\cdots,2n). \qquad (2.32)$$

The superscript j on a coefficient a_{rs} is used to identify the series to which the a_{rs} belongs.

Since each of the variables, y_j, is a function of ξ and

η only, the time derivative $\dot{y}_j$ has the form

$$\dot{y}_j = \frac{\partial y_j}{\partial \xi} \dot{\xi} + \frac{\partial y_j}{\partial \eta} \dot{\eta}. \tag{2.33}$$

If, for $j = 1$, Eqs. (2.29) and 2.32) are substituted into Eq. (2.33) in the expressions for the partial derivatives of y_1 with respect to ξ and η , the equations of motion, Eq. (2.26), may be written as

$$\left(1 + \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} r a'_{rs} \xi^{r-1} \eta^{s}\right) \dot{\xi} + \left(\sum_{r=1}^{\infty} \sum_{s=1}^{\infty} s a'_{rs} \xi^{r} \eta^{s-1}\right) \dot{\eta} - \lambda_1 \left(\sum_{r=1}^{\infty} \sum_{s=1}^{\infty} a'_{rs} \xi^{r} \eta^{s}\right) = h_1 , \tag{2.34}$$

where h_i is a multiple power series in ξ and η obtained by replacing the variables $y_j (j = 1,2,\cdots,2n)$ with the appropriate series from Eqs. (2.29-2.31). The coefficients in the series h_1, determined by the substitutions, are known. If $\dot{\xi}$ and $\dot{\eta}$ are expressed as a power series in ξ and η we can determine the coefficients of $\xi^r \eta^s$. We now make the additional assumption that the auxiliary variables ξ and η are functions of time satisfying the differential equations

$$\dot{\xi} = u \xi \tag{2.35}$$

and

$$\dot{\eta} = v \eta \tag{2.36}$$

where u is a power series in $w = \xi \eta$, which has the form

$$u = \sum_{k=0}^{\infty} u_k w^k \tag{2.37}$$

and

$$v = \sum_{k=0}^{\infty} v_k w^k, \tag{2.38}$$

for suitable choices of the coefficients u_k and v_k. In particular, we shall require that $u_0 = \lambda_1$ and $v_0 = \lambda_{n+1}$. The difficult problem of establishing the convergence of these series, Eqs. (2.37-38), has been examined by Siegel. The convergence criterion which is given in his book is not presented here because additional background material, which is included in the proof of convergence, is required to understand the criterion.

If we use Eqs. (2.35) and (2.36) to replace ξ and η in Eq. (2.34) and then substitute the appropriate power series given in Eqs. (2.37-28) for u and v, we obtain the equation

$$\left(\sum_{k=0}^{\infty} u_k \xi^{k+1}\eta^{k}\right) + \left(\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} r a^{1}_{rs}\xi^{r}\eta^{s}\right)\left(\sum_{k=0}^{\infty} u_k \xi^{k}\eta^{k}\right) + \left(\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} s a^{1}_{rs}\xi^{r}\eta^{s}\right)\left(\sum_{k=0}^{\infty} v_k \xi^{k}\eta^{k}\right) - \lambda_1\left(\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} a^{1}_{rs}\xi^{r}\eta^{s}\right) = h_1 . \quad (2.39)$$

The expressions for j = n+1 and j = 2,3,···,n,n+2,···,2n which we find in a similar manner are

$$\left(\sum_{k=0}^{\infty} v_k \xi^{k}\eta^{k+1}\right) + \left(\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} r a^{n+1}_{rs}\xi^{r}\eta^{s}\right)\left(\sum_{k=0}^{\infty} u_k \xi^{k}\eta^{k}\right) + \left(\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} s a^{n+1}_{rs}\xi^{r}\eta^{s}\right)\left(\sum_{k=0}^{\infty} v_k \xi^{k}\eta^{k}\right) + \lambda_1\left(\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} a^{n+1}_{rs}\xi^{r}\eta^{s}\right) = h_{n+1} \quad (2.40)$$

and

$$\left(\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} r a^{j}_{rs}\xi^{r}\eta^{s}\right)\left(\sum_{k=0}^{\infty} u_k \xi^{k}\eta^{k}\right) + \left(\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} s a^{j}_{rs}\xi^{r}\eta^{s}\right)\left(\sum_{k=0}^{\infty} v_k \xi^{k}\eta^{k}\right) - \lambda_j\left(\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} a^{j}_{rs}\xi^{r}\eta^{s}\right) = h_j , \quad (2.41)$$

where h_{n+1} and h_j are defined in an analogous manner to h_1.

If we now assume that when $r = s+1$ the coefficients $a_{rs}^1 = 0$ and when $s = r+1$ the coefficients $a_{rs}^{n+1} = 0$, then we will be able to find all the coefficients u_k, v_k and a_{rs}^j by equating like powers of $\xi^r \eta^s$ on both sides of Eqs. (2.39-41).

Since $a_{rs}^1 = 0$ for $r = s+1$, the only terms of the form $\xi^{k+1} \eta^k$ in Eq. (2.39) appear in the first sum on the left and since $a_{rs}^{n+1} = 0$ for $s = r+1$, the only terms of the form $\xi^k \eta^{k+1}$ in Eq. (2.40) appear in the first sum on the left. Thus we can determine all of the coefficients u_k and v_k which appear in the series for u and v.

If all the coefficients a_{rs}^j $(j = 1,2,\ldots,2n)$ have been found for $r<r'$ and $s<s'$, and the coefficients u_k and v_k for k equal to the larger of s' and r', then the coefficients $a_{r's'}^j$ can be found by equating the coefficients of $\xi^{r'} \eta^{s'}$. Except for the cases $r = s+1$ with $j = 1$ and $s = r+1$ with $j = n+1$, for which all the a_{rs}'s are zero, we have

$$s_{r's'}^j[(r-s)\lambda_1 - \lambda_j] = p, \tag{2.42}$$

where p is a function of the coefficients a_{rs}^j, u_k and v_k, already determined. If $a_{r's'}^j$ is to exist we must require that none of the ratios $\frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1}, \ldots, \frac{\lambda_n}{\lambda_1}$ be an integer. With this additional restriction on the eigenvalues, each of the coefficients a_{rs}^j $(j = 1,2,\ldots,2n)$, u_k and v_k can be computed in such a way that the series in Eqs. (2.29-31) are solutions of Eq. (2.26). The only task remaining is to determine the functional dependency of ξ and η on time.

The new Hamiltonian formed by substituting Eqs. (2.29-32) into Eq. (2.25) will be denoted by K. Since K is independent of time and therefore a constant of the motion

$$\frac{dK}{dt} = \frac{\partial K}{\partial \xi}\dot{\xi} + \frac{\partial K}{\partial \eta}\dot{\eta} = 0. \tag{2.43}$$

Using Eqs. (2.35-36), we can write

$$\frac{dK}{dt} = \dot{u}\,\xi\,\frac{\partial K}{\partial \xi} + \dot{v}\,\eta\,\frac{\partial K}{\partial \eta} = 0 \tag{2.44}$$

Both u and $\dot{v}$ are functions of w alone and by some tedious algebraic manipulation which will not be reproduced here it is possible to show that K is also a function of w alone.

By applying the chain rule for partial derivatives, Eq. (2.49) becomes

$$(\dot{u} + \dot{v})\, \dot{w}\, \frac{dK}{dW} = 0. \tag{2.45}$$

Since Eq. (2.45) is to hold for all values of $\dot{w}$ on an interval, we infer that

$$u + v = 0 . \tag{2.46}$$

Now

$$\frac{dw}{dt} = \frac{d}{dt}(\xi\eta) = (\dot{u} + \dot{v})\,\xi\eta = 0,$$

therefore w is a constant of the motion. From Eqs. (2.37-38) it is apparent that u and v are constants also; therefore, Eqs. (2.35-36) immediately integrate to yield the solutions

$$\xi = \xi_0 e^{ut} \tag{2.47}$$

and

$$\eta = \eta_0 e^{vt}, \tag{2.48}$$

where ξ_0 and η_0 are constants of integration. By a separate long proof, included in Siegel's work, he has shown that u and v are conjugate complex numbers, and since $u + v = 0$, then for some constant μ

$$u = i\mu \tag{2.49}$$

and

$$v = -i\mu \tag{2.50}$$

where $i = \sqrt{-1}$. If we substitute Eqs. (2.49-50) into Eqs. (2.47-48), we obtain

$$\xi = \xi_0 e^{iut} \tag{2.51}$$

and

$$\eta = \eta_0 e^{-iut} . \tag{2.52}$$

In essence then, if two eigenvalues say λ_1 and λ_{n+1}, of the associated linear system are pure imaginary complex conjugates and if none of the ratios $\frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1}, \ldots, \frac{\lambda_n}{\lambda_1}$ are integers, then for sufficiently small values of ξ and η, we have exhibited a method of generating a family of periodic solutions of the Hamiltoniansystem, Eq. (2.7).

III. HAMILTONIAN AND SYMPLECTIC MATRICES

Since the amount of information on symplectic and Hamiltonian matrices which one finds in the mathematics literature is quite limited, a treatment of the more important properties is given in this chapter. The goal is to make available to the reader those theorems which are fundamental to the understanding of the properties of these important matrices. Although there is little original material in the theorems and proofs given here, the theorems have been restated and the proofs expanded in the hope of achieving greater clarity.

Definition 1: A real 2n-square matrix M is called symplectic if and only if

$$M^T J M = J, \qquad (3.1)$$

where J is defined in Eq. (2.8).

Definition 2: If $q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n$ are any 2n distinct functions of the two variables (u,v) (and possibly of any number of other variables), the expression

$$\sum_{k=1}^{n} \left(\frac{\partial q_k}{\partial u} \frac{\partial p_k}{\partial v} - \frac{\partial p_k}{\partial u} \frac{\partial q_k}{\partial v} \right)$$

is called a Lagrange's bracket and is denoted by [u,v].

Theorem 1: Let $q_1, q_2, \ldots, q_n$ represent a set of n generalized coordinates and $p_1, p_2, \ldots, p_n$ represent the corresponding set of conjugate generalized moments of a Hamiltonian system. Let $q_i = q_i(Q_1, Q_2, \ldots, Q_n, P_1, P_2, \ldots, P_n)$ and $p_i = p_i(Q_1, Q_2, \ldots, Q_n, P_1, P_2, \ldots, P_n)$, $(i = 1,2,\ldots,n)$, represent equations of transformation to new variables $Q_1, Q_2, \ldots, Q_n, P_1, P_2, \ldots, P_n$. The Jacobian matrix, M, of the transformation which can be written

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A = (a_{ij}) = \frac{\partial q_i}{\partial Q_i}$, $B = (b_{ij}) = \frac{\partial q_i}{\partial P_j}$, $C = (c_{ij}) = \frac{\partial p_i}{\partial Q_j}$ and $D = (d_{ij}) = \frac{\partial p_i}{\partial P_j}$ $(i,j = 1,2,\ldots,n)$, is a symplectic matrix if and only if the transformation is canonical.

<u>Proof</u>: If we form the product M^TJM, we can write

$$M^TJM = \begin{pmatrix} A^TC-C^TA & A^TD-C^TB \\ B^TC-D^TA & B^TD-D^TB \end{pmatrix} = \begin{pmatrix} (e_{ij}) & (f_{ij}) \\ (g_{ij}) & (h_{ij}) \end{pmatrix} .$$

where $e_{ij} = [Q_i, Q_j]$, $f_{ij} = [Q_i, P_j]$, $g_{ij} = [P_i, Q_j]$ and $h_{ij} = [P_i, P_j]$ $(i,j = 1,2,\ldots,n)$. But there is a well-known theorem of analytical dynamics [5] which assures us that a transformation is canonical if and only if $[Q_i, Q_j] = 0$, $[P_i, P_j] = 0$, $[Q_i, P_j] = \delta_{ij}$ and $[P_i, Q_j] = -\delta_{ij}$ $(i, j = 1,2,\ldots n)$ where δ_{ij} is the Kronecker delta. Thus

$$\begin{pmatrix} A^TC-C^TA & A^TD-C^TB \\ B^TC-D^TA & B^TD-D^TB \end{pmatrix} = J$$

and M is symplectic if and only if the transformation is canonical.

<u>Theorem 2</u>: If a matrix M is symplectic, then it is nonsingular.

Proof: Since $M^TJM = J$, $\det(J) = 1$ and the determinant of a product of matrices is the product of the determinants of the matrices, the result is immediate.

Theorem 3: If a matrix M is symplectic, then so is M^{-1} and M^T.

Proof: Since M is symplectic

$$M^TJM = J.$$

Multiplying from the right by M^{-1} and from the left by $(M^T)^{-1}$, we obtain

$$J = (M^T)^{-1} JM^{-1} .$$

But the inverse of the transpose of a matrix is equal to the transpose of the inverse, therefore

$$(M^{-1})^T J M^{-1} = J \tag{3.2}$$

and M^{-1} is symplectic. Taking the inverse of each side of Eq. (3.2), we have

$$MJ^{-1}M^T = J^{-1}.$$

If we use the fact that $J^{-1} = -J$, then we have

$$MJM^T = J$$

and M^T is symplectic.

Theorem 4: Symplectic matrices of order n under the operation of matrix multiplication form a group.

Proof: (a) Matrix multiplication is associative on all square matrices. It is therefore associative on all 2n-square symplectic matrices.

(b) Symplectic matrices are closed under matrix multiplication. Let M and R be two symplectic matrices. The transpose of a product of two matrices is equal to the product of the transposes in reverse order, therefore we have

$$(MR)^T JMR = R^T M^T JMR \qquad (3.3)$$

But, since M is symplectic,

$$M^T JM = J \qquad (3.4)$$

Introducing Eq. (3.4) into Eq. (3.3), we have

$$(MR)^T JMR = R^T JR .$$

But R is symplectic, therefore we may write

$$(MR)^T JMR = J.$$

The product MR is symplectic and we have the closure property.

(c) The identity matrix I_{2n} is the identity element of the group, since

$$I_{2n}^T J I_{2n} = J$$

and I_{2n} is symplectic.

(d) A symplectic matrix is nonsingular, therefore each such matrix has an inverse and the inverse is symplectic by Theorem 3.

Definition 3: A real 2n-square matrix H is called Hamiltonian if and only if

$$(JH)^T = (JH) \qquad (3.5)$$

Theorem 5: A matrix H is Hamiltonian if and only if there exists a real symmetric matrix S such that

$$H = JS.$$

Proof: If H is Hamiltonian, then

$$(JH)^T = JH.$$

Let JH = -S. Then it follows immediately that

$$S^T = S,$$

i.e., S is symmetric. Now

$$JS = J(-JH),$$

where $J^2 = -I_{2n}$, therefore

$$JH = H,$$

and H has the desired representation.

If H = JS, where S is symmetric, then

$$(JH)^T = (J \cdot JS)^T$$

and, since $J^2 = -I_{2n}$ and S is symmetric,

$$(JH)^T = -S.$$

If we use the fact that $J^2 = -I_{2n}$, we may write

$$(JH)^T = J^2 S,$$

which may be factored to yield

$$(JH)^T = J(JS).$$

But, since JS = H, we have

$$(JH)^T = JH$$

and H is Hamiltonian.

Theorem 6: If M is symplectic and H is Hamiltonian, then $K = M^{-1}HM$ is also Hamiltonian.

Proof: Since M is symplectic

$$M^T JM = J. \tag{3.6}$$

If we multiply from the left by J^{-1} and then from the right by M^{-1} and note that $J^{-1} = -J$, then we have

$$M^{-1} = -JM^T J . \tag{3.7}$$

If we write

$$(JK)^T = (JM^{-1}HM)^T, \tag{3.8}$$

and then replace M^{-1} in Eq. (3.8) by Eq. (3.7) we obtain

$$(JK)^T = (-JJM^TJHM)^T.$$

If we use the fact that $J^2 = -I_{2n}$, then we may write

$$(JK)^T = (M^TJHM)^T.$$

Since H is Hamiltonian and the transpose of a product is equal to the product of the transposes in reverse order, we may write

$$(JK)^T = M^TJHM. \qquad (3.9)$$

If we multiply Eq. (3.6) from the right by M^{-1} and then from the right by J^{-1}, we have

$$M^T = JM^{-1}J^{-1}. \qquad (3.10)$$

Replacing M^T in Eq. (3.9) by its equivalent expression from Eq. (3.10), we obtain

$$(JK)^T = JM^{-1}HM.$$

But $K = M^{-1}HM$, therefore

$$(JK)^T = JK$$

and K is Hamiltonian.

Theorem 7: If λ is an eigenvalue of a Hamiltonian matrix, then so is $-\lambda$.

Proof: Let H be a 2n-square Hamiltonian matrix with eigenvalues λ_i, $(i=1,2,\ldots,2n)$. By Theorem 5 there exists a symmetric matrix S such that

$$H = JS.$$

It follows that the characteristic polynomial,

$$\varphi(\lambda) = \det(\lambda I_{2n}-H),$$

may be written

$$\varphi(\lambda) = \det(\lambda I_{2n}-JS).$$

The determinant of a matrix is equal to the determinant of the transpose of the matrix, therefore we can write

$$\varphi(\lambda) = \det(\lambda I_{2n}-JS)^T = \det(\lambda I_{2n}-SJ^T).$$

But $J^T = -J$, so we may write

$$\varphi(\lambda) = \det(\lambda I_{2n}+SJ).$$

Since $I_{2n} = -J^2$, then

$$\varphi(\lambda) = \det(-\lambda J^2-J^2SJ) = \det(J[-\lambda I_{2n}-JS]J).$$

The determinant of a product of matrices is equal to the product of the determinants of the matrices and $\det(J) = 1$, so that

$$\varphi(\lambda) = \det(-\lambda I_{2n}-JS) = \varphi(-\lambda)$$

and the characteristic polynomial is an even function. The theorem follows immediately.

Theorem 8: If there exists a matrix C with the property that $C^{-1}HC = D$, where D is a diagonal matrix whose diagonal elements are the eigenvalues of the Hamiltonian matrix H, then there exists a symplectic matrix $E = CP$, where

$$P = \begin{pmatrix} Q & O_n \\ O_n & I_n \end{pmatrix}$$

and Q is a n-square diagonal matrix whose diagonal elements are nonzero scalars, $q_i, (i=1,2,\ldots n)$, such that

$$E^{-1}HE = D.$$

Proof: Assume that there exists an invertible matrix

C such that

$$C^{-1}HC = D, \tag{3.11}$$

where H is a Hamiltonian matrix and D is a diagonal matrix whose elements are eigenvalues of H. It follows from Theorem 7 that we may write

$$D = \begin{pmatrix} L & O_n \\ O_n & -L \end{pmatrix},$$

where L is an n-square diagonal matrix whose diagonal elements are the n positive eigenvalues of H. If we multiply Eq. (3.11) from the left by C and then transpose, we obtain

$$C^T H^T = DC^T. \tag{3.12}$$

But, since H is Hamiltonian, H = JS for S a symmetric matrix. Equation (3.12) may be rewritten then, to obtain

$$C^T S J^T = DC^T,$$

and if we use the fact that $J^T = J^{-1}$

$$C^T S = DC^T J. \tag{3.13}$$

Now

$$DJ^{-1} = \begin{pmatrix} O_n & -L \\ -L & O_n \end{pmatrix}$$

is symmetric, therefore

$$DJ^{-1} = (DJ^{-1})^T = (J^{-1})^T D.$$

But $(J^{-1})^T = J$, hence it follows that

$$DJ^{-1} = JD. \tag{3.14}$$

Consider the product $(J^{-1}C^T J)JS$. Since $J^2 = -I_{2n}$ and $J^{-1} = -J$, we may write

$$(J^{-1}C^TJ)JS = JC^TS. \tag{3.15}$$

If Eq. (3.13) is introduced into Eq. (3.15), we can write

$$(J^{-1}C^TJ)JS = JDC^TJ.$$

It now follows from Eq. (3.14), that

$$(J^{-1}C^TJ)JS = D(J^{-1}C^TJ). \tag{3.16}$$

If we set $B = (J^{-1}C^TJ)^{-1}$, since $H = JS$, Eq. (3.16) may be written as

$$B^{-1}HB = D.$$

It is a straight forward matter to show that the two matrices B and C diagonalize H if and only if $C = BF$, where F is a diagonal matrix whose diagonal elements are nonzero scalars which can be written in the partitioned form

$$F = \begin{pmatrix} G & O_n \\ O_n & K \end{pmatrix}$$

where G and K are n-square diagonal matrices. From the definition of B, we can write

$$FC^{-1} = J^{-1}C^TJ$$

or

$$C^TJC = JF. \tag{3.17}$$

where, explicitly,

$$JF = \begin{pmatrix} O_n & K \\ -G & O_n \end{pmatrix} \tag{3.18}$$

If we transpose Eq. (3.17) and use the fact that $J^T = -J$, we obtain

$$C^TJC = -(JF)^T. \tag{3.19}$$

From Eq. (3.17) and Eq. (3.19) we find that

$$JF = -(JF)^T,$$

which means the matrix JF is skew-symmetric. It then follows from Eq. (3.11) that G = K.

If we let

$$P = \begin{pmatrix} G & 0_n \\ 0_n & I_n \end{pmatrix},$$

then we have

$$P^{-1}JP = JF. \tag{3.20}$$

If we equate the left-hand sides of Eqs. (3.17) and (3.20), we find that

$$C^TJC = P^TJP$$

or

$$(CP^{-1})^TJ(CP^{-1}) = J.$$

If we set $E = CP^{-1}$, then E is symplectic and

$$E^{-1}HE = D.$$

<u>Theorem 9</u>: If C is a 2n-square symplectic matrix which diagonalizes the 2n-square matrix JS, where S is symmetric, then

$$C^TSC = \begin{pmatrix} 0 & L \\ L & 0 \end{pmatrix},$$

where L is a diagonal matrix whose elements are those eigenvalues of JS which are positive.

<u>Proof</u>: By Theorem 5 the matrix JS is Hamiltonian. It follows from the hypothesis and Theorem 7 that

$$C^{-1}JSC = D \tag{3.22}$$

where D is a diagonal matrix whose elements are the eigenvalues of JS. The matrix D may be written

$$D = \begin{pmatrix} L & 0_n \\ 0_n & -L \end{pmatrix},$$

where L is an n-square diagonal matrix whose diagonal elements are the n positive eigenvalues of JS. Since C is symplectic we may write

$$C^TJC = J. \qquad (3.23)$$

If we multiply Eq. (3.21) from the left by J^{-1} and from the right by C^{-1}, we obtain

$$T^{-1} = J^{-1}C^TJ. \qquad (3.24)$$

Substituting Eq. (3.24) into Eq. (3.22) and taking note of the fact that $J^2 = I$ and $J^{-1} = -J$, we obtain

$$C^TSC = -JD.$$

But, if we form the product -JD, we can write

$$-JD = \begin{pmatrix} 0 & L \\ L & 0 \end{pmatrix},$$

and the theorem follows.

<u>Definition 4</u>: A real m-square matrix A is called orthogonal if and only if $A^T = A^{-1}$.

<u>Definition 5</u>: If $A = (a_{ij})$ is a complex m-square matrix, then the matrix $\bar{A}^T = (\bar{a}_{ji})$, where $\bar{a}_{ji}$ is the complex conjugate of a_{ji} $(i,j = 1,2,\cdots,m)$, is called the conjugate transpose of A.

Definition 6: A complex m-square matrix A is called unitary if and only if $\overline{A}^T = A^{-1}$.

Theorem 10: Let A be a 2n-square matrix of real numbers. Then A is both orthogonal and symplectic if and only if there exist real n-square matrices U and V such that

$$A = \begin{pmatrix} U & V \\ -V & U \end{pmatrix}$$

and the matrix $(U + iV)$, $(i = \sqrt{-1})$ is unitary.

Proof: Since A is symplectic then

$$JA = AJ. \qquad (3.25)$$

If we partition A into the n-square blocks

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

the right and left-hand sides respectively of Eq. (3.21) may be written explicitly as

$$AJ = \begin{pmatrix} -A_{12} & A_{11} \\ -A_{22} & A_{21} \end{pmatrix}$$

and

$$JA = \begin{pmatrix} A_{21} & A_{22} \\ -A_{11} & -A_{12} \end{pmatrix}$$

But, then Eq. (3.21) is valid if and only if $A_{21} = A_{12}$ and $A_{11} = A_{22}$. Setting $V = A_{12}$ and $U = A_{11}$, we write

$$A = \begin{pmatrix} U & V \\ & \\ -V & U \end{pmatrix}.$$

Since A is orthogonal

$$AA^T = \begin{pmatrix} UU^T + VV^T & VU^T - UV^T \\ & \\ UV^T - VU^T & VV^T + UU^T \end{pmatrix} = I_{2n},$$

so that $AA^T = I_{2n}$ if and only if $UU^T + VV^T = I_n$ and $VU^T - UV^T = 0_n$. But these are precisely the necessary and sufficient conditions for the matrix $(U + iV)$ to be unitary.

IV. EXAMPLES

Example 1: For a particle of unit mass moving in the (x,y) plane under the action of a uniform gravitational field (0,-g), we have the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + gy,$$

where p_x and p_y represent the momenta along the x and y axes, respectively. The canonical equations are

$$\dot{x} = p_x, \qquad \dot{y} = p_y, \qquad \dot{p}_x = 0, \qquad \dot{p}_y = -g. \tag{4.1}$$

To find an equilibrium solution of Eq. (4.1) we have to find values for x, y, p_x and p_y so that $\dot{x} = \dot{y} = \dot{p}_x = \dot{p}_y = 0$.

Since $\dot{p}_y = -g$, a constant, there is clearly no equilibrium solution so that Siegel's method cannot be applied.

Example 2: The Hamiton function for a simple harmonic oscillator is

$$H = \frac{1}{2}(p^2 + n^2q^2), \tag{4.2}$$

where q is the generalized coordinate, p is the generalized momentum and n is a nonzero constant. The equations of motion are

$$\dot{q} = p, \qquad \dot{p} = -n^2q. \tag{4.3}$$

The equilibrium solution is q = p = 0. It is trivially true that the expansion of the polynomial, Eq. (4.2), in a

Taylor series about an equilibrium point corresponding to the equilibrium solution is the polynomial itself hence, in matrix notation, the expanded Hamiltonian can be written

$$H = \frac{1}{2}(q,p)\begin{pmatrix} n^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}. \qquad (4.4)$$

If the column matrix $\left\{\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right\}$ is determined from Eq. (4.4), the equations of motion are

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -n^2 & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}, \qquad (4.5)$$

where the constant matrix in Eq. (4.5) is Hamiltonian. The eigenvalues of the Hamiltonian matrix are computed from Eq. (2.19). We find the eigenvalues to be $\lambda_1 = in$ and $\lambda_2 = -in$. Since we have a pair of pure imaginary eigenvalues such that $\lambda_1 = -\lambda_2$ and there are no other eigenvalues, the requisite conditions on the eigenvalues are satisfied. We know, therefore, that there is a periodic solution of Eq. (4.3).

The columns of the transformation matrix C in Eq. (2.16) are computed from Eq. (2.20) and in terms of the ρ_k's (in this case ρ_1 and ρ_2 are the elements of the first row of C), C may be written

$$C = \begin{pmatrix} \rho_1 & \rho_2 \\ in\rho_1 & -in\rho_2 \end{pmatrix}. \qquad (4.6)$$

Requiring C to be symplectic, we obtain the condition $in\rho_1\rho_2 = 1$, which leads to

$$C = \begin{pmatrix} 1 & \frac{i}{2n} \\ in & \frac{1}{2} \end{pmatrix} \tag{4.7}$$

when we choose $\rho_1 = 1$.

If we insert Eq. (4.7) into Eq. (2.16) we obtain the canonical linear transformation,

$$\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 1 & \frac{i}{2n} \\ in & \frac{1}{2} \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}, \tag{4.8}$$

to new variables Q and P. When we insert the transformation, Eq. (4.8), into Eq. (4.4), we obtain the new Hamiltonian H*. In the summation form of Eq. (2.25), we may write

$$H^* = inQP. \tag{4.9}$$

The equations of motion in the new variables may be written as

$$\begin{aligned} \dot{Q} &= inQ, \\ \dot{P} &= -inP. \end{aligned} \tag{4.10}$$

Thus all the g_j's in Eq. (2.26) are zero. Equation (4.10) immediately integrates to the solutions

$$Q = Q_o e^{int}$$

and

$$P = P_o e,^{-int} \tag{4.11}$$

which, since the g_j's are zero, is the exact solution of Eq. (4.3). In terms of the variables q, p, the solutions may be written

$$q = c_1 \cos nt + c_2 \sin nt$$

$$p = c_o \cos nt + c_y \sin nt$$

where

$$c_1 = Q_o + \frac{iP_\Omega}{2},$$

$$c_2 = iQ_o + \frac{P_o}{2n},$$

$$c_3 = inQ_o + \frac{P_o}{2},$$

$$c_4 = Q_o n - \frac{iP_o}{2}.$$

Example 3: Consider the Hamiltonian

$$H = \frac{1}{2}n(q_1{}^2 + p_1{}^2) - n(q_2{}^2 + p_2{}^2) + \frac{1}{2}\alpha(q_1{}^2 q_2 - q_2 p_1{}^2 - 2p_1 p_2), \quad (4.12)$$

where the q's are the generalized coordinates, the p's are the generalized moments and n and α are nonzero constants.

The equations of motion are

$$\begin{aligned} \dot{q}_1 &= np_1 - \alpha(q_2 p_1 + q_1 p_2), \\ \dot{q}_2 &= -2np_2 - \alpha q_1 p_1, \\ \dot{p}_1 &= -nq, - \alpha(q_1 q_2 - p_1 p_2), \\ \dot{p}_2 &= 2nq_2 - \frac{1}{2}\alpha(q_1{}^2 - p_1{}^2). \end{aligned} \quad (4.13)$$

It is readily evident that $q_1 = q_2 = p_1 = p_2 = 0$ is an equilibrium solution.

The Taylor series expansion of H about an equilibibirum point corresponding to tne equilibrium solution is

$$H(X) = H(\underline{0}) + \sum_{i=1}^{4} \left.\frac{\partial H}{\partial x_i}\right]_{(0,0,0,0)} x_i$$

$$+ \frac{1}{2!} \sum_{i=1}^{4}\sum_{j=1}^{4} \left.\frac{\partial^2 H}{\partial x_i \partial x_j}\right]_{(0,0,0,0)} x_i x_j + \cdots, \quad (4.14)$$

where X is the column matrix $\{q_1, q_2, p_1, p_2\}$. The constant and linear terms are zero and if we ignore the terms of third degree and higher, we obtain the quadratic Hamiltonian

$$H = \frac{1}{2}(q_1, q_2, p_1, p_2) \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & -2n & 0 & 0 \\ 0 & 0 & n & 0 \\ 0 & 0 & 0 & -2n \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \tag{4.15}$$

If the column matrix $\left\{ \frac{\partial H}{\partial q_1}, \frac{\partial H}{\partial q_2}, \frac{\partial H}{\partial p_1}, \frac{\partial H}{\partial p_2} \right\}$ is determined from Eq. (4.15), the equations of motion may be written

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & n & 0 \\ 0 & 0 & 0 & -2n \\ -n & 0 & 0 & 0 \\ 0 & 2n & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \tag{4.16}$$

Equation (4.16) is the associated linear system of Eq. (4.13). The constant matrix in Eq. (4.16) is Hamiltonian with eigenvalues $\lambda_1 = in$, $\lambda_2 = 2in$, $\lambda_3 = -in$ and $\lambda_4 = -2in$. Since we have a pair of pure imaginary eigenvalues λ_2 and λ_4, such that $\lambda_2 = -\lambda_4$ and the ratio $\frac{\lambda_1}{\lambda_2} = \frac{1}{2}$ is not an integer, there is a periodic solution of the Hamiltonian system .

The columns of the transformation matrix C in Eq. (2.16) are computed from Eq. (2.20) and in terms of the ρ_k's, C may be written

$$C = \begin{pmatrix} \rho_1 & 0 & i\rho_3 & 0 \\ 0 & i\rho_2 & 0 & \rho_4 \\ i\rho_1 & 0 & \rho_3 & 0 \\ 0 & \rho_2 & 0 & i\rho_4 \end{pmatrix} \tag{4.17}$$

If we require C to be symplectic, we obtain the equations

$$2\rho_1\rho_3 = 1 \tag{4.18}$$

$$2\rho_2\rho_4 = 1.$$

If we choose $\rho_1 = 1$, $\rho_2 = 1$, we obtain

$$C = \begin{pmatrix} 1 & 0 & i/2 & 0 \\ 0 & i & 0 & -\frac{1}{2} \\ i & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & -i/2 \end{pmatrix}. \tag{4.19}$$

If we substitute Eq. (4.19) into Eq. (2.16), we obtain the canonical linear transformation

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & i/2 & 0 \\ 0 & i & 0 & -\frac{1}{2} \\ i & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & -i/2 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{pmatrix}, \tag{4.20}$$

to new variables Q and P. If we introduce Eq. (4.20) into Eq. (4.14), we obtain the new Hamiltonian H* in the variables (Q_1, Q_2, P_1, P_2). In the summation form of Eq. (2.25), we have

$$H^* = inQ_1P_1 + 2inQ_2P_2 + \cdots \tag{4.21}$$

Since all fourth degree and higher terms in Eq. (4.14) vanish, we may write H* exactly in the form

$$H^* = inQ_1P_1 + 2inQ_2P_2 + \alpha\Big(\frac{1}{8}P_1{}^2P_2 + iQ_1{}^2Q_2 - \frac{i}{4}P_1{}^2Q_2 - \frac{1}{2}P_2Q_1{}^2\Big). \tag{4.22}$$

The equations of motion in the new variables are

$$\begin{aligned}
\dot{Q}_1 &= inQ_1 + \alpha(\tfrac{1}{4}P_1P_2 - \tfrac{1}{2}P_1Q_2),\\
\dot{Q}_2 &= 2inQ_2 + \alpha(\tfrac{1}{8}P_1{}^2 - \tfrac{1}{2}Q_1{}^2),\\
\dot{P}_1 &= -inP_1 + \alpha(Q_1P_2 - 2iQ_1Q_2),\\
\dot{P}_2 &= -2inP_2 + \alpha(\tfrac{i}{4}P_1^2 - iQ_1{}^2).
\end{aligned} \tag{4.23}$$

The g_j's of Eq. (2.26) are the second terms on the right-hand sides of Eq. (4.22).

If we substitute the appropriate series from Eqs. (2.29-32) into the Eqs. (4.22), and solve for the coefficients a_{rs}^j ($j = 1,2,3,4$), v_k and u_k, we find that all of the coefficients are zero. The equations of motion, Eqs. (4.30-33), reduce to

$$Q_1 = 0,$$
$$P_1 = 0,$$
$$\dot{Q}_2 = 2inQ_2,$$

and

$$\dot{P}_2 = -2inP_2,$$

which immediately integrate to the solutions

$$Q_2 = Q_o e^{2int}$$

and

$$P_2 = P_1 e^{-2int}.$$

In terms of the original variables the solutions are

$$q_1 = 0,$$
$$q_2 = c_1 \cos 2nt + c_2 \sin 2nt,$$
$$p_1 = 0,$$
$$p_2 = c_3 \cos 2nt + c_4 \sin 2nt,$$

where

$$c_1 = iQ_0 - \frac{1}{2}P_o$$

$$c_2 = \frac{i}{2} P_0 - Q_o,$$

$$c_3 = Q_o - \frac{i}{2}P_o$$

and

$$c_4 = iQ_o - \frac{1}{2}P_o.$$

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