# OPTIMAL HYBRID PROPULSION SYSTEMS THRUST TRAJECTORIES IN A PATCHED-CONIC N-BODY FORCE FIELD

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## SUMMARY

This report derives the differential equations for optimal thrust trajectories and the corresponding variational equations for power-limited (and/or thrust-limited) hybrid propulsion systems operating in a patched-conic N-body dynamical environment. The paper derives the optimal switch times for engine cut-off, planetary sphere of influence encounter, and hybrid propulsion systems' thrust exchange.

The method described herein gives an excellent approximation to the problems of obtaining realistic launch windows and overall fuel cost for maximum payload, and solves the complicated heliocentric, planetocentric interface problem in a direct and rational fashion.

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## INTRODUCTION

It is well known that the patched-conic mode offers a good approximation to trajectories in precision N-body programs. This study is offered in the expectation that the Euler-Lagrange equations for the patched-conic dynamical system will result in a good approximation to the solution of optimal thrust trajectories in an N-body dynamical system. Of importance is the solution to the problems of

a) optimal switching times for hybrid propulsion systems consisting of combinations of chemical high-thrust and exotic low-thrust/high specific impulse engines

b) matching the heliocentric and planetocentric portions of the trajectory in a realistic and simple manner.

It is felt that these problems are properly analyzed and solved through the use of optimization techniques.

## I. The Patched-Conic Equations of Motion

The theory of the patched-conic is based on the assumption that, in the neighborhood of a planet, the motion with respect to the planet can be adequately described by neglecting the gravitational acceleration influences of all other bodies. This approximation breaks down outside some pre-determined spheres of influence, where the effect of the Sun again becomes dominant. In this area, we may ignore the effects of all other bodies and take into account only the influential effects of the Sun.

Thus in a heliocentric influence region, we have

$$\ddot{R}_s = -\mu_s \frac{R_s}{r_s^3} + \frac{2p(r_s)}{c\,m} T \tag{1}$$

where $|T| = 1$ and $\frac{2p(r_s)}{c}$ is the magnitude of the thrust for limited-power systems where $p(r_s)$ may, or may not, be a function of the solar radial distance, $r_s$. For bounded thrust, the power is constant. For all cases, we have the mass flow given by

$$\dot{m} = -\frac{2p(r_s)}{c^2} \tag{2}$$

In a planetocentric influence region, we have

$$\begin{aligned} R_s &= R_p + P \\ \dot{R}_s &= \dot{R}_p + \dot{P} \\ \ddot{R}_s &= \ddot{R}_p + \ddot{P} \end{aligned} \tag{3}$$

where the accelerations are given by

$$\ddot{R}_p = -\mu_p \frac{R_p}{r_p^3} + \frac{2p(r_s)}{c\,m} T$$

$$\ddot{P} = -(\mu_p + \mu_s) \frac{P}{p^3} \tag{4}$$

Here we assume the planets are moving in two-body orbits about the Sun. This is done in order to avoid using an involved ephemeris tape at each integration step.

Thus, at the sphere of influence of each planet $(\bar{r}_p)$, we have a jump discontinuity in the gravitational portion of the acceleration. The thrust is continuous across the planetary influence sphere intercept. At the first instant of time, $\bar{t}$, for which

$$r = |R_s(t) - P(t)| - \bar{r}_p = 0 \tag{5}$$

we have for $t > \bar{t}$

$$R_s(\bar{t})^+ = R_s(\bar{t})^-$$

$$\dot{R}_s(\bar{t})^+ = \dot{R}_s(\bar{t})^-$$

$$\ddot{R}_s(\bar{t})^+ = -\mu_p \frac{R_p}{r_p^3} - (\mu_s + \mu_p) \frac{P}{p^3} + \frac{2p(r_s)}{c\,m} T \tag{6}$$

$$\ddot{R}_s(\bar{t})^- = -\mu_s \frac{R_s}{r_s^3} + \frac{2p(r_s)}{c\,m} T$$

A similar discontinuity occurs at the exit time of passage out of the planetary sphere of influence back into the heliocentric phase.

Before proceeding, we require the derivation of the jump discontinuities of the variations in the state at a point of discontinuity in some derivative of the state. This is done in the next section.

## II. A General Approach to the Variational Problem with a Discontinuity

Let r be the lowest integer corresponding to the lowest derivative of the state, in which a discontinuity first occurs. All the lower derivatives are continuous. Let the discontinuity occur at time $\bar{t}$, which may itself be a function of the state. Then, for

$$t_1 \le \bar{t}$$

$$x^{r-1}(\bar{t})^- = x^{r-1}(t_1)^- + x^r(t_1)^-(\bar{t}-t_1)$$

$$\text{for } t_2 \ge \bar{t} \text{ and } (\bar{t}-t_1) = (t_2-\bar{t}) \tag{7}$$

$$x^{r-1}(t_2)^+ = x^{r-1}(\bar{t})^+ + x^r(\bar{t})^+(t_2-\bar{t})$$

Since the r-1 derivative is continuous at $\bar{t}$, we have

$$x^{r-1}(t_2)^+ = x^{r-1}(t_1)^- + [x^r(t_1)^- - x^r(\bar{t})^+](\bar{t}-t_1) \tag{8}$$

Taking the variation of Eq. (8) and passing to the limit as $\bar{t}-t_1 = t_2-\bar{t} \to 0$, we have

$$\delta x^{r-1}(\bar{t})^+ = \delta x^{r-1}(\bar{t})^- + [x^r(\bar{t})^- - x^r(\bar{t})^+]\,\delta\bar{t} \tag{9}$$

If the discontinuity time, $\bar{t}$, is a function of the state, say

$$f(x, x', x'', \cdot\cdot\cdot\cdot, x^{r-1}) = 0 \tag{10}$$

then,

$$D f[x(r)] = 0 = \frac{\partial f}{\partial x^i}\,\delta x^i + \dot{f}\,\delta\bar{t} \tag{11}$$

and

$$\delta x^{r-1}(\bar{t})^{+} = \delta x^{r-1}(\bar{t})^{-} + [x^{r}(\bar{t})^{+} - x^{r}(\bar{t})^{-}] \frac{\frac{\partial f}{\partial x^{i}} \delta x^{i}}{\dot{f}} \tag{12}$$

This is the final derived result.

Naturally, since all the lower derivatives are continuous, their variations are continuous.

$$\delta x^{p}(\bar{t})^{+} = \delta x^{p}(\bar{t})^{-} \qquad p < r-1 \tag{13}$$

We are now in a position to develop the Euler-Lagrange equations and find the optimum control.

## III. The Optimal Thrust Trajectories in a Patched-Conic N-Body Force Field

We consider first the equations in the heliocentric phase. Here, we have the differential equations which define the optimal thrust applicable in a central force field. The reader is referred to Ref. 1 for details.

The optimal thrust direction is given by

$$T = \frac{\Lambda}{\lambda} \tag{14}$$

where $\Lambda$ is the solution of

$$\ddot{\Lambda} = -\mu_s \frac{\Lambda}{r_s^3} + 3\mu_s \frac{\Lambda \cdot R_s}{r_s^5} R_s + H(\alpha) \frac{2 \frac{\partial p}{\partial r_s} \alpha}{c\, m\, r_s} R_s \tag{15}$$

and $H(\alpha)$ is the Heaviside operator which defines the switching surface,

$$\begin{aligned} H(\alpha \leq 0) &= 0 \\ H(\alpha > 0) &= 1 \end{aligned} \tag{16}$$

where

$$\alpha = \lambda - \frac{\sigma m}{c}$$

and $\sigma$ is the Lagrange multiplier for the mass variable. The differential equation for $\sigma$ is given by

$$\dot{\sigma} = \frac{2p(r_s)}{c\, m^2} H(\alpha)\, \lambda \tag{17}$$

The optimal mass flow is given by

$$\dot{m} = -\frac{2p(r_s)}{c^2} H(\alpha) \tag{18}$$

In the event the power is constant, $\frac{\partial p}{\partial r_s} = 0$ the last term in Eq. (15) would be omitted.

The terminal condition is given by $\sigma(t_f) = 1$. The differential equation for the heliocentric position vector is given by

$$\ddot{R}_s = -\mu_s \frac{R_s}{r_s^3} + \frac{2p(r_s)}{c\,m} H(\alpha) \frac{\Lambda}{\lambda} \tag{19}$$

Within the planetary sphere of influence, again we have

$$T = \frac{\Lambda}{\lambda} \tag{14}$$

where $\Lambda$ now satisfies the variational equation of a two-body problem with the planetary mass at its center.

$$\ddot{\Lambda} = -\mu_p \frac{\Lambda}{r_p^3} + 3\mu_p \frac{R_p \cdot \Lambda}{r_p^5} R_p + H(\alpha) \frac{2p'\alpha}{c\,m\,r_s} R_s \tag{20}$$

where

$$R_s = R_p + P \tag{20a}$$

$$p' = \frac{\partial p}{\partial r_s} \tag{20b}$$

The equations for $\dot{\sigma}$ and $\dot{m}$ are given by Eqs. (17) and (18), and the optimal equations for the position of the vehicle with respect to a planet are given by

$$\ddot{R}_p = -\mu_p \frac{R_p}{r_p^3} + \frac{2p(r_s)}{c\,m\lambda} \Lambda\, H(\alpha) \tag{21}$$

We are now ready for the variational differential equations.

## IV. The Differential Equations of Variation

We have, for the variational differential equations in the heliocentric phases,

$$\frac{d^2}{dt^2}\,\delta R_s = -\mu_s \frac{\delta R_s}{r_s^3} + 3\mu_s \frac{R_s\cdot\delta R_s}{r_s^5} R_s + H(\alpha)\frac{2p' R_s\cdot\delta R_s}{c\,m\,\lambda\,r_s}\Lambda$$

$$- H(\alpha)\frac{2p}{c\,m\,\lambda^3}\left[\Lambda(\Lambda\cdot\delta\Lambda) - \lambda^2\,\delta\Lambda\right] - H(\alpha)\frac{2p\,\Lambda}{c\,m^2\lambda}\,\delta m \qquad (22)$$

$$\frac{d}{dt}\,\delta m = -H(\alpha)\frac{2p'}{c^2\,r_s} R_s\cdot\delta R_s \qquad (23)$$

$$\frac{d}{dt}\,\delta\sigma = \frac{2H(\alpha)}{c}\left\{\frac{p'\lambda}{m^2\,r_s} R_s\cdot\delta R_s + \frac{p}{m^2\lambda}\Lambda\cdot\delta\Lambda - \frac{2p\lambda}{m^3}\delta m\right\} \qquad (24)$$

$$\frac{d^2}{dt^2}\,\delta\Lambda = -\mu_s\frac{\delta\Lambda}{r_s^3} + 3\mu_s\frac{R_s\cdot\delta R_s}{r_s^5}\Lambda + 3\mu_s\frac{\delta\Lambda\cdot R_s}{r_s^5}R_s + 3\mu_s\frac{\Lambda\cdot\delta R_s}{r_s^5}R_s$$

$$+ 3\mu_s\frac{\Lambda\cdot R_s}{r_s^5}\delta R_s - 15\mu_s\frac{\Lambda\cdot R_s}{r_s^7}(R_s\cdot\delta R_s)R_s + H(\alpha)\frac{2p''\alpha}{c\,m}\frac{R_s\cdot\delta R_s}{r_s}\frac{R_s}{r_s}$$

$$+\frac{2H(\alpha)p'}{c}\left\{\frac{\frac{\Lambda\cdot\delta\Lambda}{\lambda} - \frac{1}{c}(\sigma\,\delta m + m\,\delta\sigma)}{m\,r_s}R_s + \frac{\alpha\,\delta R_s}{m\,r_s} - \frac{\alpha R_s}{m^2\,r_s}\delta m - \frac{\alpha R_s}{m\,r_s^3}R_s\cdot\delta R_s\right\} \qquad (25)$$

where

$$p'' = \frac{\partial^2 p}{\partial r_s^2} \qquad (25a)$$

In the planetocentric phase, we have, for the variational differential equations, the following:

$$\frac{d^2}{dt^2}\delta R_s = -\mu_p \frac{\delta R_s}{r_p^3} + 3\mu_p \frac{R_p \cdot \delta R_s}{r_p^5} R_p + H(\alpha) \cdot \frac{2p' R_s \cdot \delta R_s}{c\, m\, \lambda\, r_s} \Lambda$$

$$- H(\alpha) \frac{2p}{m c \lambda^3} [\Lambda(\Lambda \cdot \delta\Lambda) - \lambda^2 \delta\Lambda] - H(\alpha) \frac{2p\Lambda}{c m^2 \lambda} \delta m \qquad (26)$$

$$\frac{d}{dt}\delta m = - H(\alpha) \frac{2p'}{c^2 r_s} R_s \cdot \delta R_s \qquad (23)$$

$$\frac{d}{dt}\delta\sigma = \frac{2H(\alpha)}{c}\left\{\frac{p'\lambda}{m^2 r_s} R_s \cdot \delta R_s + \frac{p\Lambda \cdot \delta\Lambda}{m^2 \lambda} - \frac{2p\lambda}{m^3}\delta m\right\} \qquad (24)$$

$$\frac{d^2}{dt^2}\delta\Lambda = -\mu_p \frac{\delta\Lambda}{r_p^3} + 3\mu_p \frac{R_p \cdot \delta R_s}{r_p^5}\Lambda + 3\mu_p \frac{\delta\Lambda \cdot R_p}{r_p^5} R_p + 3\mu_p \frac{\Lambda \cdot \delta R_s}{r_p^5} R_p$$

$$+ 3\mu_p \frac{\Lambda \cdot R_p}{r_p^5}\delta R_s - 15\mu_p \frac{\Lambda \cdot R_p}{r_p^7}(R_p \cdot \delta R_s) R_p + \frac{2p''\alpha}{c\,m} \frac{R_s \cdot \delta R_s}{r_s} \frac{R_s}{r_s} H(\alpha)$$

$$+ H(\alpha)\frac{2p'}{c}\left\{\frac{\frac{\Lambda \cdot \delta\Lambda}{\lambda} - \frac{1}{c}(\sigma\delta m + m\delta\sigma)}{m r_s} R_s + \frac{\alpha \delta R_s}{m r_s} - \frac{\alpha R_s}{m^2 r_s}\delta m - \frac{\alpha R_s}{m r_s^3} R_s \cdot \delta R_s\right\} \qquad (27)$$

## V. Optimal Staging for Hybrid Propulsion Systems

We assume that we have a hybrid propulsion system consisting of a high-thrust chemical stage and a low-thrust, high specific impulse, power-limited stage. The problem is to find the optimal engine exchange time for the mission, if any exists at all. The method outlined here is based entirely on optimal control theory and avoids the tedious parameter search usually resorted to.

We start with the general transversality equation which must be satisfied at the terminals and at intermediate corners, such as might occur at interior discontinuities in the state.

The transversality condition is

$$0 \le \delta S = j\,\delta t\Big|_{t^-}^{t^+} + \Lambda\cdot\delta\dot{R}\Big|_{t^-}^{t^+} - \dot{\Lambda}\cdot\delta R\Big|_{t^-}^{t^+} + (\sigma - 1)\,\delta m\Big|_{t^-}^{t^+} \tag{28}$$

where $j$ is the Hamiltonian

$$j = \mu\frac{\Lambda\cdot R}{r^3} + \dot{\Lambda}\cdot\dot{R} - H(\alpha)\frac{2\,p(r_s)}{c\,m}\left(\lambda - \frac{\sigma m}{c}\right) \tag{29}$$

In considering the optimal engine exchange time from an engine with high thrust, $\frac{2\,p_1(r_s)}{c_1}$, mass $m_1$, and mass flow $\dot{m}_1$, to an engine with low thrust, $\frac{2\,p_2(r_s)}{c_2}$, mass $m_2$ (note that $m_2 = m_1$ - staging $\text{mass}_1$), and mass flow rate $\dot{m}_2$, the optimal exchange time is determined by the condition that a switch can be made only if the Hamiltonian, $j$, and $\sigma$ are continuous. Unless, of course, we run out of fuel in Stage 1 and are compelled to switch. In this case, there is no option and we are forced to take the penalty. Since $\Lambda$, $R$, $\dot{R}$ are continuous already, the optimal exchange time is provided by the time at which

$$s = \lambda\left[\frac{p_1(r_s)}{m_1 c_1} - \frac{p_2(r_s)}{m_2 c_2}\right] + \sigma\left[\frac{p_2(r_s)}{c_2^2} - \frac{p_1(r_s)}{c_1^2}\right] = 0 \tag{30}$$

In the event all the fuel is consumed during a stage and we are forced to switch, optimal trajectories for this situation are provided by a discontinuity in $\sigma$, based on the choice of keeping the Hamiltonian continuous.

$$\sigma^+ = \sigma^- \frac{c_2^2}{c_1^2}\frac{p_1(r_s)}{p_2(r_s)} - \lambda\left(\frac{c_2^2}{m_1 c_1}\frac{p_1(r_s)}{p_2(r_s)} - \frac{c_2}{m_2}\right) \tag{31}$$

In subsequent derivations, we shall use only the optimal engine exchange logic for hybrid propulsion systems and omit the logic for Eq. (31).

We are now ready to derive the jumps in the variational variables for the three types of discontinuities

a) $\alpha = 0$

b) $r = 0$

c) $s = 0$

## VI. Jump Discontinuities in the Variational Equations

We first compute the variation in the three switch parameters, $\alpha$, r and s. We have for $\alpha$,

$$\alpha = \lambda - \frac{\sigma m}{c} \tag{32}$$

and

$$\delta\alpha = \frac{\Lambda \cdot \delta\Lambda}{\lambda} - \frac{1}{c}(\sigma\delta m + m\delta\sigma) \tag{33}$$

The time derivative of $\alpha$ is given by

$$\frac{d}{dt}\left(\lambda - \frac{\sigma m}{c}\right) = \frac{\Lambda \cdot \dot{\Lambda}}{\lambda} - \frac{2p}{c^2 m}\alpha \tag{34}$$

Since $\alpha=0$ at the switch, we have

$$\frac{d}{dt}\alpha(\alpha=0) = \frac{\Lambda \cdot \dot{\Lambda}}{\lambda} \tag{34a}$$

The final result required is

$$\frac{\delta\alpha}{\dot{\alpha}} = \frac{\Lambda \cdot \delta\Lambda - \frac{\lambda}{c}(\sigma\,\delta m + m\,\delta\sigma)}{\Lambda \cdot \dot{\Lambda}} \tag{35}$$

For the planetary sphere of influence intercept parameter, r, we have

$$r = |R_s - P| - \bar{r}_p \tag{36}$$

$$\delta r = \frac{R_p \, \delta R_s}{r_p} \tag{36a}$$

$$\dot{r} = \frac{R_p \cdot \dot{R}_p}{r_p} \tag{36b}$$

and

$$\frac{\delta r}{\dot{r}} = \frac{R_p \cdot \delta R_s}{R_p \cdot \dot{R}_p} \tag{37}$$

For the hybrid propulsion engine exchange parameter s, we have

$$s = \lambda\left(\frac{p_1}{c_1 m_1} - \frac{p_2}{c_2 m_2}\right) + \sigma\left(\frac{p_2}{c_2^2} - \frac{p_1}{c_1^2}\right) \tag{38}$$

$$\begin{aligned}
\delta s = {} & \frac{\Lambda \cdot \delta\Lambda}{\lambda}\left(\frac{p_1}{c_1 m_1} - \frac{p_2}{c_2 m_2}\right) + \delta\sigma\left(\frac{p_2}{c_2^2} - \frac{p_1}{c_1^2}\right) \\
& + \left[\lambda\left(\frac{p_1'}{c_1 m_1} - \frac{p_2'}{c_2 m_2}\right) + \sigma\left(\frac{p_2'}{c_2^2} - \frac{p_1'}{c_1^2}\right)\right]\frac{R_s \cdot \delta R_s}{r_s} \\
& + \lambda\left(\frac{p_2}{c_2 m_2^2}\,\delta m^+ - \frac{p_1}{c_1 m_1^2}\,\delta m^-\right)
\end{aligned} \tag{39}$$

$$\dot{s} = \frac{\Lambda \cdot \dot{\Lambda}}{\lambda}\left(\frac{p_1}{c_1 m_1} - \frac{p_2}{c_2 m_2}\right) + \left[\lambda\left(\frac{p_1'}{c_1 m_1} - \frac{p_2'}{c_2 m_2}\right) + \sigma\left(\frac{p_2'}{c_2^2} - \frac{p_1'}{c_1^2}\right)\right]\frac{R_s \cdot \dot{R}_s}{r_s} \tag{40}$$

We are now in a position to describe the jump discontinuities in the variations of the state and their multipliers for the three critical types.

For $\alpha = 0$

$$\begin{aligned}
\delta R_s^+ &= \delta R_s^- \\
\delta \dot{R}_s^+ &= \delta \dot{R}_s^- + \frac{2p}{c\,m}\,\frac{\Lambda}{\lambda}\,[H(\alpha^+) - H(\alpha^-)]\,\frac{\delta\alpha}{\dot{\alpha}} \\
\delta \Lambda^+ &= \delta \Lambda^- \\
\delta \dot{\Lambda}^+ &= \delta \dot{\Lambda}^- \\
\delta m^+ &= \delta m^- - \frac{2p}{c^2}\,[H(\alpha^+) - H(\alpha^-)]\,\frac{\delta\alpha}{\dot{\alpha}} \\
\delta \sigma^+ &= \delta \sigma^- + \frac{2p\lambda}{c\,m^2}\,[H(\alpha^+) - H(\alpha^-)]\,\frac{\delta\alpha}{\dot{\alpha}}
\end{aligned} \tag{41}$$

For $r = 0$

$$\begin{aligned}
\delta R_s^+ &= \delta R_s^- \\
\delta \dot{R}_s^+ &= \delta \dot{R}_s^- + \left[\frac{(\mu_s + \mu_p)P}{p^3} + \frac{\mu_p R_p}{r_p^3} - \frac{\mu_s R_s}{r_s^3}\right] \operatorname{sgn}\left(R_p \cdot \dot{R}_p\right) \frac{\delta r}{\dot{r}} \\
\delta \Lambda^+ &= \delta \Lambda^- \\
\delta \dot{\Lambda}^+ &= \delta \dot{\Lambda}^- + \left[\mu_p \frac{\Lambda}{r_p^3} - 3\mu_p \frac{R_p \cdot \Lambda}{r_p^5} R_p - \mu_s \frac{\Lambda}{r_s^3} + 3\mu_s \frac{R_s \cdot \Lambda}{r_s^5} R_s\right] \operatorname{sgn}\left(R_p \cdot \dot{R}_p\right) \frac{\delta r}{\dot{r}} \\
\delta m^+ &= \delta m^- \\
\delta \sigma^+ &= \delta \sigma^-
\end{aligned} \tag{42}$$

For $s = 0$

$$\delta R_s^+ = \delta R_s^-$$

$$\delta \dot{R}_s^+ = \delta \dot{R}_s^- + 2\,H(\alpha)\,\frac{\Lambda}{\lambda}\left(\frac{p_2}{c_2\,m_2} - \frac{p_1}{c_1\,m_1}\right)\frac{\delta s}{\dot{s}}$$

$$\delta \Lambda^+ = \delta \Lambda^-$$

$$\delta \dot{\Lambda}^+ = \delta \dot{\Lambda}^- + \frac{2\,H(\alpha)\,\lambda}{r_s}\,R_s\left(\frac{p_2'}{c_2\,m_2} - \frac{p_1'}{c_1\,m_1}\right)\frac{\delta s}{\dot{s}}$$

$$\delta m^+ = \delta m^- \frac{\dot{s} + \lambda \dfrac{p_1}{c_1\,m_1^2}\left(\dfrac{2p_2}{c_2^2} - \dfrac{2p_1}{c_1^2}\right)}{\dot{s} + \lambda \dfrac{p_2}{c_2\,m_2^2}\left(\dfrac{2p_2}{c_2^2} - \dfrac{2p_1}{c_1^2}\right)} \tag{43}$$

$$- \frac{2\,H(\alpha)\left(\dfrac{p_2}{c_2^2} - \dfrac{p_1}{c_1^2}\right)}{\dot{s} + \lambda \dfrac{p_2}{c_2\,m_2^2}\left(\dfrac{2\,p_2}{c_2^2} - \dfrac{2\,p_1}{c_1^2}\right)} \left[ \frac{\Lambda \cdot \delta\Lambda}{\lambda}\left(\frac{p_1}{c_1\,m_1} - \frac{p_2}{c_2\,m_2}\right) + \left[p_1'\left(\frac{\lambda}{c_1\,m_1} - \frac{\sigma}{c_1^2}\right) - p_2'\left(\frac{\lambda}{c_2\,m_2} - \frac{\sigma}{c_2^2}\right)\right]\frac{R_s \cdot \delta R_s}{r_s} \right]$$

$$\delta\sigma^+ = \delta\sigma^- + 2\,H(\alpha)\,\lambda\left(\frac{p_2}{c_2\,m_2^2} - \frac{p_1}{c_1\,m_1^2}\right)\frac{\delta s}{\dot{s}}$$

## DISCUSSION

Whenever the engine is off and we are in a free-fall coast, closed form solutions of the equations of motion and the variational equations are available. Reference 2 contains expressions for these. Thus, it is only during burns that numerical integration must be resorted to.

A special problem occurs in estimating the time for the planetary sphere of influence intercept in a free-fall condition. It is recommended that a simple two-body search be carried out in free-fall to locate the time of $r = 0$ for the case of $R_p \cdot \dot{R}_p < 0$. For the case of free-fall within the planetary sphere of influence where we seek the planetary exit time, closed form expressions are available provided, of course, that the engine does not restart for times less than $r(t_{exit}) = 0$. It is recommended that simple closed-form two-body solutions be carried out in the free-fall case and periodic tests be made of $\alpha$, r and s, to ensure that the engine remains off and that the important critical crossings may be determined.

## REFERENCES


1. Pines, S.; "Constants of the Motion for Optimum Thrust Trajectories in a Central Force Field," AIAA Journal, Vol. 2, No. 11, November 1964, pp. 2010-2014.

2. Pines, S. and Fang, T.C.; "A Uniform Closed Solution of the Variational Equations for Optimal Trajectories During Coast," presented at the Colloquium on Advanced Problems and Methods for Space Flight Optimization, Institute of Mathematics, University of Liège, Belgium, June 19-23, 1967.