CRACK PROPAGATION UNDER VARIABLE LOAD HISTORIES IN LINEARLY VISCOELASTIC SOLIDS

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ABSTRACT

The growth of a crack in a viscoelastic solid is considered within the limitations of linear continuum mechanics. Starting from the local stress and deformation field at the tip of a crack and the first law of thermodynamics a non-linear differential equation is derived for the crack tip velocity in dependence on an arbitrary history of the crack tip stress intensity factor. Conditions for the simplification of this differential equation are discussed.

The case of cyclic loading (fatigue problem) is discussed in detail and sample calculations are given to illustrate the relative effect of maximum and minimum load during a cycle. Some experimental data is given for comparison with these calculations.

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INTRODUCTION

The fracture of viscoelastic materials is as yet only poorly understood. With the increased use of polymers in engineering designs, an estimation of failure in viscoelastic solids becomes more and more important. This is true whether one is concerned with fiber reinforced materials, including aircraft or automobile tires, or particulate composites materials such as solid propellant rocket fuel.

For a large class of materials failure occurs by crack growth rather than material flow and it is clear that for such solids an understanding of crack growth is mandatory if one wishes to examine the failure of engineering designs constructed with them.

Theories for predicting crack propagation in linearly elastic solids (brittle materials) have been advanced by Griffith [1], augmented by Irwin [2] and by Barenblatt [3]. In all these works, which today are considered fundamental in understanding the fracture of solids, the prime question to be answered is: Will the crack propagate or not? The problem of crack tip speed is usually only of importance in those cases when the high crack propagation rates invoke the inertial response of the material at the crack tip. For many engineering problems the high speed crack propagation is of little practical relevance since a rapidly spreading crack will quickly cause a structural failure, while the question of initial crack instability is much more important.

For viscoelastic solids it is, of course, also important to know whether or not a crack propagates. However, several problems arise in such materials which require a better understanding of crack growth behavior. On the one hand, cracks may propagate at such a slow rate that an observer may judge it stationary; thus interpretation of test information may be erroneous. On the other hand, it is highly desirable to know one's margin of safety in designing a structure containing inevitable cracks. Should the crack propagate, how much time remains before the structure fails totally? For very slowly propagating cracks a major portion of the structural life may contain crack growth. It is thus very important with regard to fracture of viscoelastic materials to determine the velocity of crack propagation and not only the stability condition as one normally does for brittle, rate insensitive materials.

In two previous papers crack propagation in viscoelastic solids has been considered under special conditions. One was concerned with the steady propagation of a crack in a strip [4] and the other with the monotonic growth of a crack in a large sheet [5]. In both cases the applied load was held constant to keep the problems conceptically simple and the analysis to a minimum. The main objective of this past work was to explore the feasibility of a crack propagation concept for a viscoelastic material. Having demonstrated that the calculations for these special and simple cases were well corroborated by experiment, it is natural to remove some of the assumptions pertinent to the specific nature of these problems. We shall thus consider now the problem of co-linear crack growth in an arbitrary two-dimensional geometry which is subjected to an otherwise arbitrary history of boundary loading.

The only restriction to be placed on these developments are the retention of quasi-static conditions (slow crack growth) and that the material may be approximated by a linearly viscoelastic solid possessing long time elastic behavior. This material assumption excludes from consideration those solids which exhibit ome sort of time or : ' · dependent ductility reminiscent of metal y ld.

THE POWER EQUATION

Let us consider a contour C_1 which encloses the crack tip as in Figure 1a. Its shape is arbitrary except that for co-linear crack growth it is convenient to choose one which is symmetric about the crack axis. If one permits the stresses to be singular at the crack tip and the displacements bounded then the traction $T_i(t)$ and displacements rates $\dot{u}_i(t)$ on this contour are uniquely related to the tractions on the boundary of the solid. Inertial effects are ignored. The first law of Thermodynamics requires that the rate of work done by these tractions equals the rate at which free energy inside the contour increases; plus the rate at which energy dissipated inside C_1 against viscous forces; plus the rate at which the work done against molecular forces of cohesion, $\Gamma\, dc/dt$, while the crack advances. Here $dc/dt = \dot{c}$ is the rate of crack growth or crack tip velocity and Γ is the specific surface energy. The latter is conceived as a rate independent quantity and insensitive to temperature changes. We have thus

$$\int_{C_1} T_i \dot{u}_i \, ds = \dot{D} + \dot{U} + \Gamma \dot{c} \tag{1}$$

If we replace the lower half of the material in Figure 1a by the equipollent (normal) tractions over the surface C_2 (cf. Figure 1b) the power balance becomes

$$\frac{1}{2}\int_{C_1} T_i \dot{u}_i \, ds + \int_{C_2} T_n \dot{u}_n \, dx = \frac{1}{2}(\dot{D} + \dot{U}) \tag{2}$$

Upon comparing (1) and (2) one finds readily

$$2\int_{C_2} T_n(t)\,\dot{u}_n(t)\,dx = \Gamma\dot{c} \tag{3}$$

It should be pointed out that if the normal traction vanishes on C_2 where the normal displacement is not zero and if the normal displacement vanishes on the remainder of C_2 then the integral in (3) gives, formally, zero contribution. We should therefore consider, in principle, a stress distribution on the axis as proposed by Barenblatt [3] such that there exists a section of C_2 say of length α on which both T_n and u_n do not vanish simultaneously (cf. Figure 1c). The physical nature of these tractions has been discussed previously by one of the authors [6] and with regard to the related subject of rate dependent peeling in detail by Kaelble [7,8].

In the absence of an extension of Barenblatt's solution to viscoelastic solids we shall use the classical singular solution with some appropriate modification. Two approaches are open to us: they are both explored in references [4] and [5]. Either one appears to give equally good results. In this paper we follow the approach of reference 1 and modify the power equation so that we may use the classical, singular stress distribution at the crack tip.

We consider the crack to grow from size c at time t_1 to size $c+\alpha$ at $t_1+\Delta t$ $t_1+\alpha/\dot{c}$, $\dot{c}$ being considered constant and different from zero in that time interval. For the simple problems considered previously α turned out to be a constant, independent of the rate of crack growth and on the order of 10^{-7}cm for the polyurethane rubber considered [4,5]. During the time interval Δt the tractions are to decrease at a constant rate from their maximum value to zero according to

$$T_n(x,t) = T(x,t)\left[1 - \frac{t-t_1}{\Delta t}\right] \tag{4}$$

At the end of the time interval the process starts over again. This step wise process is valid as long as the amount of crack advance α is very small. Experiments indicate that this is true [4, 5].

Equation (3) may now be re-written as

$$\int_0^{\alpha}\int_{t_1}^{t_1+\Delta t} T_n(x,t)\,\dot{u}_n(x-\alpha,t)\frac{dt}{\Delta t}\,dx = \frac{1}{2}\Gamma\dot{c} \tag{5}$$

where $\dot{u}_n(x,t)$ is the crack boundary displacement rate corresponding to $T_n(x,t)$ as given by equation (4). The evaluation of the crack propagation equation (5) hinges on determining this crack boundary displacement rate from the prescribed traction $T_n(x,t)$.

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* This equation duplicates, on a per-unit-time basis, the relation for crack growth given by Iriwn for brittle solids [2].

EVALUATION OF THE POWER EQUATION

Let the distribution of normal stresses ahead of the crack and on the line of crack propagation be given by the singular form

$$T(x,t) \;=\; K(t)\,[2\pi x]^{-1/2} \tag{6}$$

so that (4) becomes

$$T_n(x,t) = K(t)\left[1 - \frac{t-t_1}{\Delta t}\right][2\pi x]^{-1/2} \tag{7}$$

This classical stress distribution (6) applies to a large class of three-dimensional problems involving infinite domains [9], with the two-dimensional sheet containing a central crack as a special case [5]. It applies also if one assumes the material to possess a constant Poisson's ratio, an assumption which is valid only near the lower and upper extremes of the spectrum of relaxation times. For a more general material representation and geometries solutions for cracks growing in a viscoelastic solid are not available. There appears to be no reason to doubt the validity of the representation (6). In view of the approximation of the actual stress field by the simplier one, the possible difference is probably of little consequence.

It has been shown elsewhere [4, 9] that the crack boundary displacement corresponding to (6) is, for constant Poisson's ratio ν

$$u_n(x,t) = 4\chi\,[x/2\pi]^{1/2}\left\{ k(0)\,D(t) + \int_0^t D(t-\tau)\,\frac{\partial k(\tau)}{\partial \tau}\,d\tau \right\} \tag{8}$$

where

$$\varkappa = \begin{cases} 1 & \text{plane stress} \\ 1-\nu^2 & \text{plane strain} \end{cases}$$

D(t) = creep compliance in extension of the viscoelastic solid

The function K(t) is related to the stress intensity factor $K(t)\,[1-(t-t_1)/\Delta t]$ as follows. The traction $T_n(x,t)$, equation (7), is made up of two contributions, the tension $K(t)\,(2\pi x)^{-1/2}$ and a pressure $-K(t)\,(t-t_1)/\Delta t\,(2\pi x)^{-1/2}$ which causes the crack to open at the tip. It is only the latter component which enters equation (8) for the crack opening displacement. Thus

$$k(t) = K(t)\,\frac{t-t_1}{\Delta t} \tag{9}$$

Since the displacement u_n is zero for points immediately to the right of the crack tip until $t = t_1$, one finds

$$\underset{t \geq t_1}{u_n(x,t)} = 4\varkappa[x/2\pi]^{1/2}\left[k(t_1)D(t-t_1) + \int_{t_1}^{t} D(t-\tau)\,\frac{\partial k(\tau)}{\partial \tau}\,d\tau\right] \tag{10}$$

and upon substituting (9) and neglecting the first term in the brackets

$$u_n(x,t) = 4\varkappa\,[x/2\pi]^{1/2}\int_{t_1}^{t} D(t-\tau)\left[K(\tau) + \dot{K}(\tau)(\tau - t_1)\right] d\tau/\Delta t \tag{11}$$

the dot signifying differentiation with respect to τ. Denote the term in brackets by

$$F(t_1,\tau) \equiv K(\tau) + \dot{K}(\tau)(\tau - t_1) \tag{12}$$

and expand $F(t_1,\tau)$ in a Maclaurin's series about t_1 to obtain

$$F(t_1,\tau) = K(t_1) + 2\dot{K}(t_1)(\tau - t_1) + \frac{3}{2}\ddot{K}(t_1)(\tau - t_1)^2 + \frac{2}{3}\dddot{K}(t_1)(\tau - t_1)^3 + \cdots \tag{13}$$

Now (11) becomes

$$u_n(x,t) = 4\frac{\kappa}{\Delta t}[x/2\pi]^{1/2}\left\{K(t_1)\,D^{[1]} + 2\dot{K}(t_1)\,D^{[2]} + \frac{3}{2}\ddot{K}(t_1)\,D^{[3]} + \cdots\right\} \tag{14}$$

$$= 4\frac{\kappa}{\Delta t}[x/2\pi]^{1/2}\sum_{m=0}^{\infty}\frac{m+1}{m!}K_1^{(m)}\,D^{[m+1]}$$

where we have defined

$$D^{[m+1]} = D^{[m+1]}(t-t_1) = \int_{t_1}^{t}(\tau - t_1)^m\,D(t-\tau)\,d\tau$$

and

$$K_1^{(n)} = \left.\frac{\partial^n K}{\partial t^n}\right|_{t_1}$$

We may now use the displacement (14) along with the crack tip stress (7) to evaluate the power equation (5). These calculations are facilitated if we integrate (5) by parts with respect to time, taking into account that $u_n(x-\alpha, t_1) = 0$ and $T_n(x, t_1+\Delta t) = 0$. One obtains then

$$\int_0^{\alpha}\int_{\Delta t}^{t_1+\Delta t}\dot{T}_n(x,t)\,u_n(x-\alpha,t)\,dt\,dx = -\frac{1}{2}\Gamma\dot{c}\Delta t \tag{15}$$

which becomes, using (7) and (14)

$$2\int_0^{\alpha}\int_{t_1}^{t_1+\Delta t} \frac{x}{\pi}\left[\frac{x-\alpha}{x}\right]^{1/2}\left[\frac{K(t)}{\Delta t} - \dot{K}(t)\left(1-\frac{t-t_1}{\Delta t}\right)\right]\sum_{m=0}^{\infty}\frac{m+1}{m!}\,K_1^{(m)}D^{[m+1]}\;dt\,dx = \frac{1}{2}\,\Gamma\,\dot{c}\Delta t \quad (16)$$

or, after performing the integration with respect to x and noting that $\dot{c}\Delta t = \alpha$

$$\frac{2\Gamma}{\varkappa} = \int_{t_1}^{t_1+\Delta t}\left[\frac{K(t)}{\Delta t} - \dot{K}(t)\left(1-\frac{t-t_1}{\Delta t}\right)\right]\left[\sum_{m=0}^{\infty}\frac{m+1}{m!}\,K_1^{(m)}\,D^{[m+1]}\right]dt \quad (17)$$

We now expand the first bracket in the integrand in a Maclaurin series in $t-t_1$ and affect the multiplication of the resulting two series to obtain

$$\frac{2\Gamma}{\varkappa} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{n!}\left[(n+1)\frac{K_1^{(n)}}{\Delta t} - K_1^{(n+1)}\right]\frac{m+1}{m!}K_1^{(m)}\frac{1}{\Delta t}\int_{t_1}^{t_1+\Delta t}(t-t_1)^n D^{[m+1]}\,dt \quad (18)$$

Let us define the functions

$$B_{nm}(\Delta t) = \frac{(m+1)(m+n+2)}{D(\infty)\,\Delta t^{\,m+n+2}}\int_0^{\Delta t}\xi^n\int_0^{\xi}(\xi-\eta)^m D(\eta)\,d\xi\,d\eta \quad (19)$$

Then (18) may be written as

$$\frac{2\Gamma}{\varkappa} = D(\infty)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Delta t^m K_1^{(m)}\left[(n+1)\Delta t^n K_1^{(n)} - \Delta t^{n+1} K_1^{(n+1)}\right]}{m!\; n!\,(m+n+2)} B_{nm}(\Delta t) \tag{20}$$

Recalling that, $\Delta t = \alpha/\dot{c}$ and that t_1 was an arbitrary time signifying the onset of a small amount of crack propagation we may replace t_1 by the general time t to obtain a differential equation for the crack tip velocity $\dot{c}$ as a function of the time history of the stress intensity factor K(t):

$$\frac{2\Gamma E_\infty}{\varkappa} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\alpha/\dot{c})^m K^{(m)}(t)\left[(n+1)(\alpha/\dot{c})^n K^{(n)}(t) - (\alpha/\dot{c})^{n+1} K^{(n+1)}(t)\right]}{m!\; n!\,(m+n+2)} B_{nm}(\alpha/\dot{c}) \tag{21a}$$

where we have let $E_\infty = 1/D(\infty)$ denote the long time or equilibrium relaxation modulus. This equation may be cast into a different form by re-arranging the summation so as to sum the same orders of differentiation on K(t). One obtains then

$$\frac{2\Gamma E_\infty}{\varkappa} = \sum_{m=0}^{\infty} \frac{(\alpha/\dot{c})^m K^{(m)}(t)}{m!} \left\{ \frac{K(t)}{m+2} B_{om}(\alpha/\dot{c}) + \sum_{n=1}^{\infty} \frac{(n+1)(\alpha/\dot{c})^n K^{(n)}(t)}{n!\,(m+n+2)} \left[1 - \frac{n(m+n+2)}{(n+1)(m+n+1)}\right]\right\} \tag{21b}$$

ANALYSIS OF THE CRACK PROPAGATION EQUATION

Let us first examine the functions

$$B_{nm}(t) = \frac{(m+1)(m+n+2)}{D(\infty)\, t^{m+n+2}} \int_0^t \xi^n \int_0^\xi (\xi-\eta)^m D(\eta)\, d\eta\, d\xi \tag{19}$$

It is easy to show that

$$\begin{aligned} &1)\ \lim_{t\to 0} B_{nm}(t) = D(0)/D(\infty) \\ &2)\ \lim_{t\to\infty} B_{nm}(t) = \max|B_{nm}(t)| = 1 \\ &3)\ B_{nm} = B_{mn} \quad \text{as} \quad m \to \infty \end{aligned} \tag{22a}$$

Statements 1) and 2) follow from the monotonic behavior of the creep compliance D(t) and the third statement follows from the observation that the function $f(\xi,\eta)=(\xi-\eta)^m$ has the behavior $f(\xi) = \xi^m$ $0 \leq \eta \leq \xi$ as $m\to\infty$.

As a practical matter it should be observed that when $n,m \geq 1$ the difference between B_{nm} and B_{mn} is less than one percent so that B_{nm} may be considered symmetric in n and m. Furthermore, if one defines, from physical considerations a time t_m such that $D(t) = (\infty)$ whenever $t>t_m$ then it can be shown that

$$4)\ \lim_{n+m\to\infty} B_{nm}(t) = 1 \qquad t > t_m \tag{22b}$$

More specific limit behavior can be established rigorously only if more detailed behavior of the creep compliance is specified in the transition-time zone $t \sim t_m$. It appears however that there exists a limit function $B_{\infty\infty}(t)$ such that $B_{nm}(t) \geq B_{\infty\infty}(t)$ for all t. Figure 2 shows several functions $B_{nm}(t)$ along with the creep compliance of a polyurethane rubber [10].

It follows from the upper limit on B_{nm} and the finiteness of $\dot{c}$ that the double series in the crack propagation equation (21) converges always. Furthermore, because of the factorial functions, convergence is very rapid.

Immediate simplification of the crack propagation equation (21) may be achieved for some special stress histories if we interprete physically the integral (11) for the crack opening displacement and the subsequent series expansion. If during the time interval Δt, the stress intensity factor does not change significantly, then $\dot{K}(\tau) \doteq 0$ and $K(\tau)$ may be taken outside of the integral (11); the need for the series expansion (13) does then not arise. It follows that the higher derivatives of the crack tip stress intensity factor are necessary only if the stress intensity factor changes appreciably during each time interval Δt. Such would be the case primarily if the applied loading changes rapidly while the crack propagates slowly, i.e., under strongly transient conditions. It would appear, therefore, that if the boundary loading and the geometry is such as to produce a fairly steady stress intensity factor while the crack propagates through a length on the order of a micron the differential equation (21) can be very much simplified. Such would be the case in many monotonic load histories, but not in certain cyclic load histories, as we shall discuss later, (and not for cracks which are initially on the order of a micron or smaller).

Specifically, for slow changes in the stress intensity factor one would have

$$\frac{4\Gamma E_\infty}{\varkappa} \doteq K^2(t)\, B_{oo}(\alpha/\dot{c}) \tag{23}$$

which is the same as derived in [4] for the growth of a crack in a viscoelastic strip under constant strain. To the extent that $B_\infty(t) \doteq D_{cr}(at)$, $a = 3$ as may be seen from Figure 2, this result is also the same as that obtained in [5] for the growth of a single crack in a large plate under constant load.

It has been shown earlier [4, 5] that if the stress intensity factor K(t) is less than or equal to a lower critical value K^* given by

$$K^* = 2\left[\Gamma E_\infty/\varkappa\right]^{1/2} \tag{24}$$

crack propagation is not possible, i.e., then $\dot{c} = 0$. In this event the expansion leading to (21) is not valid and hence the full equation (21) cannot add any new information. Thus the crack propagation equation applies only if $K(t) > K^*$.

CRACK PROPAGATION UNDER CYCLIC LOADING

We wish now to exemplify some characteristics of crack propagation under cyclic variations of the stresses at the tip of the crack. In general this requires the solution of a mixed boundary value problem for the viscoelastic solid, in order to deduce the dependence of the crack tip stresses on the boundary loading. However, we assume the stresses at the crack tip known and of the form (6).

For illustrative purposes we choose a simple geometry, the semi-infinite crack in an infinite sheet (cf. Figure 3). This geometry has the advantage that a change in crack length does not change the stress intensity factor, so that the only time dependent changes result from the cyclic loading. If the boundary displacement u_o (strain ϵ_o cf. Figure 3) is sinusoidal then the stress intensity factor will change in a sinusoidal fashion except that the displacement and stress intensity factor will be out of phase. Under the assumption of a constant Poisson's ratio for material response near the rubbery or long time domain, the displacement loading and the crack tip stresses will be nearly in phase. In particular, for the material represented in Figure 2, this condition will be reasonsbly satisfied up to 100 cps.

Let us, therefore, consider the crack tip stress intensity history

$$K(t) = \mathrm{Re}\left\{ E_\infty [b/(1-\nu^2)]^{1/2} \epsilon_o [1 - Ai\, e^{i\omega t}]\right\} \tag{25}$$

where Re denotes the real part of the complex expression

A = some non-negative number

b = strip width (cf. Figure 3)

ν = Poisson's ratio, equal to 1/2 for our purposes.

ϵ_o = boundary strain (cf. Figure 3)

Substitution of (25) into (21) yields for plane stress ($\varkappa = 1$) and $\nu = 1/2$

$$\frac{3\Gamma}{\epsilon_0^2 E_\infty b} = \left\{1 + A[2\sin\omega t - (\alpha\omega/\dot{c})\cos\omega t] + A^2[\sin 2\omega t - (\alpha\omega/\dot{c})\cos 2\omega t]\right\} B_{oo}(\alpha/\dot{c})$$
$$- \operatorname{Re} A \sum_{m=1}^{\infty} (i\alpha\omega/\dot{c})^m \cdot \frac{ie^{i\omega t} - Ae^{2i\omega t}(1 - i\alpha\omega/\dot{c})}{m!\,(m+2)} B_{om}(\alpha/\dot{c})$$
$$- \operatorname{Re} A \sum_{n=1}^{\infty} (i\alpha\omega/\dot{c})^n \frac{ie^{i\omega t} - Ae^{2i\omega t}}{n!\,(n+2)} [n+1-(i\alpha\omega/\dot{c})] B_{no}(\alpha/\dot{c}) \tag{26}$$
$$- \operatorname{Re} A^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (i\alpha\omega/\dot{c})^{m+n} \frac{e^{2i\omega t}}{m!\,n!\,(m+n+2)} [n+1-(i\alpha\omega/\dot{c})] B_{nm}(\alpha/\dot{c})$$

We observe that if $A \to 0$, $K(t)$ becomes constant and the equation for steady crack propagation in a strip results [4]. The same is true if $\omega \to 0$. If neither A nor ω vanish it is clear from examining only the dominant term in equation (26) that the expression in curly brackets is not in phase with the applied strain, the phase shift depending on the frequency ω and the crack velocity $\dot{c}$. The same will be found true upon closer examination of the summands in the series. We should therefore expect that the crack tip velocity is not in phase with the stresses at the crack tip; furthermore since the phase shift depends inversely on the crack tip velocity, the phase shift will vary during each cycle so that for a sinusoidal strain superposed on a constant strain the crack tip velocity will not be a sinusoidal function of time superposed on an average velocity. If A is close to unity or greater, there are positions of the cycle during which $K(t) < K^*$. During these portions the crack tip will remain stationary.

Thus if we think of the strain as an input to a process and of the crack tip velocity as the result (output) of this process, then the process -- equation (26) -- acts like a highly non-linear rectifier.

Let us substitute some realistic values into (25) and (26) to calculate the time history of crack propagation. We would be interested in studying the effect of varying ϵ_o, A and ω. In order to facilitate the calculations we observe from Figure 2 that the B_{nm} for $n, m \leq 5$ fall all in a narrow range, a reasonable average being

$$B_{nm}(t) \doteq D(3t)/D(\infty) \tag{27}$$

We shall therefore approximate (26) by using the simplification (27) and terminate the series after n = m = 5. For the surface energy Γ we use the value determined on the polymer swollen in toluene [11], Γ = 0.1 lb/in. Solution of equation (26) for the velocities was achieved by Newton's method on a digital computer. Because of the automatic nature of these calculations computation of velocities under conditions of high or very low strain met with some difficulty, which are, however, not of a fundamental nature.

For this reason the results shown in Figure 4 are more indicative of crack growth behavior than complete. Nevertheless, it is clear from this figure that small sinusoidal load variations superposed on a constant load produce nearly sinusoidal velocity variations about a mean. As the variation in load increases, the sinusoidal character of the velocity history changes, the velocity history becomes less and less sinusoidal in character. Frequency has three obvious effects; it changes the shape of the velocity-time cycle as illustrated in Figure 4c; it causes a phase shift (lead) of the velocity cycle and an increase in velocity increases the peak velocity at higher frequencies. Responses to frequencies of 10 and 1 cpm gave virtually identical results as would, presumably still lower frequencies as long as ϵ_o and A remains constant. It seems, therefore, that for these low frequencies (1/ω greater than the maximum relaxation time) the simplified equation (23) is adequate.

For large variations of the strain about the mean (A=0.5, 1) the crack will temporarily stop because the stress intensity factor drops below the critical value K^*, equation (24). In this case the full non-linear character of equation (21) comes into play in producing a crack velocity history, the characteristics of which are displayed in 4c and d. Note that the increase of frequency in 4c causes a change in the asymmetry of the velocity-time cycle. The characteristics of the behavior near small strains for A=1 are quite the same as those observed in an experiment, as recorded in Figure 5. Closer comparison of theory and experiment meets primarily with instrumentation difficulties at this time. Cyclic crack propagation velocities at small strains are difficult to measure since they are on the order of one hundredth of an inch per minute. On the other hand, large strains produce higher velocities, but they are difficult to calculate and require more careful numerical work than this study seems to warrant. In addition, larger strains introduce non-linearities which are not accounted for in these calculations.

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LIST OF FIGURES

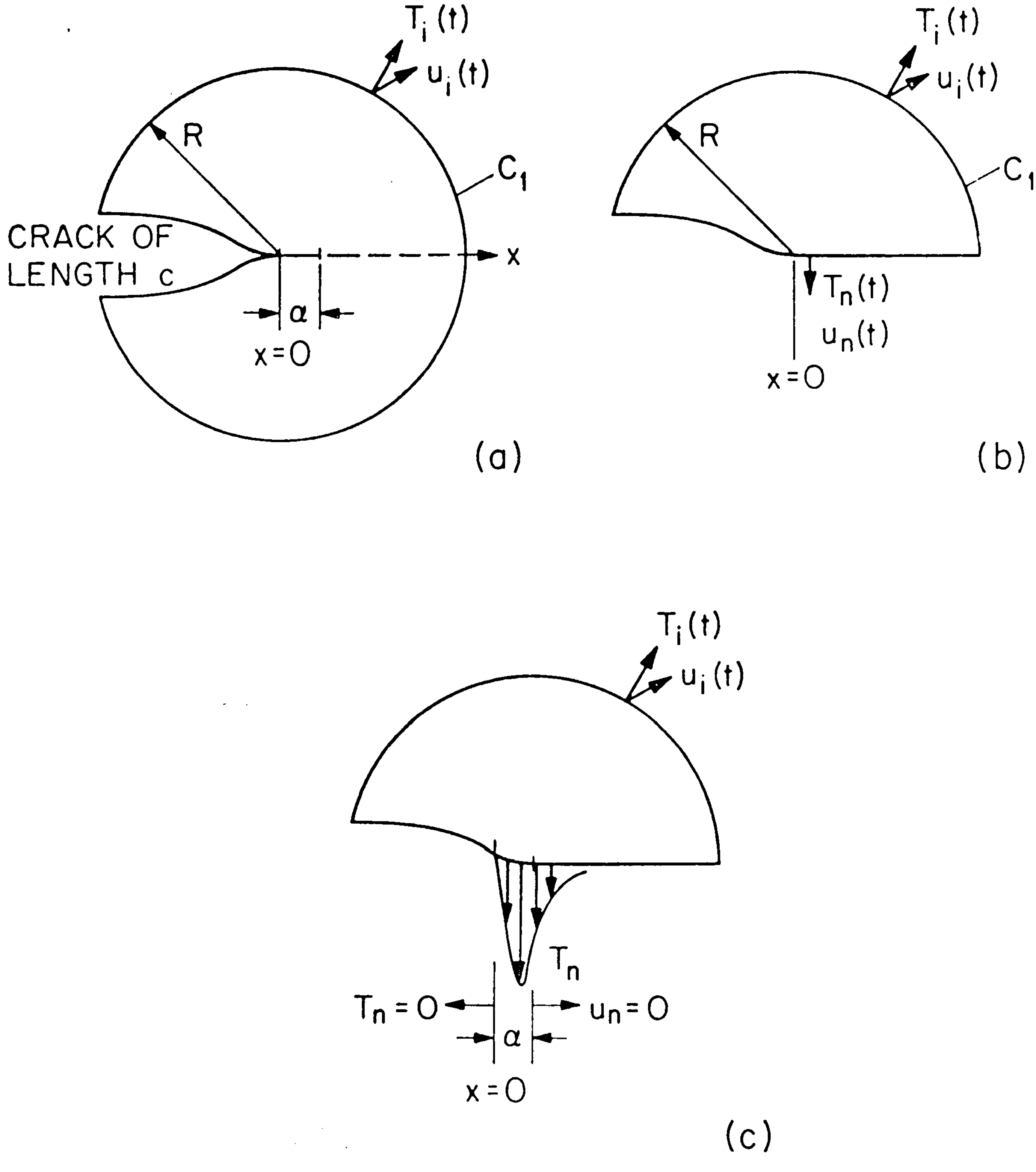

Fig. 1 Control volume around and stress distribution ahead of crack.

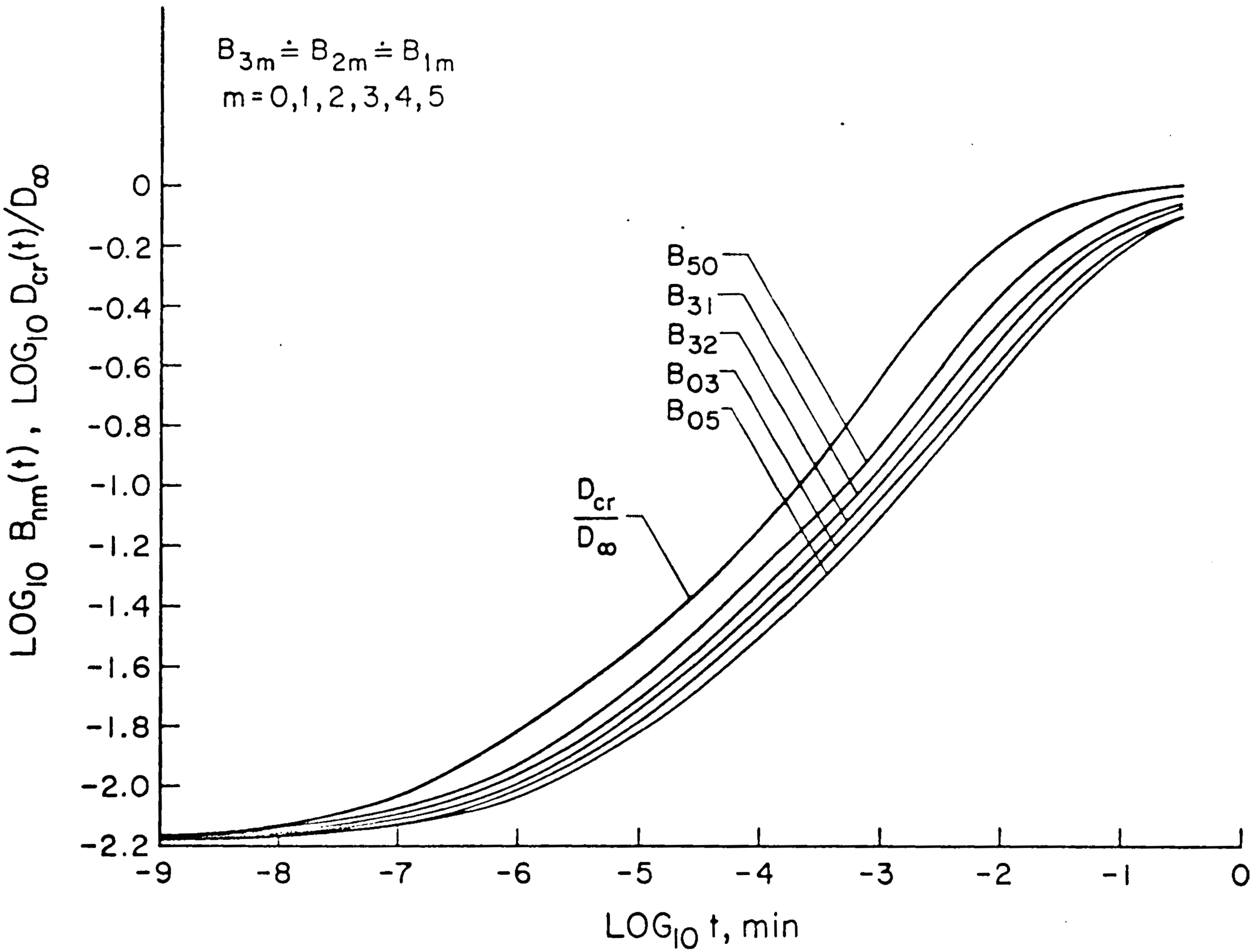

Fig. 2 Creep compliance and functions $B_{nm}(t)$ at 0 °C; for $n, m \leq 5$, $B_{05} < B_{nm} < B_{50}$.

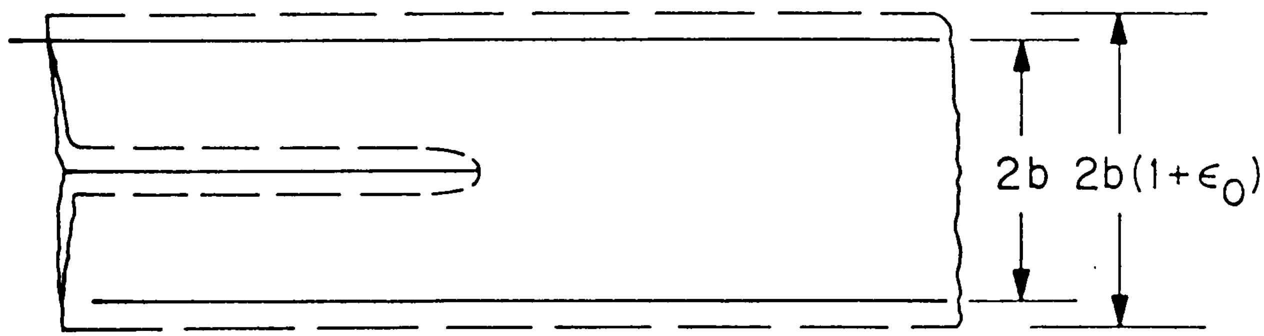

Fig. 3 Cracked strip geometry for crack propagation study.

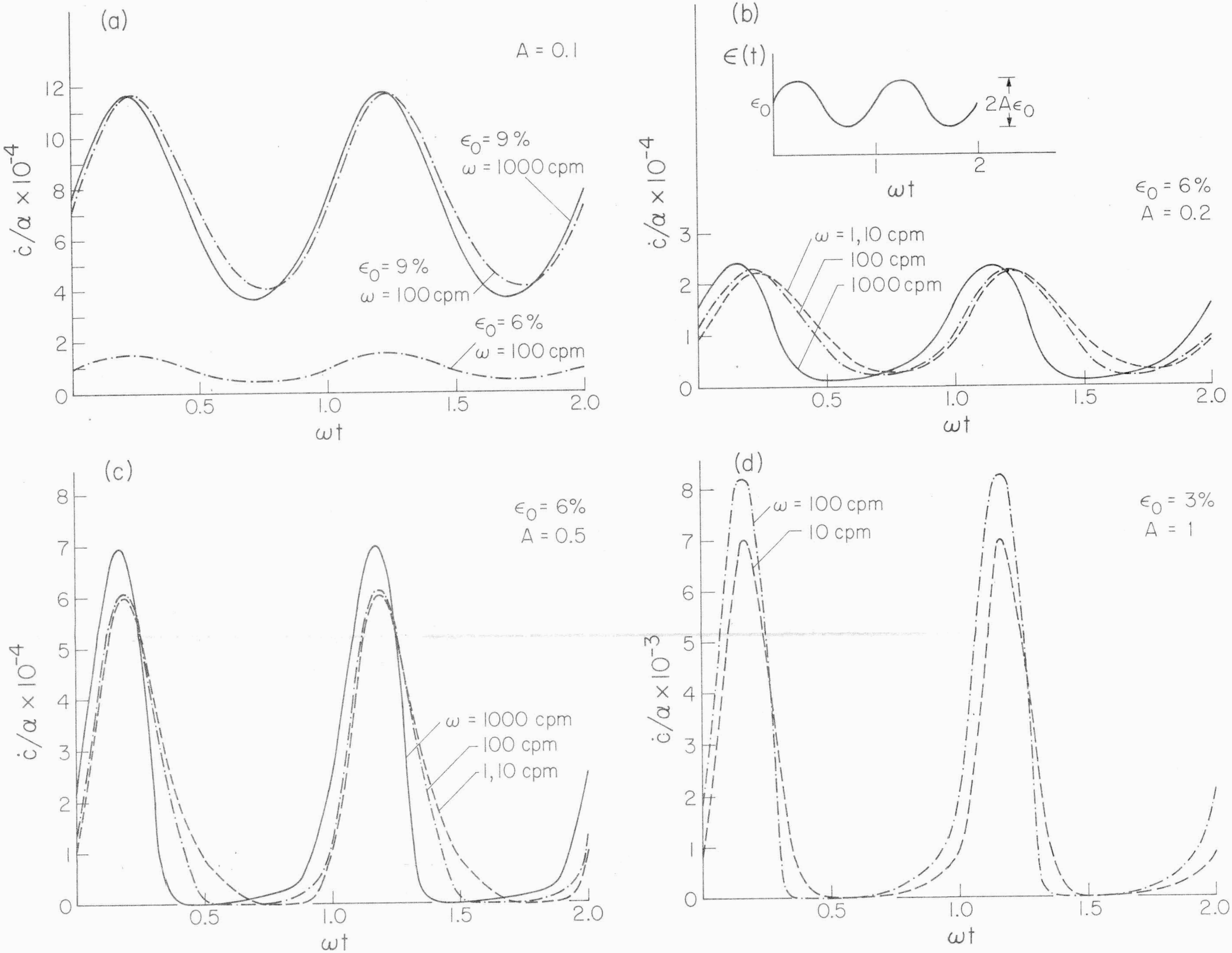
(a)
A = 0.1
ϵ_0 = 9 %
ω = 1000 cpm
ϵ_0 = 9 %
ω = 100 cpm
ϵ_0 = 6 %
ω = 100 cpm
ċ/α × 10⁻⁴
ωt
(b)
ε(t)
ϵ_0
2Aϵ_0
ωt
ϵ_0 = 6 %
A = 0.2
ω = 1, 10 cpm
100 cpm
1000 cpm
ċ/α × 10⁻⁴
ωt
(c)
ϵ_0 = 6%
A = 0.5
ω = 1000 cpm
100 cpm
1, 10 cpm
ċ/α × 10⁻⁴
ωt
(d)
ω = 100 cpm
10 cpm
ϵ_0 = 3%
A = 1
ċ/α × 10⁻³
ωt

Fig 4

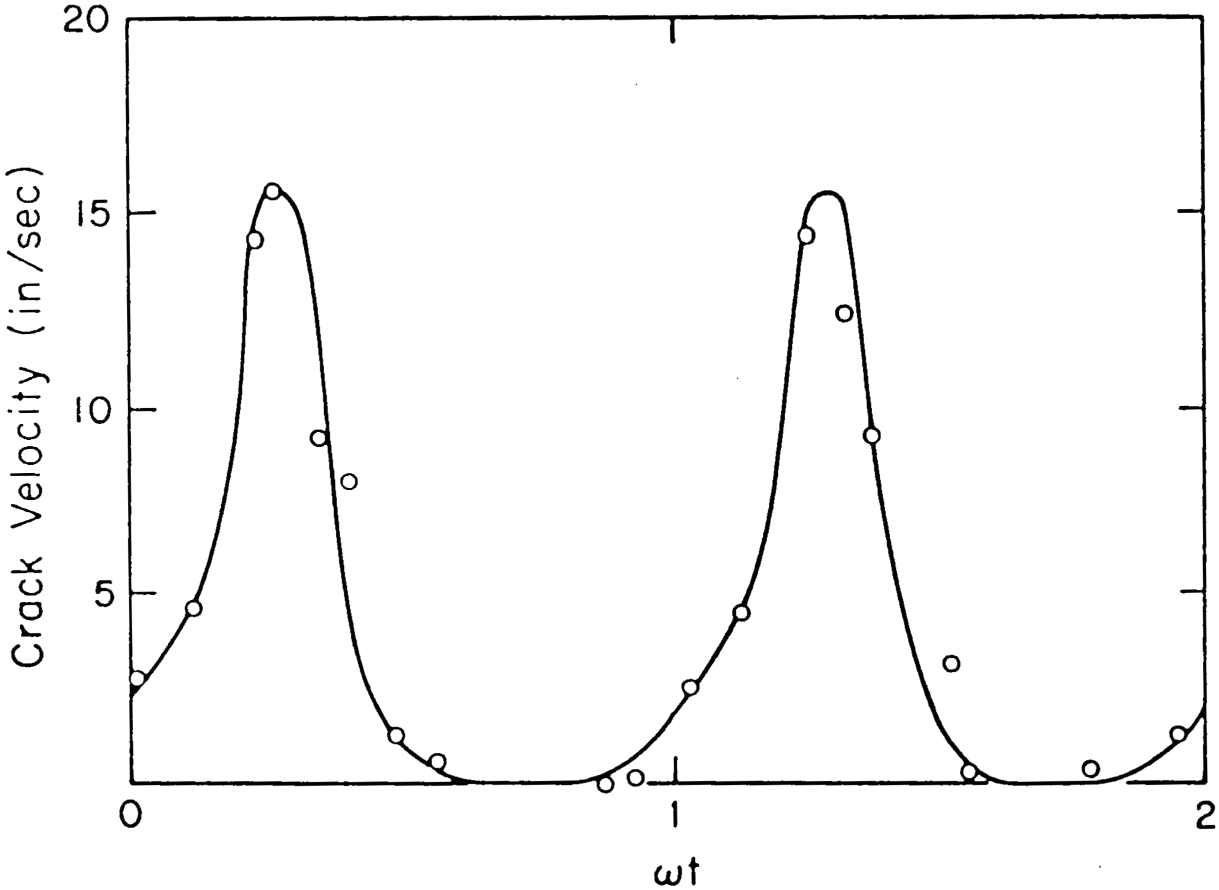

Fig. 5 Experimental crack velocity as a function of time (ω = 360 cpm, ϵ_0 = 12 %, A = 1).