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A RAPIDLY CONVERGING TRIANGULAR PLATE ELEMENT

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Finite elements have been developed for the analysis of plane stress-plane strain problems[1], bodies of revolution[2], and shells of revolution[3,4]. However, considerable difficulties have been encountered in the development of suitable plate and shell elements. These difficulties are caused by the need for both displacement and slope compatibility along the lines connecting the elements which in turn is caused by the dependence of the internal energy on the second derivatives of the normal displacement. One way to avoid the need for compatibility of first derivatives is to include transverse shear deformations thus yielding a strain energy expression as a function of the first derivatives only. This approach has been used in Refs. 5, 6, and 7. However, those researchers used linear functions for all variables which yields a solution which converges slowly with mesh refinement. The purpose of this note is to present a compatible triangular plate element based on a complete third order polynomial for the normal displacement and second order polynomials for the inplane displacements. A nine degree of freedom element

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is obtained by requiring the transverse shear deformations to be zero at the nodes and along the sides of the element.

Consider a laminate of thickness dz located at a distance z from the midsurface of the plate (Fig. 1). The displacements of this laminate are represented by

$$u = z \left[L_1(2L_1-1) \frac{\partial u_1}{\partial z} + L_2(2L_2-1) \frac{\partial u_2}{\partial z} + L_3(2L_3-1) \frac{\partial u_3}{\partial z} + 4L_1L_2 \frac{\partial u_4}{\partial z} + 4L_2L_3 \frac{\partial u_5}{\partial z} + 4L_1L_3 \frac{\partial u_6}{\partial z} \right]$$

$$v = - - - - - - - - \qquad (1)$$

$$w = L_1^2(L_1+3L_2+3L_3)w_1 + L_1^2(c_3L_2-c_2L_3) \frac{\partial w_1}{\partial x} + L_1^2(b_2L_3 - b_3L_2) \frac{\partial w_1}{\partial y} + \ . \ . \ . \ + \alpha L_1L_2L_3$$

where u, v, w = displacements in x, y, and z directions respectively

L_1, L_2, L_3 = area coordinates (Ref. 8)

$b_i = y_j - y_k$

$c_i = x_k - x_j$

α = generalized coefficient

The expression for v is of the same form as the expression for u. The additional six terms in the expression for w are obtained by cyclic permutation.

A nine degree of freedom element is obtained by requiring the transverse shear strains to be zero at the corners and along the sides of the element and, by assuming the slope normal to the element at the middle of the side is one-half the sum of the values at the corner nodes. These conditions yield

$$\frac{\partial w_i}{\partial x} = - \frac{\partial u_i}{\partial z} \qquad i = 1, 2, 3$$

$$\frac{\partial w_i}{\partial y} = - \frac{\partial v_i}{\partial z}$$

$$\frac{\partial u_4}{\partial z} = \frac{1}{b_3^2 + c_3^2} \left[\frac{3}{2} c_3 w_1 + \left(\frac{b_3^2}{2} - \frac{c_3^2}{4}\right) \frac{\partial u_1}{\partial z} + \frac{3 b_3 c_3}{4} \frac{\partial v_1}{\partial z} - \frac{3}{2} c_3 w_2 + \left(\frac{b_3^2}{2} - \frac{c_3^2}{4}\right) \frac{\partial u_2}{\partial z} + \frac{3 b_3 c_3}{4} \frac{\partial v_2}{\partial z} \right] \qquad (2)$$

$$\frac{\partial v_4}{\partial z} = \frac{1}{b_3^2 + c_3^2} \left[- \frac{3}{2} b_3 w_1 + \frac{3}{4} b_3 c_3 \frac{\partial u_1}{\partial z} + \left(\frac{c_3^2}{2} - \frac{b_3^2}{4}\right) \frac{\partial v_1}{\partial z} + \frac{3}{2} b_3 w_2 + \frac{3}{4} b_3 c_3 \frac{\partial u_2}{\partial z} + \left(\frac{c_3^2}{2} - \frac{b_3^2}{4}\right) \frac{\partial v_2}{\partial z} \right]$$

The equations for $\frac{\partial u_5}{\partial z}$, $\frac{\partial v_5}{\partial z}$, $\frac{\partial u_6}{\partial z}$, and $\frac{\partial v_6}{\partial z}$ are obtained by cyclic permutation. Interelement compatibility is still satisfied after Eqs. 2 are applied.

Neglecting the strain energy due to transverse shear the strain energy expression for the element is the same as that used in plane stress problems.

$$U = \frac{E}{1-\nu^2} \int (\varepsilon_x^2 + \varepsilon_y^2 + 2\nu\varepsilon_x\varepsilon_y + \frac{1-\nu}{2} \varepsilon_{xy}^2) \; dAdz \qquad (3)$$

where E = Young's modulus

ν = Poission's ratio

$\varepsilon_x = \frac{\partial u}{\partial x}$

$\varepsilon_y = \frac{\partial v}{\partial y}$

$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$

It is noted that w does not enter the strain energy expression but enters through Eqs. 2. The element stiffness matrix is obtained by substituting the assumed displacement functions into the strain energy expression and integrating over the volume of the element using the relation[9]

$$\int L_1^i L_2^j L_3^k dA = \frac{i!j!k!}{(n+2)!} 2A \qquad (3)$$

where $n = i + j + k$

A = area of element

The element stiffness matrix is symmetric and positive definite.

The accuracy of this element representation is demonstrated by solving simply supported and clamped plates under uniform and concentrated loadings. For the uniform pressure loading one-third of the total load was allocated to each displacement at the corners of the element. The plate idealization is shown in Fig. 2 and the results for the deflection coefficients at the center of the plate are presented in Table 1. It is noted that the basic element listed as QQ3 (quadratic u - quadratic v-third order w) is a little soft. The results listed as QQ3-3 were obtained by dividing the original element into three sub-elements with an internal node at the centroid of the original element. This gives a nine degree of freedom element with zero transverse shear along the sides and along the lines from the corners to the centroid. It is noted that the convergence characteristics of the QQ3-3 element are quite good.

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TABLE I

Deflection Coefficients For Square Plate

Mesh Size	Simply Supported Plate						Clamped Plate					
	Uniform Pressure			Concentrated Load			Uniform Pressure			Concentrated Load		
	QQ3	QQ3-3	Ref. 10	QQ3	QQ3-3	Ref. 10	QQ3	QQ3-3	Ref. 10	QQ3	QQ3-3	Ref. 10
2x2	.004162	.003666	.003446	.01248	.01100	.01378	.001890	.001533	.001480	.005669	.004598	.005919
4x4	.004056	.003942	.003939	.01169	.01128	.01233	.001547	.001452	.001403	.005856	.005506	.006134
6x6	.004064	.004012	-------	.01165	.01145	------	.001406	.001357	-------	.005763	.005569	-------
8x8	.004065	.004036	.004033	.01163	.01152	.01183	.001347	.001318	.001304	.005708	.005587	.005803
10x10	.004065	.004046	-------	.01163	.01155	------	.001319	.001299	-------	.005678	.005596	-------
12x12	.004064	.004051	.004050	.01162	.01157	.01172	.001303	.001289	.001283	.005660	.005601	.005710
14x14	.004064	.004054	-------	.01162	.01157	------	.001293	.001283	-------	.005649	.005604	-------
16x16	.004064	.004056	.004056	.01161	.01158	.01167	.001287	.001279	.001275	.005641	.005606	.005672
Exact (Ref. 11)		.00406			.01160			.00126			.00560	

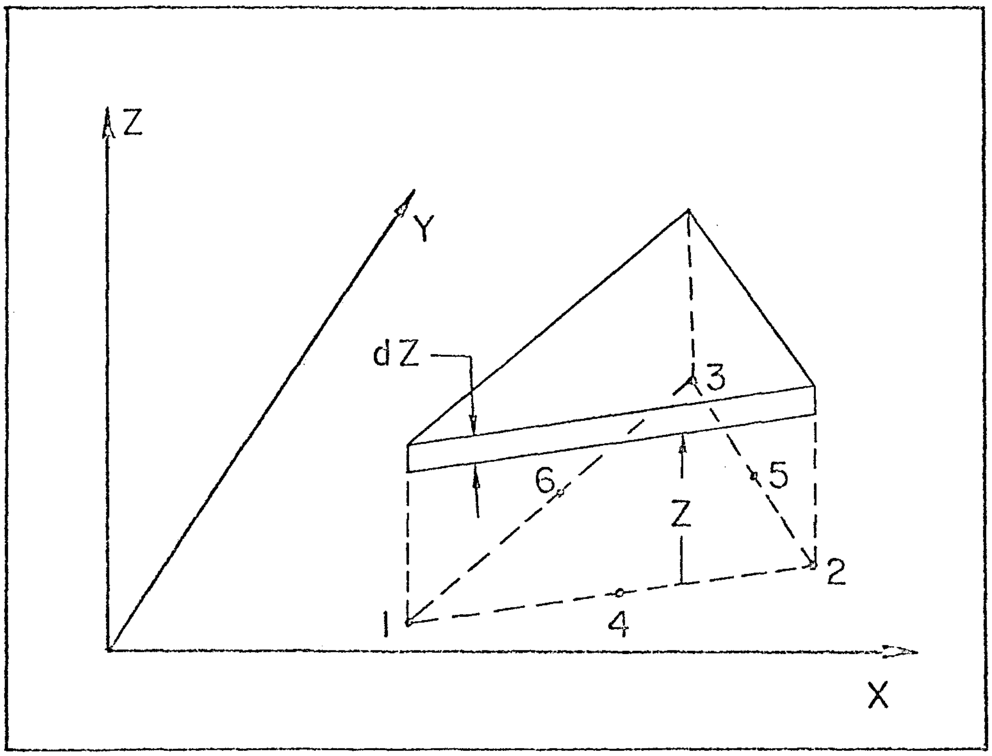

FIG. 1 ELEMENT LAMINATE AND NODES

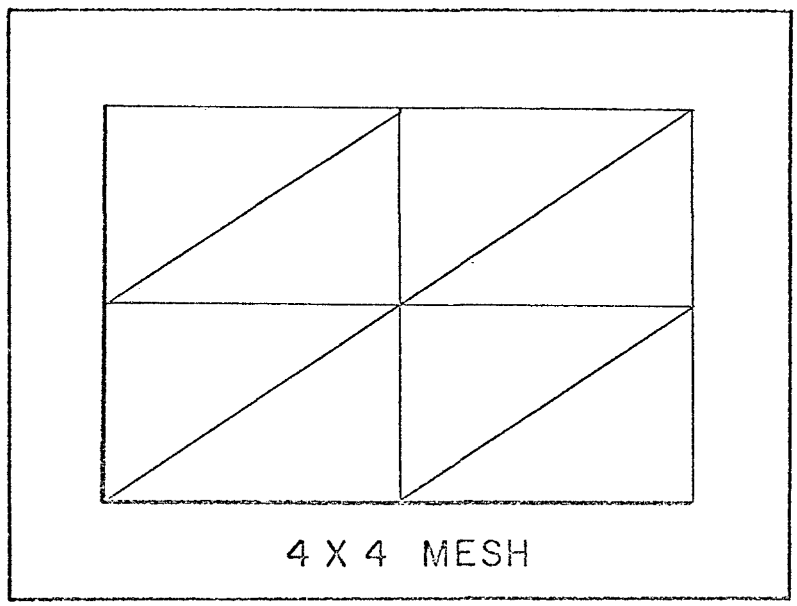

FIG. 2 PLATE IDEALIZATION FOR QUARTER OF A PLATE