NASA-CR-107929

UNIVERSITY OF CONNECTICUT

SCHOOL OF ENGINEERING

STORRS, CONNECTICUT

A STABILITY BOUND FOR RELAY CONTROL SYSTEMS IN NON-PHASE VARIABLE FORM

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November 1969

This work has been sponsored by the
National Aeronautics and Space Administration
Research Grant NGL-07-002-002

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by

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I INTRODUCTION

This paper presents an approach for determining a stability bound for relay control systems in non-phase variable form. The system under consideration is assumed to be subject to unknown but bounded parameter variations.

In a recent investigation [1], a semi-definite Lyapunov function has been employed in the design of relay control systems, subject to parameter variations and external disturbances, for systems in which the state vector is not in phase variable form. It has been noted, however, that unless the initial condition lies on the switching plane, a semi-definite Lyapunov function cannot be used to determine the region of stability in response to initial conditions. Consider for example the situtation in Figure 1. Although $\underline{x}(0)$ is within R_0, it does not follow with $V > 0$ and $\dot{V} < 0$ off the switching plane, that $\underline{x}\ (t;t>0)$ must reach the switching plane, since if $\underline{x}(t)$ leaves R_0 as shown, then the sign of $\dot{V}$ is not guaranteed outside of R_0.

In the approach of this paper, a bound inside which all trajectories will be stable can be found if a similarity transformation is used for each parameter set within the parameter space. Each resultant bound determined in the phase-variable space is then mapped into the original space where some measure of a bound on stable performance may be obtained.

II DESIGN APPROACH

Equivalence of control in Non-Phase and Phase-Variable Space

Consider the linear, time-invarient controllable system defined by the equation:

$$\dot{\underline{x}} = A\ \underline{x} + \underline{b}\ u \tag{2.1}$$

where

$\underline{x}$ = 1xn state vector

A = nxn plant matrix.

$\underline{b}$ = 1xn input control vector

u = scalar input .

The system is in non-phase variable form and coefficients in A are known to lie within some bounded region of the parameter space. In addition, A need not be a stability matrix.

A positive semi-definite Lyapunov function $V_{psd}=1/2\ [\ \gamma(x)]^2$ is defined, where $\gamma(x) = \underline{k}_0^t\ \underline{x} = 0$ constitutes a linear switching plane in $\underline{x}$ space. In order to

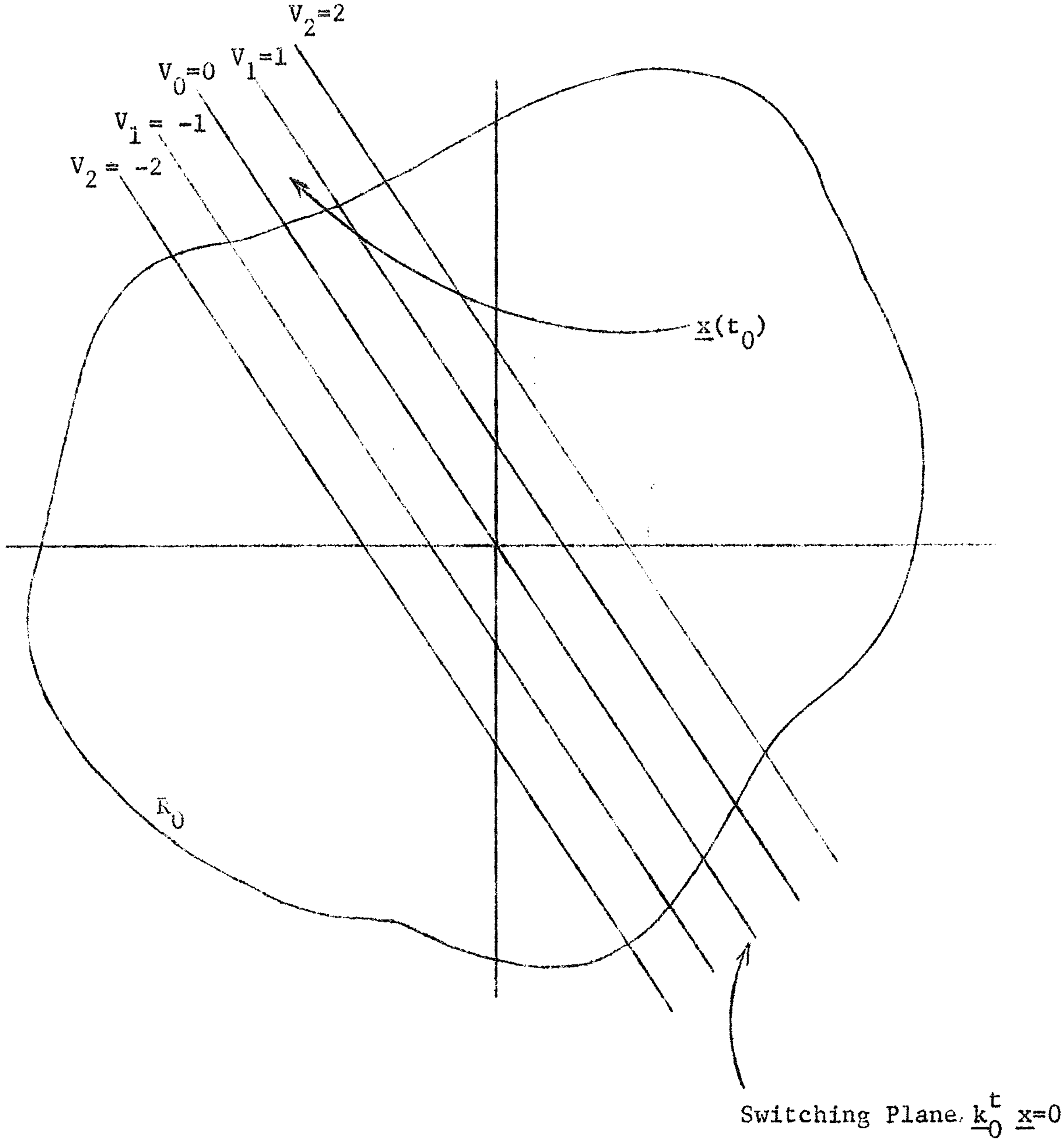

Figure 1

$\underline{x}$ Space Trajectory and Semi-definite Lyapunov Contour insure that motion on the switching plane is stable, the necessary control law has been shown [1] to be:

$$u = L \text{ sgn } [\gamma(\underline{x})] \tag{2.2}$$

where

$$L \geq \left| \frac{\underline{k}_0^t A \underline{x}}{\underline{k}_0^t \underline{b}} \right|$$

Consider now a particular parameter set $\{\alpha_i\}$ for which $A = A_i$, $\underline{b} = \underline{b}_i$. For this set there is a transformation [2]

$$\underline{y} = T_i \underline{x} \tag{2.3}$$

for which $\underline{y}_i$ is in phase- variable form and for which a control law can be defined by

$$u = L \text{ sgn } (\underline{k}^t \underline{y}) \tag{2.4}$$

For u to be equivalent in equations (2.2) and (2.4), it is required that

$$\underline{k}^t T_i = \underline{k}_0^t. \tag{2.5}$$

A system block diagram is shown in Figure 2.

Transformation to Phase-Variable Form

As shown in the design approach of [1], certain restrictions are placed on the magnitudes of the components of $\underline{k}_0$.* Hence, with $\underline{k}_0$ fixed, there exists for each parameter set $\{\alpha_i\}$ a $\{T_i, \underline{k}\}$ which describes the system in terms of phase variable coordinates.

As a result of the transformation T_i, the system may be represented in the canonic (phase variable) space as:

$$\dot{\underline{y}} = A_c \underline{y} + \underline{b}_c u$$

* In Reference 1, the model tracking problem was considered. In this paper, no model is considered since attention is focussed on the initial condition problem.

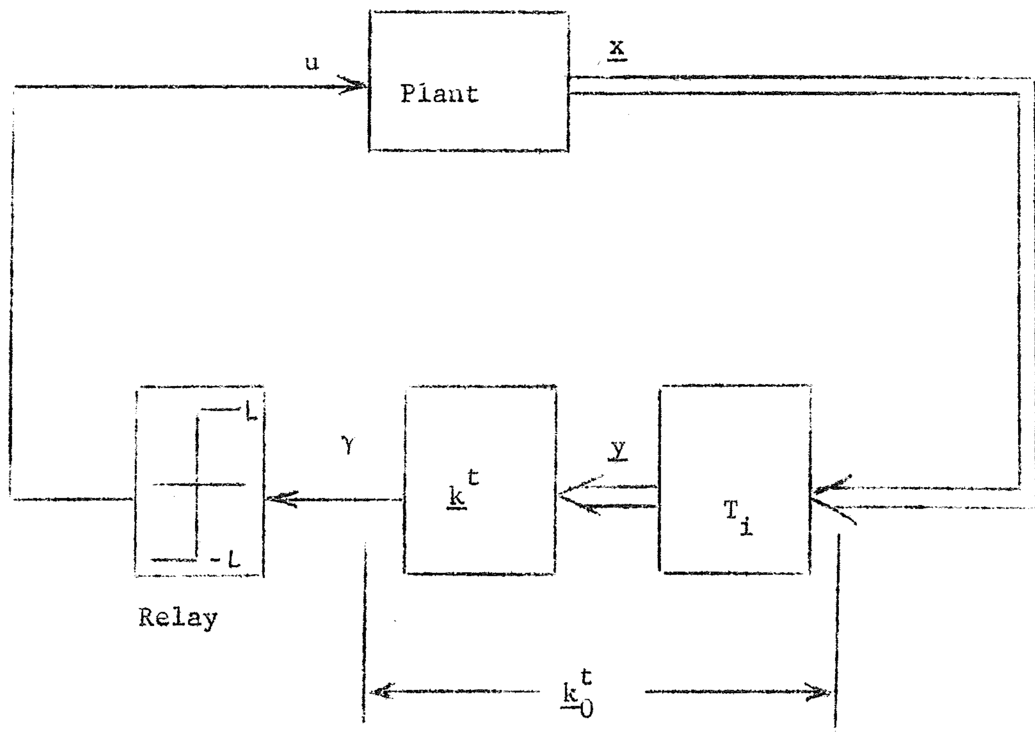

Figure 2

Diagram of Equivalent $\underline{x}$ and $\underline{y}$ Space Systems where

$$\underline{y} = 1 \times n \text{ state vector},$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 & --- & 0 \\ 0 & 0 & 1 & --- & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & 1 \\ a_1 & a_2 & - & --- & a_n \end{bmatrix}, \tag{2.6}$$

and $\underline{b}_c^t = [\ 0\ 0\ \ldots\ 1]$. In general, A_c need not be a stability matrix. This being the case, a stability matrix A_s may be defined and Equation 2.6 altered as follows:

$$\dot{\underline{y}} = A_s\, y + \underline{b}_c\, u + (A_c - A_s)\, \underline{y}\ . \tag{2.7}$$

It should be made clear that the choice of A_s is purely arbitrary, as long as A_s is a stability matrix.

Construction of Positive Definite Lyapunov Function

Consider the canonic form of the state equation (2.7). Following the method in [4], a positve definite Lyapunov function $V_{pd} = \underline{y}^t\ P\ \underline{y}$ may be found where a symmetric P is a solution to the expression:

$$- Q = A_s^t\ P + P\ A_s \tag{2.8}$$

in which Q is a positive definite symmetric matrix.

If a bound R_i is found inside which the control law is valid, then there exists a hyperellipsoid inscribed in R_i such that for any $\underline{y}\ (t_0)$ inside the hyperellipsoid $\mathring{V}$ will be negative definite, and $\underline{y} = (t;\ t > 0)$ will be asymptotically stable. This is in contrast to the semidefinite case for which a closed bound cannot be found for motion off the hyperplane.

It must be pointed out in contrast to [4] that in this paper a restriction must be placed on P; that is, the Lyapunov function V_{pd} is constrained by the fact that P must contain $\underline{k}$ as its last column in order that the control law developed in terms of $\underline{y}$ will be equivalent to that orginally developed in terms of $\underline{x}$, using the semidefinite design approach.

As previously stated, the positive-definite Lyapunov function is defined as:

$$V_{pd} = \underline{y}^t \, P \, \underline{y} \, .$$

Taking the derivative and simplifying, it is found that

$$\dot{V}_{pd} = \dot{\underline{y}}^t \, P\underline{y} + \underline{y}^t \, P\dot{\underline{y}}$$

$$= \underline{y}^t \, Q \, \underline{y} + 2\underline{y}^t \, P \, (\underline{b}_c \, u + \Delta \, A\underline{y}) \qquad (2.9)$$

where

$$\Delta A = A_c - A_s .$$

For $\dot{V}_{pd} < 0$ it is sufficient to require that

$$\underline{y}^t \, P \, [\underline{b}_c \, u + \Delta A \, \underline{y}] < 0. \qquad (2.10)$$

Expanding this term, one obtains:

$$\underline{k}^t \, \underline{y} \, (u + \sum_{j=1}^{n} c_j \, y_j) < 0 \qquad (2.11)$$

where c_j designates coefficients of the n^{th} row of ΔA.

To satisfy Equation 3.4 it is required that

$$1. \quad |u_{max}| = L > \left| \sum_{j=1}^{n} c_j \, y_j \right| \qquad (2.12)$$

$$2. \quad \text{sgn } u = \text{sgn } (\underline{k}^t \, \underline{y}).$$

Note that $L = \pm\left| \Sigma \, c_j \, y_j \right|$ defines a pair of hyperplanes which represent a region inside which the control law is valid.

The Lyapunov Contour in E^2 Space

To aid in picturing the resulting contours and bounds it is advantageous to consider an E^2 space. This is shown in Figure 3 where R_i is taken to be the bound defined by

$$L > \left| \sum_{j=1}^{n} c_j \, y_j \right| .$$

Under the assumption that the valid control region R_i completely encloses the quadratic bounds, the largest positive definite Lyapunov contour which may be inscribed in R_i is an ellipse which is tangent to at least one point of R_i.*

* Appendix A

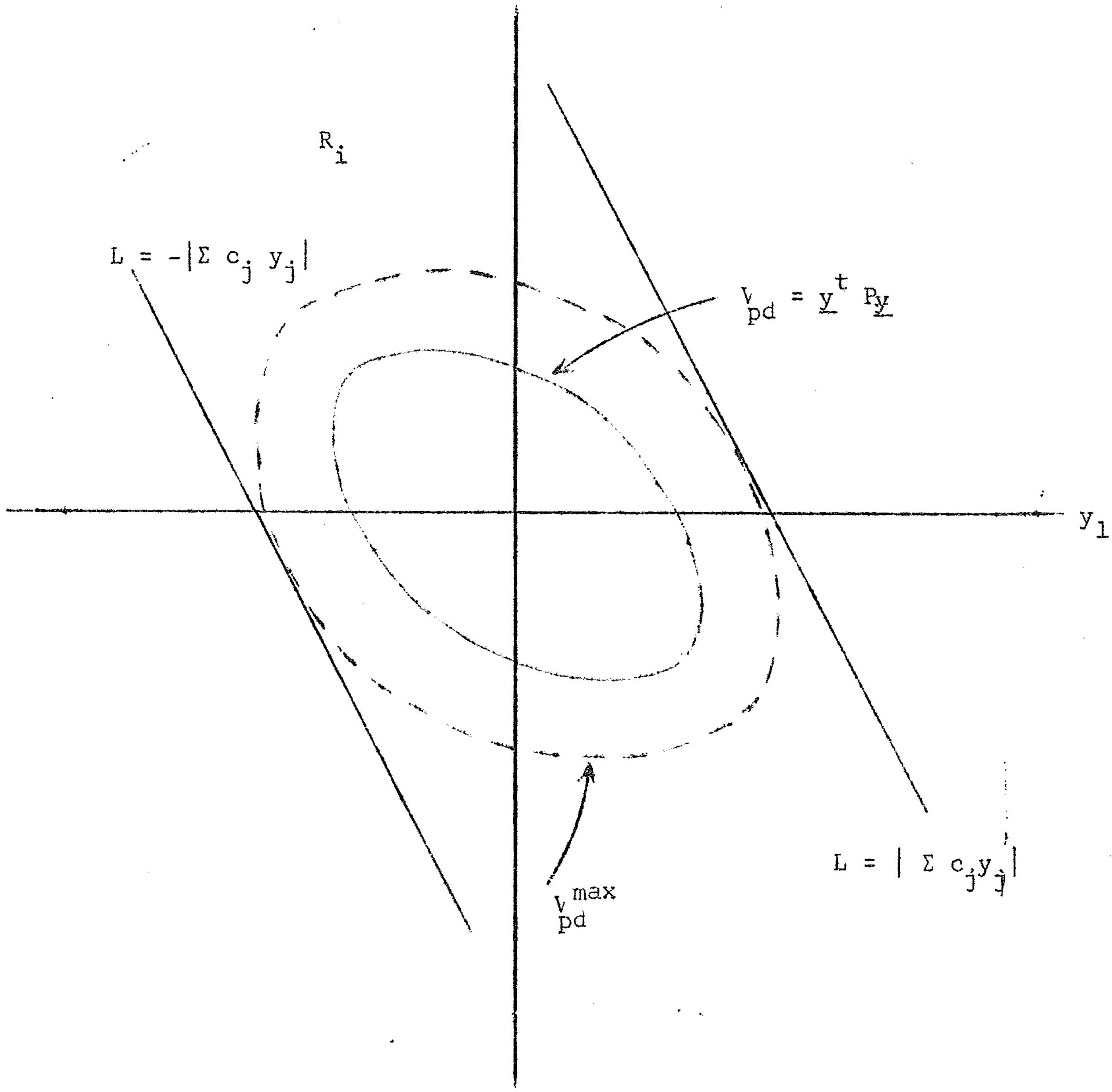

Figure 3

Positive Definite Lyapunov Contour in $\underline{y}$ Space

This is shown in Figure 3 as a dashed contour.

In order to obtain a measure of the conservativeness of the resulting positive definite elliptic contour, it is of interest to map the positive semi-definite Lyapunov contour into the canonic space. This function may be defined as

$$V_{psd} = \frac{1}{2} [\gamma (T_i \underline{x})]^2 \tag{2.13}$$

where

$$\gamma (T_i \underline{x}) = \underline{k}^t \underline{y} .$$

Taking the derivative,

$$\dot{\gamma} (\underline{y}) = \underline{k}^t \dot{\underline{y}} = \underline{k}^t A_c \underline{y} + \underline{k}^t \underline{b}_c u .$$

For $\dot{V}_{psd} \leq 0$ it is required that

$$1. \quad \text{sgn} \, \gamma (\underline{y}) = - \text{sgn} \, \dot{\gamma} (y)$$

$$2. \quad |u_{max}| = L \geq | \sum_{j=1}^{n} d_j y_j | \tag{2.14}$$

where

$$d_j = \text{coefficients of } \frac{\underline{k}^t A_c \underline{y}}{\underline{k}^t \underline{b}_c}$$

In general, the linear bounds for both the positive definite and positive semi-definite Lyapunov functions will not be parallel. This is illustrated in Figure 4. From a qualative viewpoint, if the semi-definite bound defines a region greater than the contour V_{pd}^{max}, the design approach may be considered conservative. If the situation is reversed, however, an improvement has been accomplished. It is inconsequential to consider the positive semi-definite Lyapunov bound, however, since the region of asymptotic stability, which is of primary concern, is defined only within the contour V_{pd}^{max}.*

Determination of Bound in $\underline{x}$ space

Assume that for the given set of parameters the maximum elliptic contour is found to be

$$y^t P \underline{y} = V_{pd}^{max}$$

This contour is then mapped into the $\underline{x}$ space through the transformation

$$\underline{x}^t T_i^t P T_i \underline{x} = V_{pd}^{max} \tag{2.15}$$

* For the model tracking problem considered in [1], stability is achieved only in the sense of bounded motion on the switching plane. In the regulator problem considered here, however, a region of asymptotic stability can be defined off the hyperplane.

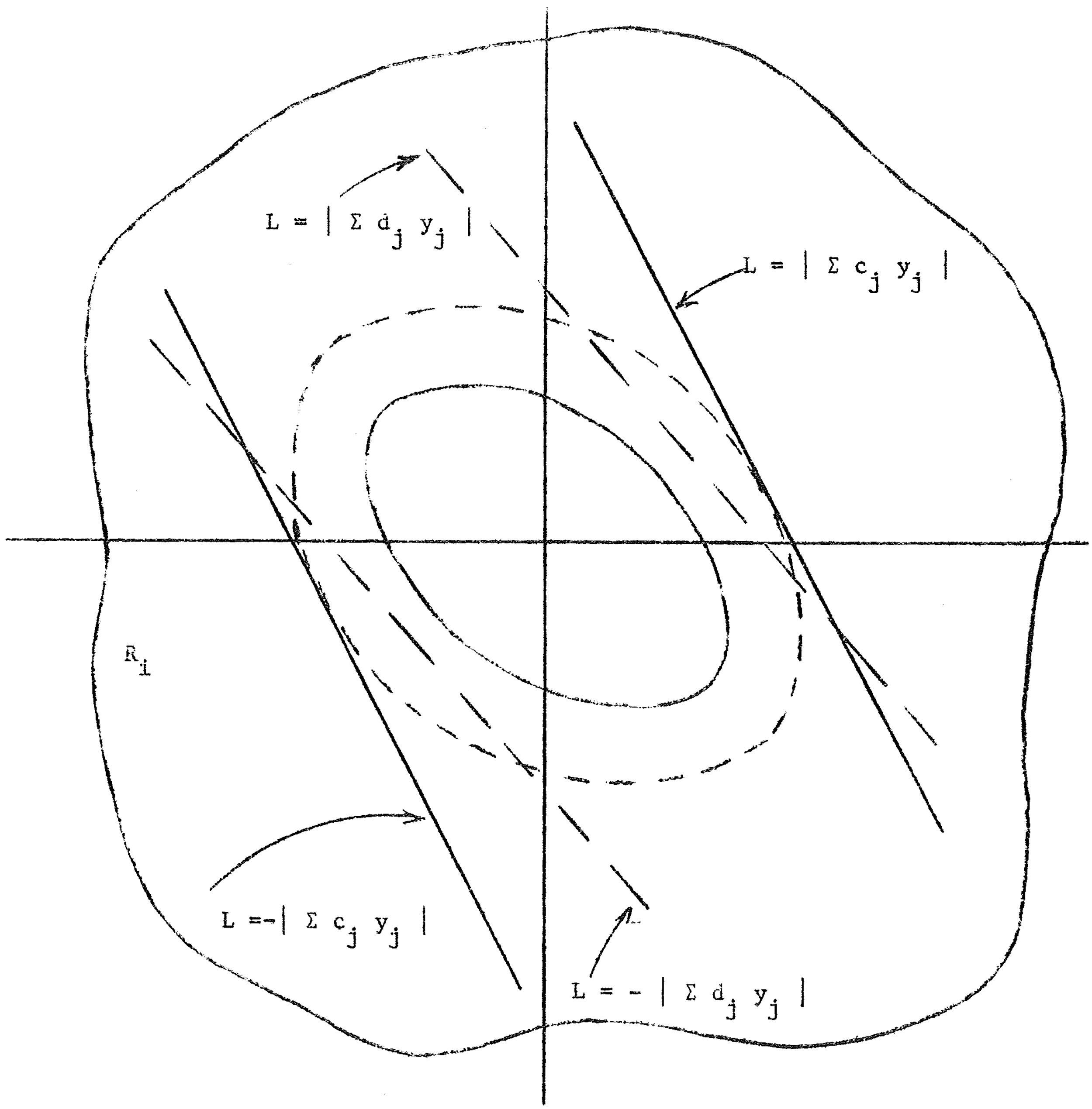

Figure 4

Positive Definite and Positive Semi-Definite Lyapunov Contours in y Space

For each set of parameters $\{\alpha_i\}$ a contour may be mapped into the $\underline{x}$ space by application of (2.15). This results in a set of intersecting ellipses (for the more general case, a set of hyperellipsoids), as is illustrated in Figure 5.

In general it is not known which set of parameters $\{\alpha_i\}$ describes the actual system. Hence, any initial condition $\underline{x}(t_0)$ within the intersection of the ellipses will result in a stable trajectory. The bound on these trajectories, however, must include the union of the ellipses. These two conditions may be stated as follows:

$$\underline{x}(t_0) = \{\underline{x} \ \varepsilon\ X_1/[\underline{x}^t P_i \underline{x} = V_i^{max}] \bigcap [\underline{x}^t\ P_j\ \underline{x}\ V_j^{max}]\ ;\ i,j=1,\ldots,\ n, i\neq j\} \quad (2.16)$$

$$\underline{x}(t;t>0) = \{x\varepsilon\ X_2/[\underline{x}^t\ P_i\ \underline{x} = V_i^{max}] \bigcup [\underline{x}^t\ P_j\ \underline{x} = V_j^{max}]\ ;\ i,\ j=1,\ldots,n,\ i\neq j\} \quad (2.17)$$

It must be recognized that a variety of parameters are utilized in the design approach, and each may assume a continuum of values. These parameters include:

1. The parameters of A and $\underline{b}$; unknown but bounded.

2. The transformation T_i, fixed by each parameter set $\{\alpha_i\}$.

3. The set of linear feedback gains $\underline{k}_0$, restricted within certain bounds by the design approach of [1].

4. Stability matrix A_s.

5. Positive definite P, selected with the restriction that $\underline{p}_n = \underline{k}$.

At first glance, it would seem that numerous contours must be generated before any approximate representation of a stability bound can be determined. This difficulty may be partially overcome, however, if it is recognized that in many practical problems, even though parameters may not be known exactly, there is a relationship between the elements of the parameter set $\{\alpha_i\}$ which reduces the parameter space considerably.

III DISCUSSION OF RECOVERABLE SETS

Additional work has been performed in order to obtain a comparison of the recoverable set suggested by the above method and the approach of Lemay [3]. For a linear, time-invarient system, the following equations are defined:

Recoverable Set

$$RRM(T) = \underline{x}:\ \underline{x} = \int_0^T e^{-A\xi}\ B\underline{u}(\xi)\ d\xi\ ;\ \underline{u}(\xi) \text{ admissible}] \quad (3.1)$$

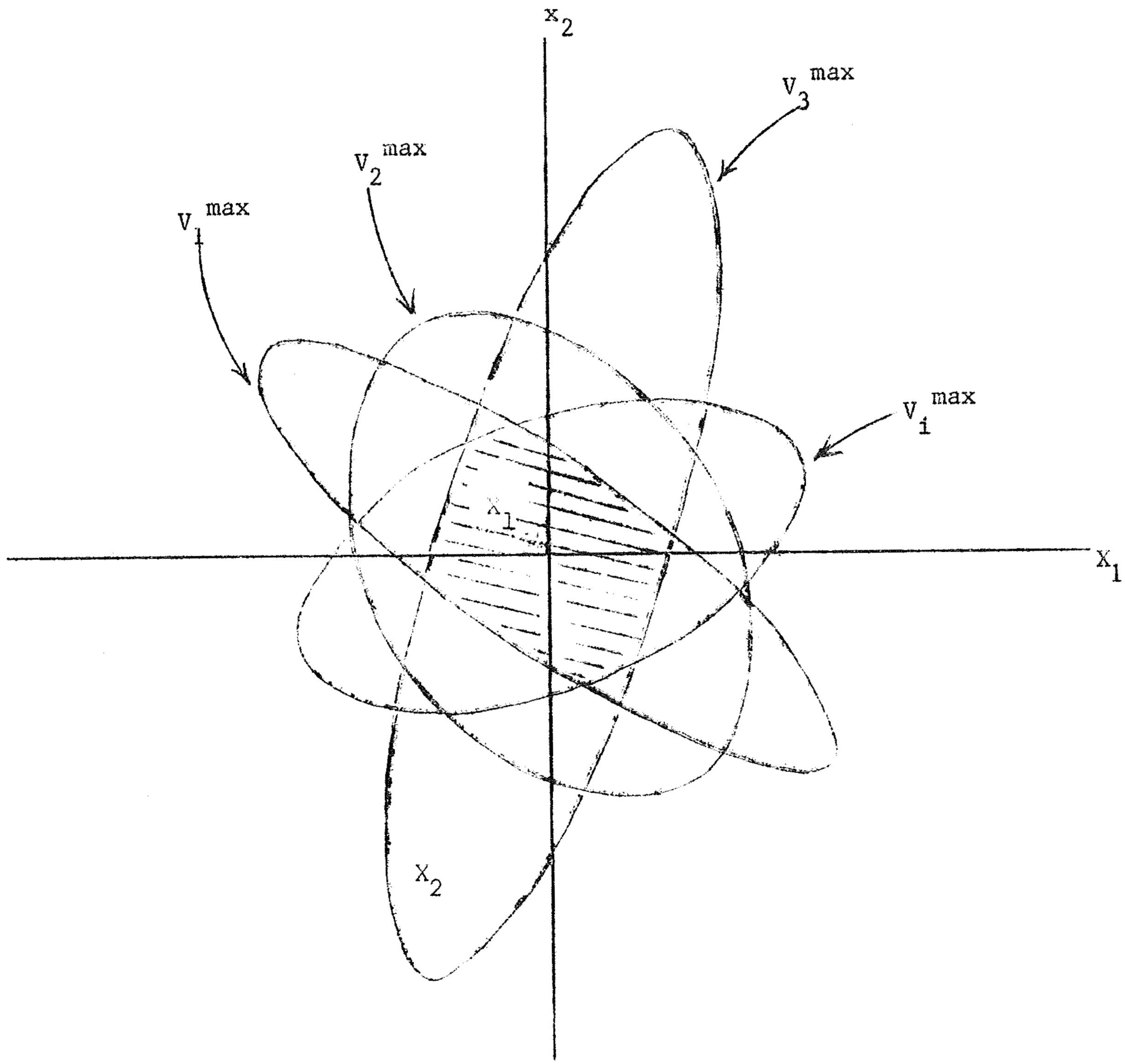

Figure 5

Intersection of Ellipses in $\underline{x}$ space for Various Parameter Values

Reachable Set

$$REM(T) = [\,x : x = \int_0^T e^{A(T-\xi)} B\underline{u}(\xi)\, d\xi;\ \underline{u}(\xi) \text{ admissible}] \tag{3.2}$$

In addition, Lemay has shown that:

$$RRM^{\pm}(T) = REM^{-}(T) \tag{3.3}$$

which says that "for a constant system, the set of states which can be driven to the origin in time T is equivalent to the set of states which can be reached by starting at the origin and running the system backwards for the same elapsed time." It is this property of the recoverable set which allows a comparison of methods.

Consider a single-input, time-invarient system

$$\dot{\underline{x}} = A\,\underline{x} + \underline{b}\,u$$

which has all positive distinct eigenvalues. Define the transformation $\underline{x} = M\,\underline{q}$ where M is a modal matrix in the following form:

$$M = (g_1\underline{e}_1, g_2\underline{e}_2, \ldots, g_n\underline{e}_n) \tag{3.4}$$

where

g_i = gains to be determined

$\underline{e}_i$ = eigenvectors

It can be shown that:

$$M^{-1} = [\,\underline{r}_i/g_1, \underline{r}_2/g_2, \ldots, r_n/g_n]^t \tag{3.5}$$

where

$\underline{r}_i$ = the rows of $M^{-1} = (r_{i1}, r_{i2}, \ldots, r_{2n})$.

Performing this transformation, one obtains the normal form:

$$\dot{\underline{q}} = \Lambda\underline{q} + \underline{b}_n u \tag{3.6}$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$\underline{b}_n = \begin{bmatrix} \underline{r}_1'\,\underline{b}/g_1 \\ \cdot \\ \cdot \\ \cdot \\ \underline{r}_n\,\underline{b}/g_n \end{bmatrix}.$$

In order to obtain the form of the state equation used by Lemay (see Equation 3.11), it is required to alter the eigenvalues of Λ. This may be accomplished by time scaling the differential expression

$$\underline{\dot{q}} = \Lambda \underline{q} + \underline{b}_n u$$

With $u = 0$

$$\begin{aligned} \dot{q}_1 &= \lambda_1 q_1 \\ \dot{q}_2 &= \lambda_2 q_2 \\ &\vdots \\ \dot{q}_n &= \lambda_n q_n \end{aligned} \tag{3.7}$$

letting $t = \beta t_1$ gives

$$\dot{q}_1(\beta t_1) = e^{\lambda_1 \beta t_1} \tag{3.8}$$

If $\beta = \frac{1}{\lambda_1}$ then

$$\begin{aligned} q_1(\beta t_1) &= e^{t_1} \\ q_2(\beta t_1) &= e^{\lambda_2/\lambda_1 t_1} \\ &\vdots \\ q_n(\beta t_1) &= e^{\lambda_n/\lambda_1 t_1} \end{aligned} \tag{3.9}$$

For vector $\underline{b}_n$ the following gains are computed

$$\begin{aligned} g_1 &= \underline{r}_1 \underline{b} \\ g_2 &= \frac{\lambda_1}{\lambda_2} (\underline{r}_2 \; \underline{b}) \\ &\vdots \\ g_n &= \frac{\lambda_1}{\lambda_n} (\underline{r}_n \; \underline{b}) \end{aligned} \tag{3.10}$$

As a result the following equation of Lemay is obtained:

$$\begin{bmatrix} \dot{q}_1 & (\beta t_1) \\ \dot{q}_2 & (\beta t_1) \\ \vdots & \vdots \\ \dot{q}_n & (\beta t_1) \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ & \lambda_2/\lambda_1 & & \\ & & \ddots & \\ 0 & & & \lambda_n/\lambda_1 \end{bmatrix} \begin{bmatrix} q_1 & (\beta t_1) \\ q_2 & (\beta t_1) \\ \vdots & \\ q_n & (\beta t_1) \end{bmatrix} + \begin{bmatrix} 1 \\ \lambda_2/\lambda_1 \\ \vdots \\ \lambda_n/\lambda_1 \end{bmatrix} u. \qquad (3.11)$$

In order to compare the recoverable set obtained by the two methods, it is necessary to map the system in phase-variable $\underline{y}$ space to the normal $\underline{q}$ space. For any contour

$$V_i^{max} = \underline{y}^t \; P \; \underline{y}$$

the corresponding contour in $\underline{q}$ is

$$V_{pd}^{max} = \underline{q}^t M^t \; T_i^t \; P \; T_i \; M \; \underline{q} \qquad (3.12)$$

where V_{pd}^{max} remains invarient under the linear transformation.

A design example will now be considered in order that a comparison of methods may be made.

IV DESIGN APPLICATION

The design technique presented in Section 2 is to be applied to a plant consisting of a cart supporting an inverted pendulum using the control law developed in [1]. Figure 6 shows the cart, which is mounted on a track, connected through a pulley and gearing to a D. C. motor.

The equations of the plant may be written in the form [1]:

$$\begin{aligned} \dot{x}_1 &= a_{11}\, x_1 + a_{12}\, x_2 + b_1\, u \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= a_{31}\, x_1 + a_{32}\, x_2 + b_3\, u \end{aligned} \qquad (4.1)$$

In order to study the effect of parameter variations, the distance (ℓ) from the pivot point to the center of gravity of the pendulum was made to be adjustable by repositioning a sliding mass on the rod. The purposes of this paper are served, however, by determining a bound for the nominal set of parameter values as listed in Table 1.

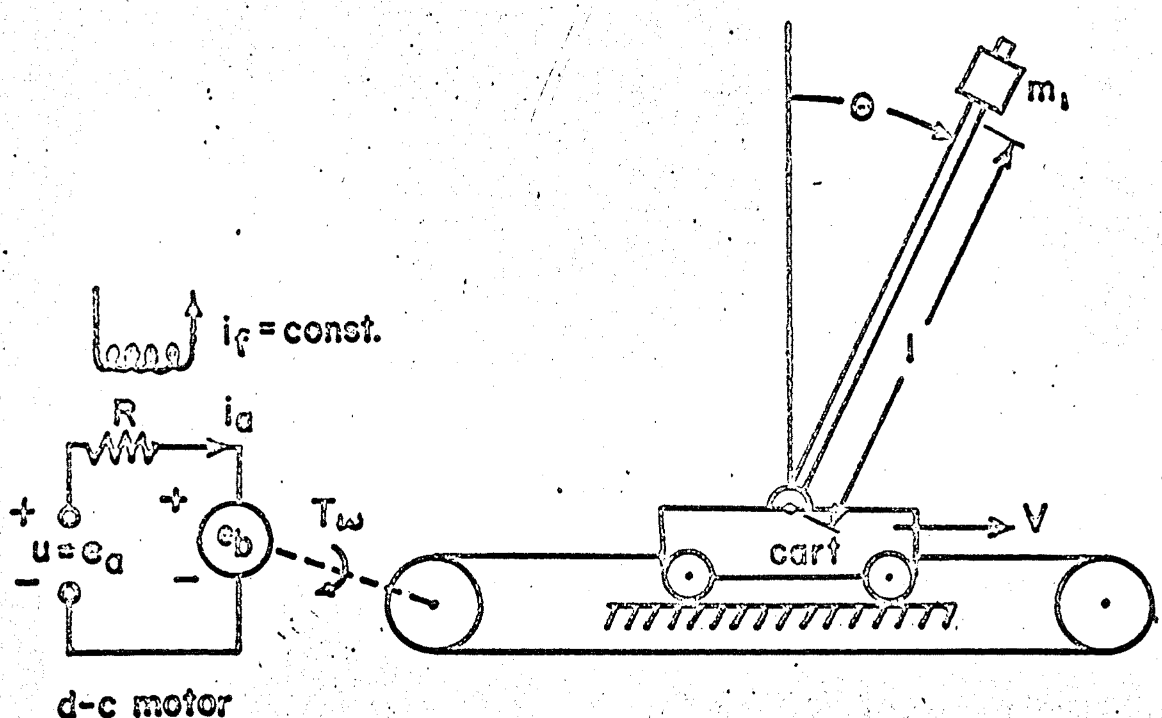

Figure 6

Cart Supported Inverted Pendulum

Table 1

Parameters	nominal	ℓ dependence
a_{11}	-800	none
a_{12}	-3	none
a_{31}	800	$1/\ell$
a_{32}	13	$1/\ell$
b_1	15	none
b_3	-15	$1/\ell$

In accordance with [1], the components of $\underline{k}_0$ are to be chosen so that

$$k_0^1 = -1$$

$$k_0^2 < 0$$

$$k_0^3 < -1$$

With $\underline{k}_0$ satisfying these conditions, and u defined by (2.2), a region of stability can be found for this set of parameters.

Transformation to Phase-Variable Form

Utilizing the method of Silverman [2], the system (with the parameters of table 1) are transformed to canonic form. The resultant matrices are:

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8000 & 13 & -800 \end{bmatrix} ; \; T = -\frac{1}{150} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} ; \; T^{-1} = \begin{bmatrix} -150 & 0 & 15 \\ 0 & -15 & 0 \\ 0 & 0 & -15 \end{bmatrix}$$

Computation of P

Since A_s is arbitrary, the stability matrix was selected as:

$$A_s = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -4 \end{bmatrix} \qquad (4.2)$$

Since $\underline{k}^t = \underline{k}_0^t T_i^{-1}$ it follows that

$$p_{13} = k^1 = -q_{11}/2a_1 \quad .$$

With $k_0^1 = -1$ and $k^1 = -150 \; k_0^1$, it follows that $q_{11} = 300$. Letting Q be a diagonal matrix to simplify the computation, we select q_{22} and q_{23} such that the conditions $k_0^2 < 0$ and $k_0^3 < -1$ are satisfied. With

$$Q = \begin{bmatrix} 300 & & 0 \\ & 40 & \\ 0 & & 80 \end{bmatrix}$$

the P matrix is equal to:

$$P = \begin{bmatrix} 660 & 700 & 150 \\ 700 & 1490 & 360 \\ 150 & 360 & 100 \end{bmatrix} . \tag{4.3}$$

Since $\underline{k}_0^t = \underline{p}_n^t T_i$ the value of $\underline{k}_0^t$ becomes :

$$\underline{k}_0^t = [-1, -24, -23/3]$$

satisfying the constraints on the elements of $\underline{k}_0$.

Determination of Linear Constraints

The linear bounds defining the validity of the control law for both the positive definite and positive semi-definite Lyapunov functions may be computed from (2.12) and (2.14), respectively. For the computed A_c and the arbitrarily selected A_s it follows that

$$\Delta A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 8001 & 15 & -796 \end{bmatrix} .$$

The bounds are then determined as:

$$|U_{max}| = L > |8001\, y_1 + 15\, y_2 + 796.43\, y_3| \;\; ; \text{p.d. case}$$

$$|U_{max}| = L \geq |8000\, y_1 + 14.5\, y_2 - 796.473\, y_3| \;\; ; \text{p.s.d. case} \tag{4.4}$$

It is seen that the sets of planes are approximately coplaner. This result is due to the large values of certain coefficients in the phase variable matrix A_c. Selection of larger negative values in A_s would only result in both sets of planes intersecting the $\underline{y}$ coordinates closer to the origin.

Determination of Maximum Ellipsoidal Contour

It is required now to determine the largest ellipsoidal contour which lies within the positive definite linear constraint, $L > |\sum_j c_j y_j|$. For L = 1.0 computer results give the tangency point as:

$$\underline{y} = \begin{bmatrix} 0.00012356 \\ -0.00009805 \\ -0.00001617 \end{bmatrix} .$$

Correspondingly $V_{pd}^{max} = 8.06 \times 10^{-6}$.

The computation to this point has involved only one set of plant parameters (the nominal), and a single stability matrix A_s and positive definite P. The resulting ellipsoidal contour will now be compared with the recoverable set obtained by Lemay [3] for one parameter set.

Comparison of Recoverable Sets

For the 3rd order example studied in this paper, the eigenvalues (nominal plant parameters) are:

$$\lambda_1 = 3.1641$$

$$\lambda_2 = 3.1604$$

$$\lambda_3 = 800.003 .$$

The model matrix M is determined from:

$$M = \text{Adj}\,[\lambda I - A] = \begin{bmatrix} \lambda_1^2 - 13 & -3\lambda_2 & -3 \\ 800 & \lambda_2(\lambda_2 + 800) & \lambda_3 + 800 \\ 800\lambda_1 & 13\lambda_2 + 8000 & \lambda_3(\lambda_3 + 800) \end{bmatrix} . \qquad (4.5)$$

The gains g_i are determined from (3.10). Utilizing (3.12), the ellipsoidal contour is then mapped into the $\underline{q}$ space.

By definition, Lemay defines the system being discussed as "3,1,1", which indicates a 3rd order system with one eigenvalue > 0, and a single control input u. It is shown in [3] that the recoverable set for this type system (with L = ± 1.0) is the open region between the planes passing through $q_1 = \pm 1$ and parallel to the $q_2 q_3$ plane. The ellipsoidal contour mapped from the $\underline{y}$ space is shown in Figure 7.

V CONCLUSIONS AND RECOMMENDATIONS

The approach taken in this paper to determine a stability bound for relay control systems has been applied to a 3rd order system as discussed in Section 4. It was found that in order to reduce the plant parameter space, the resulting mathematical manipulation becomes quite cumbersome. A comparison was made of the bound determined in this paper to the recoverable set of Lemay [3]. In general, it was concluded that the stability bound derermined in this study results in a very conservative estimate of the recoverable initial conditions.

Though preliminary results indicate a conservative design, it seems appropriate to study ways of varying the elements of P such that the overall Lyapunov contour is enlarged. The effects of such parameters as $\underline{k}_0$, Q, and the linear constraints should be considered.

Acknowledgement

The authors wish to thank Dr. Charles H. Knapp of the Univeristy of Conn. for suggesting the approach to maximizing the elliptical contour in App. A.

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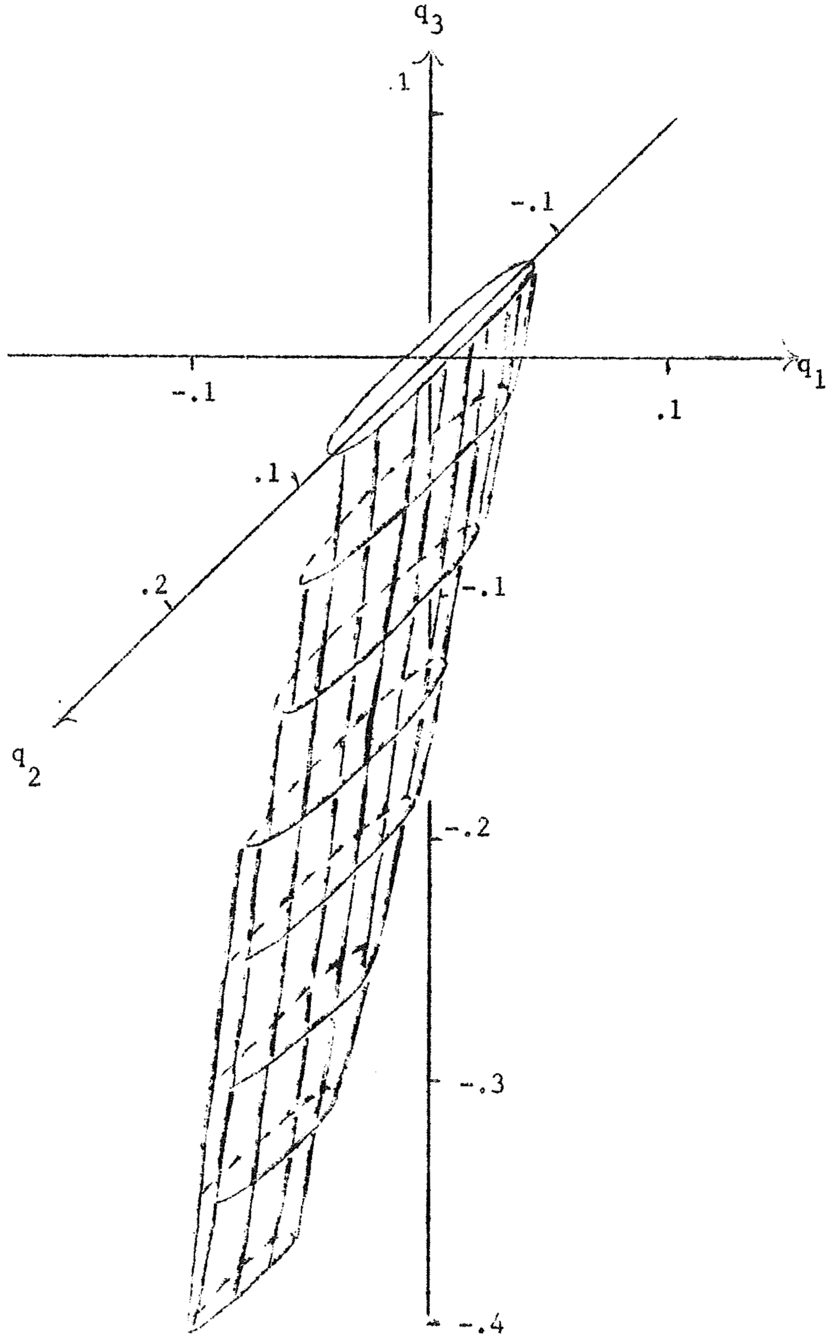

Figure 7

Ellipsoid For a Single Set of Parameters in q Space (space of Lemay)

Appendix A - Derivation of Tangency Point

To determine the maximum elliptic contour in $\underline{y}$ space, the problem may be formulated as:

$$\min J = \int_{\underline{y}} \underline{y}^t P \underline{y} \, d\underline{y}$$

subject to

$$\underline{a}^t \underline{y} \geq L$$

Define the Hamilttonian :

$$H = \lambda(L - a^t \underline{y}) + \underline{y}^t P\underline{y}$$

$$\frac{\partial H}{\partial \underline{y}} = -\lambda \underline{a} + 2 P\underline{y} = 0$$

$$\text{hence } \underline{y} = \frac{\lambda}{2} P^{-1} \underline{a}$$

Since $\underline{a}^t \underline{y} = L$, substitute $\underline{y}$ and obtain

$$\underline{a}^t \frac{\lambda}{2} P^{-1} \underline{a} = L$$

or

$$\lambda = 2L \frac{1}{\underline{a}^t P^{-1} \underline{a}}$$

Premultiplying λ by $P^{-1} \underline{a}$,

$$\lambda P^{-1} \underline{a} = \frac{2L P^{-1} \underline{a}}{\underline{a}^t P^{-1} \underline{a}}$$

or

$$\frac{x}{2} P^{-1} \underline{a} = \underline{y} = L \frac{P^{-1} \underline{a}}{\underline{a}^t P^{-1} \underline{a}}$$

$$\therefore \quad \underline{y} = L \frac{P^{-1} \underline{a}}{\underline{a}^t P^{-1} \underline{a}}$$

Appendix B - Elements of P as a Function of Q and A_s

In the solution of the simultaneous equations for P_{ij} it is assumed that $p_{ij} = P_{ij}$ for the 3 x 3 system under study, the elements are:

$$P_{11} = \frac{\left(a_2 + a_3 + a_2^2 \frac{a_3}{a_1}\right) q_{11} - 2(a_1 + a_2 a_3) q_{12} - 2a_1 a_3 q_{13} + a_1 a_3 q_{22} + a_1^2 q_{33}}{2(a_1 + a_2 a_3}$$

$$P_{12} = \frac{\frac{a_2}{a_1} a_3^2 q_{11} - 2a_2 a_3 q_{13} - a_1 q_{22} + a_1 a_2 q_{33}}{2(a_1 + a_2 a_3)}$$

$$P_{13} = -\frac{q_{11}}{2a_1}$$

$$P_{22} = \frac{\frac{a_3(a_1 + a_2 a_3) + a_3^2 [a_3^2 - a_2)}{a_1 a_3} q_{11} + 2(a_2 - a_3^2) q_{13} - (a_2 - a_3^2) q_{22} - 2(a_1 + a_2 a_3) q_{23}}{2(a_1 + a_2 a_3)}$$

$$+ \frac{\frac{a_1 a_2 (a_1 + a_2 a_3) + a_1^2 (a_3^2 + a_2) q_{33}}{a_1 a_3}}{2(a_1 + a_2 a_3)}$$

$$P_{23} = \frac{-\frac{a_3^2}{a_1} q_{11} + 2a_3 q_{13} - a_3 q_{22} - a_1 q_{33}}{2(a_1 + a_2 a_3)}$$

$$P_{33} = \frac{\frac{a_3}{a_1} q_{11} - 2q_{13} + q_{22} - a_2 q_{33}}{2(a_1 + a_2 a_3)}$$