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REALISTIC ERROR BOUNDS FOR A REDUCED-STATE MODEL-REFERENCE CONTROLLER

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October 1969

This work has been sponsored by the
National Aeronautics and Space Administration
Research Grant NGL 07-002-002

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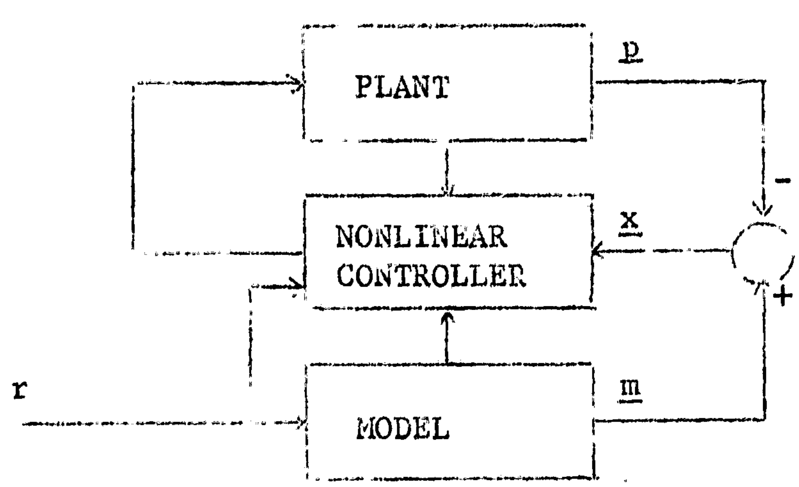
I INTRODUCTION

When confronted with the problem of controlling a plant which is imperfectly identified, it is often necessary to employ extensive simulation studies in order that desired performance is guaranteed. Although many approaches to solving this problem have been made, in particular along the lines of adaptive control, a well defined synthesis procedure cannot be said to have been defined. One approach to solving this problem involves the application of identification techniques. However, these methods require expenditure of time, and in many cases, if not ruled out by cost factors alone, are not reliable because the stability is not guaranteed due to the computation lag involved.

An alternative to this approach is to develop a control which guarantees stability over the range of parameter uncertainty involved. Although this solution to the problem will not be the most efficient, it may be justified in terms of cost, or simply because no other method is available. This method of control which has been discussed by several authors [1], [2], [3] depends upon the synthesis of a control law which guarantees stability by application of Liapunov's direct method. This is accomplished by means of a model reference control system as characterized in Figure 1.



Model-Reference Controller

Figure 1

The objec of the model-reference control scheme is to force the states, x_i, of a single-input, single-output, nonlinear, time- varying, n^{th}-order plant to track the states, m_i, of a linear, time-invariant n^{th}-order model as the model responds to some input. The main requirement on the control is one of stability. Stated in general terms, this requires that some measure of the offsets, x_i, between plant and model states, the tracking error, must either tend to zero in the limit with time (Asymptotic Stability) or eventually be contained within some small calculable bound (Lagrange Stability).

The main contribution of this report is an extension of previous work to allow filtered states to be used in formulating the control law, thereby reducing noise content in the general design and moreover providing the designer with the opportunity to use filtered derivatives of measureable signals to approximate states that cannot be measured.

The essential problem can be related to an equation of the form

$$\dot{\underline{x}} = A\underline{x} - \underline{b}(t) \operatorname{sgn} (\gamma (x)) \tag{1}$$

where the stability matrix A and $\underline{b}(t)$ are in phase-variable form

$$A = \begin{bmatrix} 0 & 1 & & & \bigcirc \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & \bigcirc & & \ddots & 1 \\ -a_1 & & \cdots & & -a_n \end{bmatrix}, \quad \underline{b}(t) = [0 \;\text{---}\; b_n(t)]^T$$

with

$$| b_n(t)| \le 2L \tag{2}$$

The Signum function is defined to be

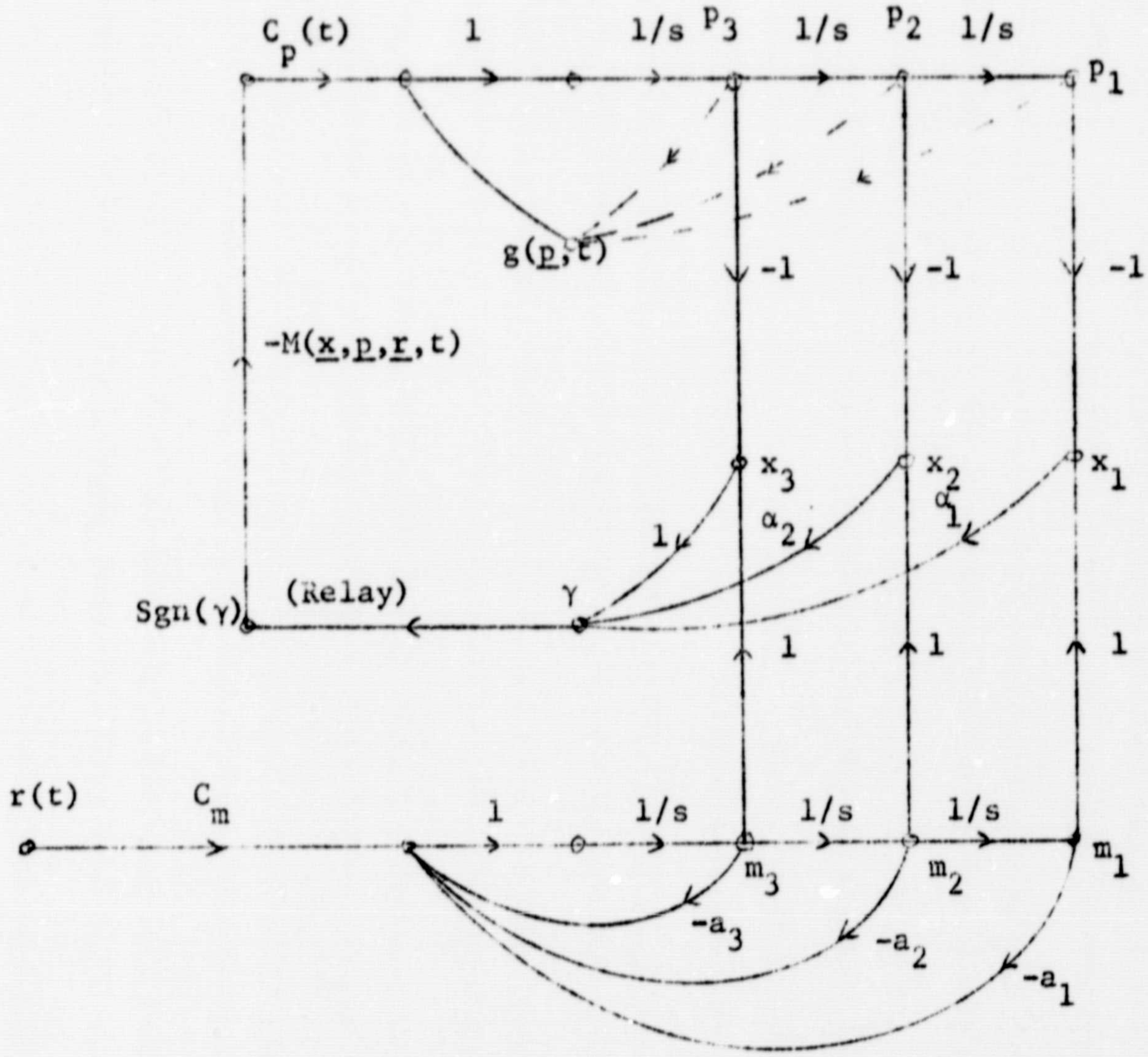
$$\operatorname{Sgn} \gamma = \begin{cases} +1 : & \gamma > 0 \\ \sigma : & \gamma = 0; \; -1 \le \sigma \le 1 \\ -1 : & \gamma < 0 \end{cases}$$

and the linear switching function γ is

$$\gamma = \underline{\alpha}^T \underline{x} = \sum_{i=1}^{n} \alpha_i x_i \tag{3}$$

where with no loss in generality we assume $\alpha_n = 1$.

That (1) is a proper representation of the controlled model-reference system can be seen by the following example. Figure 2 is the signal flow graph of a



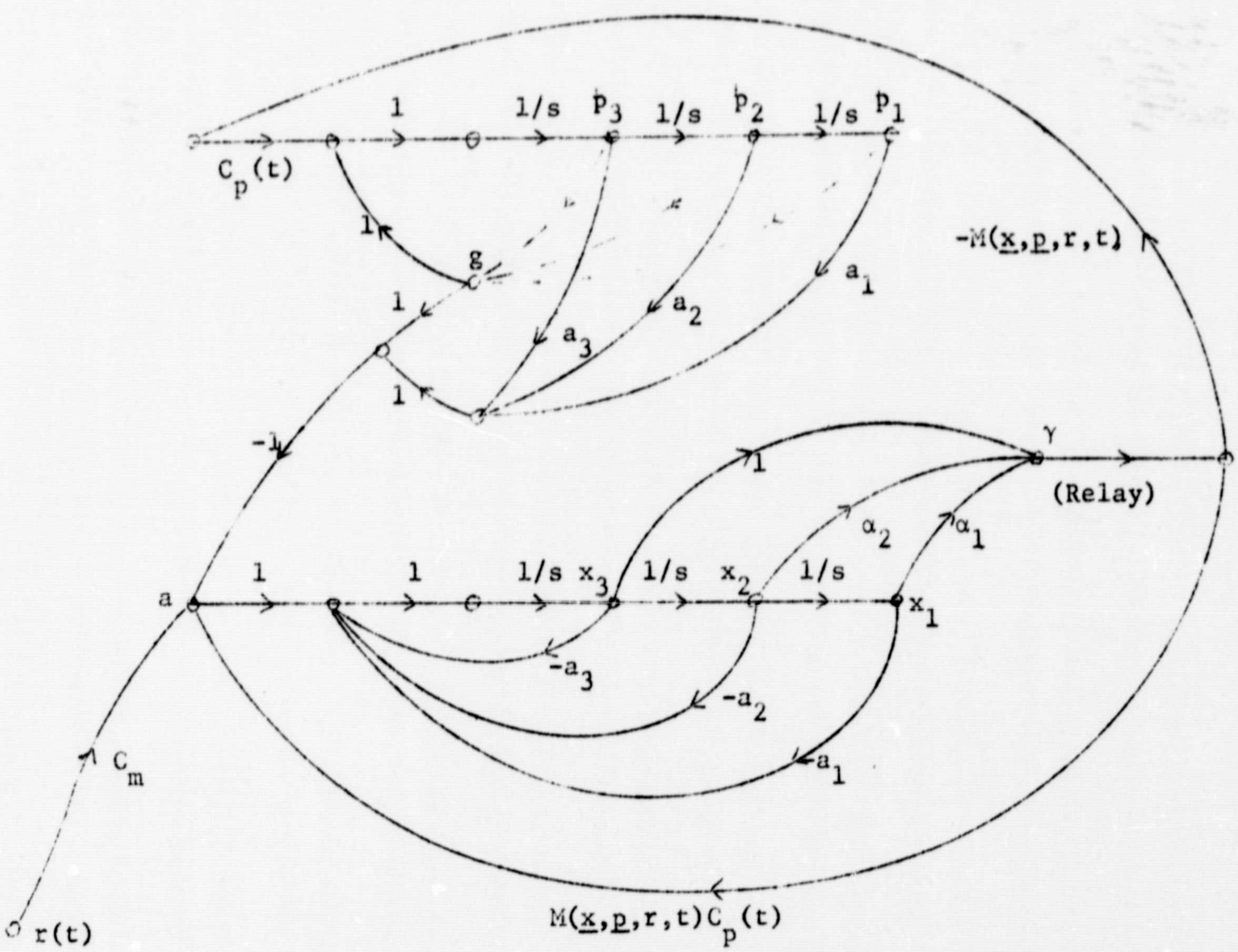
Third-Order Model Reference Controller

Figure 2

third-order nonlinear, time-varying plant having states p_i that are to follow those of the stable linear, time-invariant model reference m_i respectively. Ignorance of the plant is expressed in $g(\underline{p},t)$ and $c_p(t)$. The former contains nonlinear terms and coefficients known only within bounds; the latter is bounded away from zero and known only within bounds.

The sign of the control, u, is that of the switching function $\gamma(\underline{x})$. The gain $M(\underline{x},\underline{p},r,t)$ in the designs of references [1] and [2] is generated in the nonlinear controller to form the magnitude part of the control. To satisfy plant saturation constraints it must be shown that M is less than some number L. In so doing, one guarantees that substitution of the constant L for M would give the same stability results. This is the motivation for the design of reference [3] where M = L and the nonlinear controller is essentially a relay.

The relevance of (1) to the system of Figure 2 becomes apparent through the equivalent signal flow represented in Figure 3. A property of the synthesis



Equivalent Third-Order Model Reference Controller

Figure 3

techniques of references [1], [2], [3] is that the signal at node a has the sign of $-\gamma$ and a magnitude bounded by 2L. Inasmuch as stability of the tracking is based on stability of the error states x_i then stability of the controlled model-reference system may be based on (1).

Before going further several points should be made concerning (1) and its relevance to the model-reference controlled system, inasmuch as the remainder of this report is based entirely on (1).

First, this relation is not the only one that could have been chosen. The important qualities that must be preserved by the second term on the right of (1) are that it must be of bounded magnitude and must take the sign of $-\gamma$. Another valid representation, for example, would be the expression $c(t)\ \gamma(\underline{x})$ where $0 \leq c(t) \leq \infty$. The reason for the unboundedness of the time-variable gain $c(t)$, is explained by the second point.

Although the node a is bounded and takes the sign of $-\gamma(\underline{x})$, it is not necessarily zero when γ is zero. Thus, if a linear gain is to relate the two it must have infinite range. (Notice also that this property is present in (1) in the infinite slope of the Signum function.) It is this property which makes the stability analysis of (1) so difficult.

The third point to be made is that, due to the presence of nonlinear terms and time-variable coefficients known only within bounds, very little can be implied about the signal a. This ignorance is translated to $b_n(t)$ in (1). Consequently, no definite statement can be made concerning the time derivative of $b_n(t)$. This also confounds the analysis problem.

Another point is that due to the ignorance in $b_n(t)$ the system (1) is capable of motions that may not be possible in the actual controlled system. This is due to parameter ignorance. Of paramount importance, however, is the fact that all possible motions of the controlled system are motions of (1) and therefore any properties that can be attributed to solutions of (1) apply also to those of the model-reference controlled system.

The last point is that having assumed stability conditions to be satisfied by the relay output level, L, or varying gain $M(\underline{b},\underline{x},r,t)$ the only design freedom that remains is the choice of the switching function $\gamma(\underline{x}) = \underline{\alpha}^T\underline{x}$.

Having established the relevance of (1) to the controlled model-reference system the problem is to analyze the stability of the origin as it is affected by choice of the switching function.

II. CONTROL LAW SYNTHESIS

The purpose of this section is to derive general conditions on the linear switching function, $\gamma = \underline{\alpha}^T\underline{x}$, for which asymptotic stability is guaranteed. For convenience (1) is rewritten here as

$$\dot{\underline{x}} = A\underline{x} - \underline{b}(t)\ \mathrm{Sgn}\ (\underline{\alpha}^T\underline{x}). \qquad (4)$$

Sufficient conditions are first found by Liapunov's Direct Method. Choosing a Liapunov function

$$V = \underline{x}^T P\underline{x} \qquad (5)$$

where P is assumed symmetric, its total time-derivative is

$$\dot{V} = -\underline{x}^T Q \underline{x} - 2\underline{x}^T P \underline{b}(t) \text{ Sgn } (\underline{\alpha}^T \underline{x}) \tag{6}$$

where

$$Q = -[\ A^T P + PA] . \tag{7}$$

Choosing Q positive definite leads according to a theorem of Liapunov [5] to a positive definite P since A is a stability matrix. Furthermore P is a unique solution to (7). Therefore V is positive definite and $\dot{V}$ **will be negative definite providing**

$$\underline{x}^T P \underline{b}(t) \text{ Sgn } (\underline{\alpha}^T \underline{x}) \geq 0. \tag{8}$$

Since $\underline{b}(t)$ has only one nonzero term, $b_n(t)$ which is non-negative, then (8) is satisfied if

$$\underline{x}^T P [00 \text{-----} 1]^T \text{ Sgn } (\underline{x}^T \underline{\alpha}) \geq 0 \tag{9}$$

which implies

$$P[00\text{----}1]^T = \underline{\alpha} \tag{10}$$

Thus the coefficients of the switching function must satisfy

$$\alpha_i = p_{in}, \ i = 1,2,\ldots n \tag{11}$$

where p_{ij} is the i-j element of P and with no loss of generality, $p_{nn} = 1$. Design freedom in the choice of $\underline{\alpha}$ results from the fact that there is an infinity of positive definite matrices Q which can be chosen and each one results through (7) in a different positive definite matrix P, and this in turn leads to a different $\underline{\alpha}$.

That the set of matrices P which is mapped through (7) from the set of all positive definite matrices Q does not equal the set of all positive definite matrices P is easily demonstrated by noting that the choice of a positive definite P as the identity matrix leads through (7) to an indefinite Q, (q_{11}=0), for the case of phase-variable A. Therefore, the set of all switching functions $\underline{\alpha}^T\underline{x}$ which result in asymptotic stability of (4) is not easily determined. However, one important property of this set can be observed.

In order to eliminate a state from the control law it must be shown that a switching function satisfying the condition imposed above can be found which does not involve that particular state. The elimination of the state x_i, for example, implied that α_i=0 and through (18) that p_{in}=0. In particular, to eliminate the highest-order-derivative state x_n from the control law, α_n must be zero and thus p_{nn}=0. This last condition rules out the possibility of finding an $\underline{\alpha}$ satisfying the stability condition for the case $\alpha_n = 0$ because a necessary condition for positive definiteness of P is that all diagonal terms be greater than zero. It is therefore not possible to establish asymptotic stability via the Direct Method unless the highest-order-derivative state x_n is included in the switching function.

It is the elimination of this particular state that presents the greatest challenge. This is due to the fact that it can only be obtained by differentiating a lower-order-derivative state. The problem of eliminating any other state is synthetic in the sense that it could be obtained bt integrating another state.

In as much as the desired elimination of state cannot be shown to yield asymptotic stability by means of the Direct Method, one wonders if some other sufficiency condition might be used to establish stability. For this reason the application of a frequency domain stability criterion was investigated.

The Circle Criterion [6] applies to a system with linear part and nonlinear, time-variable feedback such as (4). However, this sufficient condition requires for the case of (4) that the loop transmission from relay output to relay input,

$$G(x) = \underline{\alpha}^T [Is-A]^{-1} [00....1]^T \tag{12}$$

must have no part of its Nyguist plot contained in the left half Nyguist plane. This condition implies indirectly that the switching function γ must involve the highest-order derivative state, x_n. This is the same condition imposed by the Direct Method.

The examples above, illustrate the difficulty of obtaining sufficient conditions for asymptotic stability of (4) for the case wherein $\alpha_n = 0$. The lack of any such condition leads one to consider the following approach to the problem.

The solution proposed here is to pass the noisy measurements of the highest-order states theough a set of filtered derivative circuits to be used in the implementation of the switching function $\gamma(\underline{x})$. In what follows, the case of x_n unavailable is treated. Generalization to the case of the r highest-order states unavailable is straightforward. The remainder of the report treats the bound of the filtered system.

In some cases the process of differentiating a lower-order state produces worse noise than was present on the original measured state. In this case it may be desirable to use the filtered measurement of the state. The following therefore treats the case wherein the state x_n is filtered and then used in the switching function.

III. DFRIVATION OF FILTERED SYSTEM

In a situation where the highest-order-derivative state x_n is not accessible but is required for asymptotic stability, a reasonable solution is to use a filtered derivative of x_{n-1} in place of x_n. Choice of the filter dynamics would be based on some knowledge of measurement noise statistics. In this case, the filter transfer function would have a denominator polynomial of order two or higher to provide adequate low pass filtering of white noise encountered in the measurement of x_{n-1}. In the case of colored noise a first-order filter might suffice but for generality we consider a second order filter. For purposes of this discussion such a filter will be represented by

$$c_2\ddot{x}_{n+1} + c_1\dot{x}_{n+1} + x_{n+1} = \dot{x}_{n-1} \tag{13}$$

This can also be expressed as

$$c_2\ddot{x}_{n+1} + c_1\dot{x}_{n+1} + x_{n+1} = x_n. \tag{14}$$

Because of the phase-variable structure, $\dot{x}_{n-1} = x_n$. However, (13) is written to indicate that the state x_{n-1} is the input to the filter rather than x_n as indicated in (14). Accordingly the derivative of noise encountered in the measurement if x_{n-1} would appear on the right side of (14).

An augmented system results when the filter (14) is incorporated into the original system (1), that is, when the state x_{n+1} is used in place of x_n in the switching function (3). It can easily be shown that the stability analysis of this augmented system is hindered in much the same manner as was that of (1). This is true in the case of both the Direct Method and the Circle Criterion.

Even though asymptotic stability cannot be guaranteed in the augmented system it is reasonable to expect that the filter could be chosen so as to minimize the ultimate bound that may result. Toward this end we consider the term

$$e = x_n - x_{n+1}. \tag{15}$$

It is this filter error that distinguishes the augmented system from the original system (1). In fact, for the following discussion it is convenient to express the augmented system as

$$\dot{\underline{x}} = A\underline{x} - b(t)\ \mathrm{Sgn}\ (\underline{\alpha}^T\underline{x} - \alpha_n e). \tag{16}$$

The problem that remains is to choose the switching function coefficients $\alpha_1, \alpha_2 \ldots \alpha_n$ and filter constants c_1, c_2 to minimize the ultimate bound on solutions of (16).

One solution is to choose $\underline{\alpha}$ so that (16) is asymptotically stable for the case $e \equiv 0$. Then treating the term $\alpha_n e$ as switching function imperfection, a realistic bound can be determined by means of a previously reported technique [4]. However, for the case of state-dependent imperfection this technique only applies to solutions having initial conditions in a certain state-space region. Inasmuch as calculation of this region is generally tedious, an alternate design is desirable.

In the following section a new design technique is developed for which the bound [4] applies to any initial condition. A natural switching function is shown to guarantee that solutions of (16) will monotonically approach a hyperplanar region, Ω, parallel to and centered about the switching plane. By nature of the fact that $\underline{x}$ will eventually enter Ω and will remain there for all subsequent time, the bound technique developed in reference [4] applies directly. This bound on $|\gamma|$ which in turn is based on worst-case magnitudes of filter error and filtered measurement noise. The trade-off that exists between the effects of these two bounds is discussed.

IV. NATURAL SWITCHING FUNCTION

Paraphasing a theorem of Lasalle's [7], it is possible to obtain asymptotic stability with the use of a semidefinite Liapunov function providing: a) that the function approaches zero asymptotically and b) that motion on the zero manifold is asymptotically stable to the origin. In terms of the problem at hand where linear switching is employed, asymptotic stability could be achieved if sufficient conditions could be found to guarantee that switching hyperplane is asymptotic to the origin. It will now be shown that choice of what will be called a natural switching function guarantees both conditions.

The linear switching function

$$\gamma = \underline{x}^T \underline{\alpha} \tag{17}$$

has a derivative based on (6) which is

$$\dot{\gamma} = \underline{x}^T A^T \underline{\alpha} - b_n(t)\ \mathrm{Sgn}\ (\underline{a}^T \underline{x} - e). \tag{18}$$

If $\underline{\alpha}$ is chosen as an eigenvector of A^T and $-\lambda$ is the corresponding eigenvalue, that is if

$$A^T \underline{\alpha} = -\lambda\ \underline{\alpha}, \tag{19}$$

then for the case $e \equiv 0$.

$$\dot{\gamma} = -\lambda\gamma - b_n(t) \cdot \mathrm{Sgn}\ (\gamma). \tag{20}$$

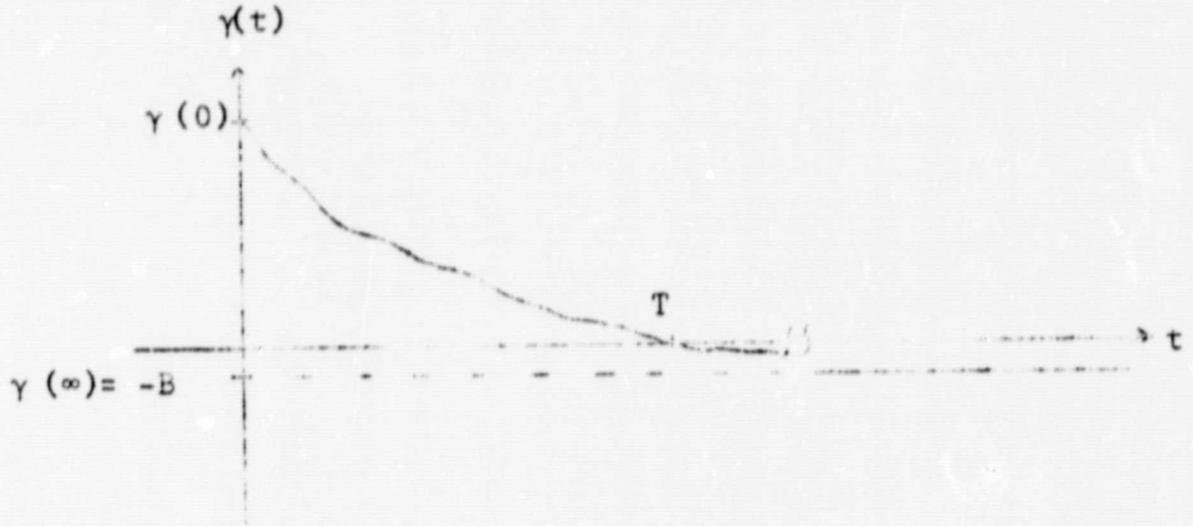
This equation is asymptotically stable since $-\lambda$ is an eigenvalue of A^T and thus of A which is a stability matrix and therefore $-\lambda < 0$, and $b_n(t)$ is non-negative. Thus γ will approach zero asymptotically. In fact γ will reach zero in finite time for the case wherein b(t) is not identically zero. This can be seen by considering γ approaching zero from above. The second term on the right of (20) has a finite, negative average value which will be designated as $-\lambda B$. The asymptote of γ is therefore $-B$. The exponential nature of the solution then gurantees that for any number v between $\gamma(0)$ and $\gamma(\infty) = -B$, there is a finite time T for which $\gamma(T) = v$; $0 \leq T \leq \infty$. Zero is such a number. This is demonstrated graphically in Figure 4.

Motion on the switching plane $\gamma = 0$ is guaranteed to be asymptotically stable since by assumption A is a stability matrix and therefore there can be no unstable manifold passing through the origin. Inasmuch as the switching hyperplane $\gamma = 0$ is a manifold of the system it must therefore be a stable one.

Another argument for this point is that, with the natural switching function design, the trajectory in finite time attains the manifold $\gamma = 0$ upon which it remains for all subsequent time. Consequently, with $e \equiv 0$ the second term on the right of (20) and (16) is identically zero. Thus (16) is represented by

$$\underline{\dot{x}} = A\underline{x};\ e \equiv 0, \tag{21}$$

whereupon subsequent motion on $\gamma = 0$ is asymptotically stable.



γ (t) Reaches Zero in Finite Time

Figure 4

The natural switching function design for the case of an n^{th}-order system is only possible when A^T and equivalently A has at least one distinct real eigenvalue. For the case of $m \leq n$ simple roots there are m possible natural switching functions. This design has several merits. The most important will now be demonstrated.

Having established asymptotic stability in the absence of filter error, the problem remains to obtain a realistic estimate of the bound on $\underline{x}$ due to a given filter. The method to be employed here is to show that the state vector is confined to a certain neighborhood of the switching plane. Then the bound estimation technique reported previously [4] may be employed. The problem therefore is to obtain a realistic bound on $\gamma(\underline{x})$, thus guaranteeing that $\underline{x}$ is contained in a hyperplane region centered about and parallel to the switching plane. The system with filter error is represented by

$$\dot{\gamma} = -\lambda\gamma - b_n(t) \cdot \text{Sgn}(\gamma - e). \quad (22)$$

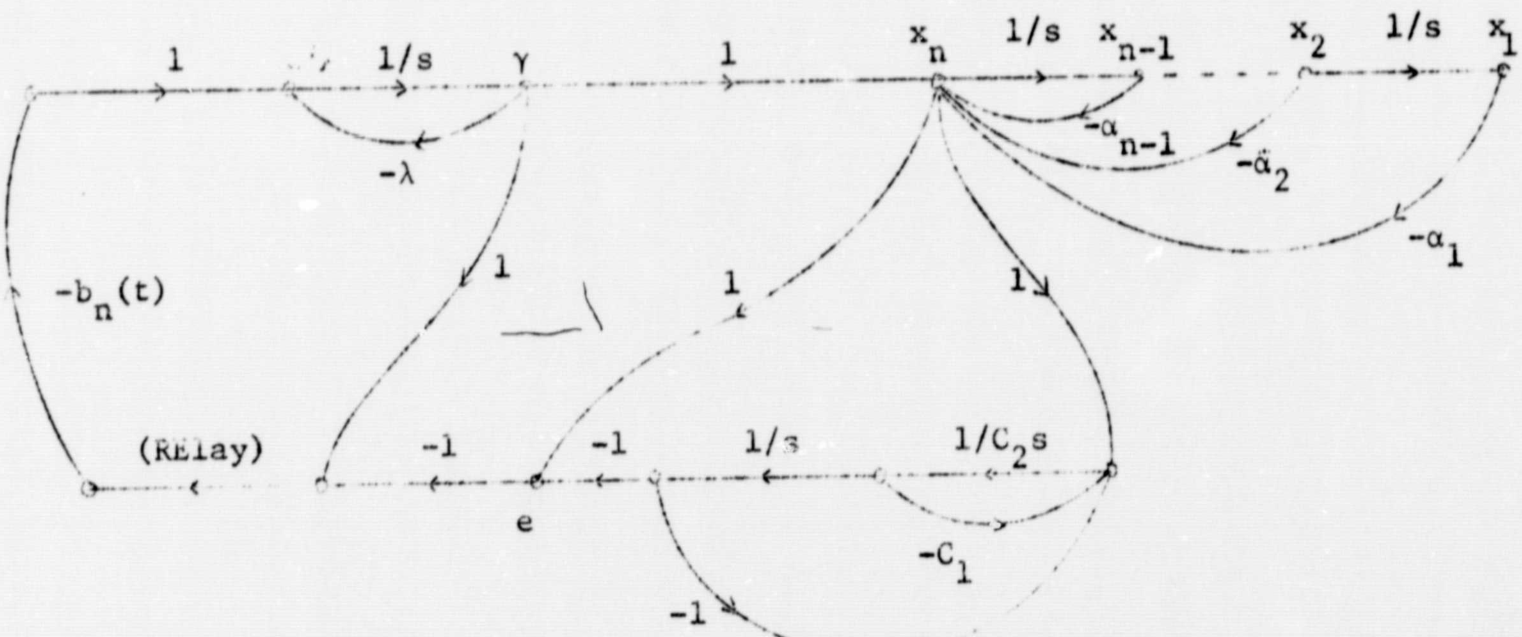
The following discussion is based on the fact that magnitude of γ will be decreasing as long as

$$|\gamma| \geq |e|. \quad (23)$$

It becomes apparent that the bound on $|e|$ becomes on upper bound on $|\gamma|$. Minimization of the bound is therefore simply based on the minimization of $|e|$. The object then in obtaining an upper bound on $|e|$ is to determine the largest value of $|\gamma|$ that can result.

The natural switching-function system is described by the following set of equations and corresponding flow graph, Figure 5.

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_{n-1} &= x_n = -\sum_{i=1}^{n-1} \alpha_i x_i + \gamma \\
\dot{x}_{n+1} &= x_{n+2} \\
c_2 \dot{x}_{n+2} &= -x_{n+1} - c_1 x_{n+2} - \sum_{i=1}^{n-1} \alpha_i x_i + \gamma \\
\dot{\gamma} &= -\partial\gamma - b_n(t)\ \mathrm{Sgn}\ (\gamma - e) \\
e &= x_n - x_{n+1}
\end{aligned}
\tag{24}
$$



Natural Switching Function System

Figure 5

The filter error responds to initial conditions of all integrator outputs and to the relay output. However it is the driven response that is of interest here for it is this response that prevails to possibly affect an eventual bound. To find the largest $|e|$ that can be reached, the system is initiated at $\underline{x} = \underline{0}$ and the relay is allowed to switch, regardless of its switching function, in an optimal fashion with $b_n(t)$ set at its maximum magnitude, 2L. Note that the system in Figure 5 does not involve plant parameters.

To achieve the optimization, it is convenient to use the transfer function relating the nodes u and e in the absence of the nonlinear feedback path. This takes the form

$$\frac{E(S)}{U(S)} = H(S) = \frac{S^n(C_2S + C_1)}{(S+\lambda)(S^{n-1} + \alpha_{n-1}S^{n-2} + ---\alpha_2S + \alpha_1)(C_2S^2 + C_1S + 1)} \quad (25)$$

for which the inverse transform h(t) obviously exists. It is important to note that the term $1/(S+\lambda)(S^{n-1} + \alpha_{n-1}S^{n-2} + --- + \alpha_1)$ relates u to x_1 and therefore the roots of this polynomial are identical with the eigenvalues of A. That is to say

$$(S + \lambda)(S^{n-1} + \alpha_{n-1}S^{n-2} + --- \alpha_2S + \alpha_1) = \det(IS-A). \quad (26)$$

Also in (25) the numerator term $S^{(n-1)}$ relates x_1 to x_n. The remaining term relates x_n to e by the filter equation (15) and the relation $e = x_n - x_{n+1}$:

$$\frac{E(S)}{x_n(S)} = \frac{S(C_2S + C_1)}{(C_2S^2 + C_1S + 1)}. \quad (27)$$

It is shown in the Appendix that the largest $|e(t)|$ that can result for $|u(t)| \le 2L$ is

$$|e|_{max} = 2L \int_0^\infty |h(\tau)|\, d\tau. \quad (28)$$

With this result, the minimization of $|e|$ and consequently of $|\gamma|$ and of the eventual bound on $\underline{x}$ can be discussed.

One method of reducing e beyond the result of (28) involves the fact that the behavior required of $b_n(t)$ and of the relay, or equivalently, of u(t), to produce the maximum $|e|$ would most likely produce a γ that is larger than e in magnitude. If a second iteration were performed wherein this constraint were placed on γ or equivalently on u(t) it is certain that a smaller e would result. This could possibly converge through several iterations to a much smaller value and possibly zero. However, the optimal problem involves a state-space constraint and its solution has not been determined.

It might be thought possible to apply the optimal technique used in obtaining (28) to the transfer function relating the transforms of γ and e, namely, $(S+\lambda)(H(S)$. However inasmuch as this has numerator and demominator of equal order it will never occur that $|e|$ would be less than $|\gamma|$.

It will now be shown for the case when A has a large negative eigenvalue that $|e|_{max}$ will be correspondingly small. When this is not the case it is shown that the filter (14) can be chosen to make $|e|_{max}$ in (28) arbitrarily small at the expense of admitting noise encountered in the measurement of x_{n-1} into the switching function. The trade-off of γ-bound due to filter error and filtered noise will also be pointed out.

Bound on $|\gamma|$

With the help of Figure 5 it is seen, since $|u| \leq 2L$, that the largest $|\gamma|$ which can occur is $2L/\lambda$. Clearly, if A were to have a distinct real eigenvalue with very large negative value, the hyperplane tangent to its corresponding eigenvector could be chosen as a natural switching plane and λ would be very large. This situtation offers two advantages. The resulting bound on γ would be very small and, by nature of the solution of (20), $|\gamma|$ would tend to converge faster from some initial condition to the bound $2L/\lambda$ than would be the case with smaller values of λ. This result agrees with intuition in that the lack of highest derivative information should not be so serious in a system which could be approximately represented by a lower-order system.

It will now be shown that when A does not have this property, the filter can be chosen so that the bound on $|e|$ and therefore on $|\gamma|$ can be designed to be arbitrarily small. If the transfer function of the general filter in (14) is replaced by a second-order filter with simple real poles so that

$$\frac{x_{n+1}(S)}{x_n(S)} = \frac{1}{(\tau_1 S + 1)(\tau_2 S+1)} , \tag{29}$$

that is if $C_2 = \tau_1\tau_2$ and $C_1 = \tau_1 + \tau_2$, then the transfer function relating x_n to e is

$$\frac{E(S)}{x_n(S)} = \frac{S[S + (\tau_1 + \tau_2)/\tau_1\tau_2]}{(S + 1/\tau_1)(S + 1/\tau_2)} . \tag{30}$$

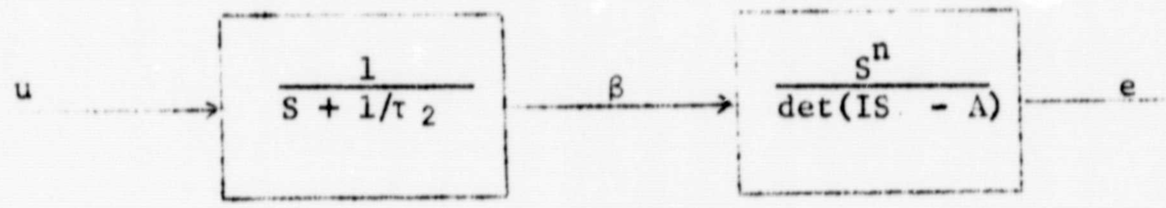
If τ_1 is chosen much less than τ_2 we have

$$\frac{E(S)}{x_n(S)} = \frac{S}{(S + 1/\tau_2)} ; \ \tau_1 \ll \tau_2 . \tag{31}$$

In light of this the transfer function relating u and e from (15) becomes

$$\frac{E(S)}{U(S)} = H(S) = \frac{S^n}{\det(IS - A)(S + 1/\tau_2)} \tag{32}$$

Which can be expressed as shown in Figure 6.



Equivalent Expression of (32)

Figure 6

A consequence of the magnitude constraint $|u| \leq 2L$ and the constraint $\tau_2 << \tau_1$ is that

$$|\beta| \leq 2\tau_2 L. \tag{33}$$

Inasmuch as e is the output of a realizable filter having all roots with negative real parts it follows that e is bounded in proportion to the filter input β. Thus e approaches zero with τ_2. However, to reduce the bound on $|e|$ and $|\gamma|$ in this way, τ_2 must be made small and as a result $S = -1/\tau_2$ becomes a far-out pole. Furthermore, the filter pole at $S = -1/\tau_1$ is even farther out by the assumption $\tau_1 << \tau_2$. These far-out poles admit high frequency measurement noise to the switching function. Treating this noise as an additive signal in the switching function and assuming a magnitude constraint on the noise $|n(t)| \leq N$ it follows that the sign of γ and therefore the relay output cannot be affected if $|\gamma| \geq N$. Thus Sgn $\gamma \neq$ Sgn $(\gamma - e)$ only if $|\gamma| < N$. Due to the fact that the switching line is approached monotonically in the absence of switching function imperfection, and since n(t) can only cause imperfection when $|\gamma| < N$, it follows that a bound on $|\gamma|$ results due to noise alone. This bound is $|\gamma| \leq N$.

It becomes apparent that the cost of choosing a wide-bandwidth filter to decrease the bound on $|\gamma|$ caused by filter error is that a greater bound on $|\gamma|$ due to the presence of high frequency noise may result. The filter would have to be based on some knowledge of the measurement noise in any particular application. To further evaluate the trade-off that exists in the absence of information about the number N, statistical properties of the bound on $|\gamma|$ due to noise could be used rather than absolute maximum values as were used in this instance.

A further advantage of the natural switching function design that becomes apparent here is that the bound on $|\gamma|$ due to both filter error and filtered noise is simply the sum of the bounds due to each acting separately. The reason is that each is based upon imperfection in the switching function and this is an additive quantity. That is to say, if filtered noise n(t) bounded in magnitude by N, and filter error e bounded in magnitude by E, are both present in the implemented signal γ, imperfect control can result only in the state space region denoted by

$$\Omega_{NE} = \{\underline{x}: \ |\gamma(\underline{x})| \leq N + E\}. \tag{34}$$

This is termed the "region of imperfect control." This region for the case of filter error or filtered noise acting alone is

$$\Omega_E = \{\underline{x}: \ |\gamma(\underline{x})| \leq E\} \tag{35}$$

and

$$\Omega_N = \{\underline{x}: |\gamma(\underline{x})| \leq N\} \tag{36}$$

respectively.

Having established the region of imperfect control for (14) and thus for (16), and having guaranteed by the natural switching function design that once $\underline{x}$ enters Ω_{NE} it will remain in this region for all subsequent time, it is possible to determine an eventual bound on the states x_i by employing a technique reported earlier. [4] This is discussed briefly in what follows.

Bound on State Vector

It will now be shown that by nature of the fact that $\underline{x}$ will eventually be confined to the region of imperfect control

$$\Omega \quad \{\underline{x}: \ |\gamma(\underline{x})| \quad \leq C\} \tag{37}$$

a reasonable bound can be determined for $\underline{x}$. This follows from the fact that $\underline{x} \in \Omega$ implies

$$-C \leq \sum_{i=1}^{n} \alpha_i x_i \leq C. \tag{38}$$

Following through with the earlier assumption that $\alpha_n = 1$, the constraint (38) implies

$$-C - \sum_{i=1}^{n-1} \alpha_i x_i \leq x_n \leq C - \sum_{i=1}^{n-1} \alpha_i x_i \tag{29}$$

which is satisfied if

$$x_n = - \sum_{i=1}^{n-1} \alpha_i x_i + \beta(t) \tag{40}$$

where

$$|\beta(t)| \leq C. \tag{41}$$

In light of (41) it is possible to represent motion of (1) confined to Ω by the n-1st -order system

$$\dot{\underline{x}}_r = A_r \underline{x}_r + \underline{b}_r \beta \tag{42}$$

where A_r is an (n-1) x (n-1) matrix of the form

$$A_r = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \bigcirc \\ & & \ddots & \ddots & \\ \bigcirc & & & \ddots & 1 \\ -\alpha_1 & \cdots & & & -\alpha_{n-1} \end{bmatrix},$$

and $\underline{b}_r$ is an (n-1)x1 vector of the form

$$\underline{b}_r = [0 \; --- \; 1]^T.$$

The reduced-order system (42) serves as a model that represents motion of (1) confined to Ω in that any solution of the latter is a solution of the former. Therefore, a bound on solutions of (42) is a bound on solutions of (1) confined to Ω.

It is important to note in (42) that the case $\beta \equiv 0$ represents motion confined to the switching hyperplane $\gamma=0$, and as pointed out earlier, the natural switching function design guarantees that this motion is asymptotically stable. Thus A_r is a stability matrix.

Calculation of the bound on $\underline{x}$ confined to Ω is now accomplished by determination of the reachable set [8] of the linear time-invariant stable reduced-order system (42). A computer estimation technique is available due to the work of Narendra [9] whereby a piecewise-planar figure which bounds the reachable set. every facet of which touches the reachable set at one point, is obtained.

A different method has been developed by the author. This method involves a computer simulation of (42) in which the impulse response is computed once during which time any number of points on the reachable set can be calculated. This scheme saves on computation time relative to the former method which requires a two-point-boundary-value problem solution for each facet of the approximation. The author's method will be submitted for publication [10].

VII. Conclusions

In this paper the design of model reference Liapunov controllers is extended to the case wherein a linear filter is incorporated in the implementation of the switching function. This broadens the scope of the design to permit filtered measurements of available states and filtered approximations of inaccessible states. It might be added that the analysis also facilitates the evaluation of the affects of transducer dynamics which heretofore have been neglected.

In the absence of any sufficient stability criteria Lagrange stability was obtained rather than Asymptotic Stability. The body of the report treats the bound estimation and the trade-off effects of filtered noise and filter error.

For purposes of facilitating the bound analysis, a new design is proposed. It has been shown that choice of a natural switching function guarantees that the state vector approaches the switching plane monotonically, until it enters the region of imperfect control, to remain there for all subsequent time. Inasmuch as the bound calculation is based on the width of the region of imperfect control, factors affecting the bound are evaluated in terms of their contribution to this width. Estimation of the state bound is made through the reachable set of a lower-order model.

Inasmuch as this is a preliminary report, simulation studies have not been included. The complete report is being prepared in the form of a doctoral dissertation which treats the reduced-state stability analysis of the general relay control system that (1) represents. Numerous examples will be included.

APPENDIX

The following pertains to a stable linear filter having input u(t) constrained according to

$$|u(t)| \leq 2L \tag{a1}$$

and output e(t). The filter is completely described by its impulse response h(t). It is to be shown that as a consequence of the input constraint the largest magnitude of filter output that can occur is

$$|e|_{max} = 2L \int_0^{\infty} |h(\tau)| d\tau \tag{a2}$$

where it is assumed e(0) = 0. Mathematically, it is to be shown that the right side of (a2) is the least upper bound on $|e|$.

It can easily be shown that (a2) is an upper bound on $|e(t)|$ for $t \leq \infty$ since

$$e(t) = \int_0^t h(\tau)\, u(t-\tau)\, d\tau . \tag{a3}$$

Thus

$$|e(t)| \leq \int_0^t |h(\tau)|\, |u(t-\tau)|\, d\tau \tag{a4}$$

which for all $t \leq \infty$ becomes in light of (a1)

$$|e| \leq 2L \int_0^{\infty} |h(\tau)|\, d\tau . \tag{a5}$$

Therefore $|e|_{max}$ given by (a2) is indeed an upper bound. To demonstrate that it is a least upper bound it must be proved that equality in (a5) can occur, otherwise, there would exist a lower upper bound.

We use a constructive proof to show that there exists a control u(t) which causes the filter output to equal $|e|_{max}$ as given in (a2). Consider the (positive) sequence $[t_n] \to \infty$, the sequence of controls $[u_n(t)]$ defined over the intervals $0 \leq t \leq t_n$ where

$$u_i(t) = 2L \operatorname{Sgn}[h(t_i - t)] \tag{a6}$$

and the sequence of resultant outputs $[e_n(t_n)] \to E$ where

$$\begin{aligned} e_i(t_i) &= 2L \int_0^{t_i} h(\tau) \operatorname{Sgn}[h(\tau)]\, d\tau \\ &= 2L \int_0^{t_i} |h(\tau)|\, d\tau . \end{aligned} \tag{a7}$$

It follows that as $[t_n] \to \infty$, $[e_n(t_n)] \to E$ and

$$E = 2L \int_0^\infty |h(\tau)| \, d\tau . \tag{a8}$$

QED

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