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THE USE OF GAUSSIAN FUNCTIONS IN RADIO ASTRONOMY SOURCE MEASUREMENTS

J. R. Cogdell

Technical Report No. NGL-006-69-2
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Submitted by

Electrical Engineering Research Laboratory
MILLIMETER WAVE SCIENCES

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Preface

The work reported herein was done by the author during several periods of time. The original ideas were developed in 1964 by the author while employed at Lincoln Laboratory* in conjunction with a source measurement program. The material was refined and expanded considerably during the Summer of 1965 under the support of The University of Texas Joint Services Contract, AF-AFOS-766-65. It appears in this report further expanded.

The approach to the source measurements problem which is criticized and improved in this report is that of Baars, et al.† Their work was done in the interest of comparing numerous source measurements which were available in the literature. Their approach, our "Method I," was used to correct the various measurements for beam size effects to allow comparison. Since they did not have access to the original data, their approach was the only possible one and was valid, appropriate, and not subject to criticism. Their results, however, have been widely used in source measurements and it is this use which we feel is not optimal.

The most commonly used method of estimating source strength involves mean-square-fitting a Gaussian curve to the data. This approach requires a

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*Operated by Massachusetts Institute of Technology. This work supported by the U. S. Air Force.

†J. W. M. Baars, P. D. Mezger, and H. Wendker, "The Spectra of the Strongest Non-thermal Radio Sources in the Centimeter-Wavelength Range," Astrophysical Journal, Vol. 142, pp. 122-134 (1965).

computer and is probably equivalent or superior to the method developed in this report. It is difficult to compare our method with mean-square-fitting due to the fact that our method is subjective to some extent. Our method is aimed at the hand reduction of data, while mean-square-fitting is best suited for complete computer reduction.

The author would like to acknowledge the aid of Jim Gort in producing Figures 5 and 6, and thank John Davis for reading the draft with care and making numerous helpful suggestions.

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I. INTRODUCTION

In the analysis of radio source observations, one frequently introduces idealizations. An idealization which is commonly used when the source size is comparable to the antenna's beamwidth is that both the source brightness distribution and the antenna's power pattern are adequately represented by Gaussian functions. This practice is founded on several considerations. For one, the respective functions (in so far as they are known) often resemble Gaussian functions to a high degree. For another, the antenna response, being the convolution of the antenna pattern with the source intensity, tends to be Gaussian in shape, since multiple convolutions tend (mathematically) toward the Gaussian form. Finally, the Gaussian form allows one a representation which has a minimum number of parameters and offers many mathematical niceties.

However, the simplifications which are suggested by the assumption of Gaussian forms, e.g., representing the antenna response by only the peak and width of the response, can lead to an oversimplification of the problem. This oversimplification can lead in turn to a neglect of the advantages which this model offers. In this report, we reexamine the basis for the common utilization of the Gaussian assumption. In Section II, we develop in a tutorial manner the benefits of the Gaussian assumption. We introduce the statistical effects of pointing errors and thus view the antenna temperature as a statistical average. In Section III, we consider the problem of estimating source strength. Here we introduce the receiver noise as a stochastic process and treat the problem as one in statistical sampling. The conventional method of determining source strength (Baars, Mezger, et al) is examined and criticized on several counts. Another method is examined and found to offer several advantages when used with discretion. In Section IV, we review briefly the basis upon which estimations of source size are made. In Section V, the effects of filtering the receiver output are examined. The distorting effects of the filter are then related to the source strength problem and corrections are given. In Section VI, we summarize our results and discuss the application of similar techniques to problems where the Gaussian assumption seems unjustified.

II. ELEMENTS OF A SOURCE MEASUREMENT

A. Antenna Temperature

The fundamental definitions of spectral intensity, antenna effective area, and antenna temperature combine to yield

$$T_a(\Omega') = \frac{1}{2k} \int_{4\pi} I(\Omega)\, A(\Omega' - \Omega)\, d\Omega \qquad (1)$$

where

Ω'	$= \Omega'(\theta', \varphi')$ is the direction in which the antenna is pointed
Ω	$= \Omega(\theta, \varphi)$, a point on the celestial sphere
$I(\Omega)$	= the intensity of the source (watts/cps/(meters)^2/steradian)

	$A(\Omega)$	= the mirrored antenna effective area function $(\text{meters})^{2*}$
	$T_a(\Omega')$	= antenna temperature (°K)
and	k	= Boltzmann's constant (Joule/°K).

In cases of interest in this report, the integrand of Equation (1) is negligible except in a region near the source. Hence we are justified in treating the integral as performed on a plane. Let us introduce a pseudo-orthogonal co-ordinate system centered on the source position. Our two coordinants, x_1 and x_2, denote right ascension and declination, respectively, and do not form a true rectangular grid since all circles of constant declination except the celestial equator are small circles on the celestial sphere. Nevertheless, in the region of the source this type of representation should be adequate, and Equation (1) becomes accordingly:

$$T_a(x_1', x_2') = \frac{1}{2k} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x_1, x_2)\, A(x_1 - x_1', x_2 - x_2')\, dx_1 dx_2 \qquad (2)$$

Thus, the antenna temperature is proportional to the two-dimensional convolution of the effective area and the source intensity:

$T_a = \frac{1}{2k} I \otimes A$, where $\otimes$ indicates the two dimensional convolution.

*The mirror image of the pattern is used to obtain a convolution relationship directly without evoking symmetry properties in either pattern or source.

B. Effects of Pointing Errors

The mapping of antenna temperature in the region of the source is most commonly done by scanning the antenna across the region at a constant rate. At a predetermined time one knows the position on the celestial sphere at which the antenna is pointed; thus, one knows the antenna's pointing direction for all time. Usually, one averages many such scans to reduce the effects of receiver noise. Errors in pointing are characterized by the fact that at the appointed time the antenna is possibly not pointed at the desired position on the celestial sphere, but is likely in error in both right ascension and declination. These errors, which by definition are unknowable except in some statistical sense, result in that the averaged map of the desired area is "smeared out" by the pointing errors. The best one can do is to estimate the smearing effect by introducing an assumed probability distribution of the errors and performing a statistical averaging of their effect. That is, if the probability density of error of x_1'' and x_2'' in right ascention and declination, respectively, is $f(x_1'', x_2'')$, the average antenna temperature measured will be

$$T_{avg}(x_1', x_2') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_a^*(x_1' + x_1'', x_2' + x_2'')\, f(x_1'', x_2'')\, dx_1'' dx_2'' \qquad (3)$$

Since we would expect that bias in the pointing would have been removed (or would be observable if an entire region is mapped) and since the errors would be symmetrically distributed, Equation (3) is equivalent to another two dimensional convolution:

$$T_{avg} = T_a \otimes f = \frac{1}{2k} \; I \otimes A \otimes f \tag{3a}$$

In words, the average antenna temperature is proportional to the two-fold convolution of the source intensity and the antenna effective area and the distribution function of the pointing errors. It is in the evaluation of this mutiple convolution that the Gaussian forms are introduced.

C. The Gaussian Form

The Gaussian curve is characterized by two parameters, conveniently the peak and the width between the peak and the half peak point. If a is the peak value and w is the width defined above, the form of the function is

$g(x) = a \exp - \left[(\ell n\, 2) \left(\frac{x}{w} \right)^2 \right]$. Since $e^{-\ell n\, 2} = \frac{1}{2}$, this can also be written as

$g(x) = a(\frac{1}{2})^{-\left(\frac{x}{w}\right)^2}$, although the first form is more convenient for calculations. In our case we must use a two-dimensional function, having possibly different widths in the two dimensions.

$$g(x_1, x_2) = a \exp - \left\{ (\ell n\, 2) \left[\left(\frac{x_1}{w_1} \right)^2 + \left(\frac{x_2}{w_2} \right)^2 \right] \right\} \tag{4}$$

In the function $g(x_1, x_2)$ the peak occurs at $x_1 = x_2 = 0$ and the half-peak curve is the ellipse described by the curve $\left(\frac{x_1}{w_1} \right)^2 + \left(\frac{x_2}{w_2} \right)^2 = 1$. But this is not the most general Gaussian form since is assumes the ellipse to be aligned with the r.a.-dec. axes. The most general form should allow for the ellipse to be oriented at an arbitrary angle. To introduce this facility, we need to change our notation.

We now treat the coordinants, x_1 and x_2, as a column matrix $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and Equation (4) becomes

$$g(\underline{x}) = a \exp\left\{-\ln 2\, \underline{x}^t \begin{bmatrix} w_1^2 & 0 \\ 0 & w_2^2 \end{bmatrix}^{-1} \underline{x}.\right\} \tag{4a}$$

The superscript t indicates the transpose of the matrix and the superscript -1 indicates the inverse matrix. We introduce the inverse matrix at this stage in order to simplify notation in later developments. We now can orient the half-peak ellipse at an angle φ with respect to the x_1, x_2 axes by introducing the following orthogonal transformation on the square matrix:

$$g(\underline{x}) = a \exp\left\{-\ln 2\, \underline{x}^t\, \underline{\Phi}^t \begin{bmatrix} w_1^2 & 0 \\ 0 & w_2^2 \end{bmatrix}^{-1} \underline{\Phi}\, \underline{x},\right\} \tag{5}$$

where

$$\underline{\Phi} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}.$$

Equation (5) now represents the most general two-dimensional Gaussian form. Some other results which will be needed later can best be stated here. Let us abbreviate Equation (5) to the form:

$$g(\underline{x}) = a \exp\left\{-\, \underline{x}^t\, \underline{A}^{-1}\, \underline{x}\right\} \tag{5a}$$

where

$$\underline{A}^{-1} = \ln 2 \, \underline{\Phi}^t \begin{bmatrix} w_1^2 & 0 \\ 0 & w_2^2 \end{bmatrix}^{-1} \underline{\Phi}.$$

The two-dimensional Fourier transform of $g(\underline{x})$ is know to be

$$\mathfrak{F}(\underline{y}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{-(\underline{y}^t \, \underline{x})\right\} g(\underline{x}) \; dx_1 dx_2$$

$$= a\pi \sqrt{\det(\underline{A})} \exp\left\{-\tfrac{1}{4} \, \underline{y}^t \, \underline{A} \, \underline{y}\right\} \qquad (6)$$

where $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and det $(\underline{A})$ indicates the determinant of $\underline{A}$. A useful special case of Equation (6) is that it gives us the volume under $g(x)$:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\underline{x}) \; dx_1 dx_2 = \mathfrak{F}(0, 0) = a\pi \sqrt{\det(\underline{A})} \qquad (6a)$$

D. The Gaussian Model Introduced

We are now in a position to present the source intensity, antenna effective area and the pointing error probability distribution as Gaussian functions.

(1) The Source Intensity - Let us say that the source is adequately characterized by a Gaussian form of volume S, the source strength, and a half-peak ellipse of semi-major and semi-minor axes of w_1 and w_2, respectively, with the major axis oriented at an angle φ_1 with respect to the right ascension axis. Such a function is of the form

$$I(\underline{x}) = \frac{\ln 2\ S}{\pi w_1 w_2} \exp\left\{-\underline{x}^t\ \underline{A}^{-1}\ \underline{x}\right\}, \tag{7a}$$

where

$$\underline{A} = \frac{1}{\ln 2}\ \underline{\Phi}_1^t \begin{bmatrix} w_1^2 & 0 \\ 0 & w_2^2 \end{bmatrix} \underline{\Phi}_1 \text{ and } \Phi_1 = \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{bmatrix}$$

The peak value of $I(\underline{x})$ is chosen such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I(\underline{x})\ dx_1 dx_2 = S \tag{7b}$$

as can be verified from Equation (6a) and the fact that

$$\det A = \frac{1}{(\ln 2)^2}\ (\det \underline{\Phi}_1)^2\ \det\begin{bmatrix} w_1^2 & 0 \\ 0 & w_2^2 \end{bmatrix} = \frac{w_1^2 w_2^2}{(\ln 2)^2}$$

(2) The Antenna Effective Area - The effective area of the antenna is proportional to the power gain pattern of the antenna. Within the scope of our Gaussian model, we can characterize it by

$$A(\underline{x}) = A_e \exp\left\{-\underline{x}^t\ \underline{B}^{-1}\ \underline{x}\right\} \tag{8}$$

where

A_e = maximum effective area of antenna

$$\underline{B} = \frac{1}{\ln 2}\ \underline{\Phi}_2 \begin{bmatrix} \theta_1^2 & 0 \\ 0 & \theta_2^2 \end{bmatrix} \underline{\Phi}_2$$

θ_1, θ_2 = semi-major and semi-minor axes of half-power ellipse of gain pattern. *

$$\underline{\Phi}_2 = \begin{bmatrix} \cos \varphi_2 & -\sin \varphi_2 \\ \sin \varphi_2 & \cos \varphi_2 \end{bmatrix}$$

φ_2 = angle between right ascension axis and major axis of half-power ellipse of antenna pattern.

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*A more general way to define the antenna beam would be to define θ_1 and θ_2, the beamwidths, such that

$$\theta_1 \theta_2 = \frac{\lambda^2}{A_e} \frac{\ln 2}{\pi} \eta_B = \frac{4 \ln 2}{G_{max}} \eta_B$$

where η_B is the main lobe efficiency of the antenna defined as

$$\eta_B = \frac{\int_{lobe} G(\Omega)\, d\Omega}{\int_{4\pi} G(\Omega)\, d\Omega} = \frac{1}{4\pi} \int_{lobe} G(\Omega)\, d\Omega$$

An additional equation defining θ_1 and θ_2 to be in the ratio of the major and minor semi-diameters of the half power ellipse is required. For the ideal beam of Gaussian shape the two definitions are equivalent. Adopting the later definition would allow the simplification of a number of equations in the following, although the equations are not necessarily clarified by this simplification.

In words, we have allowed for an unsymmetric pattern oriented at an arbitrary angle with respect to the right ascension axis. Some care would have to be exercised in the use of this expression if the antenna was controlled by an az-el mount, since in that case the angle φ_2 would in general vary with time. If the antenna beam is symmetric ($\theta_1 = \theta_2$), the preceding proviso is irrelevant. It should be noted that the θ values are not half-power beamwidths, but rather are one-half the maximum and minimum half-power beamwidth of the antenna pattern.

(3) Pointing Error Probability Distribution - The model adopted here is the Gaussian distribution. While it would not be difficult to treat r.a. and dec. errors as correlated, this seems unnecessary, and we chose to treat errors in the two directions as uncorrelated and of equal variance:

$$f(\underline{x}) = \frac{1}{2\pi\sigma^2} \exp\left\{-\underline{x}^t \underline{C}^{-1} \underline{x}\right\} \tag{9}$$

where

$$C = 2\begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}$$

Here, we have denoted the rms pointing error in r.a. and dec. as σ; the overall rms pointing error would be $\sqrt{2}\sigma$. We have thus treated errors as uncorrelated from scan to scan.

E. The Solution

We are now in a position to perform the double convolution in Equation (3a).

$$T_{avg}(\underline{x}) = \frac{(\ell n\, 2)\, S\, A_e}{4\pi^2 k\, w_1 w_2 \sigma^2} \left[\exp\left\{-\underline{x}^t\, \underline{A}^{-1}\, \underline{x}\right\} \otimes \exp\left\{-\underline{x}^t\, \underline{B}^{-1}\, \underline{x}\right\} \otimes \exp\left\{-\underline{x}^t\, \underline{C}^{-1}\, \underline{x}\right\} \right]$$

(3b)

The convolution is evaluated with the least difficulty if the Fourier transform of Equation (3b) is taken, since the convolution becomes a multiplication in the transform domain. The Fourier transform of Equation (3b) can, therefore, be expressed through the use of transform pair, given earlier (Equation (5a) and (6)) as

$$\mathfrak{T}_{avg}(\underline{y}) = \frac{\ell n\, 2\, S\, A_e}{4\pi^2 k\, w_1 w_2 \sigma^2} \left(\pi \sqrt{\det(\underline{A})}\, \exp\left\{-\tfrac{1}{4}\, \underline{y}^t\, \underline{A}\, \underline{y}\right\}\right) \left(\pi \sqrt{\det(\underline{B})}\, \exp\left\{-\tfrac{1}{4}\, \underline{y}^t\, \underline{B}\, \underline{y}\right\}\right)$$

$$\left(\pi \sqrt{\det(\underline{C})}\, \exp\left\{-\tfrac{1}{4}\, \underline{y}^t\, \underline{C}\, \underline{y}\right\}\right) = \frac{\pi\, S\, A_e\, \theta_1 \theta_2}{2k\, \ell n\, 2} \exp\left\{-\tfrac{1}{4}\, \underline{y}^t\, (\underline{A} + \underline{B} + \underline{C})\, \underline{y}\right\}$$

(3c)

The antenna temperature can now be expressed as the inverse transform of Equation (3c):

$$T_{avg}(\underline{x}) = \frac{S\, A_e\, \theta_1 \theta_2}{2\, \ell n\, 2\, k} \frac{1}{\sqrt{\det(\underline{A} + \underline{B} + \underline{C})}} \exp\left\{-\underline{x}^t\, (\underline{A} + \underline{B} + \underline{C})^{-1}\, \underline{x}\right\} \qquad (10)$$

It is to be noted that the average antenna temperature is Gaussian in form and can therefore be characterized by only four parameters, a peak value and the parameters describing the half-peak ellipse. These parameters are not relevant to our development at this stage and will not be discussed here. One property of this function can be easily obtained, however. The total volume under the function is given by the Fourier transform with zero arguments:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_{avg}(\underline{x})\, dx_1 dx_2 = \mathfrak{F}_{avg}(0,0) = \frac{\pi S A_e \theta_1\theta_2}{2 \ln 2\, k} \qquad (11a)$$

which can be written

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_{avg}(\underline{x})\, dx_1 dx_2 = \frac{S}{2k}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\underline{x})\, dx_1 dx_2 \qquad (11b)$$

In words, the integral of the antenna temperature depends upon only the source flux and the integral of the antenna's effective area, which is defined to be the antenna efficiency. It will be shown later in this report that this property can be derived from fundamentals and depends in no way upon the Gaussian assumption.

III. SOURCE STRENGTH MEASUREMENTS

A. Introduction

In this section we shall discuss the problem of estimating the strength of a radio source from a set of measurements on that source. We say "estimate" because there is inherent inaccuracy in any such measurement. Even if we ignore instrumental errors (nonlinearity, calibration error, etc.) and accept the Gaussian assumptions we still have to acknowledge the effects of system noise.

A set of measurements would consist of a set of data points, say, $T_d(\underline{x})$ where T_d is one's value for the response of the antenna looking at some point on the sky x_i. The discrete nature of the data can arise either in the data recording output of the receiver as in a digital recording system, or it can be introduced by the observer for the purpose of averaging data taken in analog form. We shall assume that the data value differs from the actual antenna temperature by a term characterizing receiver noise:

$$T_d(\underline{x}_i) = T_{avg}(\underline{x}_i) + T_{ni} \tag{12}$$

Here we conceive of T_{ni} as a set of Gaussian random variables having the following properties

$$E[T_{ni}] = 0 \qquad E[T_{ni}^2] = \left(\Delta T_{rms}\right)^2$$

where E [·] denotes the statistical expectation or average. The value of ΔT_{rms} will depend in a well-known manner upon the system temperature, the bandwidth, and the integration time going into that particular date point. It is conceivable that different data points have different accuracies, owing to different averaging times, but we shall not consider that possibility in the following. Furthermore, we shall consider T_{ni} and T_{nj} as uncorrelated, i.e.,

$$E[T_{ni}\,T_{nj}] = 0, i \neq j$$

This is not strictly true if a low pass filter is used in smoothing the data, as is common in analog recording. This difficulty will be discussed in the context where the above property is used.

The object of a source strength measurement is to take a set of data and estimate the parameter, S, in Equation (7) given that one knows the antenna properties given in Equation (8) and the pointing accuracy as used in Equation (9). We shall discuss two methods. The first, which we will for convenience call Method I, characterizes the response by its peak value and its width between half-peak values. The second, Method II, seeks to exploit the integral property presented in Equation (11).

B. Method I

1. Described.

In this method one locates the peak of the source and then scans it in two orthogonal directions, normally the directions of constant right ascension and constant declination. This results in two profiles of the source. The source strength is then computed as follows:

a. Estimate the peak response (T_p). This depends upon a subjective estimation and involves, say, three or four data points near the peak of the response.

b. Divide the peak value by two and estimate the width between the positions where this response falls below the value on the two profiles. Call these two widths $2\beta_1'$ and $2\beta_2'$.

c. Compute source strength by the formula.

$$S_{est} = \frac{2\,k\,T_p}{A_e}\;\frac{\beta_1'\,\beta_2'}{\theta_1\theta_2} \tag{13}$$

2. The Validity of the Method.

We can examine the suitability of Equation (13) by using Equation (10) to drive what the values of T_p, β_1' and β_2 should be in the absence of noise. We see no way to evaluate the effects of noise on the estimate of S owing to the subjective nature of the method.

In the absence of noise, we determine that one should obtain by the above procedure:

$$T_p = \frac{S\,A_e\,\theta_1\theta_2}{2\,\ell n\,2\,k}\;\frac{1}{\sqrt{\det\,(\underline{A} + \underline{B} + \underline{C})}},$$

$$\beta_1' \text{ such that } [\beta_1'\;0]\;[\underline{A} + \underline{B} + \underline{C}]^{-1}\begin{bmatrix}\beta_1'\\ 0\end{bmatrix} = \ell n\,2, \text{ and}$$

$$\beta_2' \text{ such that } [0\;\beta_2']\;[A + B + C]^{-1}\begin{bmatrix}0\\ \beta_2'\end{bmatrix} = \ell n\,2$$

Substitution of the above values into Equation (13) yields

$$S_{est} = S\sqrt{1 - \frac{\left[\left(w_1^2 - w_2^2\right)\sin 2\,\varphi_1 + (\theta_1^2 - \theta_2^2)\sin 2\,\varphi_2\right]^2}{D}}$$

where $D \approx \frac{1}{4}\,(\beta_1'^2 + \beta_2')^2 > 0$. Equation (13) proves to be an unbiased estimator of the source strength when the bracketed term vanishes. Sufficient conditions for this happening are that the beam and source are symmetric, or that the beam is symmetric and one takes profiles along the major and minor axes of the source.

One observes that one needs to determine more than the center of the source in order to initiate a valid measurement. One needs in addition the orientation of the maximum width of the source so that profiles can be made along and across that direction. The use of right ascension and declination scans will tend to underestimate the source strength except in the fortuitous case where the source eccentricity is aligned with the line of constant right ascension or constant declination. When there is a priori knowledge of the circular symmetry of the source, as would be the case in the observation of most planets, one can get an unbiased estimate of the strength from a right ascension profile, provided the antenna beam is symmetric.

3. Other Comments on the Method.

a. The greatest appeal of this method is perhaps its simplicity.

b. The method is subjective to a large degree. There is no guarantee that independent observers would derive the same result. For the same reason, it is difficult to assign an accuracy to the result.

c. The result depends principally on the data points near the peak and near the half-peak points. The rest of the data are largely ignored. This is all the more unfortunate if one observes the source sufficiently to determine its orientation.

d. An error in judging the peak of the response results in a corresponding (and partially compensating) error in determining the "half-peak" widths. It is easy to show that if the peak is overestimated by δ per cent then the strength will be underestimated by $.44\delta$ per cent if no errors are made in judging the widths.

e. The final result is very sensitive to errors made in judging the half-peak widths. This is especially true when circular symmetry is assumed and the estimated width is squared in the flux computation.

f. Pointing errors do not affect the estimate, subject to our assumption that they are isotropically disturbed.

g. Aside from the effects due to source and beam asymmetrics noted above, there is no reason to believe that this method biases the results, i.e., no systematic errors are in principle introduced. This is no longer true if a low pass filter is used, as will be discussed in Section IV.

C. Method II

1. Rationale. As stated earlier, Method II is based on the integral property presented in Equation (11), viz., that the integral of the response is independent of the size of the source. Of course, the data are only samples of continuous functions, and summations must ultimately replace all integrals. Nevertheless, for the sake of argument, let us assume that integrals can be performed.

Let E denote an ellipse where $\underline{x}^t (\underline{A} + \underline{B} + \underline{C})^{-1} x < \nu$, where ν is a positive number. We substitute Equation (10) into Equation (12) and integrate the result over E:

$$\int_E T_d(\underline{x})\, dx_1 dx_2 = \frac{S A_e \theta_1 \theta_2}{2 \ell n\, 2 k} \left[\frac{1}{\sqrt{\det (\underline{A} + \underline{B} + \underline{C})}} \int_E \exp - \underline{x}^t (\underline{A} + \underline{B} + \underline{C})^{-1} \underline{x}\, dx_1 dx_2 \right]$$

$$+ \int_E T_{ni}(\underline{x})\, dx_1 dx_2 \qquad (14)$$

The bracketed term has the value $\pi(1 - e^{-\nu})$, and thus Equation (14) can be re-arranged to read:

$$S_{est} = \frac{2 \ln 2 k}{\pi A_e \theta_1 \theta_2 (1 - e^{-\nu})} \int_E T_d(x) dx_1 dx_2 = S + \frac{2 \ln 2 k}{\pi A_e \theta_1 \theta_2 (1 - e^{-\nu})} \int_E T_n(\underline{x}) dx_1 dx_2 \tag{14a}$$

When Equation (12) was introduced, $T_{ni}(\underline{x})$ were viewed as a set of random variables. In the present context where we are considering the equation as of a continuous nature, we must conceive of $T_n(\underline{x})$ as a random process. The integral of T_n is therefore a random variable and thus the left side of Equation (14a) must also be considered a random variable, which we have denoted S_{est}. This is our estimate of the source strength. Since $E[T_n(\underline{x})] = 0$, we see that

$$E[S_{est}] = S,$$

that is, S_{est} is an unbiased estimator of the source strength.

The standard deviation of S_{est} will depend on the spacial correlation properties of $T_n(\underline{x})$. Before we assumed that T_{ni} were uncorrelated from point to point. If we extend that the continuous case, the autocorrelation function of $T_n(\underline{x})$ will be a two-dimensional impulse. Under this assumption, the standard deviation of the integral of T_n over E is proportional to the square root of the area of E, i.e., proportional to the square root of the number of points included in the integration. In the event that the noise is correlated over some length, this relationship is still approximately true if the dimensions of the ellipse are large comparable to the correlation length. As we shall see, this latter condition will always be true in practice. So in either case:

$$\sigma_{noise} \propto \sqrt{\int_E dx_1 dx_2} \propto \sqrt{\nu}, \tag{15a}$$

the latter proportionality following from the definition of E. We now can state that part of the standard deviation of S_{est} which comes solely from the noise term varies as

$$\sigma_{Sest} \propto \frac{\nu^{\frac{1}{2}}}{1 - e^{-\nu}} \tag{15b}$$

The right side of Relation (15b) is plotted in Figure 1, and it is seen to be minimum for $\nu = 1.25$. For that value $e^{-\nu} = .286$.

In words, Method II is rationalized as follows. The volume under the response is proportional to source strength. However, one cannot integrate the entire response because as one gets far away from the peak one gets less and less signal while the noise continues to contribute. Thus one integrates over that region where the response is greater than a certain fraction of the peak response and corrects for the signal omitted. If the fraction is selected to be 28%, the error due to receiver noise is minimized on the average.

Integration of the data also has the effect of reducing substantially the effects of misjudgment of the response widths. Of course, if the entire response were integrated, no estimate of the response width would be introduced, since no correction would be needed. This is inadvisable, as we have shown, because of the degrading effects of receiver noise. Thus, one must estimate the response widths of the response in order to determine the ellipse of integration (E) and if the widths are misjudged the correction factor will be in error for the ellipse selected. In other words, one would look at the response, estimate the center of half power ellipse, its orientation, and its major and minor axes. All of these parameters can be in error due to receiver noise. Still, one would have to take the estimated ellipse, multiply its dimension by $\frac{\nu}{\sqrt{\ell n\, 2}}$ and integrate the data, as in Equation (14), over that ellipse, say, E′. If E′ is not the correct ellipse, the bracketed term in Equation (14) no longer has the value stated and Equation (14a) must be modified to read.

$$S_{est} = \frac{2\,\ell n\,2\,k}{\pi A_e \theta_1 \theta_2 (1 - e^{-\nu})} \int_{E'} T_d\, dx_1 dx_2 = \frac{S \int_{E'} \exp - \underline{x}^t (A + B + C)^{-1} \underline{x}\, dx_1 dx_2}{\pi (1 - e^{-\nu})}$$

$$+ \frac{2\,\ell n\,2\,k}{\pi A_e \theta_1 \theta_2 (1 - e^{-\nu})} \int_{E'} T_n(x) dx_1 dx_2 \qquad (16)$$

The expected value of S_{est} will, of course, be sensitive to the errors in E′. A first order expansion of the statistical expection of Equation (16) shows that the average value of our estimation is not sensitive to first order to errors in the location and orientation of E′, but is biased only by errors in size in the following manner:

$$E[S_{est}] = S\left[1 + \left(\frac{\Delta\beta_1}{\beta_1} + \frac{\Delta\beta_2}{\beta_2}\right)\frac{\nu e^{-\nu}}{2(1 - e^{-\nu})} + \ldots\ldots\right] \tag{16a}$$

where $\Delta\beta_1$ and $\Delta\beta_2$ are the errors made in estimating β_1 and β_2, respectively. This expression can be compared to what similar errors would introduce into Equation (13).

$$S_{est} = \frac{2\,k\,T_p}{A_e}\,\frac{\beta_1\beta_2}{\theta_1\theta_2}\left[1 + \left(\frac{\Delta\beta_1}{\beta_1} + \frac{\Delta\beta_2}{\beta_2}\right) + \ldots\ldots\right]$$

When the term within the bracket is compared with the corresponding term in Equation (16a) one sees that Method II reduces the effects of misjudging the response widths by a factor of $\nu e^{-\nu}/2(1 - e^{-\nu})$ over Method I (see Figure 2). At the point which minimizes the receiver noise contribution, ($\nu = 1.25$), this factor has a value of 0.250, and decreases with increasing ν. Thus, one can reduce the effects of size misestimation by integrating one more of the response (increasing ν).

We conclude that in selecting the value of ν one must establish a compromise between minimizing the respective effects of receiver noise and size errors. In view of the broad minimum shown in Figure 1, we feel that $\nu = 2.8$, which corresponds roughly to an ellipse of twice the dimension of the estimated half power ellipse, is a good compromise value. It is appropriate to point out here that this or any reasonable choice of the integration area justifies the assertion underlying Relation (15a), namely that the dimensions of the integration area are large compared to the correlation length of the receiver noise. Any measurement where this were not true would be worthless by any method of data analysis since all but the lowest spacial frequencies of the response would be rejected by such a filter.

2. Method II Described

The flux computation must be developed from the discrete data. The following is merely the discrete version of the procedure described in the previous section.

a. Estimate the center of the response, the orientation of the major axis of the half-power ellipse (φ_3) and its major and minor semidiameters, β_1' and β_2', respectively. Considering the center of the ellipse to be the origin of the coordinant system, write down a Gaussian function having the estimated half power ellipse:

$$g_{est}(\underline{x}) = \exp\left\{-\ell n\, 2\, \underline{x}^t \underline{\Phi}_3^t \begin{bmatrix} \beta_1'^2 & 0 \\ 0 & \beta_2'^2 \end{bmatrix}^{-1} \underline{\Phi}_3 \underline{x}\right\}$$

where

$$\underline{\Phi}_3 = \begin{bmatrix} \cos\varphi_3 & -\sin\varphi_3 \\ \sin\varphi_3 & \cos\varphi_3 \end{bmatrix}$$

b. Double the size of the half-power ellipse ($\nu = 2.77$).

c. Let $\underline{x}_i$ stand for the position of the i^{th} data point within that ellipse, N points in all.

d. Equation (12) is now summed over all points within the ellipse and divided as in Equation (14a).

$$S_{est} = \frac{2\,\beta_1'\beta_2'\,k \sum_N T_d(\underline{x}_i)}{A_e\,\theta_1\theta_2 \sum_N g_{est}(\underline{x}_i)} = S\left[1 + \left(\frac{\Delta\beta_1}{\beta_1} + \frac{\Delta\beta_2}{\beta_2}\right)\left(0.250\right) + \ldots\right] + \frac{2\,\beta_1'\beta_2'\,k \sum_N T_{ni}}{A_e\,\theta_1\theta_2 \sum g_{est}(\underline{x}_i)} \qquad (17a)$$

Again, we view S_{est} as a random variable. Defined as in Equation (17a) it has the following statistical properties, assuming no bias in width errors:

$$E[S_{est}] = S \qquad (17b)$$

$$\sigma^2_{S_{est}} = \left[\frac{2\,\beta_1'\beta_2'\,\sqrt{N}\,\Delta T_{rms}}{A_e\,\theta_1\theta_2 \sum g_{est}(x_i)}\right]^2 + \left[(.250)\,S\frac{\sqrt{2}\,\sigma_\beta}{\sqrt{\beta_1'\beta_2'}}\right]^2 \qquad (17c)$$

where σ_β = rms error in judging response width (an estimate will do). Equation (17c) follows from the assumption that errors in judging response widths are statistically independent of each other and of the receiver noise.

3. The Case of Circular Symmetry

The preceding discussion is based upon the premise that one has a two-dimensional plot of the antenna response. In the event that one has valid a priori knowledge that the response will be symmetric, as when the source is a planet or is known to have dimensions much smaller than those of the antenna beam, it is possible to make a valid flux measurement with one-dimensional data, i.e., with only the right ascension profile. The approach developed above can be used in such a measurement, with the following differences:

a. One must assume, within a certain error, that one is scanning across the center of the source on the average. Other than that, random pointing errors do not bias the result.

b. Relation (15b) becomes

$$\sigma_{S_{est}} \propto \frac{\nu^{\frac{1}{4}}}{\mathrm{erf}(\nu^{\frac{1}{2}})}$$

where erf( ) is the error function. The standard deviation has a minimum at $\nu = 1.0$, corresponding to an integration length of about 1.2 times the estimated half-peak width of the response. However, here the minimum is also a rather broad one (cf. Figure 1), and the integration length can be extended in the interest of reducing the effects of errors in the estimated response width.

c. The form of the estimated Gaussian will be

$$g_{est}(x_i) = \exp\left\{-\ln 2\left(\frac{x_i}{\beta'}\right)^2\right\}$$

where β' is one half the estimated half-peak width. A one dimensional integration of this function removes only one factor of β' and the resulting flux value is sensitive to errors in β' to first order. While this is better by a factor of two than the result obtained with Method I in the case of circular symmetry, it still constitutes an argument for taking data in two dimensions even on sources of known symmetry.

With these differences in emphasis, Equation (17a) is still an unbiased estimator of the source flux. In this case, the standard deviation of the estimate would be approximately

$$\sigma_{S_{est}} = \left(\left[\frac{2\, k \sqrt{N}\, \Delta\, T_{rms}\ \beta'^2}{A_e\ \theta_1 \theta_2\ \Sigma\ g_{est}\ (x_{1i})} \right]^2 + \left[S \left(\frac{\sigma_\beta}{\beta'} \right) \right]^2 \right)^{1/2}$$

IV. SOURCE SIZE MEASUREMENTS

One feature of a source which is of interest is its size. The measurement of source size is considerably more difficult than that of source strength for the following reasons:

1. The size of a source can be measured well only by antennas whose beam dimensions are comparable to the source's angular dimensions. For strength measurements, any antenna can be used so long as confusion is not present.
2. The estimate of source size is strongly influenced by random pointing errors. This is all the more crucial if the source is smaller than the beam.
3. Even when one has a good knowledge of pointing accuracy, the estimated source size is strongly affected by errors in judging the response widths. As above, this is especially true if the source is near the limit of the antenna's resolving ability.

It is not our intention here to propose an elaborate method for estimating the source width from a given response. We shall be content rather to review the basis for the common method in use.

The parameters describing source size, within our assumed Gaussian model, are w_1 and w_2, the semi-major and semi-minor axes of the source half-power ellipse. These parameters are embedded, along with five other parameters, in the exponent of Equation (10), the response. The half-power ellipse of the response is described by the equation

$$\underline{x}^t \, (\underline{A} + \underline{B} + \underline{C})^{-1} \, \underline{x} = \ell n \, 2$$

The semi-major and semi-minor axes of this ellipse (β_1 and β_2) are related to the characteristic numbers of the matrix $\frac{1}{\ell n^2} \, (A + B + C)^{-1}$. Conventional techniques yield:

$$\beta_{\substack{1\\2}}^2 = \tfrac{1}{2} \Big\{ w_1^2 + \theta_1^2 + w_2^2 + \theta_2^2 + 4 \, \ell n \, 2 \, \sigma^2 \pm$$

$$\left[\left(w_1^2 + \theta_1^2 - w_2^2 - \theta_2^2\right)^2 - 2\left(w_1^2 - w_2^2\right)\left(\theta_1^2 - \theta_2^2\right)\left[1 - \cos 2\,(\varphi_1 - \varphi_2)\right]\right]^{1/2}\Big\}$$

The well-know relations,

$$\beta_1^2 = w_1^2 + \theta_1^2 + 2 \, \ell n \, 2 \, \sigma^2 \text{ and}$$

$$\beta_2^2 = w_2^2 + \theta_2^2 + 2 \, \ell n \, 2 \, \sigma^2 \qquad (18)$$

follow if any of the following are true:

a. Source symmetric ($w_1^2 = w_2^2$)

b. Beam symmetric ($\theta_1^2 = \theta_2^2$)

c. $\varphi_1 - \varphi_2 = 0$ (If $\varphi_1 - \varphi_2 = \pm \frac{\pi}{2}$, then θ_1 and θ_2 change roles.

In words, if the axes of the source and beam ellipses are aligned, the parallel semi-axes add to the Pythagorean manner of Equations (18)).

Equations (18) can be used to estimate the source widths since θ_1, θ_2 and σ are known and β_1 and β_2 can be estimated from the response. It is interesting to note that, so far as resolving ability is concerned, the pointing errors effectively broaden the beam in the manner

$$\theta \rightarrow \sqrt{\theta^2 + \ln 2\, \sigma^2}$$

For example, isotropic pointing errors of $1'.0$ rms, broaden a $4'.0$ (half-power width) to an effective $4'.33$. It is seen that appreciable pointing errors do not seriously impair the resolving power of an antenna, although, of course, the correction becomes significant when pointing errors are comparable to source size. It should be recalled that we are treating pointing errors statistically and that one must be averaging a sufficient number of scans to allow much credence to these results.

V. THE EFFECTS OF FILTERING

Some sort of filter must be used prior to the recording of the data. In digital data systems, a "perfect" integrator is most commonly used. Rarely is significant distortion thereby introduced, since in a digital system one is normally recording and processing data automatically and thus is not directly concerned with the data rate. Not so in analog systems. Here one is normally using a simple low pass filter and, since data are to some extent hand-processed, one tends to tolerate some distortion in the interest of reducing the data rate. The resulting distortion does not impair the accuracy of the measurement so long as proper compensation is introduced in the data analysis. It is our intention in this section to discuss the distortion effects of a low pass filter and to derive the proper corrections to remove their effects for the Gaussian model. In the following, we treat the simple case where half-peak ellipse of the antenna response is aligned with the coordinant axes. In that case, the response is represented by the Gaussian function

$$T_{avg}(x_1, x_2) = \frac{S A_e \theta_1 \theta_2}{2 k \beta_1 \beta_2} \exp\left\{-\ln 2 \left(\frac{x_1^2}{\beta_1^2} \times \frac{x_2^2}{\beta_2^2}\right)\right\} \qquad (19)$$

where β_1 and β_2 are defined in Equation (18).

A. Distortion Effects

In taking the data one allows the source to pass through the antenna pattern at a constant rate, say v. The units of v could be degrees/sec. of time, or whatever else is convenient. The input signal to the filter is thus the following function of time:

$$T_{in}(t, x_2) = T_{avg}(vt, x_2)$$

If the impulse response of the filter is h(t), the output signal is given by the convolution of h(t) and $T_{in}(t, x_2)$:

$$T_{out}(t, x_2) = h(t) \otimes T_{in}(t, x_2)$$

$$= \int_{-\infty}^{\infty} h(t - \xi)\, T_{in}(\xi, x_2)\, d\xi$$

For a simple low pass filter

$$h(t) = \frac{1}{\tau} e^{-t/\tau} \qquad t > 0$$

$$= 0 \qquad t < 0$$

where τ is the time constant of the filter.* Note that

$$\int_{-\infty}^{\infty} h(t)\, dt = 1.$$

The output signal can thus be written as

$$T_{out}(t, x_2) = \frac{S A_e \theta_1 \theta_2}{2 k \beta_1 \beta_2 \tau} \exp\left[-\ln 2 \frac{x_2^2}{\beta_2^2} - \frac{t}{\tau}\right] \int_{-\infty}^{t} \exp\left[-\ln 2 \frac{v^2}{\beta_1^2} \xi^2 + \frac{\xi}{\tau}\right] d\xi$$

$$= \frac{\sqrt{\pi}\, S A_e \theta_1 \theta_2}{4k \beta_1 \beta_2 \tau} \exp\left[-\ln 2 \frac{x_2^2}{\beta_2^2} - \frac{t}{\tau} + \frac{1}{4} \frac{\beta_1^2}{\ln 2\, v^2 \tau^2}\right] \times$$

$$\operatorname{erfc}\left[\frac{\beta_1}{2\sqrt{\ln 2}\, v \tau} - \frac{\sqrt{\ln 2}\, v t}{\beta_1}\right]$$

*This time constant is that of elementary circuit theory and is one-half the time constant in the radiometric equation.

where $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_e^{\infty} e^{-\zeta^2} \, d\zeta$, the error function complement. We can simplify the form by setting

$$x_1 = vt,$$

$$\delta = \frac{\beta_1}{2\sqrt{\ln 2}\, v\, \tau}, \text{ and}$$

$$\mu = \sqrt{\ln 2}\, \frac{x_1}{\beta_1}$$

The significance of δ is as follows: if the scan rate of the antenna is such that it takes $\Delta t_{2\beta}$ for the antenna to scan the input half-power width of the response, then

$$\delta = \frac{1}{4\sqrt{\ln 2}} \left(\frac{\Delta t_{2\beta}}{\tau}\right) \approx .3 \left(\frac{\Delta t_{2\beta}}{\tau}\right)$$

It is thus proportional to the number of filter time constants in the half-power time of the response. The parameter μ is merely the r.a. coordinant normalized to one-half the $1/e$ width of the source. In these parameters, the response would be:

$$T_{out}(x_1, x_2) = \frac{S\, A_e\, \theta_1 \theta_2}{2k\, \beta_1 \beta_2} \exp\left\{-\ln 2 \frac{x_2^2}{\beta_2^2}\right\} \left[\sqrt{\pi}\, \delta \exp(\delta^2 - 2\delta\mu) \,\mathrm{erfc}\,(\delta - \mu)\right] \quad (20a)$$

$$= \frac{S\, A_e\, \theta_1 \theta_2}{2k\, \beta_1 \beta_2} \exp\left\{-\ln 2 \left(\frac{x_1^2}{\beta_1^2} + \frac{x_2^2}{\beta_2^2}\right)\right\} \left[\sqrt{\pi}\, \delta \exp(\delta - \mu)^2 \,\mathrm{erfc}\,(\delta - \mu)\right]$$

(20b)

In Equation (20b) the bracketed term indicates the distortion introduced by the filter. In the limit $\delta \to \infty$ ($\tau \to 0$), this term approaches unity.

It is possible to generate some universal curves which display the distorting effect of the filter. Figure 3 is a plot of the bracket in Equation (20a) plotted with several values of δ. One notes that as δ is decreased the peak of the response is delayed and lowered, the response loses its Gaussian shape, and the half-peak width of the response is broadened. These effects must be removed in the estimation of the source parameters.

B. The Estimation of Source Width

The source widths are related to the input response widths by Equations (18). The problem of estimating the source widths thus reduces to that of estimating the input response widths from the output of the filter. Equation (20a) indicates that the declination width of the response is unaffected by the filter. Careful calculation shows that the right ascension width of the output response must be reduced by an amount which depends upon parameter (δ). The proper correction is given in Figure 4, Curve 2. Although the correction may appear small, it can be quite significant if the source width is smaller than the antenna beamwidth.

C. The Estimate of Source Strength

The manner in which the distortion effects of the filter affects the strength calculation depends upon the method one is using in this calculation. We shall discuss the two methods presented in Section III.

1. Method I. Here the safest procedure would be to make a direct compensation in the peak temperature and the right-ascension response width. The proper factors are given in Figure 4, Curves 1 and 2, respectively. Curve 3 shows the magnitude of the error which would be made in the worst case (circular symmetry) if no corrections were applied.

2. Method II. It can be shown that the output data, given by Equation (20a) possesses the same integral property as the input data (Equation (11)), independent of the filter time constant. Therefore, the most straightforward procedure would seem to be to replace the estimated Gaussian function $g_{est}(\underline{x}_i)$, by the distorted function $d_{est}(x_i)$, where

$$d_{est}(\underline{x}_i) = \exp\left\{- \ln 2 \frac{x_2^2}{\beta_2^2}\right\}\left[\sqrt{\pi}\,\delta \exp(\delta^2 - 2\delta\mu)\,\mathrm{erfc}\,(\delta - \mu)\right] \quad \text{(cf. Equation 20a)}$$

Of course, one would first have to correct the right ascension response width in order to estimate the parameter δ. Furthermore, one would first have to adjust the right ascension scale to make the peak of the output data correspond to the peak of d_{est}. With these adjustments, the estimator of the source would be (cf. Equation (17a))

$$S_{est} = \frac{2\,\beta_1'\beta_2'\,k\,\sum_N T_{out}(\underline{x}_i)}{A_e\,\theta_1\theta_2\,\sum_N d_{est}(\underline{x}_i)}$$

Clearly there are some difficulties with this method since $d_{est}(\underline{x}_i)$ is not easy to calculate. This is not too serious, however, since one can resort to machine computation or else make use of universal filter responses such as we have plotted in Figure 3.

VI. EXTENSION TO NON-GAUSSIAN SOURCES

A. Introduction

Throughout this report, the Gaussian form has been used to describe the source distribution in radio astronomy source measurements. The implications of this assumption have been investigated with care. Conclusions have been drawn toward improved methods of estimating source strength from a set of data.

The use of the Gaussian distribution was defended at the beginning of this report. There are a number of cases where this assumption would be inappropriate to use. The most obvious, and most common, case is where the source is know *a priori* to be non-gaussian in form, e.g., a planetary disc of known size. Nevertheless, the basic approach which we have developed can be applied to the more general problem. In this concluding section, we discuss briefly the extension of the technique to the general source measurement. We go on to relate this approach to a specific type of measurement, the measurement of the disc temperature of a planet.

B. Basic Principles

The principles upon which we have developed our analysis of the Gaussian source problem are as follows:

1. The source strength can be determined by integrating the antenna response in the data plane. This is shown for the Gaussian source in Equation (11b), page 12, but is true for a non-Gaussian source, as will be shown presently.

2. In integrating the response, one reaches a point where the noise begins to degrade the accuracy. This is surely true for any distribution.

3. One can retain an unbiased estimation of the source strength by integrating the response over a limited portion of the data plane and correcting for the omitted signal. The correction is based upon an "estimated" response, which previously was related to the Gaussian shape of the response. In the case where the source distribution is known beforehand, an "expected" response can still be formulated on the basis of this knowledge.

4. The optimum area for integrating the response for a Gaussian source corresponds to that area where the expected response is over 28% of its maximum value. However, since the S/N maximum is broad, we have advised integrating over a larger area to reduce errors introduced in formulating the expected response. Thus, we advised integrating over that area where the expected response was 5-10% of the peak response.

These criteria have application to the non-Gaussian problem. Clearly, some optimum area of integration exists. The Gaussian analysis leads us to believe that this is not a critical matter and we would still favor integrating the response over a large area.

C. A General Formulation

We begin first by proving what has been stated several times: that the integral of the antenna temperature in the data plane depends solely on the source strength and not upon the source size or shape. We being with Equation (2), page 3, rewritten in abbreviated form:

$$T_a(\underline{x}') = \frac{1}{2k} \int I(\underline{x})\, A(\underline{x}' - \underline{x})\; dx_1 dx_2$$

Changing integration variables $\underline{x} \to \underline{x}''$ where $\underline{x}'' = x' - x$, we obtain the form

$$T_a(\underline{x}') = \frac{1}{2k} \int I\,(x' - x'')\, A(x'')\; dx_1{}'' dx_2{}''$$

If we now integrate the equation in the $\underline{x}'$ variable and evoke the definition of source strength, Equation (7b), we find

$$\int T_a(\underline{x}')\; dx_1{}' dx_2{}' = \frac{S}{2k} \int A(\underline{x}'')\; dx'' dx''$$

The remaining integral on the right hand side of the equation is known from the antenna properties, so the equation can be solved for the source strength, S, thus confirming our assertion.

In order to place the equations in a convenient form for discussing data handling techniques, let us introduce the following definitions

$$I(\underline{x}) = Sf_s(\overline{x})$$

and $$A(\underline{x}) = A_e f_p(\overline{x}),$$

where S and A_e have their usual meanings and $f_s(\overline{x})$ and $f_p(\overline{x})$ are functions describing the spacial characteristics of the source and antenna pattern, respectively. Clearly, $f_s(\overline{x})$ obeys the following constraint,

$$\int f_s(\underline{x})\, dx_1 dx_2 = 1,$$

and $f_p(\overline{x})$ is the power pattern of the antenna. In this form the antenna temperature is

$$T_a(x') = \frac{SA_e}{2k}\; f_s(\underline{x}') \otimes f(\underline{x}')$$

and the data is

$$T_d(x') = \frac{SA_e}{2k}\; f_r(x') + T_n(x')$$

where $T_n(x')$ is the noise, and $f_r(x')$ is the convolution of the source distribution and pattern and is the response in the data plane. In the case of actual data, $f_r(\overline{x}')$ might be unknown as in the case of an assumed Gaussian source of unknown size, or it might be known as in the case of planetary measurement. In either event, we presumably are in a position to estimate it in some sense.

Our estimate of the source strength is based upon integration (or summation) of the data over a suitable region of the data plane, as determined by signal-to-noise considerations. Thus, our estimate would be

$$S_{est} = \frac{2k \, \Sigma \, T_d(\underline{x}')}{A_e \, \Sigma \, f_{re}(\underline{x}')} \tag{21}$$

where $f_{re}(\underline{x}')$ is the estimated response function.

Our estimate is as usual regarded as a random variable. Provided our expected response function is not biased, the estimated flux is an unbiased estimator of the actual flux. The error analysis can be handled by the same approach we used in the detailed analysis of the Gaussian problem earlier in this report. To summarize, Equation (21) is a generalized form of Equation (17a), page 23. It provides the basis for estimating the total flux of a source. The summation is performed over the source region where the response is about 10% of its peak value, assuming some sort of bell shaped response. The omitted signal is corrected for by summing the expected or estimated response over the same region. The expected response is determined by the convolution of the antenna power pattern with the *a priori* expected source distribution. We will now apply this formalism to the measurement of a planet of known size.

D. A Planetary Measurement

In order to apply the results of the previous section to a planetary measurement, we need but to derive the functions $f_{re}(x')$ based upon the planetary size and the antenna properties, suitably normalized. We have limited ourselves to modeling the planet as a round disc of radius r_o and a Gaussian beam. In cases of current interest at this Laboratory, the planet solid angle is smaller than the antenna beam, so a suitable form for the expected response is a modified Gaussian response, i.e.,

$$f_{re}(\underline{x}') = C(r_o, x')\, g(x'),$$

where $g(x')$ is the Gaussian beam function ($g(o) = 1$) and is the point source response, and $C(r_o, x')$ is the correction due to finite disc size. For the planet distribution, we have

$$f_s(\underline{x}') = \frac{1}{\pi r_o^2}, \quad |x'| < r_o$$

and for the beam the Gaussian form,

$$g(\underline{x}') = \exp\left[-\ln 2\ \underline{x}'^t \begin{bmatrix} \theta_1^2 & 0 \\ 0 & \theta_2^2 \end{bmatrix}^{-1} \underline{x}'\right]$$

was used. In this form the correction is found to depend upon the Gaussian function integrated over a circular area displaced from the origin.

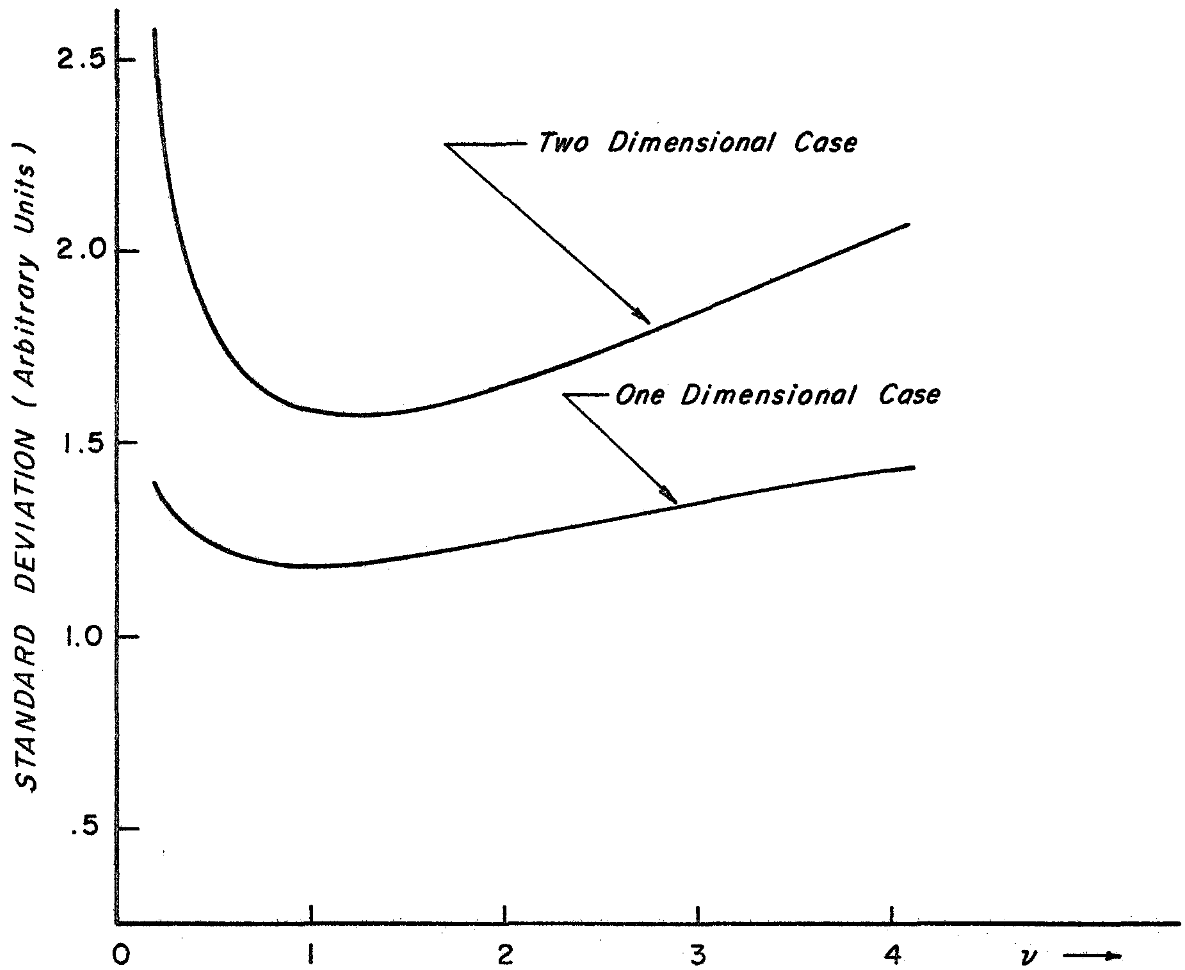
$$C(r_o, x') = \frac{1}{\pi r_o^2} \int_{x_1'-r_o}^{x_1'+r_o} e^{-\ln 2\left(\frac{x_1^2 - x_1'^2}{\theta_1^2}\right)} dx_1 \times$$

$$\int_{x_2' - \sqrt{r_o^2 - (x_1 - x_2')^2}}^{x_2' + \sqrt{r_o^2 + (x_1 - x_2')^2}} e^{-\ln 2\left(\frac{x_2 - x_2'^2}{\theta_2^2}\right)} dx_2$$

This factor has been tabulated in Figures 4 and 5 for $\theta_1 = \theta_2 = \theta$ (symmetric beam) with normalized variables

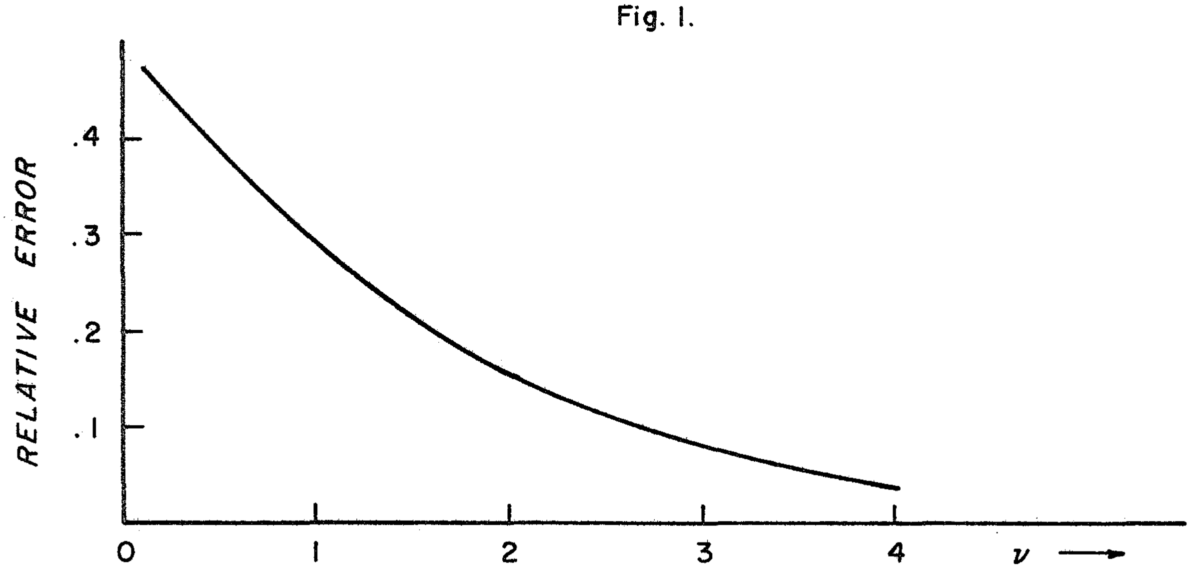
$$r_n = \frac{r_o}{\theta} \text{ and } d_n = \frac{d}{\theta} \text{ where } d = \sqrt{x_1'^2 + x_2'^2}\,.$$

In order to apply the tabulated results, one must take the planetary size from the Ephemeris and normalize it to the beamwidth. The expected response is then formulated from Figure 6. It might be noted that corrections for disc size are fairly small in cases of practical interest, of the order of ±10%. From the expected response, the source flux would be estimated from Equation (21). The disc temperature is then easily derived from the total flux and the known size.



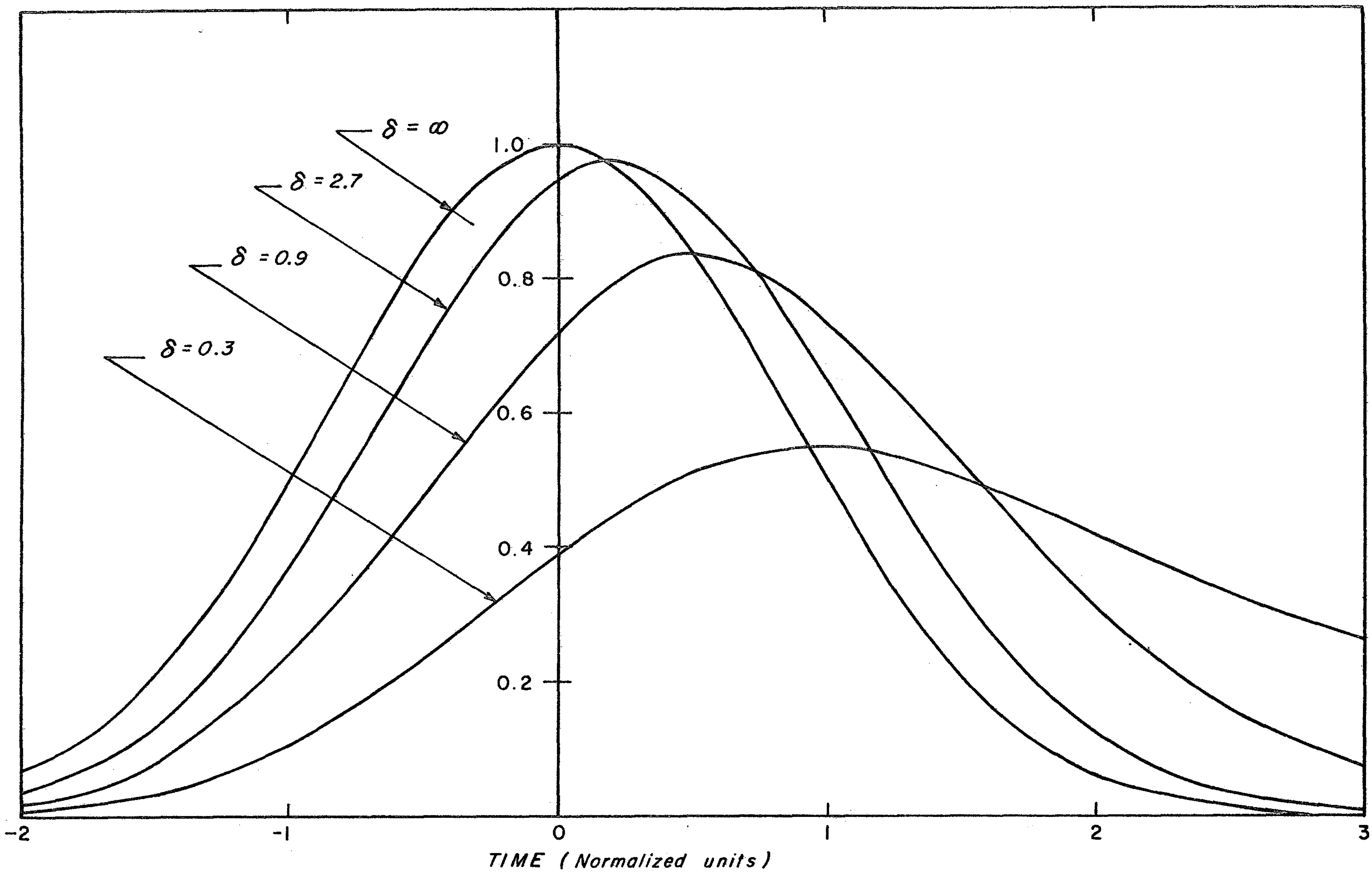
STANDARD DEVIATION OF FLUX ESTIMATE
DUE TO RECEIVER NOISE

Fig. 1.



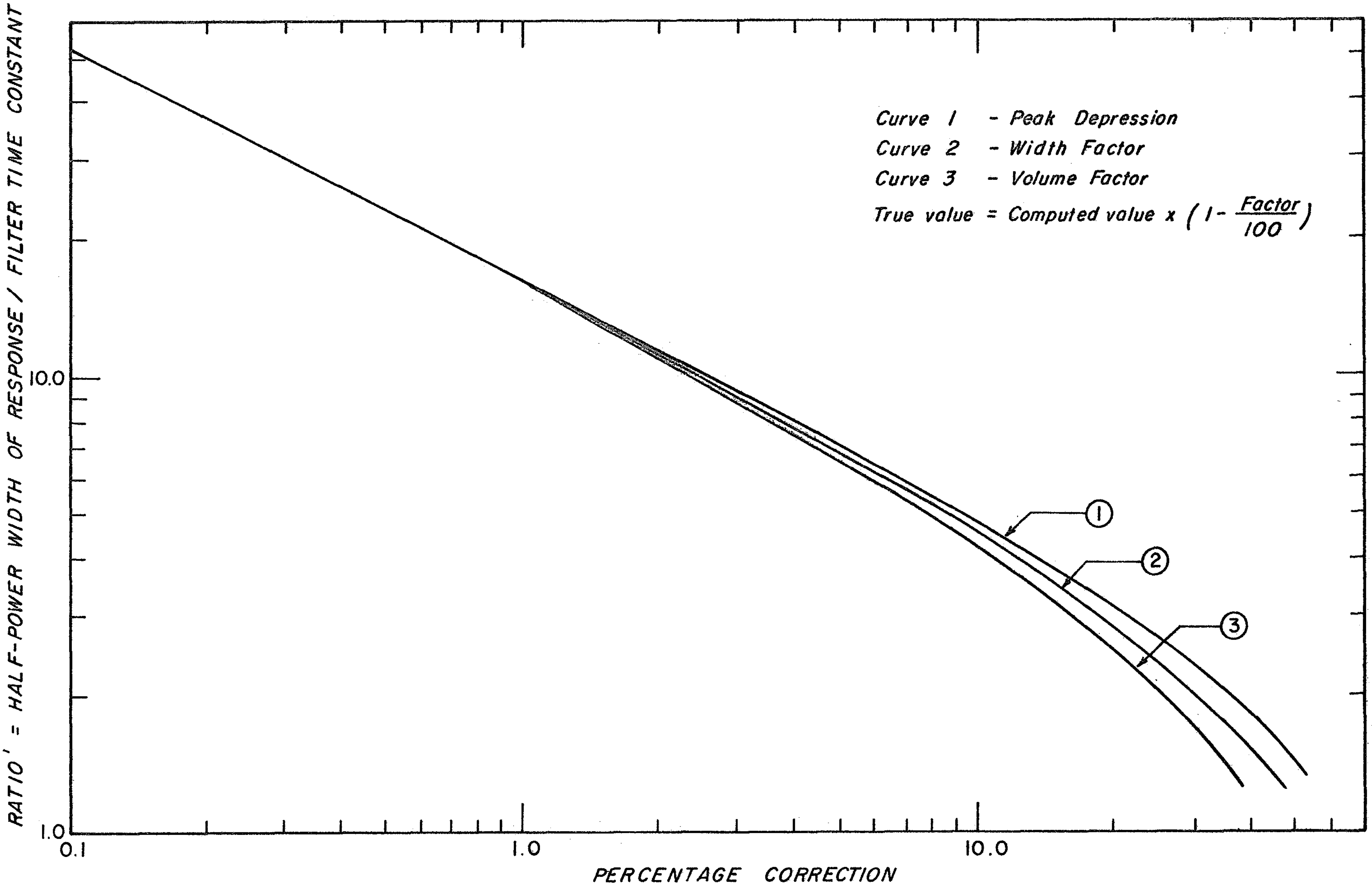
SENSITIVITY TO WIDTH ESTIMATION ERRORS:
IMPROVEMENT DUE TO INTEGRATION

Fig. 2.



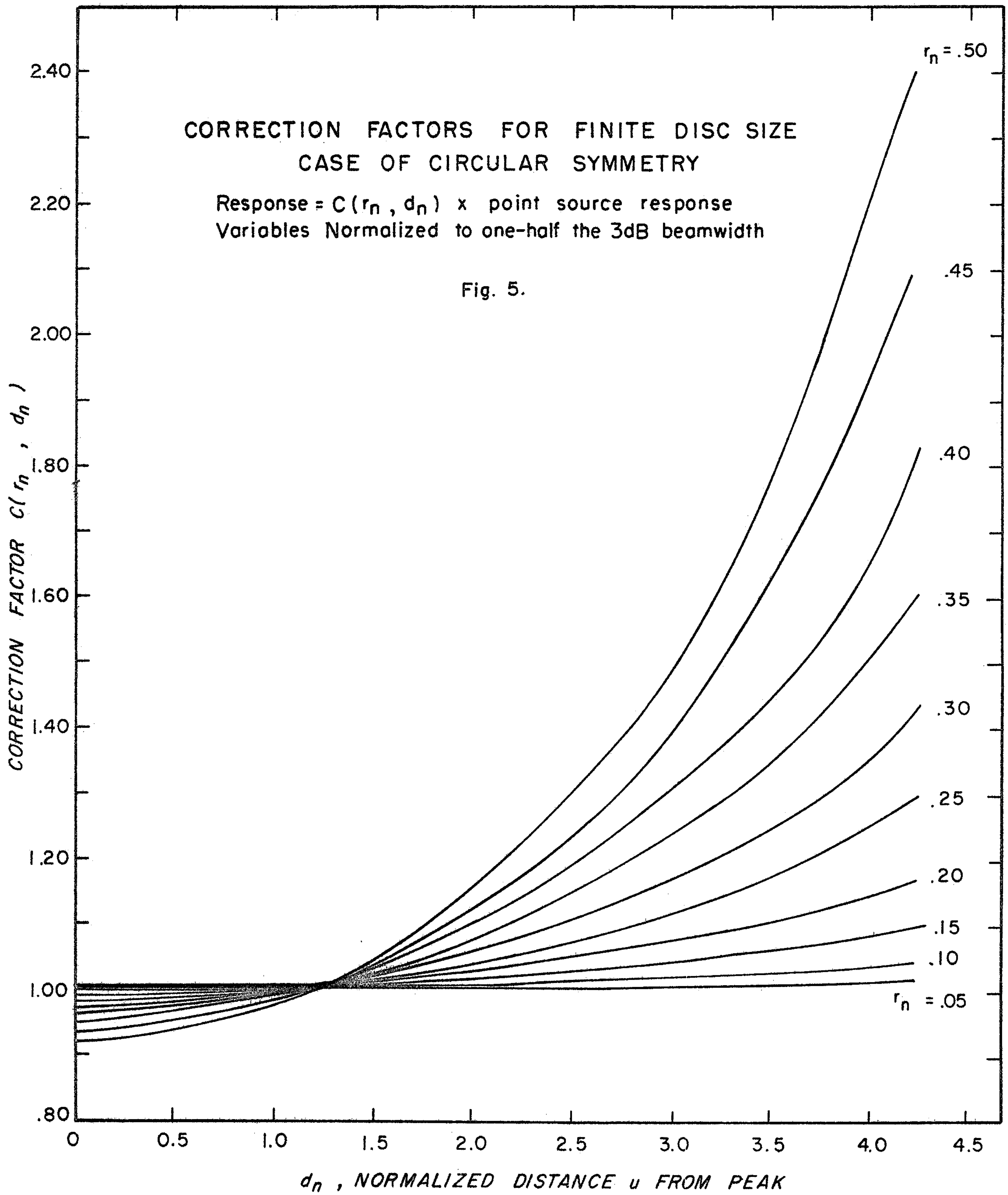
RESPONSE OF A LOW-PASS FILTER TO GAUSSIAN INPUT SIGNAL

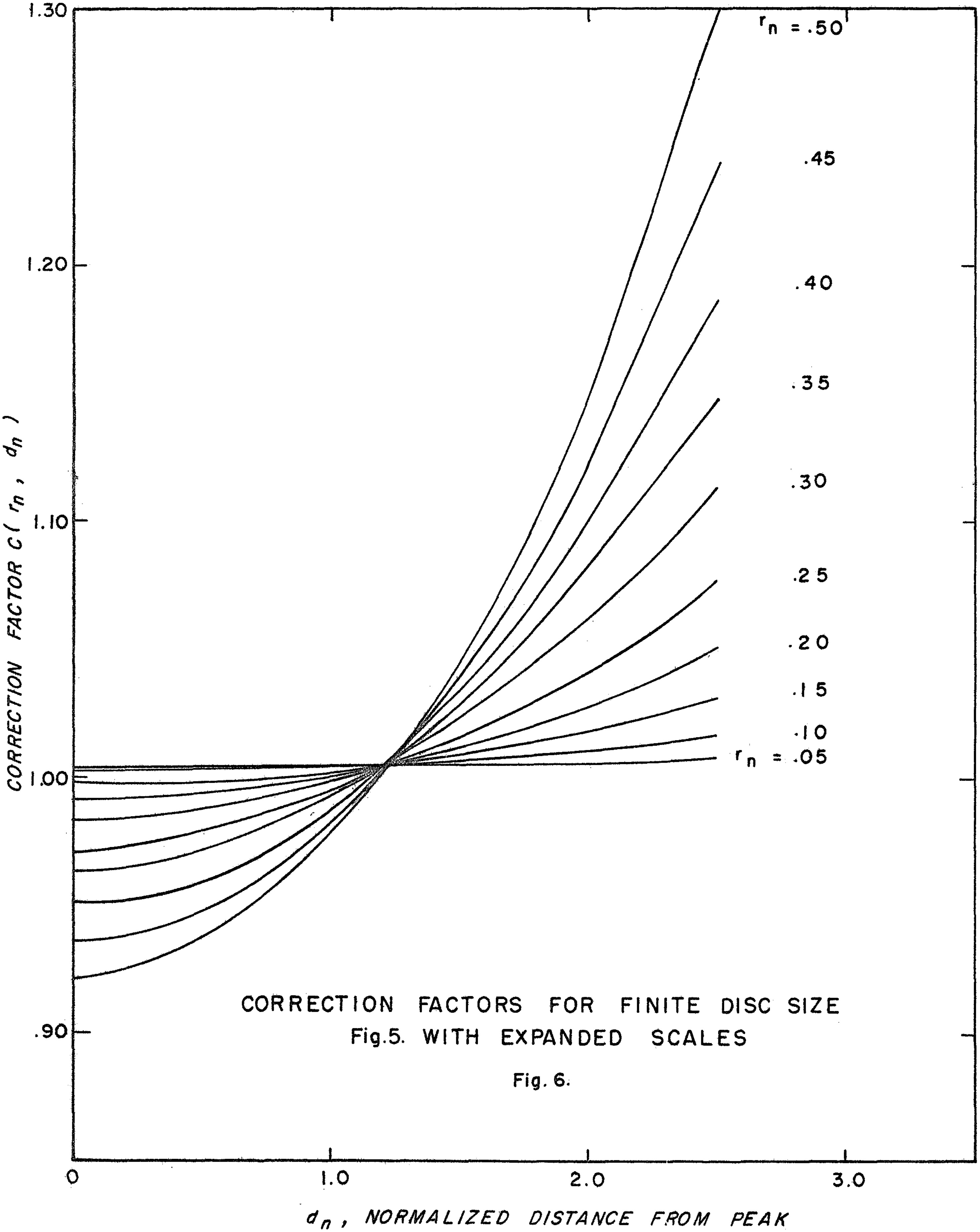
Fig. 3.



LOW PASS-FILTER CORRECTIONS FOR GAUSSIAN INPUT

Fig. 4.


CORRECTION FACTORS FOR FINITE DISC SIZE
CASE OF CIRCULAR SYMMETRY
Response = C (r_n , d_n) x point source response
Variables Normalized to one-half the 3dB beamwidth
Fig. 5.
CORRECTION FACTOR C(r_n , d_n)
d_n , NORMALIZED DISTANCE u FROM PEAK
r_n = .50
.45
.40
.35
.30
.25
.20
.15
.10
r_n = .05
2.40
2.20
2.00
1.80
1.60
1.40
1.20
1.00
.80
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5



CORRECTION FACTORS FOR FINITE DISC SIZE
Fig.5. WITH EXPANDED SCALES

Fig. 6.