CUMULATIVE PROGRESS REPORT

FOR

NGR 44-012-008

GUIDANCE METHODS FOR LOW-THRUST SPACE VEHICLES

Covering the Period from

Jan. 1, 1969 through Jan. 31, 1970

Submitted by

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21

PROGRESS REPORT

NGR 44-012-008

Contents

Page

Cumulative Progress Report 1

Interim Progress Report for Period Feb. 1, 1968 - July 31, 1968 . . . 8

Interim Progress Report for Period Aug. 1, 1968 - Jan. 31, 1969 . . . 10

Interim Progress Report for Period Feb. 1, 1969 - July 31, 1969 . . . 12

Interim Progress Report for Period Aug. 1, 1969 - Jan. 15, 1970 . . . 15

Proposed Study for Period Feb. 1, 1970 - July 31, 1970 20

Degrees Awarded During Past 5 Years 21

Students Supported During Present Year 23

Appendices 24

(Summaries of major studies reported under this grant)

References 43

NGR 44-012-008

CUMULATIVE PROGRESS REPORT

At the present time, a general study of guidance methods for the low-thrust space vehicle is being conducted by members of the Astrodynamics Research Laboratory at The University of Texas at Austin. The investigation was initiated in February, 1964, under a grant from the National Aeronautics and Space Administration (NsG-551). The study was continued under Grant NGR 44-012-008 and was approved for step funding on July 17, 1969. This study has as its objective the evaluation and comparison of existing guidance methods, and the development, where necessary, of new methods for guiding a low-thrust space vehicle on an interplanetary mission. This is a cumulative report of the results of this study from its inception through December 1969.

In the first year of the study, the major emphasis was directed toward the evaluation of the performance of optimal deterministic control schemes based on the first variation of the claculus of variations. Members of the family of control schemes based on the gradient method of trajectory optimization were compared. Several new developments were made in order to make this family of methods more adaptable to the space guidance problem. A more detailed summary of this work is given in Appendix A and is reported in full in Reference (1). Further aspects of this study are reported in References (2) and (3). In subsequent studies, the performance of guidance techniques based on the second variation has been compared with first order methods. The present comparison is being conducted for a series of optimal low-thrust Earth-Mars transfer missions.

The extension of the study into the area of higher order guidance schemes based on the second variation necessitated several new developments.

First, before a guidance study can be initiated, a second order nominal trajectory is required. Furthermore, the general nature of space missions necessitated the inclusion of effects of inequality constr[illegible]ts in the determination of the second order nominal trajectory. Numerical optimization methods to comply with these requirements were developed. A summary of this work is given in Appendix B and is reported fully in Reference (4). Further aspects of the study are reported in Reference (5) through (8).

The problem of determining optimal low-thrust interplanetary trajectories for use in the guidance studies was found to be of sufficient difficulty to warrant further study. A summary of the techniques developed for this purpose is given in Appendix B. The methods were used to develop a series of optimal low-thrust Earth-Jupiter trajectories. The launch dates for these trajectories cover a three-year period from 1983 through 1985. It was found that the initial guesses of the Lagrange multipliers, which are generally very hard to determine, exhibit a recurrent type of behavior. It is anticipated that this characteristic can be used to facilitate the determination of optimal interplanetary reference trajectories. Further study of this topic is underway. The results of the previous study are summarized in Appendix C and are reported in full in Reference (9). Further aspects of this study are reported in Reference (10) and in Reference (17).

During the early phases of the study of numerical optimization methods, several difficulties associated with the linearization procedure were encountered. It was discovered that serious numerical problems are encountered wherever the state deviations grow sufficiently large. In this situation, the linearization procedure in which higher order terms in the Taylor's series expansion are neglected is not justified. However, in order to enforce linearity restrictions, constraints on the magnitude of the state deviation vector can be handled by methods developed in Reference (4).

During the past year, the programs for the minimum time low-thrust Earth-Jupiter missions have been modified to produce prototype Earth-Jupiter trajectories which minimized a linear combination of the mission time and the fuel consumed. Coasting arcs were allowed on these trajectories. A unique approach to optimization of the coasting arc was discovered. The study in which these results were obtained is outlined in Appendix H and are discussed in detail in References (15) and (18). At present, the applications of this method to guidance along low-thrust trajectories containing coast-arcs is being examined. Low-thrust Earth-Mars trajectories on which coast-arcs occur are being used instead of Earth-Jupiter trajectories in order to economize in carrying out the computational processes.

As an outgrowth of the Earth-Jupiter studies, a nonlinear real-time guidance method based on reoptimization of the nonlinear two-point boundary value problem has been implemented. Two used are being made of the nonlinear guidance results. First, an evaluation of the reoptimization concept as a guidance method is being carried out, and second, the reoptimized guidance solutions, which are optimal, are being used as a reference solution against which the results of the guidance methods based on linearization can be compared.

The comparison of optimal deterministic guidance methods to each other and to the nonlinear guidance scheme for low-thrust interplanetary vehicles is being continued. Criteria by which such guidance methods should be compared have been developed and the comparisons are currently under way. The comparisons are being carried out as follows:

1. An optimal reference (nominal) trajectory for some prototype mission is generated. Then state perturbation vectors at various points along the nominal trajectory are introduced and the trajectory is reoptimized from each point at which a state perturbation vector was introduced. This results in a series of reoptimized trajectories which take into account known state perturbations -- i.e., this is the nonlinear guidance scheme. The reoptimized trajectories are the best which can be obtained considering the given state perturbations. These trajectories form the standards by which all approximate guidance schemes are measured. Both state and control histories are generated along each reoptimized trajectory.

2. Each approximate guidance scheme to be tested is used to generate control and state deviation histories (referenced to the optimal nominal trajectory) starting with each of the chosen state perturbation vectors. State and control deviation histories are added to the optimal nominal trajectory state and control histories and these results are compared to the state and control histories along the corresponding reoptimized trajectory.

3. The degree to which the linearization affects the state deviation computation is checked. This is done by numerically integrating the nonlinear equations of motion for the vehicle starting with a known state perturbation vector and using a control history made up of the optimal nominal control history plus the control deviation history produced by the guidance method under study. This produces a state history (R) which is the actual trajectory which would be flown by the vehicle if the control deviations predicted by the guidance method under study were implemented. Then, state history (R) is compared to (1) the reoptimized state history, and to (2) the state history predicted by the guidance method. The agreement between state history (R) and the state history from the reoptimized trajectory is a measure of the quality of the guidance method under study. The agreement between state history (R) and the state history predicted by the guidance method is a measure of the effects of the linearization (or approximation processes on the state prediction computations.

4. It must be realized that all control deviations cannot be implemented. For example, if very small changes in thrusting angles are called for, it may be impossible to carry them out. Thus, other trajectories can be calculated starting with a given state perturbation along which control deviations are implemented only if they are large enough to be implemented (i.e., $\delta u \quad \delta u_{min.}$). This produces a series of histories (S) which can be compared to all of the previously computed state histories. These final state histories are the paths that one might expect the vehicle to actually fly if the computed control deviations were implemented. Presently, various guidance schemes are being compared as explained above.

In all of the early numerical studies carried out under this grant, there have been occasions when each of the numerical methods fail to converge for unknown reasons. In an effort to better understand the behavior of the guidance procedures, a detailed study of the pathological structure of the guidance and optimization procedures was initiated to determine the conditions under which the methods can fail because of numerical problems, incorrect implementation, or incorrect problem formulation. An inquiry into this problem was made. For the numerical methods based on perturbation type techniques, it was found that the presence of a conjugate point in the optimal reference trajectory can cause the majority of the numerical problems encountered. This work is summarized in Appendix D and is fully reported in Reference (11).

The problem of numerically computing optimal reference trajectories and the associated guidance programs for space vehicles has also been studied from another point of view. Many algorithms are now available for computing these trajectories, each having certain advantages and disadvantages. However, no matter which algorithm is used, an appropriate coordinate system must be chosen to describe the physical problem. The form of the differential equations which govern the motion of the space vehicle depends on this choice of coordinate system. Hence, the choice of a coordinate system can either simplify or complicate the analytic and numerical work involved in a given optimization problem.

A study was made to determine which of two commonly used coordinate systems is most appropriate for the numerical computation of optimal low-thrust trajectories by studying a specific, representative, example problem. The problem considered was that of a low-thrust escape spiral. This problem was chosen because numerical solutions to this problem are among the most difficult of all low-thrust trajectories to calculate. The two coordinate systems investigated were a modified set of the classical orbital elements and plane polar coordinates. It was anticipated that orbital elements might comprise a preferable set of coordinates in which to conduct the computation for this problem because they change so slowly. However, the results of the study indicate that polar coordinates were somewhat better than orbital elements for this problem. A more detailed summary of this work is given in Appendix E and the complete study is reported fully in Reference (12).

In a second study of this topic, the effects of the coordinate system on the numerical integration error for systems dominated by two-body effects were studied. Five different coordinate systems were compared (Rectangular Cartesian, Circular Cylindrical, Spherical Polar, and two forms of Elliptical Cylindrical coordinates). It was found that numerical errors grow more slowly in Elliptical Cylindrical coordinates than in the other coordinates. This study is summarized in Appendix F and is reported fully in References (13) and (16).

Finally, a major effort was directed toward examing the optimal stochastic control problem. Since it is anticipated that any useful design will require that the noise in the low-thrust engine be small, the assumption of small noise has been used in the study. However, departing from the conventional assumption of most contemporary studies, the engine noise was assumed to be correlated in time. It is believed that auto-correlated noise is a more accurate model for the noise in the low-thrust engine than the "white-noise" model commonly assumed. A study of the effects of such noise on the optimal deterministic trajectory was made. The effects of the noise on the trajectory for simulated study of a low-thrust space vehicle on a minimum-time Earth-to-Mars transfer trajectory, indicate the necessity for developing an optimal stochastic control. The control procedure developed in the subsequent study is a non-random function of time and is based on a priori knowledge of the noise process. A stochastic calculus of variations approach was employed to determine an optimal control procedure for the stochastic system. The behavior of the control procedure during a numerical study of a simulated interplanetary mission was determined. The results of the numerical study indicated the necessity for modifying the procedure to obtain a scheme which will correct the control program on the basis of observational information gained during the controlling interval. The modified scheme has been applied to the interplanetary transfer problem for the case where range rate observations were available during the transfer period. The results of these studies are summarized in Appendix G and are reported in Reference (14).

Summary of Progress for February 1, 1968 — July 31, 1968

During the period from February 1, 1968 through July 31, 1968, the following specific problems have been studied:

1. The generation of a series of low-thrust nuclear-electric interplanetary trajectories has been carried out (see Appendix C). This series of trajectories provides the basis for the development of a nonlinear guidance procedure based on the reoptimization of the nonlinear two-point boundary-value trajectory problem. A comparison of this guidance method with linearized methods based on the use of precomputed data has been initiated.
2. The low-thrust reference trajectory generation computer program prepared during a previous effort supported by this grant, is limited by the study of guidance maneuvers in a small region around a minimum time nuclear-electric powered interplanetary trajectory. During the reporting period, the basic low-thrust reference trajectory generation computer program was modified to allow generation of minimum fuel nuclear-electric powered reference trajectories containing coast periods. The modified program can be used to generate nonlinear guidance information for the minimum fuel nuclear-electric trajectories just as the unmodified program produced nonlinear guidance information for the corresponding minimum time trajectories.
3. In order to be useful, a guidance method must be able to correct state errors which can be sensed. If a particular guidance method can correct state errors up to a certain maximum size, and if errors of this maximum size are too

small to be measured by existing navigational equipment, then the particular guidance method is useless. This interaction between the navigation and guidance functions cannot be ignored in the evaluation of low-thrust guidance methods.

Hence, a general deep-space orbit determination computer program has been written to determine the accuracies with which the state of a space-craft can be determined using various combinations of sensors. The information generated by this study is expected to be of direct use in establishing realistic limits on the orbit determination accuracies required for the low-thrust guidance methods.

In its present form, the program can use various combinations of earth-based and on-board sensors to determine the state of the low-thrust vehicle. Statistical estimates of the errors in the states resulting from the use of various sensor combinations are generated. These estimates are expected to provide the needed measure of how large a state error must grow before it can be sensed. Any guidance method which cannot correct errors as large as the smallest sensible error must be modified or else it must be discarded.

4. The study of the optimal control procedures for nonlinear stochastic dynamic systems described in Appendix G has been continued. In particular, an effort is being made to provide a more rigorous foundation for several formal results presented in Reference (14). The theory developed will be used to study the effects of engine noise and random measurement error on the deterministic low-thrust guidance procedures.

Summary of Progress for August 1, 1968 — January 31, 1969

During the period from August 1, 1968 through January 31, 1969, the following specific problems have been studied.

1. The major effort in the period has gone into the development of meaningful criteria for the comparison of guidance methods and the development of software to carry out these comparisons. The nonlinear reoptimizing guidance scheme produces a trajectory which is used as the proof solution for all other guidance schemes. Each guidance scheme to be tested is checked against the proof solution for precision, effect of linearization, state prediction accuracy, and whether the predicted trajectory can actually be flown. Various guidance schemes, including extremal field control, lambda matrix control, two versions of time-to-go guidance, and a guidance method based on the optimal sweep are being programmed at present. Preliminary data indicates that the criteria being developed for testing these guidance methods will be quite useful, although the criteria are as yet unrefined.

2. Research during this period was devoted primarily to the development and modification of computer programs for determining the accuracy in orbit determination required during a planetary encounter. A prototype program, STEP I, was written which allows earth based tracking of a planetary probe as well as allowing the use of on-board sensors. It is expected that the capabilities of this program will be extended to handle more than one earth based tracking station. Also, it is planned to modify the program so that it is able to handle vehicles in the Earth-Moon space as well as those on deep-space trajectories.

3. In an effort to provide a more rigorous foundation for several formal results reported in a study of the optimal control techniques for nonlinear stochastic dynamics systems (Reference 14), necessary conditions for a stochastic extremum have been derived. First, a stochastic calculus of variations was developed, based upon stochastic fundamental lemmas analogous to the fundamental lemmas used in the development of the deterministic calculus of variations. The necessary conditions derived are expressed as stochastic differential equations identical in form to the deterministic Euler-Lagrange equations.

Summary of Progress for February 1, 1969 — July 31, 1969

During the period from February 1, 1969 to July 31, 1969, the following specific problems have been studied:

1. The conversion of programs for the minimum time low-thrust Earth-Jupiter missions to a form which minimizes a linear combination of fuel and time has been completed. Prototype trajectories have been produced and production data runs have been completed. Coasting arcs are allowed on the trajectories produced. A unique approach to optimization of the coasting arc has been developed. The study in which these results were obtained has been documented in Ref. 9. The applications of this method to guidance along low-thrust trajectories containing coast-arcs is being examined.

   As an outgrowth of the Earth-Jupiter studies, a nonlinear real-time guidance method based on reoptimization of the nonlinear two-point boundary value problem has been developed. The programs to implement the nonlinear guidance algorithms have been written and checked out. Two uses are being made of the nonlinear guidance results. First, an evaluation of the reoptimization concept as a guidance method is being carried out, and second, the reoptimized guidance solutions are being used as reference solutions against which the results of the guidance methods based on linearization can be compared.

2. Research during this period was devoted primarily to the development of computer programs for the simulation of the orbit determination process during planetary encounter. An

existing program, STEP I, was modified to handle up to ten earth-based tracking stations (STEP I would handle only one). This improved version of STEP I, designated STEP II, provided a capability for simulating the OD process for satellites of the earth and moon as well as the planets. Another OD program which had been abandoned during the effort to complete STEP I was completed and checked out. This program, STEP III, solves for the gravitational parameter of the planet and the astronomical unit (STEP I and STEP II solve for a bias in the planet's position) in addition to the state of the vehicle. Further improvement of STEP II resulted in the development of STEP IV. This program includes a subroutine which computes the covariance matrix of the state estimate projected onto the $B \cdot R - B \cdot T$ targeting plane. The eigenvalues and eigenvectors of the resulting covariance matrix are used to compute a probability ellipse associated with the current estimate of the state. This capability provides a convenient geometric display of the estimated OD errors.

3. The necessary conditions for the stochastic maximum are given as stochastic differential equations which cannot be solved in an ordinary way. These necessary conditions have been applied to a regulator problem in which the state equations and measurement equations are disturbed by additional white Gaussian noise processes. The resulting stochastic differential equation is solved by a linear transformation and finally an optimal stochastic control policy has been obtained as a linear function of the best estimates (which is a conditional expectation of the state variable given all information up to the present time). This

result demonstrates the well known separability of control and estimation in the linear case. In general, the separability does not hold for the nonlinear case. The optimal control for a nonlinear system is anticipated to be a function of the conditional mean, variance and all higher order moments.

Summary of Progress for August 1, 1969 — January 15, 1970

During the period from August 1, 1969 to January 15, 1970, the following specific problems have been studied:

1. The comparison of optimal deterministic guidance methods to each other and to the nonlinear guidance scheme for low-thrust interplanetary vehicles has been continued. Criteria by which such guidance methods should be compared are being developed and the comparisons are currently being carried out. The comparisons are being carried out as follows:

   1) An optimal reference (nominal) trajectory for a prototype mission is generated. Then state perturbation vectors at specific points along the nominal trajectory are introduced and the trajectory is reoptimized from each point at which a state perturbation vector was introduced. This results in a series of reoptimized trajectories which take into account known state pertubations, i.e., this is the nonlinear guidance scheme. The reoptimized trajectories are the best which can be obtained from the perturbed states. These trajectories form the standards by which all approximate guidance schemes are measured. Both state histories and control histories are generated along each reoptimized trajectory.

   2) Each guidance scheme to be tested is used to generate control deviation histories and state deviation histories (referenced to the optimal nominal trajectory) starting with each of the chosen state perturbation vectors.

State deviation histories and control deviation histories are added to the optimal nominal trajectory state and control histories. Then, these results are compared to the nominal state and control histories and also to the state and control histories along the corresponding reoptimized trajectory.

3) The degree to which the linearizati n affects the state deviation computation is then checked. This is done by numerically integrating the nonlinear equations of motion for the vehicle starting with a known state pertubation vector and using a control history made up of the optimal nominal control history plus the control deviation history produced by the guidance method under study. This produces a state history (R) which is the actual trajectory which would be flown by the vehicle if the control deviations predicted by the guidance method under study were implemented. Then, state history (R) is compared to (1) the reoptimized state history, and to (2) the state history predicted by the guidance method. The agreement between state history (R) and the state history from the reoptimized trajectory is a measure of the quality of the guidance method under study. The agreement between state history (R) and the state history predicted by the guidance method is a measure of the effects of the linearization (or approximation) processes on the state prediction computations.

4) It must be realized that all control deviations cannot be implemented. For example, if very small changes

in thrusting angles are called for, it may be impossible to carry them out. Thus, other trajectories can be calculated starting with a given state perturbation along which control deviations are implemented only if they are large enough to be implemented (i.e., $\delta u \geq \delta u_{min.}$). This produces a series of histories (S) which can be compared to all of the previously computed state histories. These final state histories are the paths that one might expect the vehicle to actually fly if the computed control deviations were implemented.

Presently, various guidance schemes are being compared as explained above. A complex computer program has been developed in order to efficiently handle, store, and compare the state and control histories. The generated data are stored so that comparisons can be carried out with trajectories which are generated later.

2. Research during this period was devoted primarily to the analysis of orbit determination (OD) errors during planetary encounter and to the development of computer programs to evaluate guidance requirements for the Grand Tour mission. Numerous simulations of the OD process during Jupiter encounter of a 1977 Grand Tour were made and the results plotted on microfilm. From the results obtained, several general conclusions can be drawn.

   1) The OD process converges very slowly until about six days prior to encounter and then much more rapidly.

2) The process is greatly affected by the nominal trajectory up-date schedule. For large initial errors the sequential estimator sometimes diverged when the nominal trajectory was not up-dated. Results to date indicate that for best results the nominal should be up-dated (rectified) after each observation.
3) The error in the planetary position bias decreases very slowly in all cases run, but does not seem to affect the state estimation very much.
4) The onboard angle measurements are a valuable and probably necessary information source. In fact the encounter phase can probably be navigated with onboard observations as the only information source.

Another in the series of STEP programs was developed from STEP IV. The new program (STEP V) includes a simple guidance correction capability, a variable step integrator, and other improvements in program structure and output format. Two guidance options are currently available with STEP V. Guidance mode 1 corrects the targeting vector $[\,(B \cdot R)\,,\,(B \cdot T)\,,\,Te]^T$ (where Te is time of perifocal passage) and mode 2 corrects the targeting vector $\overline{V}_\infty$. (The vector $\overline{V}_\infty$ is defined as the vector with a magnitude equal to the hyperbolic excess velocity and in the direction of the departure asymptote.)

An existing IBM 360 program which computes an approximate (patched conic) Grand Tour trajectory was converted and modified to compute a heliocentric and planetocentric ephemeris for the Grand Tour. The program, designated FLYBY, gives us the capability to generate heliocentric and flyby conics which approximate the actual Grand Tour trajectory. These trajectories are used in the STEP programs to simulate the OD process.

A summary of the conclusions obtained in the completed studies is being prepared for publication.

3. Effort in studying the non-linear stochastic control problem was directed at formulating a non-linear state estimation algorithm. Applying Martingale theory and an orthogonal projection theorem, a sequential non-linear estimation technique which approximates the conditional expectation of the state variable given all information up to the present time has been obtained. At the present time, an effort is being made to demonstrate that the sequential estimation algorithm can be reduced to a non-linear continuous estimation technique by a limiting process.

Proposed Study for Period February 1, 1970 through July 31, 1970

During the period from January 1, 1970 through June 30, 1970, it is proposed that the following studies be conducted:

1. It is proposed that the study of orbit determination accuracies for deep-space missions be continued. In particular, the determination of minimum detectable error size will be emphasized in order to correlate the orbit determination process with the guidance process.
2. It is proposed that the comparison of the nonlinear guidance scheme with the various approximate guidance schemes be continued employing the recently developed comparison criteria. The various sub-optimal approximate guidance schemes will be compared for accuracy, computation times, ease of implementation, etc.
3. It is proposed that meaningful prototype missions continue to be developed in order to provide additional mission models for evaluation of promising guidance procedures.

Degrees Awarded During Past 5 Years

The degrees awarded during the past five years to students supported by this grant are as follows:

Doctor of Philosophy

1. W. T. Fowler, "First Order Control for Low-Thrust Interplanetary Vehicles", Ph.D. in E. M., June, 1965.
2. J. M. Lewallen, "An Analysis and Comparison of Several Trajectory Optimization Methods", Ph.D. in E.M., June, 1966.
3. J. F. Jordan, Jr., "Optimal Stochastic Control Theory Applied to Interplanetary Guidance", Ph.D. in E.M., August, 1966.
4. G. J. Lastman, "Optimization of Nonlinear Systems with Inequality Constraints", Ph.D. in E.M., January, 1967.
5. D. W. Childs, "Suboptimal Attitude Control of an Axisymmetric Spin-Stabilized Spacecraft", Ph.D. in E.M., August 1968.

Master of Science

1. D. G. Bettis, "Differential Corrections for Elliptical Trajectories", M.S. in E.M., August, 1964.
2. J. R. Clark, "A Solar System Ephemeris for 1950 to 2000", M.S. in A.S.E., May, 1965.
3. P. Salvato, Jr., "Analysis of the Failure of the Method of Adjoint Systems to Determine Optimal Trajectories", M.S. in A.S.E., August 1966.
4. M. McDermott, Jr., "Comparison of Coordinate Systems for Numerical Computation of Trajectories", M.S. in E.M., June 1967.
5. J.D. Hart, "Lagrange Multipliers for Optimal Low-Thrust Earth-Jupiter Transfers", M.S. in E.M., January, 1968.
6. C. A. Schwausch, "Numerical Error Comparisons for Integration of Near Earth Orbits in Various Coordinate Systems", M.S. in E.M., January, 1968.

7. W. E. Bollman, "An Ambiguity in the Orbit Determination of Planetary Flyby Trajectories", M.S. in E.M., June 1968.

8. P. M. O'Neil, "A Modified Perturbation Method for Determining Minimum Fuel Low-Thrust Earth-Jupiter Trajectories", M.S. in A.S.E., June 1969.

Students Supported During Present Year

The students who derived support during the past contract year from this grant are as follows:

NAME	TITLE	PERIOD OF APPOINTMENT	DEGREE SOUGHT
Chul Y. Choe	Res. Engr. Asst. II	2/1/69 to 1/31/70	Ph. D. in E. M.
Hamilton Hagar, Jr.	Res. Engr. Asst. II	9/1/69 to 1/31/70	Ph. D. in A. S. E.
Donald W. Jones	Res. Engr. Asst. II	6/1/69 to 1/31/70	Ph. D. in A. S. E.
Muhammed A. Rahman	Res. Engr. Asst. II	9/1/69 to 1/31/70	Ph. D. in A. S. E.
Walton E. Williamson	Res. Engr. Asst. II	4/1/69 to 4/30/69	Ph. D. in A. S. E.
Wayne R. Wright	Res. Engr. Asst. I	6/1/69 to 1/31/70	M. S. in E. M.
Jack D. Hart*	Res. Engr. Asst. II	0	Ph. D. in A. S. E.

*Formerly supported by Grant. Present effort supported by NSF Fellowship.

APPENDIX A

First Order Control for Low-Thrust Interplanetary Vehicles -- W. T. Fowler

Engineering Mechanics Research Laboratory
The University of Texas at Austin

EMRL TR-1001 May 1965

Summary

A new control scheme based on the first variation of the calculus of variations and the Weierstrass E-Function is presented. It is found that this control scheme is a member of a family of first-order control schemes. In general, a member of the control scheme family is characterized by an arbitrary weighting matrix. The choice of the arbitrary weighting matrix determines the effects of the control scheme on the system being controlled. The new control scheme, E-Function control, provides a criterion for choosing the weighting matrix.

A numerical comparison of three members of the family of first order control schemes is made. Two of the three control schemes are characterized by arbitrarily chosen constant weighting matrices. The weighting matrix for the third control scheme is chosen in the manner prescribed by the E-Function Control scheme.

A general method of predicting the state of a disturbed system in which not all of the state variables are terminally constrained is introduced. This state prediction method, used in conjunction with each control scheme, generates data which is used in the numerical comparison of the control schemes.

The system model used for the control scheme comparison is a low-thrust vehicle moving along a three-dimensional Earth-Mars trajectory. A reference trajectory is established and then each control scheme is forced to correct numerically-introduced state errors.

It is found that the new control scheme provides more satisfactory control than either of the two control schemes to which it is compared. It appears that the new control scheme gives the low-thrust analog of the impulsive correction characteristic of optimal high-thrust guidance.

APPENDIX B

Optimization of Nonlinear Systems with Inequality Constraints -- G. J. Lastman

Engineering Mechanics Research Laboratory
The University of Texas at Austin

EMRL TR-1006 October 1966

Summary

A nonlinear optimization problem with inequality constraints is formulated. Two general types of inequality constraints are examined; one which does not (state-variable constraint). It is found that a state-variable constraint can be replaced by an inequality constraint containing the control, in addition to some intermediate boundary conditions. A reformulated nonlinear optimization problem with inequality constraints (containing the control) and intermediate boundary conditions is then analyzed to obtain conditions which the optimal trajectory must satisfy. The results obtained are directly applicable to the guidance problem. Computer algorithms based on a new perturbation method for inequality-constrained problems are derived. As an example, the numerical solution of a nonlinear optimization problem with inequality constraints is presented. The example problem used is a low-thrust Earth-Mars transfer trajectory.

Two general forms of inequality constraints were examined: C(X,U,t), which involved the control explicitly, and S(X,t), the state-variable constraint, which did not explicitly involve the control. It was shown that the control could be readily determined so that the trajectory would not cross a constraint boundary of the form C(X,U,t) = 0. On a state-variable constraint boundary, S(X,t) = 0, the control was chosen to satisfy $d^qS/dt^q = 0$, where the q-th derivative os F is the first one that explicitly contains the control. If the control is chosen in this manner, the derivatives d^jS/dt^j, $j \leq q$, must be

zero at the point where the trajectory enters the boundary. Furthermore, the derivative $d^{q-1}S/dt^{q-1}$ must be zero at the point where the trajectory leaves the boundary. Thus, a state-variable constraint can be reduced to a constraint of the form $C(X,U,t)$, in addition to some intermediate boundary conditions.

A general problem involving inequality constraints, $C(X,U,t) \leq 0$, and the intermediate boundary conditions was studied to obtain the relationships which govern its solution. It was found that whenever ρ of the constraints were simultaneously zero, then ρ m, where m is the number of control variables, U_j. Furthermore, the $\rho \times m$ matrix, $\partial C_{i\kappa}/\partial U_j$, must be of full rank for $C_{i\kappa} = 0$, $\kappa = 1,2,\ldots, \rho$. It was found that the Lagrange multiplier, ρ, which was used in the analysis, could be discontinuous at the point where the trajectory entered a state-variable constraint boundary, but could be continuous at the point where the trajectory left the boundary. This is an extension of the results of Bryson et al. to the case of more than one control and more than one state-variable inequality constraint. Differential equations for the state X and the multiplier P were obtained in terms of the partial derivatives of a variational Hamiltonian, $H = Q + P^T F + M^T C$. The optimum control U and the multiplier M could be determined from a set of algebraic equations, in terms of X, P and t.

The various conditions which the solution to an inequality-constrained optimization problem must satisfy were restated in the form of a two-point boundary-value problem. A new perturbation method for inequality-constrained problems was devised to handle the two-point boundary-value problem. This method was based on the linearization of the differential equations for X and P, the optimality conditions giving M and U, and the terminal conditions, about a nominal trajectory, then calculating changes in the initial conditions and the terminal time so that the new nominal would more closely satisfy the terminal conditions. The resulting computational algorithm provided a rapidly converging procedure (if the initial approximation was "sufficiently close") for systems which are required to satisfy inequality constraints. This was demonstrated in the numerical experiments.

A simple test for the existence of points conjugate to the initial time t_0 was derived. No conjugate point existed on the trajectory if a certain matrix was positive definite.

In an optimization or open-loop control problem the state variable and the control variables are obtained as functions of time: $X = X(t)$ and $U = U(t)$. The related problem, feedback or closed-loop control, gives the control as a function of the state: $U = U(X)$. The major difficulty in closed-loop control is to determine the entering and exiting corner times. Discussions of the closed-loop control problem associated with inequality-constrained systems appearing in the literature suggest that near the corner points one would have to resort to open-loop control. A feedback control scheme based on the perturbation method of Chapter 4, where changes in the corner times are neglected, would probably give sufficiently accurate results. Further work on this topic is required.

APPENDIX C

Lagrange Multipliers for Optimal Low-Thrust Earth--Jupiter Transfers -- J. D. Hart

Engineering Mechanics Research Laboratory
The University of Texas at Austin

EMRL TR-1033 January 1968

Summary

It is likely that future exploratory probes to the planet Jupiter will be powered by constant low-thrust ion engines. A mission of this type consists of three phases; 1) a boost phase, 2) a low-thrust heliocentric interplanetary transfer phase, and 3) a terminal phase at Jupiter. Current near-Earth and interplanetary ballistic missions alos require a boost phase and a terminal phase, but the intermediate phase is a ballistic coast in which the spacecraft expends no fuel.

A continuously powered spacecraft expends fuel throughout the entire mission. Thus, it is desirable to determine, for any given launch date, the trajectory that will allow the probe to reach its destination in the minimum fuel consumption by the low-thrust engine if fuel is expended at a constant rate. A constant fuel expenditure rate is assumed in this study.

By choosing a series of launch dates at significantly large intervals, it is possible to study the full nonlinear optimal guidance problem while simultaneously obtaining a family of low-thrust interplanetary transfers. A full non-linear optimal guidance method solves the complete new nonlinear trajectory optimization problem which results each time a state deviation occurs. The major advantage of a workable nonlinear guidance scheme is that it is not based on a linearization assumption as are the linear guidance schemes and thus will work for large state deviations as well as for small state deviations.

The nonlinear trajectory optimization problem is first stated in terms of problem in the calculus of variations. This requires the specification of a performance index, a governing set of differential equations of motion, and the applicable boundary conditions. When employing the calculus of variations, the boundary conditions and the differential equations of motion are adjoined by Lagrange multipliers to the performance index. The problem can then be reduced to a two-point boundary-value problem by standard methods and then a numerical solution can be attempted by any one of several methods that are available.

Before the set of differential equations can be integrated, an initial value for each dependent variable along with a terminal time must be specified. The initial values of the velocity and position of the spececraft can be easily determined. However, the initial values of the Lagrange multipliers and the final time for the mission must be guessed. When an iterative scheme is employed to solve the two-point boundary-value problem, the rate of convergence of the iterations to the solution is highly dependent upon the initial values given the guessed quantities.

In this investigation, initial values for the Lagrange multipliers along with the corresponding total mission times for Earth-Jupiter optimal trajectories for a series of launch dates are determined. The launch dates considered lie between the years 1982 and 1985. The model used is the heliocentric portion of a low-thrust Earth-Jupiter transfer. By plotting the initial values of the Lagrange multipliers as functions of launch date, curves are obtained from which one can determine the initial values of the multipliers for other dates in and around the period under consideration.

APPENDIX D

Failure Analysis of the Method of Adjoint Systems
to Determine Optimal Trajectories -- Pete Salvato, Jr.

Engineering Mechanics Research Laboratory
The University of Texas at Austin

EMRL TR-1005 August 1966

Summary

Due to the high degree of nonlinearity and complexity encountered in many modern trajectory optimization problems, there has been a great need for, and consequently, a great upsurge of interest in the development and application of various numerical optimization techniques in recent years. Utilization of the capabilities of high speed digital computers has resulted in solutions to problems which a few short years ago were considered too cumbersome or difficult to solve.

Although numerical techniques have been successfully applied in solving a wide class of optimization problems, occasionally someone fails to optimize a particular dynamical system by the use of a particular optimization technique. This failure can occur in several ways depending upon the technique used and the particular problem, but usually an investigation of the failure is not attempted. As a result, there is very little information presently available in the literature concerning ways in which numerical optimization techniques can fail in application to dynamical systems. The need for such information is obvious.

This study analyzes one particular numerical optimization technique known as the Method of Adjoint Systems. The Method of Adjoint Systems is developed and applied to the solution of the two-point boundary value problem arising from the first necessary conditions for an optimal trajectory. A specific dynamical system is analyzed and various forms of incorrect problem

formulation are considered in order to investigate the effects on the Method of Adjoint Systems.

The sufficiency conditions for a weak minimum are established and the Matrix Riccati Equation is shown to evolve from the requirement that the second variation be positive for a minimum. The effects of a conjugate point in the nominal trajectory on the solution of the guidance optimization problem are established through the solution ot the Matrix Riccati Equation.

It is shown that the Matrix Riccati Equation can be solved by reducing it to two linear systems of differential equations, which in partitioned form are adjoint to a system of linear differential equations used in the implementation of the computational algorithm of the Method of Adjoint Systems. Through the properties of adjoint systems, a relationship is established which indicates a connection between a breakdown in the Method of Adjoint Systems due to the singularity of a matrix which must be inverted and the presence of a conjugate point in the nominal trajectory at the initial time.

APPENDIX E

Comparison of Coordinate Systems for Numerical Computation of Optimal Trajectories -- Make McDermott, Jr.

Engineering Mechanics Research Laboratory
The University of Texas at Austin

Master's Thesis June 1967

Summary

In recent years there has been considerable interest in the problem of determining optimal trajectories for powered space vehicles. In general, the formulation of a trajectory optimization problem leads to a system of first order, nonlinear, ordinary differential equations which are required to satisfy split boundary conditions. This type problem is referred to as a two-point boundary-value problem. Since analytic solutions for these problems are very difficult to obtain, most of the effort in this area has been directed toward obtaining numerical solutions. Several basic algorithms have been devised for computing numerical solutions to these two-point boundary value problems, each algorithm having certain hinerent advantages and disadvantages for a given problem or class of problems. Although singificant progress has been made in developing a capability to obtain numerical solutions to complex trajectory problems, there still exists a need to determine which of the possible approaches is best for a specific problem.

Investigations have been made which seek to determine which algorithms are best for computing numerical solutions for various classes of problems. A second factor which must be considered in seeking the best approach to a specific problem is the coordinate system used to describe the trajectory. The algebraic form of the two-point boundary-value problem which must be solved is determined by the choice of coordinate system. Hence, the choice of a coordinate system will influence:

1. the difficulty of the analytical effort required to formulate the two-point boundary-value problem,
2. the work required to implement the particular algorithm chosen,
3. the computer time required to obtain a numerical solutions, and
4. the accuracy of the numerical solution obtained.

Also, for particularly difficult problems, success or failure in obtaining a numerical solution can depend on the choice of the coordinate system.

This study compares two sets of variables commonly used for computing solutions to near-planetary trajectory optimization problems. The variables considered are plane polar coordinates and a set of modified Keplerian orbital elements. The comparison is made by studying low-thrust planetary escape trajectories.

In the comparison, the optimal planetary escape sprial problem is formulated in the two coordinate systems, thus obtaining two algebraically different boundary-value problems. It is emphasized that although the boundary-value problems in the two coordinate systems are quite different in form, they are both mathematical models for the same physical optimal trajectory. Then, a digital computer is used to solve each of the two boundary-value problems using the same numerical algorithm. Finally, a series of physically equivalent optimal trajectories is computed in each of the coordinate systems and the results of these computations are compared.

By comparing the solutions obtained in the two coordinate systems with respect to the four factors listed previously (i. e. , difficulty of problem formulation, work required to implement the algorithm, computer time required, and accuracy of results) an attempt is made to determine which of the two coordinate systems is best for computing numerical solutions to this class of problems.

The differential equations which govern the state variables and Lagrange multipliers are not difficult to formulate in either of the two coordinate systems. However, the functional forms of the derivatives of these variables are considerably more complicated in the orbital element system than the corresponding functions in the polar coordinate system. This difference in length makes the programming more tedious in orbital elements and undoubtedly accounts for the greater computing time required for forward integration of the orbital elements. The most striking difference occurs in formulating and programming the equations for the backward integration of the adjoint variables. The differential equations which govern the adjoint variables are much more difficult to formulate in the orbital element system, and the functional forms of the derivatives of these variables are much more complicated. As a result, the computer subroutine which evaluates the derivatives of the adjoint variables in orbital elements is approximately five times longer than the equivalent subroutine in polar coordinates.

Thus, it is seen that the optimization problem is easier to formulate and program in the polar coordinate system.

It was found that for any given step size, the computing time for a single forward integration or for a single backward integration is always less in polar coordinates than in orbital elements. This results from the fact that the functional forms of the derivatives are less complicated in polar coordinates than in orbital elements. (In any numerical integration procedure, most of the computer time is spent evaluating these derivatives).

By variation of the integration step size, it was found that for small step sizes the polar coordinates require much less total computing time than orbital elements. However, as the step size is increased, the number of iterations required to converge to the optimal trajectory increases much more rapidly with polar coordinates than with orbital elements. The drastic increase in the number of iterations required is believed to be caused by a rapid deterioration of the accuracy of the numerical integration in polar coordinates as the step size is increased.

Despite the increase in total computing time for large step sized in polar coordinates, the smallest total computing time for each case studied is achieved by using polar coordinates and a relatively small step size rather than orbital elements and a larger step size. The minimum computing times in two of the cases studied are achieved by using polar coordinates and a step size of approximately 500 seconds. For this step size the numerical integration should be te accurate.

A complete study of this problem would have to consider other choices of the orbital element variables, computations for lower thrust levels, and a more systematic comparison of the accuracy of the computations. However, on the bases of the factors considered here -- the ease of formulating and programming the optimization problem and the computer time required to compute a given optimal trajectory -- polar coordinates are a more favored coordinate system than orbital elements.

APPENDIX F

Numerical Error Comparisons for Integration of Near-Earth Orbits in Various Coordinate Systems -- O. A. Schwausch

Engineering Mechanics Research Laboratory
The University of Texas at Austin

EMRL 1054 January 1968

Summary

In order to accurately predict the state (i.e., position and velocity) of an interplanetary vehicle or of a near-Earth satellite, the equations governing the vehicle's motion must be integrated numerically because the perturbations produced by thrust, drag, the gravitational effects of the planets, and other factors preclude the finding of a closed form solution. Generally, the integrated states differ from the actual states because of numerical error, and these differences (state deviations or errors) increase with time. The sizes of the state deviations are influenced by the coordinate system in which the equations of motion are expressed. Since two-body effects dominate the vehicle's motion and since a closed form solution to the two-body problem is available, an analysis of how two-body motion is affected by different coordinate systems would provide some indication as to how the more complicated perturbed motion is affected by coordinate system choice.

This investigation determines the influence which the choice of coordinate systems has upon the numerical error growth in the states for two-body motion. This is accomplished by numerically integrating the equations of motion for each coordinate system under investigation. At specified time intervals an error check is made by transforming the respective position and velocity vectors to a reference coordinate system. Then a comparison of the integrated state to a reference state generated using the known closed form solution to the two-body problem is made. In this manner the relative state

errors for each system can be obtained, and respective error norms computed. By varying the integration step size and the orbit shape, the error norms are analyzed in order to determine the sensitivity of the relative error magnitudes to step size, orbit shape and coordinate system choice.

In the study, five coordinate systems (Rectangular, Spherical, Circular Cylindrical, and two forms of Elliptic Cylindrical coordinates) are compared in order to determine their efficiency for numerical integration of orbital equations. These five coordinate systems were chosen from numerous coordinate systems considered in a preliminary screening process.

APPENDIX G

Optimal Stochastic Control Theory
Applied to Interplanetary Guidance -- J. F. Jordon

Engineering Mechanics Research Laboratory
The University of Texas at Austin

EMRL TR-1004 August 1966

Summary

In the investigation presented here, the problem of the optimal control of a nonlinear dynamic system in the presence of noise is studied. In particular, the investigation is concerned with continuous autocorrelated noise which perturbs the controls of the dynamic system.

A study is made of the effects of noise in the controls on an optimal deterministic trajectory. The effects are illustrated for a simulated study of a low-thrust spacecraft on a minimum time Earth-to-Mars transfer trajectory. The characteristics of the effects of the noise illustrated in the study indicate the necessity for developing an optimal stochastic control. The control procedure developed in the investigation is a nonrandom function of time, based on a priori knowledge of the statistical behavior of the noise process, and is designed to anticipate the expected effects of the noise on the dynamic system. A stochastic calculus of variations approach is employed formally to determine the control procedure for the stochastic system. The control essentially guides the expected value of the state to meet the terminal conditions, while extremizing the expected value of the original deterministic performance index functional. The behavior of the control procedure is studied for a simulated interplanetary transfer problem.

The results of the study indicate the necessity for presenting a scheme which will correct the control program, on the basis of information gained during the controlling interval, so that the actual state comes closer to satis-

flying the terminal constraints, while preserving the optimal nature of the control program. A method is presented for replacing the mean values of the state components and the Lagrange multipliers, with which then a priori control is computed, with conditional mean values of these quantities based on the values of state observations. The scheme is applied to the interplanetary transfer problem for the case where range-rate observations are taken at discrete instances of time.

From the study reported in Reference 14 of the effects of noise on a nonlinear dynamic system, the following conclusions can be drawn:

1. Both the theory and the numerical studies of the interplanetary problem show that the occurrence of noise in a nonlinear dynamic system implies an ensemble of stochastic trajectories. The analysis shows that the mean of the ensemble differs from the deterministic trajectory.

2. In general the standard deviations of the state components increase with time indefinitely. However, the nonlinearity of the system and the optimal nature of the control strongly influence the values of the standard deviations.

3. The statistics of the ensemble of trajectories are highly dependent on both the variance and the correlation time associated with the perturbing noise. In general, the mean state deviations from the deterministic trajectory and the standard deviations both increase with increasing noise variance and/or increasing noise correlation time.

4. The result of the numerical studies on the interplanetary transfer problem show that the statistics of the ensemble of trajectories for the case in which noise occurs in the thrust magnitude and for the case in which noise occurs in the thrust direction are quite different.

Study of the application of the optimal stochastic control to the interplanetary transfer problem has led to the following conclusions:

1. In the case of the interplanetary transfer problem the difference between the a priori optimal stochastic control and the optimal deterministic control is small in comparison with the perturbing noise. It should be noted

that this may not be the case for highly nonlinear dynamic systems.

2. In the case of the interplanetary transfer problem, the implementation of an optimal stochastic control which is based only on an a priori knowledge of the statistics of the perturbing noise does not appreciably reduce the standard deviations of the state components at the final time. For this reason, it can be concluded that the control must be updated throughout the controlling interval if the terminal state is to satisfy approximately the terminal constraints.

APPENDIX H

A Modified Perturbation Method for Determining
Minimum-Fuel Low-Thrust Earth-Jupiter Trajectories -- P. M. O'Neill

Applied Mechanics Research Laboratory
The University of Texas at Austin

AMRL TR-1003 June 1969

The problem of determining the minimum fuel path that a low-thrust vehicle should fly in transferring a scientific package to the planet Jupiter has been investigated. A performance index was used that considered a linear combination of the fuel requirement and the flight time. A calculus of variations approach to the problem led to a two point boundary value problem that could not be solved by standard methods due to a discontinuity in the mass flow rate control. Necessary conditions obtained from the calculus of variations require the vehicle to coast during portions of the flight.

For the cases in which the optimal trajectory requires at most one coast period, a modified perturbation method was developed that is well suited for solving the two point boundary value problem. This method employs a unique method of handling the optimization of the coast arc which will likely be useful for guidance along trajectories containing coast arcs. First, two unknown switching times, t_1 and t_2, are guessed in addition to the unknown initial conditions. The vehicle will coast during the interval $[t_1, t_2]$. The switching function $\Gamma(Z)$ is monitored during each iterate but is not used to control the switching on and off of the engine during a given iterate. The $\Gamma(t)$ history for one iterate is used to shift the time interval $[t_1, t_2]$ for the next iterate in a manner to be described below. The ultimate goal is to choose t_1 and t_2 so that the two conditions, $\Gamma(t_1) = 0$ and $\Gamma(t_2) = 0$, are satisfied. Optimal trajectories for the November 15.0, 1983 launch date are presented for performance indices that consider various linear combinations of the fuel requirement and the flight time.

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