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The Factorization of Spectra by Discrete Fourier Transforms*

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Abstract

A discrete Fourier transform method for factoring arbitrary spectral density functions is presented. The factorization can be implemented in a straightforward and efficient manner, and it does not require that the spectra be rational. An expression for the absolute error is also presented.

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The application of spectral factorization to computing causal filters is well known. It is of recent interest in connection with the linear random process. As Blake and Thomas define it, the linear random process is constructed from a sequence of independent, identically distributed generating random variables in such a way that the generating random variables can be recovered from the past of the process by a linear operation. This operation is called inversion, and it follows immediately from the spectral factorization.(1)

For the actual computation in statistics as well as engineering, one usually assumes that the spectra are rational. Formally this is not restrictive since any factorable spectrum can be approximated by a rational spectrum.(2) The numerical techniques, however, are cumbrous and time-consuming, and precise expressions for the error are not known.(3)

The discrete-Fourier-transform method which we shall describe completely eliminates these difficulties.

I. The Discrete-Parameter Process

A discrete-parameter process has a factorable spectral density function (S.D.F.) $\varphi_P(\omega)$ if and only if it is regular,[4] i.e.

$$\int_{-\pi}^{\pi} \ln \varphi_P(\omega) \, d\omega > -\infty. \tag{1}$$

The S.D.F. $\varphi_P(\omega)$ is real, symmetric, and differentiable almost everywhere. Moreover, delta functions are explicitly excluded.

In many applications the discrete-parameter process is derived by sampling a continuous-parameter process. In that case

$$\varphi_P(\omega) = \sum_{m=-\infty}^{\infty} \varphi(\omega + 2\pi m) \tag{2}$$

is the "aliased" spectral density of the original process.[5] The subscript P is used to distinguish the discrete- and continuous-parameter S.D.F.'s.

To proceed with the factorization, note that (1) is sufficient to permit the Fourier series representation

$$\ln[\varphi_P(\omega)]^{\frac{1}{2}} = \sum_{k=-\infty}^{\infty} a_k \exp(-i\omega k). \tag{3}$$

If we define

$$\ln^+[\varphi_P(\omega)]^{\frac{1}{2}} = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \exp(-i\omega k), \tag{4}$$

it follows that

$$\varphi(\omega) = \exp\{\ln^+[\varphi_P(\omega)]^{\frac{1}{2}}\} \exp\{\ln^+[\varphi_P(\omega)]^{\frac{1}{2}}\}^*. \tag{5}$$

One can show [6] that

$$\varphi^{+}(\omega) = \exp\{\ell n^{+}[\varphi_P(\omega)]^{\frac{1}{2}}\} \tag{6}$$

has only positively indexed Fourier coefficients and that (5) is the desired spectral factorization.

Suppose now that $\varphi_P(-\pi) \neq 0$. We shall show in Section II that this assumption can be eliminated. Since $\varphi_P(\omega)$ is symmetric we can find N real numbers $\{\alpha_k\}$, where k = 0, 1, ..., N - 1, and N is even, such that

$$\ell n[\varphi_P(2\pi n/N)]^{\frac{1}{2}} = \sum_{k=0}^{N-1} \alpha_k \exp(-2\pi ink/N). \tag{7}$$

Specifically

$$\alpha_k = \frac{1}{N}\sum_{n=0}^{N-1} \ell n[\varphi_P(2\pi n/N)]^{\frac{1}{2}} \exp(2\pi ink/N). \tag{8}$$

Now if (4) is approximated by $g_N^{+}(2\pi n/N)$, where

$$g_N^{+}(2\pi n/N) = \frac{1}{2}\alpha_0 + \sum_{k=1}^{\frac{1}{2}N-1} \alpha_k \exp(-2\pi ink/N) + \frac{1}{2}(-1)^n \alpha_{N/2}, \tag{9}$$

it follows that

$$\varphi(2\pi n/N) = \exp\{g_N^{+}(2\pi n/N)\} \exp\{g_N^{+}(2\pi n/N)\}^{*} \tag{10}$$

which is the discrete analogue of (5). Hence we take as our approximation to $\varphi^{+}(\omega)$

$$\tilde{\varphi}^{+}(2\pi n/N) = \exp\{g_N^{+}(2\pi n/N)\}. \tag{11}$$

It can be shown [7] that

$$\alpha_k = \sum_{m=-\infty}^{\infty} a_{k+mN}. \tag{12}$$

If we make the definition

$$a_k^+ = \begin{cases} a_k & k > 0 \\ \frac{1}{2} a_0 & k = 0 \\ 0 & k < 0, \end{cases} \tag{13}$$

it follows from (12) that $\alpha_k = \alpha_k^+ + \alpha_{N-k}^+$, where

$$\alpha_k^+ = a_k^+ + \sum_{m=-\infty}^{\infty}{}' \; a_{k+mN}^+ . \tag{14}$$

(The primed summation indicates omission of the m = 0 term.) One can then show that

$$\epsilon_N^+(n) = \ell n[\varphi(2\pi n/N)]^{\frac{1}{2}} - g_N^+(2\pi n/N) = 2i \sum_{k=1}^{\frac{1}{2}N-1} \alpha_{N-k}^+ \sin 2\pi nk/N. \tag{15}$$

From standard Fourier theory we know that a_k^+ is $0[k^{-(K+1)}]$ whenever $\ell n \; \varphi(\omega)$ has K continuous derivatives. Hence each term in the summation in (14) is $0[(k + mN)^{-(K+1)}]$. Now for fixed k and $k \neq N$ the term $N^{-(K+1)}$ can be factored out of the summation, and we see that α_k differs from a_k^+ by a term that is $0(N^{-(K+1)})$. In (15), therefore, $\frac{1}{2} N - 1$ terms that are each $0(N^{-(K+1)})$ are being summed. We would expect $\epsilon_N(n)$ to be at least $0(N^{-K})$, and a detailed analysis verifies this conclusion.

To summarize, we have shown that

$$\tilde{\varphi}^+(2\pi n/N) = \varphi^+(2\pi n/N) \; \exp\{\epsilon_N^+(n)\}.$$

The error term $\epsilon_N^+(n)$ is purely imaginary and rapidly decreasing with increasing N for smooth S.D.F.'s. The computation requires evaluating two discrete Fourier transforms, (8) and (9), which can be efficiently implemented for $N = 2^M$, where M is a positive integer.[8]

II. The Continuous-Parameter Process

For a continuous-parameter process the regularity is guaranteed by the familiar Paley-Wiener condition[9].

$$\int_{-\infty}^{\infty} \frac{\ell n\ \varphi(\omega)}{1 + \omega^2}\ d\omega > -\infty. \tag{16}$$

Since the S.D.F. is integrable, we cannot have $\varphi(\omega) \neq 0$ at $\omega = \infty$. Hence we consider the restricted class of S.D.F.'s that satisfy the additional condition

$$\lim_{\omega \to \infty} 2^{-(K+1)}\ \varphi(\omega)\ (1 + \omega^2)^{K+1} = C, \tag{17}$$

where K is a positive integer and C is a finite, non-zero constant. The particular form of (17) is chosen for convenience. Since no S.D.F. satisfying (16) can approach zero faster than a power of ω^2, (17) excludes only S.D.F.'s that approach zero slower than ω^2.

Now if we make the coordinate transformation[10] $u = 2 \tan^{-1} \omega$ (17) guarantees that the new function

$$f(u) = (1 + \cos 2u)^{-(K+1)}\ \varphi(\tan \tfrac{1}{2} u) \tag{18}$$

is a regular S.D.F. on $(-\pi, \pi)$ that is finite at $u = -\pi$ and admits a factorization $f(u) = f^+(u)\ [f^+(u)]^*$ where

$$f^+(u) = [2^{-\frac{1}{2}}(1 + \exp(-iu))]^{-(K+1)}\ \varphi^+(\tan \tfrac{1}{2} u)$$

$$= \sum_{n=0}^{\infty} \gamma_n \exp(-inu). \tag{19}$$

Hence the method of Section I can be applied to approximate $f^+(u)$. With an additional discrete transform the coefficients γ_n can be approximated.

Once the coefficients γ_n have been computed, it is a simple matter to compute the Fourier coefficients $\gamma_n^{(K)}$ of $\varphi^+(\tan \frac{1}{2} u)$. We have the recursive relations

$$\gamma_0^{(M)} = 2^{-\frac{1}{2}} \; \gamma_0^{(M-1)}$$
$$\gamma_n^{(M)} = 2^{-\frac{1}{2}}(\gamma_{n-1}^{(M-1)} + \gamma_n^{(M-1)}) \qquad \text{for } n > 1, \tag{20}$$

with $\gamma_0^{(0)} = 2^{-\frac{1}{2}} \gamma_0$ and $\gamma_n^{(0)} = 2^{-\frac{1}{2}} (\gamma_{n-1} + \gamma_n)$ for $n > 1$. To compute $\gamma_n^{(K)}$ the relations (20) are evaluated for $M = 1, 2, \ldots, K$.

It is clear that we can use this method for the discrete-parameter process when $\varphi_p(-\pi) = 0$. We factor $\varphi_p(\omega)/(1 + \cos 2\omega)^{K+1}$, with K chosen so that $\lim_{\omega \to \pi} \varphi_p(\omega)/(1 + \cos 2\omega)^{K+1}$ is finite, and then use (20) to compute the Fourier coefficients of $\varphi_p(\omega)$.

For the continuous-parameter case the Fourier coefficients of $\varphi^+(\tan \frac{1}{2} u)$ do not lead to the most convenient representation for $\varphi^+(\omega)$. We note, however, that

$$2^{-\frac{1}{2}} \exp(- inu) \; (1 + \exp(- iu) \;) = 2^{-\frac{1}{2}}(1 - i\omega)^n/(1 + i\omega)^{n+1}, \tag{21}$$

which is the Fourier transform of the n-th Laguerre function $2^{\frac{1}{2}} e^{-t} L_n(2t)$. Hence if we compute $f(u)$ in (17), with K chosen so that (13) is satisfied, we can compute the Laguerre coefficients of $\varphi^+(\omega)$ by applying the recursive relations (20) exactly K times. If K is zero, we have the Laguerre coefficients immediately.

To demonstrate the method, the Laguerre coefficients for the factorization of $\varphi(\omega) = (1 + \omega^2/N)^{-N}$ were computed. Note that as $N \to \infty$ $\varphi(\omega) \to e^{-\omega^2}$, which does not satisfy (16). For this S.D.F., $\varphi^+(\omega)$ can be computed analytically. The Fourier transform $\varphi^+(t)$ of $\varphi^+(\omega)$ is

$$\hat{\varphi}^{+}(t) = \begin{cases} N^{N/2}\, t^{N-1}\, e^{-N^{1/2} t}/(N-1) & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{22}$$

For N = 6 the series

$$\hat{\varphi}^{+}(t) \approx \sum_{n=0}^{M} \gamma_n^K\, 2^{\frac{1}{2}}\, e^{-t}\, L_n(2t) \tag{23}$$

was computed for several values of t and compared with (22). The error, which is plotted in Fig. 1 along with $\hat{\varphi}^{+}(t)$, was less than 10^{-9}. This is nearly the smallest error we could expect since only 11 significant figures were carried in the computation. The transforms were evaluated with 128 points.

References

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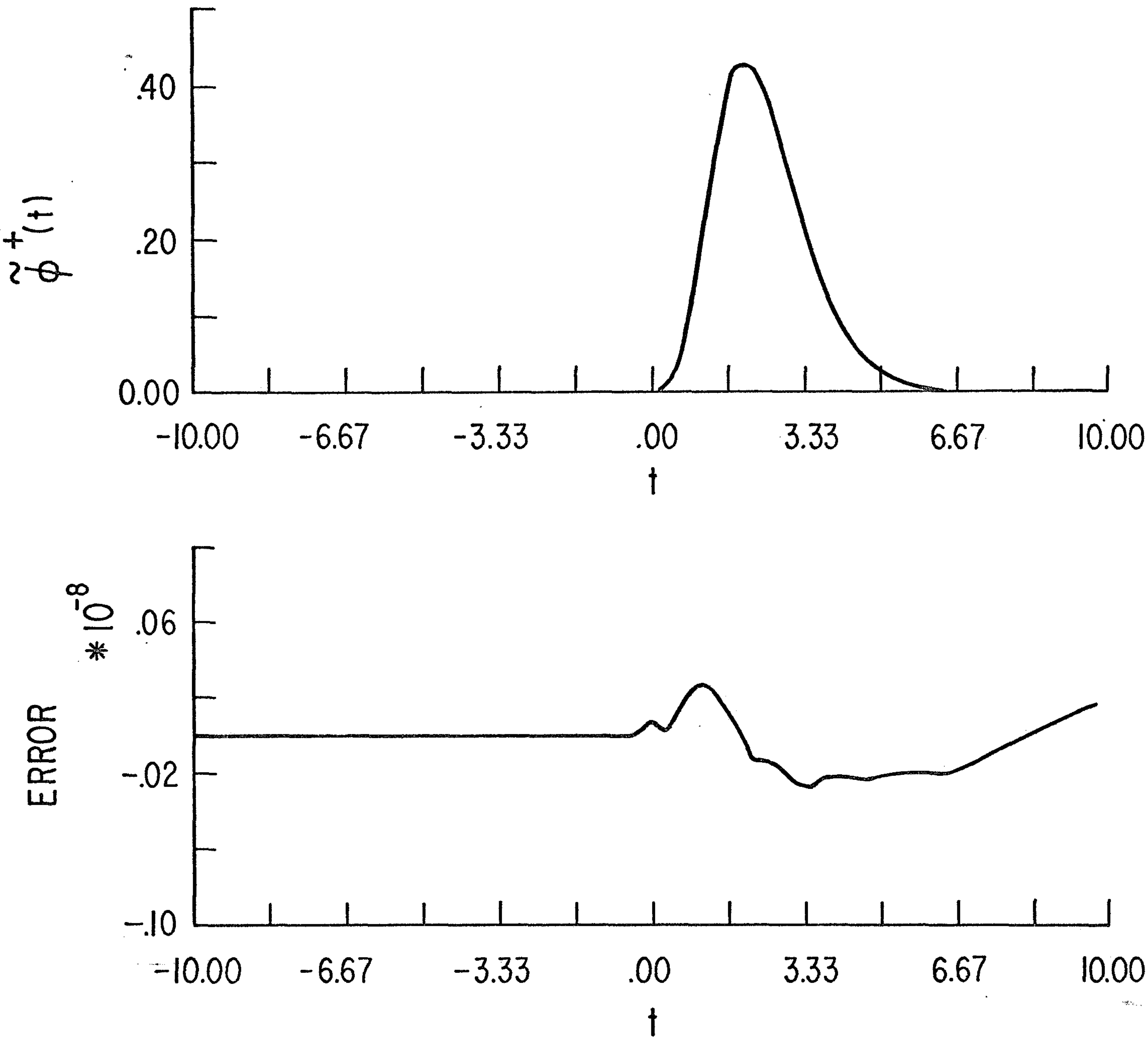

$\tilde{\phi}^+(t)$
.40
.20
0.00
-10.00
-6.67
-3.33
.00
3.33
6.67
10.00
t
ERROR $*10^{-8}$
.06
-.02
-.10
-10.00
-6.67
-3.33
.00
3.33
6.67
10.00
t

Figure Captions

1. Approximate inverse Fourier transform of $\varphi^+(\omega)$ for the spectrum $\varphi(\omega) = (1 + \omega^2/N)^{-N}$, and its absolute error. Computation performed with 128 points and $N = 6$.