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PLANAR MOTION OF A HUMAN BEING
UNDER THE ACTION OF A BODY-FIXED THRUST

by

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E R R A T A

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Page 5 -- Add the following terms to Eq. (3.2):

$$\frac{1}{2} I\dot{q}_1^{\,2} + \frac{1}{2} I' (\dot{q}_1 + \dot{\phi})^2$$

Page 17 -- First term in Eq. (5.45) should be:

$$a_2 \cos \phi_e$$

Page 22 -- Table 2, "Low Thrust-Arms Down" value of β_2 should be 7.740 instead of 7.720

Page 34 -- Last sentence should read:

As the remaining terms ....

ABSTRACT

Equations of motion are derived for a system comprised of two rigid bodies subjected to the action of a force of constant magnitude, the force being applied to one of the bodies along a line fixed in the body. Analytical and numerical solutions of these equations are then used to study effects of certain relative motions of the two parts of the system on the translation of the mass center and on the rotations of the bodies.

The purpose of this work is to assess the feasibility of providing a "weightless" astronaut with an extremely simple maneuvering device, namely, a single thruster, rigidly attached to one part of the astronaut's body, directional and attitude control to be achieved by means of limb movements.

The results obtained are encouraging: Rectilinear motions of the mass center, accompanied by negligible rotational motions of the parts of the system, turn out to be possible if the subject is capable of performing in accordance with a simple linear feed-back law. Furthermore, it is found that substantial benefits can be derived even from open loop performance of certain maneuvers.

ACKNOWLEDGEMENT

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1. Introduction

As manned space exploration progresses, the amount of extra vehicular activity required may be expected to increase. When such activity involves translational motions, thrust must be provided in some form. A number of propulsion schemes have been proposed for this purpose. The simplest method would be one requiring but a single thruster rigidly attached to a part of the body, and this would, indeed, generate rectilinear motion if the line of action of the thrust could be made to pass through the center of mass of the system. However, a certain amount of misalignment must be regarded as unavoidable due to improper initial disposition of parts of the human body; and such misalignments will inevitably give rise to undesired rotational motions. The following question thus presents itself: Is it possible to perform relative motions of parts of the body in such a way as to negate undesirable effects of thrust misalignment? It is the purpose of this report to supply a partial answer to this question by dealing with planar motions of a system comprised of two rigid bodies subjected to the action of a force of constant magnitude, this force being applied to one of the bodies along a line fixed in the body.

The system to be studied is described in detail in Sec. 2, and equations of motion are derived in Sec. 3. One of these equations is used in Sec. 4 to find equilibrium configurations, that is, configurations of the system for which no rotation occurs. The motion that results when the system remains rigid, but does not occupy an equilibrium configuration, is studied in Sec. 5. For this case, solutions are obtained in closed form, and these furnish a basis for comparisons made in the next section, which deals with motions that result when the two bodies perform small relative oscillations. Here, an approximate analytical solution is developed, and is used to discover configurations for which rotational motion is minimized. Numerical results, applicable to a model of the human body, are presented at the end of Sec. 5. Finally, Sec. 6 deals with the behavior of the system when it is assumed that relative motions proceed in accordance with a feedback control law. Two cases are investigated. In each of these, a stability analysis is made to determine permissible values of gains, and numerical results for the human model considered in Sec. 5 are presented.

2. System Description

The system to be studied consists of two rigid bodies, B and B', connected by a hinge at a point P (see Fig. 1) which is located by a position vector $\underline{p}$ relative to a point 0 that is fixed in an inertial reference frame R. The center of mass of B is designated B^* and is located relative to P by a position vector $\underline{r}$ of magnitude r. Similarly, the center of mass of B' is designated B'^* and is located relative to P by a vector $\underline{r}'$ of magnitude r'. The angle between $\underline{r}$ and $\underline{r}'$ is called φ.

Orthogonal unit vector $\underline{n}_1$ and $\underline{n}_2$ are fixed in B parallel and perpendicular, respectively, to $\underline{r}$; $\underline{N}_1$, $\underline{N}_2$, $\underline{N}_3$ are orthogonal unit vectors fixed in R; and the angle between $\underline{n}_1$ and $\underline{N}_1$ is designated θ.

Body B has a mass m and a moment of inertia I about a line passing through B^* and parallel to $\underline{N}_3$. Similarly, B' has a mass

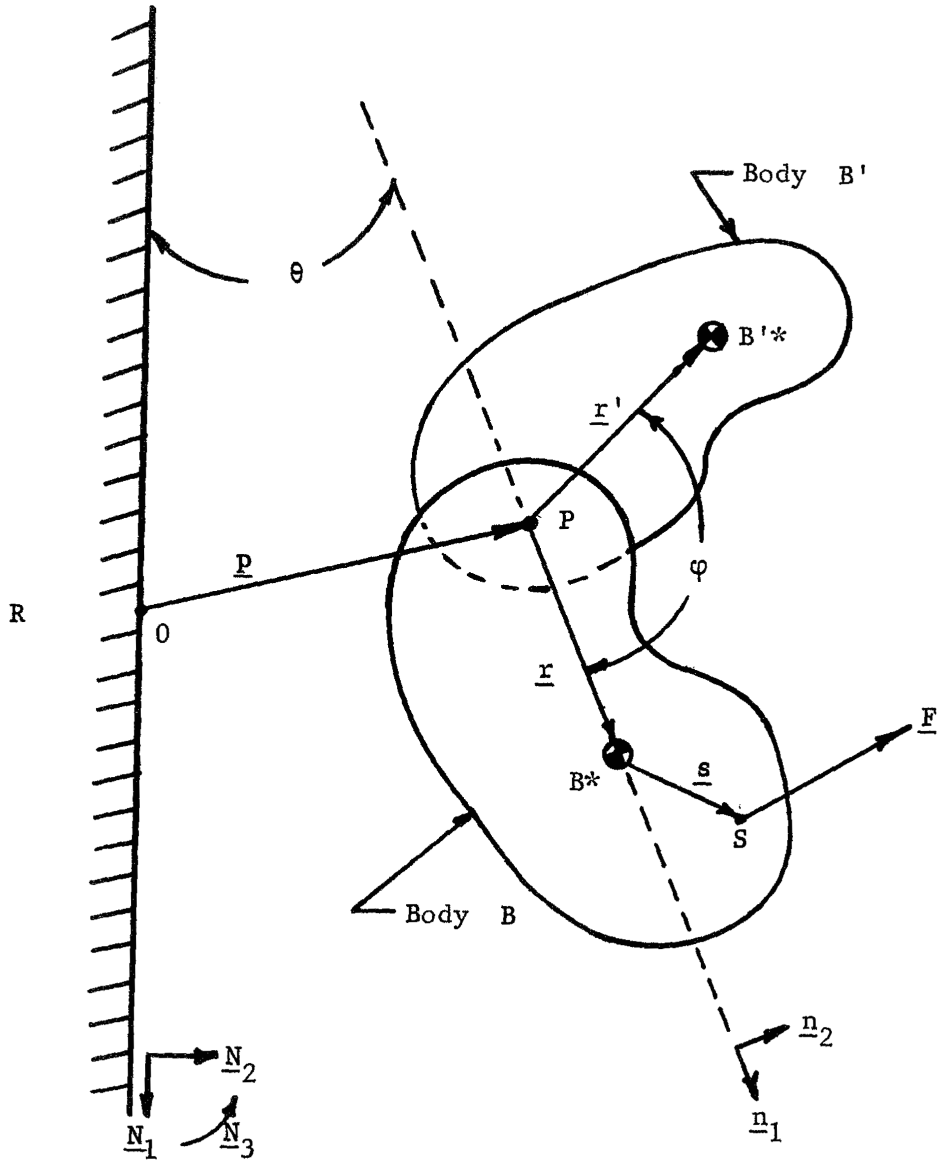

Fig. 1 Two Hinged Bodies

m' and a moment of inertia I' about a line passing through B'^* and parallel to $\underline{N}_3$.

A force $\underline{F}$ is applied to B at a point S , which is located relative to B^* by a vector $\underline{s}$.

The following scalar quantities are used in the analysis of the system:

$$x_i = \underline{p} \cdot \underline{N}_i \quad , \quad i = 1,2 \tag{2.1}$$

$$s_i = \underline{s} \cdot \underline{n}_i \quad , \quad i = 1,2 \tag{2.2}$$

$$F_i = \underline{F} \cdot \underline{n}_i \quad , \quad i = 1,2 \tag{2.3}$$

Note that the definition of x_i involves the inertially fixed unit vector $\underline{N}_i$, s_i and F_i depend on the vector $\underline{n}_i$ fixed in B . The assumption that s_i and F_i are constants thus implies that both $\underline{s}$ and $\underline{F}$ are fixed in magnitude and direction relative to B .

3. Equations of Motion

In what follows, the angle φ is assumed to be a prescribed function of time. Consequently, B and B' constitute a holonomic system with three degrees of freedom and equations of motion may be obtained by employing Lagrange's equations,

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_r}\right) - \frac{\partial K}{\partial q_r} = Q_r \quad (r = 1,2,3) \tag{3.1}$$

where K is the kinetic energy of the system, q_r is a generalized coordinate, $\dot{q}_r$ is the first time derivative of q_r , and Q_r is the generalized active force associated with the coordinate q_r . If θ , x_1 and x_2 are chosen as generalized coordinates q_1 , q_2 , and q_3 , respectively, K can be expressed as

$$K = \frac{1}{2} m \left[(\dot{q}_2 - r\dot{q}_1 \sin q_1)^2 + (\dot{q}_3 + r\dot{q}_1 \cos q_1)^2\right]$$
$$+ \frac{1}{2} m' \left\{\left[\dot{q}_2 - r'(\dot{q}_1 + \dot{\varphi}) \sin(q_1 + \varphi)\right]^2 + \left[\dot{q}_3 + r'(\dot{q}_1 + \dot{\varphi})\cos(q_1 + \varphi)\right]^2\right\} \tag{3.2}$$

and the generalized active forces are given by

$$Q_1 = -F_1 s_2 + F_2(r + s_1) \tag{3.3}$$
$$Q_2 = F_1 \cos q_1 - F_2 \sin q_1 \tag{3.4}$$
$$Q_3 = F_1 \sin q_1 + F_2 \cos q_1 \tag{3.5}$$

Substitution from (3.2)-(3.5) into (3.1) thus leads to

$$-\ddot{x}_1[mr \sin \theta + m'r' \sin(\theta + \varphi)] + \ddot{x}_2[mr \cos \theta + m'r' \cos(\theta + \varphi)]$$
$$+ (I + mr^2 + I' + m'r'^2) \ddot{\theta} + (I' + m'r'^2) \ddot{\varphi} = F_2(r + s_1) - F_1 s_2 \tag{3.6}$$

$$(m + m') \ddot{x}_1 - mr \ddot{\theta} \sin \theta - m'r'(\ddot{\theta} + \ddot{\varphi}) \sin(\theta + \varphi) - m'r' \dot{\theta}^2 \cos \theta$$
$$- m'r'(\dot{\theta} + \dot{\varphi})^2 \cos(\theta + \varphi) = F_1 \cos \theta - F_2 \sin \theta \tag{3.7}$$

$$(m + m') \ddot{x}_2 + mr \ddot{\theta} \cos \theta + m'r'(\ddot{\theta} + \ddot{\varphi}) \cos(\theta + \varphi) - m'r' \dot{\theta}^2 \sin \theta$$
$$- m'r'(\dot{\theta} + \dot{\varphi})^2 \sin(\theta + \varphi) = F_1 \sin \theta + F_2 \cos \theta \tag{3.8}$$

Equations (3.7) and (3.8) may be solved for $\ddot{x}_1$ and $\ddot{x}_2$, respectively, and the results substituted into (3.6). Furthermore, a non-dimensionalized form of the equations is obtained by introducing the following quantities:

$$\omega = (g/r)^{1/2} \tag{3.9}$$

where g is the acceleration of gravity;

$$\tau = \omega t \tag{3.10}$$

$$\beta_1 = r/r' \ , \quad \beta_2 = m/m' \tag{3.11}$$

$$\beta_3 = s_1/r \ , \quad \beta_4 = s_2/r \tag{3.12}$$

$$\beta_5 = \frac{I' + m'r'^2}{m'r'^2} \ , \quad \beta_6 = \frac{I + mr^2}{mr^2} \tag{3.13}$$

$$a_1 = \frac{F_1}{(m + m')g} \ , \quad a_2 = \frac{F_2}{(m + m')g} \tag{3.14}$$

$$a_3 = \beta_1 \left\{ \left[1 + (1 + \beta_2)\ \beta_3 \right] a_2 - (1 + \beta_2)\ \beta_4\ a_1 \right\} \tag{3.15}$$

$$a_4 = [(1 + \beta_2)\ \beta_5 - 1]/\beta_1\ \beta_2 \tag{3.16}$$

$$a_5 = a_4 + \beta_1[\beta_6(1 + \beta_2) - \beta_2] \tag{3.17}$$

In terms of these quantities, the equations of motion are

$$\frac{d}{d\tau} [\theta'(a_5 - 2\cos\varphi) + \varphi'(a_4 - \cos\varphi)] = \frac{\beta_2 + 1}{\beta_2} [a_1 \sin\varphi - a_2 \cos\varphi + a_3] \tag{3.18}$$

$$\left(\frac{x_1}{r}\right)'' = a_1 \cos\theta - a_2 \sin\theta - \frac{d^2}{d\tau^2} \left[\frac{\beta_2}{1 + \beta_2} \cos\theta + \frac{\cos(\theta + \varphi)}{\beta_1(1 + \beta_2)}\right] \tag{3.19}$$

$$\left(\frac{x_2}{r}\right)'' = a_1 \sin\theta + a_2 \cos\theta - \frac{d^2}{d\tau^2} \left[\frac{\beta_2}{1 + \beta_2} \sin\theta + \frac{\sin(\theta + \varphi)}{\beta_1(1 + \beta_2)}\right] \tag{3.20}$$

where primes denote differentiation with respect to τ. Eq. (3.18) governs the rotational motion of the system, and (3.19) and (3.20) yield the position of the hingepoint P .

4. Irrotational Motion

The system is said to be in irrotational motion when θ' and φ' are identically zero. The associated constant values of φ are called equilibrium values and are denoted by φ_e . The equation governing φ_e is obtained from (3.18) by setting θ' and φ' equal to zero and φ equal to φ_e :

$$a_1 \sin \varphi_e - a_2 \cos \varphi_e + a_3 = 0 \tag{4.1}$$

It can be shown analytically and it may be clear intuitively, that the solutions to (4.1) are the values of φ for which the line of action of $\underline{F}$ passes through the center of mass of the system. Now, when φ assumes values from 0° to 360°, the center of mass of the system moves on a circle fixed in the body B (see Fig. 1); and, since the line of action of $\underline{F}$ can intersect this circle at either one or two points, or can fail to intersect the circle, there may be one, two, or zero solutions to (4.1). Knowledge of these facts facilitates the solution of (4.1) for a specific set of a_1 , a_2 , and a_3 .

In any real system, it is very difficult to make the line of action of $\underline{F}$ pass through the system mass center; and, whenever one fails to do so, the motion involves both rotation and translation. Now, it is evident from Eqs. (3.18)-(3.20) that the behavior of φ affects the motion, and this suggests that it may be possible to vary φ in such a way as to eliminate undesirable rotational motions. The remainder of this report is concerned with this topic. Specifically, the motions that result when φ is given by certain explicit functions of time are studied in Sec. 5, and Sec. 6 deals with "closed loop" behavior of the system, i.e., with situations in which φ is prescribed as a function of θ .

5. Prescription of Internal Angle

In this section, the motion of the system when φ is described by explicit functions of time is studied for two such functions. First, φ is set equal to a constant other than one of the equilibrium values φ_e . In other words, the system behaves like a rigid body subjected to the action of a misaligned thrust. Next, φ is made to oscillate about an equilibrium value, and the resulting motion is compared to that of a rigid body.

Case I. φ = constant

If the substitutions

$$\varphi = \varphi_{RB}\ , \quad \text{a constant} \tag{5.1}$$

$$\theta = \theta_{RB}$$

$$x_1 = x_{1RB}$$

$$x_2 = x_{2RB}$$

are made in (3.18)-(3.20), then the equations of motion become

$$\theta_{RB}'' = \left(\frac{\beta_2+1}{\beta_2}\right)[a_1 \sin \varphi_{RB} - a_2 \cos \varphi_{RB} + a_3]/(a_5 - 2 \cos \varphi_{RB}) \tag{5.2}$$

$$\left(\frac{x_{1RB}}{r}\right)'' = a_1 \cos \theta_{RB} - a_2 \sin \theta_{RB} - \frac{d^2}{d\tau^2}\left[\frac{\beta_2}{\beta_2+1} \cos \theta_{RB} + \frac{\cos(\theta_{RB}+\varphi_{RB})}{\beta_1(1 + \beta_2)}\right] \tag{5.3}$$

$$\left(\frac{x_{2RB}}{r}\right)'' = a_1 \sin\theta_{RB} + a_2 \cos\theta_{RB} - \frac{d^2}{d\tau^2}\left[\frac{\beta_2}{\beta_2+1} \sin\theta_{RB} + \frac{\sin(\theta_{RB}+\varphi_{RB})}{\beta_1(1+\beta_2)}\right] \quad (5.4)$$

Equations equivalent to (5.3) and (5.4), but having a simpler form, may be obtained by studying the behavior of the system's center of mass rather than that of the hingepoint. If the position vector from point 0 in Fig. 1 to the center of mass is designated $\underline{p}^*$, and x_i^* is defined as

$$x_i^* = \underline{p}^* \cdot \underline{N}_i \ , \quad i = 1,2 \quad (5.5)$$

then (5.3) and (5.4) can be replaced with

$$\left(\frac{x_1^*}{r}\right)'' = a_1 \cos\theta_{RB} - a_2 \sin\theta_{RB} \quad (5.6)$$

$$\left(\frac{x_2^*}{r}\right)'' = a_1 \sin\theta_{RB} + a_2 \cos\theta_{RB} \quad (5.7)$$

The solutions[†] to Eqs. (5.2), (5.6) and (5.7) can be expressed in terms of the quantities, Ω and Ω' , defined as

$$\Omega = \frac{1}{2}\left(\frac{\beta_2+1}{\beta_2}\right)[a_1 \sin\varphi_{RB} - a_2 \cos\varphi_{RB} + a_3]/(a_5 - 2\cos\varphi_{RB}) \quad (5.8)$$

and

$$\Omega' = |\Omega| \quad (5.9)$$

and the Fresnel integrals (see [2])

$$C(z) = \int_0^z \cos\left(\frac{\pi}{2} s^2\right) ds \quad (5.10)$$

and

$$S(z) = \int_0^z \sin\left(\frac{\pi}{2} s^2\right) ds \quad (5.11)$$

[†] A numerical solution to the rigid body problem with a thrust misalignment appears in Ref. [1].

Specifically, if

$$\theta_{RB}(0) = \theta_{RB}'(0) = x_1^*(0) = x_2^*(0) = x_1^{*\prime}(0) = x_2^{*\prime}(0) = 0$$

the solution to (5.2) is

$$\theta_{RB} = \Omega \tau^2 \qquad (5.12)$$

and substitution into (5.6) and (5.7), followed by integration, yields

$$\left(\frac{x_1^*}{r}\right)' = \sqrt{\frac{\pi}{2\Omega'}} \left[a_1 \, C\left(\sqrt{\frac{2\Omega'}{\pi}} \, \tau\right) \mp a_2 \, S\left(\sqrt{\frac{2\Omega'}{\pi}} \, \tau\right) \right] \qquad (5.13)$$

and

$$\left(\frac{x_2^*}{r}\right)' = \sqrt{\frac{\pi}{2\Omega'}} \left[\pm a_1 \, S\left(\sqrt{\frac{2\Omega'}{\pi}} \, \tau\right) + a_2 \, C\left(\sqrt{\frac{2\Omega'}{\pi}} \, \tau\right) \right] \qquad (5.14)$$

Integrating once more, one then obtains

$$\left(\frac{x_1^*}{r}\right) = \sqrt{\frac{\pi}{2\Omega'}} \, \tau \left[a_1 \, C\left(\sqrt{\frac{2\Omega'}{\pi}} \, \tau\right) \mp a_2 \, S\left(\sqrt{\frac{2\Omega'}{\pi}} \, \tau\right) \right]$$

$$- \frac{1}{2\Omega'} \left[a_1 \sin \Omega' \tau^2 \pm a_2 (\cos \Omega' \tau^2 - 1) \right] \qquad (5.15)$$

and

$$\left(\frac{x_2^*}{r}\right) = \sqrt{\frac{\pi}{2\Omega'}} \, \tau \left[\pm a_1 \, S\left(\sqrt{\frac{2\Omega'}{\pi}} \, \tau\right) + a_2 \, C\left(\sqrt{\frac{2\Omega'}{\pi}} \, \tau\right) \right]$$

$$+ \frac{1}{2\Omega'} \left[\pm a_1 (\cos \Omega' \tau^2 - 1) - a_2 \sin \Omega' \tau^2 \right] \qquad (5.16)$$

In these equations, and in those that follow, the upper sign is to be used when $\Omega > 0$, and the lower sign when $\Omega < 0$.

Since

$$\lim_{s \to \infty} C(s) = \lim_{s \to \infty} S(s) = \frac{1}{2}$$

it follows from (5.13) and (5.14) that

$$\lim_{\tau \to \infty} \left(\frac{x_1^*}{r}\right)' = \sqrt{\frac{\pi}{2\Omega'}}\ \frac{a_1 \mp a_2}{2} \tag{5.17}$$

and

$$\lim_{\tau \to \infty} \left(\frac{x_2^*}{r}\right)' = \sqrt{\frac{\pi}{2\Omega'}}\ \frac{a_2 \pm a_1}{2} \tag{5.18}$$

This means that the center of mass approaches a straight line and that its speed approaches a constant value. Parametric equations for the straight line can be obtained by replacing the Fresnel integrals in (5.15) and (5.16) by the first two terms of their asymptotic expansion for large τ (see [2]). That is, with

$$S\left(\sqrt{\frac{2\Omega'}{\pi}}\ \tau\right) \approx \frac{1}{2} - \frac{1}{\sqrt{2\Omega'\pi}\ \tau} \cos(\Omega' \tau^2) \tag{5.19}$$

and

$$C\left(\sqrt{\frac{2\Omega'}{\pi}}\ \tau\right) \approx \frac{1}{2} + \frac{1}{\sqrt{2\Omega'\pi}\ \tau} \sin(\Omega' \tau^2) \tag{5.20}$$

(5.15) and (5.16) lead to long time expressions, denoted by $\left(\frac{x_1^*}{r}\right)_L$ and $\left(\frac{x_2^*}{r}\right)_L$, for the coordinates of the center of mass:

$$\left(\frac{x_1^*}{r}\right)_L = \sqrt{\frac{\pi}{2\Omega'}}\ \left(\frac{a_1 \mp a_2}{2}\right) \tau \pm \frac{a_2}{2\Omega'} \tag{5.21}$$

and

$$\left(\frac{x_2^*}{r}\right)_L = \sqrt{\frac{\pi}{2\Omega'}}\ \left(\frac{a_2 \pm a_1}{2}\right) \tau \mp \frac{a_1}{2\Omega'} \tag{5.22}$$

Elimination of τ from (5.21) and (5.22) yields the equation for the straight line approached by the mass center:

$$\left(\frac{x_2^*}{r}\right)_L = \left(\frac{a_2 \pm a_1}{a_1 \mp a_2}\right)\left(\frac{x_1^*}{r}\right)_L + \frac{a_1^2 + a_2^2}{2\,\Omega'(a_2 \mp a_1)} \qquad (5.23)$$

When $\Omega > 0$ and

$$a_1 = a_2$$

the limiting trajectory becomes the vertical line

$$\left(\frac{x_1^*}{r}\right) = \frac{a_2}{2\,\Omega'} \qquad (5.24)$$

It can be seen from (5.23) that the slope of the limiting trajectory depend only on the components of the thrust vector. An interesting manifestation of this fact is that the limiting trajectory is always inclined at forty-five degrees to the initial direction of the thrust vector. To see this, let $\underline{j}$ and $\underline{k}$, be unit vectors defined as

$$\underline{j} = \frac{1}{\sqrt{a_1^2 + a_2^2}}\left[a_1\,\underline{N}_1 + a_2\,\underline{N}_2\right] \qquad (5.25)$$

and

$$\underline{k} = \frac{1}{\sqrt{2\left(a_1^2 + a_2^2\right)}}\left[(a_1 \mp a_2)\,\underline{N}_1 + (a_2 \pm a_1)\,\underline{N}_2\right]. \qquad (5.26)$$

The direction of $\underline{j}$ is the same as the initial direction of the thrust vector $\underline{F}$ (see (3.14)); and, from (5.17) and (5.18), it can be seen that $\underline{k}$ has the same direction as the limiting velocity vector. Now, if α denotes the smallest angle through which $\underline{j}$ must be rotated in order to coincide with $\underline{k}$, it follows from (5.25) and (5.26) that

$$\cos\alpha = \underline{j}\cdot\underline{k} = \frac{1}{\sqrt{2}} \qquad (5.27)$$

and

$$\sin \alpha = (\underline{j} \times \underline{k}) \cdot \underline{N}_3 = \pm \frac{1}{\sqrt{2}} \tag{5.28}$$

It is recalled that the applicable sign in (5.28) is the same as the sign of Ω given by (5.8). Since a dextral coordinate system has been used in this analysis, it follows from (5.12) that positive Ω implies counterclockwise increases in θ or, equivalently, positive ($+ \underline{N}_3$ direction) torque about the mass center. The opposite is true when Ω is negative. Thus it can be concluded from (5.27) and (5.28) that, for a positive torque about the center of mass, the limiting trajectory will be oriented at $+45°$ to the initial direction of the thrust vector, and, for a negative torque about the mass center, the limiting trajectory will lie at $-45°$ to the initial direction of the thrust vector.

In summary, a rigid body subjected to a misaligned thrust of constant magnitude rotates about its mass center with an increasing angular velocity. The speed of the mass center approaches a constant value, and the mass center itself approaches a straight line oriented at forty-five degrees to the initial direction of the thrust vector.

The motion just considered can be thought of as resulting from "doing nothing" when the system is subjected to a misaligned thrust. The next case to be examined deals with motions, particularly the rotational motions, that result when the system is subjected to a misaligned thrust, but φ, instead of being kept constant, is made to vary harmonically about an equilibrium value.

Case II. $\varphi = \varphi_e + \delta \cos N \tau$

If B and B' are made to move relative to each other in such a way that

$$\varphi = \varphi_e + \delta \cos N\tau \tag{5.29}$$

where δ and N are constants, and if the associated value of θ is denoted by θ_c , then (3.18) requires that

$$\frac{d}{d\tau}\Big\{\theta_c'[a_5 - 2\cos\varphi_e \cos(\delta\cos N\tau) + 2\sin\varphi_e \sin(\delta\cos N\tau)]$$

$$- \delta N \sin N\tau[a_4 - \cos\varphi_e \cos(\delta\cos N\tau) + \sin\varphi_e \sin(\delta N\cos N\tau)]\Big\}$$

$$= \frac{\beta_2 + 1}{\beta_2}\Big\{a_1 \sin\varphi_e \cos(\delta\cos N\tau) + a_1\cos\varphi_e \sin(\delta\cos N\tau)$$

$$- a_2\cos\varphi_e \cos(\delta\cos N\tau) + a_2 \sin\varphi_e \sin(\delta\cos N\tau) + a_3\Big\} \tag{5.30}$$

This equation is linear in θ_c and has variable coefficients. Numerical (digital computer) solutions for some specific examples are presented at the end of this section. An approximate analytical solution valid for small δ , is obtained as follows.

When δ is assumed to be so small that third and higher degree terms in δ may be dropped, that is,

$$\sin(\delta\cos N\tau) \approx \delta\cos N\tau \tag{5.31}$$

$$\cos(\delta\cos N\tau) \approx 1 - \frac{\delta^2\cos^2 N\tau}{2} \tag{5.32}$$

then Eq. (5.30) can be integrated once. Letting θ_{CA} denote the approximate value of θ_c thus obtained, and using the initial condition

$$\theta_{CA}'(0) = 0$$

one obtains, after simplification with the aid of (4.1),

$$\theta_{CA}' = \left\{\delta^2\left[\frac{a_2 \cos\varphi_e - a_1 \sin\varphi_e}{4} \; \frac{\beta_2+1}{\beta_2} \; \tau + \left(\frac{a_2 \cos\varphi_e - a_1 \sin\varphi_e}{8\,N} \; \frac{\beta_2+1}{\beta_2}\right.\right.\right.$$

$$\left.\left. + \frac{N \sin\varphi_e}{2}\right) \sin 2\,N\,\tau\right] + \delta\left[\frac{a_1 \cos\varphi_e + a_2 \sin\varphi_e}{N} \; \frac{\beta_2+1}{\beta_2}\right.$$

$$\left.\left. - N(\cos\varphi_e - a_4)\right] \sin N\,\tau\right\} \Big/ \left\{a_5 - 2\cos\varphi_e + 2\delta \sin\varphi_e \cos N\,\tau\right.$$

$$\left. + \delta^2 \cos\varphi_e \cos^2 N\,\tau\right\} \qquad (5.33)$$

Next, after using the binomial theorem and dropping third and higher degree terms in δ , one can integrate (5.33) subject to the condition

$$\theta_{CA}(0) = 0$$

which gives

$$\theta_{CA} = c_1\,\delta^2\,\tau^2 + \delta\left(\frac{c_2}{N^2} - c_3\right)\left(1 - \cos N\,\tau\right) + \delta^2\left(\frac{c_4}{N^2} - c_5\right) \sin^2 N\,\tau \qquad (5.34)$$

where

$$c_1 = \frac{a_2 \cos\varphi_e - a_1 \sin\varphi_e}{8(a_5 - 2\cos\varphi_e)} \; \frac{\beta_2+1}{\beta_2} \qquad (5.35)$$

$$c_2 = \frac{a_1 \cos\varphi_e + a_2 \sin\varphi_e}{(a_5 - 2\cos\varphi_e)} \; \frac{\beta_2+1}{\beta_2} \qquad (5.36)$$

$$c_3 = \frac{\cos\varphi_e - a_4}{(a_5 - 2\cos\varphi_e)} \qquad (5.37)$$

$$c_4 = c_1 - \frac{\sin\varphi_e}{(a_5 - 2\cos\varphi_e)} \; c_2 \qquad (5.38)$$

$$c_5 = -\frac{\sin\varphi_e}{2(a_5 - 2\cos\varphi_e)} \; (1 + 2\,c_3) \qquad (5.39)$$

The leading term in (5.34) indicates that, in general, θ_{CA} is unbounded. Note that this term involves δ^2 , so that, if terms linear in δ were retained, only an oscillation about a constant mean value would be predicted. As will be seen later, such a result does not agree with the solution obtained by numerical integration of (3.18).

Consider now the limiting case $N = 0$, that is (see (5.29)) situations in which φ has a constant value that differs by an amount δ from the equilibrium value φ_e . Expanding the last two terms of (5.34) in powers of $N\tau$, letting N approach zero, and calling the result θ_{RBA} , one obtains

$$\theta_{RBA} = \lim_{N \to 0} \theta_{CA} = \left[\frac{-c_2}{2} \delta + (c_1 + c_4) \delta^2\right] \tau^2 \qquad (5.40)$$

(The same result can be deduced from (5.12) after setting φ_{RB} equal to $\varphi_e + \delta$ in (5.8) and expanding the result in powers of δ .)

We shall return to (5.40) presently. First, however, we wish to draw attention to the fact that, when N and c_1 differ from zero, θ_{CA} depends primarily on the leading term in (5.34); and, as c_1 is independent of N (see (5.35)), the growth of θ_{CA} is then relatively insensitive to the numerical value of N .

Eqs. (5.34) and (5.40) can be used to determine the configurations which involve minimum growth in θ_{CA} and θ_{RBA} in the presence of a misalignment of amount δ . Specifically, as δ is a small quantity, it follows from (5.40) that θ_{RBA} is minimized when c_2 is equal to zero, that is (see (3.11) and (5.36)), when

$$a_1 \cos \varphi_e + a_2 \sin \varphi_e = 0 \qquad (5.41)$$

Now, from Fig. 1 and Eqs. (2.3) and (3.14),

$$a_1 \cos \varphi_e + a_2 \sin \varphi_e = \frac{(\underline{F} \cdot \underline{r}')_{\varphi = \varphi_e}}{(m + m') \, gr'} \qquad (5.42)$$

To minimize the growth of θ_{RBA} , it is thus necessary to make

$$(\underline{F} \cdot \underline{r}')_{\varphi=\varphi_e} = 0 \qquad (5.43)$$

which means that the thrust vector must be perpendicular to $\underline{r}'$ when it passes through the mass center of the system (i.e., when $\varphi = \varphi_e$). It is worth noting that, even when the growth of θ_{RBA} is minimized as here indicated, the limit approached by the ratio θ_{RBA}/θ_{CA} as τ approaches infinity has the value

$$\lim_{\tau \to \infty} \left(\frac{\theta_{RBA}}{\theta_{CA}}\right) = 2 \qquad (5.44)$$

which means that θ_{RBA} eventually becomes twice as large as θ_{CA} . The efficacy of performing the oscillatory motion described by (5.29) is thus apparent.

The growth in θ_{CA} is minimized when c_1 is zero. From (5.35), this occurs when

$$a_1 \cos \varphi_e - a_1 \sin \varphi_e = 0 \qquad (5.45)$$

or, in view of (2.3) and (3.14), when

$$(\underline{r}' \times \underline{F})_{\varphi=\varphi_e} = 0 \qquad (5.46)$$

In other words, when the thrust vector passes through the system's mass center, it must be parallel to $\underline{r}'$. The conditions which minimize the growth rates in θ_{RBA} and θ_{CA} are thus mutually exclusive.

When c_1 is equal to zero, so that θ_{CA} (see (5.34)) remains small, an approximate description of the motion of the hingepoint P is obtained by replacing θ with θ_{CA} in (3.19) and (3.20), expanding in powers of δ, dropping third and higher degree terms, and integrating the resulting differential equations. With the initial conditions

$$x_{1A}(0) = x_{2A}(0) = x_{1A}'(0) = x_{2A}'(0) \tag{5.47}$$

this leads to

$$\frac{x_{1A}}{r} = G_{11}\,\tau^2 + G_{12}(\cos N\tau - 1) + G_{13}(\cos^2 N\tau - 1) + G_{14}\sin^2 N\tau \tag{5.48}$$

$$\frac{x_{2A}}{r} = G_{21}\,\tau^2 + G_{22}(\cos N\tau - 1) + G_{23}(\cos^2 N\tau - 1) + G_{24}\sin^2 N\tau \tag{5.49}$$

where

$$G_{11} = \frac{1}{2}\left\{a_1\left[1 - \frac{3}{4}\delta^2\left(\frac{c_2}{N^2} - c_3\right)^2\right] - a_2\left[\delta\left(\frac{c_2}{N^2} - c_3\right) + \frac{\delta^2}{2}\left(\frac{c_4}{N^2} - c_5\right)\right]\right\} \tag{5.50}$$

$$G_{12} = \frac{-\delta^2\left(\frac{c_2}{N^2} - c_3\right)^2}{N^2\,\beta_1(1 + \beta_2)}\left[a_1\,\beta_1(1 + \beta_2) + N^2\,\beta_1\,\beta_2 + N^2\cos\varphi_e\right] + \frac{\delta\sin\varphi_e}{\beta_1(1+\beta_2)} \tag{5.51}$$

$$G_{13} = \frac{\delta^2\left(\frac{c_2}{N^2} - c_3\right)^2}{2\beta_1(1 + \beta_2)}(\beta_1\,\beta_2 + \cos\varphi_e) + \frac{\delta^2\cos\varphi_e}{2\beta_1(1 + \beta_2)}\left(1 - \frac{2c_2}{N^2} + 2\,c_3\right) \tag{5.52}$$

$$G_{14} = \frac{\delta^2\left(\frac{c_4}{N^2} - c_5\right)}{2N^2\,\beta_1(1 + \beta_2)}\left[a_1\,\beta_1(1 + \beta_2) + 2N^2\sin\varphi_e\right] - \left(\frac{c_2}{N^2} - c_3\right)^2\frac{\delta^2\,a_1}{4N^2} \tag{5.53}$$

$$G_{21} = \frac{1}{2}\left\{a_2\left[1 - \frac{3}{4}\delta^2\left(\frac{c_2}{N^2} - c_3\right)^2\right] + a_1\left[\delta\left(\frac{c_2}{N^2} - c_3\right) + \frac{\delta^2}{2}\left(\frac{c_4}{N^2} - c_5\right)\right]\right\} \tag{5.54}$$

$$G_{22} = \frac{-\delta^2\left(\frac{c_2}{N^2} - c_3\right)^2}{N^2\ \beta_1(1+\beta_2)}\ [a_2\ \beta_1(1+\beta_2) + N^2 \sin\varphi_e] + \frac{\delta \cos\varphi_e}{\beta_1(1+\beta_2)}$$

$$+ \frac{\delta\left(\frac{c_2}{N^2} - c_3\right)}{N^2\ \beta_1(1+\beta_2)}\ [a_1\ \beta_1(1+\beta_2) + \beta_1\ \beta_2\ N^2 + N^2(\cos\varphi_e + \delta \sin\varphi_e)] \quad (5.55)$$

$$G_{23} = \frac{\delta^2 \sin\varphi_e}{2\beta_1(1+\beta_2)}\left[1 - \frac{2c_2}{N^2} + 2\ c_3\right]^2 \quad (5.56)$$

$$G_{24} = \frac{\delta^2\left(\frac{c_4}{N^2} - c_5\right)}{2N^2\ \beta_1(1+\beta_2)}\ [a_1\ \beta_1(1+\beta_2) - 2\ \beta_1\ \beta_2\ N^2 + 2N^2 \cos\varphi_e]$$

$$+ \frac{\delta^2\ a_2}{4N^2}\left(\frac{c_2}{N^2} - c_3\right) \quad (5.57)$$

The accuracy of (5.48) and (5.49) will be discussed in connection with the presentation of numerical results.

Numerical Results

In this section numerical results are presented for a human being modelled as in [3]. Body B now represents the head, torso and legs of a man in a position of "attention," and B' consists of the arms of the subject, these being required to move in unison in planes parallel to the pitch plane.

The relevant inertia properties of the system, taken from page 15 of [3], are shown in Table 1. In order to evaluate the dimensionless parameters defined in (3.11)-(3.17), one must also specify $\underline{F}$ and $\underline{s}$ (see Fig. 1). $\underline{F}$, which is to represent the action of a thruster, is chosen as

Table 1

symbol	value	units
B' (Arms)		
m'	0.576	slugs
r'	0.903	ft.
I'	0.265	slug-ft.^{2}
B (Head, Torso and Legs in position of Attention)		
m	4.458	slugs
r	1.481	ft.
I	8.150	slug-ft.^{2}

$$\underline{F} = \lambda(m + m')\ g\ \underline{n}_2 \tag{5.58}$$

where λ is a constant which determines the "thrust level." It follows from (2.3) and (3.14) that

$$a_1 = 0 \tag{5.59}$$

and

$$a_2 = \lambda \tag{5.60}$$

Two values of λ will be used: $\lambda = 1$, which means that the thrust has a magnitude equal to the weight of the subject; and $\lambda = 0.031$, which corresponds to a thrust that imparts an acceleration of one foot per second, per second to the mass center of a rigid body of mass $m + m'$. These two cases will be referred to as "high thrust" and "low thrust," respectively.

As a consequence of (5.59) and the fact that β_4 is multiplied by a_1 in (3.15), it is not necessary to determine the numerical value of β_4. This, in turn, means that s_2 (see (3.12)) need not be specified. As for s_1, we shall consider two values: The first value is selected in such a way that the line of action of $\underline{F}$ passes through the mass center of the system when $\varphi = 0$; that is, s_1 is chosen so that $\varphi_e = 0$ is a solution to (4.1). Accordingly,

$$s_1 = \frac{m'(r' - r)}{m + m'} = -\ 0.066 \qquad (5.61)$$

and we shall refer to this case as "Arms Down." The second value of s_1 is one such that the line of action of $\underline{F}$ passes through the mass center of the system when $\varphi = 90$ degrees. In this case, $\varphi_e = \pi/2$ is a solution to (4.1); that is,

$$s_1 = -\ \frac{m'\ r}{m + m'} = -\ 0.169 \qquad (5.62)$$

and we shall designate this as "Arms Up."

All of the quantities defined in (3.11)-(3.17) can now be evaluated by reference to Table 1 and (5.59)-(5.62). For the four cases to be considered, the results are shown in Table 2. The value of ω is the same in all four cases and, from (3.9) and Table 1, is found to be

$$\omega = \left(\frac{32.174}{1.481}\right)^{1/2} = 4.661\ \text{sec}^{-1} \qquad (5.63)$$

Before (3.18)-(3.20) can be integrated numerically, values must be assigned to δ and N. The value of δ, which determines both the initial thrust misalignment and the amplitude with which the arms oscillate, is chosen to be

Table 2

		β_1	β_2	β_3	β_4	β_5	β_6	a_1	a_2	a_3	a_4	a_5
HIGH THRUST	Arms Down	1.640	7.740	-0.45	-	1.564	1.834	0	1	1	.998	14.592
HIGH THRUST	Arms Up	1.640	7.740	-.114	-	1.564	1.834	0	1	0	.998	14.592
LOW THRUST	Arms Down	1.640	7.720	-0.45	-	1.564	1.834	0	.031	.031	.998	14.592
LOW THRUST	Arms Up	1.640	7.740	-.114	-	1.564	1.834	0	.031	0	.998	14.592

$$\delta = 0.1 \text{ radian} \tag{5.64}$$

and the value of N is selected so as to yield a frequency of one cycle per second. In view of (5.63), the required value of N is thus

$$N = \frac{2\pi}{4.661} = 1.348 \tag{5.65}$$

In Figs. 2 and 3, θ is plotted as a function of τ, and Figs. 4 and 5 show x_2 as a function of x_1. Figs. 2 and 4 deal with the motion associated with high thrust, whereas Figs. 3 and 5 correspond to low thrust. For each thrust level, four situations are considered. The curves labelled "Arms Oscillating" are generated by substituting the "Arms Down" and "Arms Up" values of φ_e and the values of δ and

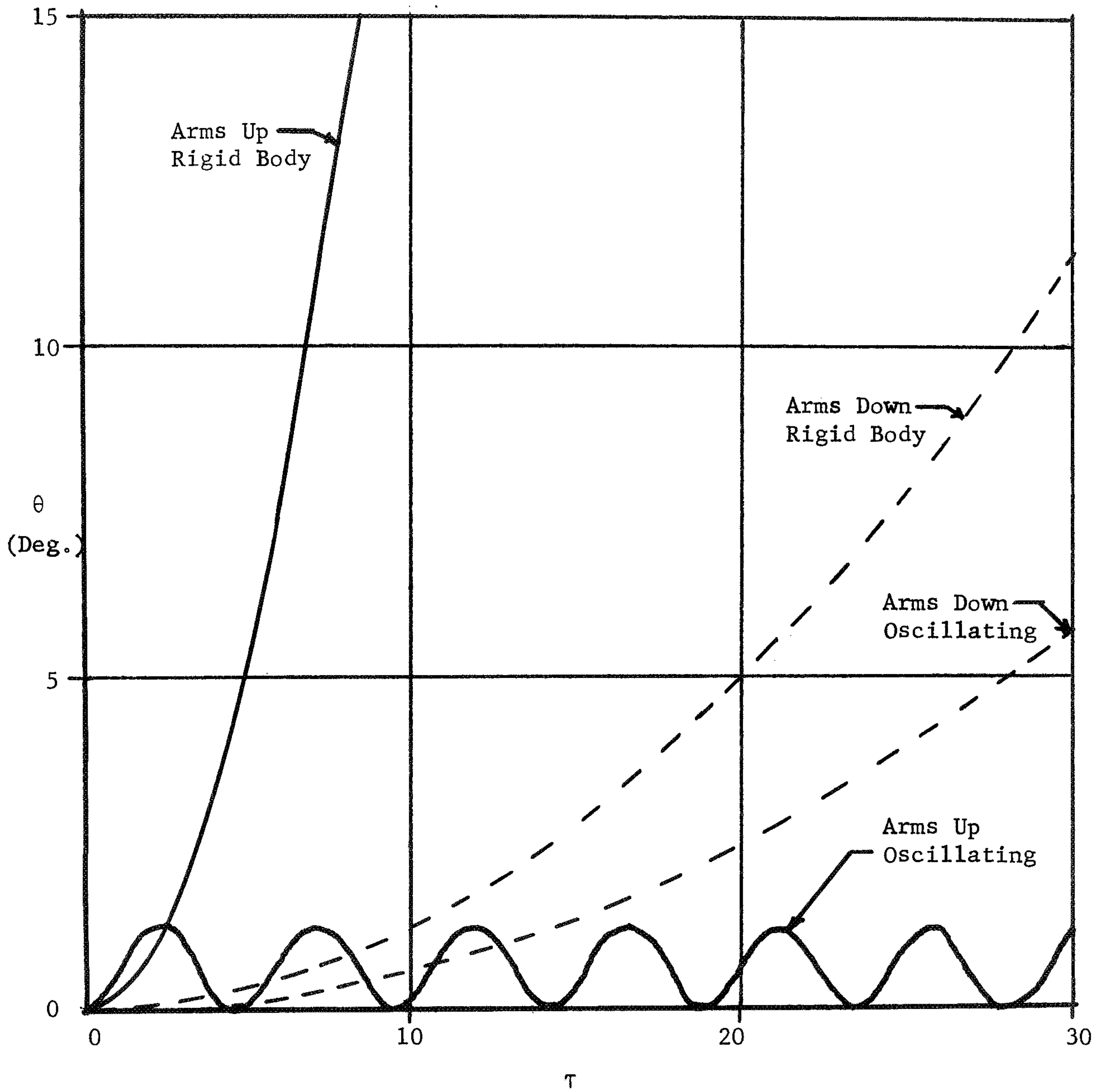

Fig. 2 Rotation Angle, High Thrust

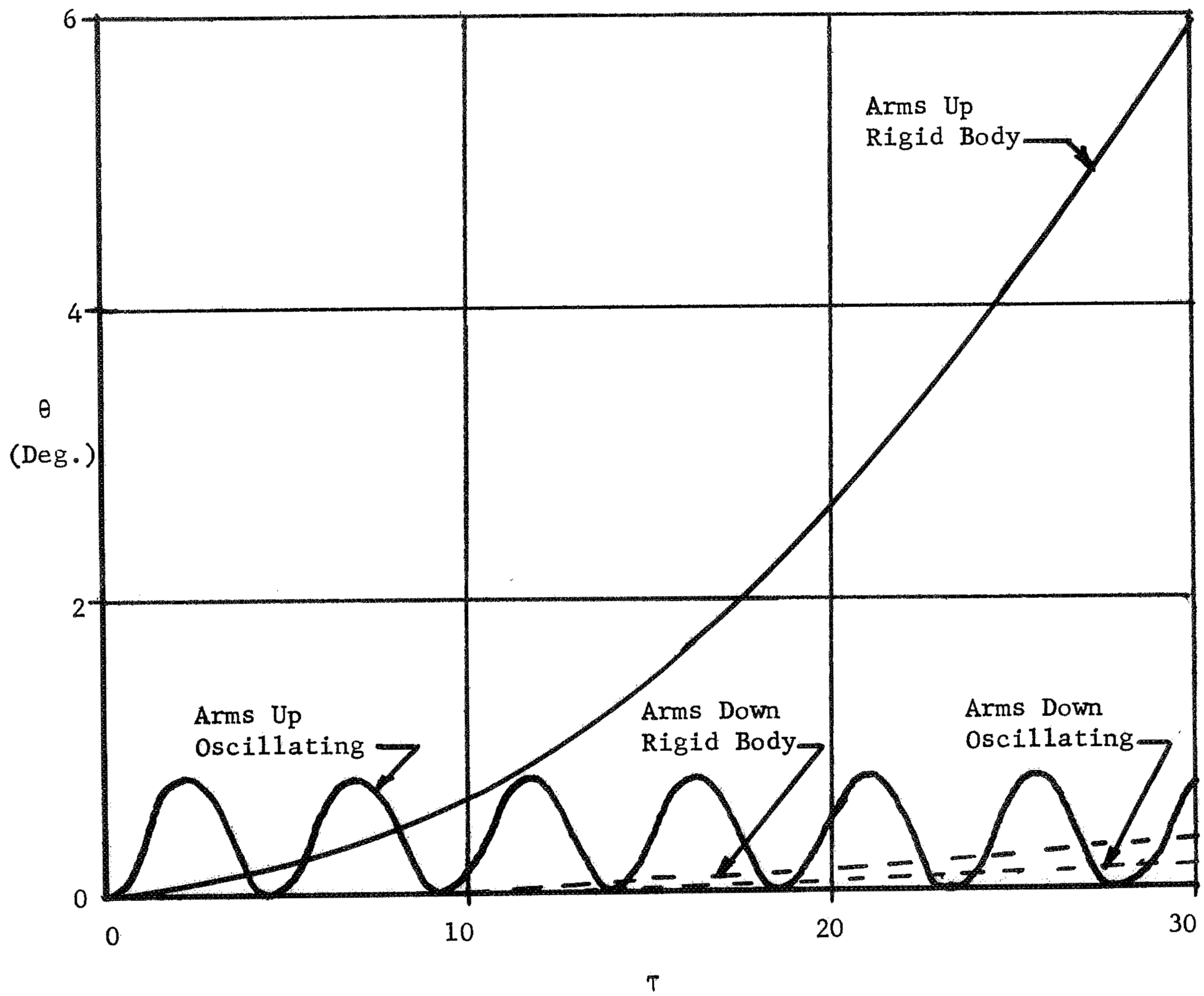

Fig. 3 Rotation Angle, Low Thrust

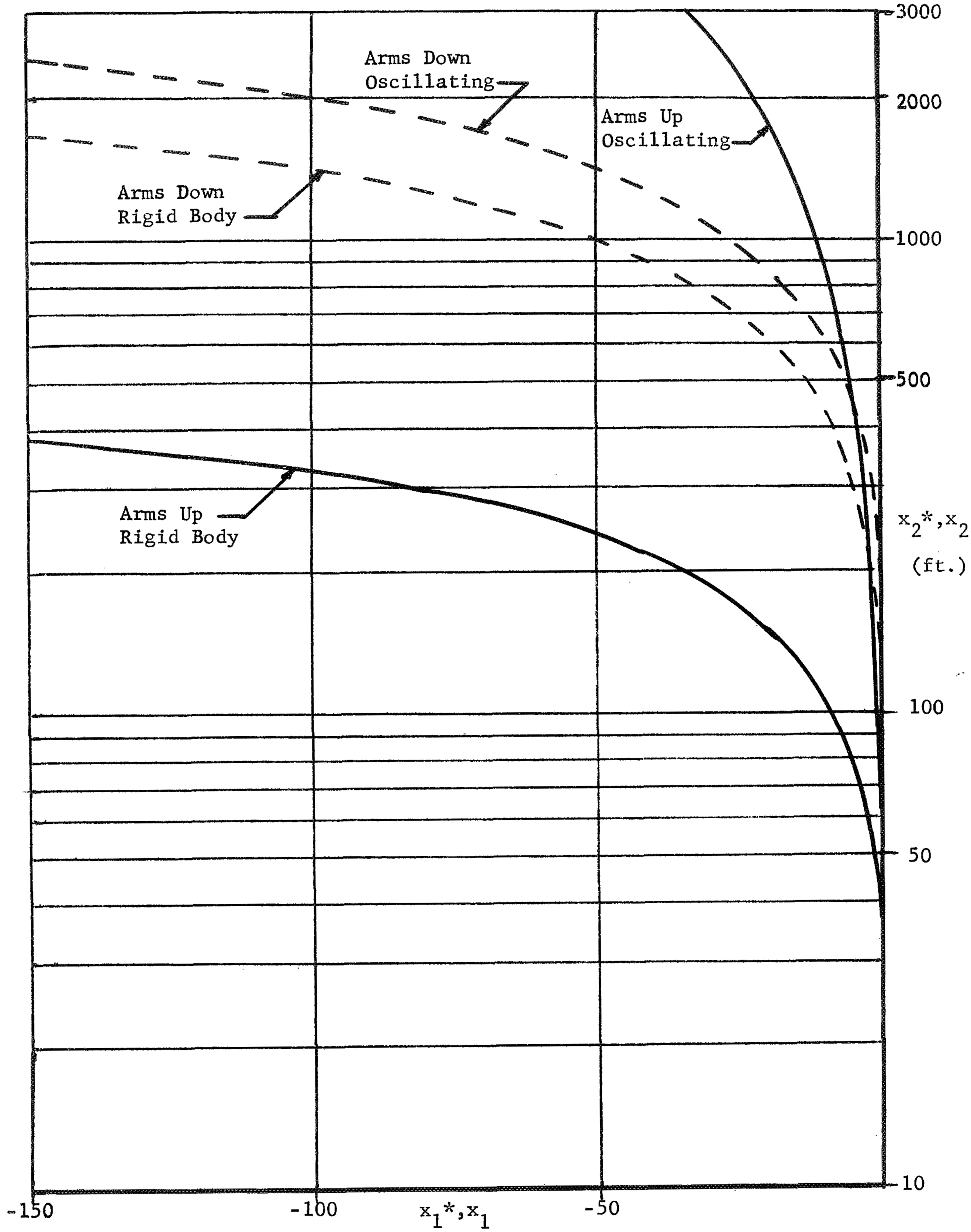

Fig. 4 Flight Path, High Thrust

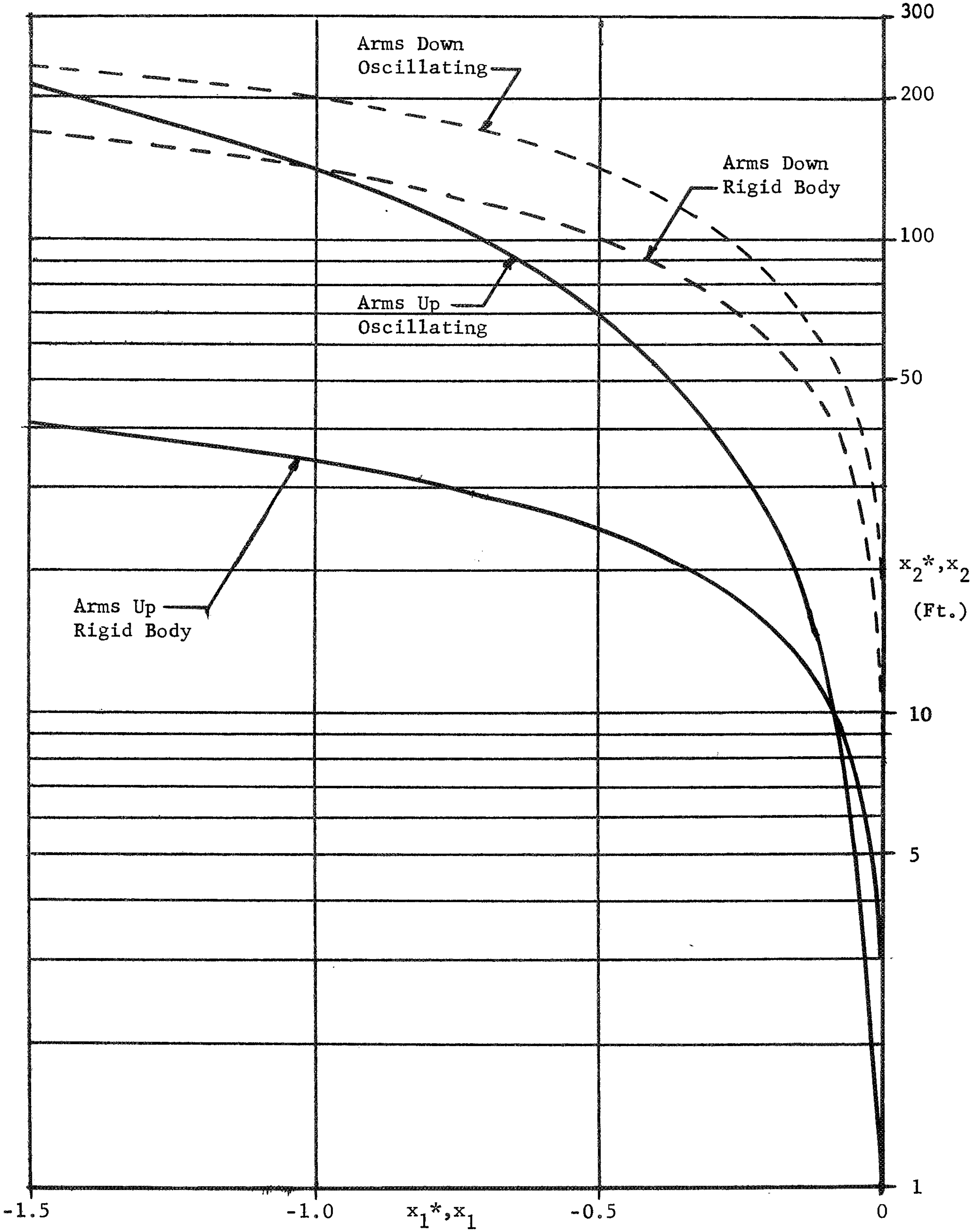

Fig. 5 Flight Path, Low Thrust

N from (5.64) and (5.65) into (5.29), which gives

$$\varphi = 0.0 + 0.1\ \cos(1.348\ \tau) \tag{5.66}$$

and

$$\varphi = \pi/2 + 0.1\ \cos(1.348\ \tau) \tag{5.67}$$

respectively, and then performing numerical integrations of (3.18)-(3.20). To obtain the curves labeled "Rigid Body," N is set equal to zero in (5.29), which leads to

$$\varphi = \varphi_{RB} = 0.0 + 0.1 \quad \text{(Arms Down)} \tag{5.68}$$

and

$$\varphi = \varphi_{RB} = \pi/2 + 0.1 \quad \text{(Arms Up)} \tag{5.69}$$

respectively. These two equations describe situations in which the arms are constantly misaligned by 0.1 radian from the "Arms Down" and "Arms Up" equilibrium positions, respectively. Substitution into (5.2), (5.6) and (5.7), and numerical integration, then yields the desired information.

Note that, in Figs. 4 and 5, the "Arms Oscillating" curves involve the coordinates of the hingepoint, x_1 and x_2 , while the "Rigid Body" curves apply to the coordinates of the mass center of the system, x_1* and x_2* .

All integrations were performed with zero initial values for the dependent variables and their first derivatives. Thus, the subject is initially at rest with the spine essentially parallel to the x_1 axis and the head on the negative side of the origin. Furthermore, the thrust vector is initially pointed in the positive x_2 direction.

Before evaluating the results in Figs. 2-5, it is worth noting that the "Rigid Body-Arms Down" case deals with a situation in which (5.43) is satisfied, so that the amount of rigid body rotation should be minimized. Further, in the "Arms Oscillating-Arms Up" mode of flight, (5.46) is satisfied. Hence, according to the approximate theory, this mode of flight should yield smaller growth rates than those resulting from "Arms Oscillating-Arms Down."

When studying the results shown in Figs. 2 and 3, it is convenient to begin by considering the broken and solid curves separately. Thus it becomes evident that, for both thrust levels and for both arm positions, less turning is associated with oscillatory arm motions than with keeping the arms at rest. Comparison of the two cases involving oscillation of the arms shows "Arms Down" eventually results in more rotation than "Arms Up." However, at both thrust levels, there is an initial time interval during which both "Rigid Body-Arms Down" and "Arms Oscillating-Arms Down" result in less rotation than "Arms Oscillating-Arms Up."

For the time interval covered in Figs. 2 and 3, the predictions of the approximate theory are verified. That is, oscillation of the arms in the "Arms Up" position is seen to result in a constant mean value for θ . The approximate theory also predicts that the amount of rotation associated with "Arms Oscillating-Arms Down" is insensitive to the frequency at which the arms oscillate. To check this, integrations were performed for a frequency of five cycles per second, and, for the "Arms Oscillating-Arms Down" configuration, this led to curves identical to those in Figs. 2 and 3. For "Arms Oscillating-Arms Up," the higher frequency resulted in a higher frequency oscillation for θ about a constant mean value that was slightly smaller than the mean value for one cycle per second. As for checks on the approximate rigid body relations (5.40) and (5.44), Figs. 2 and 3 verify that "Rigid Body-Arms Down" results in considerably less rotation than "Rigid Body-Arms Up" and that the values of θ for the former case eventually becomes twice as large as that for "Arms Oscillating-Arms Down."

Evaluation of the results shown in Figs. 4 and 5 will be made in light of the following considerations: First, keeping the initial state of the system in mind, it may be presumed that it is the subject's intention to approach a target located on the x_2 axis. Furthermore, one may expect the high thrust level to be used to traverse relatively long distances, and the low thrust level for comparatively shorter distances. Fig. 4 can then be used to determine the best mode of flight for approaching a distant target, while Fig. 5 deals with more precise, short distance approaches. It then appears from Fig. 4 that, when one wishes to approach a target at a distance of more than five hundred feet, it is advantageous to fly in the "Arms Oscillating-Arms Up" configuration. For example, a subject wishing to use the high thrust level to propel himself along the x_2 axis toward a target a thousand feet away would find himself (after moving a thousand feet in the x_2 direction) at the following distances "above" the target (i.e., negative x_1 coordinate), depending on the mode of flight:

"Rigid Body-Arms Down," $x_1{*} \approx -50$ ft.

"Arms Oscillating-Arms Down," $x_1 \approx -25$ ft.

"Arms Oscillating-Arms Up," $x_1 \approx -12$ ft.

For distances of less than one hundred feet, Fig. 5 indicates that it is best to fly in the "Arms Oscillating-Arms Down" configuration. Furthermore, for such short distances, it is better to fly "Rigid Body-Arms Down" than "Arms Oscillating-Arms Up." For example, after moving one hundred feet in the x_2 direction at the low thrust level, the following drifts in the negative x_1 direction occur:

"Arms Oscillating-Arms Up," $x_1 \approx -.75$ ft.

"Rigid Body-Arms Down," $x_1^* \approx -.50$ ft.

"Arms Oscillating-Arms Down," $x_1 \approx -.25$ ft.

The amounts of drift in these cases are much smaller than those for long distance mission at high thrust, but they are nevertheless important when one is concerned with precision maneuvering in the vicinity of a target.

As stated earlier, the curves shown in Figs. 2-5 were obtained by numerically integrating the equations of motion. However, most of these results may be reproduced without numerical integration, by using the equations developed earlier in this section. That is, the rigid body curves may be generated exactly with the aid of (5.12), (5.15), and (5.16); Eq. (5.34) produces results that agree with the "Arms Oscillating" curves of Figs. 2 and 3 to within .05 degrees; and the "Arms Oscillating-Arms Up" curves in Figs. 4 and 5 are identical to those resulting from (5.48) and (5.49).

In conclusion, it is now possible to make some general remarks regarding the planar motion of a man with a body-fixed thruster. As noted earlier, it is highly probable that there will be a small thrust misalignment when flight is initiated. If nothing is done to compensate for this misalignment, the subject will eventually move with a continually increasing angular velocity along a path which approaches a line oriented at forty-five degrees to the intended line of motion. However, by simply performing oscillatory arm motions of small amplitude and arbitrary frequency, the subject can decrease the undesirable rotation by at least a factor of two. Indeed, if the system is properly designed it is possible to eliminate the rotation almost completely. The possibility of obtaining still better results by using "feedback," that is, by making φ a function of θ or a function of both θ and θ', is examined in the next section.

6. "Closed Loop" Behavior

Assuming that the system is capable of performing relative motions more complicated than a harmonic oscillation, two problems will be studied: First, it will be assumed that the angle θ can be monitored and that the angle φ can be kept equal to the sum of a constant and a multiple of θ. Second, φ will be set equal to a linear function of both θ and θ'. Such control laws can certainly govern the behavior of a mechanical system. In dealing with a man, however, questions arise regarding the extent to which the subject can, in fact, perform as required. Some work has been done in this area ([4]), but much is still unknown about man's capability as both a sensor and an actuator in a feedback system. We shall investigate the motions associated with the aforementioned feedback laws without regard to man's actual ability to perform the necessary sensing and actuating functions.

Case I. $\varphi = \varphi_o + c\,\theta$

If

$$\varphi = \varphi_o + c\,\theta \tag{6.1}$$

where φ_o and c are constants, then (3.18) becomes

$$\frac{d}{d\tau}\left\{\theta'[a_5 + ca_4 - (2 + c)\cos(\varphi_o + c\,\theta)]\right\}$$

$$= \frac{\beta_2 + 1}{\beta_2}\,[(a_1 \sin\varphi_o - a_2 \cos\varphi_o)\cos c\,\theta + a_3$$

$$+ (a_1 \cos\varphi_o + a_2 \sin\varphi_o)\sin c\,\theta] \tag{6.2}$$

and, when $\theta = 0$, this reduces to

$$a_1 \sin\varphi_o - a_2 \cos\varphi_o + a_3 = 0 \tag{6.3}$$

Comparison with (4.1) shows that φ_o must be one of the equilibrium values φ_e.

Letting φ_o equal φ_e in (6.2), and linearizing in θ, one obtains after replacing θ with ξ_1

$$\xi_1'' + k\,\xi_1 = 0 \tag{6.4}$$

where k is defined as

$$k = \left(\frac{\beta_2 + 1}{\beta_2}\right)\frac{c(a_1 \cos\varphi_e + a_2 \sin\varphi_e)}{c(\cos\varphi_e - a_4) - (a_5 - 2\cos\varphi_e)} \tag{6.5}$$

From (6.4) it appears that θ can remain small only if

$$k > 0 \tag{6.6}$$

Furthermore, if a_6, a_7, and e are defined as

$$a_6 = \frac{\beta_2 + 1}{\beta_2} \; \frac{a_1 \cos \varphi_e + a_2 \sin \varphi_e}{\cos \varphi_e - a_4} \tag{6.7}$$

$$a_7 = \frac{a_5 - 2 \cos \varphi_e}{a_4 - \cos \varphi_e} \tag{6.8}$$

$$e = c/a_7 \tag{6.9}$$

the condition expressed in (6.6) can be re-stated as

$$a_6 \left(\frac{e}{1 + e}\right) > 0 \tag{6.10}$$

Systems for which (6.10) is satisfied will be called "stable," and those for which (6.10) is not satisfied will be termed "unstable."

When a_6 and a_7 are known, (6.9) and (6.10) permit one to determine values of c that result in a stable system.

For stable systems, the general solution of (6.4) is

$$\xi_1 = A \cos \sqrt{k}\, \tau + B \sin \sqrt{k}\, \tau \tag{6.11}$$

where A and B are arbitrary constants. In order to accommodate an initial thrust misalignment, the initial conditions

$$\xi_1(0) = \delta \tag{6.12}$$

$$\xi_1'(0) = 0 \tag{6.13}$$

are used, and this leads to

$$\xi_1 = \delta \cos \sqrt{k}\, \tau \tag{6.14}$$

With φ_o equal to φ_e , it then follows from (6.1) that

$$\varphi = \varphi_e + c\, \delta \cos \sqrt{k}\, \tau \tag{6.15}$$

Eq. (6.15) can be used to select a reasonable value of c from those that meet the requirements imposed by (6.10); that is, if the system represents a man, one can chose a value of c such that the frequency associated with (6.15) is physically realizable.

Equations governing the coordinates of the hingepoint P are obtained by substituting (6.1) into (3.19) and (3.20), replacing φ_o and θ with φ_e and ξ_1, respectively, and linearizing in ξ_1, which leads to

$$\frac{x_1''}{r} = a_1 - a_2 \xi_1 + \frac{(1 + c) \sin \varphi_e}{\beta_1(1 + \beta_2)} \xi_1'' \tag{6.16}$$

and

$$\frac{x_2''}{r} = a_2 + a_1 \xi_1 - \frac{[\beta_1 \beta_2 + (1 + c) \cos \varphi_e]}{\beta_1(1 + \beta_2)} \xi_1'' \tag{6.17}$$

Two integrations of (6.16) and (6.17) with the initial conditions

$$x_1(0) = x_1'(0) = x_2(0) = x_2'(0) = 0 \tag{6.18}$$

result in the following equations for the hingepoint coordinates:

$$\frac{x_1}{r} = \frac{a_1 \tau^2}{2} + \delta\left[\frac{a_2}{k} + \frac{(1 + c) \sin \varphi_e}{\beta_1(1 + \beta_2)}\right](1 - \cos \sqrt{k}\ \tau) \tag{6.19}$$

$$\frac{x_2}{r} = \frac{a_2 \tau^2}{2} + \delta\left[\frac{a_1}{k} + \frac{\beta_1 \beta_2 + (1 + c) \cos \varphi_e}{\beta_1(1 + \beta_2)}\right](1 - \cos \sqrt{k}\ \tau) \tag{6.20}$$

The leading term in (6.19) and in (6.20) characterizes the behavior of the hingepoint in the absence of initial thrust misalignment, that is, when $\delta = 0$. As the remaining term in (6.19) and (6.20) are bounded, it can be concluded that, for stable systems utilizing the control law given in (6.1), the distance between the intended and the actual hingepoint location remains permanently bounded. This represents an improvement over the results obtained by letting one body oscillate relative to the other at an arbitrary frequency (see Sec. 5).

Eq. (6.14) shows that the amount of turning is bounded but that the oscillations never disappear. By including θ' in the feedback law, it becomes possible to effect an attenuation of the rotational motion.

<u>Case II. $\varphi = \varphi_o + c\,\theta + d\,\theta'$</u>

If the substitution

$$\varphi = \varphi_o + c\,\theta + d\,\theta' \tag{6.21}$$

where φ_o , c and d are constants, is made in (3.18), one obtains

$$\frac{d}{d\tau}\left\{\theta'[a_5 - 2\cos(\varphi_o + c\,\theta + d\,\theta')] + (c\,\theta' + d\,\theta'')[a_5 - \cos(\varphi_o + c\,\theta + d\,\theta')]\right\}$$
$$= \frac{\beta_2 + 1}{\beta_2}\left\{a_1 \sin(\varphi_o + c\,\theta + d\,\theta') - a_2 \cos(\varphi_o + c\,\theta + d\,\theta') + a_3\right\} \tag{6.22}$$

and when $\theta = 0$, it is again necessary that

$$\varphi_o = \varphi_e$$

After replacing φ_o with φ_e , linearizing in θ , and replacing θ with ξ_2 , one now finds that

$$\xi_2''' + A_1\,\xi_2'' + A_2\,\xi_2' + A_3\,\xi_2 = 0 \tag{6.23}$$

where A_1 , A_2 , and A_3 are defined as

$$A_1 = \frac{a_7 + c}{d} \tag{6.24}$$

$$A_2 = a_6 \tag{6.25}$$

$$A_3 = \frac{c}{d} a_6 \tag{6.26}$$

The character of the solution of (6.23) depends on A_1 , A_2 , and A_3 . As before, bounded solutions will be called "stable" and unbounded solutions "unstable."

The following are necessary and sufficient conditions for stability:

$$A_2 > 0 \tag{6.27}$$

$$A_3 > 0 \tag{6.28}$$

$$A_1 A_2 - A_3 > 0 \tag{6.29}$$

When (6.24)-(6.26) are substituted into (6.27)-(6.29), the necessary and sufficient conditions for stability can be expressed as

$$a_6 > 0 \tag{6.30}$$

$$c/d > 0 \tag{6.31}$$

$$d/a_7 > 0 \tag{6.32}$$

Numerical Results

Returning to the human model in Sec. 5, it is recalled that the "Arms Up" and "Arms Down" equilibrium values are $\varphi_e = \pi/2$ and $\varphi_e = 0$, respectively. It follows from (6.7) and Table 2 that a_6 is negative for "Arms Up" and zero for "Arms Down." Hence it must be concluded from (6.30) that both configurations are unstable when φ is given by (6.21). If φ is given by (6.1), it follows from (6.10) that "Arms Down" is necessarily unstable, whereas for "Arms Up," (6.8)-(6.10) lead to

$$\frac{c}{14.621 + c} < 0$$

or

$$-14.621 < c < 0$$

so that this configuration can be stable if c is chosen properly.

It is not difficult to find configurations of the human model for which the feedback law of (6.21) results in a stable system. For example, if the direction of the thrust vector is reversed, so that the man flies backwards, then a_6 and a_7 (see (6.7) and (6.8)) are positive for the "Arms Up" configuration, and (6.30)-(6.32) show that the system is stable whenever both c and d are positive.

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