Semi-annual Status Report
for NASA Grant NGR 22-010-018
February, 1970

Summary

Efforts on NASA Grant NGR-22-010-018 for the six month period ending February 1, 1969 were directed toward two problems: (a) determining the statistical properties of the output of a Neural Pulse Frequency Modulator (NPFM) when the input is a zero mean stationary Gaussian random process which is exponentially autocorrelated; and (b) finding a suitable approach to the stability analysis problem in sampled data closed loop control systems which contain a NPF modulator.

Details of progress made on items (a) and (b) above are given in sections I and II respectively. The problem mentioned in (b) has been essentially solved. A paper on this problem is being prepared for presentation at the Midwest Symposium on Circuit Theory at the University of Minnesota in May 1970. A copy of this paper will be forwarded in the near future.

During this reporting period, two faculty members devoted 10% of their time to this project, and two graduate students spent 50% of their time on it. The projected level of activity for the next six month period is the same as that described above for this reporting period. The statistical analysis will continue and research will be started in Optimum Pulse Frequency Modulated Control Systems.

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I. STATISTICAL ANALYSIS

If the input to the NPFM unit is a random process the output will be a train of impulses where the sign of the impulse and the times of occurrence are random variables. Mathematically, this phenomenon may be described by the equation

$$y(t) = \sum_{n=-\infty}^{\infty} \alpha_n \delta(t - t_n) \tag{1}$$

where α_n is a discrete random variable with outcome $\pm$ 1, t_n is a random point process, and $\delta(t - t_n)$ is a Dirac delta function.

The problem of interest is to determine the statistical properties of output, $y(t)$, when the input, $u(t)$, is a zero mean stationary Gaussian random process which is exponentially autocorrelated.

In the study of the statistical properties of the processes typified by Eq. (1), the problem of calculating the average number, β, of impulses in unit time naturally arises [1]. To determine β the concept of the first passage time of Brownian motion [2] is found useful.

The input signal, $u(t)$, is generated by passing white noise, $w(t)$, through a single pole linear filter. Figure 1 shows the block diagram of this system, where k and c are constants, and r is the threshold level of the pulsing circuit.

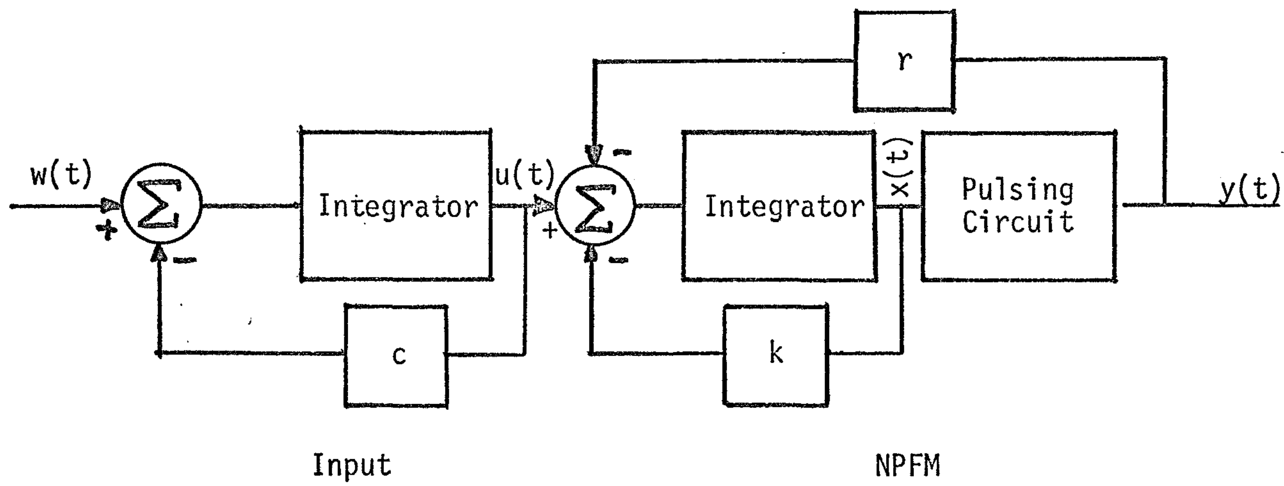

Figure 1. NPFM with exponentially autocorrelated input.

Since (x(t), u(t)) is a Markovian random vector process the joint transition probability density function $f(x, u; t | x(0) = x_0, u(0) = u_0)$ is a solution of the Kolmogorov equation

$$\frac{1}{2} s_0 \frac{\partial^2 f}{\partial u_0^2} + (-kx_0 + u_0) \frac{\partial f}{\partial x_0} - cu_0 \frac{\partial f}{\partial u_0} = \frac{\partial f}{\partial t} \tag{2}$$

where $s_0 = \mathrm{Var}[w(t)]$. From Eq. (2) a partial differential equation may be derived such that

$$\frac{1}{2} s_0 \frac{\partial^2 m}{\partial u_0^2} + (-kx_0 + u_0) \frac{\partial m}{\partial x_0} - cu_0 \frac{\partial m}{\partial u_0} = -1 \tag{3}$$

where $m = m(x_0, u_0)$ and if $p(u_0)$ is the probability density function of u_0,

$$\beta = \left[\frac{1}{\int_{-\infty}^{\infty} m(x_0, u_0)\, p(u_0)\, du_0}\right]_{x_0=0} \tag{4}$$

Some methods of solution of Eq. (3) have been tried. These methods were all forms of the standard "product solution" approach. Additional approaches will be investigated, for example, Green's function method [3] and numerical approaches [4].

Some experimental data has already been obtained to substantiate this analysis.

REFERENCES

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[3] P. M. Morse and H. Feshbach, Methods of Theoretical Physics, Part I, McGraw-Hill, New York, 1953 (Chapter 7).

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II. STABILITY ANALYSIS

Pulse frequency modulation (PFM) is a method of coding information into the time interval between similar pulses and the polarity of the pulses. One frequently used PFM scheme is known as neural pulse frequency modulation (NPFM). It is characterized by a pulse firing when the output of a lowpass filter of the form $1/(s + c)$ reaches a threshold level. The time constant of the filter, c, is a positive constant. The filter is reset to zero immediately following the pulse firing. The equations which describe NPFM are

$$\begin{aligned} \dot{p}(t) &= x(t) - cp(t) - \frac{r}{M} y(t) \\ y(t) &= M \, \mathrm{Sgn}(p) \, \delta(t - t_n) \end{aligned} \tag{1}$$

In Eq. (1), $p(t)$ is the output of the lowpass filter, c is the time constant of the filter, r is the threshold level, M is the strength of the output pulse $y(t)$, and t_n is the time the n<u>th</u> pulse is fired. The time t_n satisfies Eq. (2).

$$\pm r = \int_{t_{n-1}}^{t_n} (x(t) - cp(t)) \, dt \tag{2}$$

where t_{n-1} is the time the $(n - 1)$<u>st</u> pulse was fired.

Many authors [1, 2, 3, 4] have investigated various aspects of control systems containing NPFM, including stability. These investigators, however, assumed the signals between the NPFM output and input to be completely analog in nature. The important case of an error sampled NPFM control system has not been previously reported. It is the stability analysis of such a system which is developed in this paper.

The block diagram of the system considered is shown in Fig. 1, where g(t) is the impulse response of the linear time-invariant plant and the NPF modulator is the only nonlinear element. The system of Fig. 1 will be referred to as System I.

The NPFM model which was found most appropriate for the stability analysis is the approximate discrete model reported in [5]. A block diagram of this discrete NPFM model is shown in Fig. 2a. Some limitations of this model which were not previously reported are developed in detail. Particular attention is given to a criterion which shows that the sampling period T_2 must be small with respect to the time constants of the linear portion of System I if the discrete NPFM model is to closely approximate the model defined by Eqs. (1) and (2).

Substitution of the discrete NPFM model into System I and block diagram manipulation lead to the System I equivalent of Fig. 2b. In Fig. 2b, G(s) is the Laplace transform of the impulse response g(t) shown in Fig. 1, and $H_0(s)$ is the Laplace transform of the impulse response of the zero-order hold, $h_0(t)$. If T_1 is constrained to equal T_2, then the well-known stability techniques of either Tsypkin [6] or Jury and Lee [7] can be used to determine the stability of the System I equivalent of Fig. 2b with respect to all initial conditions. Since T_2 must be small, the constraint $T_1 = T_2$ is an extremely severe one since T_1 would normally be chosen much larger than the small value required by T_2. If T_1 does not equal T_2 neither of the above-mentioned stability techniques can be used since the transmission path for the signal between the nonlinear element output and input becomes time varying in nature.

To relax the constraint that $T_1 = T_2$, a modified version of the NPFM model of Fig. 2a is developed and is shown in Fig. 3a. The paper shows how this model allows the stability analysis of Jury and Lee to be used for the System I equivalent shown in Fig. 3b even though $T_1 \neq T_2$. In addition, it is shown that the modified NPFM model of Fig. 3a is a better approximation to a physically realizable NPF modulator than the preliminary model of Fig. 2a. This follows from the fact that a physical NPF modulator will have a maximum, finite pulse frequency. The preliminary model does not account for this fact. It can have an unlimited pulse frequency for large input signals and small sampling period, T_2. However, the modified NPFM model will have a maximum, finite pulse frequency because the hard saturation element limits the input to the lowpass filter for large modulator input signals. Assuming zero initial conditions, the maximum pulse frequency of the modified model is $1/t_{min}$, where t_{min} is the minimum time for q(t) to reach the threshold level. In terms of L_s, r and c,

$$t_{min} = \frac{1}{c} \ln \frac{L_s}{L_s - rc} .$$

To show that the modified NPFM model allows a less restrictive analysis of System I than the preliminary model, the modified model is substituted for the NPF modulator of System I. Block diagram manipulation then leads to the System I equivalent of Fig. 3b. The stability analysis technique of Jury and Lee can be used on this configuration. The constraint on the relationship between T_1 and T_2 is that T_1 must be a rational multiple of T_2. Since T_2 is small, T_1 can equal almost any desired value. Requiring T_1 to be proportional to T_2 is a much less severe restriction than resulted from the use of the preliminary model in the analysis of System I. Thus, the modified NPFM model is not only a better approximation to physical NPF modulators than the preliminary model, but it also eases the conditions for which System I can be analyzed using the technique of Jury and Lee.

An example is included in the paper to demonstrate the usefulness of the NPFM model developed. Also, a complete review of the stability theorem of Jury and Lee is given in relation to the System I equivalent of Fig. 3b.

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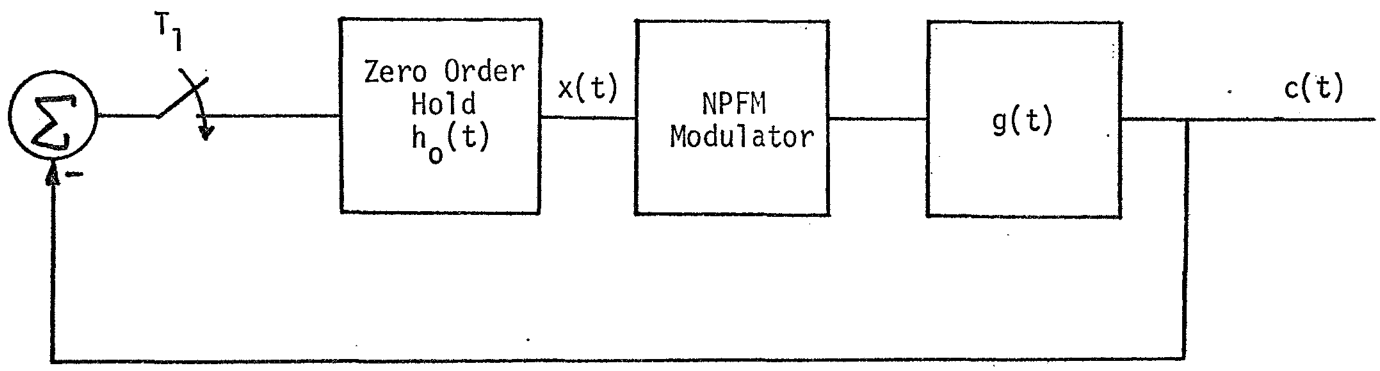

Figure 1. An error sampled NPFM control system.

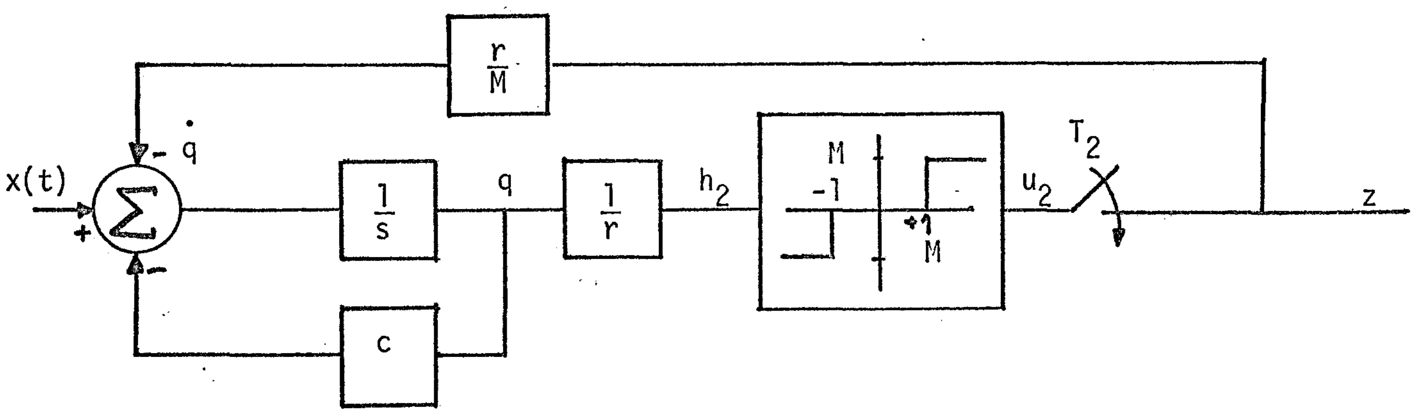

Figure 2a. A discrete NPFM model.

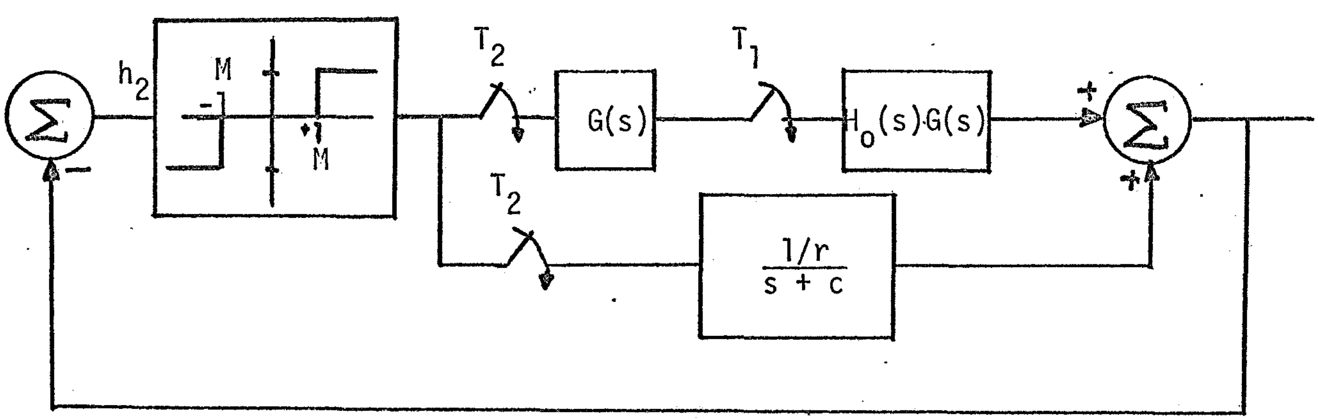

Figure 2b. A System I equivalent.

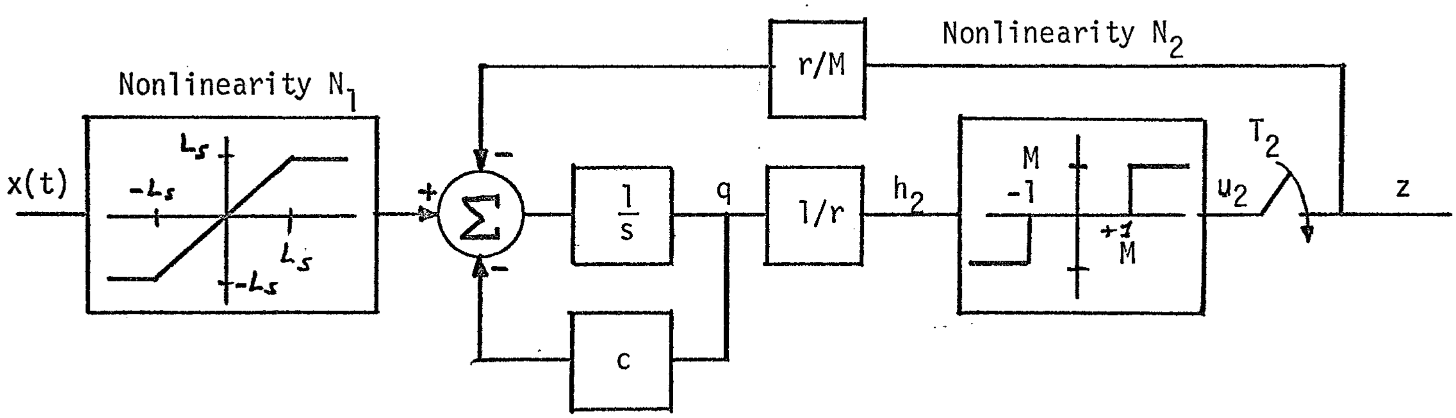

Figure 3a. A physically realistic NPFM model.

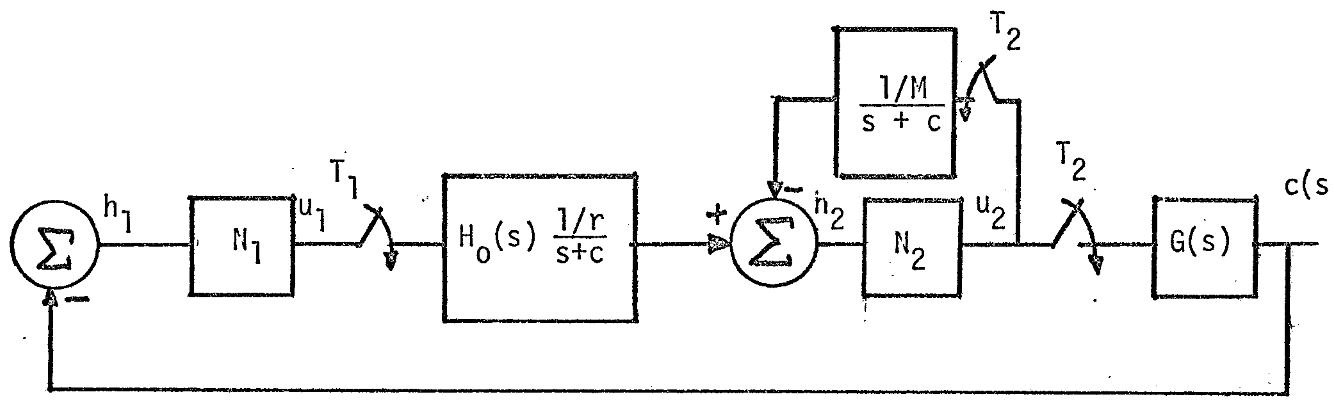

Figure 3b. A System I equivalent using Fig. 3a.