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A BALANCE FOR MEASURING THE PLANAR COMPONENTS OF SMALL FORCES

by
M. A. Messier

February 10, 1970

IONOSPHERE RESEARCH LABORATORY

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ABSTRACT

An instrument has been developed which is capable of measuring small forces along the two axes of a plane. The instrument was designed specifically for use in a molecular beam, but can be used for a variety of measurements ranging from magnetic fields to accelerations. A gravitational calibration experiment was performed and it was shown that millidyne forces were easily measured. It appears feasible to attempt detection of forces at the order of 10^{-5} dyne with this type of apparatus.

I. INTRODUCTION

With the advent of upper atmosphere and space travel, the study of rarefied gas dynamics has been given great impetus. The interaction of molecular flow with surfaces of varying shapes has received much consideration (Talbot, 1961), as has the interaction of specific molecular species with specific metallic surfaces (Hinchen, 1965; Laurmann, 1963; de Leeuw, 1966). The molecular beam detector is specifically designed for this type of work. The results of these experiments have immediate application in the design of rocket motors and reentry vehicles (Scala, 1963). The new rocket engines, such as an ion rocket, produce a low density output, which may be studied with the equipment to be described. The force detector may also be used as an accelerometer and, if appropriate magnetic materials are used, as a magnetometer in two dimensions.

1.1 Statement of Problem

An instrument must be developed which can measure the direction and magnitude of small forces, such as those exerted by a molecular beam, in a plane. It must fit the following specifications:

1) the sensor must be small enough to produce only minor perturbations in the molecular flow and to be capable of measuring small variations in flow pattern,

2) the sensor must be maneuverable enough to sample many points of the flow and to transmit signals to a remote recorder,

3) the sensor must be capable of measuring small, rapidly fluctuating forces (on the order of 10^{-5} dyne) while maintaining large signal capability,

4) the sensor must be capable of operating in vacuo or under pressure, and over a wide range of temperature and humidity conditions.

1.2 Survey of Detector Methods

There are a large number of pressure-sensitive detectors in use in high vacuum and molecular beam technology (Dushman, 1962; Ramsey, 1963). It would be far beyond the scope of this thesis to discuss them all, since most are merely omnidirectional detectors of density and do not perform functions similar to the molecular beam detector (MBD) to be described. One type of detector which is more relevant to the previously stated problem is the balance detector. A balance will be defined as a device which makes a force measurement by exerting an opposite and equal known force and brings the entire system into equilibrium.

Balances may be divided into two subclasses: those which provide the restoring force by a changing configuration, e.g., a spring balance (Type I), and those which use electronic feedback to maintain a null position (Type II). In general, the balances which will be described were not designed for use as MBD's; they usually were designed to monitor small weight changes or to measure magnetic susceptibilities. They are one dimensional and, in principle, can be adapted for use in a molecular beam apparatus. However, they are also too bulky and fragile to be moved around inside the chamber without a clever transportation system. General discussions of microbalances can be found in Gulbransen (1953), Rhodin (1953), Behrndt (1956).

Several detectors designed specifically as MBD's are of special interest. The first is a device designed by D.G.H. Marsden (1968). It was designed with a sensitivity reaching down to 10^{-5} dyne. As a type II detector, it uses a vane attached to a 100μA/100° meter for the target and a photoelectric feedback system to keep the meter movement in the null position. The current required to do this is a measure of the unknown force. A commercial microbalance using this principle is manufactured by Cahn Div., Ventron Instruments Corp., and exhibits a rugged and highly portable movement.

A type I detector is described by G.I. Skinner (1966), for use in measuring momentum accommodation coefficients with a pulsed molecular beam. The device is essentially a ballistic pendulum whose motion is detected by a differential transformer. The system is magnetically damped.

The differential transformer method was also used by Aroesty (1961) in his type I device to measure the force exerted on a sphere by a molecular beam. A quartz spring provided the restoring force.

One of the simplest detectors is a rectangular target hung on a tungsten torsion fiber with a mirror. The angle of rotation is measured by a light beam reflected from the mirror (Stickney, 1963).

Perhaps the most sensitive detectors are those which utilize a capacitor as the transducer. J.A. Smith (1968) used such a detector in his shock wave experiments. Capacitors have also been used in conjunction with beam type balances to detect changes in position and to achieve sensitivities on the order of 10^{-5} dyne (Gritsenko, 1966; Braginskii, 1964). A good discussion of mechanical measurements by electronic methods is given in the book by C. Roberts (1951).

The usual method of measuring a force vector in two or three dimensions is to use two or three one-dimensional detectors orthogonal to one another. For example, a two-dimensional detector might be made from two of Marsden's detectors at right angles. Gjessing (1969) reports a three dimensional hot wire anemometer for use in a wind tunnel. While it does not measure forces and is not designed for use with molecular beams, the principles used in its design might be used to make a MBD with the usual hot wire detector methods, e.g., a Pirani detector. A two-dimensional force detector using thermionic emission techniques has been described by Krylov (1968) for use in wind tunnels.

1.3 Comparison of Previous Detector Methods with the Author's

Only one of the detectors mentioned above meets more than two of the requirements set forth in the Statement of the Problem.

That detector is described by Marsden (1968). It is confined to one-dimensional measurements, however. All the others have either the sensitivity, the size, or the multidimensionality, but not all three. The author's detector uses a probe which can be made very small and, with the proper microcircuits, be completely isolated from the amplifiers and recorder. It is also insensitive to ambient pressure, temperature, humidity and atmospheric composition, assuming that the materials used in construction will not corrode or melt. A wide variety of these materials may be used.

This design for a MBD originated with the work (unpublished) of B.R.F. Kendall and R. Hazelton of the Ionosphere Research Laboratory. Their idea was to have the molecular beam deflect an optical fiber suspended above a concave mirror. A light ray, conducted down the fiber was to be reflected up to a bank of photocells to indicate the direction and amplitude of the displacement. An experimental model was built and tested. Its operation was found to be unsatisfactory due to low light levels (despite the use of a laser) and diffraction at the end of the fiber. The idea of a four capacitor detector system was conceived by Kendall and suggested to the author for development.

II. THEORY OF OPERATION

2.10 Basic Principles

The molecular beam detector to be described is a null device used to measure small beam forces by the change in capacitance they produce, when the beam displaces one plate of a capacitor. The voltage required to bring the capacitor back to zero displacement is the actual measure of force; resetting is done automatically by means of a feedback loop. Two identical circuits are used to make the two-dimensional detector, one for each axis. The element common to each is a sensing probe. It consists of the four capacitors needed by the two detector circuits to sense a force. Since the "x" and "y" detectors are essentially independent and identical, only one will be discussed. Fig. 1 should be consulted during the following section.

The probe will be described in chapter three; for purposes of this discussion, a simple description will suffice. It consists of two concentric cylinders; the outer one is made from thin metal foil and is deflected by the beam. The inner cylinder is divided into conducting quadrants, thereby forming four capacitors. Consider a pair on opposite sides of the cylinder. The line between them defines the x direction. In figure 1, the capacitors are labeled C_1 and C_2.

Deflection of the outer cylinder will cause C_1 to increase

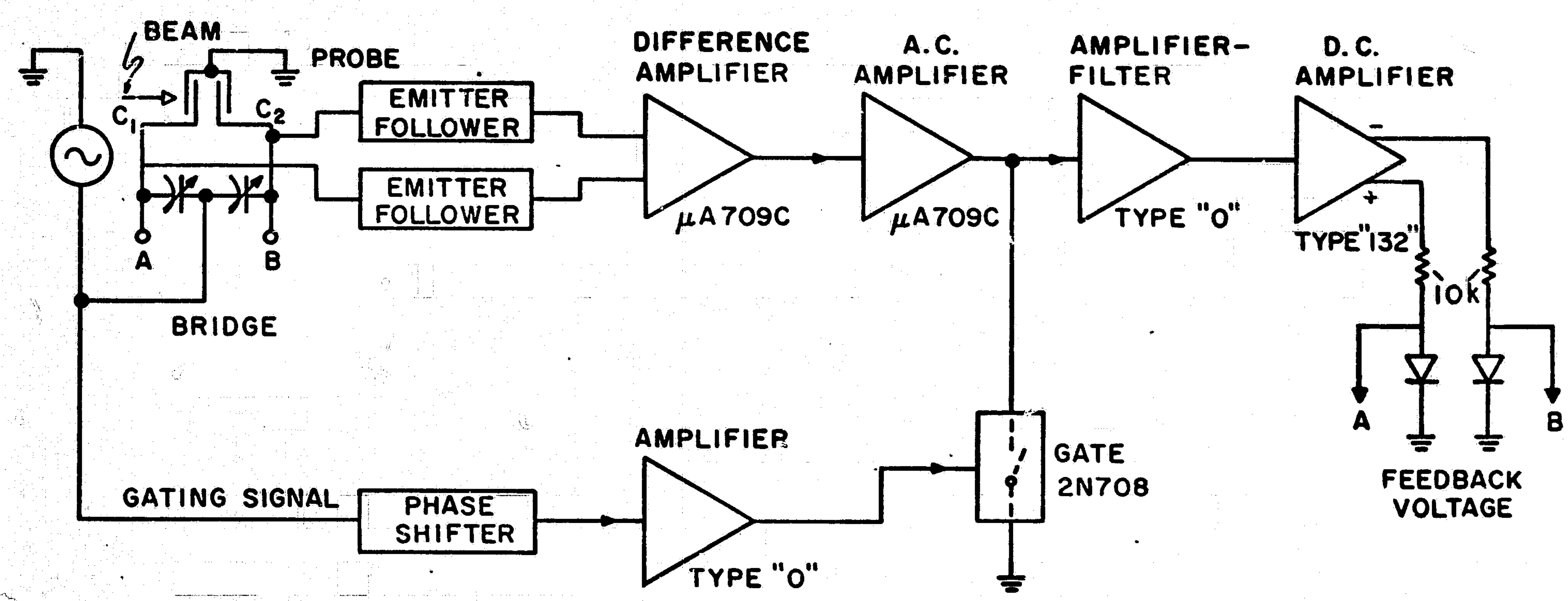

Figure 1. SIMPLIFIED CIRCUIT DIAGRAM.

and C_2 to decrease, unbalancing the capacitor bridge. An a.c. voltage proportional to the deflection is produced; a high impedance load at the bridge output is assured by field effect transistors (FET's) in common source configuration. Thus, bridge operation approaches that theoretically expected. The difference between the two voltages from each arm of the bridge is multiplied by a factor of three hundred and thirty.

In order to reset the bridge, after a deflection, some method must be used to sense the direction of that deflection. A 180° phase shift in voltage occurs as the probe capacitor is deflected from right to left, through its null position. If the output voltage is gated every half-cycle, and the phase of the gating signal is correct, only positive or negative pulses will pass, depending on the direction of deflection. In this case, the gating is done by biasing a transistor into the conducting state, and causing it to short the signal to ground. The gating voltage is supplied by the same generator which supplies the main signal voltage; in this way, the phase difference between the two is always constant. The phase relation is adjusted by a phase shifting network allowing 180° control. The phase of the gating signal is adjusted such that the above mentioned clipping takes place.

The positive or negative pulses are smoothed by an RC filter and amplified. The final amplifier has two outputs; one has the same polarity as the input signal and the other has the opposite polarity. Each is connected to ground through a 10K resistor and

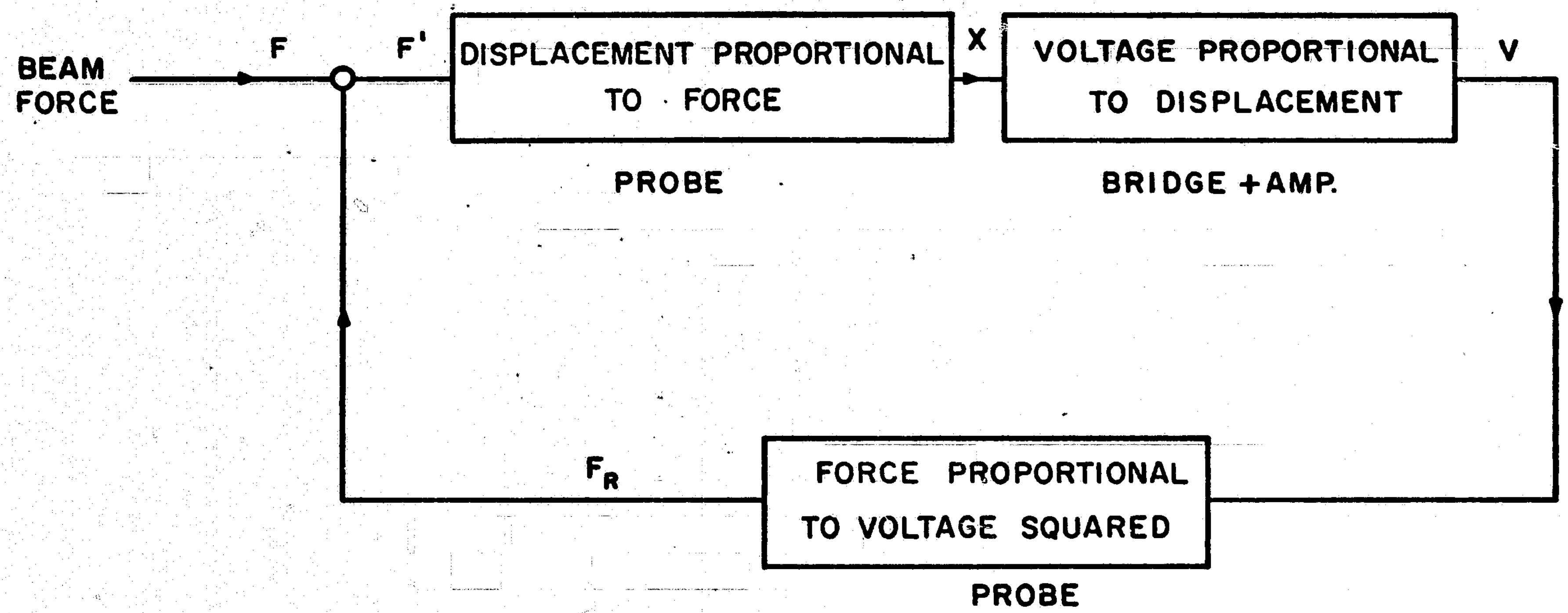

Figure 2. EQUIVALENT CIRCUIT.

a diode (see Fig. 1). Each diode is oriented in such a way that a positive potential at that amplifier output will forward bias it. The feedback voltage is taken from across the diode; each output being connected to a capacitor on opposite sides of the probe. When the diode is forward biased, its resistance is very low and most of the voltage is dropped across the resistor, providing no feedback. A negative voltage reverse biases the diode and is dropped across it because of its extremely high resistance. This appears as a feedback potential across one of the probe capacitors. Since only one diode can be forward biased by the amplifier at any given time, a feedback voltage can appear across only one probe capacitor. The force produced pulls the outer cylinder against the perturbing force, the continuous feedback loop eventually bringing it into equilibrium.

An interesting feature of the feedback system is its non-linearity. The force between the plates of a parallel plate capacitor is given by[1]

$$F' = \frac{(\varepsilon_0)}{2} A \frac{(V)^2}{S^2} \tag{21.1}$$

where

A = plate area

S = plate separation

V = voltage across the plates

ε_0 = permittivity of free space.

[1] See Corson (1962), equation 2-224.

The force is proportional to the square of the applied voltage and inversely proportional to the plate separation. However, V is proportional to the displacement of the capacitor from its null position ($V = Kx$). Let x denote this distance and x_0 the null position separation. Then $S = x + x_0$ and

$$F' = \frac{\varepsilon_0}{2} A \frac{(Kx)^2}{(x + x_0)^2} \tag{21.2}$$

Since $x << x_0$ (the restoring force keeps it near zero), we have the approximation

$$F' = \frac{\varepsilon_0}{2} \frac{K^2 A}{x_0^2} x^2 = q_0 x^2 \tag{21.3}$$

The sign of F changes as x passes through zero. This does not appear in the above equation because it is even; it will suffice, for the present, to state that q_0 bears the opposite sign to x. If q_0 is large, the effect of gravity can be ignored. Then the equation of motion of the outer cylinder can be written as

$$m\ddot{x} + q_0 x^2 - F' = 0 \tag{21.4}$$

where

F' = external force on probe

q_0 = electrical force constant

m = outer cylinder's mass

In section 2.2, this equation will be dealt with in detail. Its solution is given by the Weierstrass elliptical function. If some damping mechanism is not provided, the outer cylinder will continue to oscillate indefinitely. Damping will be considered in section 2.22.

The parallel plate approximation used here is valid because the ratio of the radius of curvature to the plate separation is negligible in the second order. The curvature in this particular apparatus actually gives an x component force about 90 percent of that of a flat capacitor of equal area.

2.20 Motion of a Particle under an X^2 Force

It was shown in section 2.1 that the outer cylinder is restored by a force proportional to the square of its displacement. In order to predict the action of the detector in a vacuum, it is helpful to solve its equations of motion.

2.21 Undamped Motion

For the undamped case, the motion is governed by the differential equation

$$m\ddot{x} + q_0 x^2 - F' = 0 \tag{22.1}$$

where

q_0 = a constant depending on probe characteristics and electronics amplification. (It was discussed in 2.1)

F' = perturbing force , normally considered a constant.

m = mass of outer cylinder

Dividing through by m, we have

$$\ddot{x} + qx^2 - F = 0 \qquad (22.1a)$$

Consider the motion from a physical viewpoint.[2] Rearranging equation 22.1

$$m\ddot{x} = -q_0x^2 + F_0' \qquad (22.2)$$

Let $F(x) \equiv -q_0x^2 + F_0'$. With F_0' a constant force, the potential energy is given by

$$V'(x) = -\int_0^x F(x)\,dx$$

$$= q_{0/3}x^3 - F_0'x \qquad (22.3)$$

Let E' = total mechanical energy, T' = kinetic energy.

Then,

$$T' + V' = E'$$

2 see Ross (1964), p. 46.

and

$$1/2\ m\dot{x}^2 + q_{0/3}x^3 - F_0{}'x = E'$$

Dividing by 1/2 m and rearranging terms,

$$(\dot{x})^2 = -\ 2/3\ qx^3 + 2\ F_0x + 2\ E \qquad (22.4)$$

Note the change in notation to indicate that the constants have been divided by m.

The term qx^2 in the equation of motion ($\ddot{x} + qx^2 + F_0 = 0$) is very special because it must change sign with x in order to describe the physical problem and yield bound solutions. To be rigorous, q might be redefined in terms of step functions, e.g.,

$q = q[2\ H(x) - 1]$, where

$H(x) = \begin{matrix} 1, & x \geq 0 \\ 0, & x < 0 \end{matrix}$, or q may be redefined as $q = q\dfrac{x}{|x|}$.

Fortunately, the special cases of motion which will be solved are such that all motion takes place in the positive half-plane, or symmetry properties allow the problem to be solved for motion in the positive half-plane and then generalized over a whole cycle. In either case, the change of sign does not need to be considered in the solution of the problem.

The equation of motion will now be attacked from a mathematical viewpoint. Rewriting equation 22.1a,

$$\ddot{x} + qx^2 - F = 0$$

The equation is easily reduced to first order. Let $y = \dot{x}$, then

$$\dot{x} = y \qquad (22.5)$$

$$\dot{y} = - qx^2 + F \qquad (22.6)$$

Eliminating time,

$$\frac{dy}{dx} = \frac{1}{y} \; (- qx^2 + F)$$

Considering F a constant (F_0) and integrating,

$$1/2\ y^2 = - 1/3\ qx^3 + F_0x + C \qquad (22.7)$$

Comparison with equation 22.4 shows the constant C to be the total energy of the system, divided by the particle mass (E). The equation may then be rewritten as

$$\left(\frac{dx}{dt}\right)^2 = -\,2/3\ qx^3 + 2\ F_0x + 2\ E \tag{22.8}$$

Multiplying by - 6/q,

$$-\,6/q\ \left(\frac{dx}{dt}\right)^2 = 4\ x^3 - \frac{12\ F_0x}{q} - \frac{12\ E}{q}$$

Define a complex time variable by

$$z = i\ (q/6)^{1/2}\ t \tag{22.9}$$

so that

$$\left(\frac{dx}{dz}\right)^2 = (\dot{x}\ (z))^2 = 4\ x^3\ (z) - \frac{12\ F_0}{q}\ x(z) - \frac{12\ E}{q} \tag{22.10}$$

It is known (Abramowitz, 1965; p. 640) that the following equation is satisfied by the Weierstrass elliptic function, P(z):

$$\dot{P}^2 = 4\, P^3(z) - g_2\, P(z) - g_3 \qquad (22.11)$$

Thus, x(z) = P(z) if

$$g_2 \equiv \frac{12\, F_0}{q} \qquad (22.12)$$

and

$$g_3 \equiv \frac{12\, E}{q} \qquad (22.13)$$

The Weierstrass function is doubly periodic, with periods 2ω and $2\omega'$. A complete listing of the function's properties is given in Abramowitz (1965) and the remaining discussion will be based on the material given there.

While the general problem can now be solved, it is more instructive to consider the two special cases:

1) lemniscatic case: a step function force hits the particle resting at equilibrium ($x = 0$, $F = F_0$, $E = 0$)

2) equianharmonic case: The particle is pulled to one side ($x = x_0$) and released ($F = 0$, $E = V$).

The names given these special cases are the names by which the corresponding Weierstrass functions are known. The function will be denoted by $P(z;\ g_2,\ g_3)$, where g_2 and g_3 are known as the invariants. Thus the two special cases correspond to $P(z;\ g_2,\ 0)$ and $P(z;\ 0,\ g_3)$ respectively.

<u>Lemniscatic Case</u>

Since $F = F_0$ and $E = 0$, the invariants are

$$g_2 = \frac{12\ F_0}{g} \quad \text{and} \quad g_3 = 0 \tag{22.14}$$

The amplitude and maximum velocity of the particle's motion can be calculated directly from equation 22.8. They are amplitude:

$$A = \left(\frac{3\ F_0}{q}\right)^{1/2} , \qquad \text{(22.15}$$

maximum velocity:

$$v_{max} = \pm\ 2 \left(\frac{F_0}{q}\right)^{1/4} \left(\frac{F_0}{3}\right)^{1/2} , \tag{22.16}$$

position of maximum velocity:

$$x_v = \left(\frac{F_0}{q}\right)^{1/2} = 3^{-1/2} A \qquad (22.17)$$

Homogeneity relations (Abramowitz, 1965; p. 658) allow the reduction of $P(z;\, g_2,\, 0)$ to $P(u;\, 1,\, 0,)$, i.e.,

$$P(z;\, g_2,\, 0) = g_2^{1/2}\, P(u;\, 1,\, 0) \qquad (22.18)$$

where

$$u = g_2^{1/4}\, z \qquad (22.19)$$

$P(u;\, 1,\, 0)$ is a well known function; its half-periods are ω and $\omega' = i\omega$, where $\omega \simeq 1.8540\ 74677$. The values at the half-periods are

$$P(\omega) = e_1 = 1/2 \qquad (22.20)$$

$$P(\omega + \omega') = e_2 = 0 \qquad (22.21)$$

$$P(\omega') = e_3 = -1/2 \qquad (22.22)$$

Calculation of the values between e_1 and e_2 by a series expansion (Abramowitz, 1965; p. 681) yields a half-cycle of the solution, which can be folded over graphically to give a full cycle. The results are given in Fig. 3, and compared with a sine wave of equal amplitude and frequency. The axes of the graph are labeled in terms of x and t.

$$x(u) = P(z;\ g_2,\ 0) = \left(\frac{12\ F_o}{q}\right)^{1/2} P(u;\ 1,\ 0) \qquad (22.23)$$

$$|u| = \left(\frac{q\ F_0}{3}\right)^{1/4} t \qquad (22.24)$$

The amplitude is given by

$$A = \left(\frac{12\ F_0}{q}\right)^{1/2} e_1 = \left(\frac{3\ F_0}{q}\right)^{1/2} \qquad (22.25)$$

The period of oscillation is such that

$$2\ \omega = \left(\frac{q\ F_0}{3}\right)^{1/4} T$$

or

$$T = 2\omega \left(\frac{q\ F_0}{3}\right)^{-1/4} \qquad (22.26)$$

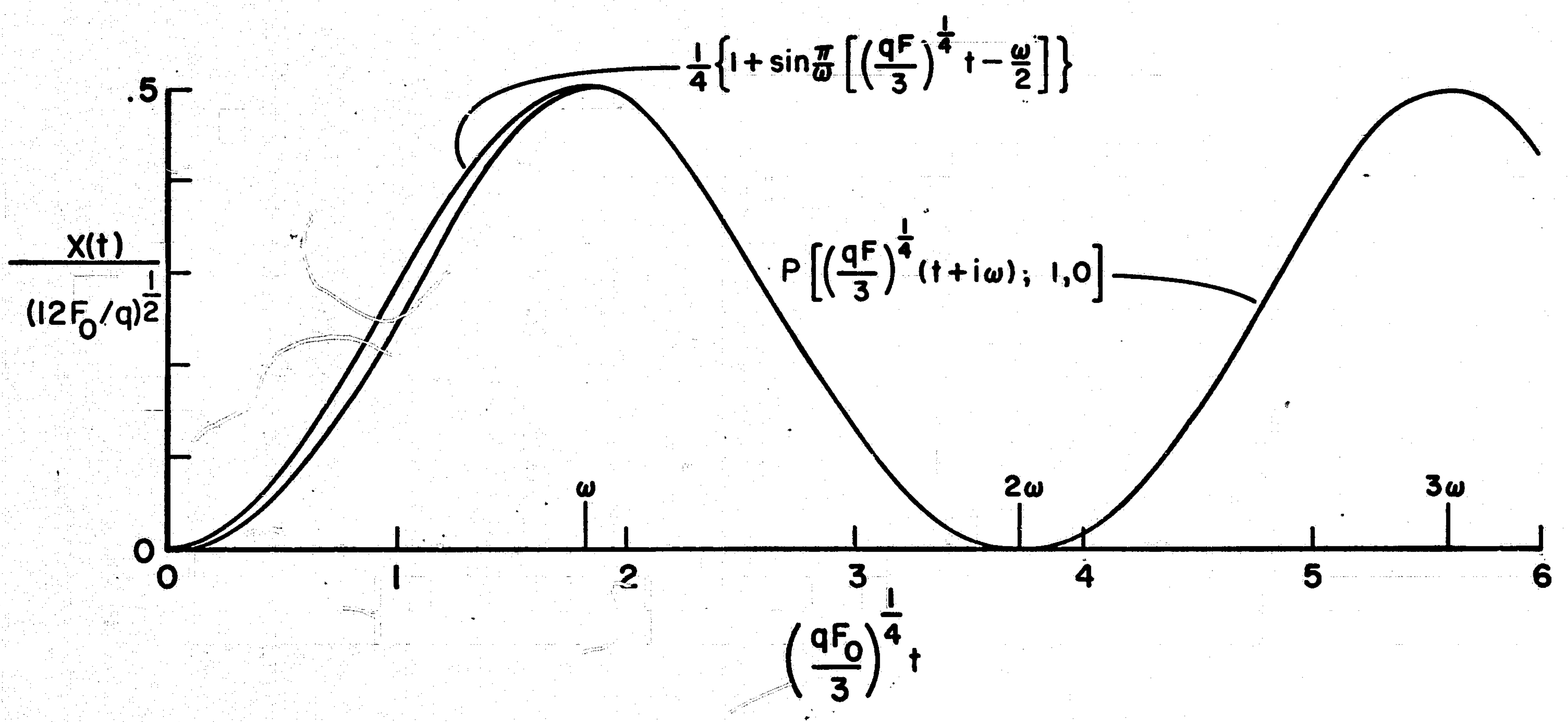

Figure 3. SOLUTION OF LEMNISCATIC CASE COMPARED WITH SINE WAVE OF SAME PERIOD AND AMPLITUDE.

Note from the figure that the peaks of oscillation are sharper and the valleys broader than the corresponding sine wave.

Equianharmonic Case

With F = 0 and E = V, the invariants are

$$g_2 = 0 \quad \text{and} \quad g_3 = \frac{12\ V}{q} \tag{22.27}$$

From the previous discussion,

$$V = -\ 1/3\ q\ x_0^{\ 3},$$

where

$$x_0 = \text{initial position.}$$

Then,

$$g_3 = -\ 4\ x_0^{\ 3}, \tag{22.28}$$

By equation 22.8, the maximum velocity (occuring at x = 0) is found to be

$$v_{max} = \left(2/3\right)\left(q\ x_0^{\ 3}\right)^{1/2}. \qquad (22.29)$$

Since the phase space trajectory is symmetrical about both axes, only a quarter-cycle of motion must be solved.

$P(z;\ 0,\ g_3)$ may be reduced to $P(u;\ 0;\ 1)$ by (Abramowitz, 1965; p. 653)

$$P(z;\ 0,\ g_3) = g_3^{\ 1/3}\ P(u;\ 0,\ 1) \qquad (22.30)$$

where

$$u = g_3^{\ 1/6}\ z.$$

The values of $P(u;\ 0,\ 1)$ at the half-periods are

$$P(\omega) = e_1 = 4^{-1/3}\ E^2 \qquad (22.31)$$

$$P(\omega_2) = P(\omega + \omega') = e_2 = 4^{-1/3} \qquad (22.32)$$

$$P(\omega') = e_3 = 4^{-1/3}\ E^{-2} \qquad (22.33)$$

$$P(\omega_2') = P(\omega - \omega') = e_2 = 4^{-1/3} \qquad (22.34)$$

where

$$E = \exp\ (i\ \pi/3) \tag{22.35}$$

and

$$\omega_2 = 1.5299\ 54037.$$

Also,

$$P(u_0) = 0 \tag{22.36}$$

where

$$u_0 = \omega_2\ (1 + i\ 3^{-1/2})\ . \tag{22.37}$$

Notice that the half-periods ω, ω' are complex, but ω_2 is purely real and $\omega_2{}' = \omega - \omega'$ is purely imaginary. The solution points are all the real values between $P(u_0) = 0$ and $P(\omega_2) = e_2$. The solution is graphed in Fig. 4. Again, the solution is compared with a sine wave of equal amplitude and frequency; the axes are relabeledin terms of x and t. The conversions are given by

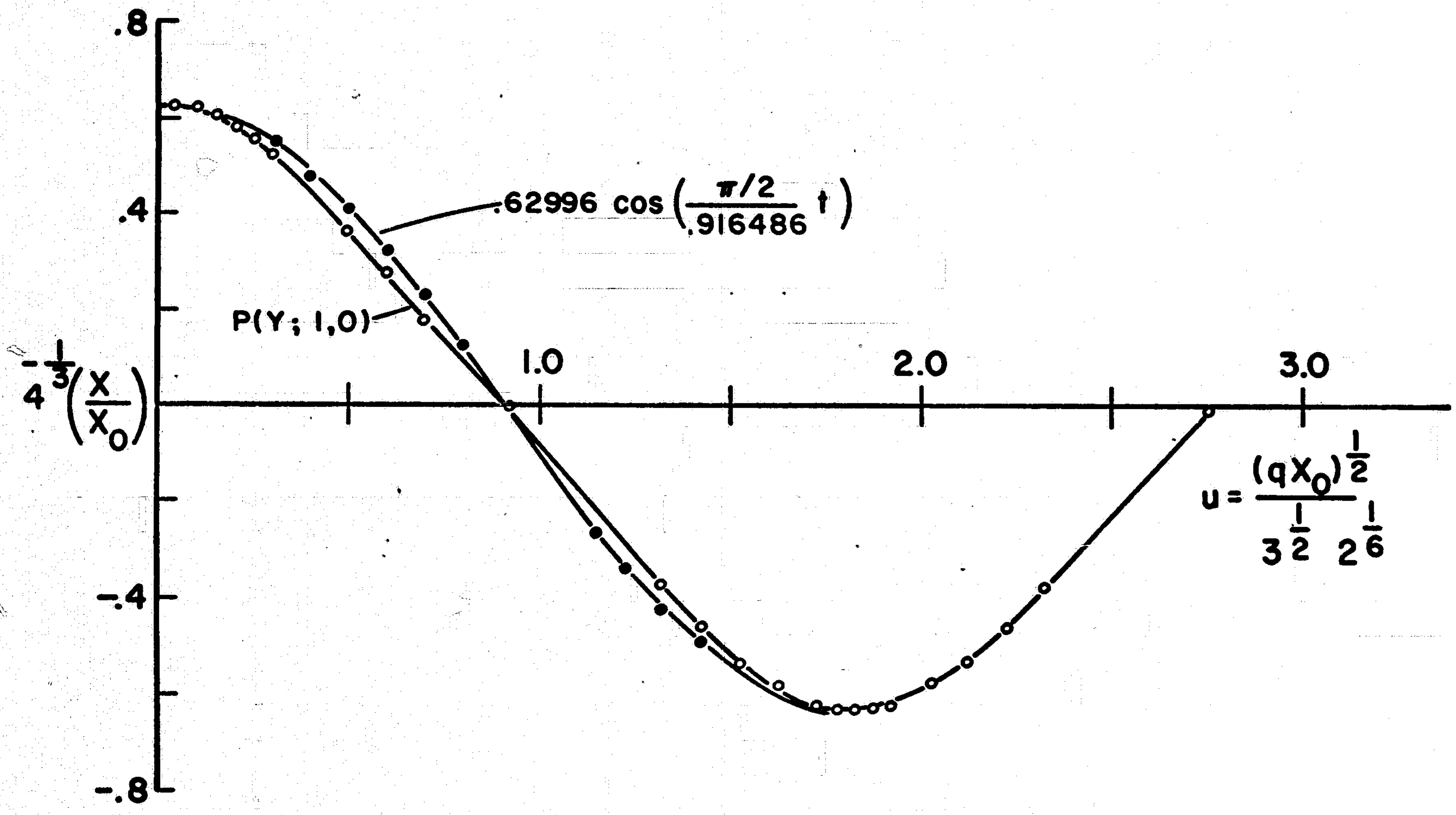

Figure 4. SOLUTION OF EQUIANHARMONIC CASE COMPARED WITH SINE WAVE OF SAME PERIOD AND AMPLITUDE.

$$|x| = |P(z;\ 0,\ g_3)| = 4^{1/3}\ x_0\ P(u;\ 0,\ 1) \qquad (22.38)$$

and

$$|u| = |i\ 4^{1/6}\ x_0^{\ 1/2}\ z| = \frac{(q\ x_0)^{1/2}}{3^{1/2}\ 2^{1/6}}\ t \qquad (22.39)$$

The period of oscillation is such that

$$4\ I_m u_0 = \frac{(q\ x_0)^{1/2}}{3^{1/2}\ 2^{1/6}}\ T$$

It can be shown from the fundamental rectangle of P(u; 0, 1) (Abramowitz, p. 652), that

$$I_m u_0 = 3^{-1/2}\ \omega_2$$

so that the period is

$$T = 2^{13/6}\ \frac{\omega_2}{(q\ x_0)^{1/2}} \qquad (22.40)$$

where

$$\omega_2 = 1.5299\ \ 54037...$$

2.22 Damped Motion

The oscillating motion, just studied, must be eliminated in a working system. The simplest method of providing damping is to differentiate a part of the position signal and add the two voltages. The restoring force is then proportional to the square of this sum. Thus

$$\ddot{x} + q(x + k\dot{x})^2 = 0 \tag{22.41}$$

Attempts to solve this equation analytically, as well as the approximation for small $k\dot{x}$,

$$\ddot{x} + 2\, qkx\dot{x} + qx^2 = 0 \tag{22.42}$$

were made with no success. The equation was solved numerically on an analog computer by S. Butler, under the direction of B.R.F. Kendall. Fig. 5 shows the problem solving circuit. An interesting feature is the method used to square the $(x + k\dot{x})$ signal and provide for the change of sign of q. The signal is used to control the X - axis of an x-y recorder, upon which is placed a graph of $y = + 1/16\ X^2$ for $X > 0$ and $y = -1/16\ X^2$ for $X < 0$. The graph scale will be discussed in a later paragraph. A photoelectric line follower is incorporated into the y-axis of the recorder, the output being proportional to the square of $(x + k\dot{x})$.

The main problem encountered in using this method is the speed of the line follower. The speed, in turn, depends upon

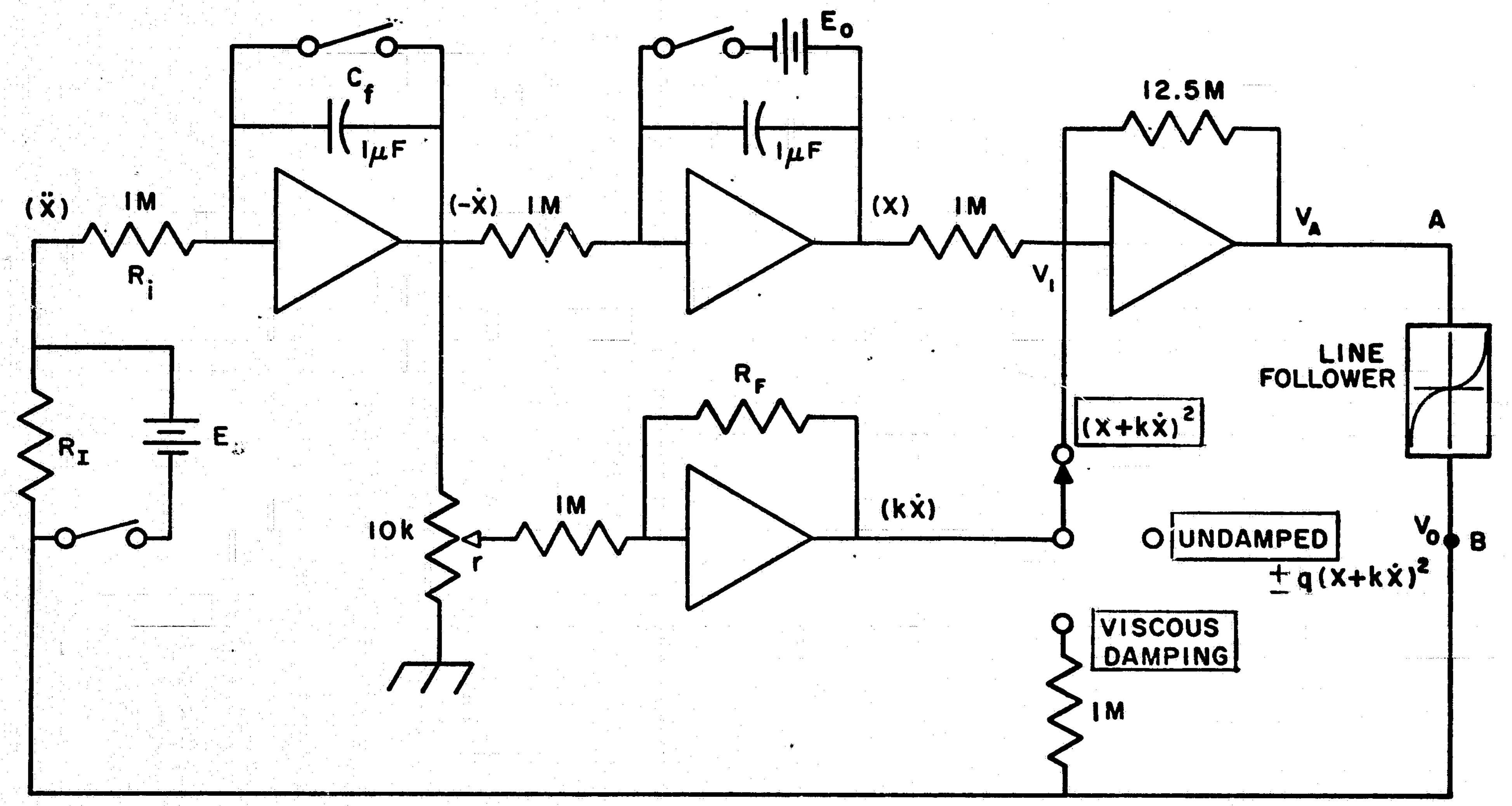

Figure 5. ANALOG COMPUTER CIRCUIT FOR DAMPED MOTION.

the slope of the curve. For example, if a plot of $y = x^2/32$ were used, it could be followed more quickly than $x^2/16$, because its slope is smaller at every point. However, near $x = 0$, the former plot is more inaccurate and introduces considerable error into the system. Conversely, a plot of $y = x^2/8$ introduces less error than $y = x^2/16$, but is harder to follow. For most of the computation, $y = x^2/16$ was used, but $y = x^2/8$ was tried on some problems. A frequency of 3 or 4 cycles per minute is the best that could be tolerated by the line follower, when using an 8 inch x 12 inch graph. This frequency is so low that the usual low frequency rejection methods cannot be used and extraneous d.c. voltages in the operational amplifiers are also integrated, providing a component proportional to time. This problem is aggravated by the low signal levels used. For example, the $\dot{x}$ signal is never higher than one volt and is usually only a few tenths. In damped problems the x signal also spends much time around one volt or less. A well designed computer circuit would take advantage of the ± 50 volt range of the Type 'O' amplifiers.

At the time the equation was solved, practical values of q had not yet been determined by experiment. Therefore, electronic component values were chosen to satisfy the line follower requirements.

Analysis of Computer Circuit

The circuit is arranged so that three classes of problems may be simulated: 1) the undamped equation, 2) the damped equation, as indicated above, and 3) the viscous damped equation, i.e.,

$$\ddot{x} + k\dot{x} + qx^2 = 0 \qquad (22.43)$$

All of these equations are of the type represented physically by a particle pulled to one side and released under the influence of the x^2 restoring force. The case of a particle initially at rest, being hit by a step function force is provided for by a voltage dropped across R_I, in series with the input to the first integrator.

The first integrator integrates $\ddot{x}$ to $\dot{x}$. The input resistance (R_i) equals 1 Meg and the feedback capacitor (C_f) is 1μf, giving a time constant of one second. The initial condition on $\dot{x}$ is $\dot{x} = 0$; this is provided for by initially shorting out C_f. The second integrator is identical with the exception of the initial conditions. For the particle drawn to one side and released, the initial condition is $x = x_0$, simulated by a voltage E_0. The problem of the step function force has the initial condition $x = 0$ and the integrator is set up like the first one.

The term $k\dot{x}$ is taken from the output of the first integrator, via a potentiometer and inverted by an operational amplifier. If k is to be greater than unity, the inverter also multiplies by a constant. When damping of the form $q(x + k\dot{x})^2$ is desired, the two signals are added at the input of the fourth operational amplifier, which multiplies by a constant, and the output is squared. If viscous damping is required, the $k\dot{x}$ term is added at the input to the first integrator.

Squaring is performed by a line follower. Since this component determines the value of q in a complicated manner, it is necessary to analyze this part of the circuit is greater detail. The first step is multiplication by $4\sqrt{10}$, performed by an operational amplifier. The multiplied signal is fed to the input of an xy-recorder and is thereby converted to a length, as determined by the volts/inch switch. The conversion is given by

$$d = V_{A/a} \tag{22.44}$$

where

d = length along x axis

a = volts/inch conversion factor (10 or 20)

V_A = multiplied voltage.

The x distance is converted to a y distance by the line follower according to the graph, i.e.,

$$e = d^2/b \tag{22.45}$$

where

e = the y distance

b = constant, 16 or 8 in these calculations.

The line follower is attached to the slide of a voltage divider, which is in a circuit designed to give an output of +3 volts at full displacement and -3 volts at full negative displacement (both x and y scales are centered about zero). Taking d and e as being measured from center zero (+ or -) and L_{max} as the maximum plus or minus excursion, the output of the line follower can be expressed as

$$V_0 = \pm 3 \, e/L_{max} \tag{22.46}$$

Substituting equations 22.44 and 22.45 into 22.46,

$$V_0 = \pm \frac{3}{L_{max}} \frac{d^2}{b}$$

$$V_0 = \frac{\pm 3}{bL_{max}} \left[\frac{V_A}{a}\right]^2 \tag{22.47}$$

For most problems

$$a = 10 \text{ volts/inch}$$

$$b = 16$$

$$L_{max} = 5 \text{ inches,}$$

then

$$V_0 = \pm \frac{3 \, V_A^{\,2}}{8000} \tag{22.48}$$

Also,

$$V_A = 4\sqrt{10} \quad V_I$$

where V_I is the input to the amplifier.

$$V_0 = 3/50\ V_I^2 \tag{22.49}$$

In the unscaled equation, q now becomes 3/50, so that

$$\ddot{V} = -(3/50)\,V^2 \tag{22.50}$$

for the undamped case.

The damped equations are

$$\ddot{V} = -\ 3/50\ (V + k\dot{V})^2 \tag{22.51}$$

and, for viscous damping

$$\ddot{V} = -(3/50)\,V^2 - k\dot{V}. \tag{22.52}$$

If a constant force is desired in the equations, it is represented by E_0, added to the right side of the equation, e.g.,

$$\ddot{V} = -\ 3/50\ (V + k\dot{V})^2 + E_o$$

Computer Solutions

Before proceeding to investigate the computer solutions, it will simplify matters to put the equations in a dimensionless form. This is a good procedure anytime, but it is particularly useful here, since the problems were run in unscaled form. Begin with the undamped equation

$$\ddot{x} = -\, qx^2 \qquad (22.53)$$

Both sides must have units of acceleration, so q has the units $L^{-1}\ T^{-2}$. Dividing both sides by q gives the unit L^2 and dividing both sides, again, by $x_0{}^2$ makes the equation dimensionless. Equation 22.53 in its dimensionless form is

$$\left(\frac{1}{qx_0}\right) \frac{\ddot{x}}{x_0} = -\left(\frac{x}{x_0}\right)^2 \qquad (22.54)$$

The analogous computer equation is

$$\frac{50}{3} \frac{d^2V}{d\tau^2} = -\, V^2$$

where $\tau = \beta t$, is the computer time variable. Converting to problem time (t), the equation becomes

$$\frac{50}{3\beta^2} \ddot{V} = -\, V^2$$

and the dimensionless form is

$$\frac{50}{3\beta^2 V_0}\left(\frac{\dot{V}}{V_0}\right) = -\left(\frac{V}{V_0}\right)^2 \tag{22.55}$$

Equation 22.53 is transformed into 22.54 by

$$\frac{x}{x_0} = \frac{V}{V_0} \tag{22.56}$$

and

$$\frac{1}{qx_0} = \frac{50}{3\beta^2 V_0} \tag{22.57}$$

The damped equation

$$\ddot{x} = -q(x + k\dot{x})^2$$

has the dimensionless form

$$\frac{1}{qx_0} \frac{\ddot{x}}{x_0} = \left(\frac{x}{x_0} + k\frac{\dot{x}}{x_0}\right)^2. \tag{22.58}$$

Transformation to the computer voltages are provided by 22.56, 22.57, and the relation between the damping coefficients

$$K = \beta k \tag{22.59}$$

where K is the computer damping coefficient and k is the problem value.

When a constant perturbing force is considered, e.g., $\ddot{x} = - qx^2 + F$, the dimensionless "force" becomes F/qx_0^2. The force is represented on the computer by an e.m.f. labeled E_0. Thus we have the relation $\frac{1}{qx_0} \frac{F}{x_0} = \frac{50}{3\beta^2 V_0} \frac{E_0}{V_0}$, which reduces to

$$\frac{F}{x_0} = \frac{E_0}{V_0} \tag{22.60}$$

Using equations 22.56, 22.57, 22.59, and 22.60, the computer results, in volts can be interpreted in terms of x, q, k, and F.

Consider the problem of a particle pulled to one side and released, with $(x + k\dot{x})^2$ damping. The rest position of the particle will be $x = 0$. Expanding the equation of motion, it is seen that two terms contribute to the damping.

$$\ddot{x} = - q(x^2 + 2\,kx\dot{x} + k^2\dot{x}^2). \tag{22.61}$$

The first order term $(kx\dot{x})$ vanishes at the rest position, leaving the second order $(k\dot{x})^2$ term. The rate of damping while high at first, will be low in general. Viscous damping would be much faster, but isn't present in practice, because experiments are done in vacuum. Figures 6 and 7 illustrate computer outputs for various values of K. Figure 8 shows viscous damping for comparison

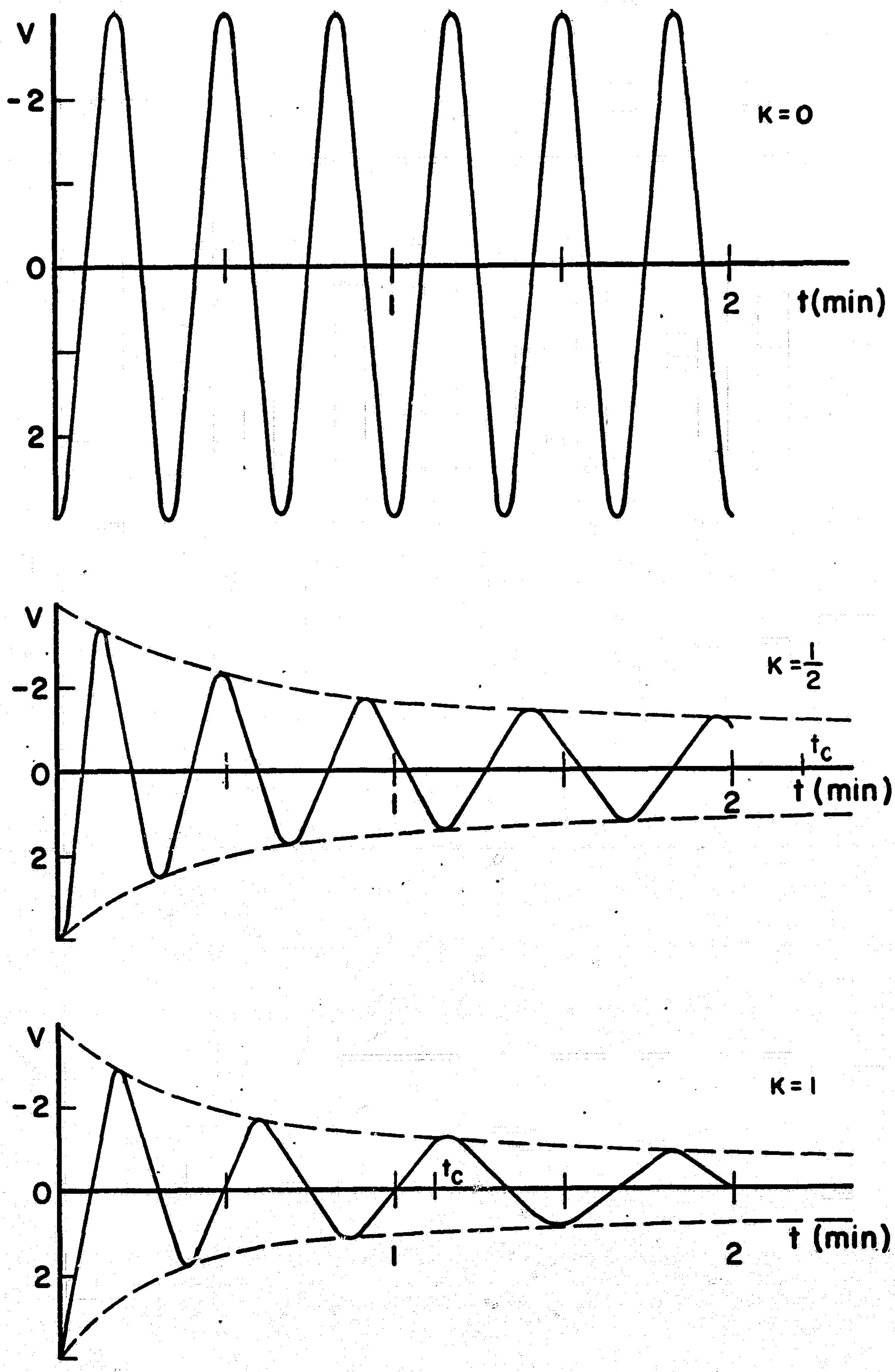

Figure 6. COMPUTER SOLUTION OF $\ddot{V} = -3/50\ (V + K\dot{V})^2$, FOR K $\leq$1.

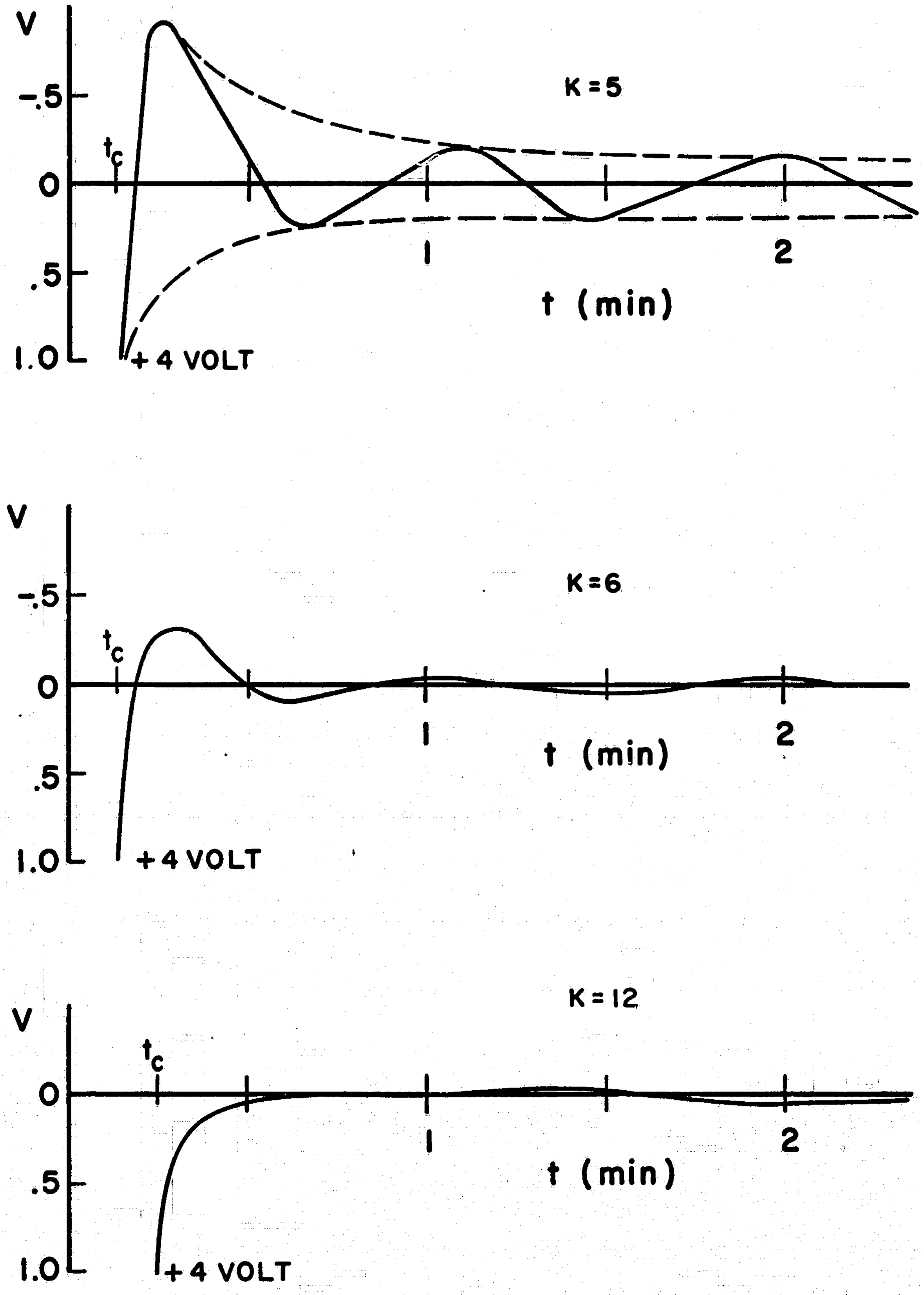

Figure 7. COMPUTER SOLUTION OF $\ddot{V} = -3/50\,(V + K\dot{V})^2$, FOR $K > 1$.

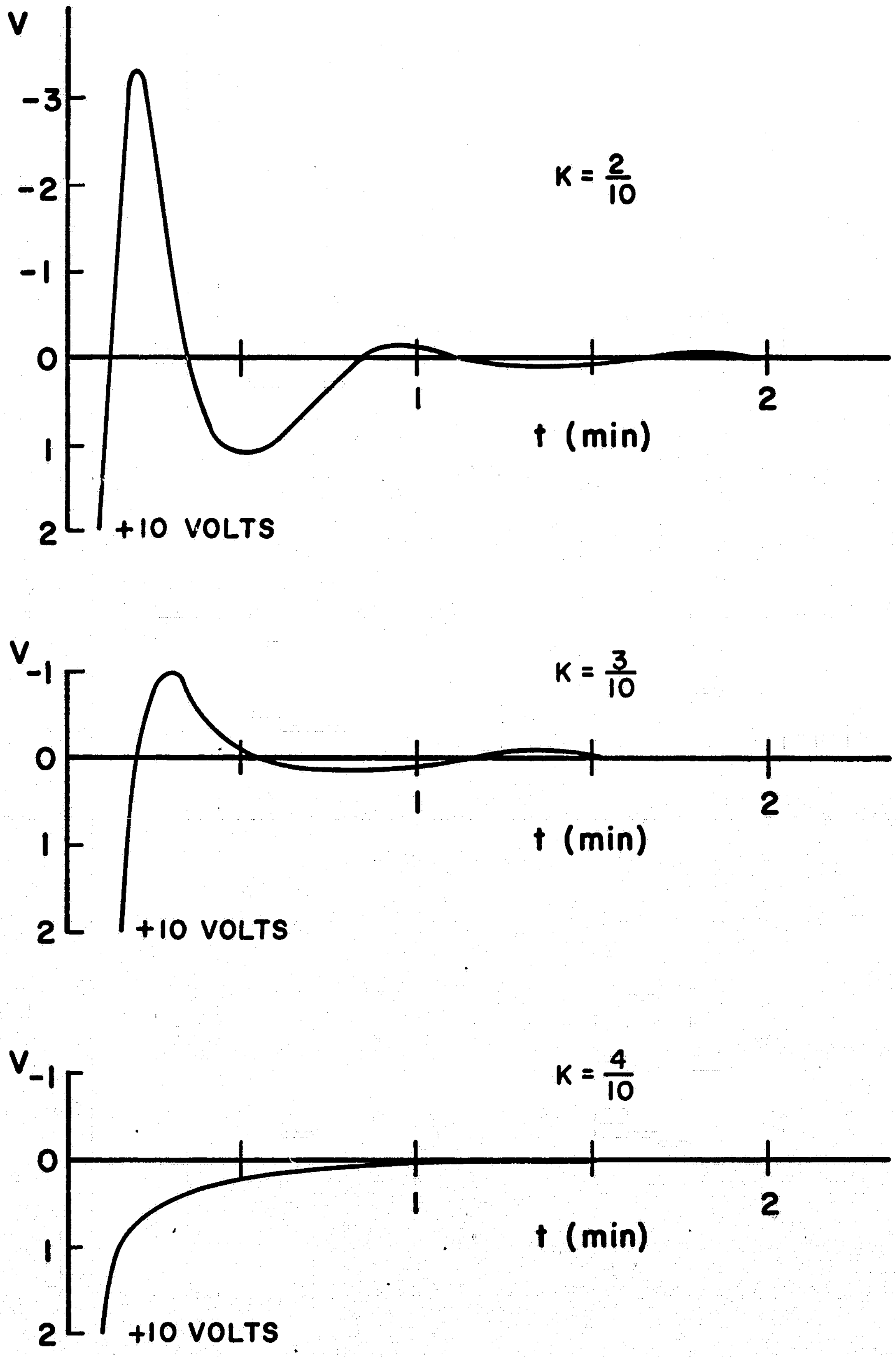

Figure 8. VISCOUS DAMPING

purposes. As a check on theory, consider the period of the undamped ($k = 0$) function as plotted in Fig. 6. Since the voltage varies from +3V to -3V, take x_0 as 3×10^{-2} cm, and $\beta = 1$, for simplicity. From 22.56 and 22.57

$$V = x/.01 \tag{22.62}$$

and

$$\frac{1}{qx_0} = \frac{16.7}{3(1)} = 5.56 \text{ sec}^2, \tag{22.63}$$

assuming c.g.s. units.

The period, as given by equation 22.40 is

$$T = 2^{13/6}\, \omega_2/(qx_0)^{1/2}$$

where

$$\omega_2 \simeq 1.53$$

Now,

$$1/qx_0 = 5.56, \quad 1/(qx_0)^{1/2} = 2.37$$

so that T = 16.2 seconds, to slide rule accuracy. The period, as given by the computer is .33 minute, or 20 seconds. The discrepancy can be attributed to the non-precision resistors employed in the multiplier and, it will be shown, to error introduced by the line follower system.

The value of q in this problem is 5.57 x_0 = 16.7. A more realistic value in a practical device would be about 16 x 10^3. Since the period varies as $q^{-1/2}$, the period for this value of q can be calculated; it is 0.52 seconds. The 16.2 seconds period was used in this calculation.

Solutions were found for the problem of a stationary particle hit by a step function force. In this case, the rest position, after damping, would not be x = 0, so that the kxx' term dominates, and damping is more rapid.

Fig. 9 illustrates the damping. To show that the mismatch between the theoretically derived period and the period exhibited by the analog computer in the last problem was due to errors introduced by the line follower system, I will compare the two periods for this problem. The "proof" comes from the fact that in this problem, a graph of y = x^2/8 was used, instead of y = x^2/16. To slow the system down, the input to the x axis was changed from 10 volts/inch to 20 volts/inch. The analog's voltage equation is changed to $\ddot{V} = -3/100\ (V + K\dot{V})^2$, and 22.57 becomes

$$\frac{1}{qx_0} = \frac{100}{3\beta^2 V_0} \qquad (22.64)$$

Letting computer time equal problem time (β = 1), and

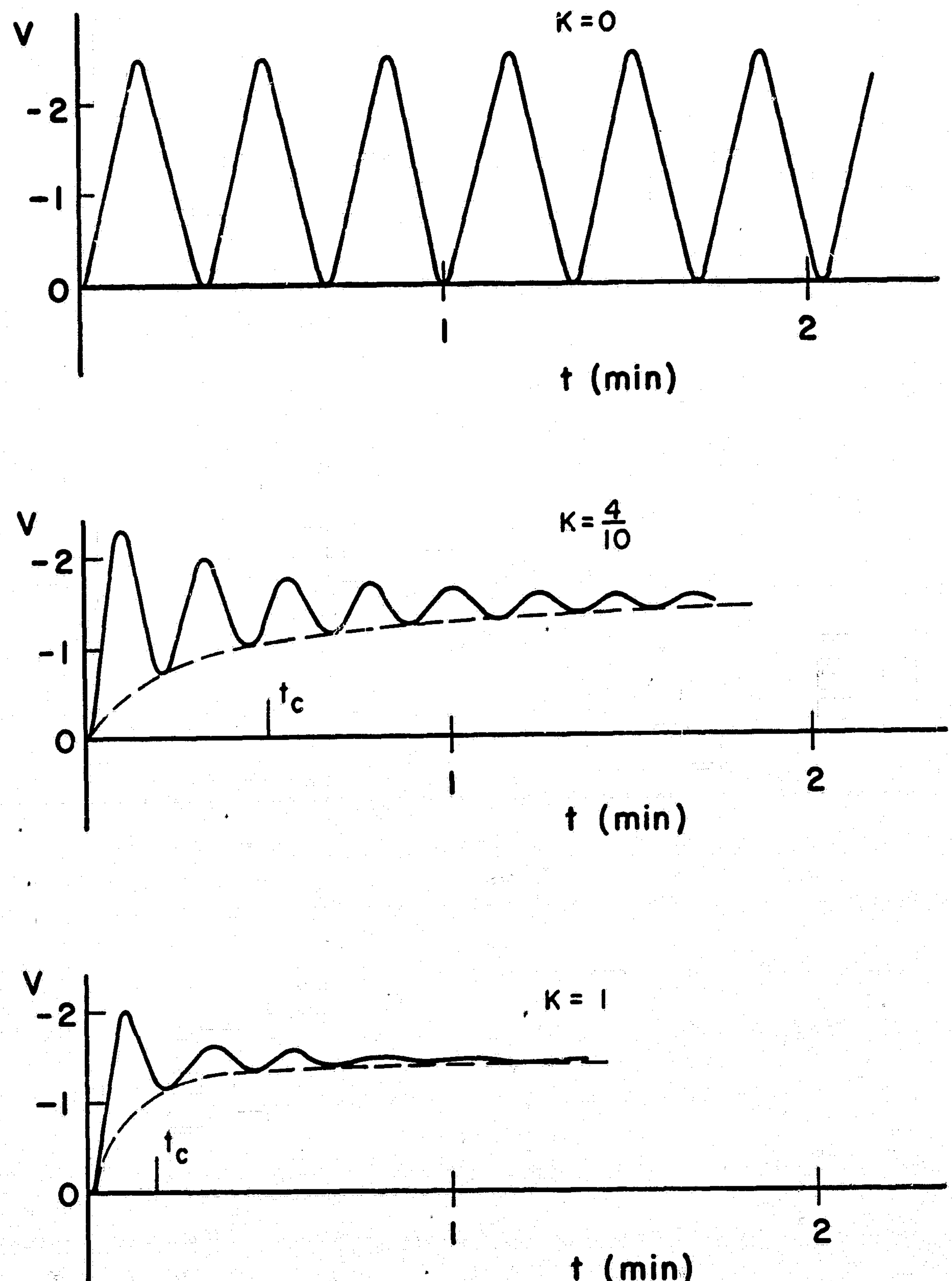

Figure 9. COMPUTER SOLUTION OF $\ddot{V} = -\ 3/50\ (V + K\dot{V})^2 + E_0$, E_0 = 0.3 VOLT.

$$\frac{x_0}{V_0} = 10^{-2} \text{ cm/V}, \qquad q = 3 \text{ dyne/cm}^2.$$

From equation 22.60, the force is

$$F = E_0 \left(\frac{x_0}{V_0}\right) = E_0 \times 10^{-2} \text{ dyne}.$$

In the example of Fig. 9, $(K = 0)$, $E_0 = 0.3$ volt, so that $F \approx 3 \times 10^{-3}$. The values of amplitude and period are given by equations 22.25 and 22.26 to be $a = (3F/q)^{1/2}$ and $T = 2\omega/(qF/3)^{1/4}$ respectively, where $\omega \simeq 1.85$. The values calculated are $a = 2.74 \times 10^{-2}$ cm and $T = 15.9$ sec, to slide rule accuracy. The values given by the computer are 2.5×10^{-2} cm and 15 sec. The agreement is much better than last time.

Having indicated the accuracy of the computer solutions, the damping characteristics may be investigated. Define a time constant, t_c, such that the height of the wave envelope decays to 25 percent of its initial value in t_c seconds. The definition usually used, i.e., the time it takes for the wave envelope to drop to 1/e times its initial value, is not indicative of the decay time. The decay is not exponential and, if the final position is near $x = 0$, becomes very slow after an initial rapid drop. Therefore, a longer time constant was chosen.

To make use of the analog computer data, it is helpful to collect the four transformation formulae; in slightly altered form, they are

$$\frac{x}{V} = \frac{x_0}{V_0} \tag{22.56}$$

$$\frac{\beta^2}{q} = I\,(x_0/V_0), \quad I = 50/3, \text{ for } y = x^2/16$$

$$I = 100/3, \text{ for } y = x^2/8 \text{ graph.} \tag{22.57}$$

$$K = \beta k \tag{22.59}$$

$$\frac{F}{E_0} = \left(\frac{x_0}{V_0}\right), \tag{22.60}$$

as well as the time transformation

$$\tau = \beta t \tag{22.65}$$

where

τ = computer time

t = problem time

There are a number of ways in which these transformations may be used, depending on what is assumed initially, and what information is desired. For example, given the values $q = 10^4$, $x_0 = 10^{-2}$cm and $K = 1$, find t_c and k for the problem of a particle displaced and released. It is known, from Fig. 6, that $V_0 = 4$ volts, so that $x_0/V_0 = 2.5 \times 10^{-3}$ cm/volt. For this set of problems, $I = 3/50$. The procedure is to solve 22.57 for β and use 22.65 and the computer value of t_c, i.e., t_c', to find the problem value. The value of t_c is calculated to be 10.3 seconds. Equation 22.59 gives k as 0.155.

It is not necessary to have a graph of the computer output for this type of calculation. All that is necessary is V_0, t_c, I, and K. For the step force problem, V_0 might be defined as the limiting value; x_0 can be calculated from the equilibrium relation $F_0 = qx_0^2$, where F_0 is the magnitude of the applied force. Fig. 10 illustrates t_c as a function of K for the two types of problems. Curve 1 is the free pendulum problem; t_c has a minimum of about 6 seconds at $K = 6$. Curve 2 is for the step function force problem. There were not enough data points to find the exact minimum, but it is in the vicinity of K = 1 or 2 with t_c less than 0.2 seconds.

2.30 Application to the Molecular Beam Detector

Before applying the results of the last two sections to a real problem, it would be wise to recall the real significance of the constant q. The undamped equation of motion is

$$m\ddot{x} + q_0 x^2 - F' = 0 \qquad (23.1)$$

Dividing through by the mass gives the equation that was worked throughout this chapter. Thus,

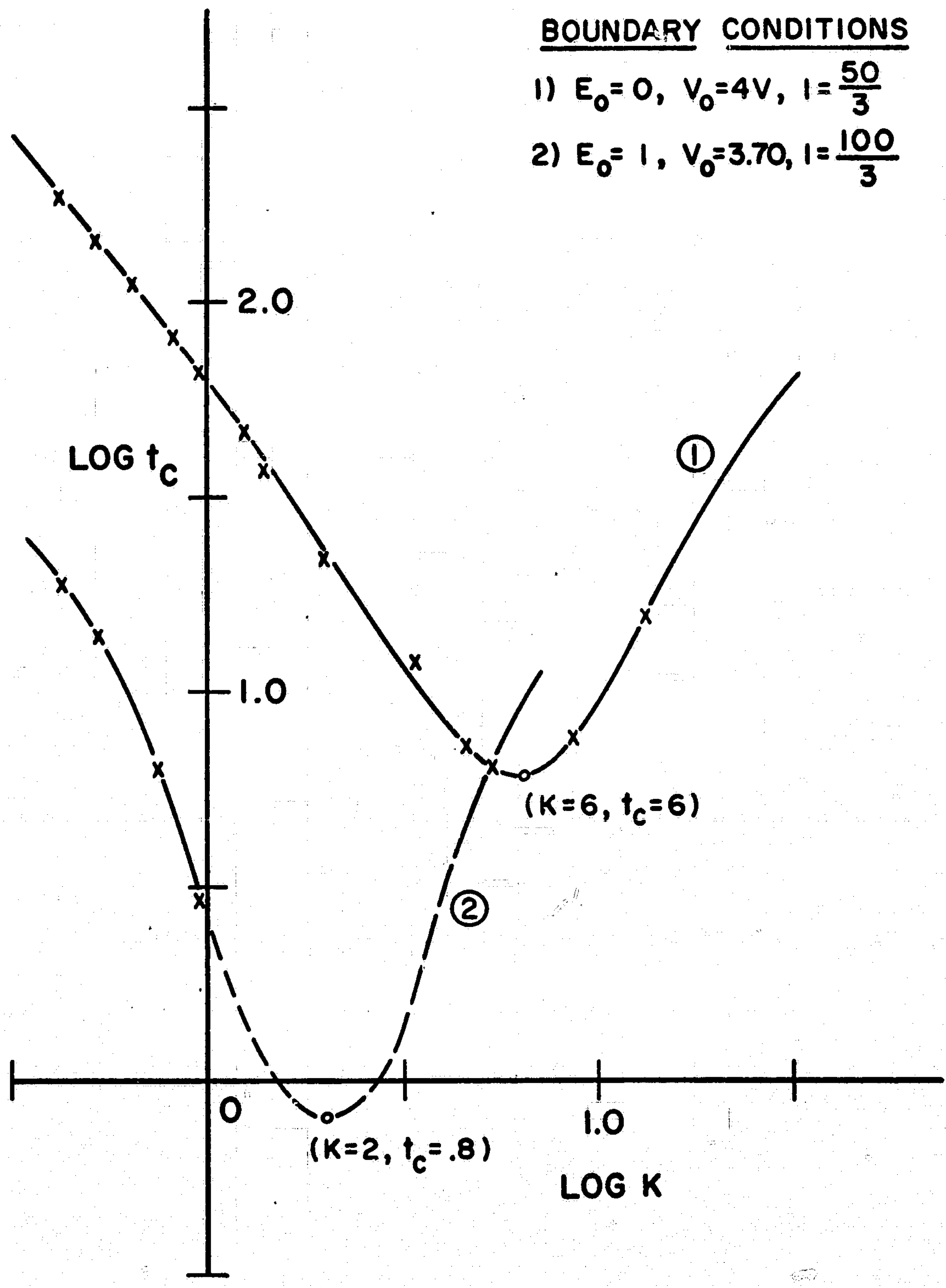

Figure 10. LOG t_c vs. LOG K.

$$q = q_0/m \tag{23.2}$$

where q is the constant used in the theoretical discussion and q_0 the constant, as it appears in the actual equation of motion. The mass of the outer cylinder denoted by m, is small in practice, on the order of one to 100 mg. In the author's apparatus, the mass was 70 mg. Much smaller masses can be attained by depositing metal on the surface of a form and dissolving that form.

The method of finding q_0 for a given system will now be discussed. q_0x^2 is the feedback force across the capacitor, so that

$$F_c = aV^2 = a(bx)^2 = ab^2x^2 \tag{23.3}$$

where

$$q_0 = ab^2 \tag{23.4}$$

$$V = \text{potential across capacitor}$$

From equation 21.2,

$$a = \frac{\varepsilon_0}{2} \frac{A}{(x + x_0)^2}$$

or

$$a = \frac{\varepsilon_0}{2} \frac{A}{x_0^2}, \quad x << x_0$$

where

x_0 = null position plate spacing

A = plate area

x = displacement from null position

The value of b comes from a gravitational calibration experiment, to be described in a later chapter. It gives the open loop output voltage as a function of the outer cylinder's displacement.

For these experiments, $a = 1.8 \times 10^{-4}$ dyne/volt^2, and b depends on the system amplification, varying from one experiment to another.

Typical values are 6.7×10^3 volt/cm, 3.4×10^4 V/cm, and 6.7×10^4 V/cm. The respective values of q_0 are 8.1×10^3 dyne/cm^2, 2.0×10^5 dyne/cm^2, and 8.1×10^5 dyne/cm^2. Dividing by the mass of the outer cylinder, to find q, we have the values 1.2×10^5 dyne/cm^2-gm, 2.9×10^6 dyne/cm^2-gm, and 1.2×10^7 dyne/cm^2-gm.

Assuming $x_0 = 10^{-2}$ cm, these values of q yield undamped pendulum periods of (by equation 22.40) .24 sec through .024 sec. Using the computer results of Fig. 10, curve 1, the minimum t_c

is calculated to be 8.6×10^{-2} second through 8.6×10^{-3} second, for k equaling 8.6×10^{-2} through 8.6×10^{-3}.

III. DESCRIPTION OF APPARATUS

3.1 Probe

The term "probe" actually refers to that part of the detector which is placed in the molecular beam, as opposed to the amplifiers and gating circuit which remain outside the test area. The probe consists of inner and outer cylinders, making up the probe capacitor, and also the housing, which supports them. If the bridge circuit (section 3.2) were to be miniaturized and mounted on the housing, it might be considered as part of the probe. For the sake of analysis, however, the term "probe" will be used to refer to just the probe capacitor, since that is the actual detecting element.

The probe (Fig. 11) consists of two concentric cylinders. The outer one is 0.394 inches in diameter, 0.59 inches in height and has a mass of 70 mg. It is suspended from a .0001 inch tungsten wire, 5.1 inches in length. The cylinder is constructed from .001 inch stainless steel foil, spot-welded into shape. The outer cylinder was constructed by J.O. Weeks of the IRL. The inner cylinder is machined from plexiglass. Its diameter is 0.330 inches. Conducting quadrants are painted on the surface by hand, using silver paint. This adds another .02 inches, making the gap between the inner and outer cylinder .032 inches. Each of the quadrants measures 0.63 inches x 0.22 inches. The outer cylinder

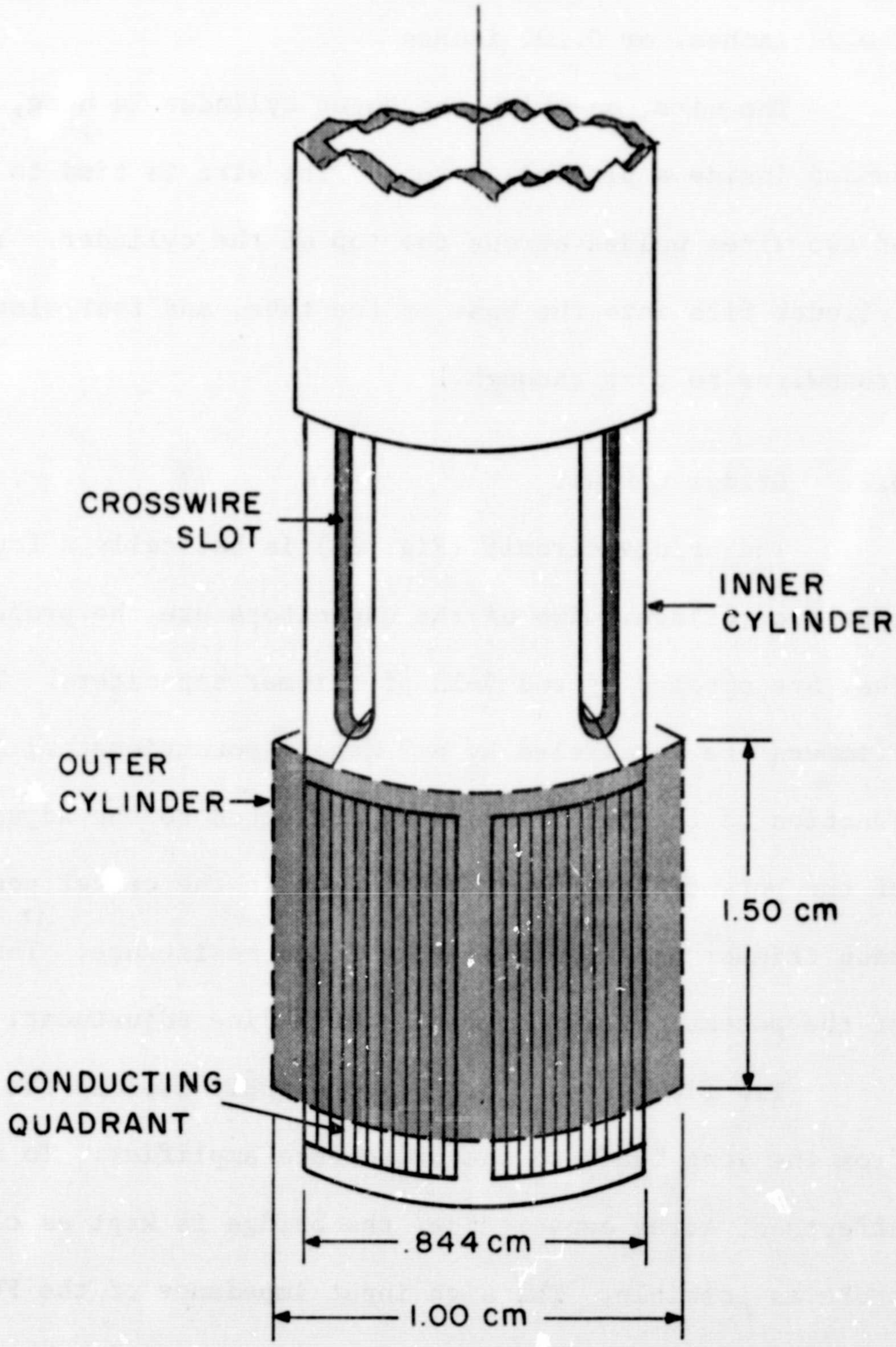

Figure 11. PROBE STRUCTURE.

only comes within 0.15 inches of the bottom, so, neglecting edge effects, the effective area of a single quadrant is 0.48 inches x 0.22 inches, or 0.106 inches^2.

The wire, on which the outer cylinder is hung, is suspended inside a plexiglass tube. The wire is tied to the junction of two wires welded across the top of the cylinder. The inner cylinder fits into the base of the tube, and four slots allow the crosswires to pass through.

3.2 Bridge Circuit

The bridge circuit (Fig. 12) is basically a four capacitor type a.c. bridge. Two of the capacitors are the probe capacitors. They are opposed by two 5-15 pf trimmer capacitors. The two trimmers are paralleled by a 2 Megohm potentiometer; the common junction of the two trimmers is connected to the adjustable arm of the potentiometer such that, when in the center position, each trimmer is paralleled by a 1 Meg resistance. The purpose of the potentiometer is to provide a fine adjustment.

Two field effect transistors (FET) isolate the bridge from the long leads to the difference amplifier. To reduce the effects of stray capacitance, the bridge is kept as close to the probe as possible. The high input impedance of the FET's also allows the bridge to perform according to simple theory, i.e., no resistive path need be considered between the two arms of the bridge. The two 10 Megohm resistors also provide a high input impedance for the bridge, looking into the feedback circuit.

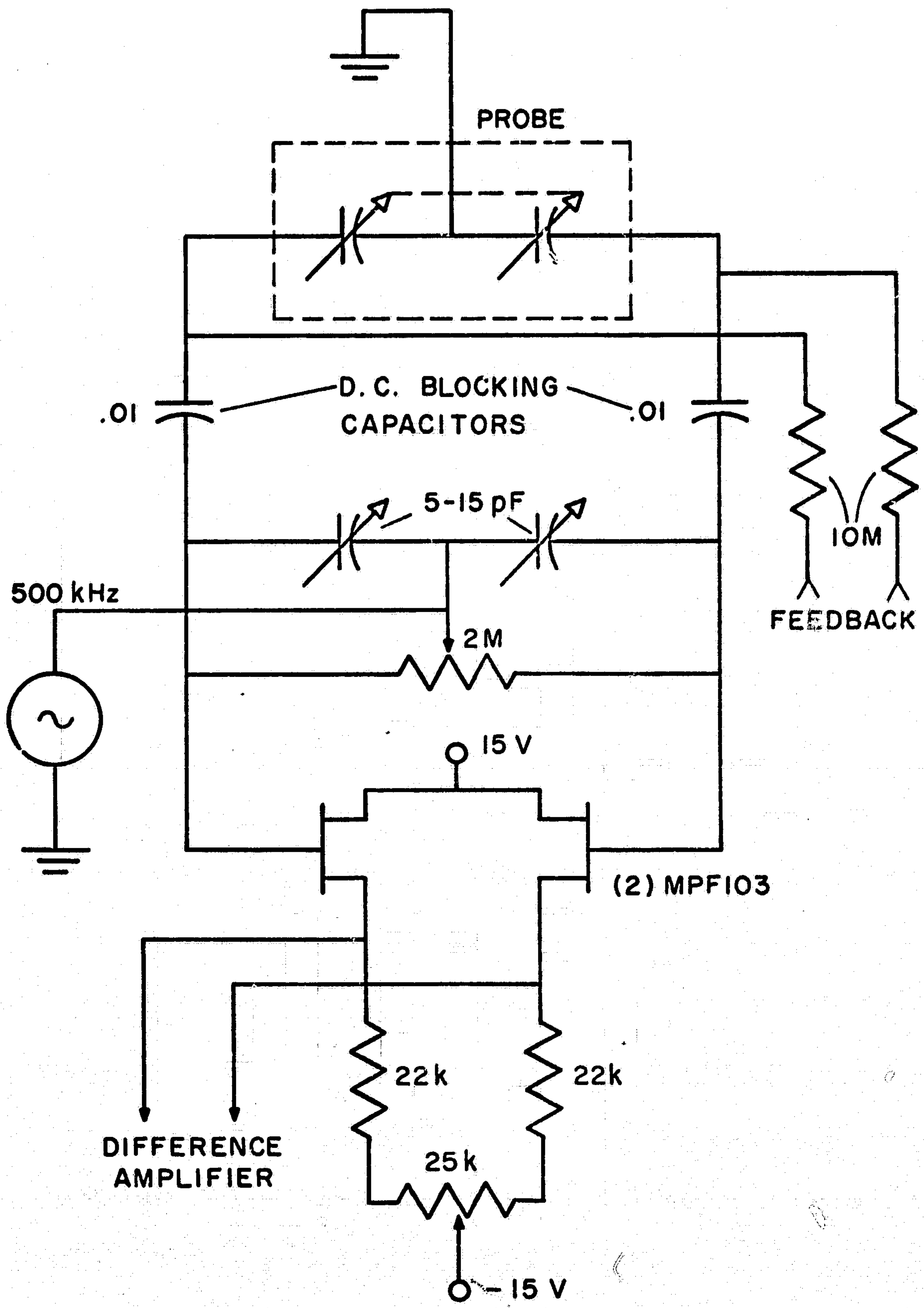

Figure 12. BRIDGE CIRCUIT.

3.3 Amplifier Circuit

The two bridge outputs are passed through high pass filters and their difference taken. A Fairchild μA709c integrated circuit (IC) amplifier is used as the difference amplifier, providing a gain of ten. This stage and the next are protected from overload by diodes. The high pass filters have a 100KHz cutoff and reduce some interfering signals, such as 60 cycle from the power lines. The filter also reduces the effects induced by mechanical vibrations. An a.c. amplifier is capacitively coupled to the difference amplifier. This is another μA709 I.C. module in a configuration giving a gain of 33 and no polarity reversal. Both I.C.s had to be compensated for operation at 500 KHz, their upper limit.

Following the a.c. amplifier is a 500 KHz tuned filter and the gating circuit. The filter is a parallel LC resonant circuit between signal path and ground. It is used mainly to reduce the 1 MHz second harmonic, which remains after the bridge is balanced. The effect of the second harmonic on the detector's performance is small. It would be negligible if the gate opened for exactly one half cycle of the first harmonic, for then a whole cycle of the second harmonic would pass and its positive and negative half-cycles would cancel each other. In reality, the gate opens for slightly less than a half-cycle so that the second harmonic contributes a small constant component. This component acts to shift the zero of the system, but can be compensated for by adding an opposite voltage at the later stage.

The gating circuit will be described in the following paragraph. A Tektronix Type 'O' operational amplifier is used for the next stage, a low frequency amplifier. The feedback loop consists of a 10 Megohm resistor and a large capacitor in parallel. The value of the capacitor is chosen by a switch on the front panel; the most commonly used values are 1μf, 0.1μf, and 0.01μf. When the .01μf capacitor is used, the cutoff frequency is approximately 2 Hz. For the other two capacitors, it is less than one Hz. The input resistance, R_1, is also selected by a panel switch, and d.c. gains from 10 to 1000 are available. This arrangement has proved superior to the usual RC smoothing filter, followed by a d.c. amplifier; much less noise is introduced into the restoring voltage for the same amount of filtering. The reduction of noise is such that it made the difference between satisfactory operation and non-operation of the author's detector.

A Tektronix Type '132' amplifier-power supply provides the voltages for the Type 'O' units and the last stage of d.c. amplification. The Type '132' has two outputs. The positive terminal gives a signal with the same polarity as the input. The negative terminal gives an output signal with the opposite polarity. Each terminal goes to ground through a 10K resistor and a 1N914 diode, in series. The diode is connected to ground with polarity such that a positive voltage will forward bias it. A detailed explanation of the operation of this resistance-diode network, with reference to the feedback voltage, is given in section 2.1, and will not be repeated. The feedback voltage is taken from across

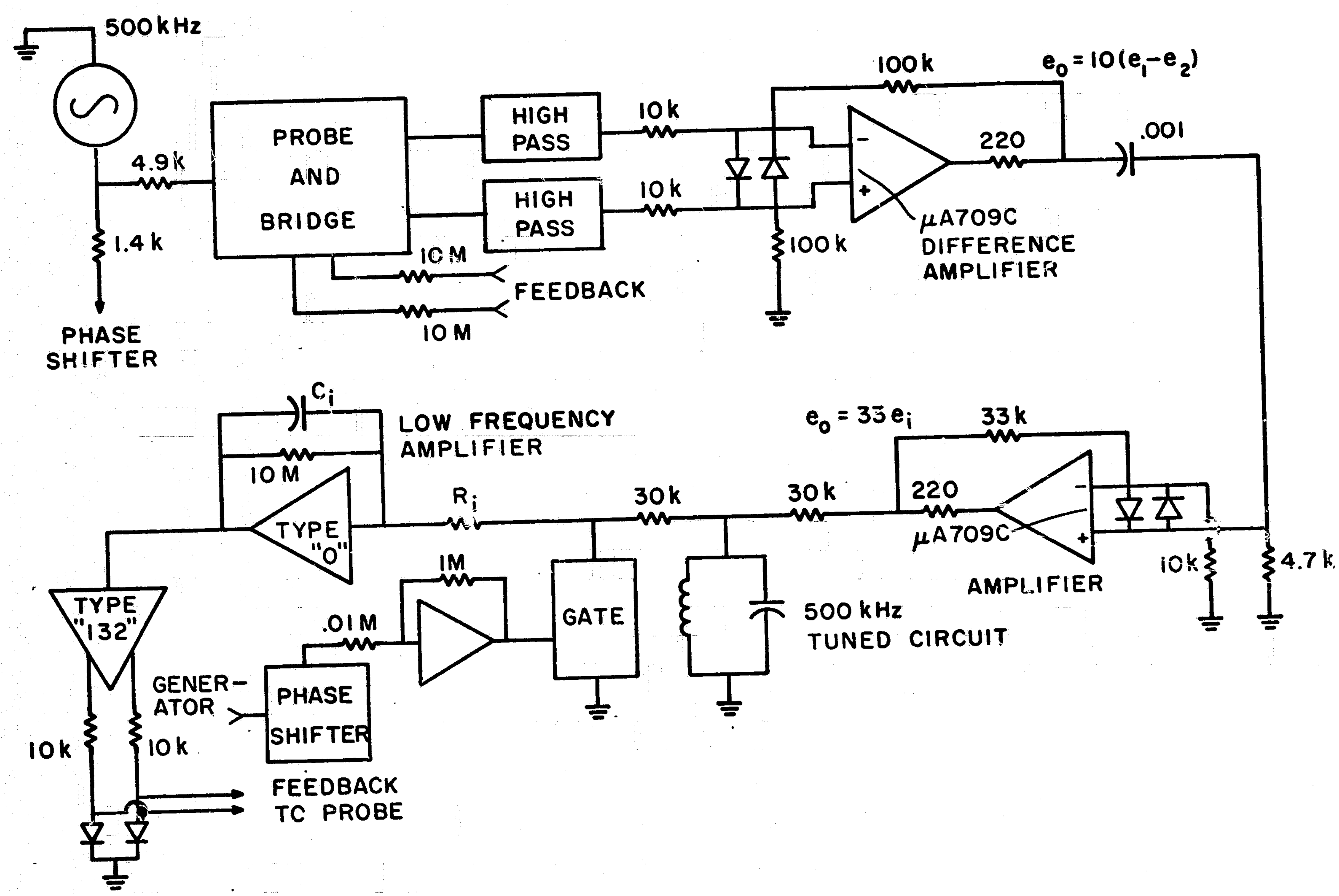

Figure 13. MOLECULAR BEAM DETECTOR.

the diodes. A 10 Megohm resistor is connected in series with the probe capacitor.

3.4 Gating Circuit

The gating signal (Fig. 14a) is taken from the same generator as the main signal, so that the phase difference will remain constant. An emitter follower circuit isolates the generator from the phase shifting network. The phase shifter is an RCL network which possesses a constant impedance at the selected frequency, and hence does not affect the amplitude of the signal. The gating signal passes to ground through a parallel RL combination, and a series capacitor. Ideally, the resistor is a rheostat variable from zero to infinity. A 10K unit was found to work well. The reactances are such that the capacitive reactance equals one half the inductive reactance at the operating frequency, 500 KHz. In this case, C = .001μf and L = .2mh. The output is taken across the capacitor. As R is increased from zero to large values, the phase shifts by 180°. A mathematical demonstration of this can be found in Chance (1965).

The shifted signal is amplified by a Tektronix Type 'O' amplifier. Its d.c. gain is set to 100 by an input resistance of .01 Meg and a feedback resistance of 1.0 Meg.

Basically, the gating circuit is a switch which shorts the main signal to ground on the positive half-cycle of the gating signal. Fig. 14b illustrates the gating circuit. The switch is a

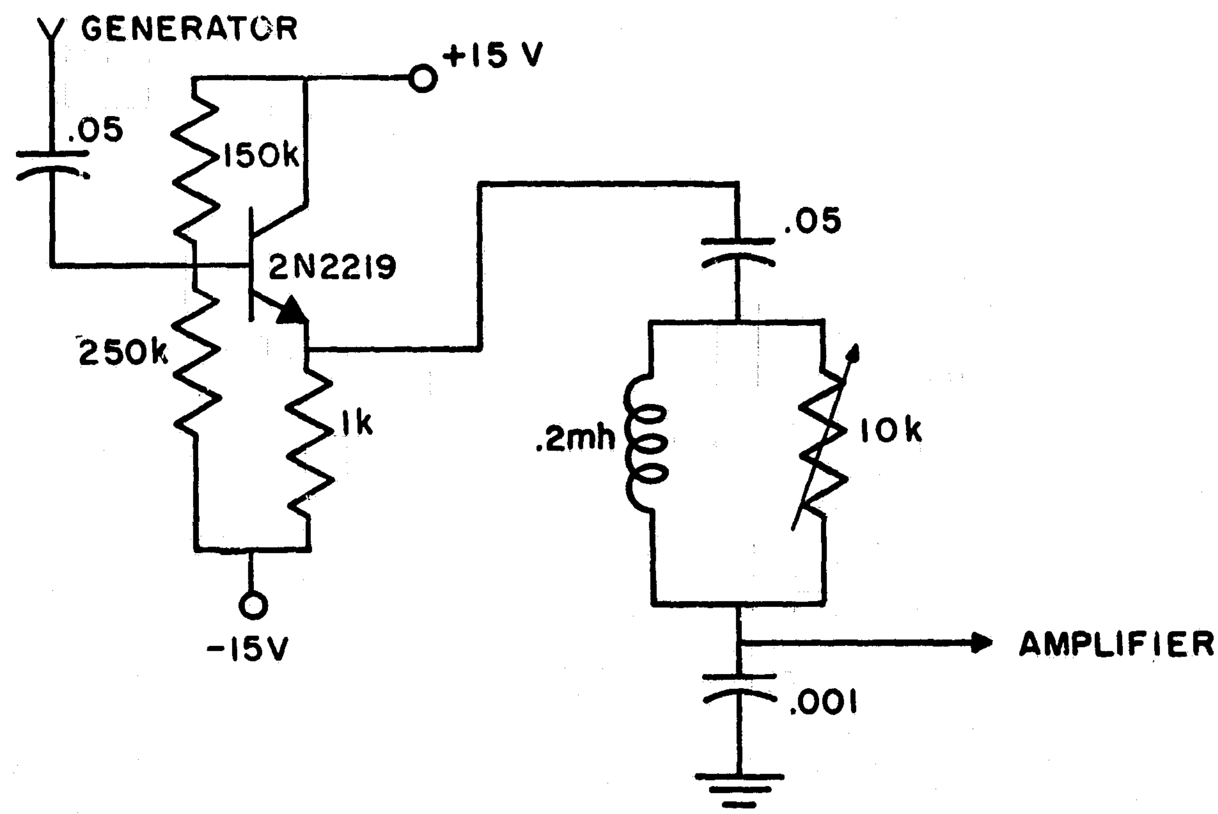

Figure 14a. PHASE SHIFTER.

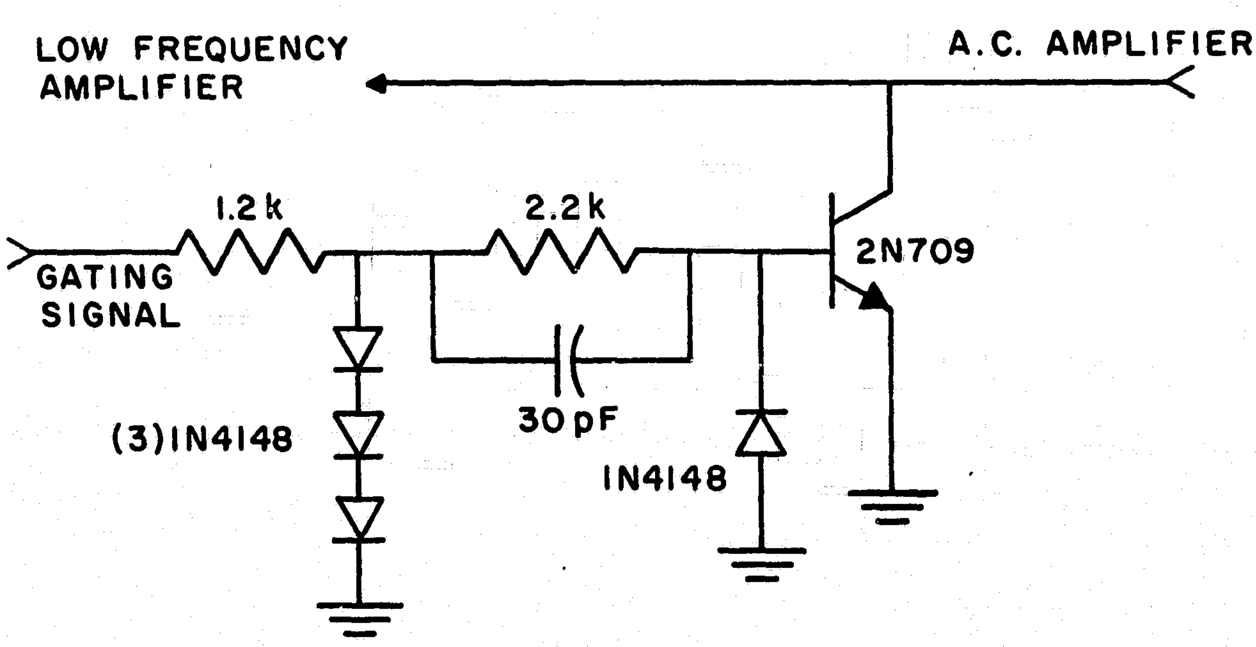

Figure 14b. GATING CIRCUIT.

2N709 transistor, which is biased into the conducting state by a positive voltage on the base. Three 1N4148 diodes clip the positive excursion of the gating signal to 2.1 volt, while another one limits the negative half-cycle to 0.7 volt. The values of the two resistances (1.2K and 2.2K) are much higher than the forward resistance of the diode. The gating circuit was designed by E. Barnes of the Ionosphere Research Laboratory.

IV. DESCRIPTION OF EXPERIMENT

Three types of experiments were performed. The first used a dummy probe capacitor in place of the true detector probe. This allowed the electronic circuit to be evaluated independently of the probe's characteristics. The second type uses the probe, but no feedback. The probe is mounted at the end of a 16.6 inch rod, which is pivoted at the opposite end. By rotating the rod, the inner cylinder is displaced a known distance, with respect to the outer cylinder, giving open loop voltage as a function of displacement. The data will be used in Chapter V to calculate the experimental value of q, so experimental results may be compared with theory. The third experiment is similar to the second, except feedback is used. Since the restoring force is pulling against the gravitational force, the experiment gives a direct recording of voltage versus force. In all experiments, the voltages were recorded on a Moseley Autograph Model 680 strip chart recorder with 2 Megohm input impedance in the voltage ranges used. The most commonly used range was ± 50 volts and ± 25 volts. During data runs, the monitoring oscilloscope was disconnected from the detector. It was normally connected at the gating point and added a significant loading effect, despite its 10 Megohm input impedance (paralleled by 14 pf).

4.1 Dummy Probe experiment

The dummy probe consists of a rod of Plexiglass fitted inside a Plexiglass cylinder. The outer surface of the cylinder is coated with conducting paint, and is grounded during operation. Except for the knob-handle, both ends of the rod are also coated for a length of 0.45 inches. Two capacitors are formed in this way. The capacitances of each are related such that, as one increases by pulling or pushing the rod, the other decreases in the same manner in which the probe capacitors react to a displacement of the outer cylinder.

Let x' be the length of the capacitor on the handle end of the rod. If end effects are ignored, the capacitance C_1, is

$$C_1 = \frac{2\ \pi \varepsilon x'}{\ell n\ (D/d)}$$

Using the values D = .384 inches, d = .252 inches, ε = 2.5, ε_0 = 22.1 x 10^{-12} farad/meter, find C_1 = 4 x' pf, where x' is in centimeters. In the experiment, the measured distance is not x', but x, i.e., the distance between the knob-handle and the cylinder. Since x = b-x' and b = 0.45 inches = 1.14 cm,

$$C_1 = 4\ (1.14-x)\ \text{pf}.$$

Now, $\Delta C_1 = -4\ \Delta x$, and the total change in capacitance of C_1

and C_2 is $2\ \Delta C_1$, so that

$$\Delta C = 8\ \Delta x$$

The bridge gives a linear voltage versus displacement curve over a range of at least .30 inches (.76 cm.). centered about the null position. In the null position, each capacitor is adjusted to half its maximum value, i.e., $x' = x = 0.57$cm. Therefore, the capacitance of one capacitor can change .38/.57 = 67 per cent of its null position value and the bridge will remain in its linear region. Measurements of x were made with a scale divided into hundredths of an inch; the scale was read with the aid of a magnifying glass.

4.2 Gravity Calibration Experiment - Open Loop

A simplified diagram of the apparatus is shown in Fig. 15. The length of the Plexiglass rod is 15.61 inches or 39.7 cm. The distance from the pivot point to the adjusting screw (l_2) is 28.3 cm.; l_1 is 13.3 cm. The length l_2 was chosen so that for one turn of the screw, x_1 would equal 0.03 cm. A screw with 40 threads per inch was chosen (standard 4-40 size), and the end turned to a point. A steel plate was glued to the rod at the point of contact to provide a flat, smooth surface. Tension is provided by a spring which forces the rod against the screw point.

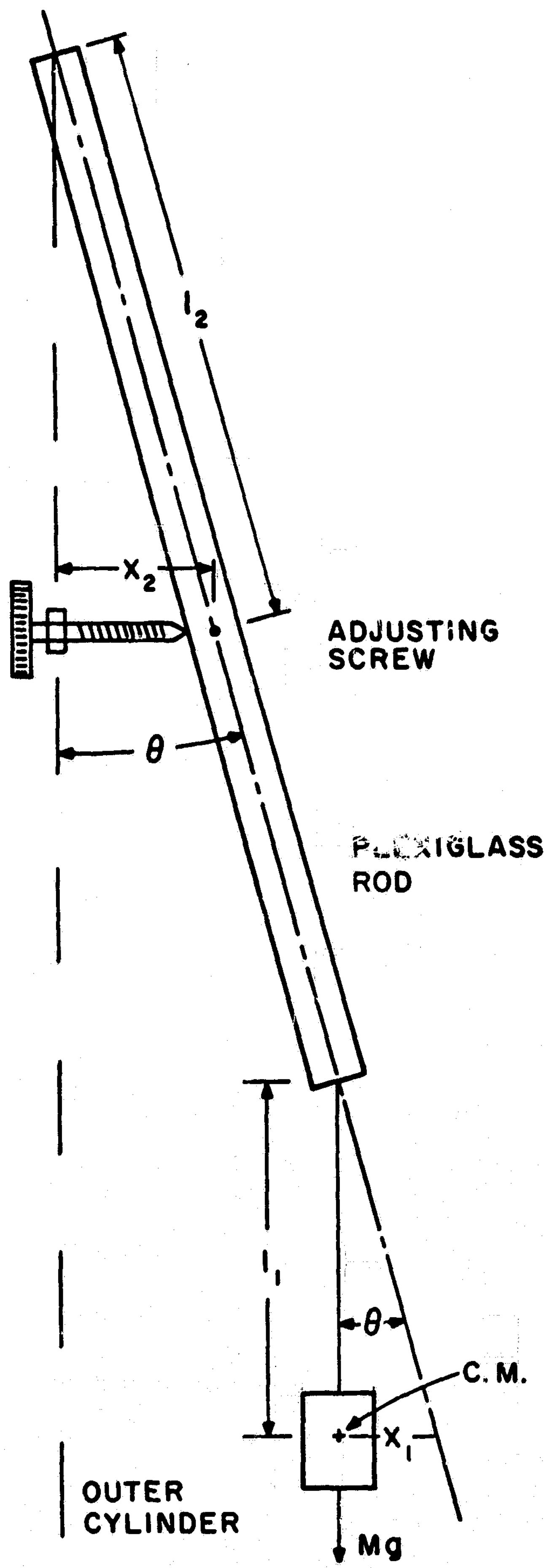

Figure 15. SIMPLIFIED DIAGRAM OF GRAVITATIONAL CALIBRATION EXPERIMENT.

The force required to pull the outer cylinder through an angle θ is given by

$$F = Mg\theta \tag{42.1}$$

The relation between θ and a rotation of the adjusting screw is

$$\theta = N/nl_2, \tag{42.2}$$

where

N = number of turns of screw

n = threads per unit length

Then,

$$F = \frac{MgN}{nl_2} \tag{42.3}$$

Using the value of 70 mg for the mass of the outer cylinder,

F = 154 N millidyne (mdn).

Since there was no rotation indicator, except the slot in the head of the screw, data points were usually taken at 1/4 turn intervals; this is because it is relatively easy to set a 1/4 turn interval by eye. This amount of rotation corresponds to a 0.0075 cm. displacement of the outer cylinder and an equivalent force of 38.6 mdn. Assume a maximum error of ± 10° in position, then the error would be 11 per cent, or .0008 cm. and 4.3 mdn, respectively. This amount of error is easily detectable. For this reason ten to twenty data points were taken for each screw position during a given run and an average computed. The data points were actually closer than an error of 11 per cent would imply, indicating the accuracy of the adjustments.

Before continuing, it will be necessary to consider the open loop system gain. The molecular beam detector is a force amplifier. An input force, a wind, displaces the outer cylinder; a voltage is produced, which in being applied across the probe capacitor, produces an opposing force. To truly measure the open loop gain of the system, one would have to break the circuit at the point where the restoring force was being added to the signal force, before being converted to a displacement. Then, for a given input force, the restoring force would be measured. This is impossible because the voltage to force conversion occurs at the same time and place as the force addition and the two processes cannot be separated. Knowing the conversion factor between voltage and restoring force, it might be suggested that the gain be measured in the following way: disconnect the restoring voltage so that an open ended circuit is obtained. Now, the wind can be applied to the probe and a voltage vs. force curve obtained; the voltage is then converted to a restoring force mathematically. The failure of this method lies in the fact that, without the restoring voltage, only gravity opposes the action of the wind on the outer cylinder, and the force-displacement relation is linear for small displacements, i.e., $Mg\theta$. The results of this experiment would depend on the value of g at the place the experiment was performed. If it was performed in a gravity-free environment, any perturbing force would cause infinite displacement. These facts are in conflict with the realization that the detector has been designed such that the restoring force is so great as to make the effects of gravity negligible.

Other schemes have been concocted by the author to measure the open loop force gain of the system, but all have failed because of the coupling which occurs at the probe capacitor. The only parameter which is actually varied from experiment to experiment, and which is related to system gain, is the voltage gain. This quantity will be chosen to characterize the system gain. The electronic signal must be considered as two components: a 500 KHz carrier and the input signal, which modulates the carrier. In the future, these two components will be referred to as the carrier and signal, respectively. The net gain for a slowly varying signal will be called the open loop system gain (G_{DC}).

The difference amplifier has a gain of ten over the 500 KHz range; the a.c. amplifier, which follows it, has a gain of 330, giving a net gain of 3300. The gate treats the signal as would a half-wave rectifier with a high resistance load. The pulsating d.c. is smoothed by a low pass filter which tends to average the signal, giving a gain of about 1/2. The low frequency amplifier has a d.c. gain variable from 10 to 1000. In its most selective state (CF = 1μf), the carrier is eliminated and the gain for a slowly varying signal can be considered the d.c. gain. The last amplifier has a gain of 1/40 to 10 and the entire voltage gain then lies in the range 400 to 16,500,000. In practice, the highest gain used is 825,000.

When high open loop gains are used (above 100,000), the data points for 1/4 turn intervals were too far apart; with a gain of 330,000, for instance, a 1/2 turn interval causes the ± 50 volt recorder to go off-scale. In these cases, 1/8 turn increments were used. A 10° error now represents 22 per cent of the increment, but again, the data point showed a smaller error spread (about 5 per cent).

A major reason for doing the open loop experiment is to evaluate the parameter b, referred to in section 2.3; b is the conversion between open loop output voltage and the outer cylinder's displacement, and is needed to calculate q_0. With a gain of 165,000, b = 4800 V/cm. This value can be considered constant only for small displacements (3/8 turn from null position). Displacements larger than this cause the bridge to exceed its linear range and becomes an increasing function of x. This problem does not occur with the closed loop system because the restoring force keeps the cylinder near the null position. The parameter b is proportional to gain.

4.3 Gravity Calibration Experiment - Closed Loop

The same procedure was used in this experiment as in the previous one, i.e., the restoring voltage is recorded at 1/4 or 1/8 turn intervals of the adjusting screw. The major difference between the two is that the restoring voltage is measured as it is being applied across the probe capacitors and while the restoring force is acting against a force (Mgθ). Knowing the magnitude of gravitational force corresponding to each setting of the adjusting screw, the voltage required to produce an equal force can be calculated and a theoretical voltage-force curve drawn for comparison with experimental results. In this manner, it was found that a voltage gain of 330,000 was required to bring the experimental data to within 5 per cent agreement with the theoretical curve at 1/8 turn. The comparison is shown in Fig. 16.

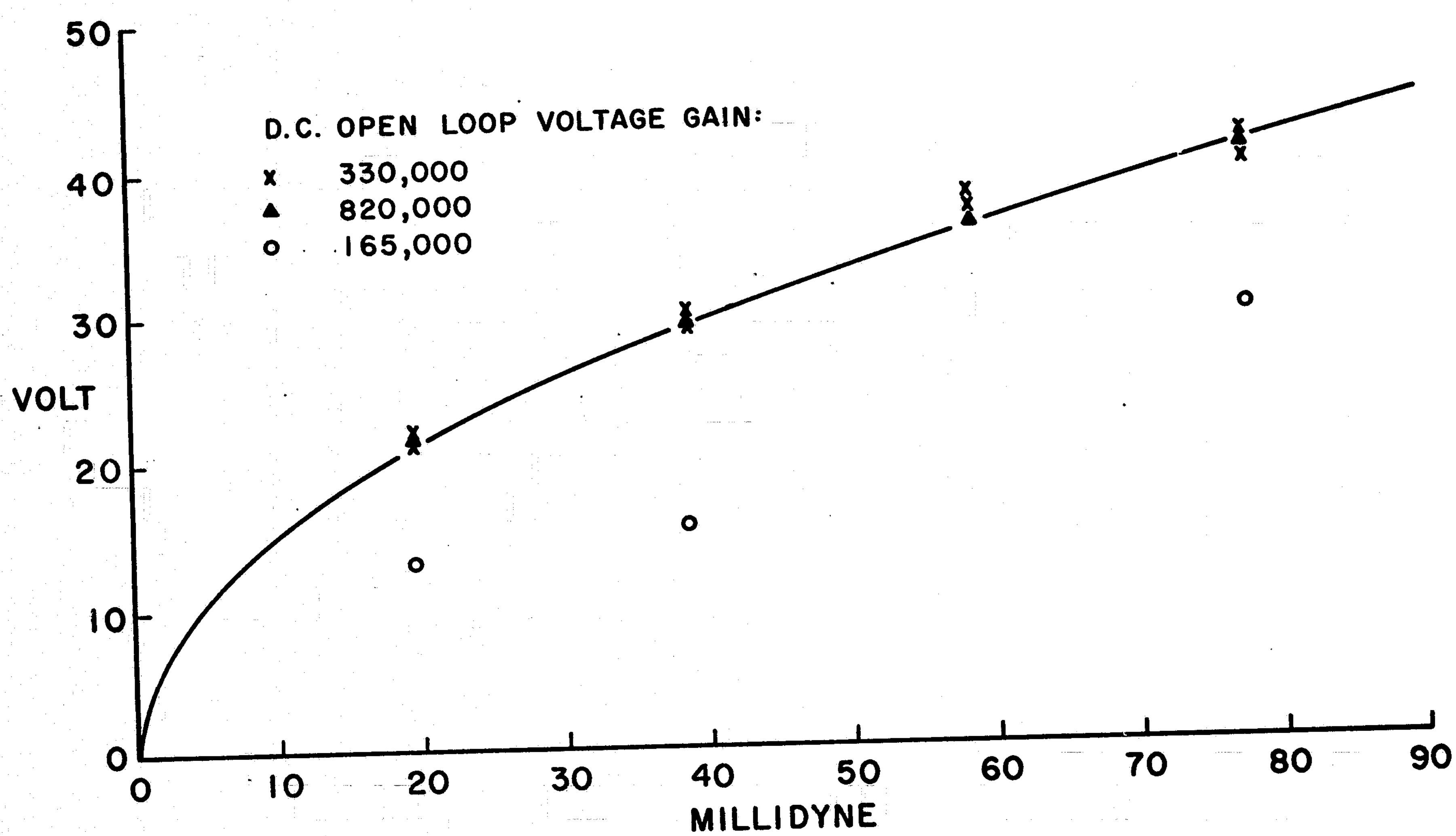

Figure 16. THEORETICAL AND EXPERIMENTAL CLOSED LOOP PERFORMANCE.

V. ANALYSIS OF EXPERIMENT

5.1 Theoretical Performance

An important characteristic of this detector is that the theoretical voltage-force response is, for high gain, independent of all circuit parameters, except the probe capacitor's dimensions. All that must be calculated is the voltage required across the capacitor to nullify the perturbing force. Equation 21.1 is used for this purpose. Solving for V as a function of F,

$$V = [\ 2/\varepsilon_0 A\ ]^{1/2}\ \ SF'^{1/2} \tag{51.1}$$

where

ε_0 = 8.87 x 10^{-12} $coul^2/nt - m$

A = area of capacitor plate (the effective area defined in section 3.1)

S = null position plate separation

F' = perturbing force

Using the probe dimensions, as given in section 3.1, the formula reduces to

$$V = 147\ F^{1/2} \tag{51.2}$$

where V is in volts, F is in dynes.
A curve is shown in Fig. 16 for the range of forces studied in the experiments. Data points are illustrated for three levels of gain. Error bars are not included for clarity, but can be assumed to be ± 2 volt at most. A gain of 330,000 is seen to give about 10 per cent error with respect to the theoretical curve, while an 825,000 gain puts the data points within 2 per cent.

To make further comparisons with theory, it will be useful to develop a formula for q as a function of d.c. open loop gain. By equation 23.4,

$$q_0 = ab^2$$

where

$$a = 1.8 \times 10^{-4} \text{ dyne/volt}^2$$

The parameter b was found from the open loop experiment to be 4800 volt/cm for a d.c. gain (G_{DC}) of 165,000.
Thus,

$$b = 2.90 \times 10^{-2} \, G_{DC} \text{ volt/cm} \qquad (51.3)$$

and

$$q_0 = 15.2 \times 10^{-8} \, G_{DC}^{\;2} \text{ dyne/cm}^2 \qquad (51.4)$$

Dividing by the mass of the outer cylinder gives q:

$$q = 2.16 \times 10^{-6} \, G_{DC}^{\;2} \text{ dyne/cm}^2 - \text{gm} \tag{51.5}$$

The values of q corresponding to the three G_{DC} used in Fig. 16, i.e., 165,000, 330,000, and 825,000, are 5.88×10^4, 23.5×10^4, and 137×10^4 respectively.

A simple relationship between the percentage error (E) of the system, with respect to the theoretical curve, and G_{DC} exists:

$$E = \alpha / \, G_{DC}^{\;2} \tag{51.6}$$

where

$$\alpha = 1.03 \times 10^{12}$$

This relationship was found empirically and holds for $G_{DC} > 150{,}000$ or $q > 47{,}000$.

A rigorous justification of the above formula would involve the theory of non-linear feedback and automatic control systems. An intuitive justification can be presented, however. If the system had infinite gain, the position error would be zero, giving credit to the inverse relation. The force which resets the outer cylinder is proportional to the square of the voltage across the capacitor, implying that the error would vary as the inverse of the voltage (or gain) squared.

Converting to q,

$$E = \beta/q \tag{51.7}$$

where

$$\beta = 2.2 \times 10^6.$$

Throughout the theoretical discussion, it has been assumed that q_0 was so large as to make the gravitational force negligible. Assume, for the moment, that this force is not insignificant when compared to the signal force F. The gravitational force on the outer cylinder is kx, where $k = Mg/l$ and l = length of the wire suspending the cylinder. Let x_0 be the equilibrium displacement of the outer cylinder with all three forces acting on it. Solving the equilibrium equation for x_0,

$$x_0 = \frac{1}{2q_0} \left[- k + (k^2 + 4\, q_0 F)^{1/2} \right]$$

Define R as the ratio

$$R = \frac{kx_0}{F} \tag{51.8}$$

The problem of interest is that of finding the q_0 necessary to make R as small as desired for a given minimum force F_M.

Solving the two equations:

$$q_0 = \frac{k^2}{F_M} \frac{(1-R)}{R^2} \tag{51.9}$$

or, for small R,

$$q_0 = \frac{k^2}{F_M R^2} \tag{51.10}$$

As an example find the minimum force which can be measured with the author's apparatus such that, with $q_0 = 9.8 \times 10^4$ ($G_{DC} = 825{,}000$), the gravitational force will be 5 per cent of the signal ($R = .05$). For this probe, $k = 5.4$ dyne/cm, then $F_M = 120$ mdn. The figure is rather high considering that an instrument capable of measuring 0.1 to 0.01 mdn is desired. There is a correction, which can be applied to the measured force F' to give the true force F:

$$F = F' \left| 1 + k/(q_0 F')^{1/2} \right| \tag{51.11}$$

An alternative is to place a square rooting circuit in the detector to give a linear voltage-force relation. The square rooter will be described in section 5.2; just its effect will be discussed here. With the above changes, the feedback force will be px instead of $q_0 x^2$. If the square rooter is just before the probe, the relation between p and q_0 is $p=q_0/b$. Using the relations between q_0 and b with G_{DC}, the restoring force becomes

$$F = px = (5.24 \times 10^{-6}\ G_{DC})\ x \qquad (51.12)$$

The largest G_{DC} used in the experiments was 825,000. The value of p is then 4.3 dyne/cm. As already seen, the value of k is 5.4 dyne/cm, or 125 times larger than p.

The solution to this problem is to move the square rooter back as many d.c. amplifier stages as possible, since p, as determined above, is multiplied by the square root of the gain of each stage. If this is done, a more complicated square root circuit must be used; otherwise the output will always be positive and the direction switching capability of the restoring voltage will be destroyed (see section 5.2).

Having shown that G_{DC} determines only the accuracy of the detector, and not its range, it is important to consider in what ways the probe capacitor dimensions affect the theoretical voltage-force curve. In all cases, the upper limit is determined by the maximum voltage output of the amplifier (± 50 volt for the Tektronix Type 132). The parallel plate approximation for the probe capacitance will continue to be used. It may not be numerically accurate in all cases, but it will illustrate what changes will occur.

Recall equation 51.1 for convenience:

$$V = \left(\frac{2}{\varepsilon_0}\right)^{1/2} \frac{S}{A^{1/2}}\ F^{1/2} \qquad (51.1)$$

Increasing the diameter of the inner cylinder, while keeping the outer cylinder constant, would leave $A^{1/2}$ almost constant; voltage then varies linearly with s. Halving s increases the potential force measuring range of the instrument by a factor of two. The value of q will increase proportionally because the more rapid change in capacitance, for a given displacement, causes the voltage output of the bridge to have a larger slope. Thus, the error of the system will vary in proportion to s.

Varying the mass of the outer cylinder causes q to change inversely. A given change in q causes E to vary inversely with the end result that E changes in proportion to the cylinder's mass (m). Evaporation techniques might be used to produce a very small m.

The consequences of reducing the size of the entire probe are too complicated to predict accurately. The value of a probe capacitor in the author's detector is about one picofarad. The effects of stray capacitance on a smaller probe would be such as to reduce the voltage output of the bridge for a given displacement. This would counteract the increased sensitivity due to closer plate spacing (S). Further analysis of the subject is beyond the scope of this thesis. It is an important consideration, however, and should not be disregarded by any potential user of the detector.

The response time of the detector is also very important in application and is mainly determined by the feedback components of the low frequency amplifier. High values of RC were used in the author's model to produce a low cutoff frequency; the resulting risetimes were on the order of 1/2 minute. This time is easily improved by reducing RC to 1 sec. or less, without significantly reducing the overall performance of the system.

5.2 Critique of Electronic Performance

The two main faults of the detector were 1) the gating circuit did not gate exactly 1/2 cycle and 2) a 1 MHz second harmonic was present with the carrier and the usual filter methods worked poorly. These two faults were coupled because, if the gate had operated properly, the second harmonic would have been of no importance (a complete cycle would be passed) and, if the harmonic had been absent, an exact half-cycle gate would not have been necessary. The data shows that given sufficient gain, the detector works as expected. The system error might have been less if these irregularities were not present, but that is in the realm of speculation.

Other methods of detection might be used, such as a lock-in amplifier. In this case, the output of one arm of the bridge would be used as the reference signal and the other output would be measured with respect to it. A P.A.R. Model 121 lock-in amplifier (Princeton Applied Research Corp.) was tested by the author in this application; while a thorough evaluation was not possible at the time, it performed at least as well as his own unit and was much simpler to operate. Additional amplification is needed.

If a gating method is to be used in a developed model, it might benefit the user to utilize a full wave rectifying version, rather than "throwing away" the half-cycle of signal. This would have the advantage of doubling the available r.m.s. signal and reducing the filter requirements. The latter feature would improve the risetime of the detector.

If a linear voltage-force output is desired, it can be obtained by using a circuit which applies the square root of the restoring voltage to the probe. The method most commonly used in analog computing is to use a squaring device in the feedback loop of an operational amplifier. The problem in most cases is that of taking the square root of a negative voltage. A switching circuit of some type must be used to provide an output with the same sign as the input to the square rooter. Alternatively, the square rooter can be placed in each of the return leads to the probe after the resistor-diode network.

System noise was not a problem with the author's apparatus, mainly because of the large time constants used. Figure 17 shows the worst case of extraneous voltage variation encountered during the experiments. In a fast response system, which would probably be used in practice, care must be taken in suppressing interference from outside vibrations.

5.3 Conclusions and Recommendations

The object of this project was to show whether or not a two-dimensional small force detector was feasible using the probe described above. It is. The device can be very useful, as indicated in the introduction, and should be developed to a higher degree. If the project was to be done over again, a commercially available

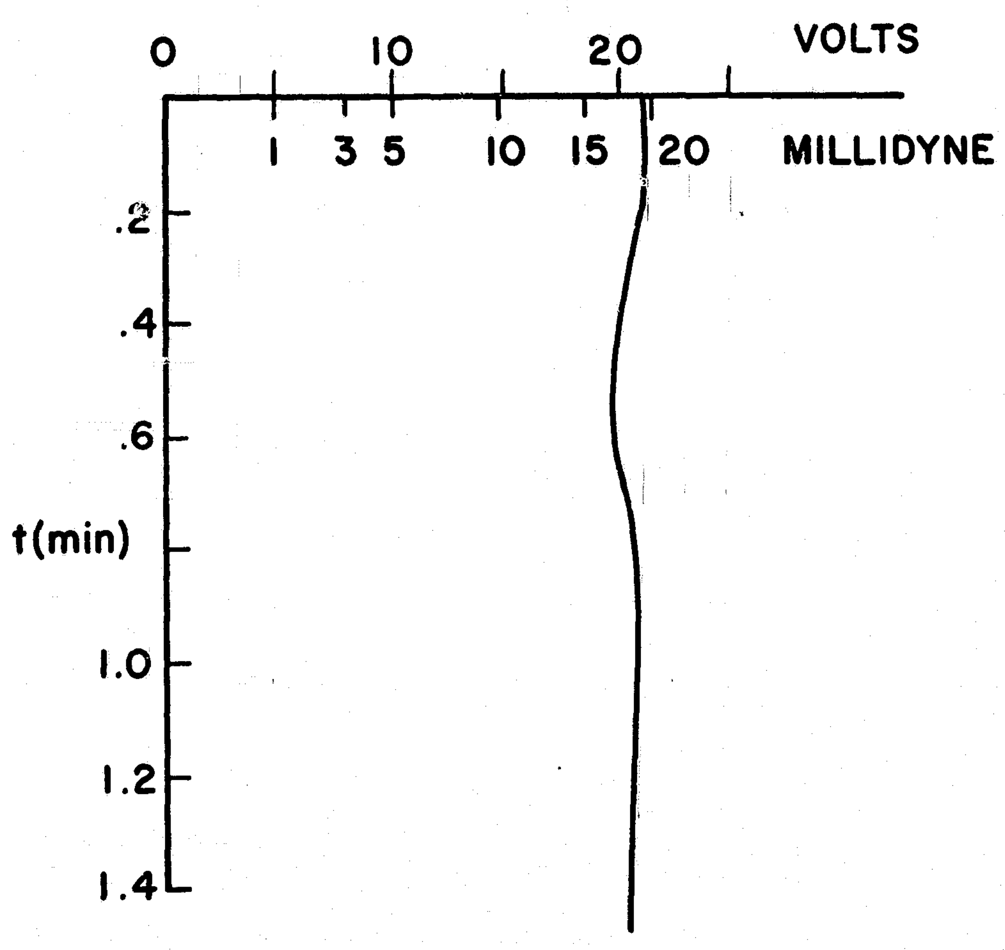

Figure 17. VOLTAGE VARIATION OF 19.6 MILLIDYNE SIGNAL (G_{DC} = 825,000; FILTER TIME CONSTANT = 10 SEC.).

detector, such as the lock-in amplifier mentioned in section 5.2, would have been used in place of the circuit described above. More time could then have been spent in developing the probe and evaluating more fully the performance of this type of system. It is regrettable that time did not allow the detector to be tested in a working environment with an actual molecular beam. The question of coupling between the x and y direction signals must be investigated. It may be advantageous to use two frequencies not harmonically related.

The biggest problem may be that of obtaining a large enough q to make reliable measurements of millidyne forces and below. A discussion of this subject was given in section 5.1. It appears that values on the order of 10^8 and above may be needed. If this is the case and larger electronic gains must be used, then noise will be of paramount importance in the determination of accuracy.

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