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CONTRIBUTION OF THE SPATIAL PART OF THE METRIC TENSOR TO THE OBSERVED FREQUENCY SHIFT IN THE MASER CLOCK EXPERIMENT

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A formula for the frequency shift of a maser clock in orbit relative to an identical maser on the earth is derived. In addition to the usual terms, due to the gravitational potential difference and special relativistic effects, this formula is shown to contain terms that depend on the spatial part of the metric tensor. These extra terms cannot be derived using only the equivalence principle. It is shown that the frequency shift given by these additional terms appears to be observable with presently attainable maser accuracies. The measurement of the additional effects will yield the same information about the metric as will the geodetic precession of a gyroscope or the deflection of light.

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TABLE OF CONTENTS

CONTRIBUTION OF THE SPATIAL PART OF THE METRIC TENSOR TO THE OBSERVED FREQUENCY SHIFT IN THE MASER CLOCK EXPERIMENT

INTRODUCTION

The orbiting maser clock experiment is generally thought of as a test of the equivalence principle. It will be pointed out in this paper that the experiment, or some modification of it, is capable, in principle, of yielding information not derivable from the equivalence principle. More precisely, the exact frequency shift observed depends not only on the gravitational potential, but on the space components of the metric tensor as well. The additional effects, which will be denoted by S, are generally small compared to those due to the gravitational potential, but we hope to show that the accuracy of maser clocks is compatible with the possibility of devising a scheme by which these additional effects are observable. The measurement of S will yield the same information about the metric as the measurement of the light deflection or the geodetic precession of a gyroscope.

DERIVATION OF S

Consider the propagation of an electromagnetic signal from an emitter situated at one point in space to a receiver situated at a different point. We wish to derive a formula for the shift in frequency of the signal as observed by the receiver. To do this, we will assume that a coordinate system is given and that the world lines of both receiver and emitter are known. The gravitational field will be described by the metric corresponding to the given coordinate system. The situation is depicted in Figure 1, in which 1 refers to the emitter and 2 to the receiver, and the connecting lines refer to the signals exchanged. Suppose n wave crests leave the emitter between t_1 and $t_1 + dt_1$. Then the

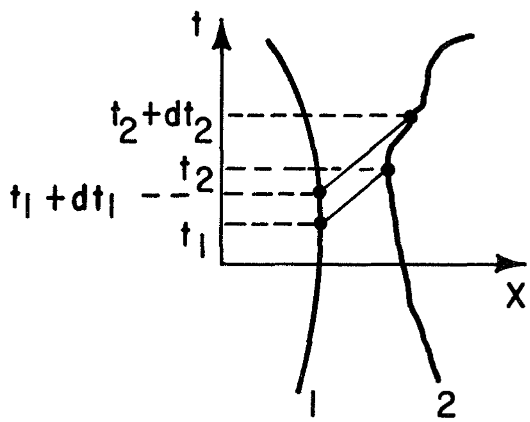

Figure 1.

frequency in the rest frame of the emitter is

$$f_1 = \frac{n}{d\tau_1} = \frac{cn/dt_1}{\left(\sqrt{g_{ij}\,\frac{dx^i}{dt}\,\frac{dx^j}{dt}}\right)_1}$$

where $d\tau_1$ is the proper time interval corresponding to the coordinate time interval dt_1. The expression $\left(\sqrt{g_{ij}\,\frac{dx^i}{dt}\,\frac{dx^j}{dt}}\right)_1$ means that the quantity in the bracket is evaluated at t_1. The quantities under the radical refer to the world line of the emitter. Now let t_2 be the time of reception of the signal emitted at t_1, and $t_2 + dt_2$ the time of reception of the signal emitted at $t_1 + dt_1$. Since n crests have arrived between t_2 and $t_2 + dt_2$, the frequency observed in the rest frame of the receiver is

$$f_2 = \frac{n}{d\tau_2} = \frac{cn/dt_2}{\left(\sqrt{g_{ij}\,\frac{dx^i}{dt}\,\frac{dx^j}{dt}}\right)_2}$$

and $d\tau_2$ is the proper time interval corresponding to dt_2. Therefore

$$\frac{f_2}{f_1} = \frac{dt_1}{dt_2}\left[\frac{\left(\sqrt{g_{ij}\,\frac{dx^i}{dt}\,\frac{dx^j}{dt}}\right)_1}{\left(\sqrt{g_{ij}\,\frac{dx^i}{dt}\,\frac{dx^j}{dt}}\right)_2}\right]$$

A similar derivation is given by Tolman [1]. We know the world lines of emitter and receiver so we can easily compute the second bracketed factor. To compute the first factor we must know t_1 as a function of t_2. That is, we must know the world line of the signal exchanged; this world line is determined by the equations of a null geodesic, and in a general coordinate system these equations must be solved to obtain t_1 as a function of t_2.

The formula above can be written as

$$\frac{f_2}{f_1} = \frac{\left(\sqrt{g_{ij}\ dx^i\ dx^j}\right)_1}{\left(\sqrt{g_{ij}\ dx^i\ dx^j}\right)_2}$$

In this form it is obvious that the result is independent of the coordinate system chosen, since $g_{ij}\ dx^i\ dx^j$ is invariant under a coordinate transformation. It must be remembered that these coordinate differentials refer to the segments of the world lines of the emitter (receiver) between t_1 and $t_1 + dt_1$ (t_2 and $t_2 + dt_2$).

The use of this formula will now be illustrated by computing the frequency shift observed in a Schwarzschild field. The metric for such a field will be taken to be

$$g_{00} = -\left(1 - \frac{2M}{r}\right) \qquad g_{\alpha\beta} = \delta_{\alpha\beta} + \frac{2Mx^\alpha x^\beta}{r^3}$$

$$g_{ij} = 0 \qquad i \neq j \qquad \text{and } M = \frac{Gm}{c^2} ,$$

where m is the mass of the attracting body. Here the metric has been expanded to first order in M/r.

We first compute t_1 as a function of t_2. To do this, we will assume the world line of the light signal to be a straight line in three-space. It is easy to see that deviations from the straight-line propagation will produce effects of order $(M/r)^2$ which can safely be neglected for present purposes. The signal propagates along the path given by

$$x^\alpha = x_1^\alpha + w\left(x_2^\alpha - x_1^\alpha\right),$$

where w is a parameter, $0 \le w \le 1$, and x_1^α, x_2^α refer to the spatial coordinates of the emitter at t_1 and the receiver at t_2, respectively. Then, along this path,

$$dx^\alpha = \left(x_2^\alpha - x_1^\alpha\right) \; dw.$$

Since the path is a null geodesic, we have

$$0 = ds^2 = g_{ij} dx^i dx^j = g_{\alpha\beta} dx^\alpha dx^\beta + g_{00} \; (dx^0)^2 \; ,$$

where the coordinate differentials here refer to the path of the signal. Integrating along this path

$$c\;(t_2 - t_1) = \int_{t_1}^{t_2} \left[\frac{g_{\alpha\beta} \; dx^\alpha \; dx^\beta}{-g_{00}}\right]^{\frac{1}{2}}$$

$$= \int_0^1 \left[\frac{\Delta x^\alpha \, \Delta x^\beta \left(\delta_{\alpha\beta} + \frac{2Mx^\alpha x^\beta}{r^3}\right)}{1 - \frac{2M}{r}}\right]^{\frac{1}{2}} dw$$

where $\Delta x^\alpha = (x_2^\alpha - x_1^\alpha)$. Expanding this expression to first order in M/r, we obtain

$$c(t_2 - t_1) = \Delta x \left[1 + M \int^1 \frac{dw}{r} + M e^\alpha e^\beta \int_0^1 \frac{x^\alpha x^\beta}{r^3} \; dw\right]$$

where

$$(\Delta x)^2 = (\Delta x^\alpha \Delta x^\alpha); \; r^2 = (x^\alpha x^\alpha); \; e^\alpha = \frac{x_2^\alpha - x_1^\alpha}{\Delta x}$$

Expressing r and Δx^2 as functions of w, x_1^α, and x_2^α and performing these integrals, the result is

$$c(t_2 - t_1) = \Delta x + M \left[2 \log \frac{r_2 + \bar{r}_2 \cdot \bar{e}}{r_1 + \bar{r}_1 \cdot \bar{e}} + \frac{\bar{r}_1 \cdot \bar{e}}{r_1} - \frac{\bar{r}_2 \cdot \bar{e}}{r_2} \right] ,$$

where

$$r_1^2 = x_1^\alpha \, x_1^\alpha \qquad \bar{r}_1 \cdot \bar{e} = x_1^\alpha \, e_1^\alpha \quad \text{etc.}$$

Since x_1^α, x_2^α are known functions of t_1, t_2, this gives t_1 as an implicit function of t_2. This equation can now be differentiated implicitly to obtain dt_1/dt_2. First note that

$$\frac{d(\Delta x)}{dt_2} = \bar{e} \cdot \bar{v}_2 - \bar{e} \cdot \bar{v}_1 \frac{dt_1}{dt_2}$$

where $\bar{v}_1 (\bar{v}_2)$ is the velocity of the emitter (receiver) at time $t_1 (t_2)$. Letting $\wedge$ denote unit vectors,

$$\frac{dr_2}{dt_2} = \hat{r}_2 \cdot \bar{v}_2 \qquad \frac{dr_1}{dt_1} = \hat{r}_1 \cdot v_1 \frac{dt_1}{dt_2}$$

$$\frac{d}{dt_2} (\bar{r}_2 \cdot \bar{e}) = (\bar{v}_2 \cdot \bar{e}) + \frac{1}{\Delta x} \left[\bar{r}_2 \cdot \bar{v}_2 - (\bar{r}_2 \cdot \bar{e}) (\bar{v}_2 \cdot \bar{e}) \right] - \frac{dt_1}{dt_2} \left\{ \frac{1}{\Delta x} \left[\bar{r}_2 \cdot \bar{v}_1 - (\bar{r}_2 \cdot \bar{e}) (\bar{v}_1 \cdot \bar{e}) \right] \right\}$$

$$\frac{d}{dt_2} (\bar{r}_1 \cdot \bar{e}) = \frac{dt_1}{dt_2} \left\{ \bar{v}_1 \cdot \bar{e} - \frac{1}{\Delta x} \left[(\bar{r}_1 \cdot \bar{v}_1) - (\bar{r}_1 \cdot \bar{e}) (\bar{e} \cdot \bar{v}_1) \right] \right\} + \frac{1}{\Delta x} \left[(\bar{r}_1 \cdot \bar{v}_2) - (\bar{r}_1 \cdot \bar{e}) (\bar{e} \cdot \bar{v}_2) \right] .$$

Using these expressions and computing dt_1/dt_2, we find

$$\frac{dt_1}{dt_2} = \left[\frac{1 - \bar{e} \cdot \bar{v}_2/c}{1 - \bar{e} \cdot \bar{v}_1/c}\right] + S ,$$

where

$$S = \frac{M}{c}\left\{ 2\left[\frac{g_1 + h_1}{r_1 + \bar{r}_1 \cdot \bar{e}}\right] - 2\left[\frac{g_2 + h_2}{r_2 + \bar{r}_2 \cdot \bar{e}}\right] - \frac{g_1}{r_1} + \frac{g_2}{r_2} + \frac{(\bar{r}_1 \cdot \bar{e})h_1}{r_1^2} - \frac{(\bar{r}_2 \cdot \bar{e})h_2}{r_2^2} \right\} ,$$

and the result has been expanded to the order $\frac{vM}{cr}$ (third order). Also,

$$g_1 = \bar{v}_1 \cdot \bar{e} + \frac{1}{\Delta x}\left[\bar{r}_1 \cdot \bar{v}_R - (\bar{r}_1 \cdot \bar{e})\ (\bar{e} \cdot \bar{v}_R)\right]$$

$$g_2 = \bar{v}_2 \cdot \bar{e} + \frac{1}{\Delta x}\left[\bar{r}_2 \cdot \bar{v}_R - (\bar{r}_2 \cdot \bar{e})\ (\bar{e} \cdot \bar{v}_R)\right] \qquad (5)$$

$$h_1 = \hat{r}_1 \cdot \bar{v}_1 \qquad h_2 = \hat{r}_2 \cdot \bar{v}_2 \qquad \bar{v}_R = \bar{v}_2 - \bar{v}_1 \quad .$$

The complete frequency shift is then given by

$$\frac{f_2}{f_1} = \left[\frac{1 - \bar{e} \cdot \bar{v}_2/c}{1 - \bar{e} \cdot \bar{v}_1/c}\right]\left[\frac{1 - \frac{v_1^2}{c^2} - \frac{2M}{r_1}}{1 - \frac{v_2^2}{c^2} - \frac{2M}{r_2}}\right]^{\frac{1}{2}} + S ,$$

valid to the order $\frac{(Mv)}{rc}$, and the two brackets can be expanded to third order if desired. The result can then be shown to be equivalent to that given by Richard [2]. The derivation given here is slightly more general and does not rely on Synge's World Function method.

After some algebra, S can be put in the form

$$S = \frac{M}{c}\left\{\frac{(\bar{v}_1 \cdot \bar{e})[r_1^2 + (\bar{r} \cdot \bar{e})^2]}{r_1^3} - \frac{(\bar{v}_2 \cdot \bar{e})[r_2^2 + (\bar{r}_2 \cdot \bar{e})^2]}{r_2^3}\right.$$

$$+ \frac{\bar{r}_1 \cdot \bar{v}_{1\perp}}{[r_1^2 - (\bar{r}_1 \cdot \bar{e})^2]\Delta x}\left[- \frac{[r_2^2 + (\bar{r}_2 \cdot \bar{e})^2]}{r_2^3}\right.$$

$$\left. + \frac{[r_1^2 + (\bar{r}_1 \cdot \bar{e})^2][r_1^2 - (\bar{r}_1 \cdot \bar{e})^2 + (\bar{r}_1 \cdot \bar{e})(\bar{r}_1 \cdot \bar{e})]}{r_1^3}\right] + \frac{\bar{r}_2 \cdot \bar{v}_{2\perp}}{[r_2^2 - (\bar{r}_2 \cdot \bar{e})^2]\Delta x}$$

$$\left.\left[- \frac{[r_1^2 + (\bar{r}_1 \cdot \bar{e})^2]}{r_1^3} + \frac{[r_2^2 + (\bar{r}_2 \cdot \bar{e})^2][r_2^2 - (\bar{r}_2 \cdot \bar{e})^2 + (\bar{r}_1 \cdot \bar{e})(\bar{r}_2 \cdot \bar{e})]}{r_2^3}\right]\right\}$$

where $\bar{v}_\perp = \bar{v} - (\bar{v} \cdot \bar{e})\ \bar{e}$. This form shows explicitly that if the position and velocity of the emitter and receiver are interchanged, the same result is obtained. This means that S is doubled upon two-way transmission.

MEASUREMENT OF S

The fact that S is doubled in a two-way transmission experiment, in contrast with the ordinary gravitational shift, which is cancelled, has two consequences. First, it is possible to observe S using a passive reflector or some sort of transponder system, rather than having two separate frequency standards involved. Secondly, S will be masked by the first-order Doppler and any double Doppler elimination scheme will also eliminate S. With these two facts in mind, several possible schemes for measuring S will now be suggested.

Let us first look at the maser clock experiment as presently conceived in which a maser in an elliptical synchronous orbit is compared with a similar one on the ground, the two clocks being in sight of each other at all times. The magnitude of the frequency shift due to S, upon two-way transmission, is plotted (Fig. 2) as a function of time for an orbit of eccentricity 1/2, and the maximum value is 4.4×10^{-15}. This is pushing the limit of the accuracy of present maser clocks which is presently quoted as from 3 to 7 parts in 10^{15}. So that even if the first order Doppler could be corrected for, it is doubtful that the effect could be observed in the present situation.

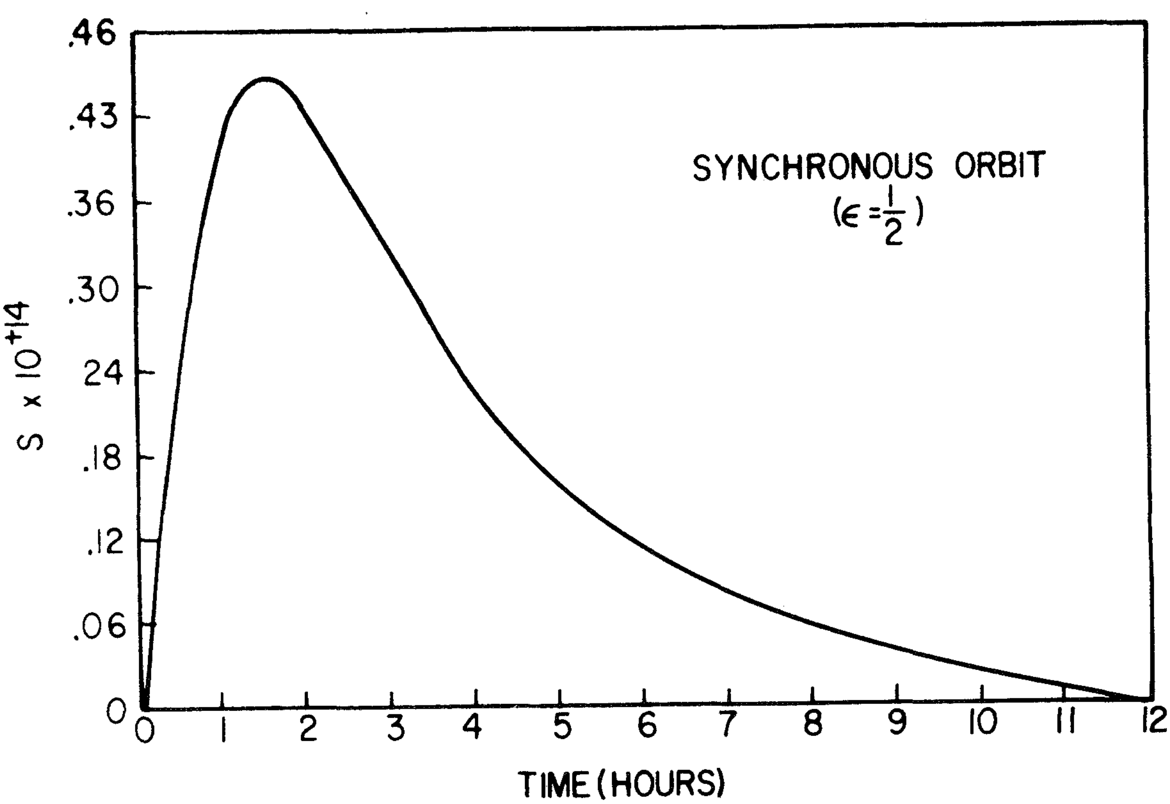

Figure 2.

It is possible, however, to augment S considerably by a scheme which exploits the fact that two-way propagation will double S. One possible arrangement would be to couple the elliptical orbit maser experiment as presently conceived with a reflector or transponder in some sort of orbit such that the situation shown in Figure 3 could be realized. If v is the velocity of the emitter and reflector, r the distance from the center of the earth, and the velocities are parallel, one finds that the frequency shift observed upon two-way transmission is

$$2S = 4\frac{Gm}{c^2 r}\left(\frac{v}{c}\right)\frac{(1 + \cos^2\theta)}{\tan\theta}$$

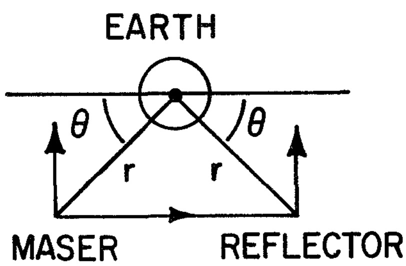

Figure 3.

Choosing values of r, v, and θ corresponding to similar elliptical synchronous orbits, one finds that

$$2S = 0.88 \times 10^{-13}$$

for $r = 24 \times 10^6$ m, $v = 5 \times 10^3$ m/sec, and $\theta = 15$ degrees. If the trajectories were adjusted such that no motion occurred along the line joining the emitter and reflector at the time of measurement, there would be no first- and second-order Doppler and no gravitational shift. Thus, 2S would be the actual observed frequency shift at the instant when the velocities were parallel and equal. Or, alternatively, we could use the tracking data and orbital calculations to correct for the first- and second-order Doppler, though this may be pushing the capability of present tracking techniques, even with laser tracking. But it appears that if we can eliminate or correct for the Doppler effects, the frequency shift 2S should be observable with presently attainable maser accuracies. In the absence of a suitable orbiting reflector, a transponder system on the moon could perhaps be used, as in the situation shown in Figure 4. Then the frequency shift observed at the maser upon two-way transmission would be approximately

$$2S \approx 2\left(\frac{GM}{c^2 h}\right)\left(\frac{-4v}{c}\right) \quad ,$$

where v is the velocity of the satellite. Letting $h = 6.5 \times 10^6$m, $v = 5 \times 10^3$m/sec, then $2S = 0.9 \times 10^{-13}$, observable with present maser accuracies. The beauty

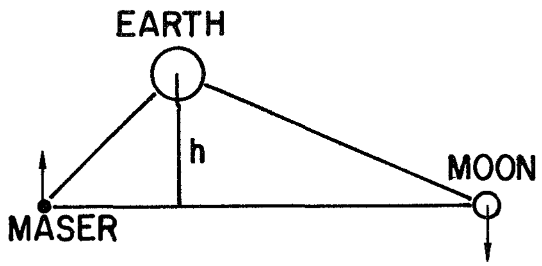

Figure 4.

of this or some similar scheme is that it would be added to the maser experiment as presently proposed without too much increase in cost or complexity, but with a large increase in results.

On the other hand, we could go to a solar orbit maser clock experiment. The gravitational shift could then be made as large as 1 part in 10^8, while S could be as great as 1 part in 10^9. We could naturally sacrifice some tracking accuracy, but the increased magnitude of the effect should more than compensate for this. We could measure the gravitational shift to the accuracy comparable to that attainable from the earth orbit experiment, and the greatly enlarged S effect ought to make it separable from the first- and second-order Doppler. In addition, a solar orbit would not require much more launch vehicle capacity than a synchronous orbit.

The complete analysis of any one of these possibilities would require much more detailed consideration. There are many factors that would have to be considered, such as propagation effects, tracking accuracies, etc. This paper is not intended to be a feasibility demonstration for any of the possibilities suggested above. The purpose here is to point out that situations exist in which S is large enough to be detectable by hydrogen masers.

SIGNIFICANCE OF S

It is evident that S depends on the spatial components of the metric tensor. That is, in spherical coordinates, the metric can be written as

$$g_{11} = \left(1 + \frac{\gamma M}{r} + \ldots\right) \qquad g_{00} = -\left(1 + \frac{\alpha M}{r} + \frac{\beta M^2}{r^2} + \ldots\right)$$

and a measurement of S would constitute a measure of γ.[1] At present, γ is known to only about 20 percent through the light deflection observations. Of

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1. A summary of the dependence of the various measurements on α, β, and γ is given by Nordvedt [3].

course, α is determined by the equivalence principle as well as the requirement that the orbits reduce to the Newtonian case to first approximation, and the usual gravitational red shift experiment will simply verify this. There is also the possibility that when S is expressed in terms of observable quantities, rather than coordinate values, it is sensitive to β, although there is some disagreement on this.

The geodetic precession of a gyroscope is also a measure of γ. So if it turns out to be possible to measure the geodetic precession to only 1 percent (0.06 arc sec/yr or greater) so that the motional or Lense-Thirring precession would not be observable, then the measurement of S to the same percent accuracy would yield exactly the same information about the metric tensor.

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APPROVAL

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The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

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