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SEMI-GROUPS, GROUPS AND LYAPUNOV STABILITY OF PARTIAL DIFFERENTIAL EQUATIONS

by

William G. Vogt   Gabe R. Buis   Martin M. Eisen

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Abstract

The stability of certain classes of partial differential equations can be rigorously investigated by extending the definition of the formal partial differential operators and obtaining operator differential equations in Hilbert spaces. Under appropriate restrictions the properties of solutions to these partial differential equations can be investigated by using the theory of one parameter semi-groups or groups of linear bounded transformations in the corresponding Hilbert spaces. This leads to a Lyapunov stability theory for partial differential equations.

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1. INTRODUCTION

The intent of this paper is to provide a mathematical and theoretical framework in which the stability of solutions to certain types of partial differential equations can be rigorously investigated. A need for doing this can be established by examining the rather substantial number of papers written on stability of partial differential equations [1-7] in which the mathematical manipulations are, at least in part, formal in nature rather than rigorously substantiated. Roughly speaking these manipulations involve integration by parts, application of certain integral inequalities, the assumption of certain "smoothness" properties of solutions to partial differential equations, and the assumption that solutions to the partial differential equations essentially satisfy the requirements of a dynamical system. Generally speaking, in the literature presently available, most of these details have not been rigorously substantiated.

It will be shown that for certain classes of partial differential equations satisfying certain types of boundary conditions, the integration by parts formula, the application of classical integral inequalities and the assumptions of sufficiently smooth solutions can be rigorously substantiated. However, the final assymption that the solutions can be regarded as characterizing a dynamical system, is not true in general even for linear partial differential equations. For certain partial differential equations this assumption is shown to be true. This class of partial differential equations generates solutions which happen to satisfy the group property which is equivalent to the dynamical system property. A much broader class of partial differential equations generate solutions which satisfy only the semi-group property. This is not the same as a dynamical system in the generally accepted terminology.

A viable Lyapunov stability theory can be rigorously developed for the class of partial differential equations which generate either groups or semi-groups. The theory for groups is "nicer" and more complete than the theory for semi-groups in that necessary and sufficient conditions can be given for asymptotic stability.

A large part of this paper is expository in nature. Most of Sections 2 through 6 and 9 are well known to mathematicians working in this particular area of functional analysis. The intent of these sections is to provide a concise treatment of the mathematical resources which are necessary to develop a Lyapunov stability theory for partial differential equations. There are certain key points in these sections which are instrumental in obtaining a suitable Lyapunov stability theory. Some of these, such as the concept of equivalent inner product, are not treated in detail in the standard references available to authors. The contents of Sections 7, 8 and 10 are thought to be relatively new and certainly important to the treatment of stability of partial differential equations.

For full understanding of this paper some background in functional analysis is essential. The best elementary reference is the book by Kolmogorov and Fomin [8] and at a somewhat more advanced level the book by Taylor [9]. In the opinion of the authors, the finest general reference for this work is the book by Yosida [10]. The standard reference for the general theory of semi-groups is the book by Hille and Phillips [11] and also, Part 1 of the work by Dunford and Schwartz [12].

For the abstract theory of partial differential operators, probably the best reference is Part 2 of the work by Dunford and Schwartz [13]. As general references on both the formal and abstract properties of partial differential operators valuable information can be found in the books by Smirnov [14] Kantorovich and Akilov [15], Petrovskii, [16], Smirnov [17], Goldberg, [18] and, of course, the pioneering work by Sobolev, well represented in the two monographs [19, 20], and the book [21]. In addition to these texts there are undoubtedly many others which the interested reader will be able to find on his own.

There is a great volume of relevant literature appearing in engineering journals, physics journals, mathematical journals and as seminar notes, lecture notes and monographs [22-28]. In sheer weight of numbers, variety of treatment, the extent of mathematical background required, and in some instances, the degree of specialization required offers to the researcher interested in this field a vast array of technical literature, not all of which can be referenced. For this reason, only a few of the most relevant technical works are referenced in this paper.

It is assumed that the reader is already familiar with some of the basic theory of functional analysis. Only a brief outline of some of the necessary topics are presented in Sections 2 through 5. Statements are made and theorems are stated without elaborating on the proofs of these fundamental concepts which may be found in many of the references. The theory here is developed in the context of real Banach and Hilbert spaces, but there is no difficulty in extending all of these results to complex Banach or Hilbert spaces. Section 2 is a brief summary of Banach and Hilbert spaces. The concept of equivalent inner product is introduced in Definition 2.2. Section 3 gives a brief summary of theory of linear operators. For applications to partial differential equations, the key concept is that of a closed, not necessarily bounded, linear operator. In addition a complete characterization of equivalent inner products is given in theorem 3.2. Section 4 is a very brief resume of the spectral theory needed in the remainder of this paper. Section 5 is a more extended treatment of the theory of semi-groups and groups. The key results in this section are the relations between dissipative operators and contractive or negative contractive semi-groups. Two elementary examples are given at the end of this section. Section 6 is a concise statement of what is meant by solutions to operator differential equations and the stability of these solutions. Most of the content of Section 7 is thought to be relatively new. The main result in this section is the choice of the form for a Lyapunov functional for studying the stability of operator differential equations. This is related to the concept of equivalent inner product and dissipative operators, and leads to the usual Lyapunov stability theory. The key results are theorems 7.3 and 7.4. It turns out that this theory developed for Hilbert spaces is a direct generalization of the usual Lyapunov stability theory for linear differential equations in finite dimensional spaces. This is shown in Section 8.

Section 9 is crucial in developing a Lyapunov stability theory for partial differential equations. It is in this section that the transition from what may be called formal partial differential equations to abstract operator differential equations is made. Mathematically, most of the content of this section is not new, in fact being taken in great part from Dunford and Schwartz [13]. The idea is to take certain types of formal partial differential operators and to extend these to closed operators in suitable Hilbert spaces. This can not be done in general. However, for a restricted class of partial differential operator this can be done. It is with this class of partial differential operators that the development here is concerned. The key results are the validity of the integral inequality formula, and Garding's inequality, which is actually fundamental in obtaining the last theorem.

In Section 10 several examples illustrate the application of the theory. Most of these are selected from the available literature in order to exemplify the relationship of this rigorous mathematical treatment to the more formal properties described earlier. Section 11 describes areas for future research along the lines described in this paper.

The content of this paper is an expansion of a previous paper [29] and is taken in part from a doctoral dissertation by G. R. Buis [30] and some other reports and papers by the authors [31, 32]. Extensions to both the linear and nonlinear theory can be found in [32-38] as described briefly in Section 11.

2. Banach and Hilbert Spaces

It is assumed that the reader is already familiar with the theory of Banach and Hilbert spaces. Only a brief outline of some of the necessary topics will be presented here. Proofs and further details may be found in the elementary book [8] and the more advanced texts [9-13]. Most of the theory will be developed in the context of real Banach and Hilbert spaces, but on occasion, complex numbers will be used; if α is complex, $\bar{\alpha}$ is the complex conjugate of α.

A normed linear space X, is denoted by $X = (E, ||\cdot||)$ where E is a linear vector space over a field of scalars, K (the real or complex numbers), and $||\cdot||$ is the norm in X satisfying for all $\alpha\varepsilon K$ and all $x, y\varepsilon E$: (a) $||x|| \geqq 0$; (b) $||\alpha x||=|\alpha|\ ||x||$; (c) $||x+y|| \leqq ||x||+||y||$; and (d) $||x|| = 0$ iff $x = 0$. A Cauchy sequence, $\{x_n\} \subseteqq X$ is a sequence such that given any $\varepsilon > 0$, there exists an integer $N = N(\varepsilon) > 0$ such that $m,n > N$ imply that $||x_m-x_n|| < \varepsilon$. If each Cauchy sequence in X converges to an element $x\varepsilon X$, the space is said to be a complete normed linear space and is called a Banach space (or B-space). The convergence is designated by $||x_n-x|| \to 0$ as $n \to \infty$ or $x_n \to x$ or $\lim x_n=x$. X is a real B-space if K is the field of the real numbers.

A Hilbert space (or H-space), H, is a special B-space, the norm of which satisfies the parallelogram law, $||x+y||^2 + ||x-y||^2 = 2(||x||^2+||y||^2)$ for all $x, y\varepsilon X$. This may be used to define an inner product $(\cdot,\cdot)$ by $(x,y)= (1/4)(||x+y||^2-||x-y||^2)$ and then H is denoted by $H = (E, (\cdot,\cdot))$. Alternatively, if H is an inner product space, the inner product, $(\cdot,\cdot)$, in H may be used to define a norm by $||x|| = (x,x)^{1/2}$. An inner (or scalar) product has the following properties for all $\alpha\varepsilon K$ and all $x,y,z\varepsilon E$: (a) $(\alpha x,y)=\alpha(x,y)$;

(b) $(x,y) = \overline{(y,x)}$; (c) $(x+y, z) = (x,z) + (y,z)$; and (d) $(x,x)>0$ whenever $x\neq 0$. Thus, an H-space is an inner product space which is also a B-space with norm $||x|| = (x,x)^{1/2}$. H is a real H-space if K is the field of the real numbers. By properties (a),(b),(c), the inner product is bilinear for a real H-space (sesqui-linear for a complex H-space).

A point $x\varepsilon X$ is said to be a limit point of a set $A \subseteq X$ iff there exists a sequence of distinct elements $\{x_n\} \subseteq A$ such that $\lim x_n = x$. The closure of a set A, denoted by $\bar{A}$, is the set comprised of A and all the limit points of A. A set A is closed iff $A = \bar{A}$. A set A is said to be dense in X if $\bar{A} = X$. If A is closed and dense in X, A=X.

Definition 2.1: If $X_1 = (E, ||\cdot||_1)$ and $X_2 = (E, ||\cdot||_2)$, then the two norms $||\cdot||_1$ and $||\cdot||_2$ are said to be equivalent iff there exist real constants, $\infty > \alpha \geq \beta > 0$ such that $\beta||x||_2 \leq ||x||_1 \leq \alpha||x||_2$ for all $x\varepsilon E$.

It is clear that all the important properties (such as convergence, denseness, etc.) holding for one norm will also hold for an equivalent norm. In such a case X_1 and X_2 are said to be topologically equivalent. Sometimes in the following no distinction will be made between X_1 and X_2 if the norms are equivalent. Based on the concept of equivalent norms, it is possible to consider the concept of equivalent inner products.

Definition 2.2: If $H_1 = (E, (\cdot,\cdot)_1)$ and $H_2 = (E, (\cdot,\cdot)_2)$, then the two inner products are said to be equivalent iff their corresponding norms are equivalent.

The equivalence of norms does not imply the equivalence of inner products, since a B-space need not be a Hilbert space. However, if each norm satisfies the parallelogram law, then the inner products are equivalent if the norms are equivalent. In the sequel, an important role is played by equivalent inner products.

Example 2.1: Let R^n be the set of all real n-tuples, $x \varepsilon R^n$ $x = \text{col}\{x_1, \cdots, x_n\}$, x_i is real and $|x_i| < \infty$ for $i=1, \cdots, n$. The inner product in R^n is

$$(x,y) = \sum_{i=1}^{n} x_i y_i = x' y \qquad (x' = \text{transpose of } x)$$

and the norm is

$$||x|| = (x,x)^{1/2} = (\sum_{i=1}^{n} x_i^2)^{1/2}$$

R^n is a Hilbert space. Any equivalent inner product is

$$(x,y)_1 = (Px,y) = \sum_{i,j=1}^{n} x_i y_j p_{ij} = x' Py$$

where P is a real, symmetric positive definite matrix. If $\lambda_1(\lambda_n)$ is the minimum (maximum) eigenvalue of P, then $||x||_1 = (x' Px)^{1/2}$ implies

$$\sqrt{\lambda_1}\ ||x|| \leqq ||x||_1 \leqq \sqrt{\lambda_n}\ ||x|| .$$

Example 2.2: Let C[0,1] be the set of continuous functions defined on [0,1] with norm

$$||f|| = \sup_{t \varepsilon [0,1]} |f(t)|$$

C[0,1] is a B-space but not a Hilbert space since $||\cdot||$ does not satisfy the parallelogram law.

Example 2.3: Let $L^2(0,2\pi)$ be the (classes of) real functions defined on $(0, 2\pi)$ such that if $f \varepsilon L^2(0,2\pi)$ the Lebesgue integral $\int_0^{2\pi} |f(t)|^2 dt < \infty$. The inner product of f, g ε $L^2(0, 2\pi)$ is

$$(f,g) = \int_0^{2\pi} f(t)\ g(t)\ dt$$

and the norm is

$$||f|| = \left(\int_0^{2\pi} f^2(t)dt \right)^{1/2}$$

$L^2\ (0,2\pi)$ is a Hilbert space. **Any elements** $f,g \varepsilon L^2(0,2\pi)$ have unique **Fourier** Series representations:

$$f(y) = \sum_{n=0}^{\infty} (a_n \sin ny + b_n \cos ny)$$

$$g(y) = \sum_{n=0}^{\infty} (c_n \sin ny + d_n \cos ny)$$

$f\ \varepsilon\ L^2(0,\ 2\pi)$ iff $\sum_{n=0}^{\infty} (a_n^2 + b_n^2) < \infty$ and

$$||f|| = [\sum_{n=0}^{\infty} (a_n^2 + b_n^2)]^{1/2}$$

$$(f,g) = \sum_{n=0}^{\infty} (a_n c_n + b_n d_n).$$

3. Linear Operators

Let X and Y be vector spaces over the same field of scalars, K. Let T be an operator (or function) which maps part of X into Y. The domain of T, $\mathcal{D}(T)$, is the set of all $x\varepsilon X$ such that there exists a $y\varepsilon Y$ for which $Tx = y$. The range of T, $\mathcal{R}(T) = \{Tx: x\varepsilon\mathcal{D}(T)\}$. The null space, (or kernel) of T is $\mathcal{N}(T) = \{x: Tx = 0\}$. If $\mathcal{D}(T_1) \subseteqq \mathcal{D}(T_2)$ and $T_1x = T_2x$ for all $x\varepsilon\mathcal{D}(T_1)$, then T_2 is called an extension of T_1 or T_1 is called a restriction of T_2 and this is denoted as $T_1 \subseteqq T_2$. If $\mathcal{D}(T_1) = \mathcal{D}(T_2)$ and $T_1x = T_2x$ for all $x\varepsilon\mathcal{D}(T_1)$, then $T_1 \equiv T_2$. The operator T is called 1:1 if distinct elements in $\mathcal{D}(T)$ are mapped into distinct elements of $\mathcal{R}(T)$. An operator T with $\mathcal{D}(T)$ a linear subspace of X and $\mathcal{R}(T)$ in Y is called linear iff for all $x,z\varepsilon\mathcal{D}(T)$ and all $\alpha,\beta\varepsilon K$, $T(\alpha x+\beta z)=\alpha Tx+\beta Tz$. A linear operator T is 1:1 iff $\mathcal{N}(T) = \{0\}$.

If X and Y are normed linear spaces and T is a linear operator with $\mathcal{D}(T) \subseteqq X$ and $\mathcal{R}(T) \subseteqq Y$, the following statements are equivalent: (a) T is continuous at a point $x_0\varepsilon\mathcal{D}(T)$; (b) T is uniformly continuous on $\mathcal{D}(T)$; (c) T is bounded; i.e., there exists a number M such that for all $x\varepsilon\mathcal{D}(T)$, $||Tx|| \leqq M||x||$. If T is bounded, the norm of T, $||T||$ is defined by $||T|| = \sup\ (||Tx|| : ||x|| \leqq 1, x\varepsilon\mathcal{D}(T))$. With this norm, [X,Y], the space of all bounded linear operators with domain X and range in Y is a normed linear space. If X is a normed linear space (not necessarily complete) and Y is a B-space [X,Y] is a B-space. For $X \equiv Y$, [X] will be used to denote [X,X].

If X and Y are normed linear spaces, the cartesian product normed linear space X x Y is defined as the normed linear space of all ordered pairs $\{x,y\}$ with $x\varepsilon X$ and $y\varepsilon Y$ with $\{x_1,y_1\} + \{x_2,y_2\} = \{x_1+x_2, y_1+y_2\}$ and

$\alpha\{x,y\} = \{\alpha x,\alpha y\}$ with norm given by $||\{x,y\}|| = (||x|| + ||y||^2)^{1/2}$. If X and Y are B-spaces, so is X x Y. If T is a linear operator with $\mathcal{D}(T) \subseteq X$ and $\mathcal{R}(T) \subseteq Y$, the graph, G(T), of T is the set $(\{x,Tx\}:x\varepsilon\mathcal{D}(T))$. Since T is linear G(T) is a subspace of X x Y.

Definition 3.1: If the graph of T is closed in X x Y, then T is said to be closed in X. When no ambiguity is possible, T is said to be closed.

Example 3.1: (See also examples 4.1, 4.3). Let X = Y = C[0,1] and let C'[0,1] be the subspace of X consisting of functions with continuous first derivatives. Define the linear differential operator T mapping C'[0,1] into Y by $(Tx)(t) = x'(t)$, $t\ \varepsilon[0,1]$. Then T is closed. However T is not continuous, since the sequence $x_n(t) = t^n$ has the properties $||Tx_n|| = 1$ [10].

Theorem 3.1: T is closed iff $x_n\varepsilon\mathcal{D}(T)$, $x_n \to x$, $Tx_n \to y$ imply $x\varepsilon\mathcal{D}(T)$ and $Tx = y$.

A bounded operator, T, need not be closed but if Y is a B-space, T has a unique extension, $\bar{T}$, to $\mathcal{D}(\bar{T}) = \overline{\mathcal{D}(T)}$ such that $||\bar{T}||=||T||$ and $\bar{T}$ is closed. If $\mathcal{D}(T)$ is dense in a B-space, X, then $\bar{T}\varepsilon[X,Y]$. Some unbounded operators have closed extensions. A linear operator T is called closable if there exists a linear extension of T which is closed in X. T is closable iff for $x_n\varepsilon\mathcal{D}(T)$, $\lim x_n = 0$ and $\lim Tx_n = y$ imply that $y=0$.

Definition 3.2: If T is a closable operator, then its closed extension T is defined as the operator whose graph $G(\bar{T})$ is the closure of the graph of T.

Let X and Y be normed linear spaces and T be a 1:1 operator with $\mathcal{D}(T) \subseteq X$ and $\mathcal{R}(T) \subseteq Y$. The inverse of T, T^{-1}, is the map from the subspace $\mathcal{R}(T)$ into X given by $T^{-1}(Tx) = x$. If T is linear, then T^{-1} is linear with domain $\mathcal{R}(T)$ and range $\mathcal{D}(T)$. T^{-1} exists and is continuous iff there exists an $m > 0$ such that $||Tx|| \geqq m||x||$ for $x \varepsilon \mathcal{D}(T)$. If this is the case, $m^{-1} \geqq ||T^{-1}||$. T^{-1} is closed iff T is closed.

In a real H-space, $H=(E, (\cdot,\cdot))$, a linear operator S with domain $\mathcal{D}(S)$ and $\mathcal{R}(S)$ both in H is called <u>positive definite</u> iff there exists a $\gamma > 0$ such that $(Sx, x) \geqq \gamma||x||^2$ for all $x \varepsilon \mathcal{D}(S)$. S is called symmetric if $(Sx,y) = (x,Sy)$ for $x,y \varepsilon \mathcal{D}(S)$. A bounded operator $S \varepsilon [H]$ is called RSPD if it is real, symmetric, positive definite. This allows a characterization of equivalent inner products by a special case of the Lax-Milgram theorem [10].

<u>Theorem 3.2</u>: The inner products in $H_1 = (E, (\cdot,\cdot)_1)$ and $H_2=(E, (\cdot,\cdot)_2)$ are equivalent iff there exists an RSPD $S \varepsilon [H_1]$ such that $(x,y)_2 = (x,Sy)_1 = (Sx,y)_1$ for all $x,y \varepsilon H_1$.

4. Spectral Theory

Let T be a linear operator with $\mathcal{D}(T)$ and $\mathcal{R}(T)$ both in a normed linear space X. The distribution of values λ for which the linear operator $(\lambda I - T)$ has an inverse and the properties of the inverse when it exists are called the spectral theory for the operator T. Additional details can be found in [10, 13].

Definition 4.1: If λ_o is such that $\mathcal{R}(\lambda_o I-T)$ is dense in X and $\lambda_o I-T$ has a continuous inverse $(\lambda_o I-T)^{-1}$, λ_o is said to be in the resolvent set, $\rho(T)$ of T; the inverse $(\lambda_o I-T)^{-1}$ is denoted by $\mathcal{R}(\lambda_o; T)$ and is called the resolvent (at λ_o) of T. All complex numbers, λ not in $\rho(T)$ form a set, $\sigma(T)$ called the spectrum of T.

Theorem 4.1: Let X be a B-space and T a closed linear operator with $\mathcal{D}(T)$ and $\mathcal{R}(T)$ both in X. Then for any $\lambda \epsilon \rho(T)$, the resolvent $R(\lambda;T)$ is an everywhere defined continuous linear operator. The resolvent set, $\rho(T)$ is an open set of the complex plane. In each component (maximal connected subset) of $\rho(T)$, $R(\lambda;T)$ is a holomorphic function of T, i.e. $R(\lambda;T)$ can be expanded in a convergent power series in $\lambda-\lambda_o$ for $\lambda_o \epsilon \rho(T)$ and $|\lambda-\lambda_o|$ sufficiently small. The coefficients of the power series are in [X].

Example 4.1: Let $H = R^n$ and let A be an nxn real matrix. $\mathcal{D}(A) = H$ and $\mathcal{R}(A) \subseteq H$. The spectrum of A, $\sigma(A) = \{\lambda_1(A), \cdots, \lambda_n(A)\}$. For any $\lambda \notin \sigma(A)$, $R(\lambda; A)$ is bounded and is defined on all of H. A, being bounded and defined on all of H, is closed.

Example 4.2: In example 3.1 the spectrum of T is the whole real line since $\frac{dx}{dt} = \lambda x$ always has a solution for every real λ.

Example 4.3: Let $H = L^2(0, 2\pi)$ (see example 2.3) and define H_p by

$$H_p = \{f \epsilon H: \sum_{n=0}^{\infty} (n^2+1)^p \quad (a_n{}^2 + b_n{}^2) < \infty\}.$$

H_p is dense in H. Define A by $\mathcal{D}(A) = H_1$ and

$$Af = - \sum_{n=0}^{\infty} (n^2 + 1) (a_n \sin ny + b_n \cos ny) \quad f \epsilon \mathcal{D}(A).$$

A is unbounded but closed. A has a continuous inverse since

$$||Af|| \geqq ||f||$$

5. Semi-Groups and Groups

In order to examine the stability of solutions to partial differential equations, it is necessary to be able to characterize the properties of solutions. This is done by considering the properties of semigroups and groups of class (C_0) or the strongly continuous semi-groups and groups. In the following, reference to a semi-group (or group) implies the strong continuity in (iii), Definition 5.1 (Definition 5.2). Further details can be found in [10,13]. In the following X is assumed to be a real B-space and H, a real H-space.

Definition 5.1: For each $t \varepsilon [0,\infty)$, let $S_t \varepsilon [X]$. The family $\{S_t; t \geqq 0\} \subseteqq [X]$ is called a semi-group iff the following conditions hold:
(i) $S_{s+t} = S_s S_t$ for $s,t \geqq 0$; (ii) $S_0 = I$; (iii) $\lim_{t \to t_o} ||S_t x - S_{t_o} x|| = 0$ for $t_o > 0$ and all $x \varepsilon X$.

Definition 5.2: If $\{G_t; -\infty < t < \infty\} \subseteqq [X]$ satisfies (i) for $-\infty < s,t < \infty$ (ii); and (iii) for $-\infty < t_o < \infty$ and all $x \varepsilon X$, $\{G_t; -\infty < t < \infty\}$ is called a group.

It is clear that if $\{G_t; -\infty < t < \infty\}$ is a group, then $\{G_t; t \geqq 0\}$ and $\{G_t; t \leqq 0\}$ are semi-groups. If $\{S_t; t \geqq 0\}$ is a semi-group, its norm satisfies for some $M \geqq 1$ and $\beta > 0$

$$||S_t|| \leqq Me^{\beta t} \qquad (t \geqq 0) \tag{5.1}$$

If $\beta = 0$, $\{S_t\}$ is said to be equibounded and if in addition $M = 1$, it is called contractive. If $\beta < 0$, $\{S_t\}$ is called negative and if in addition $M = 1$, it is called negative contractive. In general, if $\{G_t; -\infty < t < \infty\}$ is a group,

$$||G_t|| \leqq Me^{\beta |t|} \qquad (-\infty < t < \infty) \tag{5.2}$$

for some $M \geqq 1$ and some $\beta > 0$. The same terminology as above is used for the semi-group $\{G_t; t \geqq 0\}$. If $\{G_t; -\infty < t < \infty\}$ is a group, then for each t, $G_t^{-1} = G_{-t}$ and both are continuous linear transformations of X into X.

Definition 5.3: The infinitesimal generator, A, of the semi-group $\{S_t;\ t \geqq 0\}$ is defined by

$$Ax = \lim_{h \to 0^+} [h^{-1}(S_h x - x)] \tag{5.3}$$

for all $x \in X$ such that the limit exists.

It follows [10,13] that A is a closed linear operator with domain $\mathcal{D}(A)$ dense in X with $0 \in \mathcal{D}(A)$. Moreover, if $x \in \mathcal{D}(A)$, then $S_t x \in \mathcal{D}(A)$ for $t \geqq 0$ and

$$\frac{d}{dt}(S_t x) = AS_t x = S_t Ax \qquad (x \in \mathcal{D}(A)) \tag{5.4}$$

In order to characterize whether an operator A generates a semi-group the next theorem is needed.

Theorem 5.1: A closed linear operator, A, with dense domain and with range in X is the infinitesimal generator of a unique semi-group $\{S_t;\ t \geqq 0\}$ satisfying (5.1) for $M \geqq 1$ and β iff there exist real numbers M and β such that for every integer $n > \beta$, $n \in \rho(A)$ and

$$||R(n;A)^m|| \leqq M(n-\beta)^{-m} \qquad (m=1,2,\ldots) \tag{5.5}$$

In addition $\{S_t;\ t \geqq 0\}$ is

(1) equibounded iff $||(I-n^{-1}A)^{-m}|| \leqq M \qquad (n,\ m=1,2,\ldots)$

(2) contractive iff $||(I-n^{-1}A)^{-1}|| \leqq 1 \qquad (n = 1,2,\ldots)$

(3) negative iff $||(I-n^{-1}A)^{-m}|| \leqq M(1-n^{-1}\beta)^{-m} \quad (\beta < 0;\ m,\ n=1,2,\ldots)$

(4) negative contractive iff $||(I-n^{-1}A)^{-1}|| \leqq (1-n^{-1}\beta)^{-1} \quad (\beta<0, n=1,2,\ldots)$

Theorem 5.1 is easily adapted to the case of A generating a group $\{G_t;\ -\infty < t < \infty\}$ satisfying (5.2) by replacing the conditions on n by, $|n| > \beta$, $n \gtrless 0$, $n \in \rho(A)$ and replacing (5.5) by:

$$||R(n;A)^m|| \leqq M(|n| - \beta)^{-m} \qquad (m=1,2,\ldots)$$

If A generates a semigroup satisfying (5.1), then A-δI generates a semigroup, T_t satisfying

$$||T_t|| \leqq Me^{(\beta-\delta)t} \tag{5.6}$$

Conditions (1-4) hold of course for the semi-group $\{G_t;\ t \geqq 0\}$ and simple modifications hold for $\{G_t;\ t \leqq 0\}$ since (-A) generates $\{G_{-t};\ t \geqq 0\}$.

All of the conditions of Theorem 5.1 are difficult to verify in practice. The following and generalizations of these are more useful. Additional details can be found in [10,29-32].

Definition 5.4: Let A be a linear operator with $\mathcal{D}(A)$ and $\mathcal{R}(A)$ both in a real H-space. A is called dissipative with respect to the inner product $(\cdot,\cdot)$ if $(Ax, x) \leqq 0$ whenever $x\varepsilon\mathcal{D}(A)$ and strictly dissipative if there exists a $\gamma > 0$ such that $(Ax,x) \leqq -\gamma\,(x,x)$ for $x\varepsilon\mathcal{D}(A)$.

Theorem 5.2 [10]: Let A be a linear operator with domain and range in H such that $\mathcal{D}(A)$ is dense in H. If A is (strictly) dissipative and $R(I(1-\gamma)-A) = H$ where $\gamma > 0$ is a constant, then A generates a (negative) contractive semi-group in H and $\lambda\varepsilon\rho(A)$ for (Re $\lambda > -\gamma$) Re $\lambda > 0$.

The last theorem leads to a result which gives necessary and sufficient conditions for A to generate a negative contractive group in H.

Theorem 5.3 [32]: Let A be a linear operator with $\mathcal{D}(A)$ and $\mathcal{R}(A)$ in a real H-space, H=(E, $(\cdot,\cdot)$) such that $\mathcal{D}(A)$ is dense in H. Then A generates a group $\{G_t;\ -\infty < t < \infty\}$ in H such that $\{G_t;\ t \geqq 0\}$ is a negative contractive semi-group with respect to a norm, $||\cdot||_1$, induced by an equivalent inner product, $(\cdot,\cdot)_1$, iff there exist positive δ, γ with $\infty > \delta \geqq \gamma > 0$ such that

$$-\delta||x||^2 \leqq (Ax,\ x)_1 \leqq -\gamma||x||^2 \qquad (x\varepsilon\mathcal{D}(A)) \tag{5.7}$$

and

$$\mathcal{R}(I(1-\gamma)-A) = H; \mathcal{R}(I(1+\delta) + A) = H. \tag{5.8}$$

In addition $\lambda \varepsilon \rho(A)$ for Re $\lambda < -\delta$ and Re $\lambda > -\gamma$.

Example 5.1: Let $H = R^n$ and let A be a real nxn matrix and use the notations of examples 2.1 and 4.1. A generates a group. This follows from the fact that for $\beta > 0$ sufficiently large, $(A-\beta I)$ and $(-A-\beta I)$ are dissipative

$$((\pm A-\beta I)\,x,x) = (\pm Ax,x) - \beta||x||^2 \leqq (||A||-\beta)||x||^2$$

For $\beta > ||A||$, $(\pm A-\beta I)$ is dissipative and therefore by (5.6), both A and $-A$ are generators of uniquely determined semigroups, $\{T_t^+; t \geqq 0\}$ and $\{T_t^-; t \geqq 0\}$, which satisfy $||T_t^+|| \leqq e^{||A||t}$, $||T_t^-|| \leqq e^{||A||t}$ for $t \geqq 0$. Thus A generates a group $\{G_t; -\infty < t < \infty\}$ with

$$||G_t|| \leqq e^{||A|| \ |t|} \qquad (-\infty < t < \infty)$$

In fact, the group that A generates is $\{G_t = e^{At}; -\infty < t < \infty\}$.

Example 5.2: Consider the d.e.

$$\frac{\partial u}{\partial t} = a\,u + b\,\frac{\partial u}{\partial x} \qquad (-\infty < t < \infty)$$

where a and b are constants. Let Φ be the B-space of all real continuously differentiable functions defined for $x \in R$ such that $\phi(x) \to c_\phi$, a finite constant, as $|x|\to\infty$. Let $||\phi|| = \sup_{x\in R} |\phi(x)|$. For any $\phi\varepsilon\Phi$ one can define a solution $u(\phi,t)$ with $u(\phi,0) = \phi(x)$-- in fact,

$$(*) \qquad u(\phi,t) = e^{at}\,\phi(x+bt)$$

The solutions of (*) for $\phi\varepsilon\Phi$ form a one-parameter family of transformations of the space Φ into itself. G_t defined by

$$G_t\,\phi = u(\phi,t) \qquad (-\infty < t < \infty)$$

form a group of operators. If $a < 0$ then $\{G_t\ ; t \geq 0\}$ is a negative contractive semi-group.

Example 5.3: Using the results of examples 2.3 and 4.3, A is densely defined with range in $H = L^2(0, 2\pi)$. $(\lambda I - A)$ for $\lambda > 0$ has a continuous inverse on H. A is dissipative since $(Af, f) \leqq -(f,f)$. Thus A generates a negative contractive semigroup, $\{S_t;\ t > 0\}$. In fact S_t is given by

$$S_t f = \sum_{n=0}^{\infty} e^{-(n^2+1)t} (a_n \sin ny + b_n \cos ny)$$

and

$$||S_t|| \leqq e^{-t}$$

$\{S_t;\ t \geqq 0\}$ can not be extended to a group, since for fixed $t > 0$, defining $f_n(y) = \sin n y$, $||f_n||=1$. $S_t f_n = e^{-(n^2+1)t} \sin n y$ and $||S_t f_n|| \to 0$ as $n \to \infty$ and therefore a continuous inverse $(S_t)^{-1}$ does not exist. Thus S_t can not be extended to a group.

6. Operator Differential Equations and Stability

Let X be a B-space and let A be a linear operator with $\mathcal{D}(A)$ and $\mathcal{R}(A)$ in X. Consider the operator differential equation

$$dx/dt = Ax \qquad (x\varepsilon\mathcal{D}(A)) \tag{6.1}$$

with initial condition $x(0) = x_o\varepsilon\mathcal{D}(A)$. A solution to (6.1) with initial condition $x_o\varepsilon\mathcal{D}(A)$ will be designated by $\phi(t;\, x_o)$. If A is the infinitesimal generator of a semi-group $\{S_t;\ t \geqq 0\}$, then from (5.4) and (ii) of Definition 5.1, it follows that $\phi(t;\, x_o) = S_t x_o$ for $x_o\varepsilon\mathcal{D}(A)$ and $t \geqq 0$. Similarly if A is the infinitesimal generator of a group $\{G_t;\ -\infty < t < \infty\}$ $\phi(t;\, x_o) = G_t x_o$ for $x_o\varepsilon\mathcal{D}(A)$ and $-\infty<t<\infty$. In these cases, $x = 0$ is a solution of (6.1) and since (6.1) is linear, any solution may be referenced to $x = 0$ by a simple translation. The following definitions are direct generalizations from classical Lyapunov stability theory .

The null solution, $x = 0$, of (6.1) is <u>stable</u> if, given an $\varepsilon > 0$, a $\delta > 0$ can be found such that $||x_o|| < \delta$ and $x_o\varepsilon\mathcal{D}(A)$ implies $||\phi(t;x_o)|| < \varepsilon$ for $t \geqq 0$. If in addition $\lim ||\phi(t;x_o)|| = 0$ as $t\to\infty$, $x = 0$ is asymptotically stable. If in addition, there exist positive numbers M, β, T such that $||\phi(t;x_o)|| \leqq M \exp(-\beta t)\ ||x_o||$ for $t \geqq T$, then $x = 0$ is exponentially asymptotically stable.

<u>Theorem 6.1:</u> If A is the generator of a semi-group $\{S_t;\ t \geq 0\}$ (or group) then: (i) $||S_t|| \leqq M$ implies stability; and (ii) $||S_t|| \leqq M \exp(-\beta t)$ for $\beta > 0$ implies exponential asymptotic stability.

This theorem is based on knowledge of the solutions to (6.1) and thus corresponds to the "First Method" of Lyapunov. The "Second Method" or "Direct Method" of Lyapunov is based on knowledge of A and certain functions called "Lyapunov functions" in finite dimensional spaces. In B-spaces and in particular H-spaces, these become "Lyapunov functionals".

7. Lyapunov Functionals and Stability

Roughly speaking, the direct method of Lyapunov consists of finding a functional $v(x)$ such that $v(x) > 0$, $x \neq 0$ and $\dot{v}(x)$, the derivative of $v(x)$ along solutions to (6.1) satisfies $\dot{v}(x) \leqq 0$ for stability and $\dot{v}(x) \leqq -k\, v(x)$ $(k > 0)$ for exponential asymptotic stability. To obtain such a functional $v(x)$ in a real H-space, a defining bilinear functional is first obtained.

Definition 7.1: Let $H = (E, (.,.))$ be a real Hilbert Space. A defining bilinear functional, $V(.,.)$, is any inner product equivalent to $(.,.)$ in H. Thus $V(.,.) = (.,.)_1$ where $(.,.)_1$ is equivalent to $(.,.)$ in H.

Theorem 7.1: $V(.,.)$ is a defining bilinear functional in H iff there exists an RSPD linear transformation $P \varepsilon [H]$ such that $V(x,y) = (x,Py) = (Px,y)$.

This theorem is a direct result of Definition 7.1 and Theorem 3.2. From $V(x,y)$ the Lyapunov functional, $v(x)$ will be obtained and it will be a quadratic Lyapunov functional just as for a linear system in a finite dimensional space, the Lyapunov function is a quadratic form.

Definition 7.2: The function $v(x) = V(x,x)$ for $x \varepsilon H$, where $V(.,.)$ is a defining bilinear functional in H, is called a (quadratic) Lyapunov functional in H.

Definition 7.3: If $\phi(t;x)$ is a solution to (6.1), the derivative of $v(x)$, $\dot{v}(x)$ is defined by

$$\dot{v}(x) = \lim_{t \to 0^+} t^{-1} \left(v(\phi(t;x)) - v(x)\right) \qquad (7.1)$$

for all x such that this limit exists.

Theorem 7.2: Let A be the generator of a semi-group (or group); then $\dot{v}(x)$ is defined for all $x\varepsilon\mathcal{D}(A)$ and is given by

$$\dot{v}(x) = 2V(x, Ax) = 2V(Ax, x) \quad (x\varepsilon\mathcal{D}(A)) \tag{7.2}$$

Corollary: Under the hypotheses of the theorem

$$\dot{v}(x) = 2(Ax, x)_1 = 2(PAx, x) \quad (x\varepsilon\mathcal{D}(A)) \tag{7.3}$$

where $(.,.)_1$ is an inner product equivalent to (.,.) and $P\varepsilon[H]$ is RSPD.

Theorem 7.3: If A is the generator of a semi-group, a sufficient condition for stability (exponential asymptotic stability) is that there exist a Lyapunov functional v(x) the derivative of which $\dot{v}(x) \leqq 0$ ($\dot{v}(x) \leqq - \gamma ||x||^2$ $(\gamma > 0)$)for $x\varepsilon\mathcal{D}(A)$.

Theorem 7.4: If A is the generator of a group, a necessary and sufficient condition for exponential asymptotic stability is that there exist a Lyapunov functional v(x) such that $\dot{v}(x)$ satisfies for $\infty > \alpha \geqq \beta > 0$

$$-\alpha v(x) \leqq \dot{v}(x) \leqq -\beta\, v(x) \quad (x\varepsilon\mathcal{D}(A)) \tag{7.4}$$

The proof of Theorem 7.2 follows from Definition 7.3 and the fact that $V(y,y) - V(x,x) = V(y+x, y-x)$. The corollary follows from Definition 7.1. Theorem 7.3 follows from Theorem 5.2 and Definition 7.1. The sufficiency of Theorem 7.4 follows from Theorem 5.3 and Definition 7.1. The necessity follows from the fact that if $\{G_t; -\infty < t < \infty\}$ is a negative group, then there exist four positive constants $\infty > M \geqq 1 \geqq m > 0$, $\infty > \gamma \geqq \delta > 0$ such that

$$M \exp(-\gamma t)\ ||x|| \leqq ||G_t x|| \leqq M \exp(-\delta t)\ ||x|| (x\varepsilon H) \tag{7.5}$$

and the definition of V(x,y) by

$$V(x,y) = \int_0^\infty (G_t x, G_t y)dt \quad (x,y\varepsilon H) \tag{7.6}$$

where the integral may be taken as an improper Riemann integral and (.,.) is the inner product in H (see theorem 5.3 and [32] for details).

Remarks: In theorems 7.3 and 7.4, if the hypothesis that A generates a group or semigroup is replaced by ones similar to those of theorems 5.2 and 5.3, then this theory also assures existence of solutions. The extra hypotheses required, for example, are that $\mathcal{D}(A)$ is dense in H with $\mathcal{R}(A)$ in H and that $\mathcal{R}(I-A) = H$. In the case of H being finite dimensional, these always hold and consequently are never explicitly stated.

8. Applications

Example 8.1: The results of examples 2.1, 4.1 and 5.1 have shown that if $H = R^n$ and A is an n x n real matrix, A is the generator of a group $\{G_t; -\infty\ t < \infty\}$ with $G_t = e^{At}$ satisfying

$$||G_t|| \leqq e^{||A||t}$$

Theorem 7.4 gives a necessary and sufficient condition for A to generate an exponentially asymptotically stable semi-group $\{G_t; t \geqq 0\}$. A need not be dissipative with respect to the inner product $(\cdot,\cdot)$ of R^n. However if the spectrum of A is restricted to the left half complex plane, A is a stable matrix, and must generate an exponentially asymptotically stable semi-group. Thus, an RSPD matrix P is sought so that A is dissipative with respect to the equivalent inner product $(\cdot,\cdot)_1$ where

$$(x,y)_1 = (Px,y) = x'Py$$

Thus

$$2(Ax,x)_1 = 2(PAx,x) = 2(x'A'x) = x'(A'P + PA)x$$

Setting $A'P + PA = -R$ where R is RSPD yields the Lyapunov Stability Theorem for $\dot{x} = Ax$, which is A is a stable matrix iff the solution P to $A'P + PA = -R$ is uniquely determined by R and is RSPD whenever R is RSPD. The defining bilinear function is $V(x,y) = (x,y)_1 = x'Py$, the quadratic Lyapunov function is $v(x) = (x,x)_1 = x'Px$ and its derivative is $\dot{v}(x) = 2(Ax,x)_1 = -x'Rx$. This illustrates the importance of the concept of equivalent inner product in relation to Lyapunov Stability Theory.

As a specific example, let $H = R^2$ with inner product $(x,y) = x'y = x_1y_1 + x_2y_2$. In $\dot{x} = Ax$, let

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}; \qquad \sigma(A) = \{-1,-2\}$$

so A should be a stable matrix. However, A is not dissipative with respect to $(\cdot,\cdot)$ since $(Ax,x) = x'Ax = -x_1x_2-3x_2^2$ is an indefinite quadratic form. However if $A'P + PA = -2I$ is solved for P, P is unique and

$$P = \begin{bmatrix} 2.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}; \quad \sigma(P) = \{0.382,\ 2.618\}$$

so that P is RSPD implying A is a stable matrix. A is dissipative and $(Ax,x)_1 = -(x,x)$. From example 2.1, it follows that

$$-2.618(x,x)_1 \leqq (Ax,x)_1 \leqq -0.382\ (x,x)_1$$

and therefore, if $\text{Re}\ \lambda > -.382$ or $\text{Re}\ \lambda < -2.618$, $\lambda \in \rho(A)$ and $R(\lambda;A)$ is a bounded operator defined on all of H. It further follows that for $t \geqq 0$, $||e^{At}||_1 \leqq e^{-0.382t}$ and $||e^{At}|| \leqq 2.61\ e^{-0.382t}$.

Example 8.2: Letting $H = L^2(0, 2\pi)$ and using the results of examples 2.3, 4.3, 5.3, it is clear that the adequate Lyapunov functional is defined by $V(x,y) = (x,y)$ where $(\cdot,\cdot)$ is the inner product defined for $L^2(0, 2\pi)$. Thus $\dot{v}(f) \leqq -v(f)$ and therefore $||S_t|| \leqq e^{-t}$.

9. Partial Differential Operators and Sobolev Spaces

Most of the content of this section is taken from Dunford and Schwartz [13]. While there are other approaches to the final formulation of the theorems quoted here, [15-25], this particular formulation seems best suited to the specific application of the theory developed. The object is to first define what is meant by a formal PDO (partial differential operator) defined in subsets of E^n. Based on particular properties of specific formal PDO's, closed operators in appropriate Hilbert Spaces are obtained. The domains and ranges of these operators become the spaces introduced by Sobolev in 1935 and hence are usually called Sobolev spaces. These spaces can be obtained in a variety of ways, e.g. by functional completion of incomplete function spaces [15,17,19,20,21], or by the introduction of distributions [13,18,24,25]. In some instances there are subtle differences in the properties which may be imputed to these spaces using the various approaches, but there is a common theory for the restricted class considered here. For details, the interested reader should consult the references listed.

9.1. Subsets of R^n

In order to obtain a consistent notation and to avoid repetition in the theorems quoted, I, Ω, Γ, subsets of R^n, real Euclidean n-space, are defined. The essence of the approach is to define an open subset, $I \subset R^n$, in which a formal PDO is defined. I is assumed to be connected. $\Omega \subset I$ is a bounded open subset of I such that Ω, its closure, is a proper subset of I. This assures that any formal partial differential operator is defined in an open set containing $\bar{\Omega}$. Next, it is assumed that Ω has a "sufficiently smooth" boundary, Γ such that no point in Γ is interior to the closure of

Ω. The description of a domain with sufficiently smooth boundary can be made mathematically precise [13,19,20,22,24,25]. Such a domain satisfies the "cone condition" of Sobolev [15,17,19,20], is "properly regular" according to Fichera [22] or is "tres regulier" according to Lions [24]. According to [13], a sufficiently smooth boundary of this type can contain "corners", "edges", etc. as long as these configurations are locally equivalent to the intersection of a finite number of hyperplanes in E^n. Some of the definitions and theorems below will hold even if Ω <u>does not</u> have a sufficiently smooth boundary.

9.2. Formal Partial Differential Operators

Let I R^n be as described in the preceding section. Let J be an n-vector with non-negative integral components

$$J = (j_1, j_2, \ldots, j_n) \tag{9.1}$$

Designate by $|J|$ the sum

$$|J| = \sum_{i=1}^{n} j_i \tag{9.2}$$

The symbol ∂^J means partial differentiation with respect to the components of $y \in R^n$, i.e. for (9.1)

$$\partial^J = \frac{\partial^{j_1 + j_2 + \cdots + j_n}}{\partial y_1^{j_1} \partial y_2^{j_2} \cdots \partial y_n^{j_n}} . \tag{9.3}$$

If $|J| = 0$, $\partial^J = 1$. Let $a_J(y)$ be a real scalar function of $y \in I$ which is infinitely (or sufficiently) differentiable in I.

<u>Definition:</u> [13] If m is a positive integer, then a <u>real formal partial differential operator</u>, τ, defined in I is, in general, given by

$$\tau = \sum_{|J| \leq m} a_J(y) \partial^J \tag{9.4}$$

The order of τ is m. The formal adjoint of τ, designated by τ^*, is the differential operator defined by

$$(\tau^*)(\cdot) = \sum_{|J| \leqq m} (-1)^{|J|} \partial^J (a_J(y)(\cdot)) \qquad (9.5)$$

Since τ is real, τ^* is real. If $\tau = \tau^*$, then τ is called formally self adjoint. Actually, as will be made clear in a later part, τ^* is the formal adjoint of τ with respect to the $L^2(\Omega)$ inner product.

9.3. Sets of Functions

The set of $C^k(I)$ consists of all those real scalar functions, $f(y)$ $y \in I$, such that every derivative $\partial^J f$, $|J| \leqq k$ is defined and continuous in I. The set $C_o^k(I)$ consists of all functions in $C^k(I)$ such that the closure of the set with $f \neq 0$ is compact and a proper subset of I, i.e. the set of functions in $C^k(I)$ with compact support in I. The sets $C^\infty(I)$ and $C_o^\infty(I)$ are correspondingly defined. The set $C^k(\bar{I})$ is the set in $C^k(I)$ having all derivatives up to and including k in I, such that each partial derivative has a continuous extension to $\bar{I}$. If $f(y) \in C^k(\bar{I})$, then $\partial^J f(y)$ is defined, for $y \in \bar{I}$ and $|J| \leqq k$, as the extension by continuity of $\partial^J f(y)$ from I to $\bar{I}$. Then $C^\infty(\bar{I}) = \bigcap_{k=0}^{\infty} C^k(\bar{I})$, $C_o^\infty(\bar{I}) = C_o^\infty(I)$, $C_o^k(\bar{I}) = C_o^k(I)$. The sets $C^\infty(\Omega)$, $C^\infty(\bar{\Omega})$, $C^k(\Omega)$, $C^k(\bar{\Omega})$, $C_o^\infty(\Omega) = C_o^\infty(\bar{\Omega})$, $C_o^k(\Omega) = C_o^k(\bar{\Omega})$ are similarly defined. Each of these sets with the usual definitions of addition and scalar multiplication become linear vector spaces.

The Banach Space $C^k(\bar{\Omega})$

Since $\bar{\Omega}$ is compact we may define a norm for the $C^k(\bar{\Omega})$ functions for $0 \leqq k < \infty$ by

$$||f||_{C^k(\bar{\Omega})} = \{\sup|\partial^J f(y)|; \ |J| \leqq k, \ y \in \bar{\Omega}\} \qquad (9.6)$$

Endowed with this norm, $C^k(\bar{\Omega})$ is a Banach space and the $C^\infty(\bar{\Omega})$ functions are dense in $C^k(\bar{\Omega})$ with respect to $||\cdot||_{C^k(\bar{\Omega})}$.

The Hilbert Space $L^2(I)$

Let $dy = dy_1\, dy_2 \ldots dy_n$ be the Lebesgue measure in R^n. Designate by $L^2(I)$ the space of (classes of) real functions, f, which are square integrable on I. The norm and inner product are defined for f, g ε $L^2(I)$ as

$$(f,g)_o = \int_I f(y)\, g(y)\, dy \qquad (9.7)$$

$$||f||_o = ((f,f))^{1/2} \qquad (9.8)$$

As indicated, the elements of $L^2(I)$ are equivalence classes of functions; f and g belong to the same equivalence class iff f(y) = g(y) almost everywhere in I, i.e.

$$\int_I (f - g)^2 dy = 0$$

The $C_o^\infty(I)$ functions are dense in $L^2(I)$, i.e. if the $C_o^\infty(I)$ functions are completed in the $||\cdot||_o$ norm, we have $\overline{C_o^\infty(I)} = L^2(I)$. Similar definitions hold for $L^2(\Omega)$ and $\overline{C_o^\infty(\Omega)} = L^2(\Omega)$.

9.4. Integration by Parts

If τ is a formal partial differential operator of order m in I, then for any f ε $C^m(\bar{\Omega})$, τf is continuous as is $\tau^* f$. With domain Ω and its boundary Γ as defined in Section 9.1, the Green-Gauss identity [14,22] holds for the domain. This identity can be stated as follows:

Green-Gauss Identity:

Given any f, g ε $C^m(\bar{\Omega})$, and τ of order m, then

$$\int_\Omega [g(\tau f) - f(\tau^* g)]dy = \int_\Gamma H(f,g)d(\Gamma) \qquad (9.9)$$

where H(f,g) is a bilinear differential operator in f and g of order at most m - 1 and $d(\Gamma)$ is the surface area measure of Γ.

Second Order Partial Differential Operator

As a specific example [17], if τ is of order 2 in I and f(y) and g(y) are in $C^2(\bar{\Omega})$, τ f is given by

$$\tau f = \sum_{i,j=1}^{n} a_{ij}(y) \frac{\partial^2 f(y)}{\partial y_i y_j} + \sum_{j=1}^{n} b_j(y) \frac{\partial f(y)}{\partial y_j} + c(y)f(y) \quad (9.10)$$

and τ^*g is given by

$$\tau^* g = \sum_{i,j=1}^{n} \frac{\partial^2(a_{ij}g)}{\partial y_i \partial y_j} - \sum_{j=1}^{n} \frac{\partial(b_j g)}{\partial y_j} + cg \quad (9.11)$$

It may be assumed without loss of generality that $a_{ij}(y) = a_{ji}(y)$. Then, as can be verified by direct differentiation,

$$g(\tau f)-f(\tau^* g) = \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i} [a_{ij}(g \frac{\partial f}{\partial y_j} - f \frac{\partial g}{\partial y_j}) - \frac{\partial a_{ij}}{\partial y_j} fg] + \sum_{i=1}^{n} \frac{\partial(b_i fg)}{\partial y_i} \quad (9.12)$$

Integrating both sides of (9.11) as in (9.8) yields

$$H(f,g) = gP(f) - f\,P(g) + fgQ \quad (9.13)$$

where

$$P(f) = \sum_{i=1}^{n} (\sum_{j=1}^{n} a_{ij} \frac{\partial f}{\partial y_j}) \cos (\nu, y_i) \quad (9.14)$$

$$Q = \sum_{i=1}^{n} (b_i - \sum_{j=1}^{n} \frac{\partial a_{ij}}{\partial y_j}) \cos (\nu, y_i) \quad (9.15)$$

where ν is the unit outward normal to Γ and $\cos(\nu, y_i)$ is the cosine of the angle between the outward normal, ν, and the coordinate axis, y_i.

If

$$b_i = \sum_{j=1}^{n} \frac{\partial a_{ij}}{\partial y_j} \;\; (= \sum_{j=1}^{n} \frac{\partial a_{ji}}{\partial y_j}) \quad (9.16)$$

then Q = 0, and $\tau = \tau^*$, that is τ is formally self adjoint and can be written as

$$\tau f = \sum_{i=1}^{n} \frac{\partial}{\partial y_i} (\sum_{j=1}^{n} a_{ij} \frac{\partial f}{\partial y_j}) + c\; f \quad (9.17)$$

For convenience $A = A' = (a_{ij}(y))$ is the nxn symmetric matrix composed of the coefficients $a_{ij}(y)$ of the second order terms in (9.9).

Integration by Parts

It is possible under some circumstances that the integral on the right of (9.8) is zero. This happens, for example whenever $f \in C^m(\bar{\Omega})$ and $g \in C^m(\bar{\Omega}) \cap C_o^{m-1}(\Omega)$. Obviously if $g \in C_o^{m-1}(\Omega)$, g and all derivatives up to order m-1 of g vanish outside a compact subset of Ω implying H(f,g) is zero along Γ and therefore

$$\int_\Omega g(\tau f)dy = \int_\Omega f(\tau^* g)dy \qquad \left(\begin{matrix} f \in C^m(\bar{\Omega}) \\ g \in C^m(\bar{\Omega}) \cap C_o^{m-1}(\Omega) \end{matrix}\right) \quad (9.18)$$

This last formula is the usual integration by parts formula. It is to be noted that τ could be any formal PDO; in particular, τ could be of order one. If this is the case, we have

$$\int_\Omega g\left(\sum_{i=1}^n a_i(y) \frac{\partial f}{\partial y_i}\right) dy =$$

$$\int_\Gamma \sum_{i=1}^n a_i gf \, d(\Gamma) - \int_\Omega f \sum_{i=1}^n \frac{\partial(a_i g)}{\partial y_i} dy$$

Thus, if g is zero on Γ, we obtain (9.17).

Note that if $f, g \in C_o^\infty(\Omega)$,

$$\int_I g(\tau f)\ dy = \int_I f(\tau^* g)\ dy \qquad (9.19)$$

or in terms of the $L^2(\Omega)$ inner product $(\cdot,\cdot)$ defined in (9.6),

$$(g, \tau f) = (\tau^* g, f)$$

Integral Inequalities

A well known integral inequality is that of Poincar'e. If $u(y) \in C^2(\bar{\Omega}) \cap C_o(\Omega)$, then [27]

$$\int_\Omega u^2 dy \leqq \frac{1}{\lambda_1} \int_\Omega (\nabla u)' \ (\nabla u)dy \qquad (9.20)$$

where λ_1 is the smallest real number such that there is a smooth solution to

$$- \Delta u = \lambda_1 u \qquad (\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2}) \tag{9.21}$$

and

$$u(y) = 0 \qquad (y \in \Gamma) \tag{9.22}$$

An estimate for λ_1 can be obtained [27] from $\frac{1}{\lambda_1} \leqq d^2$ where d is the maximum length of the edges of any rectangle $R \supset \bar{\Omega}$.

9.5. Dirichlet Boundary Conditions

The particular PDO's considered later will be those satisfying Dirichlet boundary conditions which represent a fairly broad class of physical problems.

Definition: Let Ω and Γ be as in 9.1. Let $(\partial_\nu(\Gamma))^j$ designate the j^{th} order derivative taken in a direction normal to Γ. If $f(y) \in C^{k-1}(\bar{\Omega})$ and $\partial^J f(y)$ vanishes for all $y \in \Gamma$ and $|J| \leqq k-1$, f is said to satisfy a Dirichlet condition of order k on Γ and this is designated by

$$(\partial_\nu(\Gamma))^J f(y) = 0 \qquad y \in \Gamma \qquad 0 \leqq |J| \leqq k-1 \tag{9.23}$$

Remark: If $\bar{\Omega}$ is a closed rectangle with sides, Γ_i, perpendicular to the coordinate axes, y_i, condition (9.22) becomes the more familiar

$$\frac{\partial^j f(y)}{\partial y_i^j} = 0 \qquad y \in \Gamma_i \qquad \begin{array}{l} 0 \leqq j \leqq k-1 \\ i=1, 2, \ldots, n \end{array} \tag{9.24}$$

The formula for integration by parts (9.7) is valid if τ is of order m and either f or g or both satisfy a Dirichlet boundary condition of order m on Γ.

In the case τ is of even order 2m and f and g both satisfy a Dirichlet boundary condition of order m, then the Green-Gauss idenitity (9.8) holds where $H(f,g)$ is a bilinear differential operator in f and g of order at most m. This can be easily verified by the integration by parts formula (9.18).

9.6 Distributions

There are at least two ways of introducing Sobolev spaces: the first is by functional completion of certain sets of functions according to some norm and the second is by the introduction of distributions and then the restriction of distributions to form certain subsets which become the Sobolev spaces. We choose the latter because there is a fairly complete theory relating formal PDO's and closed operators obtained through distributions [13].

Definition: Let $\{\phi_n\}$ be a sequence of functions in $C_0^\infty(I)$ and let $\phi \in C_0^\infty(I)$. If there exists a compact subset, K, of I such that all of the functions, ϕ_n, vanish outside of K and if in addition $\phi_n \to \phi$ in the topology of $C_0^\infty(I)$ (the topology can be precisely defined [13]) then we denote this by

$$\phi_n \rightrightarrows \phi \text{ in } I.$$

Definition: A linear functional, F, defined on $C_0^\infty(I)$ such that $F(\phi_n) \to F(\phi)$ whenever $\phi_n \rightrightarrows \phi$ in I is called a distribution in I.

Definition: The class of all distributions in I will be denoted by $D(I)$.

In order to connect a distribution, F, to a Lebesgue integrable function, f, in I we have the next definition.

Definition: Let f be a function in I which is Lebesgue integrable over every compact subset of I. Then the distribution F defined by

$$F(\phi) = \int_I \phi(y)\, f(y)\, dy \qquad (\phi \in C_0^\infty(I)) \tag{9.25}$$

is called the distribution corresponding to f. A distribution, F, which corresponds to a function, f, in this sense is said to be a function. If f is in $L_2(I)$, $C^m(I)$, $C_o^\infty(I)$, etc., F will be said to be in $L_2(I)$, $C^m(I)$, $C_o^\infty(I)$, etc.

In general, we simply identify a distribution which is a function with the function to which it corresponds. There is a unique distribution associated with any two functions equal almost everywhere in a given sense, for example in $L^2(I)$. If F corresponds to any continuous function, it corresponds to a unique continuous function.

Corresponding to the concept of "generalized function" is that of "generalized derivative". We first consider the case where $f \varepsilon C^m(I)$ and τ is a formal PDO of order m. Thus τf is a function $\varepsilon\ C(I)$ and hence there is a distribution which we will call τF corresponding to τf.

$$(\tau F)(\phi) = \int_I (\tau f)(y) \quad \phi(y)dy \qquad (\phi \;\varepsilon\; C_o^\infty(I)) \tag{9.26}$$

By (9.18) it immediately follows that

$$(\tau F)(\phi) = \int_I f(y) \ (\tau^*\phi) \ (y) \ dy = F(\tau^*\phi) \tag{9.27}$$

and hence, "generalized differentiation" is defined by

$$(\tau F)(\phi) = F(\tau^*\phi) \qquad\qquad (\phi \;\varepsilon\; C_o^\infty(I)) \tag{9.28}$$

Thus, "generalized differentiation" is defined whether F corresponds to a function f or is a distribution.

In order to determine if F corresponds to a function, $f \;\varepsilon\; L^2(I)$ the next theorem holds [13].

Theorem 9.6.1: The distribution F corresponds to a function $f \;\varepsilon\; L^2(I)$ iff there is a finite constant K such that

$$|F(\phi)| \leqq K||\phi||_2 \qquad\qquad (\phi \;\varepsilon\; C_o^\infty(I)) \tag{9.29}$$

It is clear from (9.27) that if τ is a formal PDO in I and $F, G \;\varepsilon\; D(I)$ then

$$\tau(\alpha F + \beta G) = \alpha(\tau F) + \beta(\tau G)$$
$$(\alpha\tau_1 + \beta\tau_2)F = \alpha(\tau_1 F) + \beta(\tau_2 F)$$
$$(\tau_1\tau_2)F = \tau_1(\tau_2 F) \qquad (9.30)$$

9.7. The Sobolev Spaces

The Sobolev spaces are instrumental for the study of solutions to PDE's. Here, we will consider only those Sobolev spaces which are real Hilbert spaces. More details can be found in the references [13-28].

Definition: Let k be a non-negative integer. The real Sobolev space $H^k(I)$ is defined by

$$H^k(I) = \{F \in D(I);\ \partial^J F \in L^2(I),\ |J| \leqq k\} \qquad (9.31)$$

The inner product $(\cdot,\cdot)_k$ and norm $||\cdot||_k$ are defined for F, G $\in H^k(I)$ by

$$(F,G)_k = \sum_{|J|\leqq k} \int_I (\partial^J F)(y)\ (\partial^J G)(y)dy \qquad (9.32)$$

$$||F||_k = ((F, F)_k)^{1/2} \qquad (9.33)$$

Definition: The real Sobolev space $H_o^k(I)$ is defined by the closure of the $C_o^\infty(I)$ functions in the norm $||\cdot||_k$ of $H^k(I)$. In general, $H_o^k(I)$ is a proper subspace of $H^k(I)$.

Remark: $H^k(\Omega)$ can be obtained by the functional completion of the $C^k(\bar{\Omega})$ (or $C^\infty(\bar{\Omega})$) functions with respect to the norm $||\cdot||_k$. The additional elements needed to complete the space are the so called "generalized functions" or "ideal elements" which are the limits of Cauchy sequences in the $||\cdot||_k$ norm.

Theorem 9.7.1: $H^k(I)$ is a (complete) Hilbert space with inner product $(\cdot,\cdot)_k$ and norm $||\cdot||_k$ and $H_o^k(I)$ is a closed subspace of $H^k(I)$.

Theorem 9.7.2:

$$H^o(I) = H_o^o(I) = L^2(I)$$

$$H^j(I) \subseteqq H^k(I) \quad (\infty > j \geqq k \geqq 0)$$

$$H_o^j(I) \subseteqq H_o^k(I) \quad (\infty > j \geqq k \geqq 0) \tag{9.34}$$

The identity mapping (or imbedding) of $H^j(I)$ ($H_o^j(I)$) into $H^k(I)$ ($H_o^k(I)$), for $\infty > j > k \geqq 0$ is norm reducing and therefore continuous.

Theorem 9.7.3: Let τ be a formal PDO of order k with $C^\infty(I)$ coefficients. Then for $\infty > j \geqq k \geqq 0$, and $F \in H^j(\Omega)$ ($F \in H_o^j(\Omega)$); τ regarded as a mapping τ: $F \to \tau F$, is a continuous linear mapping of $H^j(\Omega)$ into $H^{j-k}(\Omega)$).

Theorem 9.7.4: Let n be a positive integer and let [n/2] be the largest integer smaller than n/2. Let j and k be integers with $\infty > k > j \geqq 0$.

(i) The natural identity mapping of $H^k(\Omega)$ into $H^j(\Omega)$ is a compact linear mapping, i.e., it takes bounded sets in $H^k(\Omega)$ into compact sets in $H^j(\Omega)$.

(ii) If there exists a non-negative integer m such that $k - [n/2] - 1 \geqq m$, then each element in $H^k(\Omega)$ is (has a representative which is) an element of $C^m(\bar{\Omega})$ and the natural identity mapping of $H^k(\Omega)$ into $C^m(\bar{\Omega})$ is a compact linear mapping with

$$||\cdot||_{C^m(\bar{\Omega})} \leqq M||\cdot||_k \tag{9.35}$$

where M is a positive constant depending only on the domain Ω and the norms in (9.35) are as defined in (9.6) and (9.33).

The following density results, already implied previously, are useful in applications.

Theorem 7.7.5: For any $p = 0,1,2,\ldots$, the subset $C^{\infty}(\bar{\Omega})$ of $H^{p}(\Omega)$ is dense in $H^{p}(\Omega)$ with respect to the $||\cdot||_p$ norm.

Theorem 9.7.6: The subspace $C_0^{\infty}(\Omega)$ of $D(\Omega)$ is dense in $D(\Omega)$. In particular $C_0^{\infty}(\Omega)$ is dense in $H_0^{p}(\Omega)$ for $p = 0,1,2,\ldots$.

Integration by Parts Formula

The integration by parts formula (9.17) is valid in the Sobolev Space [22]:

$$\int_{\Omega} g(\tau f)dy = \int_{\Omega} f(\tau^{*}g)dy \quad (f \in H^{m}(\Omega);\ g \in H_0^{m}(\Omega)) \tag{9.36}$$

Integral Inequalities

The integral inequality (9.19) is valid in the Sobolev Spaces [22,27]

$$\int_{\Omega} u^2 dy \leq \frac{1}{\lambda_1} \int_{\Omega} (\nabla u)\cdot(\nabla u)dy \quad (u \in H^2(\Omega)\cap H_0^1(\Omega)) \tag{9.37}$$

where λ_1 is as determined by (9.21).

9.8. Elliptic Partial Differential Operators

The PDO's studied here will be real, elliptic and of even order, i.e., we assume

$$\tau = \sum_{|J|\leq 2p} a_J(y)\partial^J \tag{9.38}$$

where τ is defined in I. The order of τ is 2p for some positive integer p and the $a_J(y)$ are real. For these operators, on functions satisfying Dirichlet boundary conditions, the theory is fairly complete [13].

Definition: τ is said to be elliptic in I if for each nonzero vector ξ in R^n

$$\sum_{|J|=2p} a_J(y)\,\xi^J \neq 0 \qquad y \in I \tag{9.39}$$

where ξ^J is given by

$$\xi^J = \xi_1^{j_1} \xi_2^{j_2} \dots \xi_n^{j_n} \quad (\sum_{i=1}^{n} j_i = J) \tag{9.40}$$

Definition: Let $\Omega \subset I$ be as defined in 9.1. If a positive constant c_o exists such that

$$(-1)^p \sum_{|J|=2p} a_J(y)\xi^J > c_o \, |\xi|^{2p} \qquad (y \in \Omega) \tag{9.41}$$

for every real nonzero $\xi \in E^n$, then τ is said to be strongly elliptic in Ω.

Theorem 9.8.1: (Garding's Inequality) If τ is of order 2p and is strongly elliptic in Ω, there exist two constants, $k > 0$, $K < \infty$ such that

$$(\tau\, f, f)_o + K(f, f)_o \geqq k||f||_p^2 \equiv k(f,f)_p \quad (f \in C_o^\infty(\Omega)) \tag{9.42}$$

where $(.,.)_o$ and $(\cdot,\cdot)_p$ are the inner products for $H^o(\Omega) = L^2(\Omega)$ and $H^p(\Omega)$ respectively.

Remark: It is assumed that the $a_J(y)$ are sufficiently smooth. If τ is strongly elliptic then, from the integration by parts formula, $(\tau\, f, f)_o = (f, \tau^* f)_o$ for $f \in C_o^\infty(\Omega)$ and therefore τ^* is strongly elliptic and satisfies the same Garding inequality (9.41). It is the Garding inequality which is instrumental in the establishment of stability conditions in Section 10.

The second order example of τ given in (9.9) is strongly elliptic if $A = A' = (a_{ij}(y))$ is a negative definite matrix for every $y \in \bar{\Omega}$.

9.9. Closed Operators in $L^2(\Omega)$

In order to obtain a closed operator densely defined in an appropriate Hilbert space, in this case, $L^2(\Omega)$, the following theorem taken from the contents of [13] (pp. 1730-44) is valid.

Theorem 9.9.1: Let Ω, Γ and I be as in Section 9.1. Let τ be a real, formal, strongly elliptic PDO of even order 2p in I. Let $k > 0$ and $K < \infty$ be as determined in 9.41. Let T and $\hat{T}$ be the operators in the Hilbert space, $L^2(\Omega)$, defined by

$$\mathcal{D}(T) = \mathcal{D}(\hat{T}) = \{f \in C^\infty(\bar{\Omega});\ (\partial_\nu(\Gamma))^{j-1} f(y) = 0,\ j=1,2,\dots,p,\ y \in \Gamma\}$$

$$T\,f = \tau\,f;\ \hat{T}\,f = \tau^* f \qquad (f \in \mathcal{D}(T) \equiv \mathcal{D}(\hat{T})) \qquad \textbf{(9.43)}$$

Let V and $\hat{V}$ denote the operators whose graphs are the closures of the graphs of T and $\hat{T}$, respectively. Then

(i) $V^* = \hat{V}$, $\hat{V}^* = V$; (V^* is the Hilbert Space Adjoint of V)

(ii) $\mathcal{D}(V) = \mathcal{D}(\hat{V}) = \mathcal{D}(V^*) = \mathcal{D}(\hat{V}^*)$;

(iii) $\mathcal{D}(V) = H_0^p(\Omega) \cap H^{2p}(\Omega)$;

(iv) $(Vf, f)_0 + K(f, f)_0 \geqq k(f,f)_p \qquad (f \in \mathcal{D}(V))$

(v) $(\hat{V}f, f)_0 + K(f, f)_0 \geqq k(f, f)_p \qquad (f \in \mathcal{D}(\hat{V}))$;

(vi) the spectrum of V, $\sigma(V)$, is a countable, discrete set of points in the complex plane with no finite limit points;

(vii) if $(\tau f,f) \geqq \gamma(f,f)$ for $f \in C_0^\infty(\Omega)$, then Re $\lambda < \gamma$ implies $\gamma \in \rho(V)$;

(viii) if $\tau = \tau^*$, then $V = V^* = \hat{V} = \hat{V}^*$;

(ix) if $\lambda \notin \sigma(V)$, $R(\lambda; V) \in [L^2(\Omega)]$ is a compact operator;

(x) if $\lambda \notin \sigma(V)$, $R(\lambda;V) \in [H^m(\Omega), H^{m+2p}(\Omega)]$ for every $m \geqq 0$,

(xi) if $V f \in H^m(\Omega)$, $f \in H^{m+2p}(\Omega)$ $H_0^p(\Omega)$ and for $m + 2p-[n/2]-1 \geqq j \geqq 0$, $f \in C^j(\bar{\Omega})$ and $(\partial_\nu(\Gamma))^k f(y) = 0$, $y \in \Gamma$, $0 \leqq k \leqq \min(j, p-1)$.

Remarks: Garding's Inequality is instrumental in establishing this result and the Dirichlet boundary conditions are a substantial hypothesis needed to establish the result in this way. The status of similar results for other boundary conditions is not clear from the literature available to the authors, although much mathematical literature relating to these problems is available. The essential conclusion of the theorem for this paper are (i-viii).

Through a convenient perversion of mathematical terminology, if τ is a real, formal, strongly elliptic PDO of even order 2p in I, V obtained through theorem 9.9.1 will be called the closed extension of τ in the remainder of this paper. This means that the intermediate process of defining T is assumed to have been done. Such a formulation is valid for boundary value problems with Dirichlet boundary conditions, but as remarked, is not valid in general for other boundary conditions. $\mathcal{D}(V)$, since V* exists, is dense in $L^2(\Omega)$ and in fact $C_0^\infty(\Omega) \subseteq \mathcal{D}(V)$ and is dense in $L^2(\Omega)$.

10. Applications to Partial Differential Equations

The general procedure in the remainder of the paper is, for example, to consider a partial differential equation of the type $u_t(y,t) + \tau u(y,t) = 0$ satisfying Dirichlet boundary conditions, where τ is a formal PDO as described. V becomes the closed extension of τ in $L^2(\Omega)$ and $u(\cdot,t) = x(t) \in L^2(\Omega)$ for each t. Thus the partial differential equation is formulated as an operator differential equation $\dot{x}(t) = -Vx(t)$ in $L^2(\Omega)$ or replacing $-V$ by A, this becomes $\dot{x}=Ax$ as described in Section 6. If A can be shown to generate a group or semigroup as in Section 5 and in addition satisfy the stability theorems in Section 7, then the stability of the solution $x=0$ of $\dot{x} = Ax$ is assured.

In every case, the crucial point in the stability analysis is whether or not A is strictly dissipative with respect to some inner product, i.e. whether a relation of the form $(Ax,x)_1 \leqq -\gamma(x,x)_1$ for $x \in \mathcal{D}(A)$ can be obtained. Since the $C_0^\infty(\Omega)$ functions are dense in $H_0^p(\Omega)$ and are in $H^{2p}(\Omega)$, then they are dense in $H_0^p(\Omega) \cap H^{2p}(\Omega)$. From this fact it can be shown that if $(-\tau x,x)_1 \leqq -\gamma(x,x)_1$ for $x \in C_0^\infty(\Omega)$ then $(Ax,x)_1 \leqq -\gamma(x,x)_1$ for $x \in \mathcal{D}(A)$. Note that the evaluation of $(-\tau x,x)_1$ for $x \in C_0^\infty(\Omega)$ proceeds formally, but the explicit density results assure that the same evaluation holds for $(Ax,x)_1$ for $x \in \mathcal{D}(A) = H_0^p(\Omega) \cap H^{2p}(\Omega)$. The rigorous mathematical structure for solutions then depends on the semi-group or group generated by A, and the corresponding stability theory developed for the semigroup or group structure.

The specific stability results are with respect to the norm of the base Hilbert space, in this case, $L^2(\Omega)$. This is not a pointwise stability result, but such a result may be possible by using the Sobolev imbedding theorems as pointed out in the last Section.

To illustrate the use of Theorem 7.3 consider first the class of parabolic partial differential equations (evolution equations)

$$u_t(y,t) + \tau u(y,t) = 0 \qquad (t \geqq 0) \tag{10.1}$$

where τ is a strongly elliptic PDO of even order 2p in I and suppose u(y,t) is subject to the Dirichlet boundary conditions of order p:

$$(\partial_\nu(\Gamma))^j\, u(y,t) = 0, \quad 0 \leqq j \leqq p-1 \qquad y \in \Gamma,\ t \geqq 0 \tag{10.2}$$

Equations (10.1) and (10.2) do not define an operator differential equation (6.1). However, identification by $u(\cdot,t) = x(t) \in L^2(\Omega)$ for each $t \geqq 0$ and using Theorem 9.9.1 allows the following formulation:

$$\dot{x} = -V\, x = Ax \qquad x \in \mathcal{D}(V) = \mathcal{D}(A)$$

$$\mathcal{D}(V) = \mathcal{D}(A) = H_0^p(\Omega) \cap H^{2p}(\Omega) \subseteqq L^2(\Omega) \tag{10.3}$$

where V is the closed extension of τ in $L^2(\Omega)$

Theorem 10.1: A sufficient condition for the null solution x=0 of the system (10.3) to be the only equilibrium solution of $\dot{x} = Ax$ in (10.3), and to be exponentially asymptotically stable with respect to the L^2-norm is that there exist a $c \geqq 1$ such that:

$$\text{(i)} \quad ||x||_p^2 \geqq c||x||^2 \qquad x \in \mathcal{D}(V)$$

and

$$\text{(ii)} \quad ck - K > 0 \tag{10.4}$$

where k and K are the two constants satisfying Garding's Inequality for τ in (9.41).

The proof of this theorem follows from Theorems 5.2, 7.3 and 9.9.1. Since τ is strongly elliptic $A = -V$ satisfies from (iv) of Theorem 9.9.1

$$(Ax,\ x) \leqq -k||x||_p^2 + K(x,x) \qquad x \in \mathcal{D}(A)\ .$$

Then by (vii) of Theorem 9.9.1 for $ck-K > 0$, $\mathrm{Re}\lambda > -(ck-K)$ implies $\lambda \in \rho(A)$, then A and thus $-V$ generates a semi-group and satisfies all of the conditions of Theorem 7.3 with $v(x) = (x,x)$ and hence assures the asymptotic stability of $x=0$ (or $u=0$). It is also true that A, since $0 \in \rho(A)$, has a continuous inverse which assures that $x=0$ is the only equilibrium solution.

Furthermore it can be shown that the imbedding of the closed subspace $H_o^p(\Omega)$ in $L^2(\Omega)$ implies that there exists a constant $c \geqq 1$ such that (i) of Theorem 10.1 is satisfied.

The objective in the stability analysis becomes thus to determine (i) the maximum value of c (often from well known integral inequalities) and (ii) the maximum k and minimum K such that Garding's Inequality in (iv) of Theorem 9.9.1 is satisfied.

Example 10.1. As a first example let $\Omega = (0,1) \subseteqq R^1$ and let

$$\tau u = -\alpha\ u_{yy} + \beta\ u \tag{10.5}$$

For τ to be strongly elliptic α must be positive, thus $\alpha > 0$. The Dirichlet boundary conditions are $u(0,t) = u(1,t) = 0$.

Note that τ is formally self-adjoint. The Lyapunov functional $v(u)$ can be taken as $v(u) = (u,u) = \int_o^1 u^2\, d\, y$ and the evaluation of (Vu,u) for $u \in C_o^\infty(\Omega)$ proceeds formally as follows:

$$(Vu,u) = \int_o^1(-\alpha u\ u_{yy} + \beta\ u^2)dy = \int_o^1(\alpha u_y^2 + \beta\ u^2)dy = \alpha(u,u)_1 + (\beta - \alpha)(u,u) \tag{10.6}$$

Using the well known inequality, valid here,

$$\int_0^1 u_y^2 \, dy \geqq \pi^2 \int_0^1 u^2 \, dy \tag{10.7}$$

there follows

$$(u,u)_1 = \int_0^1 (u_y^2 + u^2) dy \geqq (\pi^2 + 1) \; (u,u)$$

or in other words $c = \pi^2 + 1$ and this is a maximum. The condition (ii) of Theorem 10.1 becomes now

$$(\pi^2 + 1)\,\alpha + (\beta - \alpha) = \pi^2 \alpha + \beta > 0$$

A sufficient condition for asymptotic stability is $\alpha > 0$ and $\beta > -\pi^2 \alpha$.

From the above example and the formal manipulations, it becomes apparent that the inequality (10.7) can directly be used to find a sufficient condition for asymptotic stability by evaluating

$$(Vu,u) = \int_0^1 (\alpha \; u_y^2 + \beta \; u^2) dy \geqq (\pi^2 \alpha + \beta) \int_0^1 u^2 \, dy =$$

$$= (\pi^2 \alpha + \beta) \quad (u,u) \qquad u \in \mathcal{D}(V) \tag{10.8}$$

The condition that is imposed on the coefficients of Garding's Inequality for τ is thereby implemented.

The second example was studied by Eckhaus [6] using approximate methods. This particular example will show the important use of equivalent inner products in choosing a Lyapunov functional, i.e. in evaluating (Vu,u).

<u>Example 10.2</u>: Take again $\Omega = (0,1) \subseteqq R^1$ and let

$$\tau \, u = -\frac{1}{R}\frac{\partial^2 u}{\partial y^2} - \frac{2}{\sqrt{R}}\, y \,\frac{\partial u}{\partial y} - (y^2 + \frac{2}{\sqrt{R}})\, u \tag{10.9}$$

where R is a positive constant. The Dirichlet boundary conditions are $u(0,t) = u(1,t) = 0$.

τ is for $R > 0$ a strongly elliptic partial differential operator; however, τ is not formally self-adjoint. If a Lyapunov functional $v(u) = (u,u)$ is chosen, the evaluation of (Vu,u) results in:

$$(Vu, u) \geqq \left(\frac{\pi^2}{R} - \frac{1}{\sqrt{R}} - y^2 \right) (u,u) \qquad (10.10)$$

where the inequality (10.7) has been used. A sufficient condition for the asymptotic stability of the null solution u=0 is thus

$$0 < R < \frac{1}{2} \left(1 + 2\pi^2 - \sqrt{1 + 4\pi^2}\right) . \qquad (10.11)$$

However τ is equivalent to τ_e;

$$\tau_e = - \frac{1}{w(y)} \frac{\partial}{\partial y} \left(p(y) \frac{\partial}{\partial y}\right) - \left(y^2 + \frac{2}{\sqrt{R}}\right) \qquad (10.12)$$

where

$$w(y) = R\, e^{\sqrt{R}\, y^2} \quad \text{and} \quad p(y) = e^{\sqrt{R}\, y^2} \quad . \qquad (10.13)$$

And τ_e is strongly elliptic, and both τ_e and V are self-adjoint with respect to the equivalent inner product

$$(f,g)_w = \int_0^1 f\, g\, w(y) dy \qquad f,\ g \in L^2(\Omega). \qquad (10.14)$$

The sufficient condition for asymptotic stability of the null solution, u=0, follows from evaluating

$$(Vu,u)_w = \int_0^1 \left\{ e^{\sqrt{R}\, y^2} \left(\frac{\partial u}{\partial y}\right)^2 - R\, e^{\sqrt{R}\, y^2} \left(y^2 + \frac{2}{\sqrt{R}}\right) u^2 \right\} dy > 0 \qquad (10.15)$$

Application of the integral inequality (10.7) to $e^{(1/2)\sqrt{R}\, y^2}$ u, rather than to u gives

$$(Vu,u)_w \geqq \left(\frac{\pi^2}{R} - \frac{1}{\sqrt{R}}\right) (u,u)_w > 0 \qquad (10.16)$$

Thus a sufficient condition for asymptotic stability of the null solution u=0 becomes now

$$0 < R < \pi^4 \qquad (10.17)$$

which is a considerable improvement over (10.11).

The above example shows the importance of selecting the "optimum" Lyapunov functional, i.e. the inner product for the space. The general procedure is to introduce τ_e in such a way that the highest order odd derivative of τ is eliminated. This is once more illustrated in the following example. Again taken from Eckhaus [6].

Example 10.3: For $\Omega = (0,1) \subsetneqq R^1$ and $R > 0$ let

$$\tau\, u = \frac{1}{R^2}\frac{\partial^4 u}{\partial y^4} + \frac{1}{R\sqrt{R}}\frac{\partial^3 u}{\partial y^3} + \frac{5}{4R}\frac{\partial^2 u}{\partial y^2} + \frac{1}{\sqrt{R}}\frac{\partial u}{\partial y} + \frac{1}{4}u \tag{10.18}$$

and with Dirichlet boundary conditions

$$u(t,0) = u(t,1) = \left.\frac{\partial u}{\partial y}\right|_{y=0} = \left.\frac{\partial u}{\partial y}\right|_{y=1} = 0 \ .$$

For τ to be a strongly elliptic formal PDO of order 2p, p=2, R > 0. Evaluation of (Vu,u) on the $C_0^\infty(\Omega)$ functions results in a sufficient condition for asymptotic stability of the null solution u=0 of

$$0 < R < \frac{4}{5}\pi^2. \tag{10.19}$$

However, τ is equivalent to τ_e

$$\tau_e = + \frac{1}{w(y)}\ \frac{\partial^2}{\partial y^2}(p(y))\ \frac{\partial^2}{\partial y^2}) + \frac{1}{R}\frac{\partial^2}{\partial y^2} + \frac{1}{\sqrt{R}}\frac{\partial}{\partial y} + \frac{1}{4} \tag{10.20}$$

with $p(y) = e^{\frac{1}{2}\sqrt{R}\,y}$. The subsequent evaluation of $(Vu,u)_w = (Vu,w(y)u)$ gives the sufficient condition for asymptotic stability of the null solution as

$$0 < R < \frac{16}{15}\pi^2 \tag{10.21}$$

Next consider the class of wave equations:

$$u_{tt}(y,t) + a\, u_t(y,t) + \tau\, u(y,t) = 0 \qquad (t \geqq 0) \tag{10.22}$$

with a being a positive constant and τ now a strongly elliptic self-adjoint partial differential operator of even order 2p in I. Again let u(y,t) satisfy the Dirichlet boundary condition of order p (10.2).

By employing Theorem 9.9.1, (10.22) can be reformulated in terms of a closed self-adjoint operator, V, where the following holds:

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + Vx = 0 \qquad (x \in \mathcal{D}(V)) \tag{10.23}$$

$$V \text{ extends } \tau \tag{10.24}$$

$$\mathcal{D}(V) = H_o^p(\Omega) \cap H^{2p}(\Omega) \subseteqq L^2(\Omega) \tag{10.25}$$

Equation (10.23) can be written in the form (6.1) by transforming to

$$\frac{d\underline{x}}{dt} = \underline{A}\,\underline{x}\,, \quad \mathcal{D}(\underline{A}) = H_o^p(\Omega) \cap H^{2p}(\Omega) \times H_o^p(\Omega) \tag{10.26}$$

where

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \quad ; \quad \underline{A} = \begin{bmatrix} 0, & 1 \\ -V, & -a \end{bmatrix} \tag{10.27}$$

Since

$$\underline{A}\,\underline{x} = \begin{bmatrix} x_2 \\ -V\,x_1 - a\,x_2 \end{bmatrix}$$

there also follows $R(\underline{A}) = H_o^p(\Omega) x R(V) = H_o^p(\Omega) \times L^2(\Omega)$.

The following theorem can now be proven:

Theorem 10.2: A necessary and sufficient condition for the asymptotic stability of the null solution of the system given by (10.26) and (10.27), where V is the closed extension of the strongly elliptic formally self-adjoint PDO τ, is that there exist a $c \geqq 1$ such that

$$\text{(i)} \quad ||x_1||_p^2 \geqq c||x_1||^2 \qquad (x_1 = x \in \mathcal{D}(V))$$

$$\text{(ii)} \quad k - \frac{K}{c} \geqq \varepsilon > 0 \tag{10.28}$$

$$\text{(iii)} \quad a > 0$$

where k and K are the constants satisfying Garding's Inequality for τ.

The proof of this theorem is based on the Theorems 5.3, 7.4 and 9.9.1 by constructing the bilinear functional

$$(\underline{x},\ \underline{\hat{P}}\ \underline{x}) \qquad (\underline{x}\ \varepsilon\ C_o^\infty(\Omega)\ \text{x}\ C_o^\infty(\Omega)) \tag{10.29}$$

where

$$\underline{\hat{P}} = \begin{bmatrix} 2V + a^2 & , a \\ a & , 2 \end{bmatrix} \tag{10.30}$$

An evaluation of $(\underline{x},\ \underline{\hat{P}}\ \underline{x})$ together with the condition (i), (ii) and (iii) of Theorem 10.2 gives

$$(\underline{x},\ \underline{\hat{P}}\ \underline{x}) \geqq 2\varepsilon(x_1,x_1)_p + 2c(k - \frac{K}{c} - \varepsilon)(x_1,x_1) + a^2(x_1,x_1) + 2a(x_1,x_2)$$

$$+ 2(x_2,x_2) \geqq d[(x_1,x_1)_p + (x_2,\ x_2)_o] = d||\underline{x}||^2_{p,o} \tag{10.31}$$

where $d=d(\varepsilon)$ is some positive constant.

Since the coefficients of τ and thus V are uniformaly bounded on Ω there exists a constant D, $0 < D < \infty$ such that

$$(\underline{x},\ \underline{\hat{P}}\ \underline{x}) \leqq D[(x_1,x_1)_p + (x_2,x_2)] = D||\underline{x}||^2_{p,o} \tag{10.32}$$

The bilinear **form** $(\underline{x},\underline{\hat{P}x})$ being defined on a dense subset of $H_o^p(\Omega)xL^2(\Omega)$ and being bounded can be extended by continuity [10] to the form $(\underline{x},\underline{Px})$ which also satisfies (10.31) and (10.32) and $\underline{P}\ \varepsilon\ [H_o^p(\Omega)\ \text{x}\ L^2(\Omega)]$ is RSPD, or in other words $(\underline{x},\underline{Py})$ is an equivalent inner product in the Hilbert space $H_o^p(\Omega)\ \text{x}\ L^2(\Omega)$.

Similarly an evaluation of $(\underline{A}\ \underline{x},\ \underline{P}\ \underline{x})$ for $\underline{x}\ \varepsilon\ C_o^\infty(\Omega)\ \text{x}\ C_o^\infty(\Omega)$ gives

$$(\underline{A}\ \underline{x},\ \underline{P}\ \underline{x}) = -a(Vx_1,\ x_1) - a(x_2,x_2)$$

$$\leqq -a\varepsilon(x_1,x_1)_p - ac(k - \frac{K}{c} - \varepsilon)(x_1,x_1) - a(x_2,x_2)$$

$$\leqq -e\ [(x_1,x_1)_p + (x_2,x_2)] = -e||\underline{x}||^2_{p,o} \qquad (e=e(\varepsilon)) \tag{10.33}$$

for some $e > 0$. It can also be shown that there exists a constant E, $0 < E < \infty$ such that

$$(\underline{A}\ \underline{x},\ \underline{P}\ \underline{x}) \geqq -E||\underline{x}||_p \tag{10.34}$$

Thus $\underline{A}$ generates a group if $\underline{A}$ satisfies the conditions (5.8) which can be shown [30,31]. Furthermore let

$$v(\underline{x}) = (\underline{x},\ \underline{P}\ \underline{x})$$

Then it follows after combining (10.31), (10.32), (10.33) and (10.34) that there exist an α and β, $\infty > \alpha \geqq \beta > 0$ such that

$$-\alpha\ v(\underline{x}) \leqq \dot{v}(\underline{x}) \leqq -\beta\ v(\underline{x}) \qquad \underline{x} \in \mathcal{D}(\underline{A}) \tag{10.35}$$

by letting $\alpha = \frac{2E}{D}$ and $\beta = \frac{2e}{d}$. Thus the conditions of Theorem 7.4 are also satisfied and the null solution $\underline{x} = \underline{0}$ (and thus $u=0$) of (10.26) is asymptotically stable.

Thus for $a > 0$ the stability analysis of (10.22) requires only an evaluation of (Vx,x), where V is the extension of τ. This evaluation must however proceed as follows

$$(Vx,x) \geqq \varepsilon(x,x)_p + c(k-\frac{K}{c} - \varepsilon)\ (x,x). \tag{10.36}$$

In order to establish bounds on the system parameters one can take ε sufficiently small and thus require $ck - K > 0$. The particular choice of the matrix $\underline{P}$, (10.30), is motivated in [7].

<u>Example 10.4</u>. Let τ be defined as in Example 10.1. The conditions for stability follow immediately from the evaluation in Example 10.1 as $a > 0$, $\alpha > 0$ and $\beta > -\pi^2\alpha$. However, the Lyapunov functional $v(\underline{u})$ must be chosen as

$$v(\underline{u}) = \int_0^1 \{2\alpha\ u_y^2 + (2\beta + a^2)u^2 + 2a\ u\ u_t + 2u_t^2\}dy \tag{10.37}$$

and the stability is with respect to the norm for the space $H_o^1(\Omega) \times L^2(\Omega)$.

Example 10.5 Let τ be defined as in Example 10.2, then since Theorem 10.2 requires τ to be self-adjoint the evaluation of the Lyapunov functional and its derivative must be done with respect to the inner product as defined by (10.14). The conditions for stability follow from $(Vu, u)_w > 0$ as, $a > 0$ and $0 < R < \pi^4$.

In Theorem 10.2 and Examples 10.4 and 10.5 only self-adjoint strongly elliptic PDO's have been considered. In all these cases the stability analysis can be based on an evaluation of (Vu, u) only, where V is the closed extension of τ.

However, one is certainly not formally limited to self-adjoint PDO's. Consider again equation (10.22) and let τ just be a strongly elliptic PDO of even order 2p in I. And let the Dirichlet boundary conditions (10.2) be satisfied. Then the natural coice of Lyapunov functional is:

$$v(u) = \int_\Omega [\bar{u}(\tau + \tau^*)u + a^2|u|^2 + a\,\bar{u}\,u_t + a\,u\,\bar{u}_t + 2|u_t|^2]dy \quad (10.38)$$

where $\bar{u}$ designates the conjugate of u. The derivative $\dot{v}(u)$ becomes now:

$$\dot{v}(u) = -a\int_\Omega [\bar{u}(\tau + \tau^*)u + \bar{u}(\tau - \tau^*)u_t - \bar{u}_t(\tau - \tau^*)u + 2|u_t|^2]dy \quad (10.39)$$

In order to formally derive sufficient conditions for asymptotic stability, $v(u)$ and $\dot{v}(u)$ must be considered in their totality as given by (10.38) and (10.39). The term $\int_\Omega \bar{u}(\tau + \tau^*)u\,dy$ can again be evaluated with Garding's Inequality, since both τ and τ^* satisfy (9.42) for identical coefficients k and K. The second and third term in (10.39) necessitate a further evaluation as will be shown in the following example.

Example 10.6 Consider the panel flutter problem as for example studied by Parks [5]. From the nondimensional equation for the panel motion of [5] the following partial differential equation can be derived:

$$\frac{\partial^2 u}{\partial t^2} + \frac{1}{\mu}\frac{\partial u}{\partial t} + \frac{d}{\mu}\frac{\partial^4 u}{\partial y^4} - \frac{f}{\mu}\frac{\partial^2 u}{\partial y^2} + \frac{M}{\mu}\frac{\partial u}{\partial y} = 0 \tag{10.40}$$

with $\Omega = (0,1) \subseteq R^1$. Here, d, the flexural stiffness parameter, μ, the panel-air mass ratio, and M, the Mach number, are essentially positive; f, the tension parameter, may be positive or negative. The boundary conditions are $u(0,t) = u(1,t) = 0$ and $u_y(0,t) = u_y(1,t) = 0$. In this case the Hilbert space would be $H_0^2(\Omega) \times L^2(\Omega)$.

For $d > 0$ and $\mu > 0$,

$$\tau = \frac{d}{\mu}\frac{\partial^4}{\partial y^4} - \frac{f}{\mu}\frac{\partial^2}{\partial y^2} + \frac{M}{\mu}\frac{\partial}{\partial y} \tag{10.41}$$

is a strongly elliptic PDO. Similarly for the formal adjoint of τ, τ^*, given by

$$\tau^* = \frac{d}{\mu}\frac{\partial^4}{\partial y^4} - \frac{f}{\mu}\frac{\partial^2}{\partial y^2} - \frac{M}{\mu}\frac{\partial}{\partial y} . \tag{10.42}$$

The evaluation of the Lyapunov functional (10.38) for this case with $a = \frac{1}{\mu} > 0$ requires the evaluation of $\int_\Omega \bar{u}(\tau+\tau^*)u\, dy = \int_\Omega \bar{u}\, \tau\, u\, dy = 2(\tau u,u)$ which should proceed as in (10.36). Carrying out the integration by parts, applying the integral inequality (10.7) and letting $\varepsilon \to 0$ gives

$$\mu > 0\ , \quad f + \pi^2 d > 0 \tag{10.43}$$

as the conditions for $v(u) > 0$.

Similarly for the derivative $\dot{v}(u)$ as given by (10.39) can be written:

$$\dot{v}(u) \leqq -2a[\varepsilon(u,u)_2 + \{(\frac{d}{\mu} - \varepsilon)\pi^2 + (\frac{f}{\mu} - 2\varepsilon)\}(u_y,u_y) + 2\frac{M}{\mu}(u_y, u_t) +$$

$$+ (u_t, u_t) + \varepsilon(\pi^2 - 1)(u,u)] \tag{10.44}$$

If $M^2 < \mu(f + \pi^2 d)$, then from (10.44) it follows that

$$\dot{v}(u) \leqq -2\, a\, k[(u, u)_2 + (u_t, u_t)] \tag{10.45}$$

where $k = k(\varepsilon) > 0$ for $\varepsilon > 0$ sufficiently small. Thus the conditions for $\dot{v}(u)$ to be negative definite follow as:

$$f + \pi^2 d > 0 \quad \text{and} \quad M^2 < \mu (f + \pi^2 d). \qquad (10.46)$$

It can be shown that formally, all the conditions of Theorems 5.3 and 7.4 are satisfied, so that the conditions for the asymptotic stability of the null solution of (10.40) are:

$$\mu > 0,\ d > 0,\ f + \pi^2 d > 0 \quad \text{and} \quad M^2 < \mu(f + \pi^2 d) \qquad (10.47)$$

These results, formally derived, are compatible with those obtained by Parks [5].

11. Additional Results and Suggestions for Further Research

The theory presented in this paper has been extended in several directions. A logical direction, of course, is to extend this to partial differential equations which are nonlinear. This has been done making use of results originally obtained by Kato [33,34,35,36]. Essentially, instead of having a semi-group or group of linear operators, one assumes the existence of a semi-group of nonlinear operators $\{T_t;\ t\geq 0\}$ where T_t for each $t \geq 0$ is nonlinear and defined on a Hilbert space, H. As a result of this assumption, the infinitesimal generator A, is defined on a subset of the Hilbert space H and is a nonlinear operator. If the nonlinear semi-group is contractive then (-A) is an m-monotone operator. If (-A) is an m-monotone operator and linear, then A is a dissipative operator, which connects this theory to the theory encompassed under that of linear semi-groups defined on a Hilbert space H. Since the theory of nonlinear semi-groups and nonlinear infinitesimal generators has been developed only recently, some of the fine structure of this theory has not yet been established. For example, the domain of a nonlinear operator A need not be dense in the Hilbert space H. It turns out that if A is a linear operator, then the domain is dense if it generates a linear semi-group. In addition, solutions to the operator equation $\dot{x} = Ax$ no longer have all the nice properties that exist in the case of linear semi-groups. However, even with all these limitations, the classical Lyapunov stability theorem on differential equations with an asymptotically stable linear approximation and a nonlinear part carries forward into the partial differential equation case; that is, if the linear approximation is asymptotically stable, then under the proper assumptions regarding the nonlinear term, the nonlinear equation is asymptotically stable also in some neighborhood of the origin. There exists a great deal to be done in the nonlinear cases.

Another direction in which extensions have been made to the present theory is in the choice of Lyapunov functionals. Essentially, in this paper, Lyapunov functionals have been restricted to the class of equivalent inner products for a Hilbert space. It turns out that there is a natural extension to the concept of inner product, which is called a semi-inner product [10,31,32,37,38]. This natural extension leads to a much broader class of possible Lyapunov functionals, which in turn lead to significant developments. The first of these is that the Lyapunov stability theory can be extended to Banach spaces, that is, spaces which do not necessarily have an inner product structure. The second of these is that a theorem similar to Theorem 7.4 can be proved for semi-groups using the semi-inner product formulation in either Hilbert space or Banach space [31,32].

An additional limitation in the present paper which should provide an opportunity for further research is that the formal PDO, τ and the associated boundary value problem consists of the Dirichlet problem. If one attempts an extension of theorem 9.9.1, which is the basic theorem to connect partial differential equations with operator differential equations, one is faced with an extremely difficult problem even in the linear case. Essentially, what is required is additional mathematical analysis. It should be emphasized, however, that the main difficulty does not lie with the general approach to the problem as detailed in this paper, but a specific result that is required, which in this paper, for the Dirichlet problem, is emcompassed in theorem 9.9.1. Once a more general result of this type is available, the general theory for partial differential equations with non-Dirichlet boundary conditions should proceed in much the same way as indicated in this paper.

Another direction for further research, is to determine the stability with respect to different norms. In this paper the main stability result is with respect to the L^2-norm. There are many physical problems where stability with respect to norms other than the L^2-norm is important. Once again however the difficulty is in the details of the proof for a theorem such as theorem 9.9.1. What one would try to do, as a conjecture, would be to close the operator T, not in L^2 but in a Sobolev space such as $H^m(\Omega)$ which becomes the base Hilbert space and stability is with respect to $||\cdot||_m$ of $H^m(\Omega)$. If m is sufficiently large, the Sovolev embedding theorem from (xi) of Theorem 9.9.1 can be applied which states in essence that elements of $H^m(\Omega)$ are in $C^j(\Omega)$ if $m >> j$. If I is the imbedding operator from $H^m(\Omega)$ to $C^j(\Omega)$, then from the norm relationship

$$||I\ T_t x||_{C^j(\Omega)} \leqq M||T_t x||_m$$

it follows, $I\ 0 \varepsilon C^j(\Omega)$ is asymptotically stable with respect to the $C^j(\Omega)$-norm if $0 \varepsilon H^m(\Omega)$ is asymptotically stable with respect to the $H^m(\Omega)$ norm. This is not strictly pointwise stability, but if one is willing to ignore, at each $t \geqq 0$, the distinction between equivalence classes of functions which are the elements of $H^m(\Omega)$ and a $C^j(\Omega)$ function which is a representative of such an equivalence class, one has "almost everywhere pointwise stability".

An additional possibility is the proof of LaSalle's Theorem [39] in these Sobolev spaces. An essential requirement in LaSalle's Theorem is compactness of a set which in $H = R^n$ is assured if the set is closed and bounded. In a Sobolev space, $H^m(\Omega)$ a closed bounded set, say $V_q = \{x \in H^m(\Omega);\ ||x||_m \leqq q\}$, is not compact, but if V_q is imbedded in $H^{m-j}(\Omega)$ for $j=1,2,\cdots,m$, $V_q \subseteqq H^{m-j}(\Omega)$ is compact and perhaps from this, LaSalle's Theorem can be proved.

Of course, this mathematical formulation for solutions to partial differential equations can be pursued in directions other than stability theory, for example optimal control, numerical approximation, etc.

12. Summary

The purpose of this paper has been to present a rigorous approach to the stability of partial differential equations. The required mathematical machinery has been explored and applications to a class of partial differential equations have been given. Much of the formal manipulation of Lyapunov functionals for these types of partial differential equations has been rigorously justified. The mathematical treatment to attain this is sophisticated but well within reach of doctoral level engineers. This is a beginning--much is left to be done.

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