Parametric Wave Amplification and Mixing in the Ionosphere

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ABSTRACT

An analysis is presented of nonlinear interaction between right- or left-hand polarized transverse electromagnetic waves and longitudinal (Langmuir) plasma waves propagating in the ionosphere. The object is to determine the feasibility of observing parametric amplification or frequency mixing due to such interactions, using ground-based transmitters and receivers. It is shown theoretically that wave coupling should occur in three situations: first, where the highest frequency component is the plasma wave and the two lower frequencies are right- and left-hand polarized waves, respectively; second, where the highest frequency component and one other are right-hand polarized and the third is a plasma wave, and third, where the highest frequency component and one other are left-hand polarized and the third is a plasma wave. The first case shows the weakest coupling, and seems unlikely to be observable. The second and third give rise to somewhat stronger effects, but for typical transmitter and ionospheric parameters suggest the necessity of measuring increments in wave amplitudes due to nonlinear coupling only of order 0.01 percent. It is concluded that although under the most favorable conditions such phenomena may be observable, they will generally be insignificant.

1. INTRODUCTION

Small-signal electromagnetic wave propagation in the ionosphere can usually be described adequately by cold plasma Appleton-Hartree theory, which predicts dispersion characteristics independent of signal amplitude. If the source power is raised progressively, however, nonlinear propagation phenomena not included in this theory, such as harmonic generation and intermodulation effects, are to be expected. There is consequently considerable interest attached to predictions concerning their observability. The authors are engaged in a systematic study of the specific class of nonlinear wave propagation phenomena which involves three-wave interaction [Harker and Crawford 1969a,b], and the present contribution concerns coupling between two transverse (left- or right-hand polarized) waves and a longitudinal (Langmuir) plasma wave.

Two examples of experimental situations to which the work is relevant are illustrated in Figure 1. Case I assumes that a large-amplitude plasma wave has been excited in the ionosphere. In the auroral zones, for example, this might result from beam/plasma interaction due to precipitation of high-energy electrons. It will be shown in section 3 that this pump wave (p) can amplify two lower-frequency transverse waves (s,i) by the nonlinear mechanism of traveling-wave parametric amplification, over an interaction region in which certain synchronism conditions can be satisfied. Alternatively, it can cause mixing between two right- or two left-hand polarized waves, of which at least one has a higher frequency than the pump. In the mixing situation, the small-amplitude signal (s) might be supplied by a ground-based transmitter. If a component at the idler (i) frequency were already present in the interaction region, it would be amplified, otherwise it would be generated there. Thus a suitable experiment to obtain evidence of nonlinear interaction would be to pulse-modulate the signal, and study the modulation of the idler by means of a ground-based receiver. By its longitudinal wave nature, the pump wave could

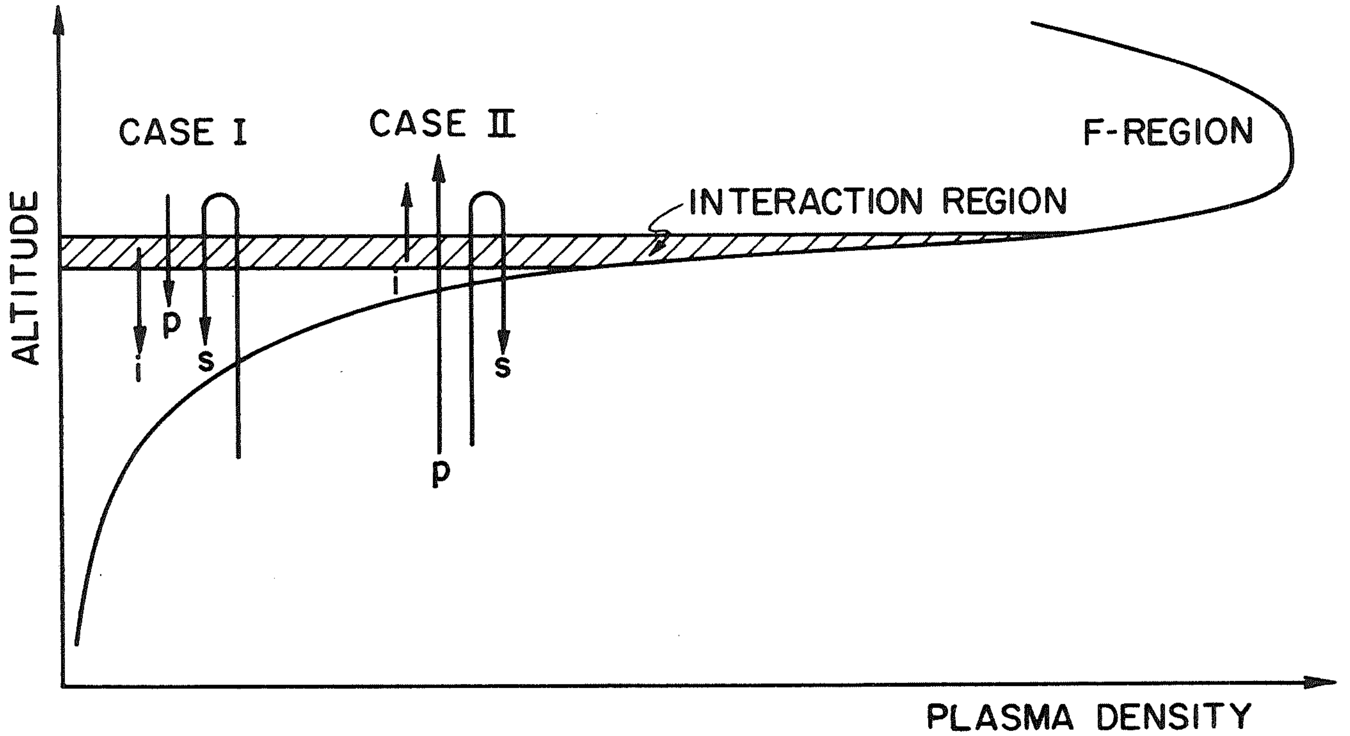

FIG. 1--Possible experiments on wave interaction in the ionosphere. (Case I: Plasma wave pump (p) causes amplification or mixing of transverse signal (s) and idler (i) waves, the signal being supplied by a ground-based transmitter. Case II: Transverse wave pump (p) causes amplification of a transverse wave signal (s) and a plasma wave idler (i), the pump and signal being supplied by ground-based transmitters.)

not propagate all the way down to the ground. It could be studied best *in situ* by a space-probing vehicle.

Case II assumes that a strong right- or left-hand polarized pump wave has been directed at the ionosphere by a high-power transmitter. A small-amplitude signal of the same polarization but at a lower frequency is also transmitted from the ground, and reflected from the ionosphere to a receiver located close to the source. In the interaction region, where appropriate synchronism conditions are satisfied, a plasma wave idler of lower frequency than the pump is generated and/or amplified. Evidence of amplification occurring could be sought by pulse-modulating the pump, and looking for modulation of the received signal wave. Its amplitude should show corresponding pulsed increments, delayed by the propagation time of the signal from the interaction region up to the reflection height and then down to the receiver.

The main purpose of the present paper is to assess the likelihood of success of experiments such as those described, by computing the relevant nonlinear coupling and parametric growth rates. For simplicity, a plane stratified ionosphere will be assumed, and only a vertical static magnetic field component will be included. Various facets of the basic wave coupling problem to be treated have already been examined in the literature, but no detailed numerical results appear to have been presented so far for the synchronism conditions and corresponding growth rates. This previous work will be referred to as the present analysis unfolds.

The plan of the paper is as follows: Section 2 examines the plasma and magnetic field parameter ranges over which the synchronism conditions for three-wave interaction can be satisfied. Section 3 develops expressions for the coupling and growth rates for the various combinations of right- and left-hand polarized waves and plasma waves that can interact in a homogeneous plasma. Section 4 presents comprehensive numerical results for each of these combinations. In practice, effects of plasma inhomogeneity in the direction of propagation, and of momentum transfer collisions, may operate to degrade the nonlinear interaction predicted in this section. These factors are discussed in Appendices A and B. Section 5 closes the paper with a brief discussion, based on the numerical results, of the feasibility of demonstrating nonlinear interaction in the ionosphere.

2. SYNCHRONISM CONDITIONS

The dispersion equation for plane transverse waves propagating parallel to the static magnetic field in a cold infinite magneto-plasma may be written in normalized variables as [Stix 1962]

$$K^2 = \Omega^2 - \Omega/(\Omega \pm \Omega_c) \quad , \tag{1}$$

where

$$K = kc/\omega_0 \quad , \quad \Omega = \omega/\omega_0 \quad , \quad \Omega_c = \omega_c/\omega_0 \quad , \tag{2}$$

propagation is as $\exp j(\omega t - kz)$; ω and k are the frequency and wavenumber; $\omega_0 [\equiv (n_0 e^2/\epsilon_0 m)^{1/2}]$ and $\omega_c = [eB_0/m]$ are the electron plasma and cyclotron frequencies; e is the magnitude of the electronic charge; m is the electron mass, and c is the speed of light. The static magnetic field, B_0 , is oriented along the z-axis. Positive ion motions have been neglected.

Figure 2 illustrates the dispersion characteristics described by (1). The right- and left-hand polarized waves, corresponding to the '-' and '+' signs, both have one fast wave branch $(\Omega/K > 1)$, and have cutoff frequencies $\Omega_{U,L} = (1 + \Omega_c^2/4)^{1/2} \pm \Omega_c/2$. The right-hand polarized wave also has a slow wave branch $(\Omega/K < 1)$, commonly referred to as the 'whistler' mode, for propagation below Ω_c .

The dispersion equation for plane longitudinal waves propagating parallel to the static magnetic field in a warm infinite magneto-plasma may be written in normalized variables as [Stix 1962]

$$\Omega^2 = 1 + K^2 S^2 \quad , \tag{3}$$

where

$$S^2 = 3eV_e/mc^2 \qquad (\approx 6 \times 10^{-6}\, V_e \quad \text{for} \quad V_e \text{ in volts}), \tag{4}$$

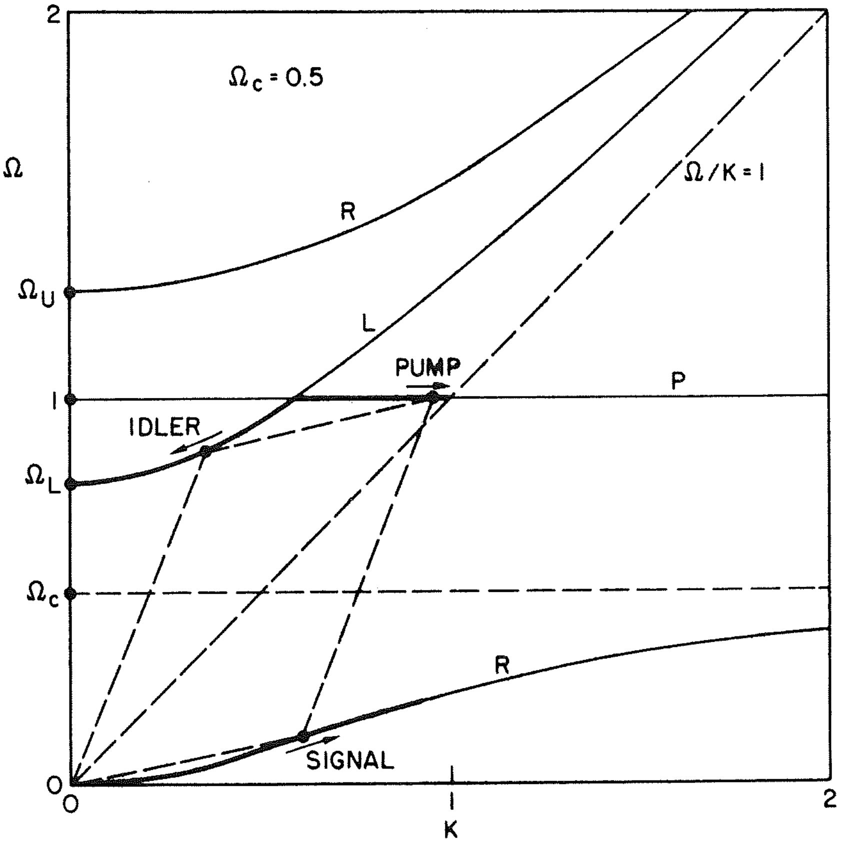

FIG. 2--Synchronism conditions for a plasma (P) pump wave, a right-hand polarized (R) signal wave, and a left-hand polarized (L) idler wave. (Heavy portions of the curves show ranges of variation over which the synchronism conditions can be satisfied.)

and V_e is the electron temperature. The dispersion characteristic is illustrated in Figure 2, and has a cutoff at the plasma frequency ($\Omega = 1$). The hydrodynamic approximation expressed by (3) will be valid only for $KS \lesssim 0.5$. For higher values, Landau damping should be taken into account. As we shall show later, we shall be interested only in cases with KS substantially smaller than 0.5.

It will be noted that electron temperature effects have been included only for the longitudinal plasma wave. Collisionless cyclotron damping can occur in the whistler mode close to Ω_c, but it turns out that the synchronism conditions for strong nonlinear interaction are never satisfied in the vicinity of this frequency for any situations of interest to us. For other frequencies, the cold plasma approximation is adequate for both the right- and left-hand polarized waves.

Our task in this section is to determine over what plasma parameter ranges the synchronism conditions necessary for traveling-wave parametric amplification [Louisell 1960],

$$\Omega_p = \Omega_s + \Omega_i \quad , \qquad K_p = K_s + K_i \quad , \tag{5}$$

can be satisfied by three collinearly propagating waves, which we shall refer to as in section 1 as the 'pump' (Ω_p, K_p), 'signal' (Ω_s, K_s) and 'idler' (Ω_i, K_i). We specify that one of these will be a plasma wave. At first sight, there are the seven possibilities listed in Table 1. The theory of section 3 shows that four of them can be excluded as they do not lead to coupling, so in fact only three

Table 1. Possible Cases for Investigation of Synchronism

Pump	Signal and Idler	Comment
P	R, L	Acceptable
R	R, P	
L	L, P	
P	R, R; L, L	No Coupling
R	L, P	
L	R, P	

possibilities exist. These will now be treated individually:

Pump P-Signal and Idler R, L. In this case, the synchronism conditions are,

$$\Omega_{Pp} = \Omega_{Rs} + \Omega_{Li} , \qquad K_{Pp} = K_{Rs} + K_{Li} , \tag{6}$$

where the clear though somewhat cumbersome convention has been adopted that the first subscript denotes the type of wave (P, R or L), and the second its role (p, s or i). The labeling of the signal and idler is arbitrary, and the designations may be interchanged if desired.

Suppose now that we specify Ω_{Rs} , so that by (1)

$$K_{Rs} = \left[\Omega_{Rs}^2 - \Omega_{Rs}/(\Omega_{Rs} - \Omega_c)\right]^{1/2} . \tag{7}$$

To determine Ω_{Pp} , Ω_{Li} , K_{Pp} , and K_{Li} in terms of Ω_{Rs} , we make use of (1), (3) and (6), and note that $K_{Pp}^2 S^2 \ll 1$ for all cases of interest to us. After expansion in powers of S^2 , the following approximate results may be obtained

$$\Omega_{Li} = \Omega_{Li0} + \Omega_P S^2 , \qquad \Omega_{Pp} = 1 + \Omega_P S^2 ,$$

$$K_{Pp} = K_{Pp0} + K_P S^2 , \qquad K_{Li} = K_{Li0} + K_P S^2 , \tag{8}$$

where

$$\Omega_{Li0} = 1 - \Omega_{Rs} , \quad K_{Li0} = \left[\Omega_{Li0}^2 - \Omega_{Li0}/(\Omega_{Li0} + \Omega_c)\right]^{1/2} , \quad K_{Pp0} = K_{Rs} + K_{Li0} ,$$

$$\Omega_P = K_{Pp0}^2/2 , \quad K_P = (K_{Pp0}^2/4K_{Li0})\left[2\Omega_{Li0} - \Omega_c/(\Omega_{Li0} + \Omega_c)^2\right] . \tag{9}$$

The wavenumber approximations are valid for $K_{Pp0}^2 S^2/4K_{Li0}^2 \ll 1$, which is sufficient for our purposes.

Figure 2 illustrates how synchronism is achieved for any nonzero value of Ω_c . Equation (6) is satisfied if a parallelogram can be drawn with one vertex at the origin, and the other three lying on appropriate branches of the dispersion characteristics. A typical solution is shown, and the ranges of variation for the three waves are illustrated by the heavy portions of the curves. In particular, the signal always lies in the approximate range $0 \leq \Omega_{Rs} \leq 1 - \Omega_L$. For Ω_c small, Ω_{Rs} has an upper limit of $\Omega_c/2$. For Ω_c large, Ω_{Rs} lies just below unity. In no case can it approach Ω_c , so electron temperature effects on the right-hand polarized wave can be neglected, as assumed earlier. The pump frequency varies within the range

$$1 + \left[\Omega_c/(1 + \Omega_c) \right] S^2/2 \leq \Omega_{Pp} \leq 1 + (1 - \Omega_L + \Omega_L^2)S^2/2 \quad , \quad (10)$$

a total range of variation which can be expressed as

$$\left[\Omega_L^3(2 - \Omega_L)/(1 + \Omega_L - \Omega_L^2)\right] S^2/2 \quad ,$$

and is very small for the cases of interest to us. We shall return to this point in section 3 and Appendix A, in our discussion of the effects of plasma inhomogeneity.

Pump R-Signal and Idler R,P. In this case, the synchronism conditions are,

$$\Omega_{Rp} = \Omega_{Rs} + \Omega_{Pi} \quad , \quad K_{Rp} = K_{Rs} + K_{Pi} \quad . \quad (11)$$

As before, we suppose that we specify Ω_{Rs} , so that K_{Rs} is again given by (7). For $K_{Pi}^2 S^2 \ll 1$, the following approximate results can be obtained, satisfying (11)

$$\begin{aligned} \Omega_{Rp} &= \Omega_{Rp0} + \Omega_R S^2 \quad , \quad \Omega_{Pi} = 1 + \Omega_R S^2 \quad , \\ K_{Rp} &= K_{Rp0} + K_R S^2 \quad , \quad K_{Pi} = K_{Pi0} + K_R S^2 \quad , \end{aligned} \quad (12)$$

where

$$\Omega_{Rp0} = 1 + \Omega_{Rs} \; , \quad K_{Rp0} = \left[\Omega_{Rp0}^2 - \Omega_{Rp0}/(\Omega_{Rp0}-\Omega_c)\right]^{1/2} \; , \quad K_{Pi0} = K_{Rp0}-K_{Rs} \; ,$$

$$\Omega_R = K_{Pi0}^2/2 \quad , \quad K_R = (K_{Pi0}^2/4K_{Rp0})\left[2\Omega_{Rp0} + \Omega_c/(\Omega_{Rp0}-\Omega_c)^2\right] \quad . \quad (13)$$

There are apparently three cases to consider: two with both the right-hand polarized waves on the same branch, either upper or lower, and the third with one on the upper and the other on the lower. Simple algebraic arguments can be used to show that for S^2 small the synchronism conditions cannot be satisfied in the third case, however, for any value of Ω_c . We can also ignore the case with two waves on the lower (whistler) branch since this requires $\Omega_c > 1$ for synchronism. The only likely places in the ionosphere for this condition to be satisfied are in the D- or E-regions, where very strong collisional damping should be taken into account. Signals in the whistler mode can, of course, penetrate the ionosphere and reach regions of lower collision frequency. Outside, in the magnetosphere, Ω_c probably exceeds unity only beyond the plasmapause, however. Although the mechanisms under discussion might be of interest there, it seems unlikely that a ground-based transmitter could produce the pump electric field strengths necessary to cause strong nonlinearity.

With these considerations, only the case shown in Figure 3 is of immediate interest. Synchronism is possible for all values of Ω_c . Ranges of variation are shown for the pump and idler by the heavy portions of the curves. The signal ranges from Ω_U to ∞ .

<u>Pump L-Signal and Idler L,P.</u> Algebraic expressions for the synchronism conditions can be obtained directly from (11) - (13) by changing all subscripts R to L , and Ω_c to $-\Omega_c$. Synchronism is possible for all values of Ω_c . Typical conditions are illustrated in Figure 3.

An important point to be noted for the two types of interactions just discussed (R-R,P and L-L,P) is the limited range of variation of the plasma wave. This is readily shown to be

$$1 + S^2/2 \leq \Omega_{R,Li} \leq 1 + \left(\Omega_{U,L}^2 + \Omega_{U,L} + 1\right)S^2/2 \quad , \qquad (14)$$

so that the total idler frequency range over which interaction can occur is $\Omega_{U,L}(\Omega_{U,L}+1)S^2/2$, which will be very small for the cases of interest to us. Further comment on this will be delayed to the discussion in section 3 of the effects of plasma inhomogeneity.

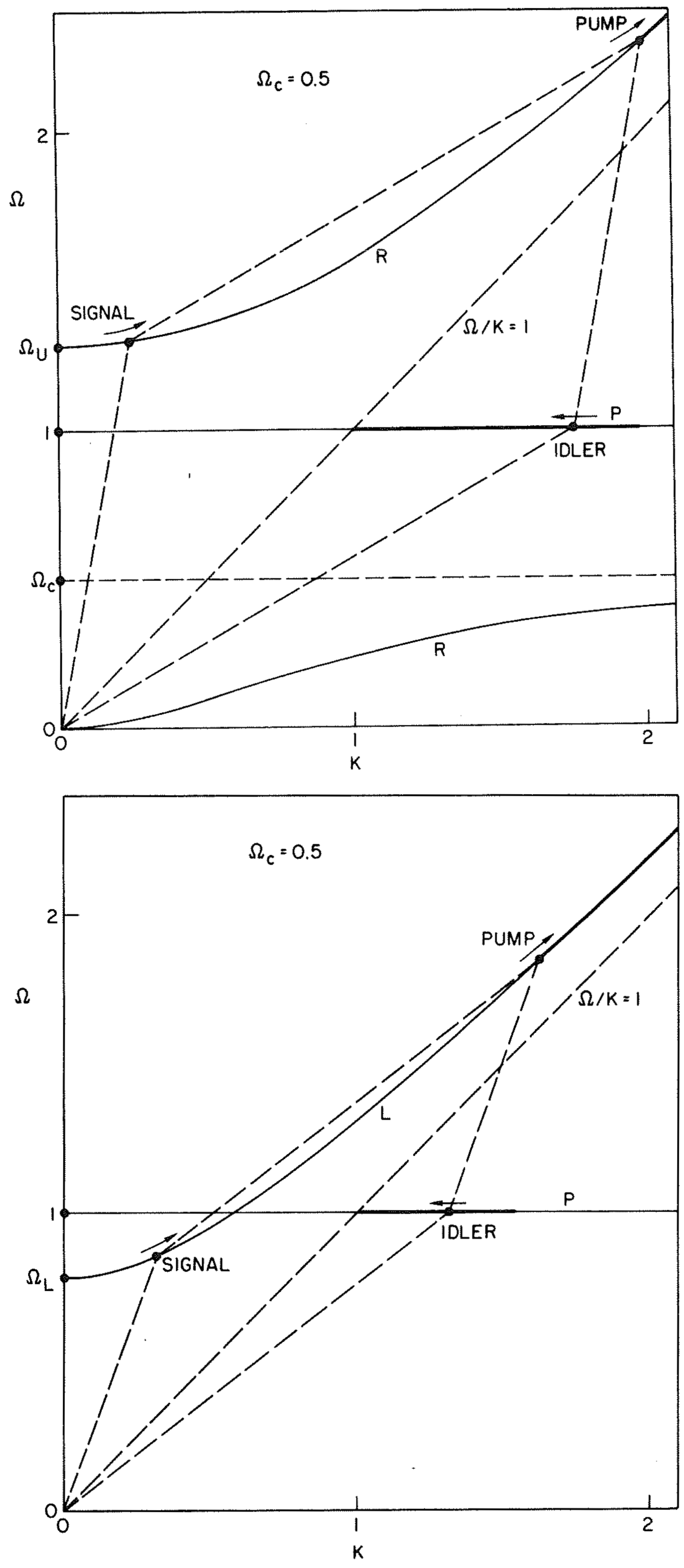

FIG. 3--Synchronism conditions for two right-hand or two left-hand polarized waves and a plasma wave. (Heavy portions of the curves show ranges of variation over which the synchronism conditions can be satisfied.)

3. THEORY

Basic equations. The system may be described by Maxwell's equations,

$$\nabla \times \vec{\mathcal{E}} + \frac{\partial \vec{H}}{\partial T} = 0 \ , \quad \nabla \times \vec{H} + \vec{V} - \frac{\partial \vec{\mathcal{E}}}{\partial T} = - N\vec{V} \ , \quad \nabla \cdot \vec{\mathcal{E}} + N = 0 \ , \tag{15}$$

and the force equation for electrons

$$\frac{\partial \vec{V}}{\partial T} + S^2 \nabla N + \vec{\mathcal{E}} + \vec{V} \times \vec{\Omega}_c = S^2 N \nabla N - [(\vec{V} \cdot \nabla)\vec{V} + \vec{V} \times \vec{H}] \ , \tag{16}$$

where $\vec{\Omega}_c$ is oriented in the z-direction, and $|N| < 1$ for (16) to be valid. Equations (15) - (16) have been written in terms of the normalized variables,

$$\vec{\mathcal{E}} = \frac{e\vec{E}}{mc\omega_0} \ , \quad \vec{H} = \frac{e\vec{B}}{m\omega_0} \ , \quad \vec{V} = \frac{\vec{v}}{c} \ ,$$

$$N = \frac{n}{n_0} \ , \quad T = \omega_0 t \ , \quad \vec{X} = \frac{\omega_0 \vec{x}}{c} \ . \tag{17}$$

If we take the temporal Fourier transforms of (15) - (16), we obtain

$$\nabla \times \vec{H}_\alpha + \vec{V} - j\Omega_\alpha \mathcal{E}_\alpha = - \sum N_\beta \vec{V}_\gamma \ , \quad \nabla \times \vec{\mathcal{E}}_\alpha + j\Omega_\alpha \vec{H}_\alpha = 0 \ , \quad \nabla \cdot \vec{\mathcal{E}}_\alpha + N_\alpha = 0 \ ,$$

$$j\Omega_\alpha \vec{V}_\alpha + S^2 \nabla N_\alpha + \vec{\mathcal{E}}_\alpha + \vec{V}_\alpha \times \vec{\Omega}_c = \sum S^2 N_\beta \nabla N_\gamma - \sum \left[(\vec{V}_\beta \cdot \nabla)\vec{V}_\gamma + \vec{V}_\beta \times \vec{H}_\gamma\right] . \tag{18}$$

Here, subscripts α , β , γ refer to the Fourier components varying as $\exp j\Omega_\alpha T$, etc. The summations are made over all values of β and γ satisfying the synchronism condition,

$$\Omega_\alpha = \Omega_\beta + \Omega_\gamma \ . \tag{19}$$

The right-hand sides of the expressions in (18) represent the nonlinear coupling terms between the waves. The left-hand sides are the terms appropriate to small-signal linear theory. An iterative solution to the nonlinear problem will be obtained by first assuming the right-hand sides to be zero and obtaining small-signal expressions for the various field quantities. These will then be substituted in the right-hand sides to obtain a first approximation to the nonlinear equations.

To carry out this detailed procedure, we must specialize the basic equations to the various combinations of transverse and longitudinal waves listed in Table 1. When this is done, it is found that the non-linear driving terms are nonzero only for the three cases tabulated as 'acceptable'. The ranges of synchronism for these were established in section 2. The relevant coupling coefficients and parametric growth rates will now be determined.

<u>Pump P - Signal and Idler R,L.</u> First, the expressions in (18) are written in component form. They are then simplified by introduction of right- and left-hand polarized variables defined so that the electric fields, for example, are given by

$$\mathcal{E}_{Rs} = \mathcal{E}_{Rsx} + j\mathcal{E}_{Rsy}, \qquad 0 = \mathcal{E}_{Rsx} - j\mathcal{E}_{Rsy},$$

$$0 = \mathcal{E}_{Lix} + j\mathcal{E}_{Liy}, \qquad \mathcal{E}_{Li} = \mathcal{E}_{Lix} - j\mathcal{E}_{Liy}. \tag{20}$$

We obtain the following nonlinear equations and small-signal approximations to them,

$$\frac{\partial \mathcal{E}_{Rs}}{\partial z} + \Omega_{Rs}H_{Rs} = 0, \qquad -jK_{Rs}\mathcal{E}_{Rs} + \Omega_{Rs}H_{Rs} \approx 0,$$

$$\frac{\partial \mathcal{E}_{Li}}{\partial z} - \Omega_{Li}H_{Li} = 0, \qquad -jK_{Li}\mathcal{E}_{Li} - \Omega_{Li}H_{Li} \approx 0, \tag{21}$$

$$\frac{\partial \mathcal{E}_{Pp}}{\partial z} + N_{Pp} = 0, \qquad -jK_{Pp}\mathcal{E}_{Pp} + N_{Pp} \approx 0, \tag{22}$$

$$j\frac{\partial H_{Rs}}{\partial Z} + V_{Rs} - j\Omega_{Rs}\mathcal{E}_{Rs} = -N_{Pp}V^{*}_{Li}\ , \quad K_{Rs}H_{Rs} + V_{Rs} - j\Omega_{Rs}\mathcal{E}_{Rs} \approx 0\ ,$$

$$-j\frac{\partial H_{Li}}{\partial Z} + V_{Li} - j\Omega_{Li}\mathcal{E}_{Li} = -N_{Pp}V^{*}_{Rs}\ , \quad -K_{Li}H_{Li} + V_{Li} - j\Omega_{Li}\mathcal{E}_{Li} \approx 0\ ,$$

$$V_{Pp} - j\Omega_{Pp}\mathcal{E}_{Pp} = 0\ , \tag{23}$$

$$j\Omega_{Pp}V_{Pp}+\mathcal{E}_{Pp}+S^2\frac{\partial N_{Pp}}{\partial Z} = -\frac{j}{2}\left(V_{Rs}H_{Li}-V_{Li}H_{Rs}\right), j\Omega_{Pp}V_{Pp}+\mathcal{E}_{Pp}-jK_{Pp}S^2N_{Pp} \approx 0\ ,$$

$$j(\Omega_{Rs}-\Omega_c)V_{Rs}+\mathcal{E}_{Rs} = -jV_{Pp}(K_{Li}V^{*}_{Li}+H^{*}_{Li})\ , \quad j(\Omega_{Rs}-\Omega_c)V_{Rs} + \mathcal{E}_{Rs} \approx 0\ ,$$

$$j(\Omega_{Li} + \Omega_c)V_{Li} + \mathcal{E}_{Li} = -jV_{Pp}(K_{Rs}V^{*}_{Rs}-H^{*}_{Rs})\ , \quad j(\Omega_{Li}+\Omega_c)\ V_{Li}+\mathcal{E}_{Li} \approx 0\ , \tag{24}$$

where '*' denotes a complex conjugate.

The next step is to eliminate all velocity components, and to introduce the convenient variables,

$$A_{Pp} = \left(\mathcal{E}_{Pp} - j\frac{N_{Pp}}{K_{Pp}}\right)/2\ , \qquad A_{Pp} \approx \mathcal{E}_{Pp} \approx -j\frac{N_{Pp}}{K_{Pp}}\ ,$$

$$A_{Rs} = \left(\mathcal{E}_{Rs} - j\frac{\Omega_{Rs}H_{Rs}}{K_{Rs}}\right)/2\ , \qquad A_{Rs} \approx \mathcal{E}_{Rs} \approx -j\frac{\Omega_{Rs}H_{Rs}}{K_{Rs}}\ ,$$

$$A_{Li} = \left(\mathcal{E}_{Li} + j\frac{\Omega_{Li}H_{Li}}{K_{Li}}\right)/2\ , \qquad A_{Li} \approx \mathcal{E}_{Li} \approx j\frac{\Omega_{Li}H_{Li}}{K_{Li}}\ . \tag{25}$$

After some tedious algebra, (21) - (24) reduce to

$$\frac{\partial A_{Pp}}{\partial Z} + jK_{Pp}A_{Pp} = - C_P\left(\frac{1}{4K_{Pp}S^2}\right) A_{Rs}A_{Li} \quad ,$$

$$\frac{\partial A_{Rs}}{\partial Z} + jK_{Rs}A_{Rs} = C_P\left(\frac{\Omega_{Rs}^2}{2K_{Rs}}\right) A_{Li}^{*}A_{Pp} \quad ,$$

$$\frac{\partial A_{Li}}{\partial Z} + jK_{Li}A_{Li} = C_P\left(\frac{\Omega_{Li}^2}{2K_{Li}}\right) A_{Pp}A_{Rs}^{*} \quad , \tag{26}$$

where the coupling coefficient, C_P , is given by

$$C_P = \frac{K_{Rs}}{\Omega_{Rs}(\Omega_{Li} + \Omega_c)} + \frac{K_{Li}}{\Omega_{Li}(\Omega_{Rs} - \Omega_c)} \quad . \tag{27}$$

The general solution of a set of equations in the canonical coupled mode form of (26) can be given in terms of elliptic integrals [Sagdeev and Galeev 1969]. The problem is considerably simplified, however, if one or two of the waves are weak. For example, the mixing problem in which strong signals A_{Rs} and A_{Li} are applied, and give rise to a component A_{Pp} , can be solved using only the first equation from (26). We simply assume that A_{Pp} is of the form $\hat{A}_{Pp}(Z) \exp(-jK_{Pp}Z)$, where $\hat{A}_{Pp}$ is a slowly-varying function of Z . The coupled mode equation becomes,

$$\frac{\partial \hat{A}_{Pp}}{\partial Z} = - C_P\left(\frac{1}{4K_{Pp}S^2}\right) A_{Rs}A_{Li} \quad , \tag{28}$$

so that if $\hat{A}_{Pp} = 0$ at $Z = 0$, then there is linear growth with Z

$$|\hat{A}_{Pp}| = \kappa_{RL}Z \quad , \qquad \kappa_{RL} = \left(\frac{1}{4K_{Pp}S^2}\right) |C_P A_{Rs} A_{Li}| \quad . \tag{29}$$

If the effects of collisional damping on the plasma wave are taken into account by including a phenomenological loss term $\kappa_{P\nu}\hat{A}_{Pp}$ on the left-hand side of (28), it is easily shown that (29) is simply an approximation to the expression

$$|\hat{A}_{Pp}| = \kappa_{RL}[1 - \exp(-\kappa_{P\nu}Z)]/\kappa_{P\nu} \quad . \tag{30}$$

It follows that the saturation amplitude of $\hat{A}_{Pp}$ is $\kappa_{RL}/\kappa_{P\nu}$.

Next, consider the parametric amplification problem in which the pump is strong compared with the signal and idler, so that it suffers negligible attenuation as a result of energy transfer to them. We now use the second and third equations from (26) and assume that the signal and idler have the spatial variations $\exp(\kappa_P - jK_{Rs})Z$ and $\exp(\kappa_P - jK_{Li})Z$ respectively. This yields

$$\kappa_P = C_P\left(\frac{\Omega_{Rs}^2}{2K_{Rs}}\right)\frac{A_{Li}^*A_{Pp}}{A_{Rs}} \quad , \quad \kappa_P = C_P\left(\frac{\Omega_{Li}^2}{2K_{Li}}\right)\frac{A_{Pp}A_{Rs}^*}{A_{Li}} \; . \tag{31}$$

These must be consistent, of course. Taking the complex conjugate of one expression and multiplying by the other gives an expression for the growth rate, κ_P , due to parametric amplification,

$$\kappa_P = |C_P| \frac{\Omega_{Rs}\Omega_{Li}}{2(K_{Rs}K_{Li})^{1/2}} |\mathcal{E}_{Pp}| \quad , \tag{32}$$

involving only the pump amplitude, $|\mathcal{E}_{Pp}|$ [$\approx |A_{Pp}|$ by (25)]. It should be understood that the spatial variations assumed ignore boundary conditions, and that the resulting growth rate is consequently only effective for distances such that $\kappa_P Z > 1$. A boundary value problem is solved in detail in Appendix B, including the effects of collisions, for the combination R-R,P .

<u>Pump R - Signal and Idler R,P.</u> The analysis for this case can be carried through along similar lines to that described in the previous subsection and we shall simply quote here the results analogous to (25) - (27), (29), (30) and (32). We have

$$\frac{\partial A_{Rp}}{\partial Z} + jK_{Rp}A_{Rp} = - C_R\left(\frac{\Omega_{Rp}^2}{2K_{Rp}}\right) A_{Rs}A_{Pi} \quad ,$$

$$\frac{\partial A_{Rs}}{\partial Z} + jK_{Rs}A_{Rs} = C_R\left(\frac{\Omega_{Rs}^2}{2K_{Rs}}\right) A_{Pi}^{*}A_{Rp} \quad ,$$

$$\frac{\partial A_{Pi}}{\partial Z} + jK_{Pi}A_{Pi} = C_R\left(\frac{1}{4K_{Pi}S^2}\right) A_{Rp}A_{Rs}^{*} \quad , \tag{33}$$

$$A_{Rp} = \left(\mathcal{E}_{Rp} - j\,\frac{\Omega_{Rp}H_{Rp}}{K_{Rp}}\right)/2 \approx \mathcal{E}_{Rp} \quad , \quad A_{Pi} = \left(\mathcal{E}_{Pi} - j\,\frac{N_{Pi}}{K_{Pi}}\right)/2 \approx \mathcal{E}_{Pi} \quad , \tag{34}$$

$$C_R = \frac{K_{Rp}}{\Omega_{Rp}(\Omega_{Rs} - \Omega_c)} - \frac{K_{Rs}}{\Omega_{Rs}(\Omega_{Rp} - \Omega_c)} \quad , \tag{35}$$

$$|\hat{A}_{Rp}| = \kappa_{RP}[1 - \exp(-\kappa_{R\nu}Z)]/\kappa_{R\nu} \quad , \quad \kappa_{Rp} = \left(\frac{\Omega_{Rp}^2}{2K_{Rp}}\right) |C_R A_{Rs} A_{Pi}| \quad , \tag{36}$$

$$\kappa_R = |C_R|\,\frac{\Omega_{Rs}}{2^{3/2}S(K_{Rs}K_{Pi})^{1/2}}\,|\mathcal{E}_{Rp}| \quad . \tag{37}$$

Pump L - Signal and Idler L,P. Again, we shall simply quote the results analogous to (25) - (27), (29), (30), and (32). We have

$$\frac{\partial A_{Lp}}{\partial Z} + jK_{Lp}A_{Lp} = - C_L\left(\frac{\Omega_{Lp}^2}{2K_{Lp}}\right) A_{Ls}A_{Pi} \quad ,$$

$$\frac{\partial A_{Ls}}{\partial Z} + jK_{Ls}A_{Ls} = C_L\left(\frac{\Omega_{Ls}^2}{2K_{Ls}}\right) A_{Pi}^{*}A_{Lp} \quad ,$$

$$\frac{\partial A_{Pi}}{\partial Z} + jK_{Pi}A_{Pi} = C_L\left(\frac{1}{4K_{Pi}S^2}\right) A_{Lp}A_{Ls}^{*} \quad , \tag{38}$$

$$A_{Lp} = \left(\mathcal{E}_{Lp} + j\,\frac{\Omega_{Lp}H_{Lp}}{K_{Lp}}\right)/2 \approx \mathcal{E}_{Lp} \;, \quad A_{Ls} = \left(\mathcal{E}_{Ls} + j\,\frac{\Omega_{Ls}H_{Ls}}{K_{Ls}}\right)/2 \approx \mathcal{E}_{Ls} \;, \tag{39}$$

$$C_L = \frac{K_{Lp}}{\Omega_{Lp}(\Omega_{Ls} + \Omega_c)} - \frac{K_{Ls}}{\Omega_{Ls}(\Omega_{Lp} + \Omega_c)} \;, \tag{40}$$

$$|\hat{A}_{Lp}| = \kappa_{Lp}[1 - \exp(-\kappa_{L\nu}Z)]/\kappa_{L\nu} \;, \quad \kappa_{Lp} = \left(\frac{\Omega_{Lp}^2}{2K_{Lp}}\right)|C_L A_{Ls} A_{Pi}| \;, \tag{41}$$

$$\kappa_L = |C_L|\,\frac{\Omega_{Ls}}{2^{3/2}\;S(K_{Ls}K_{Pi})^{1/2}}\,|\mathcal{E}_{Lp}| \;. \tag{42}$$

Expressions equivalent to (33) and (38) have been derived by Sjölund and Stenflo [1967a]. These authors have also considered the situation with zero magnetic field, to which the results of this and the previous subsection tend as $\Omega_c \to 0$ [Sjölund and Stenflo 1967b], and have derived an expression identical to the limiting form $(\Omega_c = 0)$ of (37) or (42). This is simply

$$\kappa = \frac{1}{2^{3/2} S\,\Omega_p}\left(\frac{K_i}{K_s}\right)^{1/2}|\mathcal{E}_p| \;, \tag{43}$$

where the R and L subscripts are no longer necessary and have been dropped.

4. COMPUTATIONS

Section 3 has provided explicit forms for the parametric growth rates κ_P, κ_L and κ_R, expressed in terms of the pump, signal and idler frequencies and wavenumbers, and the relevant pump electric field strengths. We now wish to investigate the numerical values of the coupling coefficients and parametric growth rates, consistent with satisfaction of the synchronism conditions discussed in section 2.

Pump P - Signal and Idler R,L. It is convenient to introduce into (32) the simplified expressions of (8) and (9). Expanding κ_P in powers of S^2, and retaining only the first term, gives as a very good approximation

$$\kappa_P = \frac{\Omega_{Rs}\Omega_{Li0}}{2(K_{Rs}K_{Li0})^{1/2}} \left| \frac{K_{Rs}}{\Omega_{Rs}(\Omega_{Li0} + \Omega_c)} + \frac{K_{Li0}}{\Omega_{Li0}(\Omega_{Rs} - \Omega_c)} \right| |\mathcal{E}_{Pp}| \quad . \quad (43)$$

It is interesting to note that this expression is independent of the precise value of S, i.e., it is a cold plasma effect. We shall work with the gain, G_P, defined by

$$G_P = 8.69 \frac{\kappa_P \omega_0}{c} = \Gamma_P |E_{Pp}| \quad \text{dB/unit length} \quad , \quad (44)$$

where E_{Pp} is in V/unit length and

$$\Gamma_P = q \frac{\Omega_{Rs}\Omega_{Li0}}{(K_{Rs}K_{Li0})^{1/2}} \left| \frac{K_{Rs}}{\Omega_{Rs}(\Omega_{Li0} + \Omega_c)} + \frac{K_{Li0}}{\Omega_{Li0}(\Omega_{Rs} - \Omega_c)} \right| \quad ,$$

$$q = 4.34 \frac{e}{mc^2} = 8.50 \times 10^{-6} \text{ dB/V} \quad . \quad (45)$$

Note that 'V' is being used from here on for 'volts'.

Figure 4 gives information on growth and synchronism conditions for $\Omega_c = 0.1$ and 1.0. In each case, the lowest box shows the

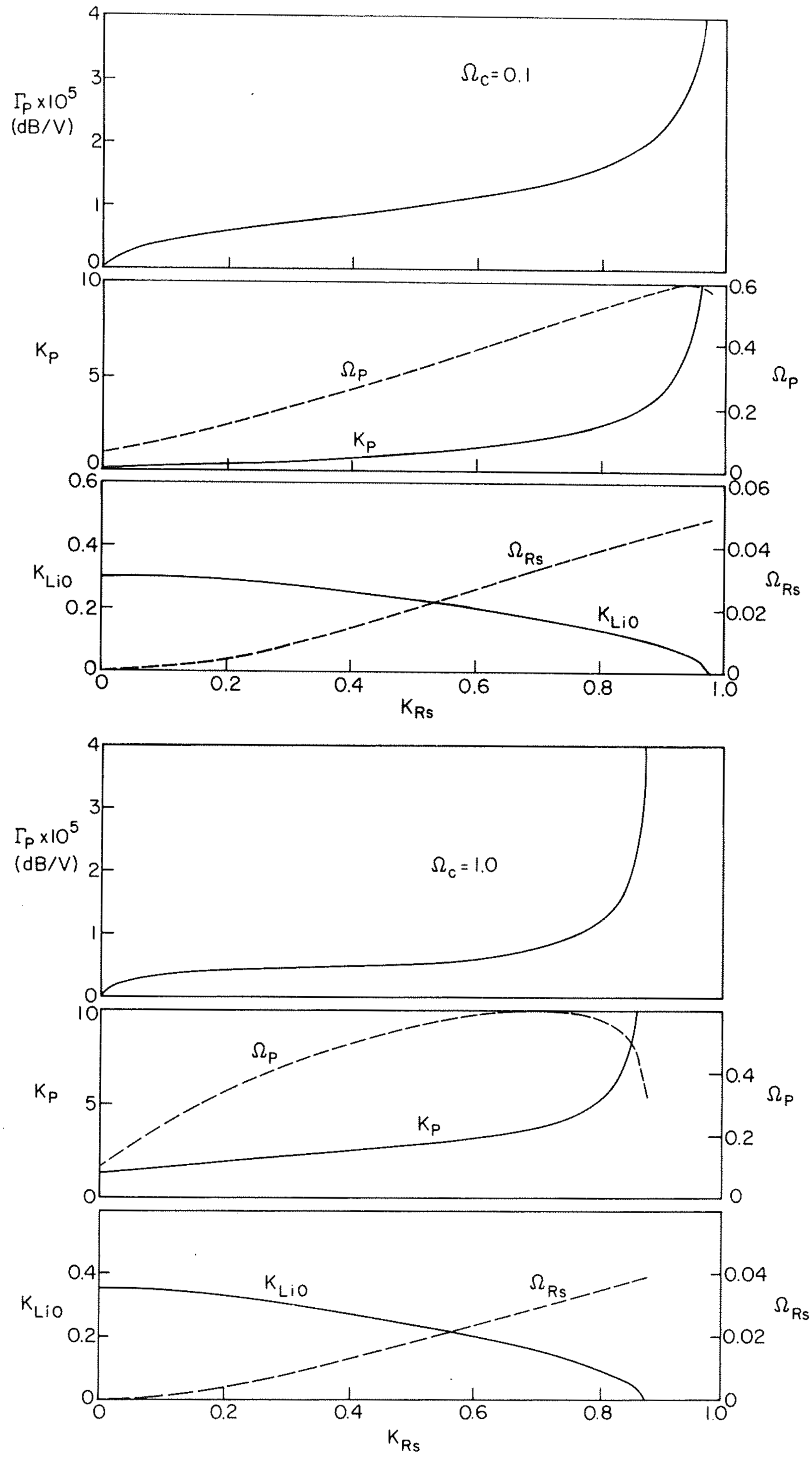

FIG. 4--Synchronism conditions and growth parameter (Γ_P) for nonlinear interaction between a plasma wave pump (Ω_{Pp}, K_{Pp}), a right-hand polarized signal wave (Ω_{Rs}, K_{Rs}), and a left-hand polarized idler wave (Ω_{Li}, K_{Li}).

variation of Ω_{Rs} and K_{Li0} with K_{Rs}. The middle box shows the coefficients Ω_P, K_P, used in (8). Since an ionospheric electron temperature of 0.2 eV corresponds to $S \approx 10^{-3}$, the correction terms $\Omega_P S^2$ and $K_P S^2$ in these equations are generally negligible. The value of $\Omega_P S^2$ is significant, however, as an indication of the plasma density inhomogeneity that could be tolerated over the interaction region. This point is dealt with in Appendix A, and is also discussed in section 5. The upper box shows the variation of Γ_P. Its order of magnitude is 10^{-5}, for $\Omega_c = 0.1$, i.e., the same as that of q, and somewhat lower for $\Omega_c = 1.0$. If we assume arbitrarily that the plasma wave pump has an electric field strength of 0.1 V/m, growth of only $G_P \sim 10^{-6}$ dB/m is predicted. It is clear that very much higher field strengths would have to be postulated for measurable growth to occur.

Pump R - Signal and Idler R,P. With similar procedure to that of the preceding subsection, we obtain,

$$G_R = 8.69 \frac{\kappa_R \omega_0}{c} = \Gamma_R |E_{Rp}| \text{ dB/unit length}, \quad (46)$$

$$\Gamma_R = q \frac{\Omega_{Rs}}{S(2K_{Rs}K_{Pi0})^{1/2}} \left| \frac{K_{Rp0}}{\Omega_{Rp0}(\Omega_{Rs} - \Omega_c)} - \frac{K_{Rs}}{\Omega_{Rs}(\Omega_{Rp0} - \Omega_c)} \right| \text{dB/V}, \quad (47)$$

where E_{Rp} is the pump electric field strength in V/unit length.

Figure 5 shows data on growth and synchronism conditions analogous to those of Figure 4. The most significant point to note is the greatly increased growth rate now possible due to the presence of S in the denominator of (47). For an ionospheric electron temperature of 0.2 eV and $E_{Rp} \approx 0.1$ V/m, we would have $G_P \sim 10^{-3}$ dB/m. This could easily be measured even if it could only be sustained over interaction distances as short as 100 m, i.e., if a total amplification of ~ 0.1 dB occurred. We note, however, that the range of frequency for synchronism is extremely narrow, so that plasma inhomogeneity may easily make the effect unobservable. From Appendix A,

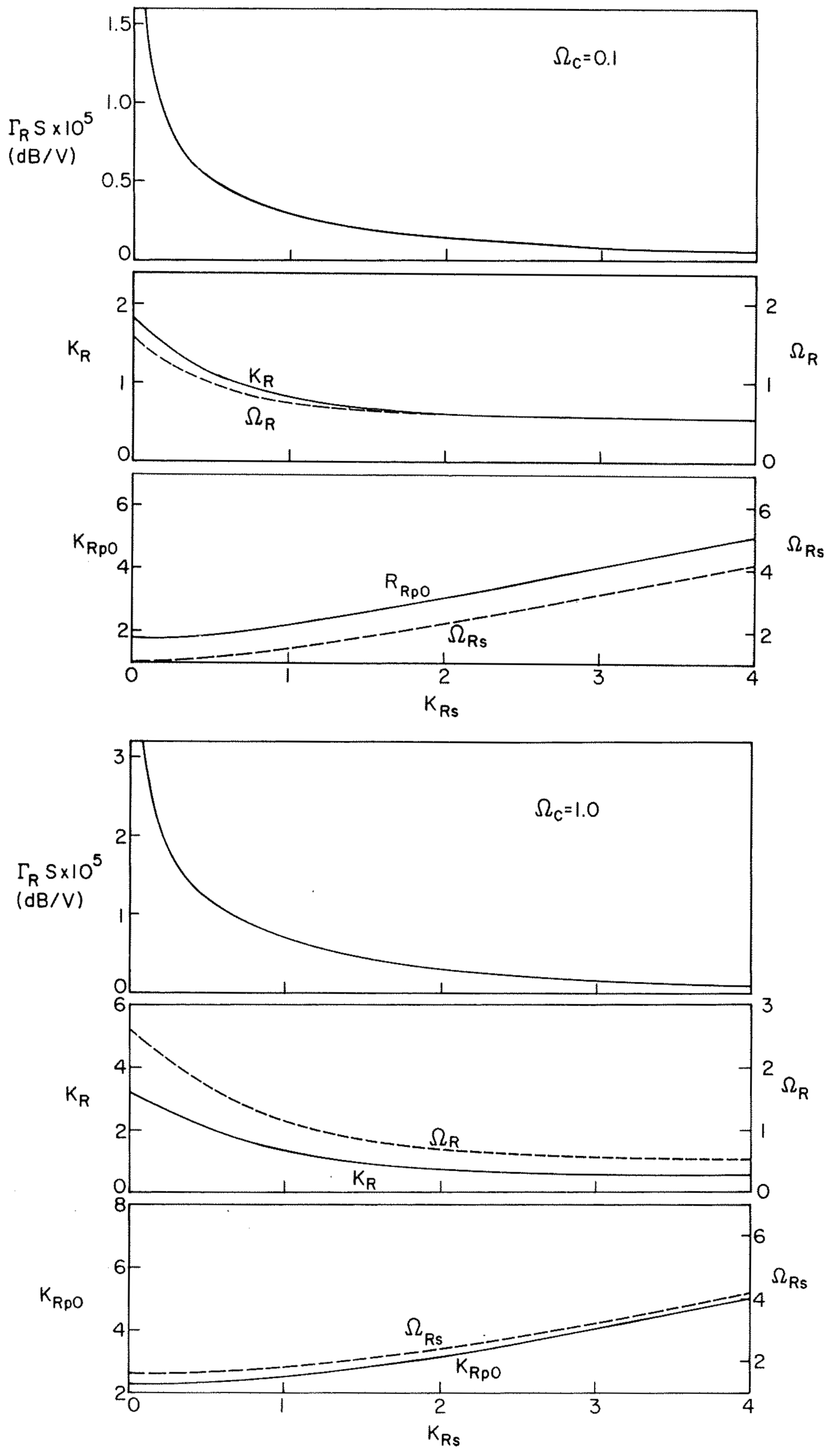

FIG. 5--Synchronism conditions and growth parameter (Γ_R) for nonlinear interaction between a right-hand polarized pump wave (Ω_{Rp}, K_{Rp}), a right-hand polarized signal wave (Ω_{Rs}, K_{Rs}), and a plasma wave idler (Ω_{Pi}, K_{Pi}).

we have the results that

$$\frac{\delta E_{Rs}}{E_{Rs}} = \kappa_R^2 \frac{\pi}{\beta} \ , \quad \text{or} \quad \kappa_R^2\left(\frac{\pi}{\beta}\right) \bigg/ \left(\frac{\delta E_{Pi}}{E_{Pi}}\right) \quad , \tag{48}$$

depending on whether the initial value of the idler is small or large compared with the total increment through the interaction region. The second result is clearly the larger, and it is of interest to estimate it. Assuming that the electron density varies as $n_0(1 + \alpha Z)$, we have from (A.2)

$$\beta = \frac{\alpha}{2}\left[- \frac{\Omega_{Rp}}{K_{Rp}(\Omega_{Rp} - \Omega_c)} + \frac{\Omega_{Rs}}{K_{Rs}(\Omega_{RS} - \Omega_c)} + \frac{1}{K_{Pi}S^2}\right]$$

$$\approx \frac{\alpha}{2K_{Pi}S^2} \quad \left(\text{for} \ K_{RS} \gg S^2\right) \quad . \tag{49}$$

Using (37), and assuming for example $\delta E_{Rs}/E_{Rs} = \delta E_{Pi}/E_{Pi}$, gives

$$\frac{\delta E_{Rs}}{E_{Rs}} = |C_R| \frac{\Omega_{Rs}}{2} \left(\frac{\pi}{K_{Rs}}\right)^{1/2} \frac{\varepsilon_{Rp}}{\alpha^{1/2}} \quad ,$$

$$\sim \frac{\varepsilon_{Rp}}{\alpha^{1/2}} \quad , \tag{50}$$

which we note is independent of S. For a pump electric field strength of 0.1 V/m at a region where $n_0 \sim 10^6/\text{cm}^3$, then $\varepsilon_{Rp} \sim 10^{-6}$. For an increment of $\delta E_{Rs}/E_{Rs} = 10^{-2}$, say, then $\alpha \sim 10^{-8}$ would be required. What this means in terms of ionospheric homogeneity is that over a distance where the density can be expressed as $n_0\left(1 + (\alpha\omega_0/c)\ z\right)$, we require $(\alpha\omega_0/c) \sim 10^{-9}/\text{m}$. This is three or four orders of magnitude smaller than would be expected over most of the ionosphere. Clearly, the best solution would be to work close to the peak of the F-layer, but even there the inhomogeneity within about 1 km of the peak is only likely to be of order 10^{-7}/m so the interaction distances would be extremely small. The general conclusion is that only small increments could be expected, of order $\delta E_{Rs}/E_{Rs} \sim 10^{-4}$, say.

Pump L - Signal and Idler L, P. The considerations for this case are identical to those of the previous subsection. Analogous to (46) and (47) we have,

$$G_L = 8.69 \frac{\kappa_L \omega_0}{c} = \Gamma_L |E_{Lp}| \text{ dB/unit length} , \tag{51}$$

$$\Gamma_L = q \frac{\Omega_{Ls}}{S(2K_{Ls}K_{Pi0})^{1/2}} \left| \frac{K_{Lp0}}{\Omega_{Lp0}(\Omega_{Ls} + \Omega_c)} - \frac{K_{Ls}}{\Omega_{Ls}(\Omega_{Lp0} + \Omega_c)} \right| \text{dB/V}, \tag{52}$$

and the relevant computed data are presented in Figure 6. For $\Omega_c = 0.1$, the amplification is similar to that in Figure 5 for right-hand polarized modes. For $\Omega_c = 1.0$, the growth is about a factor of two higher than for that case.

The analysis and comments on the effects of inhomogeneity made above apply equally to the left-hand polarized modes with appropriate changes of subscripts.

Effects of Collisions. So far, collisions have only been taken into account crudely in (30), (36) and (41) to estimate the nonlinear saturation levels to be expected in mixing experiments. Apart from such a limiting influence, the inclusion of collisions might affect our predicted results in two somewhat different ways. First, it might be expected to reduce the parametric growth rate. Appendix B shows, however, that this is not the case: Although there may no longer be net gain, there is a reduction in attenuation for the signal and idler equal to the gain predicted in this section. From the point of view of measuring an increment in received signal, there will be

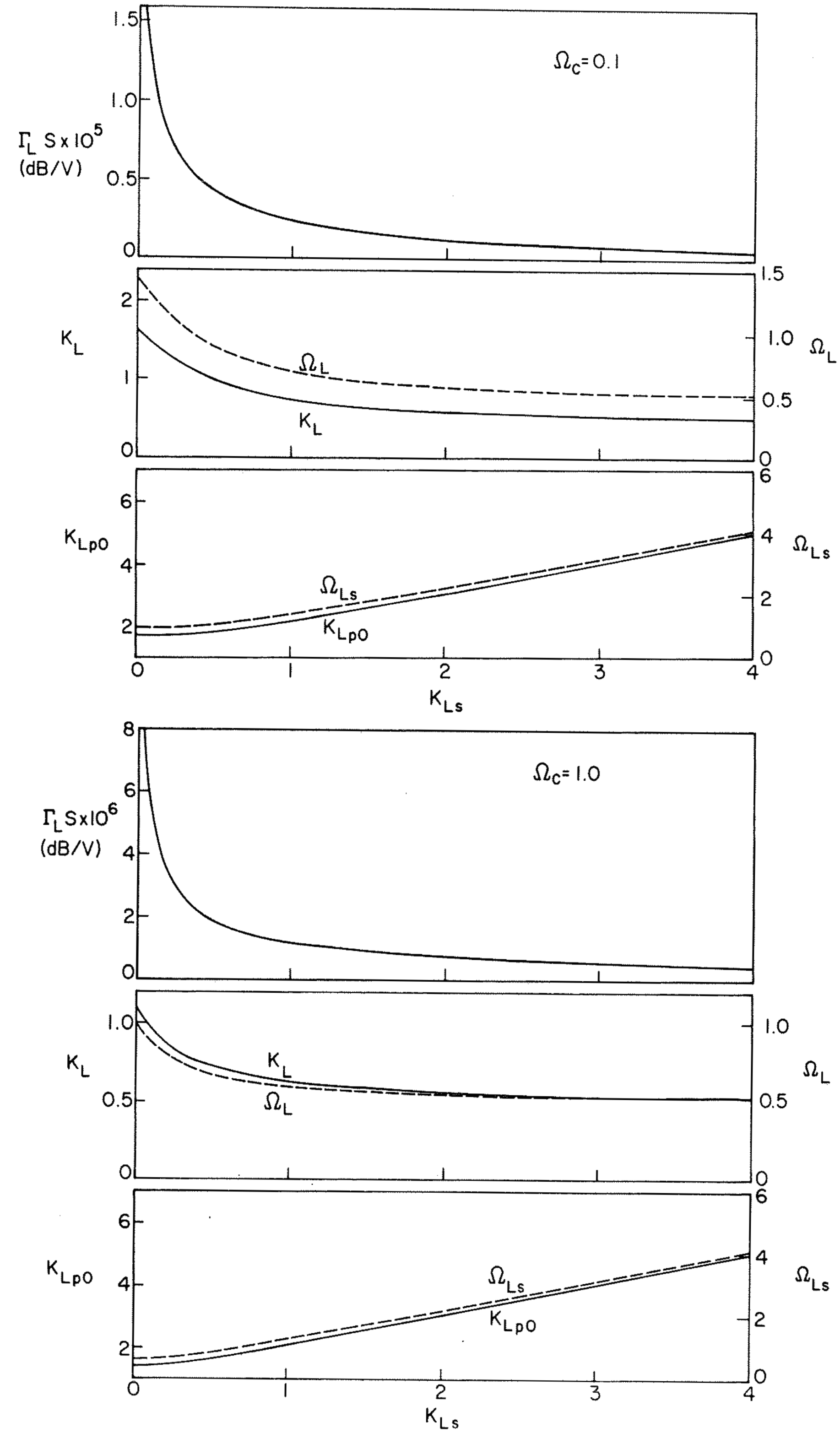

FIG. 6--Synchronism conditions and growth parameter (Γ_L) for nonlinear interaction between a left-hand polarized pump wave (Ω_{Lp}, K_{Lp}), a left-hand polarized signal wave (Ω_{Ls}, K_{Ls}), and a plasma wave idler (Ω_{Pi}, K_{Pi}).

no change. Second, collisions will modify the shapes of the real branches of the various wave dispersion characteristics, particularly in regions of low group velocity. This implies that the synchronism conditions should be recomputed for the modified curves. This has not been done, but would be expected to broaden the range of synchronism, and perhaps offset somewhat the effects of inhomogeneity.

5. DISCUSSION

Our purpose in this paper has been to discuss some of the nonlinear wave/wave interactions which might be susceptible to experimental study in the ionosphere using ground-based equipment, and it is now appropriate to reconsider the several types of experiments suggested in section 1. We begin with mixing, and will then discuss amplification experiments.

Although three cases have been analyzed throughout the paper, i.e., with plasma wave, right-hand polarized wave, or left-hand polarized wave as the highest frequency, the properties of the last two are somewhat similar, and for purposes of discussion we need only consider the first and second as separate cases. As pointed out in section 3, there is an important difference in the mixing formulas that makes interaction of the first type much weaker than that of the second, in which S appears in the denominator. Even so, the idea of being able to excite longitudinal plasma waves by two higher frequency transverse waves has long been considered an attractive possibility. For example it was discussed several years ago from the point of view of an experiment in which two laser beams were intended to interact with longitudinal oscillations in a plasma by Kroll, Ron, and Rostoker [1964], and Montgomery [1965]. As was pointed out by Sjölund and Stenflo [1967a], however, numerical estimates lead to the conclusion that parametric growth would be unobservable for the conditions discussed by these authors. In the ionospheric case being discussed here, the situation is not much better, though there is the important point that the frequencies do not differ by large factors, which operates to strengthen the interaction.

Although the effects are not strong, this does not imply that they could not be measured at all. For example, the coupling of various combinations of extraordinary and ordinary waves propagating across a magnetoplasma perpendicular to the static magnetic field suggests equally weak coupling coefficients [Cano et al., 1968; Harker and Crawford, 1969a], yet the effect seems to have been observed in the laboratory [Cano et al., 1969]. It may also have been observed in the ionosphere by Green [1965], who has reported a difference-frequency return from the ionosphere when illuminated by two strong signals. In view of this demonstrated success, a speculative series of experiments seems justified in which attempts are first made to excite sum- or difference-frequency components, and then to determine precisely the mechanisms involved. At first sight, the results of this paper might not seem directly relevant to such experiments, since they would involve a plasma wave as the mixing product. When plasma inhomogeneity is present, however, such a wave might easily be converted to a propagating transverse wave. In this connection, it may be remarked that our present theory is only capable of handling inhomogeneity in the direction of propagation. Inhomogeneity perpendicular to the ray path may itself be a very strong source of coupling, and deserves considerable attention.

As pointed out in section 1, mixing is not the only manifestation of wave coupling, and parametric amplification measurements offer an intriguing possibility for observing nonlinearity in the ionosphere. The results of section 4 show that a plasma wave pump excited by instability is unlikely to have an electric field strength high enough to amplify right- and left-hand polarized waves appreciably. The second case, in which the plasma wave is the idler, offers a much stronger effect, though this is degraded by the inevitable inhomogeneity of the ionosphere. The best results could probably be obtained by working with a signal frequency reflected near the peak of the F-layer. Sophisticated measuring techniques would be required to measure the weak amplification caused even by a very powerful pump. As demonstrated by (A.5) and (A.8), it is highly desirable for success of such experiments that a component at the frequency of the plasma wave idler should already be present in the ionosphere. For this reason, reflection from the auroral zones might produce the best results. It is perhaps significant in this respect that the results of Green [1965] seem to depend on such a propagation direction. His observed difference frequency component may well have been a transverse wave coupled from an amplified plasma idler.

Acknowledgments. The research described in this paper was supported by the NASA and the NSF.

APPENDIX A: EFFECT OF PLASMA INHOMOGENEITY ON PARAMETRIC GROWTH

The analysis of section 3 has shown that the nonlinear interaction is described by coupled mode equations of the form given in (33). In the inhomogeneous case, these will be valid locally. We now wish to calculate the total growth through a region where the plasma density varies. Assume solutions to (33) of the form

$$\hat{A}_{Rp} \exp j(\Omega_{Rp}T - \int K_{Rp}dZ),\ \hat{A}_{Rs} \exp j(\Omega_{Rs}T - \int K_{Rs}dZ),\ \hat{A}_{Pi} \exp j(\Omega_{Pi}T - \int K_{Rs}dZ),$$

and that the frequency synchronism condition $(\Omega_{Rp} = \Omega_{Rs} + \Omega_{Pi})$ is satisfied. Substitution in the signal and idler equations yields,

$$\frac{\partial \hat{A}_{Rs}}{\partial Z} = C_R \left(\frac{\Omega_{Rs}^2}{2K_{Rs}}\right) \hat{A}^*_{Pi}\hat{A}_{Rp} \exp\left[-j\int (K_{Rp} - K_{Rs} - K_{Pi})\, dZ\right] ,$$

$$\frac{\partial \hat{A}_{Pi}}{\partial Z} = C_R \left(\frac{1}{4K_{Pi}S^2}\right) \hat{A}_{Rp}\hat{A}^*_{Rs} \exp\left[-j\int (K_{Rp} - K_{Rs} - K_{Pi})\, dZ\right] . \quad (A.1)$$

We now expand $(K_{Rp} - K_{Rs} - K_{Pi})$ about the point where synchronism is satisfied, to obtain

$$K_{Rp} - K_{Rs} - K_{Pi} \approx \beta Z , \quad \beta = \left[\frac{\partial}{\partial Z}(K_{Rp} - K_{Rs} - K_{Pi})\right]_{K_{Rp}=K_{Rs}+K_{Pi}} . \quad (A.2)$$

Substitution in (A.1) yields

$$\frac{\partial \hat{A}_{Rs}}{\partial Z} = C_R \left(\frac{\Omega_{Rs}^2}{2K_{Rs}}\right) \hat{A}^*_{Pi}\hat{A}_{Rp} \exp(-j\beta Z^2/2) , \quad (A.3)$$

$$\frac{\partial \hat{A}_{Pi}}{\partial Z} = C_R \left(\frac{1}{4K_{Pi}S^2}\right) \hat{A}_{Rp}\hat{A}^*_{Rs} \exp(-j\beta Z^2/2) .$$

The next step is to integrate (A.3), subject to appropriate boundary conditions. We consider first the case in which the signal has nonzero amplitude and the idler zero amplitude for $Z \rightarrow -\infty$. We can then consider $\hat{A}_{RP}$ and A^{*}_{Rs} as constant in the right-hand sides of (A.3). Integrating, we obtain

$$\hat{A}_{Pi} = C_R \left(\frac{1}{4K_{Pi}S^2}\right) \hat{A}_{Rp}\hat{A}^{*}_{Rs} \int_{-\infty}^{Z} \exp(-j\beta Z^2/2)\, dZ \quad ,$$

$$\delta\hat{A}_{Rs} = C_R \left(\frac{\Omega^2_{Rs}}{2K_{Rs}}\right) \hat{A}_{Rp} \int_{-\infty}^{\infty} \hat{A}^{*}_{Pi} \exp(-j\beta Z^2/2)\, dZ \quad , \qquad (A.4)$$

where $\delta\hat{A}_{Rs}$ is the increment due to parametric amplification. Carrying out the integration leads to

$$\frac{\delta\hat{A}_{Rs}}{\hat{A}_{Rs}} = C_R^2 \left(\frac{1}{4K_{Pi}S^2}\right)\left(\frac{\Omega^2_{Rs}}{2K_{Rs}}\right) \frac{\pi}{\beta} |\hat{A}_{Rp}|^2 = \kappa_R^2 \frac{\pi}{\beta} \quad . \qquad (A.5)$$

Next we consider the case in which both the signal and idler have nonzero initial amplitudes. We can then consider A_{Rp} , A^{*}_{Rs} , and A^{*}_{Pi} as constant in the right-hand sides of (A.3) and calculate the increments in them. Integrating gives

$$\delta\hat{A}_{Pi} = C_R\left(\frac{1}{4K_{Pi}S^2}\right) \hat{A}_{Rp}\hat{A}^{*}_{Rs} \int_{-\infty}^{\infty} \exp(-j\beta Z^2/2)\, dZ$$

$$= C_R\left(\frac{1}{4K_{Pi}S^2}\right) \hat{A}_{Rp}\hat{A}^{*}_{Rs} \left(\frac{\pi}{\beta}\right)^{1/2} \quad , \qquad (A.6)$$

and similarly

$$\delta\hat{A}_{Rs} = C_R \left(\frac{\Omega^2_{Rs}}{2K_{Rs}}\right) \hat{A}_{Rp}\hat{A}^{*}_{Pi} \left(\frac{\pi}{\beta}\right)^{1/2} \quad . \qquad (A.7)$$

Hence we have

$$\left(\frac{\delta\hat{A}^{*}_{Pi}}{\hat{A}^{*}_{Pi}}\right) \left(\frac{\delta\hat{A}_{Rs}}{\hat{A}_{Rs}}\right) = \kappa_R^2 \frac{\pi}{\beta} \quad , \qquad (A.8)$$

where we note that $\delta\hat{A}_{Pi}/\hat{A}_{Pi} \approx \delta E_{Pi}/E_{Pi}$, etc.

APPENDIX B: EFFECT OF COLLISIONS ON PARAMETRIC GROWTH

To assess the effect of collisions, we shall consider the case with pump R, and signal and idler R, P. At the relatively high frequency and group velocity of the pump, collisional attenuation will be much weaker than for the signal and idler, so we will ignore it here. Assume that A_{Rs} and A_{Pi} are of the form $\hat{A}_{Rs} \exp(-jK_{Rs}Z)$, $\hat{A}_{Pi} \exp(-jK_{Pi}Z)$, where $\hat{A}_{Rs}$ and $\hat{A}_{Pi}$ are slowly-varying functions of Z, and substitute in the last two equations of (33). With the inclusion of phenomenological loss terms, $\kappa_{R\nu}\hat{A}_{Rs}$ and $\kappa_{P\nu}\hat{A}_{Pi}$, to represent collisional attenuation, we obtain

$$\frac{\partial \hat{A}_{Rs}}{\partial Z} + \kappa_{R\nu}\hat{A}_{Rs} = C_R\left(\frac{\Omega_{Rs}^2}{2K_{Rs}}\right)\hat{A}_{Pi}^{*}A_{Rp} ,$$

$$\frac{\partial \hat{A}_{Pi}}{\partial Z} + \kappa_{P\nu}\hat{A}_{Pi} = C_R\left(\frac{1}{4K_{Pi}S^2}\right)A_{Rp}\hat{A}_{Rs}^{*} . \tag{B.1}$$

To demonstrate the effect of the pump wave clearly, it is convenient to work in terms of the variables $\bar{A}_{Rs} = \hat{A}_{Rs} \exp \kappa_{R\nu}Z$, $\bar{A}_{Pi} = \hat{A}_{Pi} \exp \kappa_{P\nu}Z$, which are constants in the absence of the pump. Equation (B.1) then becomes

$$\frac{\partial \bar{A}_{Rs}}{\partial Z} = C_R\left(\frac{\Omega_{Rs}^2}{2K_{Rs}}\right)\bar{A}_{Pi}^{*}A_{Rp} \exp(\kappa_{R\nu} - \kappa_{P\nu})Z ,$$

$$\frac{\partial \bar{A}_{Pi}}{\partial Z} = C_R\left(\frac{1}{4K_{Pi}S^2}\right)A_{Rp}\bar{A}_{Rs}^{*} \exp(\kappa_{P\nu} - \kappa_{R\nu})Z . \tag{B.2}$$

For illustration, we shall solve (B.2) with the boundary conditions $\bar{A}_{Rs} = \bar{A}_{Rs0}$, $\bar{A}_{Pi}^{*} = 0$ at $Z = 0$. We then have,

$$\frac{\bar{A}_{Rs}}{\bar{A}_{Rs0}} = \frac{m_2 \exp m_1 Z - m_1 \exp m_2 Z}{m_2 - m_1} ,$$

$$m_{1,2} = \left(\frac{\kappa_{R\nu} - \kappa_{P\nu}}{2}\right) \pm \left[\kappa_R^2 + \left(\frac{\kappa_{R\nu} - \kappa_{P\nu}}{2}\right)^2\right]^{1/2} . \tag{B.3}$$

We note first that for $\kappa_{R\nu} = 0 = \kappa_{P\nu}$,

$$\frac{\hat{A}_{Rs}}{\hat{A}_{Rs0}} = \frac{\bar{A}_{Rs}}{\bar{A}_{Rs0}} = \cosh \kappa_R Z \quad . \tag{B.4}$$

In the special case $\kappa_{R\nu} = \kappa_{P\nu} \neq 0$, (B.3) reduces to

$$\frac{\hat{A}_{Rs}}{\hat{A}_{Rs0}} \exp \kappa_{R\nu} Z = \frac{\bar{A}_{Rs}}{\bar{A}_{Rs0}} = \cosh \kappa_R Z \quad , \tag{B.5}$$

implying that even though the signal is now attenuating, its parametric growth factor ($\cosh \kappa_R Z$) through the region $0 \to Z$ is exactly the same as in the absence of collisions. Of much greater interest is the fact that for $\kappa_R Z$ and $(\kappa_{R\nu} - \kappa_{P\nu})Z$ small expansion of (B.3) gives

$$\frac{\hat{A}_{Rs}}{\hat{A}_{Rs0}} \exp \kappa_{R\nu} Z = \frac{\bar{A}_{Rs}}{\bar{A}_{Rs0}} \approx 1 + \frac{\kappa_R^2 Z^2}{2} \quad , \tag{B.6}$$

which is just the small - Z expansion of (B.5), i.e., even in the presence of loss, the growth factor for Z small is independent of this loss.

This simple result is not preserved at large Z . We then have

$$\frac{\hat{A}_{Rs}}{\hat{A}_{Rs0}} \exp \kappa_{R\nu} Z = \frac{\bar{A}_{Rs}}{\bar{A}_{Rs0}} \approx \frac{m_2}{m_2 - m_1} \exp m_1 Z \quad , \tag{B.7}$$

which is larger or smaller than the growth factor without collisions according to whether $\kappa_{R\nu} > \kappa_{P\nu}$ or $\kappa_{R\nu} < \kappa_{P\nu}$.

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