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VALIDITY OF THE ELECTROSTATIC APPROXIMATION

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ABSTRACT

Criteria are derived for the existence of normal modes in a magnetized plasma for which $|\nabla \times \vec{E}| \ll |\nabla \cdot \vec{E}|$. The condition is different from the frequently quoted condition of thermal pressure $\ll$ magnetic pressure.

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I. INTRODUCTION

The familiar electrostatic approximation (assuming the electric field curl-free) often greatly simplifies the analysis of small-amplitude wave propagation in a plasma.[1,2,3] Little attention, though, has been given to establishing criteria for the validity of the approximation (cf., however, References 4-6). It seems to be generally believed, but with little foundation, that the approximation is justified when $\beta \ll 1$ ($\beta \equiv$ thermal pressure/magnetic pressure).

Here, we derive a criterion for the existence of at least some nearly electrostatic modes as solutions. We do not address ourselves to the more nebulous question of whether only electrostatic modes can be unstable at low β (at least one counter-example seems to exist[7]), but it will be noted that β itself does not enter directly into the criterion at all.

In any spatially uniform linear medium in which the electric current can be related linearly to the components of the electric field through a conductivity tensor, $\vec{j} = \overset{\rightrightarrows}{\sigma} \cdot \vec{E}$, Maxwell's equations can be reduced, for the situations in which all perturbations from uniformity vary as $\exp[i(\vec{k} \cdot \vec{x} + \omega t)]$, to

$$\overset{\rightrightarrows}{R} \cdot \vec{E} = 0, \tag{1}$$

where the dyadic $\overset{\rightrightarrows}{R} = c^2 \vec{k}\,\vec{k} + (\omega^2 - c^2 k^2)\overset{\rightrightarrows}{1} - 4\pi i \omega \overset{\rightrightarrows}{\sigma}$. If we divide all vectors into components perpendicular and parallel to $\vec{k}$, so that, for example $\vec{j} = \vec{j}_\perp + \vec{j}_\parallel$, the electrostatic approximation is well justified if the corresponding normal mode solution of Equation (1) satisfies $|\vec{E}_\perp| / |\vec{E}_\parallel| \ll 1$.

We first assume that $\vec{E} = \vec{E}_\parallel$ only in Equation (1), then calculate recursively an expression for $\vec{E}_\perp$. If this $|\vec{E}_\perp|$ is in fact $\ll |\vec{E}_\parallel|$, the approximation is valid; otherwise it is not.

Setting $\vec{E}_\perp = 0$ in Equation (1) gives, as a necessary condition that $\vec{E}_\parallel \neq 0$,

$$\omega^2 - 4\pi\, i\omega(\vec{k} \cdot \overleftrightarrow{\sigma} \cdot \vec{k})/k^2 = 0. \tag{2}$$

This is the "electrostatic dispersion relation." Calculating the first order $\vec{E}_\perp$ gives

$$\vec{E}_\perp = \frac{4\pi\, i\omega\, (\overleftrightarrow{\sigma} \cdot \vec{E}_\parallel)_\perp}{\omega^2 - c^2k^2}$$

or

$$\frac{|\vec{E}_\perp|}{|E_\parallel|} = \left| \frac{4\pi\, i\omega\, (\overleftrightarrow{\sigma} \cdot \vec{k})_\perp}{(\omega^2 - c^2k^2)k} \right|$$

$$= \left| \frac{4\pi\omega}{\omega^2 - c^2k^2} \quad \frac{\vec{k} \times (\overleftrightarrow{\sigma} \cdot \vec{k})}{k^2} \right| , \tag{3}$$

with ω given by (2) in terms of $\vec{k}$. If we choose Cartesian coordinates with $\vec{k} = k\,(\sin\theta, 0, \cos\theta)$, the condition that the ratio $|\vec{E}_\perp|/|\vec{E}_\parallel|$ be $\ll 1$ becomes equivalent to the two conditions

$$\left| \frac{4\pi\omega}{\omega^2 - c^2k^2} (\sin\theta\; \sigma_{yx} + \cos\theta\; \sigma_{yz}) \right| \ll 1, \tag{4a}$$

$$\left| \frac{4\pi\omega}{\omega^2 - c^2k^2} \left\{ \cos^2\theta\; \sigma_{xz} - \sin^2\theta\; \sigma_{zx} + \sin\theta \cos\theta\, (\sigma_{xx} - \sigma_{zz}) \right\} \right| \ll 1. \tag{4b}$$

As estimates of the elements of the conductivity tensor $\overset{\rightrightarrows}{\sigma}$ necessary for evaluating Equations (4), we use expressions derived from a multi-species fluid model with locally adiabatic equations of state. It is known that for waves with weak damping or growth, such models often give frequencies and polarizations similar to Vlasov solutions, but with much less cumbersome arithmetic.

If, in the same Cartesian coordinates, $\vec{B}_o = B_o(0,0,1)$, the elements of $\overset{\rightrightarrows}{\sigma}$ are $\overset{\rightrightarrows}{\sigma} = \sum_j \overset{\rightrightarrows}{\sigma}(j)$, where the sum is over all types of particle present, and

$$4\pi \overset{\rightrightarrows}{\sigma}(j) = (\omega_{pj}^2/\xi_j^2)\overset{\rightrightarrows}{\eta}^j . \tag{5}$$

$\omega_{pj}^2 = 4\pi n_{oj} e_j^2/m_j$ is the plasma frequency of the jth charge species, and

$$\xi_j^2 \equiv \Omega_j^2\left(1 - \frac{k^2 C_j^2 \cos^2\theta}{\omega^2}\right) - \omega^2 + k^2 C_j^2 ,$$

where

$$C_j^2 \equiv \gamma_j P_{oj}/m_j n_{oj}$$

is the speed of sound for the jth species. The γ_j, P_{oj}, and n_{oj} are the equilibrium adiabatic compression constant, pressure and number density for the jth species, which has charge e_j, mass m_j. $\Omega_j \equiv e_j B_o/m_j c$ is the gyrofrequency of the jth species. The elements of the dyadic $\overset{\rightrightarrows}{\eta}^j$ are:

$$\eta_{xx}^j = i\omega\left(1 - \frac{k^2 C_j^2}{\omega^2}\cos^2\theta\right)$$

$$\eta^j_{xy} = - \eta^j_{yx} = \Omega_j \left(1 - \frac{k^2 C_j^2}{\omega^2} \cos^2\theta\right)$$

$$\eta^j_{xz} = \eta^j_{zx} = i\omega \frac{k^2 C_j^2}{\omega^2} \sin\theta \cos\theta$$

$$\eta^j_{yy} = i\omega \left(1 - \frac{k^2 C_j^2}{\omega^2} \cos^2\theta\right)$$

$$\eta^j_{yz} = - \eta^j_{zy} = - \Omega_j \frac{k^2 C_j^2}{\omega^2} \sin\theta \cos\theta$$

$$\eta^j_{zz} = i\omega \left(1 - \frac{\Omega_j^2}{\omega^2} - \frac{k^2 C_j^2}{\omega^2} \sin^2\theta\right) . \tag{6}$$

It now remains to substitute the expression of Equations (6) into Equations (4). Since the conductivities are additive for the various species, the conditions (4) must apply either to each $\overset{\Rightarrow}{\sigma}(j)$, or to the dominant term in $\overset{\Rightarrow}{\sigma}$, if there is one (the electron term often dominates at high frequencies because of the lighter electron mass). Cancellations among the species cannot occur, because Equation (1) would then just reduce to the equations of vacuum electrodynamics (an exception to this statement is the limit $\omega \to 0$, $k \to 0$, ω/k = const., which will always have to be treated separately). Therefore interpreting j as either applying to every species or to the dominant species in the conductivity tensor, Equations (4) and (6) lead to

$$|\omega_{pj}^2 \, \omega \, \Omega_j \sin\theta| \ll | \, \xi_j^2 \, (\omega^2 - c^2 k^2) \, | \tag{7}$$

and

$$|\omega_{pj}^2 \, \Omega_j^2 \sin\theta \cos\theta \,| \ll |\, \xi_j^2 \, (\omega^2 - c^2 k^2) \,|, \tag{8}$$

with ω and k related by (2).

For nearly parallel propagation, $\sin\theta \cong 0$, both (7) and (8) are automatically satisfied. For nearly perpendicular propagation, only (7) need be considered. For in-between angles, with both $\sin\theta$ and $\cos\theta$ treated as of $O(1)$, Equations (7) and (8) may be combined with the definition of ξ_j^2 to give

$$\omega_{pj}^2 \left[1 + \frac{\omega^2}{\Omega_j^2} \right]^{\frac{1}{2}} \ll \left| (\omega^2 - c^2k^2)\left(1 - \frac{k^2C_j^2}{\omega^2}\cos^2\theta - \frac{\omega^2}{\Omega_j^2} + \frac{k^2C_j^2}{\Omega_j^2}\right) \right|. \tag{9}$$

Equation (9) is our main result: the condition that a mode obeying (2) shall have $\vec{E} \parallel \vec{k}$, to a good approximation. It has almost innumerable special cases of interest, and we shall only note one of these here. In some modes of interest[2,3] the electrons dominate $\vec{\vec{\sigma}}$. If this is assumed to be the case and it is further assumed that $\omega^2 \ll \Omega_e^2$, $\omega_{pe}^2 \ll \Omega_e^2$, use of (2) gives

$$\omega^2 \simeq \omega_{pe}^2 \cos^2\theta \left[\frac{1 + k^2C_e^2/\omega_{pe}^2}{1 + k^2C_e^2/\Omega_e^2 + \omega_{pe}^2/\Omega_e^2} \right]$$

which is the long-wavelength limit just gives $\omega \cong \omega_{pe} \cos\theta$. The condition (9) becomes, setting $\omega \cong \omega_{pe} \cos\theta \ll |\,\Omega_e\,|$, $1 \ll \left|\,\frac{c^2k^2}{\omega_{pe}^2} - \cos^2\theta\,\right|$, or

$c^2 >> \omega_{pe}^2/k^2$.

It is also interesting to observe that (9) is always satisfied near a resonance ($k \to \infty$, ω = const.). But it should also be noted that the parameter β, per se, does not enter in a fundamental way.

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