N70 - 25369

AD703660

Office of Naval Research
Contract N00014-67-A-0298-0006 NR-372-012

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
Grant NGR 22-007-068

CONTROL OF NONLINEAR SYSTEMS IN REGIONS OF STATE SPACE

N70-25369

By

Stanley B. Gershwin and David H. Jacobson

January 1970

Technical Report No. 600

Division of Engineering and Applied Physics
Harvard University • Cambridge, Massachusetts

Office of Naval Research

Contract N00014-67-A-0298-0006

NR-372-012

National Aeronautics and Space Administration

Grant NGR 22-007-068

CONTROL OF NONLINEAR SYSTEMS IN REGIONS OF STATE SPACE

By

Stanley B. Gershwin and David H. Jacobson

Technical Report No. 600

January 1970

The research reported in this document was made possible through support extended the Division of Engineering and Applied Physics, Harvard University by the U. S. Army Research Office, the U. S. Air Force Office of Scientific Research and the U. S. Office of Naval Research under the Joint Services Electronics Program by Contracts N00014-67-A-0298-0006, 0005, and 0008 and by the National Aeronautics and Space Administration under Grant NGR 22-007-068.

Division of Engineering and Applied Physics

Harvard University · Cambridge, Massachusetts

CONTROL OF NONLINEAR SYSTEMS IN REGIONS OF STATE SPACE

By

Stanley B. Gershwin and David H. Jacobson

Division of Engineering and Applied Physics

Harvard University · Cambridge, Massachusetts

ABSTRACT

Recently, a nonlinear controllability theory based upon Liapunov-like notions was developed. In this paper the theory is generalized and strengthened, and a wider class of nonlinear systems is considered. In particular, conditions for controllability of a dynamic system which is subject to state variable inequality constraints are obtained. It is shown that initial conditions which are interior to a certain ellipse can be made to generate trajectories which remain in that ellipse and which reach the desired terminal state. When the ellipse is a subset of the feasible region of state space the trajectory clearly remains in this region (i. e. the state variable inequality constraints are satisfied). A design procedure for finding the largest such ellipse is given, and illustrative examples are presented. In addition, stabilization of constrained dynamic systems is considered.

I. Introduction

Consider the problem of finding a control policy $u(x, t)$ to satisfy

$$x(t_o) = x_o \tag{1.1}$$

$$x(t_f) = 0 \tag{1.2}$$

$$\dot{x}(t) = f(x(t), u(x, t), t) \tag{1.3}$$

where t_o and t_f are finite, specified, and $t_o < t_f$. Here, $x(t)$ is an n-dimensional state vector, and $u(x, t)$ is an m-dimensional control vector.

We call this problem A, and, when a solution exists, we say that (1.3) is controllable from (x_o, t_o) to $(0, t_f)$†.

The case where $f(\cdot)$ is a linear function of x and u, i.e., where (1.3) is

$$\dot{x}(t) = F(t)x(t) + G(t)u(t) \tag{1.4}$$

was solved by Kalman [2] and is well known. Solutions have been obtained by the authors [1] for several cases where $f(\cdot)$ is nonlinear. The following theorem was used. It is assumed that $u(t)$, $t_o \leq t \leq t_f$ is restricted to some constraint set Ω.

Theorem 1 [1]*

If a scalar function $V(x, t)$ exists such that:

(i) $V_x(x, t)$ and $V_t(x, t)$ exist for all x, $t \in [t_o, t_f)$

(ii) for all continuous $c(t)$ (n-vector function of t)

$$\lim_{t \to t_f} c(t) \neq 0 \implies \lim_{t \to t_f} V(c(t), t) = \infty \tag{1.5}$$

---

† Note that this definition of controllability (see also [1]) is different from Kalman's [2].

* Some minor changes have been made in this theorem.

(iii) $V(x_o, t_o) \leq B < \infty$

and if a control function $u^*(x,t) \in \mathcal{U}$ exists such that:

(iv) along the trajectory of (1.3) starting at (1.1), the full time derivative $\dot{V}$ of $V(x,t)$ satisfies:

$$\dot{V}(x,t) = V_t(x,t) + V_x(x,t)f(x, u^*(x,t), t) \leq M < \infty \qquad \forall t \in [t_o, t_f) \tag{1.6}$$

and the following limit exists:

$$V(x(t_f), t_f) = \lim_{t \to t_f} V(x(t), t)$$

(v) the solution to (1.3), (1.1) (with $u(x,t) = u^*(x,t)$) exists, is unique, and satisfies

$$x(t_f) = \lim_{t \to t_f} x(t) \tag{1.7}$$

then system (1.3) is controllable from (x_o, t_o) to $(0, t_f)$ and $u^*(x,t)$ accomplishes this transfer.

In each of the examples treated by this theorem in [1],

$$V(x,t) = x^T S(t) x \tag{1.8}$$

where $S(t)$ is an $n \times n$ positive definite symmetric matrix satisfying

$$\lim_{t \to t_f} S^{-1}(t) = 0 \tag{1.9}$$

and

$$\dot{S}(t) + S(t)F(t) + F(t)^T S(t) - S(t)G(t)G(t)^T S(t) = 0 \tag{1.10}$$

(F and G are chosen in some appropriate manner such that (1.4) is completely controllable) and $u^*(x,t)$ is such that

$$\dot{V}(x,t) = \Gamma(x,t) \leq 0 \tag{1.11}$$

i.e., Theorem 1 is satisfied with $M = 0$. In fact, in [1], the function $\Gamma(x,t)$ is made negative or zero for all x (for all $t \in [t_o, t_f)$), although the theorem only requires that $\Gamma \leq 0$ along a trajectory.

Extensions of the results given in [1] are contained in the present paper:

1. A technique is developed which allows the condition that $\Gamma(x, t) \leq 0$ for all x to be weakened. More precisely, the region $\mathcal{R} = \{x \mid \Gamma(x, t) \leq 0$ for all $t \in [t_o, t_f]\}$ is defined and the state constrained problem of finding $u(x, t)$ to satisfy Problem A such that $x(t) \in \mathcal{R}$ for $t \in [t_o, t_f]$ is solved. See the next paragraph.

2. Let $\mathcal{R}$ be some region of n-space, and denote the following problem by Problem B: find $u(x, t)$ to satisfy

$$x(t_o) = x_o$$

$$x(t_f) = 0$$

$$\dot{x}(t) = f(x(t), u(x(t), t), t)$$

$$x(t) \in \mathcal{R} \tag{1.12}$$

The first three conditions form Problem A; condition (1.12) is a state variable constraint. A method is developed below which is applicable to the case where $\mathcal{R}$ is a region bounded by inequality constraints: in particular where $\mathcal{R}$ contains the ellipse

$$\mathcal{E}(t_o) = \{x \mid x^T S(t_o) x \leq x_o^T S(t_o) x_o\} \tag{1.13}$$

It is shown that if F and G are constant and $\mathcal{E}(t_o) \subset \mathcal{R}$, the methods developed in [1] for Problem A also solve Problem B.

3. That there is an analogy between Theorem 1 and Liapunov's stability theory is pointed out in [1]. Here, we show a precise mathematical relationship between the control of a system using Theorem 1 and the stabilization of that system using Liapunov techniques. Under certain conditions (which include $V \triangleq x^T S(t) x$), if a control exists which solves Problem A, a slight modification of that control stabilizes (1.3). The

Liapunov function of the stabilized system is just $V(x, t_1)$, where t_1 is a fixed time ($t_o \leq t_1 < t_f$).

This technique is then used for the problems discussed in the preceeding two paragraphs for cases where $x_o \in \mathcal{R}$ but $\mathcal{E}(t_o) \not\subset \mathcal{R}$. These cases are handled as follows: if some t_1 exists such that $\mathcal{E}(t_1) \subset \mathcal{R}$, then "stabilize" (1.3) from $t = t_o$ until $t = t_1$ (i.e., apply the stabilizing control in that time interval). From $t = t_1$ until $t = t_f$, apply the control obtained from Theorem 1.

In Section II of this paper are found the analytic results upon which paragraphs 1 and 2 above are based. It is shown that if F and G are constant matrices, the solution to (1.9), (1.10) satisfies $\dot{S}(t) \geq 0$ for all $t < t_f$. As a consequence, if $\dot{V} = \frac{d}{dt}(x^T(t)S(t)x(t)) \leq 0$ and $t_o \leq t_1 < t_2 < t_f$, then $\mathcal{E}(t_2) \subset \mathcal{E}(t_1)$. Then as t increases, x(t) is found in (or on the boundary of) an ever-shrinking ellipse. Section III shows how these results can be applied to A- and B-type problems.

In the course of applying these techniques, it is often necessary to ask: Is $\mathcal{E}(t_o) \subset \mathcal{R}$? and What is the largest ellipse of the form

$$E(S_o, \epsilon) = \{x \,|\, x^T S_o x \leq \epsilon\} \tag{1.14}$$

such that $E(S_o, \epsilon) \subset \mathcal{R}$? Section IV supplies techniques to solve these subsidiary problems when $\mathcal{R}$ is a region bounded by linear inequalities.

Section V contains specific examples to illustrate the techniques. These examples are modifications of problems treated in [1].

The material summarized above in paragraph 3 is contained in Section VI.

II. Analytic Results

The new control theory techniques that appear in succeeding sections are derived from the mathematical results of this section. By means of

Lemma 1, it is shown in Theorem 2 that the solution to Riccati equation (2.2) with boundary condition (2.3) has the property that $\dot{S} \leqslant 0$. Using this result and Lemma 2, Theorem 3 proves that if $V(x(t), t) = x(t)^T S(t)x(t)$ and $\dot{V} \leqslant 0$ then x(t) is found in or on the boundary of an ever-shrinking ellipse.

Lemma 1: Let

$$\dot{x} = F(t)x + G(t)u \tag{2.1}$$

be a linear time-varying dynamic system where x(t) is an n-vector and u(t) is an m-vector. Assume that (2.1) is completely controllable at time t to $(0, t_f)$. Let $S(t, t_f)$ be the unique n x n matrix that satisfies the matrix Riccati equation

$$\frac{\partial}{\partial t} S(t, t_f) + S(t, t_f)F(t) + F^T(t)S(t, t_f) - S(t, t_f)G(t)G^T(t)S(t, t_f) = 0 \tag{2.2}$$

with boundary condition

$$\lim_{t \to t_f} S^{-1}(t, t_f) = 0 \tag{2.3}$$

Then

$$\frac{\partial}{\partial t_f} S(t, t_f) \leqslant 0 \tag{2.4}$$

Proof: Define

$$Z(t, t_f) = S^{-1}(t, t_f) . \tag{2.5}$$

$Z(t, t_f)$ satisfies

$$\frac{\partial}{\partial t} Z(t, t_f) - F(t)Z(t, t_f) - Z(t, t_f)F(t) + G(t)G^T(t) = 0 \tag{2.6}$$

$$Z(t_f, t_f) = 0 \tag{2.7}$$

and thus [3] can be expressed as

$$Z(t, t_f) = \int_t^{t_f} \Phi(t, \tau)G(\tau)G^T(\tau)\Phi^T(t, \tau)d\tau \tag{2.8}$$

where $\Phi(t, \tau)$ (the "transition matrix") satisfies

$$\frac{\partial}{\partial t}\Phi(t,\tau) = F(t)\Phi(t,\tau) \tag{2.9}$$

$$\Phi(\tau,\tau) = I \quad . \tag{2.10}$$

The controllability assumption on (2.1) implies that $Z(t,t_f)$ is positive definite and therefore invertible for every $t < t_f$, [2].

From (2.5) and (2.8),

$$\frac{\partial}{\partial t_f} Z(t,t_f) = -S^{-1}(t,t_f)[\frac{\partial}{\partial t_f} S(t,t_f)]S^{-1}(t,t_f)$$

$$= \Phi(t,t_f)G(t_f)G^T(t_f)\Phi^T(t,t_f) \tag{2.11}$$

or,

$$\frac{\partial}{\partial t_f} S(t.t_f) = -S(t,t_f)\Phi(t.t_f)G(t_f)G^T(t_f)\Phi^T(t,t_f)S(t,t_f) \tag{2.12}$$

The right-hand side of (2.12) is clearly negative semidefinite, so that the lemma is proved.

<u>Theorem 2</u>: If in addition to the hypotheses of Lemma 1, F and G are constant matrices, then

$$\frac{\partial}{\partial t} S(t,t_f) \geq 0 \tag{2.13}$$

<u>Proof</u>: Because F is constant, we can write [3]

$$\Phi(t,\tau) = \Psi(\tau - t) \tag{2.14}$$

and (2.8) can be written:

$$Z(t,t_f) = \int_t^{t_f} \Psi(\tau - t)GG^T\Psi^T(\tau - t)d\tau \tag{2.15}$$

Let $\sigma = \tau - t$. Then

$$Z(t,t_f) = \int_0^{t_f - t} \Psi(\sigma)GG^T\Psi^T(\sigma)d\sigma \tag{2.16}$$

From (2.16) it is clear that $Z(t,t_f)$ is a function only of $t_f - t$. Therefore $S = Z^{-1}$ is a function only of $t_f - t$, which implies that

$$\frac{\partial}{\partial t} S(t,t_f) = -\frac{\partial}{\partial t_f} S(t,t_f) \quad . \tag{2.17}$$

The proof follows from Lemma 1.

Henceforth we will only consider the time invariant Riccati equation; i.e. F and G will be constant matrices. In addition, t_f will be suppressed as an argument of S and Z.

<u>Lemma 2</u>: Let $V(x,t) = x^T S(t)x$, where x is an n-vector. Let y(t) be a function of t such that

$$\dot{V} = \frac{d}{dt} V(y(t),t) = V_y \dot{y} + V_t \leqslant 0 \quad . \tag{2.18}$$

Then for $t_1 < t_2 < t_f$,

$$V(y(t_2),t_1) = y^T(t_2)S(t_1)y(t_2) \leqslant y^T(t_1)S(t_1)y(t_1) = V(y(t_1),t_1) . \tag{2.19}$$

<u>Proof</u>: Inequality (2.18) implies

$$V(y(t_2),t_2) = y^T(t_2)S(t_2)y(t_2) \leqslant y^T(t_1)S(t_1)y(t_1) = V(y(t_1),t_1) . \tag{2.20}$$

Theorem 2 implies

$$S(t_1) \leqslant S(t_2) \tag{2.21}$$

which means that

$$V(y(t_2),t_1) = y^T(t_2)S(t_1)y(t_2) \leqslant y^T(t_2)S(t_2)y(t_2) = V(y(t_2),t_2) \quad . \tag{2.22}$$

The lemma is proved by combining (2.20) and (2.22).

Inequality (2.19) implies that $y(t_2)$ is contained in the ellipse

$$y^T S(t_1)y \leqslant y^T(t_1)S(t_1)y(t_1) \quad . \tag{2.23}$$

Or, defining

$$\mathcal{E}(t) = \{z \mid z^T S(t)z \leqslant y(t)^T S(t)y(t)\} \quad , \tag{2.24}$$

then (2.19) implies that $y(t_2) \in \mathcal{E}(t_1)$. A stronger result is proved in Theorem 3: that for each $z \in \mathcal{E}(t_2)$, $z \in \mathcal{E}(t_1)$.

Theorem 3: Under the hypotheses of Lemma 2, $\mathcal{E}(t_2) \subset \mathcal{E}(t_1)$.

Proof: Let $x \in \mathcal{E}(t_2)$. Then

$$x^T S(t_2) x \leq y(t_2)^T S(t_2) y(t_2) \tag{2.25}$$

From (2.21)

$$x^T S(t_1) x \leq x^T S(t_2) x \tag{2.26}$$

and from (2.18)

$$y(t_2)^T S(t_2) y(t_2) \leq y(t_1)^T S(t_1) y(t_1) \tag{2.27}$$

Combining (2.25), (2.26), and (2.27),

$$x^T S(t_1) x \leq y(t_1)^T S(t_1) y(t_1) \quad . \tag{2.28}$$

Inequality (2.28) is equivalent to $x \in \mathcal{E}(t_1)$. Since x may be any point in $\mathcal{E}(t_2)$, the theorem is proved.

These results are illustrated in Figure 1. Ellipses $\mathcal{E}(t_i)$, $i = 0, \ldots, 4$ are shown ($t_o = 0$, $t_1 = .2$, $t_2 = .4$, $t_3 = .6$, $t_4 = .8$) where $S(t)$ satisfies (2.2), (2.3) with

$$F = \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad t_f = 1 .$$

Three trajectories $x(t)$ are shown such that $\dot{V}(x(t), t) = 0$.

III. Applications to Control Theory

Consider the dynamic system

$$\dot{x} = f(x, u, t) \tag{3.1}$$

Let $V(x, t) = x^T S(t) x$ be defined as in Theorems 1, 2, and 3. From Theorem 1, if $u(x, t)$ is such that $\dot{V} \leq 0$, then $x(t_f) = 0$. Techniques for finding such control policies are given in [1] and below. From Lemma 2, if $t_o < t < t_f$, $x(t) \in \mathcal{E}(t_o)$. Theorem 3 implies that the ellipse $\mathcal{E}(t)$ is shrinking with increasing t. These results are significant for the following reasons.

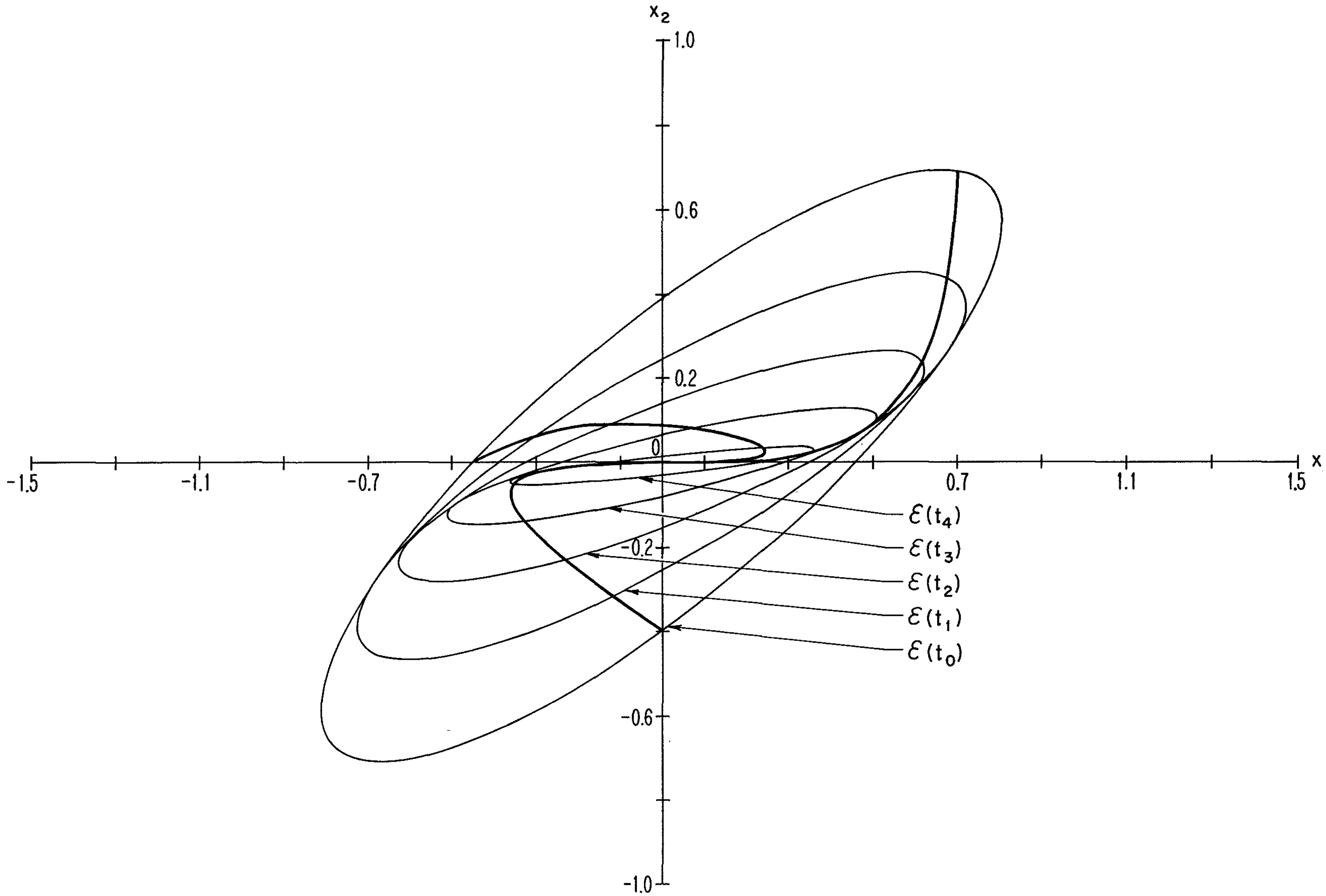

FIG. 1 TRAJECTORIES OF (2-1) AND EVOLUTION OF ELLIPSE $\mathcal{E}(t)$ FROM $t = 0$ TO $t = 1$.

(1) Let $W(x, t)$ be the minimum energy function of (2.1)

$$W(x_o, t_o) = \min_{u(\cdot)} \int_{t_o}^{t_f} u^T(t)u(t)dt \tag{3.2}$$

where $x(t_o) = x_o$, $x(t_f) = 0$; t_o and t_f are given; and $x(t)$, $u(t)$ satisfy (2.1). Then $W(x, t) = x^T\overline{S}(t)x$, where $\overline{S}(t)$ satisfies

$$\dot{\overline{S}} + \overline{S}F + F^T\overline{S} - \frac{1}{2}\overline{S}GG^T\overline{S} = 0 \tag{3.3}$$

$$\lim_{t \to t_f} \overline{S}(t)^{-1} = 0 \tag{3.4}$$

Clearly Lemma 1 and Theorem 2 apply to $\overline{S}(t)$. The minimum energy control is $u = -\frac{1}{2}G^T\overline{S}x$. Then

$$\begin{aligned}\dot{W}(x, t) &= x^T(\dot{\overline{S}} + \overline{S}F + F^T\overline{S} - \overline{S}GG^T\overline{S})x \\ &= -\frac{1}{2}x^T\overline{S}GG^T\overline{S}x \leq 0\end{aligned} \tag{3.5}$$

so that Lemma 2 and Theorem 3 are satisfied. Thus the results of Section II apply to the time-invariant linear-quadratic optimal control problem with constrained terminal state.

(2) Some state-variable inequality-constrained problems can now be solved. Consider the following problem: region $\mathcal{R}$ contains the initial point $x(t_o) = x_o$ and the terminal point $x(t_f) = 0$. Find a control for (3.1) which solves Problem A (see Section I) and is such that $x(t) \in \mathcal{R}$ for all $t \in [t_o, t_f]$ (Problem B). A solution is: if $\mathcal{E}(t_o) \subset \mathcal{R}$, use a control $u(x, t)$ such that $\dot{V}(x, t) \leq 0$. From Lemma 2, $x(t) \in \mathcal{E}(t_o)$ for all $t_o \leq t \leq t_f$ and therefore $x(t) \in \mathcal{R}$. From Theorem 1, $x(t_f) = 0$.

(3) By using this technique for B-problems, the material in [1] for A-problems may be generalized. In Theorem 1 and in Section II, $\dot{V}(x, t) \leq 0$ is required merely along a trajectory $x = x(t)$. In each of

the examples of Theorem 1 in [1] some inequality of form $p(x,t) \geq 0$ (equivalent to $\dot{V}(x,t) \leq 0$) is required to be satisfied for all x, $t \in [t_o, t_f]$.

If it happens that $p(x,t)$ is not positive or zero for all x, convert this problem to a B-problem in the following way. Define

$$\mathcal{R} = \{x \mid p(x,t) \geq 0 \text{ for all } t \in [t_o, t_f]\} \tag{3.6}$$

and solve the B-problem in the manner described in the previous paragraph. The solution to this state-constrained problem clearly also solves the original, unconstrained A-problem.

IV. The Ellipse of Controllability

Define

$$E(S, \epsilon) = \{z \mid z^T S z \leq \epsilon\} \quad . \tag{4.1}$$

In the notation of Section II,

$$\mathcal{E}(t_o) = E(S(t_o), y^T(t_o)S(t_o)y(t_o)) \quad . \tag{4.2}$$

To apply the techniques of Section III, it is necessary to determine whether or not $\mathcal{E}(t_o) \subset \mathcal{R}$. Also, it is of interest to find the largest ϵ such that $E(S(t_o), \epsilon) \subset \mathcal{R}$ (because for any positive definite symmetric S, $\epsilon_2 > \epsilon_1 \Longrightarrow E(S, \epsilon_1) \subset E(S, \epsilon_2)$). The latter problem is related to a problem of Julich [5] on acceptable motions of stable systems. See Section VI.

In the following, $\mathcal{R}$ is assumed to be a region with linear boundaries, i.e.,

$$\mathcal{R} = \{x \mid a_i^T x + b_i \leq 0, \ i = 1, \ldots, \ell\} \tag{4.3}$$

where a_i, $i = 1, \ldots, \ell$ are n-vectors and b_i, $i = 1, \ldots, \ell$ are scalars. We assume that $x = 0$ is an interior point of $\mathcal{R}$, so $b_i < 0$, $i = 1, \ldots, \ell$.

IV. 1. Is the Ellipse in $\mathcal{R}$?

The statement $E(S, \epsilon) \subset \mathcal{R}$ is equivalent to

$$\max_{i=1,\ldots,\ell} \ \max_{\substack{x \\ x^T S x \leq \epsilon}} \ a_i^T x + b_i \leq 0 \tag{4.4}$$

The maximum of $a_i^T x + b_i$ occurs on the boundary of $E(S, \epsilon)$. Therefore, (4.4) is equivalent to

$$\max_{i=1,\ldots,\ell} \; \max_{\substack{x \\ x^T S x = \epsilon}} \; a_i^T x + b_i \leq 0 \quad . \tag{4.5}$$

To determine if (4.5) is satisfied, calculate

$$c_i = \max_{\substack{x \\ x^T S x = \epsilon}} \; a_i^T x + b_i \quad . \tag{4.6}$$

If $c_i \leq 0$, $i = 1, \ldots, \ell$, then (4.5) is satisfied. To find c_i, let

$$J_i = c_i + \lambda_i (x^T S x - \epsilon) \tag{4.7}$$

where λ_i is a scalar Lagrange multiplier, and maximize J_i subject to

$$x^T S x = \epsilon \quad . \tag{4.8}$$

Rewrite

$$J_i = a_i^T x + b_i + \lambda_i (x^T S x - \epsilon) \tag{4.9}$$

Then

$$J_{i_x} = a_i^T + 2\lambda_i x^T S \tag{4.10}$$

so the maximizing value of x (where $J_{i_x} = 0$) is

$$x_i = -\frac{1}{2\lambda_i} S^{-1} a_i \tag{4.11}$$

Since x_i is maximizing,

$$0 \geq J_{i_{xx}} = 2\lambda_i S \tag{4.12}$$

and since $S > 0$, we expect $\lambda_i \leq 0$.

From (4.8),

$$\epsilon = x_i^T S x_i = \frac{1}{4\lambda_i^2} a_i^T S^{-1} a_i \tag{4.13}$$

so

$$\lambda_i = -\frac{1}{2} \sqrt{\frac{a_i^T S^{-1} a_i}{\epsilon}} \tag{4.14}$$

where the sign is chosen to satisfy (4.12). Equation (4.11) implies

$$x_i = \sqrt{\frac{\epsilon}{a_i^T S^{-1} a_i}} \quad S^{-1} a_i \tag{4.15}$$

which clearly satisfies (4.8). From (4.6),

$$c_i = a_i^T x_i + b_i = \sqrt{\epsilon a_i^T S^{-1} a_i} + b_i \tag{4.16}$$

Note that the positive square root is chosen in (4.16) and that $b_i < 0$, $i = 1, \ldots, \ell$.

The procedure for ascertaining whether $E(S, \epsilon) \subset \mathfrak{R}$ is a simple one: evaluate c_i for each $i = 1, \ldots, \ell$. If any is positive, $E(S, \epsilon)$ is not a subset of $\mathfrak{R}$ and if none are positive, $E(S, \epsilon)$ is contained in $\mathfrak{R}$.

IV. 2. <u>The Largest Ellipse in $\mathfrak{R}$</u>

It is clear that the largest ellipse in $\mathfrak{R}$ touches one or more of the linear constraints at one point and does not touch the others at any points. This is equivalent to

$$\max_{i=1,\ldots,\ell} c_i = 0 \tag{4.17}$$

where c_i is given by (4.6) and (4.16). Write (4.16) as

$$c_i(\epsilon) = \sqrt{\epsilon a_i^T S^{-1} a_i} + b_i \quad . \tag{4.18}$$

Define

$$\epsilon_i = \frac{b_i^2}{a_i^T S^{-1} a_i} \tag{4.19}$$

$$\epsilon_{min} = \min_{i=1,\ldots,\ell} \epsilon_i \tag{4.20}$$

Note that

$$c_i(\epsilon_i) = 0 \quad . \tag{4.21}$$

Thus ellipse $E(S, \epsilon_i)$ is tangent to the ith constraint; in fact $E(S, \epsilon_i)$ is the largest ellipse such that for every $x \in E(S, \epsilon_i)$, $a_i^T x + b_i \leq 0$. $E(S, \epsilon_{min})$ is therefore the largest ellipse such that for every $x \in E(S, \epsilon_{min})$, $a_i^T x + b_i \leq 0$ for all $i = 1, \ldots, \ell$.

V. Examples

V.1. Example 1

Consider the dynamic system

$$\dot{x} = Fx + Gu + h(x, t) \tag{5.1}$$

where

$$G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$h = \begin{pmatrix} 0 \\ -p(x, t)x_2 \end{pmatrix}$$

$p(x, t)$ is a scalar function of x and t and F is a constant matrix such that the linear system

$$\dot{x} = Fx + Gu \tag{5.2}$$

is completely controllable (i.e., $F_{21} \neq 0$).

Find a control $u(x, t)$ to drive the state of (5.1) from its initial value $x(t_o) = x_o$ to the origin at $t = t_f > t_o$. What conditions on $p(x, t)$ and x_o guarantee that (5.1) will be controllable from (x_o, t_o) to $(0, t_f)$?

This example was considered by Gershwin and Jacobson [1, Example 2.2.3] who found that if $p(x, t) \geq 0$ for all x and all $t \in [t_o, t_f]$, then (5.1) is completely controllable from t_o to $(0, t_f)$ and a control that drives the state to the origin is

$$u(x, t) = -\frac{1}{2}G^T S(t)x - p(x, t)x_1 \tag{5.3}$$

where $S(t)$ satisfies

$$\dot{S} + SF + F^T S - SGG^T S = 0 \tag{5.4}$$

$$\lim_{t \to t_f} S^{-1}(t) = 0 \quad . \tag{5.5}$$

Control (5.3) is known to satisfy Problem A because it satisfies Theorem 1 with

$$V(x,t) = x^T S(t) x \tag{5.6}$$

and

$$\dot{V}(x,t) = -2p(x,t)x^T S x \quad . \tag{5.7}$$

Clearly $\dot{V} \leq 0$ because S is positive definite and $p(x,t)$ is positive or zero by hypothesis.

Now, relax the hypothesis that $p(x,t) \geq 0$ for all x. Define

$$\mathcal{R} = \{x \mid p(x,t) \geq 0 \quad \forall t \in [t_o, t_f]\} \quad . \tag{5.8}$$

Assume that the origin is an interior point of $\mathcal{R}$ (i.e., that $p(0,t) > 0$ for all $t \in [t_o, t_f]$). Assume also that $x_o \in \mathcal{R}$.

Consider control (5.3). If the trajectory generated by this control stays in $\mathcal{R}$, then $x(t_f) = 0$ because if $x \in \mathcal{R}$, $p \geq 0$ and (5.7) implies that Theorem 1 holds.

By Lemma 2, if ellipse $\mathcal{E}(t_o) \subset \mathcal{R}$, the trajectory stays in $\mathcal{R}$. To find out whether $\mathcal{E}(t_o) \subset \mathcal{R}$ or to find the largest ϵ such that $E(S(t_o), \epsilon) \subset \mathcal{R}$, the methods of Section IV apply (if $\mathcal{R}$ has linear boundaries).

Now let $t_o = 0$, $t_f = 1$, and

$$F = \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} \tag{5.9}$$

and consider the following special cases of (5.1).

V.1.1. <u>Case 1</u>

$$p_1 = 1 - x_2^2 \quad .$$

Then (5.1) becomes

$$\dot{x}_1 = u \tag{5.10}$$

$$\dot{x}_2 = -x_1 - 2x_2 + x_2^{3*} \tag{5.11}$$

Then

$$\mathcal{R}_1 = \{x \mid 1 - x_2^2 \geq 0\} = \{(x_1, x_2) \mid |x_2| \leq 1\}$$

$$= \{(x_1, x_2) \mid x_2 - 1 \leq 0, \ -x_2 - 1 \leq 0\} \tag{5.12}$$

The latter form for expressing $\mathcal{R}_1$ is chosen to conform with the notation of Section IV. Comparing (5.12) with (4.3), it is readily seen that $\ell = 2$,

$$a_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad b_1 = -1$$

$$a_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad b_2 = -1 \quad .$$

The solution of Riccati equation (5.4) with boundary condition (5.5) is such that

$$S^{-1}(t = 0) = \begin{pmatrix} 1 & .718 \\ .718 & .758 \end{pmatrix} \quad .$$

It is a simple matter to calculate ϵ_{min} from (4.19), (4.20):

$$\epsilon_{min} = 1.32 \quad .$$

---

*It should be pointed out that in [1, Example 2.2.3], $\dot{x}_2 = -x_1 - x_2 - x_2^3$ so that $p(x, t) = x_2^2$. If we had chosen the F-matrix differently in the present case, i.e., if instead of (5.9),

$$F = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

then (5.11) would have become $\dot{x}_2 = -x_1 - x_2 + x_2^3$. Thus, by a tricky redefinition of the system matrix, we can solve a problem that might appear to the reader of [1, Example 2.2.3] to be impossible.

Figure 2 displays $\mathcal{R}_1$ and the ellipse $\mathcal{E} = E(S(0), 1.32)$. Three trajectories of (5.1) (solid lines) and three trajectories of (5.10)-(5.11) (dashed lines) are shown.

V.1.2. Case 2

$$p_2 = 1 - x_1 \tag{5.13}$$

In this case, system (5.1) is

$$\dot{x}_1 = u \tag{5.14}$$

$$\dot{x}_2 = -x_1 - 2x_2 + x_1 x_2 \quad . \tag{5.15}$$

Equation (5.13) implies

$$\mathcal{R}_2 = \{(x_1, x_2) | x_1 - 1 \leq 0\} \quad . \tag{5.16}$$

From the notation of (4.3), $\ell = 1$,

$$a = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \qquad b = -1$$

which implies that $\epsilon_{min} = 1$. Figure 3 is similar to Figure 2, displaying $\mathcal{R}$, $\mathcal{E} = E(S(0), 1)$ and trajectories of (5.2) and (5.14)-(5.15).

V.2. Example 2

Consider system (5.1) with

$$F = \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}, \qquad G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

((5.2) is completely controllable) and

$$h(x, t) = \begin{pmatrix} 0 \\ -p(x, t)x_1 \end{pmatrix} \tag{5.17}$$

where $p(x, t)$ is a scalar function. In Example 2.2.5 of [1] it is found that if $V(x, t) = x^T S(t)x$ (where $S(t)$ satisfies (5.4), (5.5)) and

$$u(x, t) = -\frac{1}{2}G^T S(t)x + (S_{22}(t)/S_{12}(t))p(x, t)x_1 \tag{5.18}$$

then

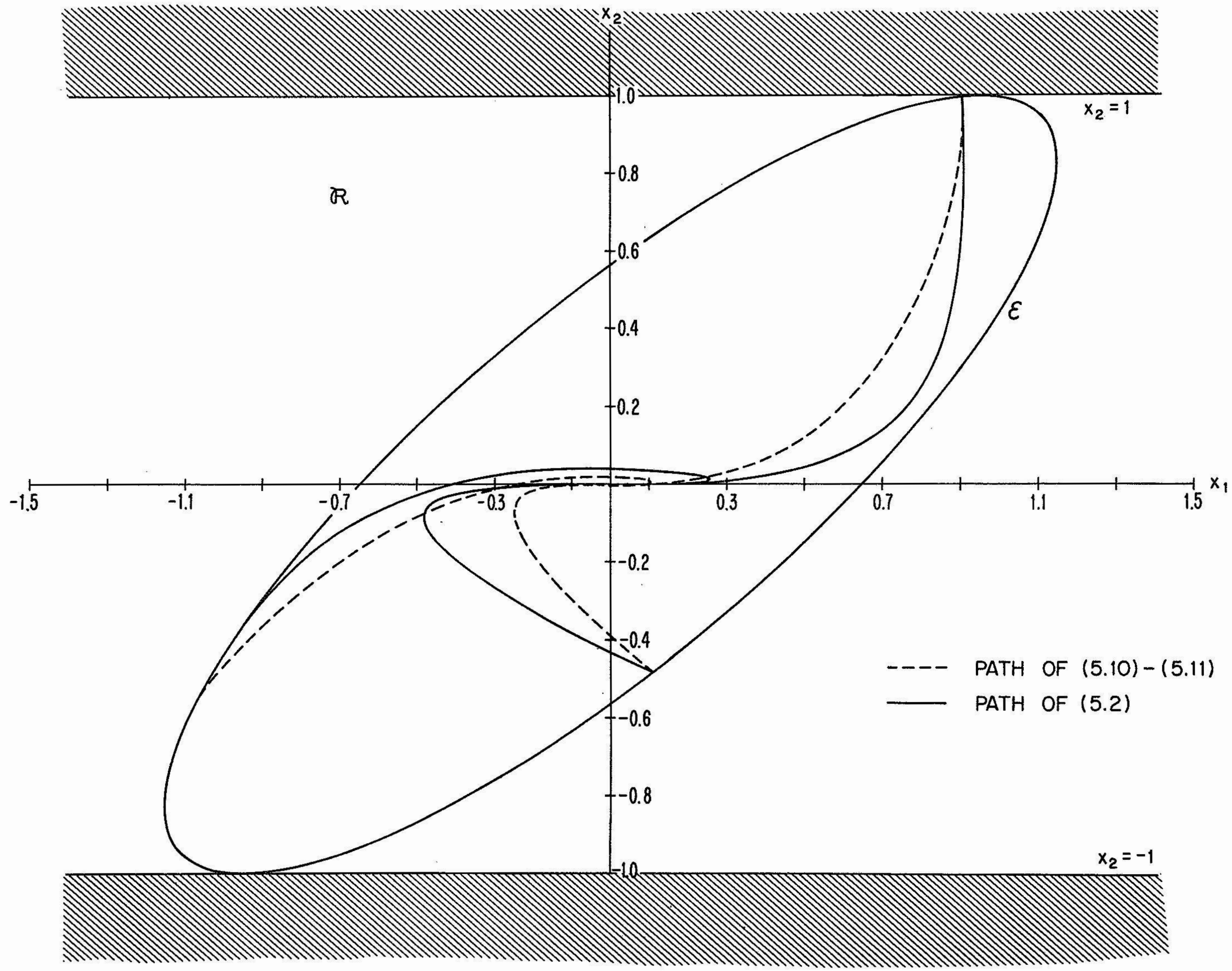

FIG. 2 REGION $\mathcal{R}$, ELLIPSE $\mathcal{E}$ AND TRAJECTORIES OF NONLINEAR SYSTEM (5.10) - (5.11) AND LINEAR SYSTEM (5.2) EXAMPLE 1 , CASE 1 .

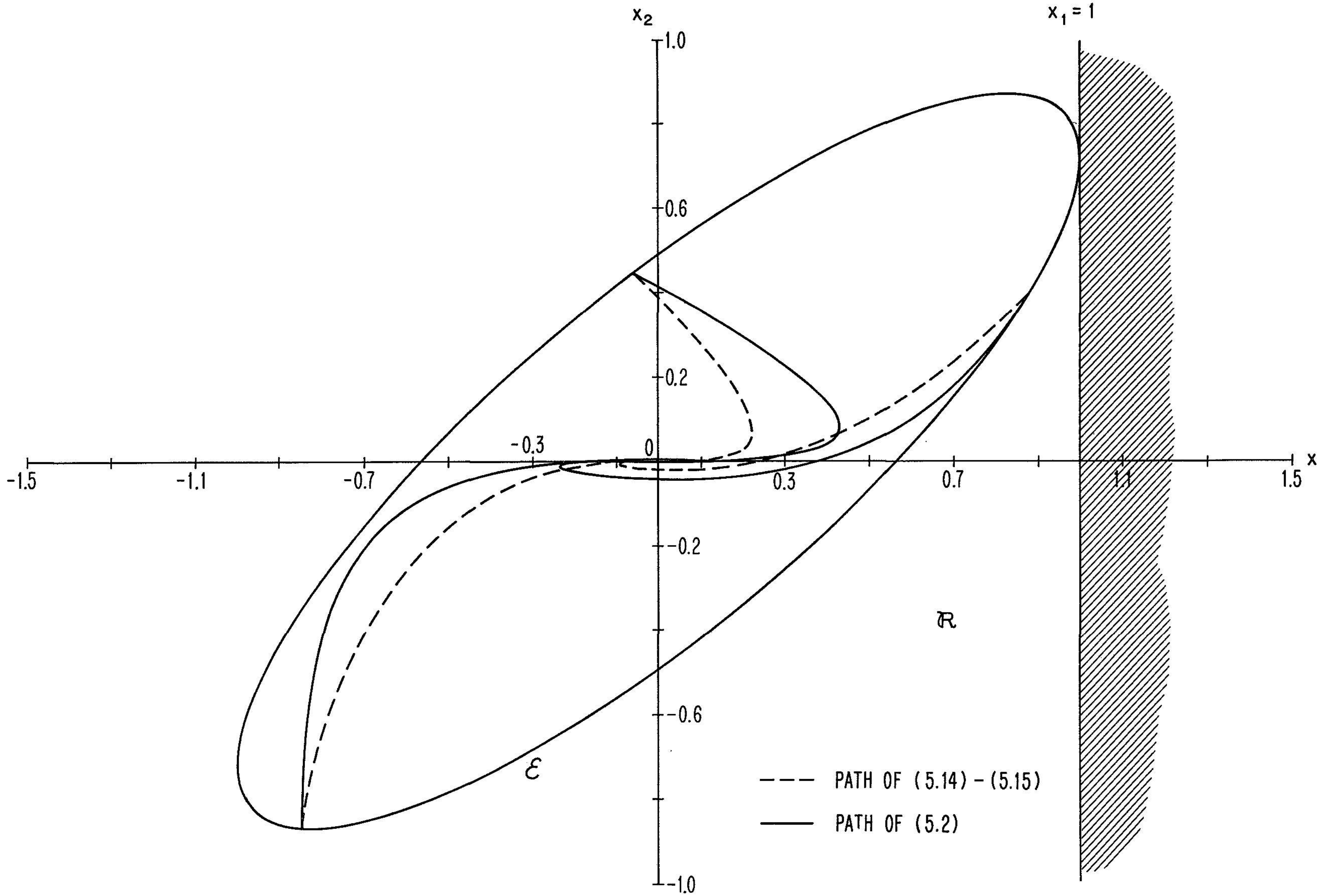

FIG. 3 REGION $\mathcal{R}$, ELLIPSE $\mathcal{E}$ AND TRAJECTORIES OF NONLINEAR SYSTEM (5.14)-(5.15) AND LINEAR SYSTEM (5.2) — EXAMPLE 1, CASE 2

$$\dot{V}(x,t) = 2\,\frac{\det S}{S_{12}}\,x_1^2 p(x,t) \tag{5.19}$$

where $\det S = S_{11}S_{22} - S_{12}^2 > 0$.

In that example $p(x,t) \geq 0$ and $S_{12}(t) < 0$ for all x and t so that Theorem 1 applies and $x(t_f) = 0$.

For the present, let $t_o = 0$, $t_f = 1$, and for illustration purposes let

$$p(x,t) = 1 + \min(x_1, x_2) \quad . \tag{5.20}$$

and let us find a set of initial conditions that can be driven to the origin in the time interval $[0, 1]$. In this case, system (5.1) becomes

$$\dot{x}_1 = u \tag{5.21}$$

$$\dot{x}_2 = -2x_1 - x_2 - x_1 \min(x_1, x_2) \tag{5.22}$$

It is certainly not true that $p \geq 0\ \forall x$. Following the method outlined in Section III, define

$$\mathcal{R} = \{x \mid p(x,t) \geq 0\}$$

$$= \{x \mid a_i^T x + b_i \leq 0,\ i = 1, 2\} \tag{5.23}$$

(in the notation of Section IV), where

$$a_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad b_1 = -1$$

$$a_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad b_2 = -1 \quad .$$

From (4.19) and (4.20), $\epsilon_1 = 1.32$, $\epsilon_{min} = \epsilon_2 = 1$.

Figure 4 shows region $\mathcal{R}$, ellipse $\mathcal{E} = E(S(0), 1)$ and a set of trajectories of (5.21)-(5.22) with control law (5.18).

V.3. Example 3

Consider the dynamic system

$$\dot{x} = Fx + \widetilde{G}(x,t)u \tag{5.24}$$

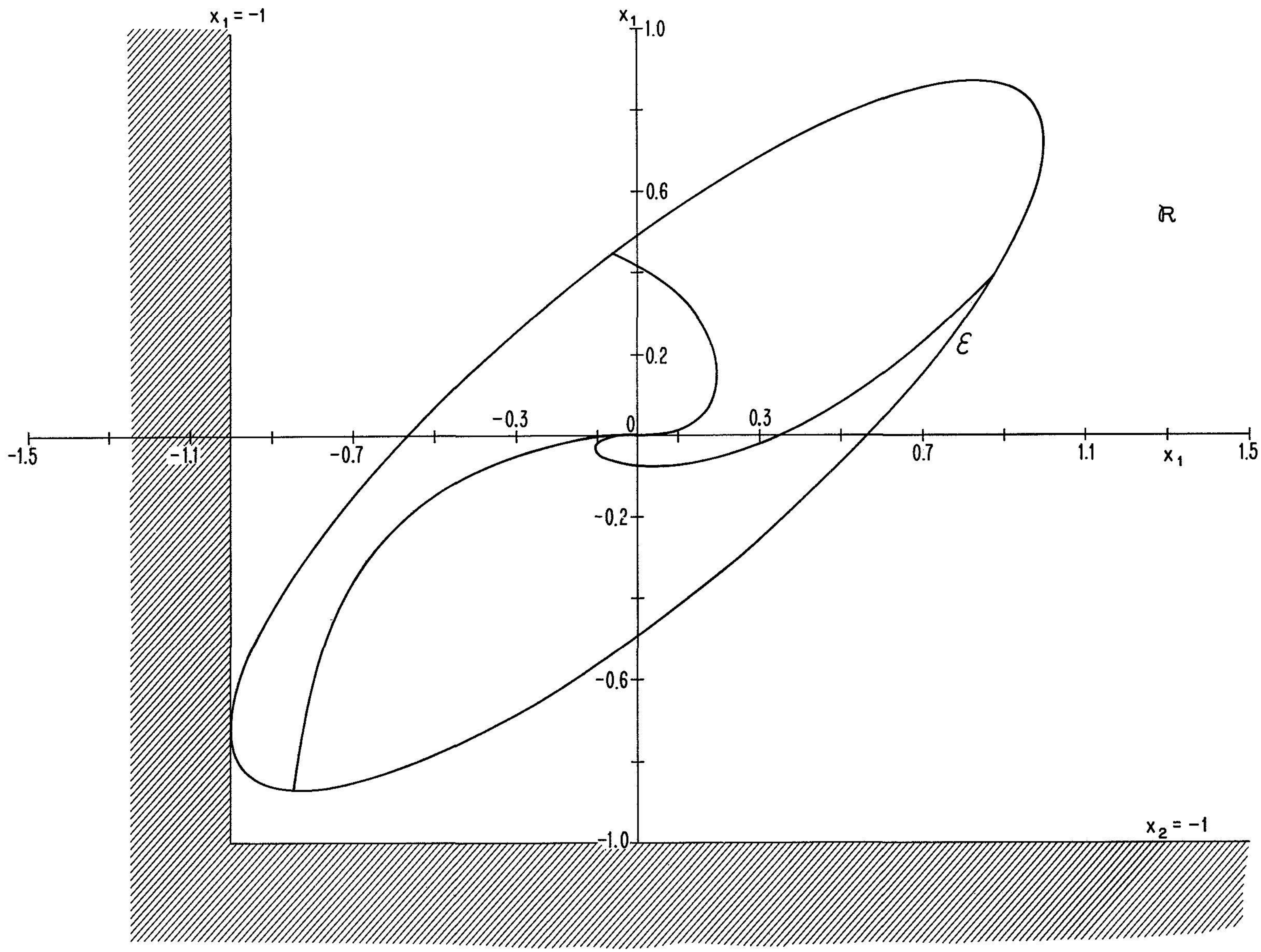

FIG. 4 REGION $\mathcal{R}$, ELLIPSE $\mathcal{E}$ AND SET OF TRAJECTORIES OF EXAMPLE 2.

where F is a constant matrix and $\widetilde{G}(x,t)$ is a matrix function of x and t. Let G be a constant matrix such that the linear system

$$\dot{x} = Fx + Gu \tag{5.25}$$

is completely controllable and define S(t) as the solution to (5.4), (5.5). In [1, Example 2.2.5] Theorem 1 is used to show that if the control

$$u = -\frac{1}{2}G^T S(t)x \tag{5.26}$$

is used over the interval $[t_o, t_f]$ and $\widetilde{G}$, G satisfy

$$\Delta = 2GG^T - \widetilde{G}G^T - G\widetilde{G}^T \leqslant 0 \quad . \tag{5.27}$$

For all x, t, then $x(t_f) = 0$. In this case, $V = x^T S(t)x$ and

$$\dot{V} = \frac{1}{2}x^T S(2GG^T - \widetilde{G}G^T - G\widetilde{G}^T)Sx \quad . \tag{5.28}$$

Now, if matrix Δ is not negative semidefinite for all x and t, the methods of the previous sections may apply. Define region $\mathcal{R}$

$$\mathcal{R} = \{x \,|\, \Delta(x,t) \leqslant 0 \text{ for all } x, t\} \quad . \tag{5.29}$$

According to Section III, if $x_o \in E(S(t_o), \epsilon)$ and ϵ is such that $E(S(t_o), \epsilon) \subset \mathcal{R}$, then controller (5.26) transfers system (5.24) from $x(t_o) = x_o$ to $x(t_f) = 0$.

As a specific case, let $t_o = 0$, $t_1 = 1$,

$$F = \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}$$

$$\widetilde{G} = \begin{pmatrix} 2 + x_1 + x_2 \\ 0 \end{pmatrix} \tag{5.30}$$

System (5.24) becomes

$$\dot{x}_1 = (2 + x_1 + x_2)u \tag{5.31}$$

$$\dot{x}_2 = -x_1 - x_2 \tag{5.32}$$

Let

$$G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

From (5.27),

$$\Delta = -2\begin{pmatrix} 1+x_1+x_2 & 0 \\ 0 & 0 \end{pmatrix} \leqslant 0 \tag{5.33}$$

which is satisfied whenever

$$1 + x_1 + x_2 \geqslant 0 \quad . \tag{5.34}$$

Therefore $\mathcal{R}$ is defined by a linear inequality and the methods of Section IV apply.

$$\mathcal{R} = \{x \,|\, a^T x + b \leqslant 0\}$$

where

$$a = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \qquad b = -1 \quad .$$

From (4.19), (4.20), $\epsilon_{min} = .31$. Figure 5 displays region $\mathcal{R}$, ellipse $\mathcal{E} = E(S(0), .31)$, and trajectories of (5.31)-(5.32).

Note that system (5.31)-(5.32) is a bilinear system, of the form discussed by Rink and Mohler [4].

V.4. Example 4

Consider the problem of driving

$$\dot{x}_1 = (2 + x_1 + x_2)u \tag{5.35}$$

$$\dot{x}_2 = -x_1 - 2x_2 + x_1 x_2 \tag{5.36}$$

from $x = x_o$ to $x = 0$ in the time interval $[0, 1]$. This system combines the least desirable features of systems (5.14)-(5.15) and (5.31)-(5.32).

Let

$$F = \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}, \qquad G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and let S satisfy (5.4), (5.5).

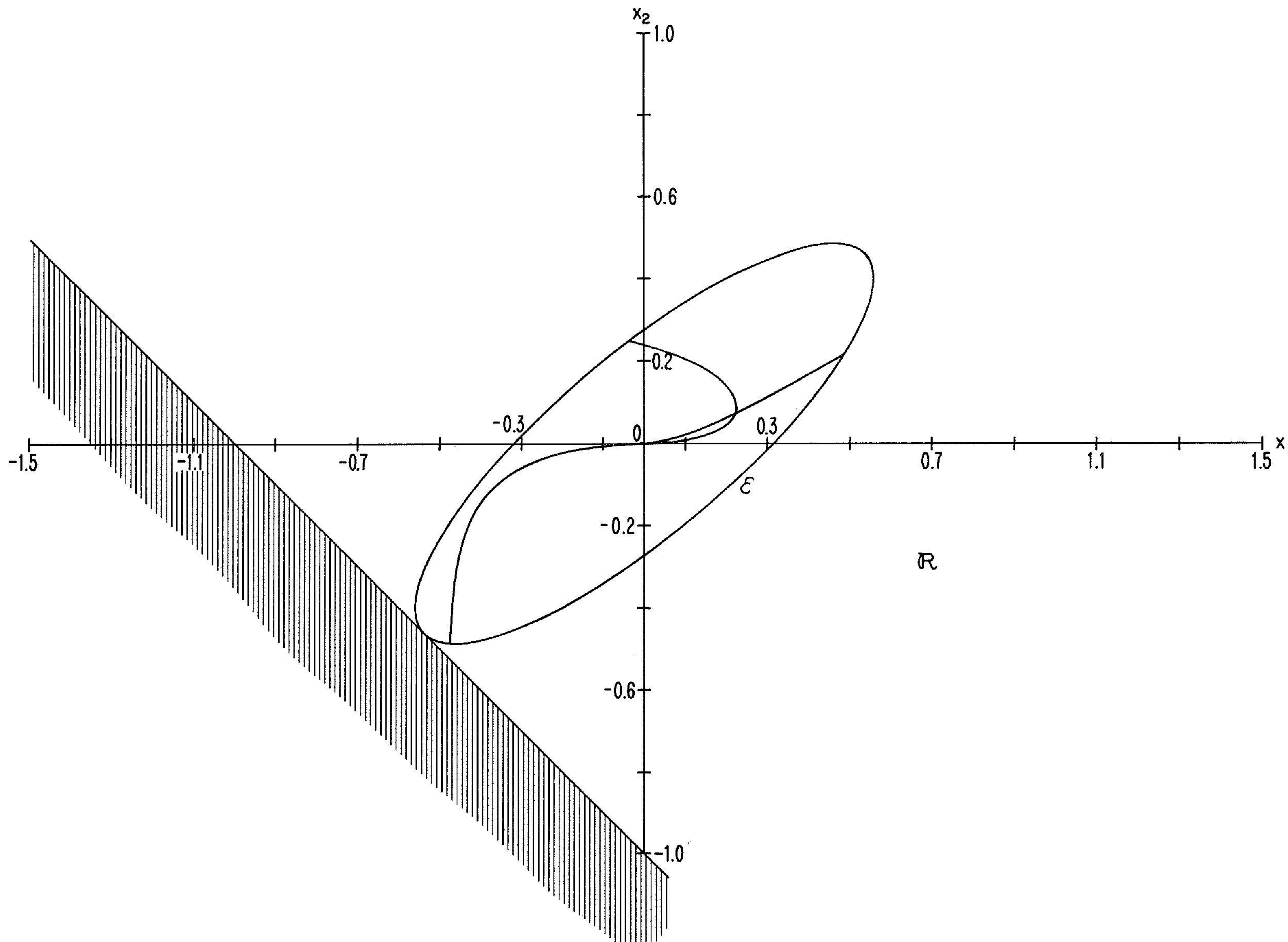

FIG. 5 REGION $\mathcal{R}$, ELLIPSE $\mathcal{E}$ AND SET OF TRAJECTORIES OF EXAMPLE 3 .

Rewrite (5.35)-(5.36) as

$$\dot{x} = Fx + \widetilde{G}u + h(x,t) \tag{5.37}$$

where h is given in Example 1, case 2 and $\widetilde{G}$ is given in Example 3. Let $V(x,t) = x^T S(t)x$ and choose

$$u = -\frac{1}{2}G^T Sx + q(x,t) \quad . \tag{5.38}$$

Then

$$\dot{V} = \frac{1}{2}x^T S\Delta Sx + 2x^T S(\widetilde{G}q + h) \tag{5.39}$$

where

$$\Delta = 2GG^T - \widetilde{G}G^T - G\widetilde{G}^T$$

$$= -2\begin{pmatrix} 1 + x_1 + x_2 & 0 \\ 0 & 0 \end{pmatrix} \tag{5.40}$$

and

$$\widetilde{G}q + h = \begin{pmatrix} (2 + x_1 + x_2)q \\ -x_2(1 - x_1) \end{pmatrix} \quad . \tag{5.41}$$

Consider the following nonlinear control term:

$$q = \frac{-x_1(1 - x_1)}{2 + x_1 + x_2} \quad . \tag{5.42}$$

Ignore, for the moment, the difficulty that arises when the denominator of q is zero. Equation (5.41) becomes

$$\widetilde{G}q + h = -(1 - x_1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{5.43}$$

so the second term of (5.39) is

$$-(1 - x_1)x^T Sx \quad . \tag{5.44}$$

Define region $\mathcal{R}$ in the following way:

$$\mathcal{R} = \{x \mid 1 + x_1 + x_2 \geq 0 \text{ and } 1 - x_1 \geq 0\} \quad . \tag{5.45}$$

$\dot{V} \leq 0$ for all $x \in \mathcal{R}$. The largest ellipse $E(S(0), \epsilon) \subset \mathcal{R}$ can be found by the methods of Section IV. Rewrite

$$\mathcal{R} = \{x | a_i^T x + b_i \leqslant 0, \ i = 1, 2\} \tag{5.46}$$

where

$$a_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \qquad b_1 = -1$$

$$a_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad b_2 = -1$$

From (4.19), (4.20), $\epsilon_1 = .31$, $\epsilon_2 = 1$, $\epsilon_{min} = \epsilon_1$. Then if $x_o \in E(S(0), .31)$ and $u(x,t)$ is given by (5.38) and (5.42), $x(t) \in E(S(0), .31)$, $0 \leqslant t \leqslant 1$ and $x(1) = 0$.

What about the denominator of (5.42)? Clearly, if $x \in E(S(0), .31)$, then $x \in \mathcal{R}$, and $1 + x_1 + x_2 \geqslant 0$. Therefore $2 + x_1 + x_2 \geqslant 1$, so the denominator of (5.42) is never zero.

Figure 6 shows region $\mathcal{R}$, ellipse $\mathcal{E} = E(S(0), .31)$ and several trajectories of (5.35)-(5.36).

V.5. Example 5

There are cases where the states of linear, time-varying systems may be bounded by the techniques used in Section III, even though those results seem to apply only to autonomous systems.

Consider the system

$$\dot{x} = \widetilde{F}(t)x + \widetilde{G}(t)u \quad . \tag{5.47}$$

Find a control to drive the state from $x(t_o) = x_o$ to $x(t_f) = 0$.

Let F and G be constant matrices such that the constant coefficient system

$$\dot{x} = Fx + Gu \tag{5.48}$$

is completely controllable, and define S(t) as the solution to (5.4), (5.5).

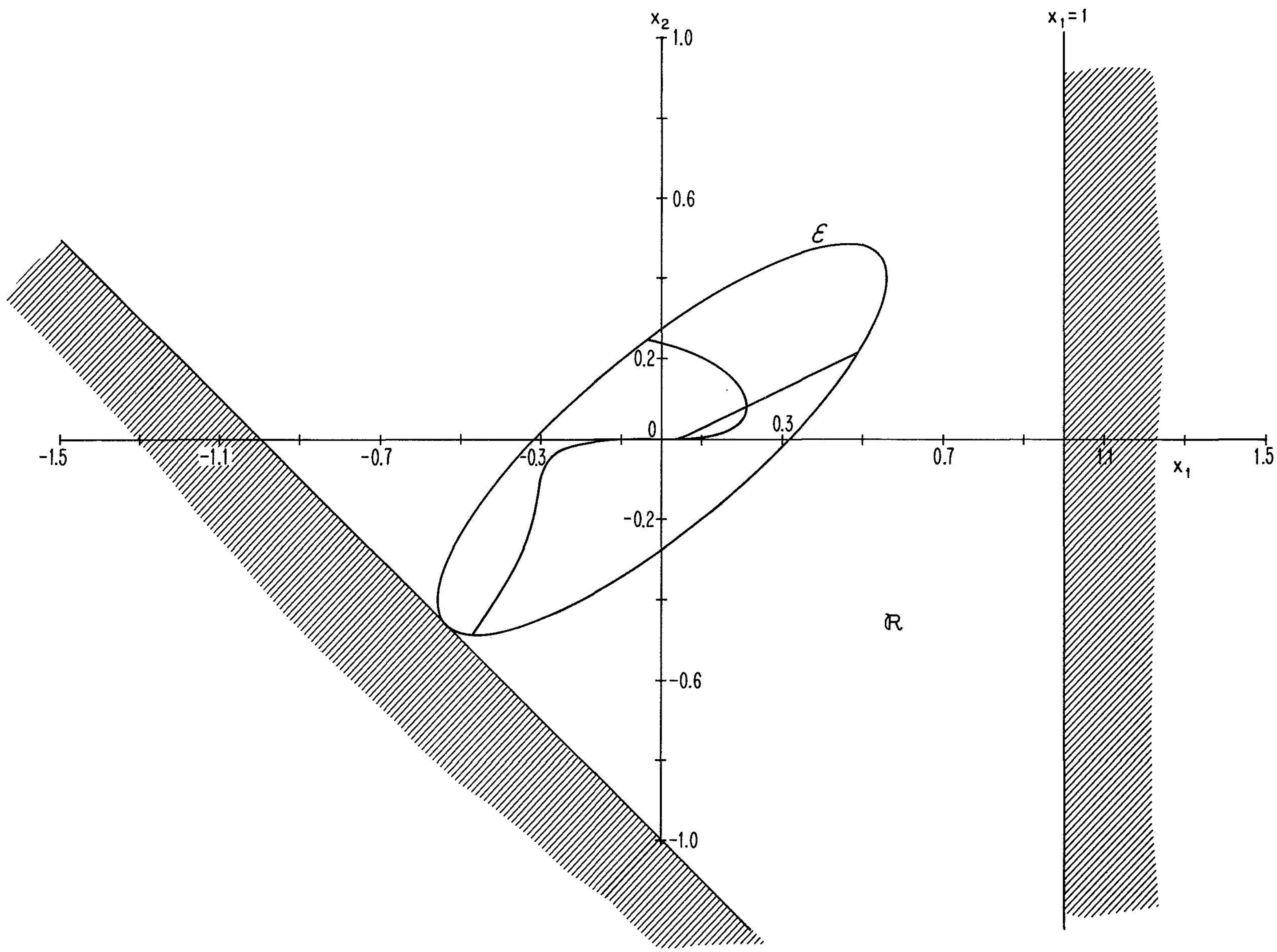

FIG. 6 REGION $\mathcal{R}$, ELLIPSE $\mathcal{E}$ AND SET OF TRAJECTORIES OF EXAMPLE 4 .

where we have used the asymptotic expression for I, and $\langle |E_L(t')|^2\rangle$ is the time average of the laser intensity. The maximum of the exponent occurs at $(t' - t'') = \kappa_1\kappa_2\langle |E_L|^2\rangle z/\Gamma^2$, and the result is essentially the steady state power gain $\exp G_{SS}$ with G_{SS} given by Eq. (14).

If the input Stokes signal $E_S(0,t')$ has a constant phase, and does not follow the phase variations in the laser pump, the integral in Eq. (11a) is reduced by $G_{SS}\Delta\omega/\Gamma$, compared to the case that the phases of the laser and Stokes are in sychronism. This may be seen from the fact that the exponential in Eq. (18) has a 1/e width of G_{SS}/Γ around its maximum, and the laser phases reverse sign about $G_{SS}(\Delta\omega/\Gamma)$ times. The Stokes gain coefficient is thus $G_{SS} - \ell n(G_{SS}\Delta\omega/\Gamma)$. For large steady state gain G_{SS} this reduction is insignificant. The amplified Stokes field "automatically" assumes the correct phase variation for maximum gain.

In figure 12, the result of a numerical calculation is shown for the dispersionless case in which these considerations are confirmed. A Gaussian envelope with a random spectral distribution is taken. The half width of the power spectrum is $\Delta\omega = 20\Gamma$. This corresponds to a stationary random process switched on at $t' = 0$. Since the numerical calculation is possible only for a finite number of Fourier components, the laser pulse shown in figure 12 is assumed to repeat itself with a period of about $800/\Gamma$. The Stokes gain coefficient is calculated from Eq. (11a) as a function of time at a point z, for which $G_{SS} = 46$. The broken line in figure 12 shows the Stokes gain coefficient for a laser with no phase modulation or frequency broadening, switched on at $t' = 0$. The Stokes gain coefficient for the random laser pulse follows essentially the same curve except for a constant factor of about $\ell n(G_{SS}\Delta\omega/\Gamma)$, and except near $z = 0$. Figures 12b and c show that the Stokes amplitude structure follows the variations in the

VI. Extensions

VI. 1. Enlarging the Region of Controllability

Suppose $x_o \in \mathcal{R}$ but x_o is not in ellipse $E(S(t_o), \epsilon)$ for any ϵ such that $E(S(t_o), \epsilon) \subset \mathcal{R}$. This case is not covered by the methods discussed above. In this section we shall develop a method to drive the state from some such x_o to the origin within the time interval $[t_o, t_f]$.

Define

$$\bar{\epsilon}(t) = \arg\max_{\epsilon} \{E(S(t), \epsilon) \subset \mathcal{R}\} \tag{6.1}$$

$$\bar{\mathcal{E}}(t) = E(S(t), \bar{\epsilon}(t)) \tag{6.2}$$

i.e., $\bar{\mathcal{E}}(t)$ is the largest ellipse of the form $\{x | x^T S(t) x \leqslant \epsilon\}$ which is a subset of $\mathcal{R}$. S(t) satisfies (5.4), (5.5) with some appropriately chosen F, G matrices.

Consider the case where $x_o \notin \bar{\mathcal{E}}(t_o)$ but $x_o \in \bar{\mathcal{E}}(t)$ for some t, $t_o < t < t_f$. Define t_1 as the smallest value of t for which $x_o \in \bar{\mathcal{E}}(t)$. Consider the function

$$V(x, t) = x^T \bar{S}(t) x \tag{6.3}$$

where $\bar{S}(t)$ satisfies

$$\lim_{t \to t_f} \bar{S}(t)^{-1} = 0 \tag{6.4}$$

$$\dot{\bar{S}} + \bar{S}F + F^T\bar{S} - \bar{S}GG^T\bar{S} = 0, \qquad t_1 < t < t_f \tag{6.5}$$

$$\bar{S}(t) = \bar{S}(t_1), \qquad t_o \leqslant t \leqslant t_1 \tag{6.6}$$

Note that $\bar{S}(t) = S(t)$, $t_1 \leqslant t < t_f$ and $\bar{S}(t) = S(t_1)$, $t_o \leqslant t \leqslant t_1$.

It is clear that V(x, t) and x_o satisfy conditions (i), (ii), and (iii) of Theorem 1. Consider the control

$$u(x, t) = w(x, \bar{S}(t), t) \tag{6.7}$$

where $w(\cdot)$ is a function such that the control

$$u(x, t) = w(x, S(t), t) \tag{6.8}$$

satisfies Theorem 1 with $V = x^T S(t)x$ for all $x \in \overline{\mathcal{E}}(t_o)$.* Clearly condition (v) of Theorem 1 is satisfied by (6.7) if $x(t)$ is sufficiently close to 0 for $t > t_1$, because for $t > t_1$, (6.7) and (6.8) coincide. Note that the state is sufficiently near the origin if $x(t_1) \in \overline{\mathcal{E}}(t_1)$.

Condition (iv) is satisfied as follows:

$$\dot{V} = x^T(\dot{\overline{S}} + \overline{S}F + F^T\overline{S} - \overline{S}GG^T\overline{S})x + Q(x, t) \tag{6.9}$$

where the derivative is taken with the control given by (6.7), and $Q(x, t)$† is a negative semidefinite function in $\mathcal{R}$.

When $t > t_f$, (6.9) reduces to $\dot{V} = Q(x, t)$ and if $x \in \mathcal{R}$, $\dot{V} \leq 0$. Therefore, if $x(t_1) \in \overline{\mathcal{E}}(t_1)$, $\dot{V} \leq 0$.

For $t_o \leq t \leq t_1$, (6.9) becomes

$$\dot{V} = x^T(\overline{S}F + F^T\overline{S} - \overline{S}GG^T\overline{S})x + Q(x, t) \tag{6.10}$$

From Theorem 2 (2.13) $\dot{\overline{S}}(t) \geq 0$, $t_1 < t < t_f$. Then, from (6.5)

$$\overline{S}F + F^T\overline{S} - \overline{S}GG^T\overline{S} \leq 0 \tag{6.11}$$

for $t_1 < t < t_f$. Inequality (6.11) also holds for $t_o \leq t \leq t_1$ because $\overline{S}$ is defined to be continuous at $t = t_1$ and constant on $[t_o, t_1]$. Therefore $\dot{V} \leq 0$ on $t_o \leq t \leq t_1$ as well as $t_1 < t < t_f$, so condition (iv) of Theorem 1 holds.

By hypothesis, $x_o \in \overline{\mathcal{E}}(t_1)$ and therefore $x_o^T S(t_1)x_o \leq \overline{\epsilon}(t_1)$. For $t \geq t_1$, $\overline{S}(t) = S(t)$ so $V(x_o, t_o) = x_o^T\overline{S}(t_o)x_o = x_o^T\overline{S}(t_1)x_o \leq \overline{\epsilon}(t_1)$. Because

---

*In other words, $w(x, S(t), t)$ is the general form of the control function. For the linear system $\dot{x} = Fx + Gu$, $w(x, S, t) = -\frac{1}{2}G^T Sx$. In Example 1, $w(x, S, t) = -\frac{1}{2}G^T Sx - p(x, t)x_1$ (from (5.3)). Equation (6.7) says to use $\overline{S}$ instead of S.

†$Q(x, t)$ is due to the nonlinear part of the dynamics. For a linear system, $Q = 0$. In Example 1, $Q = -2p(x, t)x^T\overline{S}x$. In Example 3, $Q = \frac{1}{2}x^T\overline{S}(2GG^T - G\widetilde{G}^T - \widetilde{G}G^T)\overline{S}x$.

$\dot{V} \leqslant 0$, $V(x(t), t) \leqslant V(x_o, t_o) \leqslant \bar{\epsilon}(t_1)$ for all $t \in [t_o, t_f]$. In particular, $V(x(t_1), t_1) = x(t_1)^T S(t_1) x(t_1) \leqslant \bar{\epsilon}(t_1)$. Therefore $x(t_1) \in \bar{\mathcal{E}}(t_1)$, and Theorem 1 holds.

Also, Lemma 2 and Theorem 3 hold for $t \in [t_1, t_f)$.

To summarize, consider the problem of driving the system

$$\dot{x} = f(x, u, t)$$

from $x(t_o) = x_o$ to $x(t_f) = 0$ such that $x(t) \in \mathcal{R}_1$ for all t. Define S(t) to be the solution to (5.4), (5.5) for some F, G matrices. Define $V = x^T S(t) x$ and let $u = w(x, S(t), t)$ be such that

$$\dot{V} \leqslant 0 \tag{6.13}$$

for all $x \in \mathcal{R}_2$ for all $t \in [t_o, t_f]$. Let $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$ and define $\bar{\mathcal{E}}(t)$ as in (6.1), (6.2). If there exists some $t_1 \in (t_o, t_f)$ such that $x_o \in \bar{\mathcal{E}}(t_1)$, define $\bar{S}(t)$ as in (6.4), (6.5), (6.6). The control $u = w(x, \bar{S}(t), t)$ is such that $x(t_f) = 0$ and $x(t) \in \mathcal{R}$ for all $t \in [t_o, t_f]$.

The region of controllability, i.e. the set of all x_o such that a control exists to drive the system from x_o to 0 in $[t_o, t_f]$ and $x(t) \in \mathcal{R}_1$, is thus a set that contains the following set as a subset

$$C = \bigcup_{t \in [t_o, t_f)} \bar{\mathcal{E}}(t) \tag{6.14}$$

because if $x_o \in C$, some t_1 exists such that $x_o \in \bar{\mathcal{E}}(t_1)$.

VI. 2. Example 6

Consider Example 1, case 1. From Figure 2, it is apparent that the point

$$x_o = \begin{pmatrix} a \\ 0 \end{pmatrix} \qquad a > .65 \tag{6.15}$$

is not covered by the analysis of that example. However, we shall construct a controller to drive system (5.10)-(5.11) from x_o to the origin in the time interval [0, 1] using the method of Section VI. 1.

The first step is to find t_1 such that $x_o \in \overline{\mathcal{E}}(t_1)$ and $\overline{\mathcal{E}}(t_1) \subset \mathcal{R}$, where $\mathcal{R}$ is given by (5.12). This is equivalent to the problem discussed and solved in Section IV.1. We ask "Is $E(S(t), x_o^T S(t) x_o) \subset \mathcal{R}$?" for each t starting at $t = t_o$. Define the first value of t for which the answer is "yes" as t_1.

From (4.8),

$$\epsilon = x_o^T S(t) x_o = S_{11}(t)\alpha^2 \quad . \tag{6.16}$$

Note that $a_1 = -a_2 = (0,1)^T$ and $b_1 = b_2 = -1$. Then if $Z(t) = S^{-1}(t)$, $a_i^T S^{-1}(t) a_i = Z_{22}(t)$ and, from (4.16), $C_1(t) = C_2(t) =$

$$C(t) = \sqrt{\alpha^2 S_{11}(t) Z_{22}(t)} - 1 \tag{6.17}$$

As long as $C(t) > 0$, the answer to the above question is "no". The first time* at which $C(t) \leq 0$ is the time t_1.

If $\alpha = 5$, $t_1 = .84$ (when $[0,1]$ is discretized into 100 subintervals). The control law for system (5.2) is

$$u = -\frac{1}{2} G^T \overline{S}(t) x \tag{6.18}$$

where $\overline{S}(t)$ is given by (6.4), (6.5), (6.6). In this case, of course, the restriction that $x(t) \in \mathcal{R}$ is a state constraint; it need not be satisfied to guarantee that $x(t_f) = 0$. The solid trajectories in Figure 7 are from system (5.2). The trajectory that leaves $\mathcal{R}$ is controlled by

$$u = -\frac{1}{2} G^T S(t) x \tag{6.19}$$

where $S(t)$ is given by (5.4), (5.5). The solid line that stays inside $\mathcal{R}$ is the trajectory of (5.2) that starts at $x_o^T = (5, 0)$ and is controlled by (6.18).

The dashed line is the trajectory of (5.10)-(5.11) driven by

$$u = -\frac{1}{2} G^T \overline{S}(t) x - (1 - x_1^2) x_1 \tag{6.20}$$

---

*Note that t_1 is not critical. Any $t_1^* \geq t_1$ will work equally well in the control law described in Section IV.1.

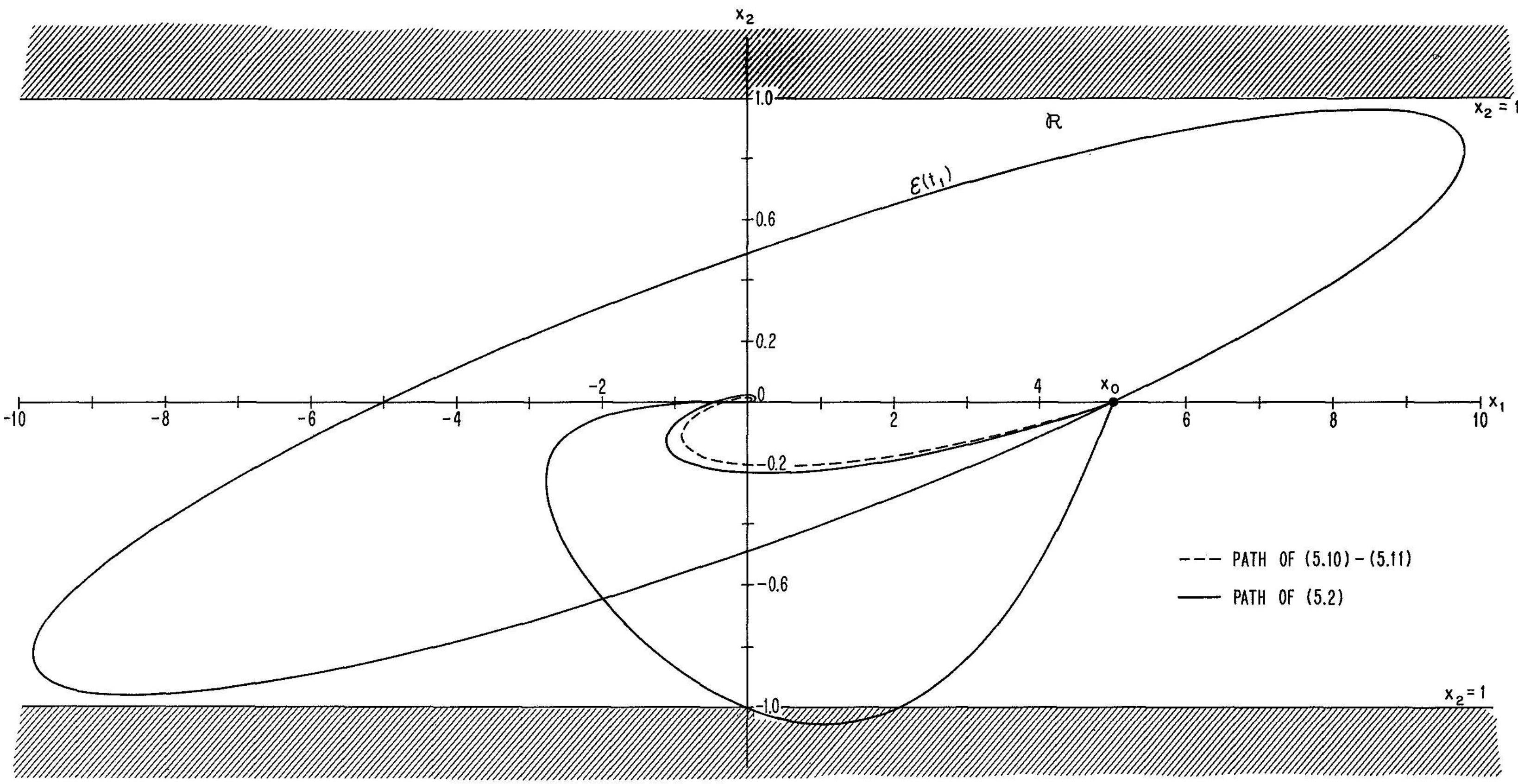

FIG. 7 REGION $\mathbb{R}$, ELLIPSE $\mathcal{E}(t_1)$, CONSTRAINED TRAJECTORIES OF (5.2), (5.10) - (5.11), AND AN UNCONSTRAINED TRAJECTORY OF (5.2).

starting at $x(0)^T = (5, 0)$.

It appears that it is not necessary to use this method for all initial conditions in set C (6.14) which are not in $\overline{\mathcal{E}}(t_o)$. For instance, Figure 2 and Figure 7 both seem to indicate that if $x_o^T = (\alpha, 0)$ and $|\alpha| < 4$, the trajectory of (5.2) stays inside $\mathcal{R}$ even though the control used is (6.19). Furthermore, this also may be true for the trajectory of (5.11)-(5.12) with control $u = -\frac{1}{2}G^T S(t)x - (1 - x_1^2)x_1$. Further research is required in order to develop methods for characterizing these initial conditions.

VI. 3. Application to Stability Theory

Theorem 4: Let $V(x, t)$ and $u^*(x, t)$ satisfy Theorem 1 for

$$\dot{x} = f(x, u) \tag{6.21}$$

such that $\dot{V} \leq 0$ over the interval $[t_o, t_f)$ for all $x \in \mathcal{R}$. Let $t_1 \in [t_o, t_f)$ be such that

$$V(0, t_1) = 0 \tag{6.22}$$

$$x \in \mathcal{R} \implies V(x, t_1) > 0 \iff x \neq 0 \tag{6.23}$$

$$\frac{\partial V}{\partial t}(x, t_1) \geq 0 \quad \text{for all } x \in \mathcal{R} \quad . \tag{6.24}$$

Then the system of differential equations

$$\dot{x} = f(x, u^*(x, t_1)) \tag{6.25}$$

is stable about $x = 0$.

Proof: Define $W(x) = V(x, t_1)$. Then

$$\dot{W} = W_x \dot{x} = V_x(x, t_1) f(x, u^*(x, t_1)) \quad . \tag{6.26}$$

Also,

$$\dot{V}(x, t) = V_t(x, t) + V_x(x, t) f(x, u^*(x, t)) \leq 0 \tag{6.27}$$

so

$$V_x(x, t) f(x, u^*(x, t)) \leq -V_t(x, t) \quad . \tag{6.28}$$

Evaluating (6.28) at $t = t_1$,

$$V_x(x, t_1)f(x, u^*(x, t_1)) \leqslant -\frac{\partial V}{\partial t}(x, t_1) \leqslant 0 \tag{6.30}$$

Comparing (6.30) and (6.26), we see that $\dot{W} \leqslant 0$, so that W is a Liapunov function and therefore (6.25) is stable [7].

Note that if $\dot{V}(x, t) - \partial V(x, t)/\partial t < 0$ at $t = t_1$ for all non-zero $x \in \mathcal{R}$, then (6.25) is asymptotically stable.

Theorem 5: Let $V(x, t) = x^T S(t)x$ satisfy Theorem 1 for

$$\dot{x} = f(x, u, t) \tag{6.31}$$

$$x(t_o) = x_o, \qquad x(t_f) = 0 \tag{6.32}$$

with

$$u = w(x, S(t), t) \tag{6.33}$$

for any x in some region $\mathcal{E}$, where S(t) satisfy the usual Riccati equation (5.4) and boundary conditions (5.5). Assume F and G are constant. Assume that condition (iv) of Theorem 1 is satisfied as follows:

$$\dot{V} = x^T(\dot{S} + SF + F^T S - SGG^T S)x + Q(x, S(t), t) \tag{6.34}$$

where the first term is zero because S(t) satisfies the Riccati equation and where it is required that

$$Q(x, S(t_1), t) \leqslant 0 \tag{6.35}$$

for some $t_1 \in [t_o, t_f)$, for all $x \in \mathcal{E}$. Then system (6.31) is stabilized by

$$u = w(x, S(t_1), t) \quad . \tag{6.36}$$

A Liapunov function for the stabilized system is

$$W(x) = V(x, t_1) = x^T S(t_1)x \quad . \tag{6.37}$$

Proof: In the problem of solving (6.31), (6.32), the full time derivative of V is

$$\dot{V} = x^T \dot{S}x + 2x^T Sf(x, w(x, S, t), t) \quad . \tag{6.38}$$

Comparing (6.38) and (6.34),

$$Q(x, S, t) = 2x^T Sf(x, w(x, S, t), t) - x^T(SF + F^T S - SGG^T S)x \quad . \tag{6.39}$$

Comparing (6.39) and (6.35),

$$Q(x, S_1, t) = 2x^T S_1 f(x, w(x, S_1, t), t) - x^T(S_1 F + F^T S_1 - S_1 GG^T S_1)x$$
$$\leq 0 \tag{6.40}$$

(where $S_1 = S(t_1)$), or

$$2x^T S_1 f(x, w(x, S_1, t), t) \leq x^T(S_1 F + F^T S_1 - S_1 GG^T S_1)x \quad . \tag{6.41}$$

Now calculate $\dot{W}$ for the system with u given by (6.36):

$$\dot{W} = 2x^T S_1 f(x, w(x, S_1, t), t) \quad . \tag{6.42}$$

From Theorem 2,

$$S_1 F + F^T S_1 - S_1 GG^T S_1 \leq 0 \quad . \tag{6.43}$$

Then, by comparing (6.41), (6.42), (6.43), we see that

$$\dot{W} \leq 0$$

and the theorem is proved.

Consider a problem of Julich [5]: to find out if a given system is stable and if all trajectories that start in a region $\mathfrak{R}'$ stay in that region for all t. System (6.25) is stable in that way if $W(x) = V(x, t_1) \leq \epsilon \Longrightarrow x \in \mathfrak{R} \cap \mathfrak{R}'$ for some real number ϵ (i.e., $E(S(t_1), \epsilon) \subset \mathfrak{R} \cap \mathfrak{R}'$).

We have already (in Sections VI. 1, VI. 2) made use of Theorem 5 in the interval $[t_o, t_1]$. When F and G are constant, it is clear that Theorem 4 is satisfied by $V = x^T S(t)x$. All the systems in this paper that are controlled (by some control $u = w(x, S(t), t)$) can be stabilized (by $u = w(x, S(t_1), t)$ for any t_1).

Kalman [2] has shown conditions under which it is possible to stabilize

$$\dot{x} = Fx + Gu \tag{6.44}$$

by integrating a Riccati equation

$$\dot{S} + SF + F^T S - SGB^{-1} G^T S = -A \tag{6.45}$$

backwards from $t = \infty$ to a finite value of t. (In practice, one integrates

(6.45) backwards from a finite value of t to a value of t such that S(t) has settled down to a "steady state" solution* -- i.e., $\dot{S} = 0$.) A and B are positive definite matrices. (Actually, the conditions on A and B are somewhat more restrictive.)

By contrast, we stabilize (6.44) in the following manner: integrate

$$\dot{Z} - FZ - ZF^T + GG^T = 0 \tag{6.46}$$

with boundary condition

$$Z(t_f) = 0 \tag{6.47}$$

until $t = t_1 < t_f$, where t_1 and t_f are finite times. Define $S(t_1) = Z(t_1)^{-1}$. Use control

$$u = -\frac{1}{2}G^T S(t_1)x \quad . \tag{6.48}$$

It should be pointed out that we have only stabilized (6.44); we may not have rendered it asymptotically stable. Theorem 2 only guarantees that $\dot{S}(t_1) \geq 0$, so that $S(t_1)F + F^T S(t_1) - S(t_1)GG^T S(t_1) \leq 0$. Then if $V = x^T S(t_1)x$, $\dot{V} = x^T(S(t_1)F + F^T S(t_1) - S(t_1)GG^T S(t_1))x \leq 0$. If some t exists such that $\dot{S}(t) > 0$, we may use that as t_1 and thereby guarantee that $\lim_{t \to \infty} x(t) = 0$.

As Kalman did, we calculate our stabilizing control by integrating an n x n matrix differential equation. However, we integrate over a finite interval, and thereby save computer time. The shortcomings of this method are that (1) it is only guaranteed for autonomous systems and (2) it may only result in non-asymptotic stability unless the user verifies that $SF + F^T S - SGG^T S < 0$.

---

*We restrict ourselves to the constant coefficient case. Kalman allows F, G, A, and B to vary with time.

Of course, we are not restricted to linear systems. We may apply the techniques of this paper and [1] to control some time-varying systems and some nonlinear systems. Similar results have been obtained by Barnett and Storey [6].

VII. Conclusion

In [1], Theorem 1 was used to solve the following problem (Problem A) for various cases of system dynamics (7.3): find a control law $u(x, t)$ such that

$$x(t_o) = x_o \tag{7.1}$$

$$x(t_f) = 0 \tag{7.2}$$

$$\dot{x} = f(x, u, t) \quad . \tag{7.3}$$

Theorem 1 requires a function $V(x, t)$ to exist and have certain properties. The control function $u(x, t)$ is such that

$$\dot{V}(x, t) \leqslant M < \infty \tag{7.4}$$

(where M is a constant) on the trajectory of (7.3) starting at $x(t_o) = x_o$.

In all the applications of Theorem 1 in [1],

$$V = x^T S(t) x \tag{7.5}$$

where $S(t)$ satisfies

$$\lim_{t \to t_f} S(t)^{-1} = 0 \tag{7.6}$$

$$\dot{S} + SF + F^T S - SGG^T S = 0 \quad . \tag{7.7}$$

The analogy between Theorem 1 and Liapunov stability theory [7] is clear. In the latter, the uncontrolled system

$$\dot{x} = g(x, t) \tag{7.8}$$

is stable (asymptotically stable) about the origin if a function $V(x, t)$ exists which is positive definite and whose derivative satisfies $\dot{V} \leqslant 0$ ($\dot{V} < 0$).

The results of [1] have been extended in the following ways.

1. Consider Problem B: satisfy (7.1), (7.2), (7.3) and

$$x(t) \in \mathfrak{R} \ , \qquad t_o \leqslant t \leqslant t_f \tag{7.9}$$

where $\mathfrak{R}$ is some region in n-space.

When (7.4) is satisfied with M = 0 and (7.7) is satisfied with constant F and G matrices, then it is shown in Section II that for $t_o < t \leqslant t_f$, $x(t) \in \mathcal{E}(t_o)$, where

$$\mathcal{E}(t_o) = \{z \,|\, z^T S(t_o) z \leqslant x_o^T S(t_o) x_o\} \tag{7.10}$$

so that if $\mathcal{E}(t_o)$ is a subset of $\mathfrak{R}$ the control function found for Problem A solves Problem B.

2. Controls are found above to solve Problem A for more general systems (7.3) then in [1]. In [1], some inequality $p(x, t) \geqslant 0$ is required to hold for all x and all t, $t_o \leqslant t \leqslant t_f$. By contrast in this paper, $p(x, t) \geqslant 0$ (for all t) defines a region $\mathfrak{R}$. If (7.9) as well as (7.1), (7.2), (7.3) is required, we have a type B problem, to which the above technique may be applied. If it has a solution, the type A problem ((7.1), (7.2), (7.3)) has the same solution.

Several examples were performed to illustrate these techniques on type A and B problems. The type A problems considered in this paper cannot be solved by the methods given in [1].

3. In Section VI, the condition $\mathcal{E}(t_o) \subset \mathfrak{R}$ is weakened as follows. With a small modification in the control law, any x_o that satisfies

$$\mathcal{E}(t) = \{z \,|\, z^T S(t) z \leqslant x_o^T S(t) x_o\} \subset \mathfrak{R} \tag{7.11}$$

for <u>some</u> $t \in [t_o, t_f)$ can be driven to the origin. If some ellipse $x^T S(t) x = \epsilon$ passes through x_o and lies inside $\mathfrak{R}$ then (with the suitably modified control function) $x(t_f) = 0$, and the trajectory satisfies $x(t) \in \mathfrak{R}$.

Also in Section VI the relationship between Theorem 1 and Liapunov stability theory is demonstrated. For a large class of V-functions satisfying Theorem 1 (including $V = x^T S(t) x$) and a large class of systems (7.3), a control function $\bar{u}(x,t)$ to stabilize the system is closely related to $u^*(x,t)$, a control function obtained from Theorem 1. In that case a Liapunov function for (7.3) with $u = \bar{u}(x,t)$ is $V(x, t_1)$, where t_1 is some fixed time.

Several areas of further research present themselves. Among them are the following questions.

1. Can any further statements be made about the trajectories beyond those of Section II? As pointed out in Section VI.2, it is probably not necessary to resort to the technique of VI.1 for all $x_o \in \mathcal{R}$ where $\mathcal{E}(t_o)$ is not a subset of $\mathcal{R}$. In other words, there are probably many $x_o \in \mathcal{R}$ where the trajectory generated by naively applying Theorem 1 behaves properly. How can these x_o be characterized?
2. For the type A problems as generalized herein, under what conditions can the state venture out of $\mathcal{R}$ (i.e., go to where $p(x,t) < 0$) and still satisfy $x(t_f) = 0$?
3. Generally, the F, G matrices chosen to form the "linear part" of the dynamics and thus to enter the Riccati equation are not unique. In most cases, any of a large set of such matrices would be appropriate. Is it possible to be "best" in some sense? For example, can we choose the F, G matrices to maximize the volume of $E(S(t_o), \epsilon)$ (which must be a subset of $\mathcal{R}$)? Note that this is not the same problem as in Section IV, where ϵ was chosen. Here we would like to manipulate $S(t_o)$.
4. For a given region $\mathcal{R}$, how can set C (6.14) be characterized? For instance, if $\mathcal{R} = \{x \,\big|\, |x_n| \leq 1\}$, under what conditions, if any, does C

include <u>all</u> of $\mathfrak{R}$? Also, because C is a union of ellipses, if $x \in C$, then $-x \in C$. What other properties of C can be determined?

REFERENCES

[1] S. B. Gershwin and D. H. Jacobson, A Controllability Theory for Nonlinear Systems with Applications, Harvard University Technical Report No. 592, July, 1969.

[2] R. E. Kalman, Contributions to the Theory of Optimal Control, Boletin de la Sociedad Matematica Mexicana, 2nd Series, Vol. 5, 1960, pp. 102-119.

[3] A. E. Bryson and Y. C. Ho, Applied Optimal Control, Blaisdell, 1969.

[4] R. E. Rink and R. R. Mohler, Completely Controllable Bilinear Systems, SIAM J. Control, Vol. 6, No. 3, 1968.

[5] P. M. Julich, Region of Acceptable Motions, IEEE Trans. Automatic Control, Vol. AC-13, No. 6, December 1968.

[6] S. Barnett and C. Storey, Some Result on the Sensitivity and Synthesis of Asymptotically Stable Nonlinear Systems, Automatica, Vol. 4, pp. 187-194, 1968.

[7] J. P. Lasalle and S. Lefschetz, Stability by Liapunov's Direct Method with Applications, Academic Press, 1961.

Academy Library (DFSLB)
U. S. Air Force Academy
Colorado Springs, Colorado 80840

Hq. AEDC (AETS)
Attn: Library/Documents
Arnold AFB, Tenn. 37389

Aeronautics Library
Graduate Aeronautical Laboratories
California Institute of Technology
1201 E. California Blvd
Pasadena, California 91109

Aerospace Corporation
P. O. Box 95085
Los Angeles, California 90045
Attn: Library Acquisitions Group

Airborne Instruments Laboratory
Deerpark, New York 11729

AFAL (AVTA/R. D. Larson)
Wright-Patterson AFB
Ohio 45433

AFCRL (CRMPLR)
AFCRL Research Library, Stop 29
L. G. Hanscom Field
Bedford, Mass. 01731

ADTC (ADBPS-12)
Eglin Air Force Base
Florida 32542

AFETR Technical Library
(ETV, MU-135)
Patrick AFB, Florida 32925

Atmospheric Sciences Office
Atmospheric Sciences Laboratory
White Sands Missile Range
New Mexico 88002

ADTC (ADBPS-12)
Eglin AFB
Florida 32542

Case Institute of Technology
Engineering Division
University Circle
Cleveland, Ohio 44106

AULJT-9663
Maxwell AFB
Alabama 36112

Commanding Officer
Naval Weapons Center
Corona Laboratories
Attn: Library
Corona, California 91720

Commander
U. S. Naval Missile Center
Point Mugu, California 93041

Deputy for Research and Engineering
(AMSWE-DRE)
U. S. Army Weapons Command
Rock Island Arsenal
Rock Island, Illinois 61201

Hollander Associates
P. O. Box 2276
Fullerton, California 92633

Carnegie Institute of Technology
Electrical Engineering Dept.
Pittsburg, Pa. 15213

Central Intelligence Agency
Attn: OCR/DD Publications
Washington, D. C. 20505

Dr. G. M. Janney, AMSEL-HL-NVOR
Night Vision Laboratory, USAECOM
Fort Belvoir, Virginia 22060

Director, Electronic Programs
Department of the Navy
Washington, D. C. 20360
Attn: Code 427 (3)

Commandant
U. S. Army and General Staff College
Attn: Acquisitions, Lib. Div.
Fort Leavenworth, Kansas 66370

Commander (ADL)
Naval Air Development Center
Johnsville,
Warminster, Pa. 18974

Technical Director (SMUFA-A2000-107-1)
Frankford Arsenal
Philadelphia, Pa. 19137

Commandant
U. S. Army Air Defense School
Attn: Missile Sciences Div. C & S Dept.
P. O. Box 9390
Fort Bliss, Texas 79916

U. S. Army Mobility Equipment Research
and Development Center
Attn: Technical Document Center, Bldg. 315
Fort Belvoir, Virginia 22060

Commanding General
Attn: STEWS-RE-L, Technical Library
White Sands Missile Range
New Mexico 88002 (2)

Commanding General
U. S. Army Electronics Command
Fort Monmouth, New Jersey 07703

ATTN: AMSEL-SC
NL-P-2 (Mr. D. Harats)
NL-SM (Drs. Schiel/
RD-GF Hieslmair)
RD-MT
XL-D
XL-E
XL-C
XL-S (Dr. R. Buser)
HL-CT-DD
HL-CT-R
VL-F (Mr. R. J. Niemela)
HL-CT-L (Dr. W. S.
HL-CT-O McAfee)
HL-CT-I
HL-CT-A
NL-D
NL-A
NL-P
NL-R (Mr. R. Kulinyi)
NL-S
KL-D
KL-E
KL-S (Dr. H. Jacobs)
KL-T
VL-D
WL-D

Commanding General
U. S. Army Material Command
Attn: AMCRD-TP
Washington, D. C. 20315

Commanding General
U. S. Army Missile Command
Attn: Technical Library AMSMI-RBX
Redstone Arsenal, Alabama 35809

Commanding Officer
Naval Avionics Facility
Indianapolis, Indiana 46241

Commander
Naval Ordnance Laboratory
White Oak, Maryland 21502 (2)
Attn: Librarian

Commander (Code 753)
Naval Weapons Center
China Lake, California 93555
Attn: Technical Library

Commanding Officer
Naval Training Device Center
Orlando, Florida 32811

Deputy Director and Chief Scientist
Office of Naval Research Branch Office
1030 East Green Street
Pasadena, California 91101

Commanding Officer
Office of Naval Research Branch Office
219 South Dearborn Street
Chicago, Illinois 60604

Commanding Officer
Office of Naval Research Branch Office
495 Summer Street
Boston, Massachusetts 02210

U. S. Post Office Department
Library - Room 6012
12th & Pennsylvania Ave., N. W
Washington, D. C. 20260

Commanding Officer
Office of Naval Research Branch Office
Box 39, FPO
New York, New York 09510 (2)

Project Manager
Common Positioning and Navigation
Systems
Attn: Harold M. Bahr (AMCPM-NS-TM)
Building 439
U. S. Army Electronics Command
Fort Monmouth, New Jersey 07703

Commanding Officer
U. S. Army Electronics R & D Activity
White Sands Missile Range
New Mexico 88002

Director of Faculty Research
Department of the Air Force
USAF Academy
Colorado Springs, Colorado 80840

Richard O. Ulsh (CRDARD-IPO)
U. S. Army Research Office (Durham)
Box CM, Duke Station
Durham, North Carolina 27706

Commanding General
U. S. Army Strategic Comm. Command
Attn: SCC-CG-SAE
Fort Huachuca, Arizona 85613

Commanding Officer
Harry Diamond Laboratories
Attn: Dr. Berthold Altman (AMXDO-TD)
Connecticut Avenue and Van Ness St. NW
Washington, D. C. 20438

Commanding Officer
Human Engineering Laboratories
Aberdeen Proving Ground
Maryland 21005

Commanding Officer
U. S. Army Ballistics Research Lab
Attn: AMXRD-BAT
Aberdeen Proving Ground
Aberdeen, Maryland 21005

Director, USAF Project RAND
Via: Air Force Liaison Office
The RAND Corporation
1700 Main Street
Santa Monica, California 90406
Attn: Library D

Director
U. S. Army Engineer Geodesy,
Intelligence and Mapping
Research and Development Agency
Fort Belvoir, Virginia 22060

Dr. G. M. R. Winkler
Director, Time Service Division
U. S. Naval Observatory
Washington, D. C. 20390

Commander, U.S. Naval Security Group Com
Attn: G43
3801 Nebraska Avenue
Washington, D. C. 20390

Division of Engineering and Applied Physics
130 Pierce Hall
Harvard University
Cambridge, Massachusetts 02138

Hq. ESD (ESTI)
L. G. Hanscom Field
Bedford, Mass. 01731 (2)

European Office of Aerospace Research
APO
New York 09667

Prof. J. J. D'Azzo
Dept. of Electrical Engineering
Air Force Institute of Technology
Wright-Patterson AFB, Ohio 45433

Lincoln Laboratory
Massachusetts Institute of Technology
Lexington, Mass. 02173

Professor Nicholas George
California Institute of Technology
Pasadena, California 91109

Dr. John C. Hancock, Head
School of Electrical Engineering
Purdue University
Lafayette, Indiana 47907

Dr. H. Harrison, Code RRE
Chief, Electrophysics Branch
NASA
Washington, D. C. 20546

Head, Technical Services Division
Naval Investigative Service Hqs
4420 North Fairfax Drive
Arlington, Virginia 22203

Headquarters
Defense Communications Agency (340)
Washington, D. C. 20305

Dr. L. M. Hollingsworth
AFCRL (CRN)
L. G. Hanscom Field
Bedford, Massachusetts 01731

Hunt Library
Carnegie-Mellon University
Schenely Park
Pittsburgh, Pa. 15213

The Johns Hopkins University
Applied Physics Laboratory
8621 Georgia Avenue
Silver Spring, Maryland 20910
Attn: Document Librarian

Lt. Col. Robert B. Kalisch (SREE)
Chief, Electronics Division
Directorate of Engineering Sciences
Air Force Office of Scientific Research
Arlington, Virginia 22209 (5)

U. S. Army Limited War Laboratory
Attn: Technical Director
Aberdeen Proving Ground
Aberdeen, Maryland 21005

Commanding Officer
Army Materials Research
Watertown Arsenal
Watertown, Massachusetts 02172
Attn: Dr. H. Priest

Commanding General
U. S. Army Security Agency
Arlington Hall Station
Arlington, Virginia 22212
Attn: IARD-T

Commanding Officer and Director
U. S. Naval Underwater Sound Lab.
Fort Trumbull
New London, Connecticut 06840

Defense Documentation Center
Attn: DDC-TCA
Cameron Station
Alexandria, Virginia 22314 (20)

Det. No. 6, OAR
Air Force Unit Post Office
Los Angeles, California 90045

Director
Advanced Research Projects Agency
Department of Defense
Washington, D. C. 20301

Director for Materials Sciences
Advanced Research Projects Agency
Department of Defense
Washington, D. C. 20301

Director
Columbia Radiation Laboratory
Columbia University
538 West 120th Street
New York, New York 10027

Director
Coordinated Science Laboratory
University of Illinois
Urbana, Illinois 61801

Director
Electronics Research Laboratory
University of California
Berkeley, California 94720

Director
Electronic Sciences Laboratory
University of Southern California
Los Angeles, California 90007

Director
Microwave Physics Laboratory
Stanford University
Stanford, California 94305

Mr. Robert O. Parker, AMSEL-RD-S
Executive Secretary, JSTAC
U. S. Army Electronics Command
Fort Monmouth, New Jersey 07703

Director
National Security Agency
Fort George G. Meade
Maryland 20755
Attn: TDL

Director, Naval Research Laboratory
Technical Information Officer
Washington, D. C. 20360
Attn: Code 2027 (6)
Dr. W. C. Hall, Code 7000 (1)
Dr. A. D. Brodzinsky, Supt. (1)

Director
Research Laboratory of Electronics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

Director
Stanford Electronics Laboratories
Stanford University
Stanford, California 94305

Hqs. USAF (AFRDDD)
The Pentagon
Washington, D. C. 20330

Illinois Institute of Technology
Dept. of Electrical Engineering
Chicago, Illinois 60616

Los Alamos Scientific Laboratory
Attn: Reports Library
P. O. Box 1663
Los Alamos, New Mexico 87544

Commander
U. S. Naval Electronics Laboratory Cent
San Diego, California 93152 (2)
Attn: Library

The University of Arizona
Dept. of Electrical Engineering
Tucson, Arizona 85721

Dr. I. R. Mirman
AFSC (SCT)
Andrews Air Force Base
Maryland 20331

Director, Aerospace Mechanics Div.
Frank J. Seiler Research Laboratory
U. S. Air Force Academy
Colorado Springs, Colorado 80912

Dr. G. J. Murphy
The Technological Institute
Northwestern University
Evanston, Illinois 60201

Director
Air Force Avionics Laboratory
Wright-Patterson AFB, Ohio 45433

NASA Lewis Research Center
Attn: Library
21000 Brookpark Road
Cleveland, Ohio 44135

Dr. Robert E. Fontana
Systems Research Laboratories Inc.
7001 Indian Ripple Road
Dayton, Ohio 45440

Nuclear Instrumentation Group
Bldg. 29, Room 101
Lawrence Radiation Laboratory
University of California
Berkeley, California 94720

Sylvania Electronic Systems
Applied Research Laboratory
Attn: Documents Librarian
40 Sylvan Road
Waltham, Mass. 02154

Naval Air Systems Command
AIR 03
Washington, D. C. 20360 (2)

Naval Electronics Systems Command
ELEX 03, Rm. 2046 Munitions Bldg
Dept. of the Navy
Washington, D. C. 20360 (2)

Utah State University
Dept. of Electrical Engineering
Logan, Utah 84321

Naval Ship Systems Command
SHIP 035
Washington, D. C. 20360

Naval Ship Systems Command
SHIP 031
Washington, D. C. 20360

New York University
College of Engineering
New York, New York 10019

Dr. H. V. Noble (CAVT)
Air Force Avionics Laboratory
Wright-Patterson AFB, Ohio 45433

Office of Deputy Director
(Research and Information Rm. 3D1037)
Department of Defense
The Pentagon
Washington, D. C. 20301

Polytechnic Institute of Brooklyn
55 Johnson Street
Brooklyn, New York 11201
Attn: Mr. Jerome Fox
Research Coordinator

Rome Air Development Center
Griffiss AFB, New York 13440
Attn: Documents Library (EMTLD)

Raytheon Co.
Bedford, Mass. 01730
Attn: Librarian

AFSC (SCTSE)
Andrews Air Force Base
Maryland 20331

Dr. A. A. Dougal
Asst. Director of Research
Office of Defense Res. and Eng.
Department of Defense
Washington, D. C. 20301

Union Carbide Corporation
Electronic Division
P. O. Box 1209
Mountain View, California 94041

Dr. H. Robl, Deputy Chief Scientist
U. S. Army Research Office (Durham)
Box CM, Duke Station
Durham, North Carolina 27706

Dr. Sheldon J. Welles
Electronics Properties Information Center
Hughes Aircraft Company
Mail Station E-175
Culver City, California 90230

School of Engineering Sciences
Arizona State University
Tempe, Arizona 85281

Hqs. SAMSO (SMTTA/Lt. Nelson)
AF Unit Post Office
Los Angeles, California 90045

Raytheon Company
Research Division Library
28 Seyon Street
Waltham, Mass. 02154

USAFSAM (SMKOR)
Brooks Air Force Base
Texas 78235

Syracuse University
Department of Electrical Engineering
Syracuse, New York 13210

Government Publications Section
University of California
Santa Barbara, California 93106
Attn: Library

Engineering and Math. Sciences Library
University of California at L. A.
405 Hilgard Avenue
Los Angeles, California 90024

Prof. Joseph E. Rowe, Chairman
University of Michigan
Electrical Engineering Department
Ann Arbor, Michigan 48104

U. S. Army Munitions Command
Attn: Technical Information Branch
Picatinney Arsenal, SMUPA-VA6
Dover, New Jersey 07801

U. S. Army Research Office
Physical & Engineering Sciences Div
1045 Columbia Pike
Arlington, Virginia 22204

Dr. A. D. Schnitzler, AMSEL-HL-NVII
Night Vision Laboratory, USAECOM
Fort Belvoir, Virginia 22060

Department of Electrical Engineering
Texas Technological College
Lubbock, Texas 79409

U. S. Naval Weapons Laboratory
Dahlgren, Virginia 22448

Hq. USAF (AFRDSD)
The Pentagon
Washington, D. C. 20330

Director
The Walter Reed Institute of Research
Walter Reed Medical Center
Washington, D. C. 20012

Missile Electronic Warfare
Technical Area, AMSEL-WT-MT
White Sands Missile Range
New Mexico 88002

The University of Iowa
The University Libraries
Iowa City, Iowa 52240

Lenkurt Electric Co, Inc.
1105 County Road
San Carlos, California 94070
Attn: Mr. E. K. Peterson

Yale University
Engineering Department
New Haven, Connecticut 06520

Philco-Ford Corporation
Communications and Electronics Div
Union Meeting and Jolly Roads
Blue Bell, Pa. 19422

Dr. Leo Young
Stanford Research Institute
Menlo Park, California 94025

Hq. AMD (AMR)
Brooks Air Force Base
Texas 78235

Morton M. Pavane, Chief
AFSC Scientific & Technical Liaison Office
26 Federal Plaza, Rm. 1313
New York, New York 10007

Library (Code 2124)
U. S. Navy Post Graduate School
Monterey, California 93940
Attn: Technical Report Section

Director
Electronics Research Center
University of Texas at Austin
Austin, Texas 78712

Mr. H. E. Webb (EMIIB)
Rome Air Development Center
Griffiss Air Force Base
New York 13440

Col. E. P. Gaines, Jr.
ACDA/FO
1901 Pennsylvania Ave., N.W.
Washington, D.C. 20451

Mr. Billy Locke
Technical Adviser, Requirements
USAF Security Service
Kelly Air Force Base, Texas 78241

Weapons Systems Evaluation Group
Attn: Colonel Blaine O. Vogt
400 Army-Navy Drive
Arlington, Virginia 22202

Hq. USAF (AIRDDG)
The Pentagon
Washington, D. C. 20301

W. A. Eberspacher, Assoc. Head
Systems Integration Division
Code 5340A, Box 15
Naval Missile Center
Point Mugu, California 93041

Glen A. Myers (Code 52Mv)
Assoc. Prof. of Electrical Engineering
Naval Postgraduate School
Monterey, California 93940

Mr. Norman J. Field, AMSEL-RD-S
Chief, Office of Science and Technology
R and D Directorate
U. S. Army Electronics Command
Fort Monmouth, New Jersey 07703

Redstone Scientific Information Center
Attn: Chief, Document Section
U. S. Army Missile Command
Redstone Arsenal, Alabama 35809

Federal Aviation Administration
Attn: Admin Stds Div (MS-110)
800 Independence Avenue, S. W.
Washington, D. C. 20590

Mr. M. Zane Thornton, Chief
Network Engineering, Communications
and Operations Branch
Lister Hill National Center for
Biomedical Communications
8600 Rockville Pike
Bethesda, Maryland 20014

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Division of Engineering and Applied Physics Harvard University Cambridge, Mass. 02138	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

CONTROL OF NONLINEAR SYSTEMS IN REGIONS OF STATE SPACE

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Interim technical report

5. AUTHOR(S) *(First name, middle initial, last name)*

Stanley B. Gershwin
& David H. Jacobson

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
January 1970	47	7

8a. CONTRACT OR GRANT NO. N00014-67-A-0298-0006 and NASA b. PROJECT NO. Grant 22-007-068 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Report No. 600 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited. Reproduction in whole or in part is permitted by the U. S. Government.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Office of Naval Research

13. ABSTRACT

Recently, a nonlinear controllability theory based upon Liapunov-like notions was developed. In this paper the theory is generalized and strengthened, and a wider class of nonlinear systems is considered. In particular, conditions for controllability of a dynamic system which is subject to state variable inequality constraints are obtained. It is shown that initial conditions which are interior to a certain ellipse can be made to generate trajectories which remain in that ellipse and which reach the desired terminal state. When the ellipse is a subset of the feasible region of state space the trajectory clearly remains in this region (i.e. the state variable inequality constraints are satisfied). A design procedure for finding the largest such ellipse is given, and illustrative examples are presented. In addition, stabilization of constrained dynamic systems is considered.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Control						
Controllability						
Nonlinear dynamics						
State constraint						
Constraint						
Riccati equation						
Stability						
Liapunov						
Feedback control						