TRAJECTORY COMPUTATIONAL TECHNIQUES EMPHASIZING EXISTENCE, UNIQUENESS, AND CONSTRUCTION OF SOLUTIONS TO BOUNDARY PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

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Summary

The objective of the investigation was to study the existence, uniqueness, and construction of solutions for the two-point boundary value problem of nonlinear ordinary differential equations. The text of this report is divided into six parts, each part possessing its own list of references.

In part I we have attempted to unify the existing methods for solving linear boundary value problems. Many of the techniques discussed here have not been discussed in the same publication, but have been scattered throughout the literature. Since most of the techniques for solving nonlinear boundary value problems involves solving several linear problems, the importance of these methods cannot be overemphasized.

Part II continues the techniqes in part I to nonlinear problems. The parallel shooting method is discussed in some detail, and should prove to be the most fruitful general purpose technique for solving, boundary value problems.

Parts III and IV survey the recent developments in existence and uniqueness theory, in particular the sub and super function approach.

Part V applies Liapunov and perturbation theory to the problem of determining interval length in the parallel shooting method. Estimates are obtained on the interval length which are easily obtained <u>without</u> actual computation of solutions.

Part VI developes Liapunov theory for existence and uniqueness of solutions to boundary value problems. The Liapunov conditions for

uniqueness are of a different form than those for initial value problem uniqueness. The results of Hartman are obtained as a special case of our theory by a suitable choice of the Liapunov function.

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CHAPTER I

THE SOLUTION OF LINEAR BOUNDARY VALUE PROBLEMS

1. Introduction. Many techniques have been proposed for solving boundary value problems. Excellent sources are Keller [1], Osbourne [2], Lee [3], and Bellman and Kalaba [4]. To consolidate and unify many of the more promising techniques, we shall develop these techniques for a common equation and boundary condition. This should provide the advantages and disadvantages of each procedure, since one may be better than another when used on a particular problem.

We shall discuss the linear problem here, since any nonlinear problem is usually solved by some sort of linearization process. That is, the solutions of a sequence of linear boundary value problems approach in some sense the solution of the original nonlinear problem.

We shall discuss the nonlinear techniques after developing thoroughly the linear methods. It should also be mentioned that more general boundary conditions such as multipoint conditions or mixed conditions could be imposed, but for simplicity and clarity, we shall not develop the theory for these conditions, [2].

2. Preliminaries. Consider the ordinary differential equations

$$u' = A(t)u + B(t)v + h(t), \tag{2.1a}$$

$$v' = C(t)u + D(t)v + g(t), \tag{2.1b}$$

where $A(t), B(t), C(t)$ and $D(t)$ are $n \times n$ matrices with continuous elements on $[a,b]$, $u, v, h(t)$ and $g(t)$ are n-vectors, and $h(t)$ and $g(t)$ are continuous on $[a,b]$. Let us assume that (2.1) is subject to the two point boundary conditions

(2.2a) $$u(a) = \alpha,$$

(2.2b) $$B_1 u(b) + B_2 v(b) = \beta,$$

where B_1 and B_2 are constant $n \times n$ matrices such that $B_1 + B_2$ is nonsingular, and α and β are constant n-vectors.

Let $U(t)$ and $V(t)$ be $n \times n$ matrices satisfying

(2.3a) $$U' = A(t)U + B(t)V,$$

(2.3b) $$V' = C(t)U + D(t)V,$$

and the initial conditions

(2.4a) $$U(a) = 0,$$

(2.4b) $$V(a) = I \quad \text{(unit matrix)}.$$

3. <u>Reduction to an initial value problem by direct substitution</u>. Let us denote a solution of (2.1) by the pair $(u(t), v(t))$. Let $(x(t), y(t))$ be the solution of (2.1) satisfying the initial conditions $(x(a), y(a)) = (\alpha, 0)$ on the interval $a \leq t \leq b$.

Remark. This solution can be obtained numerically by any of several standard routines.

Theorem 3.1. *Let* $(u(t),v(t))$ *be a solution of* (2.1) *satisfying* $(u(a),v(a)) = (\alpha,d)$, *where* d *is obtained from*

$$[B_1U(b) + B_2V(b)]d = \beta - B_1x(b) - B_2x(b). \tag{3.1}$$

Then $(u(t),v(t))$ *satisfies the boundary conditions* (2.2).

Proof. Consider

$$u(t) = x(t) + U(t)d, \tag{3.2a}$$

$$v(t) = y(t) + V(t)d, \tag{3.2b}$$

where $(x(t),y(t))$ is a solution of (2.1) satisfying $(x(a),y(a)) = (\alpha,0)$ and $U(t)$ and $V(t)$ are given by (2.3) and (2.4). It is easily verified that $(u(t),v(t))$ satisfies $(u(a),v(a)) = (\alpha,d)$ and that $(u(t),v(t))$ satisfies (2.1) for all d. If (2.2b) is to be satisfied, d must be the solution of (3.1).

Remark. For this technique to be effective, $B_1U(b) + B_2V(b)$ *must* be not only nonsingular but also computable. A method suggested by Conte [5], (see also [4]), orthogonalizes the solution $(U(t),V(t))$ of (2.3) at each integration step when certain criteria are violated.

Corollary 3.1 (Hartman [6]). *The homogeneous boundary value problem* (2.1) *and* (2.2) *with* $h(t) \equiv g(t) \equiv 0$ *and* $\alpha = \beta = 0$ *has only* *the trivial solution* $(u(t),v(t) \equiv (0,0)$ *if and only if* $[B_1U(b) + B_2V(b)]$ *is nonsingular*.

Proof. (3.1) becomes

$$[B_1U(b) + B_2V(b)]d = 0 \tag{3.3}$$

which has only the solution $d = 0$ if and only if $[B_1U(b) + B_2V(b)]$ is nonsingular.

Then $(u(t),v(t)) = (U(t)d,V(t)d) \equiv (0,0)$ on $[a,b]$ and since this solution is unique [6] the homogeneous problem has only the trivial solution. To prove the converse, $(u(t),v(t)) = (U(t)d,V(t)d) \equiv (0,0)$ implies $d = 0$. (Since $(u(a),v(a)) = (0,d) = (0,0)$). Thus from (3.3) $[B_1U(b) + B_2V(b)]$ must be nonsingular.

Corollary 3.2. If $(x(t),y(t))$ satisfies (2.2b) in addition to the assumptions of Theorem 3.1 and $[B_1U(b) + B_2V(b)]$ has rank $n - k$, then k linearly independent solutions can be found.

Proof. (3.1) becomes (3.3) which has k linearly independent solutions for d.

Remarks. 1. If the linear system (2.1) is unstable, (for example, if A,B,C,D are constant $n \times n$ matrices and $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ has eigenvalues in the right half plane), then $[B_1U(b) + B_2V(b)]^{-1}$ is extremely difficult to compute, (see Osbourne [2] and Bailey and Shampine [7]).

2. In remark 1, parallel shooting has been used by Osbourne [2] and others to alleviate the problem.

4. Reduction to an initial value problem by adjoint equations. The formal adjoint of (2.1) can be written

(4.1a) $$x' = -A^*(t)x - C^*(t)y,$$

(4.1b) $$y' = -B^*(t)x - D^*(t)y,$$

where $*$ denotes the complex conjugate transpose. Let $X(t)$ and $Y(t)$ be $n \times n$ matrix solutions to (4.1) on $[a,b]$. That is,

(4.2a) $$X^{*\prime} = -X^*A(t) - Y^*C(t),$$

(4.2b) $$Y^{*\prime} = -X^*B(t) - Y^*D(t),$$

Let us assume also that $X(t)$ and $Y(t)$ satisfy the initial conditions

(4.3a) $$X(b) = B_1^*,$$

(4.3b) $$Y(b) = B_2^*.$$

Multiplying (4.2a) by $u(t)$ and (4.2b) by $v(t)$, where $(u(t),v(t))$ is a solution of (2.1), and multiplying (2.1a) by X^* and (2.1b) by Y^* and adding the resulting expressions yields

$$X^{*'}u + Y^{*'}v + X^{*}u' + Y^{*}v' = X^{*}h + Y^{*}g$$

or

$$\frac{d}{dt}[X^{*}u + Y^{*}v] = X^{*}h + Y^{*}g. \tag{4.4}$$

Integrating (4.4) from a to b yields

$$X^{*}(b)u(b) + Y^{*}(b)v(b) - X^{*}(a)u(a) - Y^{*}(a)v(a) = \int_a^b [X^{*}(t)h(t) + Y^{*}(t)g(t)]dt$$

and from (4.3) and (2.2) we obtain

$$Y^{*}(a)v(a) = \beta - X^{*}(a)\alpha - \int_a^b [X^{*}(t)h(t) + Y^{*}(t)g(t)]dt. \tag{4.5}$$

We have proved the following result.

Theorem 4.1. *Let* $v(a) = d$ *be obtained from* (4.5). *Then the solution of* (2.1) *with the initial conditions* $(u(a),v(a)) = (\alpha,d)$ *satisfies the boundary conditions* (2.2).

Remarks. 1. It may not be possible to solve for $v(a)$ if $Y^{*}(a)$ is singular and the right side of (4.5) is nonzero.

2. If $Y^{*}(a)$ is singular, it may be possible that the method of section 3 would yield a solution for d.

3. If the right side of (4.5) is zero, it would be possible, as in section 3, to obtain a solution for $v(a)$ if $Y^{*}(a)$ is singular.

5. *Reduction to an initial value problem by the method of factorization* [8]. Let

$$v = Ju + z, \tag{5.1}$$

where $(v(t),u(t))$ is a solution of (2.1), J is an $n \times n$ matrix and z is an n-vector, (J and z both functions of t to be determined). Differentiating (5.1) we obtain

$$v' = J'u + Ju' + z'$$

or, from (2.1a) and (5.1),

$$v' = Ju + J(A(t)u + B(t)Ju + B(t)z + h(t)) + z'. \tag{5.2}$$

To insure that (5.2) and (2.1b) are equivalent, let us define J and z as follows:

$$J' + JA(t) + JB(t)J = C(t) + D(t)J, \tag{5.3a}$$

$$z' = (D(t) - JB(t))z - Jh(t) + g(t). \tag{5.4a}$$

To prescribe appropriate initial conditions, from (5.1)

$$v(b) = J(b)u(b) + z(b).$$

Substitution into (2.2b) yields

$$[B_1 + B_2J(b)]\ u(b) = \beta - B_2z(b) \tag{5.5}$$

which we would like satisfied for all $u(b)$. Assume B_2 is nonsingular. If

(5.3b) $$J(b) = - B_2^{-1}B_1$$

(5.4b) $$z(b) = B_2^{-1}\beta$$

are considered as initial conditions for (5.3a) and (5.4b) it is possible to solve (5.3) for a solution $J(t)$ on the interval $a \leq t \leq b$, and then solve (5.4) for $z(t)$ on $[a,b]$. In this manner $J(a)$ and $z(a)$ would be obtained, and from (5.1) and (2.2a)

(5.6) $$d = v(a) = J(a)\alpha + z(a)$$

would give the missing initial condition in (2.1). This can be summarized as follows:

Theorem 5.1. *Let* $(u(t),v(t))$ *be a solution of* (2.1) *satisfying* $(u(a),v(a)) = (\alpha,d)$ *where* $d = J(a) + z(a)$ *from* (5.6). *Then* $(u(t),v(t))$ *satisfies the boundary conditions* (2.2).

6. *Reduction to initial value problem by invariant imbedding.* As discussed by Bailey and Wing [9], Lee [3], and Bellman and Kalaba [4], invariant imbedding is a concept rather than a formal technique, and hence a collection of several disjoint procedures. Included in this would be the method of factorization discussed in section 6. A recent paper by Meyer [10] relates the invariant imbedding principle to the

formal method of characteristics. Following Meyer, let $v(t,u)$ satisfy

(6.1) $$v_t(t,u) + v_u(t,u)\ [A(t)u + B(t)v(t,u) + g(t)] = C(t)u + D(t)v(t,u) + h(t)$$

subject to the initial condition (initial manifold)

(6.2) $$B_1u + B_2v(b,u) = \beta$$

for $a \le t \le b$, u arbitrary.

Theorem 6.1 (Meyer [10]). The solution of (6.1) is generated by the characteristics $(t,u(t),v(t))$ satisfying $t = b$, $u(b) = u$, $B_1u + B_2v(b) = \beta$.

Since the characteristic equations (2.1) are linear it is easily shown [10] that

(6.3) $$v(t,u) = J(t)u + z(t).$$

Substitution of (6.3) into (6.1) yields an equation which, when satisfied for all u, gives (5.3a) and (5.4a). Since from (6.3), $v(b,u) = J(b)u + z(b)$, (6.2) becomes

(6.4) $$[B_1 + B_2J(b)]u = \beta - B_2z(b)$$

which must be satisfied for all u.

Thus, the quantity multiplying u in (6.4) and the right side of

(6.4) must both be identically zero, yielding (5.3b) and (5.4b). Obtaining theorem 5.1 now proceeds exactly as in section 5. The method of factorization as developed in section 5 is seen to be a special case of the invariant imbedding procedure as developed by Meyer [10].

7. Difference methods. Let $\{t_j\}$ be a uniform net on $[a,b]$, where $t_j = a + j\delta \quad j = 0,1,\ldots,N+1,$

$$\delta = \frac{b-a}{N+1} .$$

Let u_j and v_j be "approximations" in some sense to $u(t_j), v(t_j)$ respectively. By replacing the derivatives in (2.1) by a difference scheme, it is possible to obtain a linear system of equations which, when solved, has a solution "approximating" the solution to (2.1) and (2.2). To apply the idea to (2.2) consider

(7.1a) $$\frac{u_j - u_{j-1}}{\delta} = A(t_j)u_j + B(t_j)v_j + h(t_j), \quad j = 1,2,\ldots,N+1,$$

(7.1b) $$\frac{v_{j+1} - v_j}{\delta} = C(t_j)u_j + D(t_j)v_j + g(t_j), \quad j = 1,2,\ldots N$$

and the boundary conditions

(7.2a) $$u_o = \alpha,$$

(7.2b) $$B_1 u_{N+1} + B_2 v_{N+1} = \beta,$$

Let

$$a_j = \delta A(t_j), \quad b_j = \delta B(t_j),$$

$$c_j = \delta C(t_j), \quad d_j = \delta D(t_j),$$

$$w_N = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_N \\ v_N \\ u_{N+1} \\ v_{N+1} \end{pmatrix}, \quad \gamma_N = \begin{pmatrix} \alpha + \delta h(t_1) \\ \delta g(t_1) \\ \delta h(t_2) \\ \delta g(t_2) \\ \vdots \\ \delta h(t_N) \\ \delta g(t_N) \\ \delta h(t_{N+1}) \\ \beta \end{pmatrix}$$

and

$$L_N = \begin{pmatrix} I-a_1, & -b_1, & 0, & 0, & 0, & 0, & \cdots & , 0 \\ -c_1, & -I-d_1, & 0, & I, & 0, & 0, & \cdots & , 0 \\ -I, & 0, & I-a_2, & -b_2, & 0, & 0, & \cdots & , 0 \\ 0, & -I, & -c_2, & -I-d_2, & 0, & I, & \cdots & , 0 \\ \vdots & \vdots & \vdots & \vdots & & & & \vdots \\ 0 & \cdots & -I, & 0, & I-a_N, & -b_N, & 0 & , 0 \\ 0 & & 0, & -I, & -c_N, & -I-d_N, & 0 & , I \\ 0 & & 0, & 0, & -I, & 0, & I-a_{N+1} & , -b_{N+1} \\ 0 & \cdots & 0, & 0, & 0, & 0, & B_1 & , B_2 \end{pmatrix}$$

Then (7.1) and (7.2) can be written as

(7.3) $$L_N W_N = \gamma_N.$$

Now if L_N is nonsingular it is possible to solve (7.3) and obtain $W_N = L_N^{-1}\gamma_N$. To conclude, the relation between (7.3) and (2.1), (2.2) can be summarized by the Lax equivalence theorem.

Lax's equivalence theorem [11]. $L_N^{-1}\gamma_N$ is uniformly bounded (as a function of N) if and only if $u_j \to u(t_j)$ and $v_j \to v(t_j)$ for all $j \leq N+1$ as $N \to \infty$

Remarks. 1. "Consistency" [11] is usually a condition required in this theorem, but we have imposed this condition by our choice of difference scheme in (7.1).

2. Numerically, the matrices may become quite large, causing error.

3. See [1] for conditions implying uniform boundedness of $L_N^{-1}\gamma_N$.

8. Green's function. For the discussion here, let us consider the vector equation formed by (2.1),

(8.1) $$\underline{u}' = \underline{A}(t)\underline{u} + \underline{h}(t),$$

where $\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \underline{A}(t) = \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix}, \underline{h}(t) = \begin{pmatrix} h(t) \\ g(t) \end{pmatrix},$

subject to the boundary conditions

(8.2) $$\underline{A}_1\underline{u}(a) + \underline{A}_2\underline{u}(b) = \underline{\alpha},$$

where $\underline{A}_1$ and $\underline{A}_2$ are 2n x 2n constant matrices and $\underline{\alpha}$ is a constant 2n-vector. The boundary conditions (2.2) are clearly a special case of (8.2).

Let $\underline{U}(t)$ be the $2n \times 2n$ matrix satisfying

$$\underline{U}' = \underline{AU}. \tag{8.3}$$

Also consider

$$\underline{u}' = \underline{A}(t)\underline{u}, \tag{8.4a}$$

$$\underline{A}_1\underline{u}(a) + \underline{A}_2\underline{u}(b) = 0. \tag{8.4b}$$

Lemma 8.1. *(8.4) has a nontrivial ($\not\equiv 0$) solution if and only if $[\underline{A}_1\underline{U}(a) + \underline{A}_2\underline{U}(b)]$ is singular.*

Proof. Since $u = U(t)\underline{d}$ is the general solution of (8.4a) substitution into (8.4b) yields $[\underline{A}_1\underline{U}(a) + \underline{A}_2\underline{U}(b)]\underline{d} = 0$ which has a solution $\not\equiv 0$ if and only if $[\underline{A}_1\underline{U}(a) + \underline{A}_2\underline{U}(b)]$ is singular.

Theorem 8.1. *(8.1) has a solution $\underline{u}(t)$ satisfying (8.2) of the form*

$$\underline{u}(t) = \int_0^p \underline{G}(t,s)\underline{h}(s)ds \tag{8.5}$$

where $\underline{G}(t,s)$ is a $2n \times 2n$ matrix function of t and s such that $\underline{G}(t,s)\underline{h}(s)$ is an integrable function of s, if and only if (8.4) has no nontrivial solution.

Proof. Assume (8.4) has only a trivial solution. Then by lemma 8.1,

$\underline{N} = \underline{A}_1\underline{U}(a) + \underline{A}_2\underline{U}(b)$ is nonsingular. The general solution of (8.1) is

(8.6) $$\underline{u}(t) = \underline{U}(t)[\underline{d} + \int_a^t \underline{U}^{-1}(s)\underline{h}(s)ds]$$

which satisfies (8.2) if and only if

(8.7) $$[\underline{A}_1\underline{U}(a) + \underline{A}_2\underline{U}(b)]\underline{d} = \beta - \underline{A}_2\underline{U}(b)\int_a^b \underline{U}^{-1}(s)\underline{h}(s)ds$$

or

(8.8) $$\underline{d} = \int_a^b \frac{\underline{N}^{-1}\beta ds}{b-a} - \underline{N}^{-1}\underline{A}_2\underline{U}(b)\int_a^b \underline{U}^{-1}(s)\underline{h}(s)ds.$$

By substituting (8.8) into (8.6) the function $\underline{G}(t,s)$ can be identified and (8.5) obtained. To prove the converse, the <u>alternative theorem</u> is needed (see Hartman [6] and Stakgold [12]). The Green's function gives an extremely important representation of the solution, one which can be used to obtain integral representations of nonlinear systems.

9. <u>General matrix Ricatti equation</u>. Consider

(9.1a) $$L(J) = J' + JA(t) + JB(t)J - D(t)J - C(t) = 0,$$

(9.1b) $$J(b) = J_o,$$

along with (2.3) and the initial conditions

(9.2a) $$U(b) = U_o,$$

(9.2b) $$V(b) = V_o,$$

where U_o is nonsingular and $J_o = V_o U_o^{-1}$.

Lemma (Reid [13]). *If (2.3) and (9.2) have solutions U(t) and V(t) on [a,b] and V(t) is nonsingular, then $J(t) = V(t)U^{-1}(t)$ is a solution of (9.1) on [a,b].*

Proof. Since $U(t)U^{-1}(t) = I$, differentiating both sides with respect to t yields

(9.3) $$U'(t)U^{-1}(t) + U(t)[U^{-1}(t)]' = 0$$

(9.4) $$J'(t) = V'(t)U^{-1}(t) + V(t)[U^{-1}(t)]'.$$

Substitution of (9.3) into (9.4) yields

$$\begin{aligned} J'(t) &= V'(t)U^{-1}(t) - V(t)U^{-1}(t)U'(t)U^{-1}(t) \\ &= (CU + DV)U^{-1} - VU^{-1}(AU + BV)U^{-1} \\ &= C(t) + D(t)J - JA(t) - JB(t)J. \end{aligned}$$

An interesting idea proposed by Bellman [14] for solving the matrix Ricatti equation consists of replacing JB(t)J by an upper and a lower estimate involving only J, B(t) and an arbitrary matrix S in a linear combination. By solving the resulting linear equations, estimates are obtained for the actual solution.

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CHAPTER II

THE SOLUTION OF NONLINEAR BOUNDARY VALUE PROBLEMS

1. Introduction. Nonlinear boundary value problems can be reduced to the solution of transcendental equations. It is then possible to apply techniques from numerical analysis, such as successive approximation, Newton's method and the method of false position (Collatz [2]) to obtain a solution. Many sufficient conditions have been developed to insure convergence of a given iteration procedure [3,4]. However, solutions of boundary value problems and transcendental equations can be obtained by iteration procedures without having formal convergence criteria. Thus, in practice, iteration schemes are used even though formal convergence criteria are not satisfied.

Any method for solving nonlinear boundary value problems relies rather heavily on initial value problems. For example, if a solution does not exist on an interval $[a,b]$, it could not satisfy two point boundary conditions at a and b. We shall assume the standard theory of initial value problems such as developed by Hartman [1].

One of the more important techniques for solving boundary value problems is the parallel shooting procedure [3,5]. Many problems not previously solvable by shooting techniques can be treated by this method.

2. Preliminaries and basic results. Consider the system of differential equations

$$u' = F(t,u,v) \tag{2.1a}$$

$$v' = G(t,u,v) \tag{2.1b}$$

subject to the boundary conditions

$$u(a) = \alpha \tag{2.2a}$$

$$H(u(b),v(b)) = 0 \tag{2.2b}$$

Here u,v,F,G,H,α are n-vectors, F and G are continuous functions defined on a set $[a,b] \times D$, $D \subseteq R^{2n}$, and H is defined on D.

Let $(u(t,d),v(t,d))$ be a solution of (2.1) existing on $[a,b]$ and satisfying $(u(a,d),v(a,d)) = (\alpha,d)$. We have the following result.

Theorem 2.1. (2.1) has a solution satisfying (2.2) if and only if

$$\phi(d) = H(u(b,d),v(b,d)) = 0 \tag{2.3}$$

for some d.

This procedure for obtaining a solution of boundary value problem is usually referred to as a shooting method and is solved by numerical procedures such as Newton's method and its variants, and multipoint "false position" methods which can often be shown to converge [2,3].

Theorem 2.2. *Let (2.1) be such that solutions exist and are unique on $[a,b] \times R^{2n}$. Then the boundary value problem (2.1) and (2.2) has as many solutions as there are distinct roots $d = d^{(v)}$ of (2.3).*

Proof. If $\phi(d) = 0$ for some d, then $(u(t,d),v(t,d))$ satisfies (2.1) and (2.2), let d_1 and d_2 be distinct points, $d_1 \neq d_2$. Then $(u(t,d_1),v(t,d_1)) \neq (u(t,d_2),v(t,d_2))$, since if not, uniqueness would be violated. Thus each distinct root of $\phi(d) = 0$ yields a solution to (2.1) and (2.2).

Corollary 2.2. *If (F,G) satisfies a Lipschitz condition on $[a,b] \times R^{2n}$ then the conclusion of theorem 2.2 follows.*

3. *Parallel shooting.* [3] We shall develop the parallel shooting technique for

$$y' = f(t,y) \tag{3.1}$$

subject to the *linear* boundary conditions

(3.2) $$B_1y(a) + B_2y(b) = \gamma$$

where y, f are 2n-vectors, f is defined and continuous on $[a,b] \times D$, $D \subset R^{2n}$, B_1 and B_2 are constant $2n \times 2n$ matrices, $B_1 + B_2$ nonsingular, and γ is a constant 2n-vector. Let the interval $[a,b]$ be subdivided into N subintervals with the points t_j, $j = 0,1,2, \ldots ,N$, $a = t_o < t_1 < \ldots < t_{N-1} < t_N = b$.

Let $\delta_j = t_j - t_{j-1}$ and on each interval $[t_{j-1}, t_j]$ let

$r = \dfrac{t - t_{j-1}}{\delta_j}$, $y_j(r) = y(t) = y(r\delta_j + t_{j-1})$ and

$f_j(r, y_j(r)) = \delta_j f(r\delta_j + t_{j-1}, \; y_j(r))$.

Using these changes of variables, (3.1) becomes

(3.3) $$\frac{dy_j}{dr} = f_j(r, y_j) \qquad 0 < r < 1 \quad j = 1,2, \ldots ,N$$

The boundary conditions (3.2) become

(3.4) $$B_1y_1(0) + B_2y_N(1) = \gamma$$

Assume also that solutions to initial value problems for (3.1) exist on every interval $|t - t'| < \delta = \max\limits_{0 \leq n \leq N} \delta_n$, $t' \in [a,b]$.

In addition, the solution of (3.1) must be continuous, requiring that

(3.5) $\quad y_{j+1}(0) = y_j(1) \qquad j = 1,2, \ldots ,N-1$

let $\underline{y} = \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_N \end{pmatrix} \qquad \underline{f}(r,\underline{y}) = \begin{pmatrix} f_1(r,y_1) \\ \cdot \\ \cdot \\ \cdot \\ f_N(r,y_N) \end{pmatrix} \qquad \underline{\gamma} = \begin{pmatrix} \gamma \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$

Then (3.3) can be written as

(3.6) $\quad \underline{y}' = \underline{f}(r,\underline{y}) \qquad 0 < r < 1$

and (3.4) and (3.5) become

(3.7) $\quad P\underline{y}(0) + Q\underline{y}(1) = \underline{\gamma}$

where

$$P = \begin{pmatrix} B_1 & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & I \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & B_2 \\ -I & 0 & 0 & \cdots & 0 & 0 \\ 0 & -I & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -I & 0 \end{pmatrix}$$

Parallel shooting, technique I. Consider along with (3.6) the initial conditions

(3.8)
$$\underline{y}(0) = \underline{d} = \begin{pmatrix} d_1 \\ \cdot \\ \cdot \\ \cdot \\ d_N \end{pmatrix}$$

Then if $\underline{y}(r,\underline{d})$ is the corresponding solution of (3.6) we attempt to find a $\underline{d}$ such that (3.7) is satisfied. This will be true if

(3.9)
$$\phi(\underline{d}) = P\underline{d} + Q\underline{y}(1,\underline{d}) - \underline{\gamma} = 0$$

If Newton's method is used to solve (3.9) for $\underline{d}$, the variational system of (3.6) will be useful. Let $W(r,\underline{d}) = \frac{\partial \underline{y}(r,d)}{\partial \underline{d}}$

where $W(r,\underline{d})$ is the $2nN \times 2nN$ Jacobian matrix of $\underline{y}(r,\underline{d})$ with respect to $\underline{d}$. (Here f must be assumed sufficiently smooth)

Then from (3.6) and (3.8),

(3.10a) $$\frac{dW}{dr} = \frac{\partial \underline{f}}{\partial \underline{y}} (r,\underline{y}(r,\underline{d}))W$$

(3.10b) $$W(0,\underline{d}) = \underline{I}$$ ($\underline{I}$ is the $2nN \times 2nN$ unit matrix)

An iteration procedure for solving (3.9) can now be given by the following:

(3.11a) $$\underline{d}_{v+1} = \underline{d}_v + \Delta\underline{d}_v$$ where

(3.11b) $$[P + QW(1,\underline{d}_v)]\ \Delta\underline{d}_v = -\ \phi(\underline{d}_v)$$

To solve this equation, note that (3.6) and (3.8) involve the solution of N systems of $2n$ equations where each system is independent of the others.

Define the $2n \times 2n$ matrix W_j as the solution of

(3.12a) $$\frac{dW_j}{dr} = \frac{\partial f_j}{\partial y_j} (r,y_j(r,d_j))W_j$$

(3.12b) $W_j(0) = I$, (I is the $2n \times 2n$ unit matrix) $j = 1,2, \ldots ,N$

Then

$$W(r,\underline{d}) = \text{diag}\ \{W_j(r,\underline{d}_j)\}$$

Let us compute the inverse of $[P + QW(1,\underline{d}_v)]$ allowing a solution for $\Delta\underline{d}_v$ to be computed in (3.11). Let

$$T_K = \quad N+1-K \begin{pmatrix} I & 0 & 0 & & \cdot & \cdot & \cdot & 0 \\ 0 & I & 0 & & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & I & & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & W_{N-K} & & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & I \end{pmatrix}$$
$$\begin{matrix} \cdot \\ \cdot \\ N-K \end{matrix}$$

That is, in T_K W_{N-K} occupies the $N + 1 - K$ now and the $N - K$ column. (T_K is a $2nN \times 2nN$ matrix). Then

(3.13) $$[P + QW]T_1T_2 \ldots T_{N-1} = R$$

where

$$R = \begin{pmatrix} B_1 + B_2W_N \cdots W_1, & B_2W_NW_{N-2} \cdots W_1, & B_2W_NW_{N-3} \cdots W_1, & \cdots & B_2W_N \\ 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 0 & & \cdots & & I \end{pmatrix}$$

If

$$R = \begin{pmatrix} R_1 & R_2 & \cdots & R_N \\ 0 & I & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & I \end{pmatrix}$$

Then

$$R^{-1} = \begin{pmatrix} R_1^{-1}, & -R_1^{-1}R_2 & \cdots & -R_1^{-1}R_N \\ 0, & I, & \cdots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & & \cdots & I \end{pmatrix}$$

Let

$$R_1 = B_1 + B_2W_N \cdot \cdot \cdot W_1$$

$$R_2 = B_2W_NW_{N-2} \cdot \cdot \cdot W_1$$

$$R_3 = B_2W_NW_{N-3} \cdot \cdot \cdot W_1$$

.

.

.

$$R_N = B_2W_N$$

Then

$$(3.14) \qquad [P + QW] = RT_{N-1}^{-1} \dots T_1^{-1}$$

$$[P + QW]^{-1} = T_1 \dots T_{N-1}R^{-1}$$

<u>Parallel shooting, technique</u> II. Assume N is even and that the solution originates at the odd points $1,3, \dots ,N - 1$ in <u>both</u> directions. Assume the net is fine enough so that initial value solutions of (3.1) exist in both directions up to the even points.

Let $y_{2j-1}(t)$, $y_{2j}(t)$ be solutions of (3.3) originating at t_{2j-1} to the left and right respectively such that at t_{2j-1}

$$(3.15) \qquad y_{2j-1}(0) = y_{2j}(0) = d_j \qquad j = 1,2, \dots ,N/2$$

and $y_{2j-1}(r)$ extends to t_{2j-2} and $y_{2j}(r)$ extends to t_{2j}.

At the points with even indices, t_{2n}, we must have

$$(3.16) \qquad y_{2j}(1) = y_{2j+1}(1) \qquad j = 1,2, \dots ,[N - 1]/2$$

to insure continuity.

The boundary conditions (3.2) become $Ay_1(1) + By_N(1) = \gamma$.

Now suppose the solutions initiating at T_{2j-1} satisfying (3.15) are denoted by $y_{2j-1}(r,d_j)$, $y_{2j}(r,d_j)$. Then (3.15) and (3.2) become

$$(3.17a) \qquad y_{2j}(1,d_j) = y_{2j+1}(1,d_{j+1}) \qquad j = 1,2, \dots ,N/2 + 1$$

$$(3.17b) \qquad B_1y_1(1,d_1) + B_2y_N(1,d_{N/2}) = \gamma$$

(3.17) is a system of $N/2$ equations in the $N/2$ unknowns $d_1, \dots d_{N/2}$ and any of the standard numerical techniques such as Newton's method can now be used to obtain a solution.

Remark. For computational purposes, after the points $\{t_j\}$ have been specified, a solution is obtained on the interval of interest for an arbitrary initial condition ξ initiating at t_j. If, for example, $||y(t_{j+1})|| > R||\xi||$ where R is some preassigned constant depending on accuracy requirements, the grid should be refined and this procedure repeated.

4. Quasilinearization [6,7]. Let $y(t)$ be a solution of (3.1) and (3.2). Let us assume that f is sufficiently smooth, so that

$\frac{\partial f}{\partial y}(t,y)$ exists and is continuous. Let $y_v(t)$ be an "approximation" to $y(t)$ such that (3.2) is satisfied by $y_v(t)$. Let $x_v(t) = y(t) - y_v(t)$. Then if $y_v(t)$ is "sufficiently close" to $y(t)$

(4.1a) $$x_v' = \frac{\partial f}{\partial y}(t,y_v(t))x_v + f(t,y_v(t)) - y_v'(t)$$

(4.1b) $$B_1x_v(a) + B_2x_v(b) = 0 \qquad v = 1,2, \ldots$$

This is a linear problem which can be solved for x_v if a solution exists.

SCHEME. Compute $y_{v+1}(t) = x_v(t) + y_v(t)$ and replace $y_v(t)$ in (4.1) by $y_{v+1}(t)$, and x_v by x_{v+1}. An iteration is thus established. Convergence of the sequence generated by this technique falls in the general category of Newton's method in a function space [2,4].

5. Iteration techniques closely related to quasilinear equations. Goodman and Lance [8] have devised an iterative technique for obtaining the missing initial condition. In (4.2) let

$$\frac{\partial f}{\partial y}(t,y_v(t)) = A_v(t)$$

$$f(t,y_v(t)) - y_v'(t) = r_v(t)$$

Then (4.1a) becomes

(5.1) $$x_v' = A_v(t)x_v + r_v(t)$$

subject to the boundary conditions

(4.2b) $$B_1x_v(a) + B_2x_v(b) = 0$$

Consider the adjoint equation

(5.2) $$X_v' = - X_vA_v(t), \quad X_v(b) = B_2$$

where X_v is a $2n \times 2n$ matrix continuously differentiable with respect to t.

Then from (5.1) and (5.2)

(5.3) $$\frac{d}{dt} [X_vx_v] = X_v(t)r_v(t)$$

or, integrating from a to b,

(5.4) $$X_v(b)x_v(b) - X_v(a)x_v(a) = \int_a^b X_v(t)r_v(t)dt$$

If $d_v = x_v(a)$, d_1 is given, and $X_v(a)$ is nonsingular, then (5.4) and (4.2b) give

(5.5) $$d_{v+1} = - X_v^{-1}(a)[B_1d_v + \int_a^b X_v(t)r_v(t)dt]$$

$$v = 1,2, \ldots$$

An alternative method for obtaining the missing initial condition would be to compute the matrix solution to

$$(5.6) \qquad Y_v' = A_v(t)Y_v \qquad Y_v(a) = I$$

where Y_v is a $2n \times 2n$ matrix continuously differentiable with respect to t.

The general solution of (5.1) can be written

$$(5.7) \qquad x_v(t) = v_v(t) + Y_v(t)d$$

where d is an arbitrary constant and $v_v(t)$ is an arbitrary initial value solution to (5.1). If (5.7) satisfies the boundary conditions (4.2b),

$$[B_1 + B_2Y_v(b)]\ d = - B_1v_v(a) - B_2v_v(b)$$

and an iteration for d could be

$$d_{v+1} = [B_1 + B_2]^{-1}\ \{B_2[I - Y_v(b)]d_v - B_1v_v(a) - B_2v_v(b)\}$$

Roberts and Shipman [9] have shown that the iteration described by (5.5) is equivalent to Newton's method. Thus the Kantorovich Theorem [4] can be used to give convergence criteria.

6. Continuity methods. Roberts and Shipman [9] develop the following procedure for solving (3.1) and (3.2). Let $y(a) = d_1$ and integrate (3.1) as an initial value problem until the solution becomes excessively large. (For example, $||y(t_1)|| > R||d_v||$

where R is given). Let us assume that at t_1 the solution of (3.1) is sufficiently "well behaved". Solve the boundary value problem

$$y' = f(t,y) \tag{6.1a}$$

$$B_1 y(a) + B_2 y(t_1) = \gamma \tag{6.1b}$$

for $y_1(t)$ using any of the techniques mentioned in sections 5 and 6. For this solution $y_1(t)$ let $d_2 = y_1(a)$. Now for (6.1a) and the initial condition $y(a) = d_2$ integrate past t_1 until the solution becomes excessively large, and assume it is "well behaved" at $t_2 > t_1$. Then replace t_1 by t_2 in (6.1b) and solve the boundary value problem (6.1) for $y_2(t)$. Letting $d_3 = y_2(a)$ the procedure is continued until b is reached.

The Poincare continuity method [3] involves introducing a new system

$$z' = \sigma f(t,z) \tag{6.2a}$$

$$B_1 z(a) + B_2 z(b) = \gamma \tag{6.2b}$$

Here, if $\sigma = 1$ we are back to (3.1) and (3.2). Let $z(t,d,\sigma)$ be a solution of (6.2a) satisfying $z(a) = d$. To solve (6.2b) we must have

$$\phi(d,\sigma) = B_1 d + B_2 z(b,d,\sigma) - \gamma = 0 \tag{6.3}$$

Now clearly at $\sigma = 0$ the solution of (6.2) is $z(t) = (B_1 + B_2)^{-1}\gamma$, a constant. Also $\det \left[\frac{\partial\phi(d,0)}{\partial d}\right] = \det [B_1 + B_2] \neq 0$. We would like to obtain a solution of (6.3) when $\sigma = 1$. Since $\det \left[\frac{\partial\phi(d,0)}{\partial d}\right] \neq 0$, continuity implies $\det \left[\frac{\partial\phi(d,\sigma)}{\partial d}\right] \neq 0$ for $|\sigma| < \varepsilon$. By the implicit function theorem, there exists a continuous function $d(\sigma)$ on $|\sigma| < \varepsilon$ such that

$$\phi(d(\sigma),\sigma) = 0 \tag{6.4}$$

for all σ such that $|\sigma| < \varepsilon$

By assuming suitable conditions on f and the boundary conditions, it is possible to insure that $\varepsilon > 1$ so that $\sigma = 1$ is a satisfactory solution.

7. Galerkin's method. For (3.1) and (3.2) assume a system of approximating functions $\{\psi_k(t)\}$ to the solution of (3.1) and (3.2), where $\psi_k(t)$ are orthonormal and piecewise continuously differentiable

$$\int_a^b \psi_j(t)\psi_k(t)dt = \delta_{jk} \qquad i,j = 1,2, \dots$$

Let

$$u_N(t) = \sum_{j=1}^{N} \xi_j\phi_j$$

where $\xi_1, \dots ,\xi_N$ are arbitrary. To determine these numbers, compute

$$\int_a^b [u_N'(t) - f(t,u_N(t))]\ \psi_k(t)dt = 0 \qquad K = 1,2, \dots ,N - 1$$

and

$$B_1u_N(a) + B_2u_N(b) = \gamma$$

which is again a nonlinear system of N equations for the N unknowns $\xi_1, \dots, \xi_N$.

8. Power series methods. If $f(t,y)$ in (3.1) is analytic, a solution would be of the form

$$y(t) = d + \sum_{k=1}^{\infty} a_k(t - a)^k \tag{8.1}$$

and conditions (3.2) become

$$B_1d + B_2 \sum_{k=1}^{\infty} a_k(b - a)^k = \gamma \tag{8.2}$$

Since the a_k are functions of d, (8.2) is a transcendental system of equations in d. Of course convergence becomes a problem, since the series (8.1) is not known to converge for all t, $a \leq t \leq b$. If we assume that

$$u_N(x) = d + \sum_{k=1}^{N} a_k(t - a)^k$$

is an approximation to $y(t)$ then (8.2) could be "approximately" solved [3].

Leavitt [10] has considered a solution of the form

$$y(t) = \sum_{k=0}^{N} \sum_{j=0}^{M} \alpha_{kj}t^kd^j \tag{8.3}$$

where d is the initial condition, j, M are multiindices, and $d^j = d_1^{j_1} d_2^{j_2} \ldots d_n^{j_n}$. Then substitution into (3.2) yields

$$(8.4) \qquad B_1 d + B_2 \sum_{k=0}^{N} \sum_{j=0}^{M} \alpha_{kj} t^k d^j = \gamma$$

which is a nonlinear system of equations in $d_1, \ldots, d_n$.

9. Miscellaneous methods. Several of the techniques discussed earlier in chapter 1 for linear systems can be applied to nonlinear systems.

Difference methods can be used if a solution of both the difference equation and the original equation, is known to exist. The theory of "approximate" systems developed by Kantorovich and Akilov [4] could be used to show convergence. Since the formal mechanics of obtaining the difference equations are the same in the linear and the nonlinear case, it will not be done here. See Keller [3] for a derivation of the equations. The technique of invariant imbedding is discussed from the characteristic surface standpoint by Meyer [11] for nonlinear systems. Since invariant imbedding has already been discussed briefly in chapter 1 for linear equations, we shall not go further into this subject.

A novel technique discussed by Bellman and Kalaba [6] is based on backwards and forward integration. For example, assume (2.1) and (2.2), with (2.2b) replaced by $u(b) = \beta$. Then an initial condition $v(a) = d_1$ is chosen, (2.1) is integrated forward to b, and $(u_1(b),v_1(b))$ is obtained. Then the problem is integrated from $(\beta,v_1(b))$ backwards to a, and $(u_2(a),v_2(a))$ is obtained. Then from $(\alpha,v_2(a))$, (2.1) is integrated forward to b and an iteration procedure is established. Convergence is claimed in some instances [6].

Summarizing, inventing techniques for solving boundary value problems is limited only by the imagination of the researcher.

10. Convergence. Convergence must be mentioned in any discussion of iteration techniques for solving equations. By reducing the boundary value problem to that of solving a system of nonlinear equations, many standard techniques are available. Among these are fixed point theorems, such as the contraction principle and Schauder's theorem, [2,3,4]. Since criteria for convergence are given in these references [2,3,4] we shall not repeat them here. Again it should be mentioned that often a given iteration procedure will converge *without* satisfying any of the known sufficient conditons for convergence.

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CHAPTER III

SUBFUNCTION APPROACH TO THE TWO-POINT BOUNDARY VALUE PROBLEM

1. Introduction. We consider the two-point boundary value problem (BVP),

(1.1) $$y''(x) = f(x,y(x),y'(x))$$

(1.2) $$y(a) = A, y(b) = B$$

The subfunction approach to the boundary value problem is to develop the properties of subfunctions, and then to use these to extend local existence theorems to global existence theorems. This approach originated in the work of Perron [7], where he uses subharmonic functions in the study of the Dirichlet problem for harmonic functions with bounded plane domains. The actual application of this approach to the BVP is relatively recent, and it is primarily due to the efforts of Bebernes [1], Fountain and Jackson [4], and Jackson [5].

This chapter will deal with the definition of and elementary properties of subfunctions, local existence theorems and their generalization to global existence theorems, and the relationship between subfunctions and functions satisfying differential inequalities. Throughout we will assume that $f(x,y,y')$ is continuous on

$[a,b] \times (-\infty,\infty) \times (-\infty,\infty)$, $[a,b]$ a compact interval.

2. Preliminaries. A function $\phi(x)$ is said to be a subfunction with respect to solutions of $y'' = f(x,y,y')$ on an interval I in case for any $[x_1,x_2] \subset I$ and any solution $y \in C^{(2)}[x_1,x_2]$, $y(x_i) \geq \phi(x_i)$ for $i = 1,2$ implies $y(x) \geq \phi(x)$ on $[x_1,x_2]$. A function $\psi(x)$ is said to be a superfunction with respect to solutions of $y'' = f(x,y,y')$ on an interval I in case for any $[x_1,x_2] \subset I$ and any solution $y \in C^{(2)}[x_1,x_2]$, $y(x_i) \leq \psi(x_i)$ for $i = 1,2$ implies $y(x) \leq \psi(x)$ on $[x_1,x_2]$.

We will give our results in terms of subfunctions, although there will be exactly analogous results in terms of superfunctions.

To give the reader an intuitive idea of a subfunction, we list some properties.

Remark. First note that a subfunction need not be continuous, (as required by some earlier authors).

Lemma 2.1. *If ϕ is a bounded subfunction on $J \subset I$, then ϕ has at most a countable number of discontinuities on J.*

Lemma 2.2. *If ϕ is a bounded subfunction on $J \subset I$, then ϕ has a finite derivative almost everywhere (a.e.) on J.*

Theorem 2.3. *Assume that the collection of subfunctions $\{\phi_\alpha : \alpha \in A\}$ on the interval $J \subset I$ is bounded above at each point of J.*

Then $\phi_o(x) = \sup_{\alpha\epsilon A}\phi_\alpha(x)$ is a subfunction on J.

Proof: Assume $[x_1,x_2] \subset J$ and assume that $y(x) \epsilon C^{(2)}[x_1,x_2]$ is a solution on $[x_1,x_2]$ with $\phi_o(x) \leq y(x)$ at $x = x_1,x_2$. Then from the definition of $\phi_o(x)$ it follows that $\phi_\alpha(x) \leq y(x)$ at $x = x_1,x_2$ for each $\alpha \epsilon A$. Since each ϕ_α is a subfunction on J, we conclude that $\phi_\alpha(x) \leq y(x)$ on $[x_1,x_2]$ for each $\alpha \epsilon A$. This implies $\phi_o(x) \leq y(x)$ on $[x_1,x_2]$ and ϕ_o is a subfunction on J.

From the definition of a subfunction, it is natural to consider the relationship between subfunctions and differential inequality theory. Necessary and sufficient conditions for subfunctions to satisfy the differential inequality

$\underline{D}\phi'(x) \equiv \lim_{\delta\to 0}\inf \frac{\phi'(x+\delta) - \phi'(x-\delta)}{2\delta} \geq f(x,\phi(x),\phi'(x))$ have been derived. These will be discussed later.

3. *Local Existence*. The local existence that we will need is summarized as the following:

Theorem 3.1. *Let* $M > 0$ *and* $N > 0$ *be given real numbers and let* q *be the maximum of* $|f(x,y,y')|$ *on the compact set* $\{(x,y,y'):a \leq x \leq b, |y| \leq 2M, |y'| \leq 2N\}$. *Then, if*

$\delta = \min[(\frac{8M}{q})^{1/2},\frac{2N}{q}]$, *any* BVP $y' = f(x,y,y')$, $y(x_1) = y_1$, $y(x_2) = x_2$

with $[x_1,x_2] \subseteq [a,b]$, $x_2 - x_1 \le \delta$, $|y_1| \le M$, $|y_2| \le M$, $\left| \frac{y_1-y_2}{x_1-x_2} \right| \le N$ *has a solution* $y(x) \in C^{(2)}[x_1,x_2]$. *Furthermore, given* $\varepsilon > 0$ *there is a solution* $y(x)$ *such that* $|y(x) - w(x)| < \varepsilon$ *and* $|y'(x) - w'(x)| < \varepsilon$ *on* $[x_1,x_2]$ *provided* $x_2 - x_1$ *is sufficiently small where* $w(x)$ *is the linear function with* $w(x_1) = y_1$, $w(x_2) = y_2$.
Essentially, Theorem 3.1 says that on a sufficiently small interval with admissible boundary conditions, the boundary value problem can be solved, and that, furthermore, the solution can be made arbitrarily close to the straight line connecting the boundary points.

Now, using Theorem 3.1 along with some of the properties of subfunctions and superfunctions, we can obtain the following theorem about properties of bounded functions which are simultaneously subfunctions and superfunctions.

Theorem 3.2. *Assume that* $f(x,y,y')$ *is such that* $C^{(2)}$ *solutions of boundary-value problems, when they exist, are unique. That is, assume that, if* $[x_1,x_2] \subset I$ *and* $y_1,y_2 \in C^{(2)}[x_1,x_2]$ *are solutions* of $y'' = f(x,y,y')$ on $[x_1,x_2]$ *with* $y(x_1) = y_2(x_1)$ *and* $y_1(x_2) = y_2(x_2)$, *then* $y_1(x) \equiv y_2(x)$ *on* $[x_1,x_2]$. *Assume that* $z(x)$ *is bounded on each compact subinterval of* $J \subset I$ *and that* $z(x)$ *is simultaneously a subfunction and a superfunction on* J. *Then* $z(x)$ *is a solution of* $y'' = f(x,y,y')$ *on an open subset of* J *the complement of which has measure zero. Furthermore, if* $x_0 \in J^o$ *is a point of continuity of* $z(x)$ *at which* $z(x)$ *does not have a finite derivative, either* $Dz(x_0+) = Dz(x_0-) = +\infty$ *or*

$Dz(x_o+) = Dz(x_o-) = -\infty$. *If* $z_o(x_o + 0) > z(x_o - 0)$, $Dz(x_o+) = Dz(x_o-) = +\infty$, *and if* $z_o(x_o + 0) < z(x_o - 0)$, $Dz(x_o+) = Dz(x_o-) = -\infty$.

These properties of $z(x)$ will be needed later in the study of boundary value problems by the Perron method.

4. *Study of boundary-value problems by subfunction methods.* A bounded real-valued function ϕ defined on $[a,b]$ is said to be an *underfunction* with respect to the boundary-value problem

$$y'' = f(x,y,y'),\ y(a) = A,\ y(b) = B \tag{4.1}$$

in case $\phi(a) \leq A$, $\phi(b) \leq B$, and ϕ is a subfunction on $[a,b]$ with respect to solutions of $y'' = f(x,y,y')$. The bounded function $\psi(x)$ defined on $[a,b]$ is said to be an *overfunction* with respect to the BVP in case $\psi(a) \geq A$, $\psi(b) \geq B$, and ψ is a superfunction on $[a,b]$ with respect to solutions of $y'' = f(x,y,y')$.

Theorem 4.1. *Assume that* $C^{(2)}$ *solutions of BVP's for* $y'' = f(x,y,y')$ *on subintervals of* $[a,b]$ *are unique in the sense of Theorem* 3.2. *Assume that there exists both an overfunction* ψ_o *and an underfunction* ϕ_o *with respect to the BVP* (4.1) *and that* $\phi_o(x) \leq \psi_o(x)$ *on* $[a,b]$. *Let* Φ *be the collection of all underfunctions* ϕ *such that* $\phi(x) \leq \psi_o(x)$ *on* $[a,b]$. *Then* $z(x) = \sup_{\phi\in\Phi}\phi(x)$ *is simultaneously a subfunction and a superfunction on* $[a,b]$.

Definition 4.3: The function $z(x)$ defined in Theorem 4.1 depends on the BVP (4.1) and on the overfunction $\psi_o(x)$. It will be designated by $z(x,\psi_o)$ and will be called a generalized solution of the BVP.

Remark. The justification of the title "generalized solution" follows from Theorem 3.2, which is applicable since $z(x,\psi_o)$ is both a subfunction and a superfunction.

The other properties given in Theorem 3.2, also apply to $z(x,\psi_o)$. The behavior of $z(x,\psi_o)$ at the endpoints of $[a,b]$ is given by the following theorem.

Theorem 4.2. Assume hypotheses of Theorem 4.1 are satisfied, and let $z(x,\psi_o) = z(x)$ be the corresponding generalized solution of BVP (4.1). Then $z(a) = A$. If $Dz(a+) = +\infty$, $z(a+0) \leq z(a)$. If $z(a+0) < A$, $Dz(a+) = -\infty$. Hence, if $Dz(a+)$ is finite, $z(a+0) = z(a) = A$. Similar statements apply at $x = b$.

With the above results, we can divide the study of the BVP by the subfunction approach into two parts: First, to establish the existence of an overfunction ψ_o and an underfunction ϕ_o such that $\phi_o(x) \leq \psi_o(x)$ on $[a,b]$ (this gives us a candidate for a solution, $z(x,\psi_o)$); second, to establish conditions under which the generalized solution $z(x,\psi_o)$ is of class $C^{(2)}[a,b]$ and is a solution of the boundary value problem on $[a,b]$. Theorems 3.2, 4.1, and 4.2 play a major role in achieving the second part, since they tell us it is sufficient to show $Dz(x+)$ is finite on $[a,b)$ and $Dz(x-)$ is finite on $(a,b]$.

A function $\alpha(x)$ is called a lower solution of the differential equation $y'' = f(x,y,y')$ on an interval I in case $\alpha(x) \in C(I) \cap C^{(1)}(I^o)$, I^o the interior of I, and

$$\underline{D}\alpha'(x) \equiv \lim_{\delta\to 0}\inf \frac{\alpha'(x+\delta) - \alpha'(x-\delta)}{2\delta} \geq f(x,\alpha(x),\alpha'(x)) \quad \text{on} \quad I^o.$$

Similarly, $\beta(x)$ is an upper solution if

$$\overline{D}\beta'(x) \equiv \lim_{\delta\to 0}\sup \frac{\beta'(x+\delta) - \beta'(x-\delta)}{2\delta} \leq f(x,\beta(x),\beta'(x)) \quad \text{on} \quad I^o.$$

Proceeding along the line of argument (in the two parts) mentioned above, we obtain first Lemma 4.3, and then Theorem 4.4.

Lemma 4.3. Assume that $f(x,y,y')$ is non-decreasing in y on $[a,b] \times (-\infty,+\infty) \times (-\infty,+\infty)$ for fixed x,y' and is such that lower and upper solutions of the differential equation are subfunctions and superfunctions, respectively. Further assume that there is a $k > 0$ such that $|f(x,0,y') - f(x,0,0)| \leq k|y'|$ on $a \leq x \leq b$ for all y'. Then there exists overfunctions and underfunctions with respect to every BVP on $[a,b]$. With the aid of this lemma, we can then show

Theorem 4.4. Assume that $f(x,y,y')$ is non-decreasing in y on $[a,b] \times (-\infty,+\infty) \times (-\infty,+\infty)$ for fixed x,y' and assume that $f(x,y,y')$ satisfies a Lipschitz condition with respect to y' on each compact subset of $[a,b] \times (-\infty,+\infty) \times (-\infty,+\infty)$ or that solutions of initial-value problems are unique. In addition assume that there is a $k > 0$ such that $|f(x,0,y') - f(x,0,0)| \le k|y'|$ on $[a,b]$ for all y'. Then for any boundary-value problem on $[a,b]$ with an associated overfunction $\psi_o(x)$ the generalized solution $z(x) = z(x,\psi_o)$ belongs to $C^{(2)}(a,b)$ and $z'' = f(x,z,z')$ on (a,b).

Proof: (sketch). Lemma 4.3 guarantees us that with respect to a given BVP on $[a,b]$, there is an overfunction $\psi_o(x)$ and an underfunction $\phi_o(x)$ with $\phi_o(x) \le \psi_o(x)$ on $[a,b]$. Consequently, the generalized solution $z(x) = z(x,\psi_o)$ is defined. Furthermore, the hypotheses imply that solutions of BVP's when they exist, are unique; hence, the conclusions of Theorem 3.2 apply to $z(x)$. Thus, to complete the proof, it is sufficient to show that $Dz(x_o+)$ and $Dz(x_o-)$ are finite at every point of (a,b).

Other representative theorems which can be obtained in this manner are the following:

Theorem 4.5. Assume that $f(x,y,y')$ is nondecreasing in y on $[a,b] \times (-\infty,+\infty) \times (-\infty,+\infty)$ for fixed x,y' and satisfies a uniform Lipschitz condition with respect to y' on $[a,b] \times (-\infty,+\infty) \times (-\infty,+\infty)$. Then for any A,B, the boundary-value problem (4.1) has a unique solution $y(x) \in C^{(2)}[a,b]$.

Corollary 4.6. If $f(x,y)$ is continuous on $[a,b] \times (-\infty,+\infty)$ and is nondecreasing in y for fixed x, then for any A,B, the

<u>boundary-value problem</u> $y'' = f(x,y)$, $y(a) = A$, $y(b) = B$ <u>has a unique solution</u> $y \in C^{(2)}[a,b]$.

Using the subfunction approach, Jackson was able to show a result first proved by Opial and Lasota in 1967, which is that uniqueness of solutions of BVP's implies their existence.

<u>Theorem</u> 4.7. <u>Assume that</u> $I \subset$ Reals <u>is an interval and that</u> $f(x,y,y')$ <u>is continuous on</u> $I \times R^2$. <u>Assume that for every</u> $(x_o,y_o,y_o') \in I \times R^2$, <u>the initial-value problem</u> $y'' = f(x,y,y')$, $y(x_o) = y_o$, $y'(x_o) = y_o'$ <u>has a unique solution</u> $y(x) \in C^{(2)}(I)$. <u>Further, assume that, if for any</u> $[x_1,x_2] \subset I$ <u>and any</u> A,B, <u>the</u> BVP (*) $y'' = f(x,y,y')$, $y(x_1) = A$, $y(x_2) = B$ <u>has a solution</u> $y(x) \in C^{(2)}[x_1,x_2]$, <u>then that solution is unique</u>. <u>Then for any proper subinterval</u> $[x_1,x_2] \subsetneq I$ <u>and any</u> A,B, <u>the</u> BVP (4.1) <u>has a solution</u>.

The basic idea behind the subfunction approach of working in the 'small', and then using these results to obtain some in the 'large', is quite well established and produces fruitful results. So far, practically all theorems obtained in this manner have been already known. In this sense, little new is being added, and it seems certain that any result which has been shown otherwise could also be shown using subfunctions. The important thing to keep in mind is that the subfunction theory is relatively new and may well be leading the development in the field of BVP's in the coming years.

5. <u>Relation between subfunctions and differential inequalities</u>

An interesting question is, what is the relationship between subfunctions and differential inequalities? A function $\alpha(x)$ is called a lower solution of the differential equation $y'' = f(x,y,y')$ on an interval I in case $\alpha(x) \in C(I) \cap C^{(1)}(I^o)$ and

$$\underline{D}\alpha'(x) \equiv \liminf_{\delta\to 0} \frac{\alpha'(x+\delta) - \alpha'(x-\delta)}{2\delta} \geq f(x,\alpha(x),\alpha'(x)) \quad \text{on } I^o.$$

(Similar definition for upper solution).

Theorem 5.1. *Assume that* $\phi \in C(I) \cap C^{(1)}(I^o)$ *is a subfunction on* I *with respect to solutions of* $y'' = f(x,y,y')$. *Then* ϕ *is a lower solution of the differential equation on* I.

That is, a sufficiently smooth subfunction is a lower solution.

The converse, that a lower solution is a subfunction, is not true under just the assumption of continuity of f. To see this, assume so, i.e. that every lower solution is a subfunction. Then, since a solution is a lower solution, we have that every solution is a subfunction. By the definition of a subfunction, this implies that the solution to the BVP must be unique. It is easy to think of a BVP with non-unique solutions. Thus, we have that a theorem which gives sufficient conditions for a lower solution to be a subfunction is automatically a theorem giving sufficient conditions for solutions of BVP's to be unique.

Along these lines, we have

Theorem 5.2. *Let* $f(x,y,y')$ *be non-decreasing in* y *for fixed* x,y' *and satisfy a Lipschitz condition with respect to* y' *on each compact subset of* $[a,b] \times R^2$. *Then a lower solution* ϕ *on a subinterval* $I \subset [a,b]$ *is a subfunction on* I.

If we retain the non-decreasing assumption and alter the other to assuming that solutions of IVP for $y'' = f(x,y,y')$ are unique, then the conclusion is still valid.

As a last result, we have

Theorem 5.3. Assume that solutions of BVP for $y'' = f(x,y,y')$, when they exist, are unique (i.e., if $y_1, y_2 \in C^{(2)}[x_1,x_2]$ are solutions on $[x_1,x_2]$ $[a,b]$ with $y_1(x_i) = y_2(x_i)$, $i = 1,2$, then $y_1(x) \equiv y_2(x)$ on $[x_1,x_2]$). Assume also that each IVP for $y'' = f(x,y,y')$ has a solution which extends throughout $[a,b]$. Then, if $I \subseteq [a,b]$ and $\phi \in C^{(1)}(I)$ is a lower solution on I, then ϕ is a subfunction on I.

As an example of results obtained using lower and upper solutions, we have

Theorem 5.4. There exists a solution y of $y'' = f(x,y,y')$, $y(a) = A$, $y(b) = B$, which is in $C^{(2)}[a,b]$ provided the following conditions hold:

(i) There exists $\alpha,\beta \in C^{(1)}[a,b] \cap C^{(2)}[a,b]$ with α a lower solution and β an upper solution on $[a,b]$. Also $\alpha(x) \le \beta(x)$ for $x \in [a,b]$ and $\alpha(a) \le A \le \beta(a)$, $\alpha(b) \le B \le \beta(b)$.

(ii) f satisfies the Nagumo condition on set $E = \{(x,y) : a \le x \le b,\ \alpha(x) \le y(x) \le \beta(x)$, where $\alpha,\beta \in C[a,b]\}$; that is, there is a positive continuous function h such that $|f(x,y,y')| \le h(|y'|)$ for all $(x,y) \in E$, and $|y'| < +\infty$, where

$$\int_\lambda^\infty \frac{s}{h(s)}\,ds > \max_{x \in [a,b]}\beta(x) - \min_{x \in [a,b]}\alpha(x)$$ with

$\lambda = \max\{|\alpha(b) - \beta(a)|/(b-a), \frac{|\beta(b) - \alpha(a)|}{(b-a)}\}$. _Furthermore, the solution is such that_ $\alpha(x) \leq y(x) \leq \beta(x)$, _and_ $|y'(x)| \leq M$ _on_ $[a,b]$, _where_ $\int_{\lambda}^{M} \frac{s}{h(s)}\, ds = \max_{x \in [a,b]} \beta(x) - \min_{x \in [a,b]} \alpha(x)$.

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CHAPTER IV

LITERATURE SURVEY OF EXISTENCE AND UNIQUENESS THEOREMS

1. Introduction. The literature survey conducted by Mr. York centered on the investigation of techniques used to prove uniqueness and existence theorems for boundary value problems. Basically, the survey dealt with three main areas: (1) contraction mappings, which yield both existence and uniqueness; (2) distance between zeroes, which resulted in improved estimates of uniqueness intervals; and (3) comparison theorems and differential inequalities, which yield bounds on solutions along with existence and uniqueness results. Of the three, the one that seems to offer the most promise for future investigation is the last. Many of the more recent papers in the field of boundary value problems utilize a subfunction or superfunction approach which is in reality part and parcel of the comparison theorems and differential inequalities.

2. Preliminaries. Before proceeding with the actual findings, it will be advantageous to introduce some terminology which we will employ throughout. Consider the differential equation:

$$(2.1) \qquad y''(t) + f(t,y(t),y'(t)) = 0 \quad t \in [a,b].$$

By a solution to the first boundary value problem (denoted 1st BVP), we will mean a solution of (2.1) satisfying the imposed boundary condition, $y(a) = A$ and $y(b) = B$, A, B real numbers. By a solution to the second boundary value problem (denoted 2nd BVP), we will mean a solution of (2.1) satisfying the boundary condition $y'(a) = m$ and $y(b) = B$. Included in this case, of course, is the boundary condition $y(a) = A$, $y'(b) = m$.

Throughout, we will always assume that $f(t,y(t),y'(t))$ is a continuous function on $[a,b] \times (-\infty,\infty) \times (-\infty,\infty)$, unless otherwise stated. Additional assumptions on the function f will be stated fully when needed. Frequently, we assume f to be Lipschitzian, i.e., there exist two non-negative constants K and L such that whenever (t,y,y') and (t,x,x') are in the domain of f, then the inequality $|f(t,y,y') - f(t,x,x')| \leq K|y - x| + L|y' - x'|$ holds. Remark: if $f(t,y,y')$ is linear in y and y', then $f(t,y,y')$ is Lipschitzian for t confined to some finite closed interval. More generally, if $f(t,y,y')$ has bounded partial derivatives, $\frac{\partial f}{\partial y}(t,y,y')$, $\frac{\partial f}{\partial y'}(t,y,y')$, then $f(t,y,y')$ is Lipschitzian with

$$K = \sup_{(t,y,y')} \left|\frac{\partial}{\partial y} f(t,y,y')\right| \quad \text{and} \quad L = \sup_{(t,y,y')} \left|\frac{\partial}{\partial y'} f(t,y,y')\right|.$$

By a solution to the initial value problem (IVP), we mean a solution of (2.1) satisfying $y(a) = A$, $y'(a) = m$ or $y(b) = B$, $y'(b) = m$.

3. Linear and nonlinear boundary value problems.

Before proceeding to the three main areas, let us give some preliminary results concerning nonlinear vs. linear problems, and boundary value problem vs. initial value problems. By studying such simple BVP's as the linear problem

$$y''(t) + y(t) = 0$$
$$y(0) = 0 \quad y(b) = B$$

and the nonlinear problem

$$y''(t) + |y(t)| = 0$$
$$y(0) = 0, \; y(b) = B,$$

we find the following: (1) for linear problems, for fixed a, existence and/or uniqueness fail for exceptional values of b; while (2), for nonlinear problems, both existence and uniqueness may fail for all b greater than or equal to a certain b_o.

4. Application of initial value theory to boundary value problems.

Frequently, much use is made of the theory of the initial value problem in obtaining theorems for the boundary value problem. The most used results are that continuity of $f(t,y,y')$ guarantees existence of a solution, and the added assumption of the Lipschitz condition implies uniqueness and continuability of the solution. Also, under the same assumptions, we have the continuous dependence on initial conditions and parameters. As an example, we can use these results of IVP theory to prove

Theorem 4.1. *If* $f(t,y,y')$ *satisfies a Lipschitz condition on* $[a,b] \times (-\infty,+\infty) \times (-\infty,\infty)$ *and is bounded*, i.e., $|f(t,y,y')| \leq N$ *for every* (t,y,y'), *then the 1st BVP has a solution*.

Remark: The assumption of boundedness, here, is rather restrictive, but the theorem does illustrate how knowledge of IVP's implies that of BVP's.

Also, we can make certain assertions concerning the relation between existence and uniqueness intervals for the two BVP's.

Theorem 4.2. *Let* $a < c < b$. *If uniqueness holds for all 2nd BVP* $y(a) = A$, $y'(c') = m$ *whenever* $c' \in (a,c]$, *and if uniqueness holds for all 2nd BVP* $y(b) = B$, $y'(c') = m$ *whenever* $c' \in [c,b)$, *then uniqueness holds for the 1st BVP on* $[a,b]$.

Let $a < c < b$. Then we have that: if all IVP on $[a,b]$ have unique solutions, and if both 1st and 2nd BVP have unique solutions on $[a,c]$ and also on $[c,b]$, then the 1st BVP has a unique solution on $[a,b]$. This result is of importance later in establishing the best uniqueness interval for the 1st BVP.

5. *Contraction mappings*. We are now ready to study the contraction mapping approach. Let S be a normed linear space S with norm denoted by $||\cdot||$. The space will be called *complete* if every Cauchy sequence converges to a point in S. An operator T mapping S into S will be called a *contraction mapping* if there is a number α, $0 < \alpha < 1$, such that, for all $x,y \in S$, $||Tx - Ty|| \leq \alpha||x - y||$. The whole idea behind the contraction mapping approach is contained in the following theorem.

Theorem 5.1. *Every contraction mapping* T *defined on a complete normed linear space* S *has one and only one fixed point* (i.e., y = Ty has exactly one solution). In this one theorem, we have both the assertion of existence and uniqueness of a solution. Our problem then is how to view the BVP as a map.

We can accomplish this end by employing a Green's function to rewrite our differential equation as an integral equation. The Green functions we will use are

$$G(t,s) = \begin{cases} \dfrac{(b-t)(s-a)}{b-a} & a \le s \le t \le b \\[2ex] \dfrac{(b-s)(t-a)}{b-a} & a \le t \le s \le b. \end{cases}$$

and

$$H(t,s) = \begin{cases} s-a & a \le s \le t \le b \\ t-a & a \le t \le s \le b. \end{cases}$$

Hence the 1st BVP with zero boundary conditions is equivalent to

$$y(t) = \int_a^b G(t,s)\ f(s,y(s),y'(s))\ ds \quad a \le t \le b$$

and

$$y'(t) = \int_a^b \frac{\partial G}{\partial t}(t,s)\ f(s,y(s),y'(s))\ ds.$$

For non-zero boundary conditions, we would add to the right-hand side of the first equation the linear function

$$\ell(t) = \frac{bA - aB + (B - A)t}{b - a}$$

which satisfies $\ell(a) = A$, $\ell(b) = B$, and is a solution of $y''(t) = 0$. For this reason, boundary conditions may be taken to be zero without loss of generality. Similarly, we would add $\ell'(t)$ to the right-hand side of the second equation.

When $A = 0$ and $m = 0$, the 2nd BVP is equivalent to

$$y(t) = \int_a^b H(t,s)\ f(s,y(s),y'(s))\ ds$$

and

$$y'(t) = \int_a^b \frac{\partial H}{\partial t}(t,s)\ f(s,y(s),y'(s))\ ds.$$

For A and m not zero, we add to the right-hand side of the first equation, the solution of $y''(t) = 0$ which satisfies $y(a) = A$, $y'(b) = m$, namely, $A + m(t - a)$, and its derivative to the second. The properties of the Green's functions that we will need are
$\int_a^b G(t,s) \le \frac{(b-a)^2}{8}$ and $\int_a^b \left|\frac{\partial G}{\partial t}(t,s)\right| ds \le \frac{b-a}{2}$.

Just as Picard did some seventy years ago, we can define an iterative procedure as follows: starting with any continuously differentiable function, we define $y_n(t)$ as the solution of

$$y_n''(t) + f(t,y_{n-1}(t),y'_{n-1}(t)) = 0 \quad n = 1, 2, \ldots$$

$$y_n(a) = A \quad y_n(b) = B.$$

Here, in effect, we have defined a map T, $y_n = Ty_{n-1}$, on the space of all continuously differentiable functions into itself. We need now to investigate under what assumptions will T be a contraction mapping. If T is, then the fixed point y, $y = Ty$, will be the unique solution to the original BVP.

We first consider the special case of the 1st BVP in which y' does not appear. Take S to be the space of all continuous functions on $[a,b]$ with norm $||u|| = \max_{a\leq t\leq b} |u(t)|$. S is then a complete normed linear space. We arrive at

Theorem 5.2. Let $f(t,y)$ satisfy a Lipschitz condition. Then the 1st BVP has a unique solution whenever $b - a < \sqrt{8/K}$. This result is not best possible meaning that existence and/or uniqueness may not fail when $b - a = \sqrt{8/K}$.

To obtain a sharper estimate, we change the norm by introducing a non-negative weight function $w(t)$, which we later choose. Define a new norm $||u||_1 = \max_{a\leq t\leq b} \frac{|u(t)|}{w(t)}$. With the appropriate choice of $w(t)$, we find that

Theorem 5.3. Let $f(t,y)$ satisfy a Lipschitz condition. Then the 1st BVP has a unique solution whenever $\frac{K(b - a)^2}{\pi^2} < 1$. This result is best possible. The important point here is that in this special case, i.e., no y', the Picard iterations do converge on the best possible uniqueness and existence interval. This result does not hold in the general case.

For the more general case with y' included, we introduce the space S of continuously differentiable functions on $[a,b]$ with norm $||u|| = \max_{a\leq t\leq b} (K|u(t)| + L|u'(t)|)$, where K and L are the Lipschitz constants.

__Theorem__ 5.4. __Let__ $f(t,y,y')$ __satisfy a Lipschitz condition. If__ $\frac{K(b-a)^2}{8} + \frac{L(b-a)}{2} < 1$, __then the 1st BVP has one and only one solution. This result, however, is not best possible.__

To obtain results for the 2nd BVP, we let S consist of the space of all continuously differentiable functions on $[a,b]$, with norm

$$||u|| = \max \left| \max_{a\leq t\leq b} \frac{|u(t)|}{w(t)}, \max_{a\leq t\leq b} \frac{|u'(t)|}{v(t)} \right|$$

in which we introduce weight functions for both the function and its derivative. Let us introduce the following notation. If $u(t)$ is any non-trivial solution of $u''(t) + Lu'(t) + Ku(t) = 0$ which vanishes at $t = a$, then its derivative vanishes at $t = a + \alpha(L,K)$, where

$$\alpha(L,K) = \begin{cases} \frac{2}{(4K - L^2)^{\frac{1}{2}}} \cos^{-1} \frac{L}{2\sqrt{K}} & \text{if } 4K - L^2 > 0 \\ \frac{2}{(L^2 - 4K)^{\frac{1}{2}}} \cosh^{-1} \frac{L}{2\sqrt{K}} & \text{if } 4K - L^2 < 0,\ L > 0,\ K > 0 \\ \frac{2}{L} & \text{if } 4K - L^2 = 0,\ L > 0 \\ +\infty & \text{otherwise} \end{cases}$$

Then we can show

Theorem 5.5. *If* $f(t,y,y')$ *satisfies a Lipschitz condition, then, if* $b - a < \alpha(L,K)$, *then the 2nd BVP has one and only one solution. This result is best possible.*

In conclusion, Picard's iteratives converge on largest possible interval for the 2nd BVP, but only in the special case for the 1st BVP. By employing results on relations between uniqueness intervals for BVP's, we get

Theorem 5.6. *Let* $f(t,y,y')$ *satisfy a Lipschitz condition. If* $b - a < 2\alpha(L,K)$, *then the 1st BVP has one and only one solution. Result is best possible.*

In some sense, the above can be considered a global result. By requiring that T be a contraction mapping not on the whole space but on some ball, we can obtain local existence and uniqueness of solutions.

Theorem 5.7. *Let* $f(t,y,y')$ *be continuous on* $[a,b] \times [-N,N] \times [-\frac{4N}{b-a}, \frac{4N}{b-a}]$, *and satisfy a Lipschitz condition there. Let* $m = \max_{a \leq t \leq b} |f(t,0,0)|$, $M = \max |f(t,y,y')|$ *for* $|y| \leq N$, $|y'| \leq \frac{4N}{b-a}$, $t \in [a,b]$. *Then if* $\alpha = \frac{K(b-a)^2}{8} + \frac{L(b-a)}{2} < 1$ *and either* $\frac{m(b-a)^2}{8} \leq N(1-\alpha)$ or $\frac{M(b-a)^2}{8} \leq N$, *then the 1st BVP has one and only one solution* $y(t)$ *such that* $|y(t)| \leq N$, $|y'(t)| \leq \frac{4N}{b-a}$ *for* $t \in [a,b]$. If $f(t,y,y')$ is continuous and bounded on $[a,b] \times D$, $D \subset R^{2n}$, then by the Shauder fixed point theorem, local solutions exist. A more general fixed point theorem might well result in existence and uniqueness theorems with less conditions imposed on the function $f(t,y,y')$.

6. Estimates of uniqueness intervals.

The second main area of consideration was obtaining better estimates of the uniqueness intervals. To this end, we introduce a generalized Lipschitz condition. Instead of just the two constants K and L, we now have four constants K_1, K_2, L_1, L_2 and linear functions G_1 and G_2. Then

$$G_1(y - x, y' - x') \le f(t,y,y') - f(t,x,x') \le G_2(y - x, y' - x')$$

where

$$G_1(y,y') = \begin{cases} K_1 y + L_1 y' & \text{if } y \ge 0 \text{ and } y' \ge 0 \\ K_1 y + L_2 y' & \text{if } y \ge 0 \text{ and } y' \le 0 \\ K_2 y + L_2 y' & \text{if } y \le 0 \text{ and } y' \le 0 \\ K_2 y + L_1 y' & \text{if } y \le 0 \text{ and } y' \ge 0 \end{cases}$$

and

$$G_2(y,y') = \begin{cases} K_2 y + L_2 y' & \text{if } y \ge 0 \text{ and } y' \ge 0 \\ K_2 y + L_1 y' & \text{if } y \ge 0 \text{ and } y' \le 0 \\ K_1 y + L_1 y' & \text{if } y \le 0 \text{ and } y' \le 0 \\ K_1 y + L_2 y' & \text{if } y \le 0 \text{ and } y' \ge 0. \end{cases}$$

Note that if we have $-K_1 = K_2 = K$, and $-L_1 = L_2 = L$, then we just have the usual Lipschitz condition. The advantage of this approach is simply that more information is contained in four constants than two. Now, we can distinguish between such differential equations as $y'' - y = 0$ and $y'' + y = 0$. The 1st BVP for the first equation has unique solutions on all finite intervals $[a,b]$; whereas, the 1st BVP for the second equation has a unique solution only on intervals of length less than π. With the old Lipschitz condition, these two equations fell into the same class.

One of the most fundamental results is what might be called an 'alternative'

Theorem 6.1. The maximum interval on which all of the equations in the family (K_1, K_2, L_1, L_2 specified) have unique solutions to all first boundary value problems coincides with the minimum interval on which none of the "unforced" equations in the family has a non-trivial solution with two zeroes. We say that an equation $y''(t) + f(t,y(t),y'(t)) = 0$ is unforced if $f(t,0,0) \equiv 0$. Similarly, uniqueness holds for all the second boundary value problems in the class, if and only if, none of the unforced equations has a non-trivial solution such that both it and its derivative have a zero on the interval. The main idea here is that without loss of generality, when considering the question of uniqueness, it is sufficient to study the distance between zeroes of unforced equations.

When considering the family of all differential equations associated with given Lipschitz constants (K_1, K_2, L_1, L_2), it is

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$$\beta(L,K) = \begin{cases} \frac{2}{(4K - L^2)^{\frac{1}{2}}} \cos^{-1} \frac{-L}{2\sqrt{K}} & \text{if } 4K - L^2 > 0 \\ \frac{2}{(L^2 - 4K)^{\frac{1}{2}}} \cosh^{-1} \frac{-L}{2\sqrt{K}} & \text{if } 4K - L^2 < 0,\ L < 0, \quad > 0 \\ -\frac{2}{L} & \text{if } 4K - L^2 = 0,\ L < 0 \\ +\infty & \text{otherwise} \end{cases}$$

By studying the relationships between $u(t)$, $v(t)$, and $y(t)$, we arrive at the following uniqueness theorem.

<u>Theorem</u> 6.2. <u>Suppose</u> $f(t,y,y')$ <u>is continuous and satisfies a generalized Lipschitz condition</u>.

(1) <u>If</u> $0 < b - a < \alpha(L_2,K_2)$, <u>then the</u> 2nd BVP

$$y''(t) + f(t,y(t),y'(t)) = 0$$

$$y(a) = A,\ y'(b) = m$$

<u>has one solution at most</u>.

(2) <u>If</u> $0 < b - a < \beta(L_1,K_2)$, <u>the</u> 2nd BVP

$$y''(t) + f(t,y(t),y'(t)) = 0$$

$$y'(a) = m,\ y(b) = B$$

<u>has one solution at most</u>.

(3) *If* $0 < b - a < \alpha(L_2,K_2) + \beta(L_1,K_2)$, *then the 1st BVP*

$$y''(t) + f(t,y(t),y'(t)) = 0$$

$$y(a) = A, \; y(b) = B$$

has one solution at most. These results, (1), (2), (3), *are best possible.*

7. *Comparison theorems.* Chapters 5 and 6 of Bailey, Shampine, and Waltman's book develop comparison theorems based on differential inequality theory and use them to prove existence of solutions to boundary value problems. A sample comparison theorem would be:

Theorem 7.1. *Let* $v(t)$ *be a twice continuously differentiable function on* $[a,b]$ *satisfying*

$$v''(t) + f(t,v(t),v'(t)) > 0.$$

(*Assume* f *continuous on* $[a,b] \times (-\infty,+\infty) \times (-\infty,+\infty)$ *and that all IVP's and all* BVP's *have unique solutions existing throughout the interval* $[a,b]$).

(1) *If* $u(t)$ *is a solution of* $u''(t) + h(t,u(t),u'(t)) = 0$ (7.1) *which agrees with* $v(t)$ *in both value and slope at some point* $t_o \in [a,b]$, *then* $v(t) > u(t)$ *for* $t \neq t_o$.

(2) *If* $u(t)$ *is a solution of* (7.1) *which agrees with* $v(t)$ *in value at* a *and at* b, *then* $v(t) < u(t)$ *for* $t \neq a,b$.

Although of interest in itself as Theorem 7.1 relates solutions of the differential equation with functions satisfying the associated differential inequality, this comparison theorem can be used to prove

Theorem 7.2. Suppose $f(t,y,y')$ is continuous and satisfies a general Lipschitz condition. Then the second boundary value problem

$$y''(t) + f(t,y(t),y'(t)) = 0, \ y(a) = A \quad y'(b) = m$$

has a unique solution whenever $0 < b - a < \alpha(L_2,K_2)$. Result best possible. Note that this result is not new, as it has been obtained before using fixed point theorems.

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PERTURBATION THEORY USEFUL IN PARALLEL SHOOTING METHODS[1]

By John H. George*

1. Introduction. Shooting methods are techniques for solving boundary value problems by reduction to initial value problems. They have been the subject of numerous recent papers, (see Roberts and Shipman [1], Osbourne [2], Bailey and Shampine [3] and Keller [4]). The main computational advantage of shooting methods is the availability of sophisticated numerical procedures for integrating initial value problems. The difficulties in the use of shooting methods occur because (i) the initial value problem is "unstable", (for now, unstable means a small variation in the initial conditions gives rise to large variations in the corresponding solution) and (ii) it is difficult to obtain "good" starting values for most iterative techniques used to solve nonlinear problems. Osbourne [2] shows how the parallel shooting technique (see Keller [4] for a comprehensive explanation of parallel shooting) makes positive contributions to both of these problems.

Bailey and Shampine [3] have given several concepts closely related to stability and boundedness (as in Hahn [5]) of initial value problems. In their treatment a Lipschitz condition is assumed on the differential equation. Using differential inequalities obtained from the Lipschitz condition, solution bounds are obtained.

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In this paper Liapunov theory will be applied to the shooting methods to obtain solution estimates. These estimates will then be used to determine a suitable interval length in the parallel shooting technique.

2. Preliminaries. Let us consider the system of n differential equations

$$(2.1) \qquad y' = f(t,y), \quad ' = \frac{d}{dt} ,$$

where $f(t,y)$ is defined and continuous on $[a,b] \times D$, $D \subset R^n$, subject to the boundary conditions

$$(2.2) \qquad B_1 y(a) + B_2 y(b) = c ,$$

where B_1 and B_2 are $n \times n$ matrices, $B_1 + B_2$ is nonsingular, and c is a constant n vector. Let us assume that (2.1) and (2.2) has a unique solution and that the initial value problem (2.1) and

$$(2.3) \qquad y(a) = \alpha$$

has a unique solution on $[a,b]$.

A real valued function $\phi(r)$ belongs to class K $(\phi \in K)$ if it is defined, continuous, and strictly increasing on $0 \leq r < \infty$, and $\phi(0) = 0$. The solution $y(t)$ of (2.1) and (2.2) can always be transferred to the solution $u(t) \equiv 0$ of a new equation as follows:

Let $z(t)$ be any solution of (2.1) and (2.3). Then if $u(t) = z(t) - y(t)$,

$$(2.4) \qquad u' = f(t,u + y(t)) - f(t,y(t)) = g(t,u)$$

and $g(t,0) \equiv 0$. Thus, $u(t) \equiv 0$ is a solution of (2.4).

The solution $u(t) \equiv 0$ of (2.4) is uniformly stable if there exists a function $\phi \in K$ such that if $u(t,a,\alpha)$ is a solution of (2.1) and (2.3), then $||u(t,a,\alpha)|| \leq \phi(||\alpha||)$, $a \leq t \leq b$. If $\phi(r) = Lr$ then L is called a growth factor. Obtaining growth factors by Liapunov theory can then be used to estimate the interval length in the parallel shooting technique.

3. Liapunov theory. A Liapunov function $v(t,u)$ is a real valued continuous function which is locally Lipschitzian on $[a,b] \times D$, $D \subset R^n$. Let

$$v_g'(t,u) = \limsup_{h \to 0^+} \frac{1}{h}[v(t + h, x + hg(t,x)) - v(t,x)] .$$

Then, as in Yoshizawa [6], it can be shown that

$$v_g'(t,u) = \frac{dv}{dt}(t,u(t,a,\alpha)) .$$

Theorem 3.1. If there exists a Liapunov function $v(t,u)$ such that

$$\phi_1(||u||) \leq v(t,u) \leq \phi_2(||u||), \quad \phi_1,\phi_2 \in K$$

and $v'(t,u) \leq 0$, then $u(t) \equiv 0$ is uniformly stable.

Proof. This is a standard theorem [5], but since the proof indicates how growth factors could be determined, it will be included.

Since $v_g'(t,u) \leq 0$, $v(t_1,u(t_1,a,\alpha)) \geq v(t_2,u(t_2,a,\alpha))$ for $t_1 \leq t_2$. Then

$$\phi_2(||\alpha||) \geq v(a,\alpha) \geq v(t,u(t,a,\alpha)) \geq \phi_1(||u(t,a,\alpha)||) .$$

By the properties of class K, ϕ_1^{-1} exists and $\phi_1^{-1}\phi_2 \in K$, [5]. Then

$$||u(t,a,\alpha)|| \leq \phi_1^{-1}\phi_2(||\alpha||) = \phi(||\alpha||), \quad \phi = \phi_1^{-1}\phi_2 \in K .$$

Remark. If an L can be found so that $\phi(||\alpha||) \leq L||\alpha||$ then L would be a growth factor. For example, if $\phi_1(r) = c_1r^2$, $\phi_2(r) = c_2r^2$, $c_1,c_2 > 0$, then $L = \sqrt{c_1/c_2}$.

4. Perturbation theory. In the study of perturbations, the most widely used methods involve the construction of Liapunov functions for the perturbed system [5,7,8]. Other methods are based on the

variation of parameter technique [9,10]. Several of the more useful theorems will be given here.

Let $A(t)$ be an $n \times n$ matrix with continuous elements on $[a,b]$, let $A^T(t)$ be the transpose of $A(t)$, and let $\lambda(A(t))$ denote the largest eigenvalue of $\frac{1}{2}(A(t) + A^T(t))$ on $[a,b]$.

Lemma 4.1 (Wazewski [11]). Every solution $z(t)$ of the linear system

$$z' = A(t)z \tag{4.1}$$

satisfies

$$||z(t)|| \leq ||z(a)|| \exp\left[\int_a^t \lambda(A(s))ds\right],\ a \leq t \leq b\ . \tag{4.2}$$

Proof. Let $v(z) = z^Tz = ||z||^2$. Then

$$v'_{Az}(z) = [z']^Tz + z^Tz' = z^T[A(t) + A^T(t)]z \leq 2\lambda(A(t))v.$$

Thus by solving this differential inequality, (4.2) is obtained.

Let $u(t,a,\alpha)$ be a solution of (2.4) through (a,α) and let $g_u(t,u)$ represent the Jacobian matrix. Let $\lambda(g_u(t,u))$ denote the largest eigenvalue of $\frac{1}{2}[g_u(t,u) + g_u^T(t,u)]$ and suppose

$$\lambda(g_u(t,u)) \leq h(t) \quad \text{for} \quad a \leq t \leq b,\ u \in D, \tag{4.3}$$

where $h(t)$ is a continuous function defined on $[a,b]$. Then in an analogous manner to Lemma 4.1 we have

<u>Lemma</u> 4.2 (Brauer [9]). If $\alpha \varepsilon$ a convex subset $\hat{D}$ of D then for all t for which all solutions with initial values in $\hat{D}$ remain in D,

(4.4) $$||u(t,a,\alpha)|| \leq ||\alpha|| \exp [\int_a^t h(s)ds] .$$

Consider

(4.5) $$z' = A(t)z + F(t,z) ,$$

where A and f are matrix and vector functions respectively. Assume $A(t)$ is continuous on $a \leq t \leq b$ and $F(t,z)$ is continuous on $a \leq t \leq b$, $z \varepsilon D$. Let $Z(t)$ be the fundamental solution of (4.1) satisfying $Z(a) = I$.

<u>Theorem</u> 4.1 (Coppel [10]). Let the solution $z(t) \equiv 0$ of (4.1) be uniformly stable, and let F satisfy

$$||F(t,z)|| \leq \gamma(t) \; ||z|| ,$$

where $\gamma(t)$ is a continuous non-negative function satisfying $\int_a^b \gamma(t)dt < \infty$. Then there exists a positive constant L such that for any solution $z(t,a,\alpha)$ of (4.5) through (a,α),

$$||z(t,a,\alpha)|| \leq L||\alpha||, \; a \leq t \leq b .$$

Remark. If $||B|| = \sup_{||x||=1} ||Bx||$, and if $||Z(t)Z^{-1}(s)|| \le K$ for $a \le s \le t \le b$, then $L = K \exp [K \int_a^b r(s)ds]$ is a growth factor [10]. These results should give conservative estimates of L.

5. Applications. Holt [12] and Osbourne [2] consider the differential equation

$$\frac{d^2y}{dt^2} - (1 + t^2)y = 0 \; , \tag{5.1a}$$

$$y(0) = 1, \; y(b) = 0 \; . \tag{5.1b}$$

According to Holt, the solution of (5.1) cannot be obtained for $b > 3.5$ by conventional shooting methods. Reverting to (4.1),

$$z = \begin{pmatrix} y \\ y^1 \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & 1 \\ 1 + t^2 & 0 \end{pmatrix},$$

$$\lambda(A(t)) = 2 + t^2, \text{ and from (4.2)},$$

$$||z(t)|| \le ||z(a)|| \exp \left[2t + \frac{t^3}{3}\right], \quad 0 \le t \le b \; .$$

If $L = 10^5$, then $\exp [2t + \frac{t^3}{3}] \le L = 10^5$ holds for $t \le 2.7$. If $L = 10^3$ as is suggested by Keller [4, p. 68], then $t \le 2$.

The interval length of 2.7 is a reasonable estimate to the interval length 3.5 obtained by Holt by numerical computation.

Consider the problem

$$\text{(5.2a)} \qquad y'' + \sin y = \sin \frac{\pi t}{b} ,$$

$$\text{(5.2b)} \qquad y(0) = 0,\ y(b) = 0 ,$$

considered for $b = 3.1$ by Bailey and Shampine [3].

Suppose $y(t)$ is a solution of (5.2) and $z(t)$ is a solution of (5.2a) and

$$\text{(5.3)} \qquad y(0) = 0,\ y(a) = \alpha .$$

Then letting $v(t) = z(t) - y(t)$ we have

$$v'' = - \sin(v + y(t)) + \sin y(t) .$$

Writing

$$h = \begin{pmatrix} v \\ v' \end{pmatrix}, \quad g(t,u) = \begin{pmatrix} v' \\ - \sin(v + y(t)) + \sin y(t) \end{pmatrix},$$

we have

$$g_u(t,u) = \begin{pmatrix} 0 & 1 \\ - \cos(v + y(t)) & 0 \end{pmatrix}$$

and

$$\lambda(g_u(t,u)) = 1 - \cos(v + y(t)) \leq 2 ,$$

From 4.4, if $u(t,0,\alpha)$ is a solution of $u' = g(t,u)$ through $\begin{pmatrix} 0 \\ \alpha \end{pmatrix}$ at $t = 0$, we have

$$||u(t,0,\alpha)|| \leq |\alpha| \exp 2t , \tag{5.4}$$

and if $\exp 2t \leq L = 10^2$, then $t \leq 2.3$.

Remark. A disadvantage of this method is that singularities must be known beforehand. For example, in (5.2) there is a singularity at $b = \pi$ which would not appear in (5.4). In any case, because of the ease of application, it is felt that the described methods should yield useful information in many parallel shooting problems as, for example, in the method for determining the parallel shooting interval length as described by Keller [4, p. 68].

Use of a quadratic Liapunov function and the theory of first approximation would yield other estimates of a similar nature to those already given.

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J. R. GEORGE and W. G. SUTTON, Application of Liapunov theory to boundary value problems.

Abstract: The theory of Liapunov's direct method is developed for boundary value problems occurring in ordinary differential equations. Conditions are given in terms of a Liapunov function which are sufficient to insure uniqueness and existence of solutions to boundary value problems. A suitable Liapunov function is obtained to give conditions obtained by Hartman as special cases.

Application of Liapunov Theory to Boundary Value Problems

By J. H. George and W. G. Sutton[1]

1. Introduction. Many techniques and theories developed for boundary value problems of ordinary differential equations originated as initial value concepts. For example, fixed point theorems [1], Picard's iteration [4] and differential inequalities [2,3,4] are commonly used techniques in both initial and boundary value problems.

A theoretical technique that has proved extremely useful in initial value theory [5], but does not seem to be given its due in boundary value theory, is the direct method of Liapunov. In initial value problems, since necessary and sufficient Liapunov function conditions are obtained for many types of solution behavior, the theory can be considered as a unifying concept. That is, all known sufficient conditions can be obtained by choosing the proper Liapunov function, as is done by Yoshizawa [5, p.10] for the Lipschitz condition as a uniqueness criterion. (See George [7] for a Liapunov function for more general uniqueness theorems). Yoshizawa [3] has obtained a Liapunov result for boundary value problems, giving necessary and sufficient conditions for the boundary value solution to remain between two estimates obtained by differential inequalities.

We shall develop a Liapunov theory for existence and uniqueness of solutions of boundary value problems. Also the existing theory of Hartman will be shown to be included in our theory by a suitable Liapunov function selection.

2. Preliminaries and notation. Let us consider the system of ordinary differential equations

$$(1) \qquad x'' = f(t,x,x')$$

where x and f are n-vectors, $' = \frac{d}{dt}$ and f is a function defined and continuous on a domain $D = [a,b] \times \bar{D}$, where $[a,b]$ is an interval on the real line and $\bar{D} \subset R^{2n}$.

The boundary value problem is that of finding a solution $x(t)$ of (1) on $[a,b]$ satisfying for $b > a$,

$$(2) \qquad x(a) = A, \quad x(b) = B.$$

The corresponding initial value problem is obtaining a solution $x(t)$ of (1) satisfying the initial values

$$(3) \qquad x(a) = A, \quad x'(a) = \alpha.$$

A Liapunov function $V(t,x,x')$ is a continuous, locally Lipschitzian with respect to (x,x'), real valued function. Corresponding to $V(t,x,x')$ define

$$V_f'(t,x,x') = \lim_{h \to 0^+} \inf h^{-1}[V(t+h,x+hx',x'+hf(t,x,x')) - V(t,x,x')].$$

Lemma 2.1. (Yoshizawa, [5, p. 4]). If $V(t,x,x')$ is a Liapunov function and $x(t)$ is a solution of (1), then $V(t,x(t),x'(t))$ is nonincreasing (nondecreasing) if and only if $V_f'(t,x,x') \leq 0$ $(V_f'(t,x,x') \geq 0)$.

Lemma 2.2. Let $x(t)$ be a solution of (1) satisfying

$$x(a) = 0, \; x'(a) = 0 \tag{4}$$

and either $x(t) \not\equiv 0$ or $x'(t) \not\equiv 0$ on $[a,b]$. Then there exists an open interval $I \subset [a,b]$ such that both $x(t) \neq 0$ and $x'(t) \neq 0$ on I.

Proof. Suppose $x(t) \not\equiv 0$ on $[a,b]$. Then by continuity of $x(t)$ there exists an open interval $I_1 = (t_o,t_1)$ such that $x(t) \neq 0$ on I_1, and $x(t_o) = 0$. Assume $x'(t) \equiv 0$ on I_1. Then $x(t) \equiv c$ on I_1, where c is a constant, and $c \equiv 0$ since $x(t_o) = 0$. Since $x'(t)$ is continuous there exists an open interval $I_2 \subset I_1$ where $x'(t) \neq 0$. Thus on $I = I_1 \cap I_2$, $x(t) \neq 0$ and $x'(t) \neq 0$. If $x'(t) \not\equiv 0$ on $[a,b]$ a similar argument concludes the proof.

Let $\langle x,y \rangle$ be the standard inner product in a Hilbert space and let $||x||^2 = \langle x,x \rangle$ be the corresponding norm.

3. Uniqueness and continuability. Let $u(t)$ be a solution of the boundary value problem (1) and (2). What conditions on f insure that $u(t)$ is the only solution of (1) and (2)? Many criteria on f are given to insure uniqueness; for example, the Lipschitz condition [4] and nondecreasing properties [2, p. 317] are standard sufficient conditions. We shall develop a Liapunov theory for boundary value problems which gives sufficient conditions for uniqueness.

Suppose $v(t)$ is another solution of (1) and (2). If $x(t) = u(t) - v(t)$, then x must satisfy

(5) $$x'' = f(t,x + v, x' + v') - f(t,v,v') = F(t,x,x')$$

(6) $$x(a) = 0, x(b) = 0$$

Now $F(t,0,0) \equiv 0$ and hence $x(t) \equiv 0$ is a solution satisfying (5) and (6). We have proved the following:

Lemma 3.1. $x(t) \equiv 0$ is the only solution of (5) and (6) if and only if $u(t)$ is the only solution of (1) and (2).

Theorem 3.1. For F defined in (5), if there exists a Liapunov function $V(t,x,x')$ defined on D such that

(i) $V(t,x,x') = 0$ if $x = 0$

(ii) $V(t,x,x') > 0$ if $x \neq 0$

(iii) $V'_F(t,x,x') \geq 0$ in the interior of D,

then there is at most one solution of (1) and (2).

Proof. By Lemma 3.1 it suffices to show $x(t) \equiv 0$ is the unique solution of (5) and (6). Suppose there exists a solution $\phi(t)$ of (5) such that $\phi(a) = 0$, $\phi(b) = 0$ and $\phi(t_1) \neq 0$ for some $t_1 \in (a,b)$. Then there exists $[t_2,t_3] \subset [a,b]$ such that $t_1 \in (t_2,t_3)$, $\phi(t_2) = \phi(t_3) = 0$, and $\phi(t) \neq 0$ on (t_2,t_3). Thus $V(t,\phi(t),\phi'(t)) > 0$ on (t_2,t_3). Since $V'_F(t,x,x') \geq 0$, $V(t,\phi(t),\phi'(t))$ is nondecreasing along the solution $\phi(t)$ and thus $V(t_3,\phi(t_3),\phi'(t_3)) > 0$, a contradiction.

Corollary 3.1. If there exists a Liapunov function as in Theorem 3.1 <u>except</u> that (ii) holds when <u>both</u> x and x' are $\neq 0$, then a solution of (1) and (2) is unique.

Proof. Follows as in Theorem 3.1 by using Lemma 2.2.

Example. In Hartman, [1, p. 427], the condition $\langle x,F\rangle + ||x'|| > 0$ if $x \neq 0$, $\langle x,x'\rangle = 0$ is given to insure uniqueness of $x \equiv 0$. By choosing $V(t,x,x') = \langle x,x\rangle$ all conditions of Theorem 3.1 are satisfied since Hartman's condition insures V does not have a maximum, and hence $V_F'(t,x,x') \geq 0$.

Because it may be convenient to give continuability conditions, such as are required by Jackson [2] in the theory of sub and super-functions, as Liapunov conditions, it will simply be mentioned that the necessary and sufficient conditions for continuability are given by Yoshizawa [5, pp. 11-17].

4. Existence. If f is bounded, then it is possible to give local existence results such as the following: (see also [1, p. 424]).

Theorem 4.1. (Jackson, [2, p. 309]). Let $M > 0$ and $N > 0$ be given real numbers and let $q = \max ||f(t,x,x')||$ on $[a,b] \times \{x: ||x|| \leq 2M\} \times \{x': ||x'|| \leq 2N\}$. Let $\delta = \min [(8M/q)^{1/2},(2N/q)]$. Then for any $[t_1,t_2] \subset [a,b]$ such that $t_2 - t_1 \leq \delta$,

$$x(t_1) = x_1,\ x(t_2) = x_2,\ t_1 < t_2 \tag{7}$$

where $||x_1|| \leq M$, $||x_2|| \leq M$, $||(x_2 - x_1)/(t_2 - t_1)|| \leq N$, the boundary value problem (1) and (7) has at least one solution.

Hartman [1, p. 435] has introduced the following concept, where $\ell = t_2 - t_1$.

A solution $x(t)$ of (1) satisfying $||x(t)|| \leq M$ on $[t_1,t_2]$ has property A_ℓ if there exists a constant $N(\ell)$ such that $||x'(t)|| \leq N$ on $[t_1,t_2]$.

Lemma 4.1. (Hartman [1, p. 435]). Let there exist an M such that every solution of (1) satisfies $||x(t)|| \leq M$ and has property A_ℓ. Then (1) and (7) has at least one solution.

We are now in a position to give Liapunov sufficient conditions to insure the hypothesis of Lemma 4.1.

Theorem 4.2. Let $x(t)$ be a solution of the boundary value problem (1) and (7), where $||x_1|| \leq M$, $||x_2|| \leq M$. Let there exist a Liapunov function $V(t,x,x')$ defined on $D_1 = [a,b] \times \{x: ||x|| \geq M\} \times R^n$ such that

(i) $V(t,x,x') = 0$ whenever $||x|| = M$

(ii) $V(t,x,x') > 0$ whenever $||x|| > M$

(iii) $V_f'(t,x,x') \geq 0$ in the interior of D_1.

Then $||x(t)|| < M$ on $[t_1,t_2]$.

Proof. Follows as in Theorem 3.2.

Theorem 4.3. Let $x(t)$ be a solution of (1) and (7) satisfying $||x(t)|| \leq M$ on $[t_1,t_2]$. Let there exist a Liapunov function $V(t,x,x')$ defined on $D_2 = [a,b] \times \{x: ||x|| \leq M\} \times \{x': ||x'|| \geq K\}$, where K is sufficiently large, satisfying

(i) $V(t,x,x') \geq a(||x'||)$ where $a(r)$ is a positive continuous function defined on $[K,\infty)$ such that $a(r) \to \infty$ as $r \to \infty$.

(ii) $V'_f(t,x,x') \leq 0$ in the interior of D_2.

Then $x(t)$ has property A_ℓ.

Proof. Let $x(t)$ be such a solution, where $x'(t_1) \geq K$. We can then find a constant K_1 such that $V(t_1,x(t_1),x'(t_1)) \leq K_1$ and an $N > K$ such that $a(||x'||) > K_1$ when $||x'|| > N$. We shall now show $||x'(t)|| \leq N$ on $[t_1,t_2]$. For if not, there exists a $t_3 \in (t_1,t_2)$ where $||x'(t_3)|| > N$. But

$$K_1 \geq V(t_1,x(t_1),x'(t_1)) \geq V(t_3,x(t_3),x'(t_3)) \geq a(||x'(t_3)||) > K_1,$$

a contradiction. Hence $x(t)$ has property A_ℓ.

Theorem 4.4. Suppose there exist two Liapunov functions having the properties given in Theorems 4.2 and 4.3 respectively. Then the boundary value problem (1) and (7) has at least one solution.

Proof. Since every solution initiating in

$$D_3 = [t_1,t_2] \times \{x: ||x|| \leq M\} \times \{x': ||x'|| \leq N\}$$

remains in D_3 by Theorems 4.2 and 4.3, the function f can be restricted to the set D_3. Since f is bounded on this compact set, a solution exists to the boundary value problem.

Example. Hartman [1, p. 433] gives the following condition to insure $||x(t)|| \leq M$.

$$\langle x,f\rangle + ||x'||^2 > 0 \text{ if } \langle x,x'\rangle = 0 \text{ and } ||x|| \geq M.$$

If $V(t,x,x') = \langle x,x \rangle - M^2$, then Hartman's condition implies V evaluated along a solution $x(t)$ of (1) does <u>not</u> have a maximum at any point $t \in [t_1,t_2]$ where $||x(t)|| \geq M$. Also this V satisfies all conditions of Theorem 4.2, thus insuring $||x(t)|| \leq M$ on $[t_1,t_2]$.

5. Obtaining existence from uniqueness. This interesting concept was introduced by Lasota and Opial [6] and Jackson [2]. We shall restrict our considerations in this section to second order differential equations where $f(t,x,y)$ is defined, continuous and real valued on the strip $D_4 = (a,b) \times R^2$. Let $D_5 = [t_1,t_2] \times R^2$ where $a < t_1 < t_2 < b$.

Theorem 5.1. (Lasota and Opial [6, p. 2]). Assume solutions to initial value problems through any point of D_5 are unique. If there exists at most one solution of (1) and (7) for every pair (t_1,x_1), $(t_2,x_2) \in (a,b) \times R$ then there exists one and only one solution of this problem.

Theorem 5.2. If solutions to initial value problems through any point of D_5 are unique and there exists a Liapunov function as in Theorem 3.1, then there exists one and only one solution of (1) and (7) as in Theorem 5.1.

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Footnotes

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