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Technical Report 32-1459

*Reduction of the Viscous Shock-Layer Equations to
Boundary-Layer Equations*

W. A. Schneider

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**JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA**

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Preface

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Abstract

The shock layer at a blunt body in hypersonic flow is studied in the Reynolds number regime where strong (nonlinear) vorticity interaction occurs between the viscous and the inviscid parts of the shock layer. The differential equations of a thin shock layer are reduced to classical boundary-layer equations on the basis of the assumption that the pressure across the viscous part of the shock layer varies only slightly. The transformations of variables that accomplish this reduction are given for various coordinate systems.

Reduction of the Viscous Shock-Layer Equations to Boundary-Layer Equations

I. Introduction

The classical boundary-layer theory applies only if the Reynolds number is sufficiently high. At lower Reynolds numbers, so-called "second-order effects" (vorticity interaction, displacement, curvature, slip, and temperature jump) are important (Ref. 1). With a further decrease of the Reynolds number, the whole shock layer becomes viscous and merges with the thickened shock to a "merged layer" (Ref. 2).

The following four methods have been used to extend the boundary-layer theory to lower Reynolds numbers:

- (1) On the basis of certain qualitative arguments, classical boundary-layer equations have been applied to the viscous shock layer in the stagnation region where the pressure is approximately constant (Refs. 3, 4, and 5).
- (2) The method of matched asymptotic expansions has led to a second-order boundary-layer theory, which is applicable if the deviations from the classical boundary-layer equations are small (Ref. 6).

- (3) The Navier–Stokes equations have been integrated numerically in the stagnation region by the use of a similarity assumption (Ref. 7).
- (4) The Navier–Stokes equations have been reduced to a parabolic system of equations on the basis of a thin shock-layer approximation (Refs. 7, 8, and 9).

The purpose of this report is to show that the differential equations of a thin viscous shock layer in the nonlinear vorticity interaction regime can be reduced to the classical boundary-layer equations (with pressure being constant across the whole layer) by means of new variables. The boundary conditions, however, differ from their classical counterparts in some aspects, depending on the coordinate system that is used.

II. Basic Equations

Two-dimensional and axisymmetric steady flows are studied by the use of a boundary-layer coordinate system x, y , as indicated in Fig. 1. Let us assume that the shock layer is thin when compared to $1/K$, where K is the longitudinal curvature of the body (defined as positive if the body is convex). The conservation equations of mass,

momentum, and energy can then be written in the following form (see Ref. 9):

$$\frac{\partial}{\partial x} (r^j \rho u) + \frac{\partial}{\partial y} (r^j \rho v) = 0 \quad (1a)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{1}{\rho} \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) \quad (1b)$$

$$-Ku^2 + \frac{1}{\rho} \frac{\partial p}{\partial y} = 0 \quad (1c)$$

$$u \frac{\partial H}{\partial x} + v \frac{\partial H}{\partial y} = \frac{1}{\rho} \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} + \mu u \frac{\partial u}{\partial y} \right) \quad (1d)$$

The velocity components in x - and y -directions are denoted by u and v , respectively; j equals 0 for two-dimensional flow and 1 for axisymmetric flow; p is the pressure, ρ is the density, T is the absolute temperature, r is the distance of a point at the body surface from the axis of symmetry (not used in case of two-dimensional flow), μ is the shear viscosity, k is the thermal conductivity, and H is the total enthalpy, which is given in the present approximation by

$$H = h + \frac{u^2}{2} \quad (2)$$

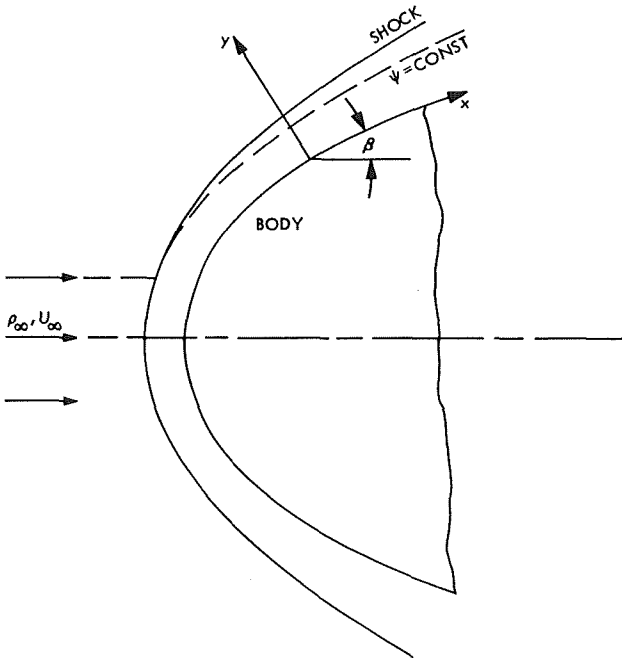


Fig. 1. Coordinate systems

with h as specific enthalpy of the gas. Equation (1) applies to a general fluid.

To satisfy the continuity equation, Eq. (1a), a stream function ψ is introduced by means of the equations

$$\frac{\partial \psi}{\partial x} = -r^j \rho v \quad (3a)$$

and

$$\frac{\partial \psi}{\partial y} = +r^j \rho u \quad (3b)$$

With ψ and $\bar{x} = x$ as new independent variables (von Mises coordinates), the following equations of momentum and energy are obtained from Eq. (1):

$$u \frac{\partial u}{\partial \bar{x}} + \frac{1}{\rho} \frac{\partial p}{\partial \bar{x}} = r^{2j} u \frac{\partial}{\partial \psi} \left(\mu \rho u \frac{\partial u}{\partial \psi} \right) \quad (4a)$$

$$\frac{\partial p}{\partial \psi} - \frac{K}{r^j} u = 0 \quad (4b)$$

$$u \frac{\partial H}{\partial \bar{x}} = r^{2j} u \frac{\partial}{\partial \psi} \left[\rho u \left(k \frac{\partial T}{\partial \psi} + \mu u \frac{\partial u}{\partial \psi} \right) \right] \quad (4c)$$

Partial derivatives with respect to x are taken with y held constant. Derivatives with respect to \bar{x} , however, are taken with ψ fixed. Inverting Eq. (3) yields

$$\frac{\partial y}{\partial \bar{x}} = \frac{v}{u} \quad (5a)$$

$$\frac{\partial y}{\partial \psi} = \frac{1}{r^j \rho u} \quad (5b)$$

Equation (5a) can be used for determining v if necessary, and Eq. (5b) gives a relationship between ψ and y , which is needed for transforming the solution of Eq. (4) back to the boundary-layer coordinates x, y .

The stream function ψ defined by Eq. (3) contains a constant of integration. This constant is chosen so that $\psi = 0$ at the body surface. Then ψ represents the mass flow between the streamline $\psi = \text{const}$ and the body surface (per unit depth for two-dimensional flow and per unit azimuthal angle for axisymmetric flow). Therefore,

the value ψ_s of the stream function at the shock point given by the radius r is (Fig. 1)

$$\psi_s = \rho_\infty U_\infty \frac{r^{1+j}}{1+j} \quad (6)$$

Velocity, density, pressure, and specific enthalpy of the uniform, free stream are denoted by U_∞ , ρ_∞ , p_∞ , and h_∞ , respectively. The density ratio across the shock is defined by

$$\epsilon = \frac{\rho_\infty}{\rho_s} \quad \left(= \frac{\gamma-1}{\gamma+1} \text{ for a perfect gas} \right) \quad (7)$$

where γ is the ratio of specific heats and ϵ is assumed to be very small in accordance with the thin-layer approximation. With terms of order ϵ neglected, the flow quantities immediately behind the shock are

$$\left. \begin{aligned} u_s &= U_\infty \cos \beta \\ v_s &= -U_\infty \epsilon \sin \beta \\ p_s &= p_\infty + \rho_\infty U_\infty^2 \sin^2 \beta \\ h_s &= h_\infty + \frac{1}{2} U_\infty^2 \sin^2 \beta \end{aligned} \right\} \quad (8)$$

where β is the body inclination angle (Fig. 1). It is implied in Eq. (8) that viscosity and heat-conduction effects are negligibly small immediately behind the shock (as well as in front of the shock, of course) which means that the Reynolds number is assumed to be sufficiently high to justify the concept of an adiabatic shock wave. Thus, we exclude the so-called merged layer regime from consideration, and direct attention to the (linear and nonlinear) vorticity interaction regime.

According to the Eqs. (6) and (8), the outer boundary conditions for the shock-layer equations of Eq. (4) for $\psi = \psi_s$ are:

$$\left. \begin{aligned} u &= u_s \\ p &= p_s \\ h &= h_s \end{aligned} \right\} \quad (9)$$

Slip and temperature jump at the body surface should be considered (Ref. 1) in relation to the inner-boundary conditions. Cheng (Ref. 9) pointed out that the velocity u and the temperature jump ΔT at the body surface are of

the following orders of magnitude:

$$\begin{aligned} \frac{u}{u_s} &\sim \left(\epsilon \frac{T_w}{T_0} \right)^{1/2} C_F \\ \frac{\Delta T}{T_0} &\sim \left(\epsilon \frac{T_w}{T_0} \right)^{1/2} C_H \end{aligned} \quad (10)$$

where T_0 is the (adiabatic) stagnation temperature, T_w is the wall temperature, and C_F and C_H are the coefficients of skin friction and heat transfer, respectively. Hence, the effects of velocity slip and temperature jump are negligibly small for a highly cooled body; therefore, the following nonslip and nontemperature-jump conditions are used in the present analysis for $\psi = 0$:

$$\begin{aligned} u &= 0 \\ T &= T_w \end{aligned} \quad (11)$$

However, the use of the simple conditions (nonslip and nontemperature-jump) is just a matter of convenience. The problem of reducing the shock-layer equations to boundary-layer equations is not affected by the boundary conditions at the body surface.

III. Reduction to Boundary-Layer Equations in von Mises Coordinates

Equation (4), which describes the flow field of a thin viscous shock layer, is similar to the well-known system of boundary-layer equations. There is, however, an essential difference. The pressure is taken to be constant across the layer in boundary-layer theory, whereas Eq. (4b) shows that in the thin shock layer there is a nonzero pressure gradient in the direction normal to the body surface. This pressure gradient yields a pressure difference between the shock and the body surface of the order of, according to Eq. (4),

$$p_s - p_w \sim \frac{K}{r^j} u_s \psi_s \sim \rho_\infty U_\infty^2 Kr \cos \beta \quad (12)$$

This pressure difference must not be neglected (the stagnation region and special body shapes excepted). Despite this fact, the viscous shock-layer equations can be reduced to boundary-layer equations by means of the following analysis.

This analysis is based on the assumption that the pressure variations *across the viscous part* of the shock layer are negligibly small. In this case, *viscous part* means the part of the shock layer in which viscosity and heat conduction have an essential effect on the flow field. The assumption is obviously justified if the total pressure variation across the whole shock layer is small, as is the case in the stagnation region. But, more generally, it is also justified if the viscous part of the shock layer is sufficiently thin. This does not necessarily mean that the viscous part of the shock layer should be very thin in comparison with the total thickness of the shock layer. The primary contribution to the pressure difference given in Eq. (12) comes from the outer parts of the shock layer where the centrifugal forces are high; that is, from the parts of the shock layer where viscosity is least important. Thus, we can say that the following results apply in a Reynolds number regime that covers the classical boundary-layer regime, the linear-vorticity interaction regime (where second-order effects can be taken into account as small perturbations), and—most important—the nonlinear vorticity interaction regime (which cannot be treated by second-order boundary-layer theory).

The assumption of small pressure variations across the viscous part of the shock layer has three obvious consequences. First, as in boundary-layer theory, the pressure p_w at the body surface can be calculated by methods for inviscid flow fields. In consistence with the thin-layer approximation, the uniformly valid solution given in Ref. 10 might be used, for instance. Second, the viscosity and heat conduction terms of Eq. (4) can be taken with local pressure $p(\bar{x}, \psi)$ replaced by surface pressure $p_w(\bar{x})$. Third, in the viscous part of the shock layer, also the pressure gradient term $(1/\rho)\partial p/\partial \bar{x}$ can be taken with p_w substituted for p ; that is, this term can be replaced by

$$\frac{1}{\rho(p_w, h)} \frac{dp_w}{d\bar{x}}$$

It will now be shown that the last substitution applies not only in the viscous part, but also in the inviscid part of the shock layer.

The pressure-gradient term $(1/\rho)\partial p/\partial \bar{x}$ is an order of magnitude smaller (in terms of the density ratio ϵ) than the leading term $u\partial u/\partial \bar{x}$. Despite this fact, however, the pressure gradient term should be retained to keep the equations uniformly valid (Ref. 7). The reason for this can be seen in the following estimate. When the viscous term is omitted, and when the inviscid form of

the tangential momentum equation is integrated, we obtain

$$u^2 - u_*^2 + 2 \int_{p_*}^p \frac{dp}{\rho(p, s)} = 0 \quad (13)$$

where the subscript * refers to conditions immediately behind the shock at the point where the streamline $\psi = \text{const}$ intersects the shock. The entropy s remains constant on a streamline in inviscid flow without shock waves. This is incorporated in Eq. (13). Because the pressure variation dp/p with entropy held constant is of the same order of magnitude as the density variation $d\rho/\rho$, the magnitude of the integral term in Eq. (13) can be evaluated as

$$\int_{p_*}^p \frac{dp}{\rho(p, s)} \sim \frac{p}{\rho} \ln \frac{\rho_*}{\rho} \sim \epsilon U_\infty^2 \ln \frac{p_*}{p} \quad (14)$$

Since the gas may expand strongly on streamlines coming from the stagnation region, so that $\ln(p_*/p)$ can become very large, it is now obvious that the integral term in Eq. (13) and, correspondingly, the tangential pressure-gradient term in Eq. (4) must not be neglected, although $\epsilon \ll 1$ has been assumed (Ref. 10). However, following the method that led to a uniformly valid solution of the inviscid problem (Ref. 10), we can show that the pressure $p(\bar{x}, \psi)$ in the tangential pressure-gradient term can be replaced by the surface pressure $p_w(\bar{x})$. The integral in Eq. (13) is divided into two parts to give

$$u^2 - u_*^2 + 2 \int_{p_*}^{p_w} \frac{dp}{\rho(p, s)} + 2 \int_{p_w}^p \frac{dp}{\rho(p, s)} = 0 \quad (15)$$

The first integral is of the same order of magnitude as the integral shown in Eq. (14) and, therefore, should not be neglected. The second integral can be similarly estimated to be

$$\int_{p_w}^p \frac{dp}{\rho(p, s)} \sim \epsilon U_\infty^2 \ln \frac{p}{p_w} \quad (16)$$

Results of inviscid flow-field calculations for the smooth body shapes, which are of practical interest, show that

the pressure somewhere in a thin shock layer is of the same order of magnitude as the pressure on the body surface at the same value of x (Ref. 10). Since effects of viscosity certainly cannot change the wall pressure by an order of magnitude, we may assume that $\ln(p/p_w)$ is of order 1 in all cases of practical interest. Hence, according to Eq. (16), the second integral in Eq. (15) can be neglected as a higher order term; Eq. (15) then becomes

$$u^2 - u_*^2 + 2 \int_{p_*}^{p_w} \frac{dp}{\rho(p, s)} = 0 \quad (17)$$

Differentiating Eq. (17) with respect to \bar{x} , and observing that p_* , u_* , and s do not depend on \bar{x} , we obtain

$$u \frac{\partial u}{\partial \bar{x}} + \frac{1}{\rho(p_w, s)} \frac{dp_w}{d\bar{x}} = 0 \quad (18)$$

Since the entropy s is not a convenient variable in viscous flow (in contrast to inviscid flow), we substitute for s by means of the thermodynamic equation of state $s = s(p, h)$. According to the general thermodynamic relations,

$$\begin{aligned} \left(\frac{\partial s}{\partial h}\right)_p &= \frac{1}{T} \\ \left(\frac{\partial s}{\partial p}\right)_h &= -\frac{1}{\rho T} \end{aligned} \quad (19)$$

the entropy variation due to a pressure change is an order of magnitude smaller (in terms of the density ratio ϵ) than the entropy variation due to an enthalpy change; therefore, we may write

$$\rho(p_w, s) = \rho[p_w, s(p, h)] = \rho[p_w, s(p_w, h)] + \dots = \tilde{\rho}(p_w, h) \quad (20)$$

Thus, with the pressure $p(\bar{x}, \psi)$ replaced by $p_w(\bar{x})$ in the inviscid terms as well as in the viscous terms, Eqs. (4a) and (4c) become

$$u \frac{\partial u}{\partial \bar{x}} + \frac{1}{\tilde{\rho}} \frac{dp_w}{d\bar{x}} = r^{2j} u \frac{\partial}{\partial \psi} \left(\tilde{\mu} \tilde{\rho} u \frac{\partial u}{\partial \psi} \right) \quad (21a)$$

and

$$u \frac{\partial H}{\partial \bar{x}} = r^{2j} \frac{\partial}{\partial \psi} \left[\tilde{\mu} \tilde{\rho} u \left(\frac{1}{\tilde{P}r} \frac{\partial H}{\partial \psi} + \frac{\tilde{P}r - 1}{\tilde{P}r} u \frac{\partial u}{\partial \psi} \right) \right] \quad (21b)$$

where new variables are defined by

$$\left. \begin{aligned} \tilde{\rho} &= \rho(p_w, h) \\ \tilde{\mu} &= \mu(p_w, h) \\ \tilde{k} &= k(p_w, h) \\ \tilde{c}_p &= c_p(p_w, h) \end{aligned} \right\} \quad (22a)$$

and a Prandtl number (which does not necessarily have to be a constant) has been introduced as

$$\tilde{P}r = \frac{\tilde{c}_p \tilde{\mu}}{\tilde{k}} \quad (22b)$$

with c_p as specific heat at constant pressure. The boundary conditions, Eqs. (9) and (11), can simply be taken from Section II of this report. Equation (21), which consists of two equations for the two dependent variables u and H , is completely identical with the classical system of boundary-layer equations for u and H in von Mises coordinates. However, $\tilde{\rho}$, $\tilde{\mu}$, etc., are not the real, local values of density, viscosity, etc. To obtain the physically real density, viscosity, etc., we should calculate the pressure in the shock layer. This calculation may be done by integrating Eq. (4b), which has been disregarded so far. Then we obtain

$$p = p_w + \frac{K}{r^j} \int_0^\psi u d\psi \quad (23)$$

Finally, the solution obtained in von Mises coordinates \bar{x}, ψ can be transformed back into the usual boundary-layer coordinates x, y by the integration of Eq. (5), which yields

$$y = \frac{1}{r^j} \int_0^\psi \frac{d\psi}{\rho u} \quad (24)$$

with $\rho = \rho(p, h)$. However, the skin friction and heat transfer at the wall can be determined without having transformed the solution. By using Eq. (5), we obtain the following expressions for the heat flux q_w and the shear stress τ_w at the wall:

$$\begin{aligned} q_w &= \left(k \frac{\partial T}{\partial y} \right)_w = r^j \tilde{k} \tilde{\rho} \left(u \frac{\partial T}{\partial \psi} \right)_w \\ \tau_w &= \left(\mu \frac{\partial u}{\partial y} \right)_w = r^j \tilde{\mu} \tilde{\rho} \left(u \frac{\partial u}{\partial \psi} \right)_w \end{aligned} \quad (25)$$

IV. Transformed Boundary-Layer Coordinates

It was shown in the previous section that the viscous shock-layer equations, if they are written in von Mises coordinates, can be reduced to boundary-layer equations by the replacement of the local pressure with the wall pressure. However, this substitution is not permitted in the transformation equation (Eq. 24), which is equivalent to the continuity equation. Therefore, the original shock-layer equations of Eq. (1), written in boundary-layer coordinates x and y , cannot be reduced to the corresponding boundary-layer equations if we simply neglect the pressure variations across the layer. The following transformation, however, will accomplish the reduction:

$$\left. \begin{aligned} \tilde{x} &= x \\ \tilde{y} &= \int_0^y \frac{\rho}{\tilde{\rho}} dy \\ \tilde{v} &= v \frac{\rho}{\tilde{\rho}} + u \frac{\partial}{\partial x} \int_0^y \frac{\rho}{\tilde{\rho}} dy \end{aligned} \right\} \quad (26)$$

where $\rho = \rho(p, h)$ and $\tilde{\rho} = \rho(p_w, h)$ in accordance with the definition in Eq. (22a). From Eqs. (3) and (26) we obtain, by straightforward calculations,

$$\left. \begin{aligned} \frac{\partial \psi}{\partial \tilde{x}} &= -r^j \tilde{\rho} \tilde{v} \\ \frac{\partial \psi}{\partial \tilde{y}} &= +r^j \tilde{\rho} u \end{aligned} \right\} \quad (27) \quad \tilde{\tau} = \tilde{\mu} \frac{\partial u}{\partial \tilde{y}} \quad (29)$$

Thus, the original flow field [$u = u(x, y)$, $v = v(x, y)$, ...] and the transformed flow field [$u = u(\tilde{x}, \tilde{y})$, $\tilde{v} = \tilde{v}(\tilde{x}, \tilde{y})$, ...] have the same stream function ψ . That is to say, the transformation from the old variables (x, y, ρ, u, v) to the new variables ($\tilde{x}, \tilde{y}, \tilde{\rho}, u, \tilde{v}$) has been chosen so that the auxiliary coordinates $\tilde{x} = x$ and ψ , which are related to the old coordinates by a von Mises transformation in terms of the old dependent variables, are also related to the new coordinates by a von Mises transformation, but in terms of the new dependent variables. It follows that

Eq. (21) is equivalent to

$$\frac{\partial}{\partial \tilde{x}} (r^j \tilde{\rho} u) + \frac{\partial}{\partial \tilde{y}} (r^j \tilde{\rho} \tilde{v}) = 0 \quad (28a)$$

$$u \frac{\partial u}{\partial \tilde{x}} + \tilde{v} \frac{\partial u}{\partial \tilde{y}} + \frac{1}{\tilde{\rho}} \frac{dp_w}{d\tilde{x}} = \frac{1}{\tilde{\rho}} \frac{\partial}{\partial \tilde{y}} \left(\tilde{\mu} \frac{\partial u}{\partial \tilde{y}} \right) \quad (28b)$$

$$u \frac{\partial H}{\partial \tilde{x}} + \tilde{v} \frac{\partial H}{\partial \tilde{y}} = \frac{1}{\tilde{\rho}} \frac{\partial}{\partial \tilde{y}} \left\{ \frac{\tilde{\mu}}{\tilde{P}r} \left[\frac{\partial H}{\partial \tilde{y}} + (\tilde{P}r - 1) u \frac{\partial u}{\partial \tilde{y}} \right] \right\} \quad (28c)$$

which is the classical system of boundary-layer equations in terms of the new variables defined by Eq. (26).

V. Crocco's Transformation

The general features of the present theory, and in particular the effect of the outer-boundary conditions on the flow-field characteristics, can be shown by the application of Crocco's transformation to Eq. (28). For simplicity, let us assume two-dimensional flow. (In case of axisymmetric flow, we can first apply Mangler's transformation by which the equations of axisymmetric boundary layers are reduced to the equations of two-dimensional flow, and then we can use Crocco's transformation.) In terms of the variables used in Eq. (28), Crocco's transformation is based on $\xi = \tilde{x} = x$ and u as independent variables and the use of the transformed shear stress

as one of the dependent variables. When Eq. (28) is transformed and when \tilde{v} is eliminated (Ref. 11), the following equations are obtained:

$$-u \frac{\partial}{\partial \xi} \left(\frac{\tilde{\mu} \tilde{\rho}}{\tilde{\tau}} \right) + \frac{dp_w}{d\xi} \frac{\partial}{\partial u} \left(\frac{\tilde{\mu}}{\tilde{\tau}} \right) = \frac{\partial^2 \tilde{\tau}}{\partial u^2} \quad (30a)$$

$$\begin{aligned} \tilde{\mu} \tilde{\rho} u \frac{\partial H}{\partial \xi} - \tilde{\mu} \frac{dp_w}{d\xi} \frac{\partial H}{\partial u} &= \tilde{\tau}^2 \frac{\partial}{\partial u} \left(\frac{1}{\tilde{P}r} \frac{\partial H}{\partial u} \right) + \frac{1 - \tilde{P}r}{\tilde{P}r} \tilde{\tau} \frac{\partial \tilde{\tau}}{\partial u} \frac{\partial H}{\partial u} \\ &\quad - \tilde{\tau} \frac{\partial}{\partial u} \left(\frac{1 - \tilde{P}r}{\tilde{P}r} \tilde{\tau} u \right) \end{aligned} \quad (30b)$$

These differential equations are, of course, formally identical with the classical boundary-layer equations for Crocco's variables. The nonslip and no-temperature jump conditions at the wall, together with the tangential momentum equation, yield the following inner-boundary condition for $u = 0$:

$$H = h_w, \quad \tilde{\tau} \frac{\partial \tilde{\tau}}{\partial u} = \tilde{\mu} \frac{dp_w}{d\xi} \quad (31)$$

These equations, too, are formally identical with the corresponding inner-boundary condition of the classical boundary-layer theory. The outer-boundary conditions, however, will show a difference between the present formulation and the classical boundary-layer theory. In boundary-layer theory, the asymptotical transition from the viscous boundary layer to the inviscid flow field is formulated as $\tau = 0$ for $u = u_\infty$ (with u_∞ as velocity of the inviscid flow at the wall). In the viscous shock layer, however, the outer-boundary condition is controlled by the shock wave. We formulate the outer boundary condition for $u = u_s$:

$$\tilde{\tau} = \tilde{\tau}_s, \quad H = H_s = H_\infty \quad (32)$$

where u_s is the tangential velocity component at the shock and $\tilde{\tau}_s$ is the transformed shear stress immediately behind the shock.

Let us recall that viscosity and heat-conduction effects are very small immediately behind the shock in the vorticity interaction regime (including the nonlinear vorticity interaction regime). Hence, $\tilde{\tau}_s$ may be calculated by the use of inviscid flow equations together with ordinary Rankine-Hugoniot conditions. The result is

$$\tilde{\tau}_s = -\tilde{\mu} \frac{\tilde{\rho}_s}{\rho_s} \frac{u_s}{v_s} \frac{du_s}{d\xi} \quad (33)$$

Applying the thin-layer approximation, we finally obtain the following outer-boundary condition (with β denoting body inclination angle) for $u = U_\infty \cos \beta$:

$$\tilde{\tau} = \frac{\rho_\infty U_\infty^2}{\epsilon Re_s} \frac{\tilde{\rho}_s}{\rho_s} \frac{K \cos \beta}{K_0}, \quad H = H_\infty \quad (34)$$

where Re_s is a Reynolds number defined by

$$Re_s = \frac{\rho_\infty U_\infty}{\mu_s K_0} \quad (35)$$

with K_0 as longitudinal curvature at the stagnation point. Again, the density ratio across the shock is assumed to be very small.

The boundary condition of Eq. (34) is of the same form as the outer-boundary condition in Crocco's classical boundary-layer equations. Equation (34) differs from its classical counterparts only by the fact that $\tilde{\tau}$ is nonzero at the outer edge of the layer. The classical boundary-layer limit is reached when $\tilde{\tau} \rightarrow 0$ with $Re_s \rightarrow \infty$. The effect of viscosity at the outer edge of the layer is controlled by the parameter ϵRe_s , which is essentially the parameter K^2 introduced by Cheng (Ref. 8).

VI. Conclusion

It has been shown that the equations of a thin shock layer in the nonlinear vorticity interaction regime can be reduced to classical boundary-layer equations in terms of von Mises coordinates \bar{x}, ψ . Then a new variables transformation has been given by which the shock-layer equations are transformed into classical boundary-layer equations in terms of a boundary-layer type of coordinate system \tilde{x}, \tilde{y} . Finally, Crocco's transformation has been applied to show the main features of the analysis.

The following is a list of three suggested applications:

- (1) Determination of heat transfer and skin friction on a blunt body in the linear and nonlinear vorticity interaction regime by the use of existing boundary-layer calculation schemes.
- (2) Transfer of existing analytical results and known features of boundary-layer flow to viscous shock-layer flow.
- (3) Generalization of the present reduction method to more complex problems, such as three-dimensional shock layers and shock layers with nonequilibrium flow.

Nomenclature

<p>C_F coefficient of skin friction</p> <p>C_H coefficient of heat transfer</p> <p>H total enthalpy</p> <p>h specific enthalpy</p> <p>j 0 for two-dimensional flow; 1 for axisymmetric flow</p> <p>K longitudinal body curvature</p> <p>k thermal conductivity</p> <p>Pr Prandtl number</p> <p>p pressure</p> <p>Re Reynolds number</p> <p>r distance of a point at body surface from axis of symmetry</p> <p>s specific entropy</p> <p>T absolute temperature</p> <p>U_∞ free-stream velocity</p> <p>u, v velocity components in x- and y-directions, respectively</p>	<p>x, y boundary-layer coordinate system, see Fig. 1</p> <p>β body inclination angle, see Fig. 1</p> <p>γ ratio of specific heats</p> <p>ϵ density ratio across the shock, $\epsilon = \rho_\infty/\rho_s$</p> <p>$\mu$ shear viscosity</p> <p>ρ density</p> <p>τ shear stress</p> <p>ψ stream function</p> <p style="text-align: center;">Subscripts</p> <p>∞, s, w refer to conditions in the free stream, immediately behind the shock, and at the wall, respectively</p> <p>$*$ refers to conditions immediately behind the shock at the point where the streamline $\psi = \text{const}$ intersects the shock</p> <p style="text-align: center;">Superscript</p> <p>\sim refers to new variables, see Eqs. (22a), (22b), (26), and (29).</p>
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