NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report 32-1267

Prediction of the Position and Velocity of a Satellite After Many Revolutions

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Preface

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Acknowledgments

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Abstract

A systematic study on the prediction of the position and velocity of a satellite after many revolutions about the central mass is presented. It is shown that the methods of general perturbations are ideally suited for the prediction of orbits extending over many periods. The mathematical foundations for a theory of general perturbations using either the variation of coordinates or the variation of parameters, each with multivariable asymptotic expansions, as well as a theory of general perturbations using the variation parameters with Taylor's series expansions, are presented.

Five theories of general perturbations are applied to the elementary problem of the damped linear harmonic oscillator. These applications illustrate the mathematical processes involved as well as the advantages and disadvantages of each theory. It is shown that the variation of coordinates and the variation of parameters, each with multivariable asymptotic expansions, are the only theories from this group that will yield the superior, uniformly valid approximation.

Two satellite theories are exhibited: the theory of the variation of coordinates with multivariable asymptotic expansions, and the theory of the variation of parameters with Taylor's series expansions applied to the differential equations governing the motion of a satellite in an equatorial orbit about an oblate body (second harmonic only). In addition, a complete set of exact equations is developed for this restricted satellite motion; that is, equations that yield the position and velocity of the satellite. The right ascension of the satellite is found to consist of constants and a Legendre normal elliptic integral of the first kind, whereas the time associated with the position of the satellite is found to consist of constants, powers of the radius distance, and Legendre normal elliptic integrals of the first, second, and third kinds.

Extensive numerical accuracy studies are made in the case of the damped linear harmonic oscillator and in the case of the equatorial satellite, using the exact analytic solutions as the standards of comparison. It is concluded that only the application of the variation of parameters with multivariable asymptotic expansions to the set of ordinary nonlinear differential equations governing the motion of a satellite predicts the position and velocity of the satellite to a sufficiently high accuracy after many revolutions about the central mass.

A more realistic mathematical model of the motion of a satellite is qualitatively discussed; that is, the case where the satellite is perturbed by the asphericity of the central mass (second harmonic only), atmospheric drag, and the gravitational attraction of the sun, simultaneously, and where the initial conditions are not restricted to yield an equatorial orbit.

Finally, the bibliography provides one of the largest (1376 citations) lists of references on satellite theory.

Prediction of the Position and Velocity of a Satellite After Many Revolutions

I. Introduction

One of the more difficult problems confronting astrodynamicists today is the prediction of the position and velocity of a celestial body orbiting a much more massive celestial body. The complexity of this problem can be seen by the extent of subject material developed during the last two hundred years by many generations of scientists, among them some of the most brilliant and ingenious minds of history (see the Bibliography, Refs. 1–1370).

The orbit prediction problem set forth above can be expressed as a mathematical problem that consists of the integration of a set of ordinary nonlinear differential equations with given initial conditions and given numerical values for the physical constants appearing in the differential equations (initial value problem). The type and structure of these ordinary differential nonlinear equations depend on the formulation of the problem and on the character of the forces acting on the orbiting celestial body. The successful mathematical representation of this physical problem is entirely dependent upon the ability of the astrodynamicist to mathematically represent and include all of the important physical forces of the problem.

Depending upon the character of the forces acting on the orbiting celestial body, the theory of solution may be either of two possibilities: planetary theory or satellite theory, as described below.

Planetary theory is concerned with the motion of a point mass moving around a central point mass so massive as to dominate the system but under the disturbing influence of other point masses, relativity, etc. In general, the point mass under investigation moves in a nearly circular orbit that lies in a slightly varying plane. The planets and current interplanetary spacecraft, in motion around the sun, are examples of orbiting celestial bodies whose motions are determined by using planetary theory.

Satellite theory is concerned with the motion of a point mass moving around a central mass (not necessarily a point mass) so massive as to dominate the system but under the disturbing influence of other point masses, asphericity of the central mass, atmospheric resistance, etc. In general, the point mass moves in an osculating elliptical orbit that lies in a moderately varying plane. Artificial satellites in motion around a planet or a moon of a planet, as well as natural satellites in motion around a planet, are examples of orbiting celestial bodies whose motions are determined by using satellite theory.

Classical planetary theory is concerned with the motion of a planet under the gravitational attraction of the sun and a disturbing planet, where all three bodies are treated as point masses, while classical satellite theory is concerned with the motion of the moon under the gravitational attraction of the earth and the disturbing sun, where all three bodies are treated as point masses (sometimes called the main problem of the lunar theory, Refs. 213 and 215). Classical satellite theory is better known as classical lunar theory, since the moon was the primary natural satellite under investigation in the seventeenth, eighteenth, and nineteenth centuries.

The investigation herein is concerned primarily with satellite theory.

A. The Problem of Satellite Orbit Prediction

The objective of this investigation is to obtain a satellite theory that will predict the position and velocity of a satellite to a sufficiently high accuracy even after many revolutions about the central mass. The satellite theory should not be dependent upon the disturbing forces acting on the satellite; that is, once the disturbing forces acting on the satellite are defined, the satellite theory is then used to develop an orbit prediction process for that particular force field.

1. Description of the problem. The problem as previously stated is concerned with orbit prediction, the computation of satellite orbits from given initial conditions, and should not be confused with the more general problems in astrodynamics of

(1) Orbit determination, the computation of satellite orbits from observational data,

(2) Orbit selection, the choosing from sets of precomputed satellite orbits particular satellite orbits that satisfy the orbit design constraints of a given mission, and

(3) Determination of physical constants, the calculation of the constants that appear in the differential equations of motion of the satellite (mass of the central body, coefficients of the harmonics of the central body, etc.) using satellite orbits computed from observational data.

A careful examination of these three general problems of astrodynamics in Fig. 1, represented by simplified flow diagrams, will show that orbit prediction is the foundation for orbit determination, orbit selection, and determination of physical constants and therefore that the achievement of the objective of this investigation will benefit each of the three general problems.

The terminology "satellite theory" used herein is used differently by some astrodynamicists. Clemence (Ref. 277) defines the complete solution of problems in satellite theory as being an indeterminate number of theories, containing the minimum number of degrees of freedom that represent all the observations of every known satellite within limits fixed by the errors of the observations. Furthermore, Clemence states that, for a complete solution, the maximum number of theories is equal to the number of satellites and the minimum number of theories is 1, but the minimum will not be reached, since the motions of different satellites differ so much in character that a single theory including all of them would be unmanageably complex. Satellite theory as used by Clemence is force-field-dependent, since the character of the motion is determined by the disturbing forces acting. It is quite clear that if a listing of different satellite theories appearing in Refs. 1–1370 is made using Clemence's interpretation of a satellite theory, there would be a very large number of possible satellite theories. However, if a listing of different satellite theories is made using the interpretation of Section I-A, there would be ten or twenty possible satellite theories. Clemence's interpretation of a "satellite theory" will be referred to herein as a "solution to the motion of a satellite" (the solution to the motion of a satellite is force field dependent).

2. Importance of the problem. The importance of this investigation is attributable to the National Aeronautics and Space Administration program for the exploration of the solar system, which is based upon the three following primary scientific problems confronting scientists today: the origin and evolution of the planets, the moon, and the sun; the origin and evolution of life; and the dynamic processes that shape man's terrestrial environment (Ref. 1371). As a result, the NASA program includes the exploration of the planets and the moon by placing artificial satellites in orbit about these celestial bodies. In the past decade, the initial objectives of placing satellies in orbit about the earth and the moon have been achieved with overwhelming success. In the present decade, the explorations of Mars and Venus are of high priority because of their relevance to the three primary problems above (both planets are believed to be uncontaminated by foreign matter). Because Mars is similar to earth in many important respects, the study of Mars will contribute significantly to the solutions of these problems. For this reason, the exploration of Mars is of higher priority than the exploration of Venus.

Placing an artificial satellite in an orbit about Mars or Venus, a much more difficult task than placing an arti-

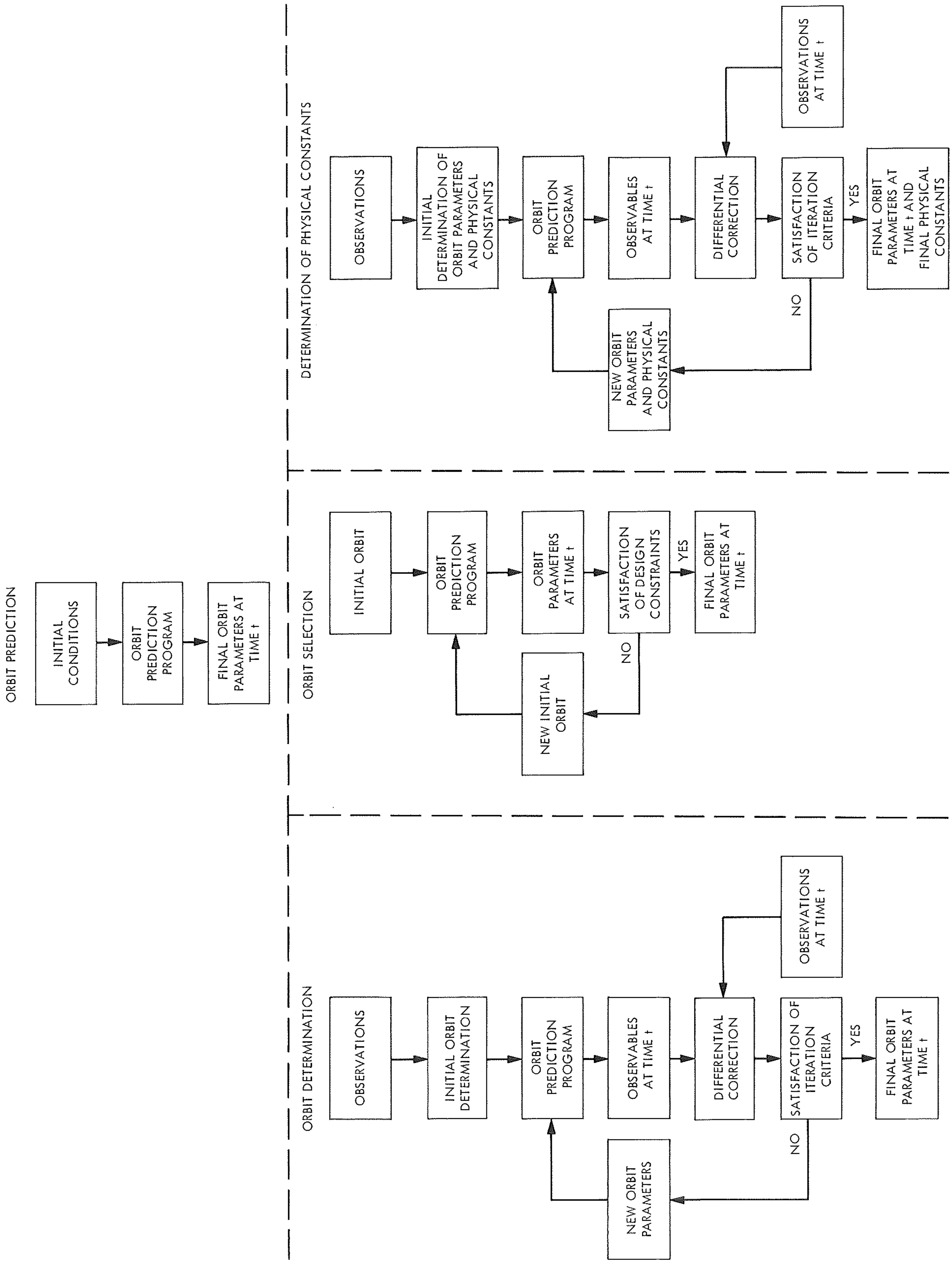

Fig. 1. Flow diagrams of four related problems in astrodynamics

ficial satellite about the earth or the moon primarily because Mars and Venus are at large distances from the earth (the point of launch and observation), requires knowledge beforehand of how an artificial body will move about the planet (what its orbit will be), how inevitable disturbances will influence its motion, and what regions of the planet it can explore. Such requirements imply that preliminary predictions of orbits must be made and that the accuracy of such predictions may be quite exacting. Furthermore, in order to avoid contaminating the surface of Mars or Venus with an unsterilized spacecraft, NASA has established the design constraint that all unsterilized spacecraft (of which several are planned) that orbit the planet must have a lifetime of several years. Hence, sufficiently accurate long-term orbit prediction is desirable (long-term orbit prediction is not necessary for artificial satellites of the earth or the moon).

The previous discussion has been concerned with the importance of orbit prediction in the general problem of orbit selection. However, as pointed out in Section I-A-1, orbit prediction also enters into the remaining two general problems of orbit determination and the determination of physical constants, both of which are very important once an artificial satellite is placed in an orbit about Mars or Venus. Thus, the problem under investigation is not academic.

The importance of the problem having been established, the question of a suitable satellite theory arises. Fortunately, orbit theory has been well investigated by many generations of scientists, as mentioned earlier. It should be quite simple to review the extensive literature (Refs. 1–1370) and choose a suitable satellite theory according to a specified set of selection criteria. Paradoxically, this is not the case.

First, the various theories that were developed in classical celestial mechanics for the solution and study of orbiting celestial bodies were concerned with classical lunar or planetary theory as defined above. For example, the works of Euler, Clairaut, D'Alembert, Lagrange, and Laplace in the eighteenth centrury were concerned with the motion of the moon and the motions of the planets (Refs. 410 and 411; memoirs by Clairaut and D'Alembert[1]; Refs. 786 and 787; and 793). Furthermore, the works of Poisson, Gauss, Hansen, Delaunay, Leverrier, Hill, Newcomb, Gylden, Lindstedt, and Poincaré in the nineteenth century were also concerned with the motions of the moon and the planets (memoirs by Poisson[1]; Refs. 468; 537–542; 343; 806; 567–578; 984–986; 518; 818 and 819; and 1056).

Although satellite theory is closely related to classical orbit theory (one of the oldest and most developed branches of celestial mechanics) satellite theory is inherently different from classical orbit theory. In the latter, the point mass moves only in a nearly circular orbit that lies in a slightly varying plane, and, in the former, the point mass can move in an osculating elliptical orbit that lies in a rapidly varying plane. Hence, classical orbit theory is primarily concerned with natural celestial bodies whose orbits are relatively stationary and have existed for a long time. Each orbit is investigated separately, as an independent phenomenon. However, satellite theory can be concerned with orbits that are complex and rapidly changing. Furthermore, it may be necessary to study a whole class of feasible orbits rather than just one orbit. In satellite theory, phenomena can occur that have no analogies in classical orbit theory—for example, zero eccentricity, eccentricities near unity, inclinations near 180 deg or the critical 63.4 deg,[2] nonconservative forces, etc.

In addition, the time scale in classical orbit theory (months or years) can be so different from that of satellite theory (hours or months) that many of the classical expansions that converge for intervals of time on the order of hundreds of years in classical orbit theory would converge for intervals of time on the order of months at most in satellite theory. Finally, with the advent of the high-speed electronic computer, the use of a classical theory in the form in which it was originally developed would not be desirable. Because computational schemes in the classical theories were adapted to the technology of manual calculations, the simplicity of the algorithm was considered as second to the reduction of the volume of the calculations. However, with the use of high-speed electronic computers, the volume of the calculation is not the dominant factor. As a result, simplicity and universality of the algorithm play a much greater role (of great importance is the amount of information that must be stored in the memory of the computer for a transition from one stage of calculation to another). In conclusion, the widely used classical orbit theories are insufficient for use as satellite theories and have to be substantially developed and augmented.

[1]Specific citations may be found in Refs. 936 and 1320.

[2]For an explanation of the critical inclination, see the Glossary.

Second, many of the investigators of satellite theory in the twentieth century have been neophytes who have not taken the time to study classical orbit theory or the work that had already been done in satellite theory. Hence, the important contributions to satellite theory have been made by a relatively small number of investigators and, as a consequence, much work still remains to be done. For example, additional investigations are desirable in

(1) The use of a perturbed reference orbit as a basis for a satellite theory,

(2) The use of variables and parameters that yield a solution to the motion of a satellite free from small divisors and mathematical singularities, and

(3) The use of expansions that are valid for extremely long times, or, better, for all time.

Third, the objective of astrodynamics has not been attained when a solution to the motion of a satellite is only more or less approximately developed, unless the explicit degree of approximation (numerical accuracy) is known. It is therefore important to determine an upper limit for the error committed, a factor with which most investigators of satellite theory have not been sufficiently concerned.

Finally, many solutions to the motion of a satellite that have been developed are force-field-dependent (Clemence's interpretation of a satellite theory). These solutions may yield excellent results for a particular force field, but when they are used with another force field, unacceptable results are obtained.

3. ***Historical synopsis.*** Since satellite theory utilizes many of the results originally developed in classical orbit theory, it is perfectly correct to consider Sir Isaac Newton's geometrical studies of the problem of three bodies, written in the seventeenth century (Ref. 991, Book I, Section XI), as the origin of satellite theory (Kepler's work is not considered here since it consisted of three postulates deduced from the observations of the planets rather than an entire mathematical theory).

Subsequent investigations were conducted by Euler, Clairaut, and D'Alembert in the middle of the eighteenth century using integration by series. During this time, Euler also began the theory of the perturbations of the planets and developed the analytical method of the variation of parameters. Lagrange and Laplace developed and augmented the work of Euler in the latter part of the eighteenth century. They succeeded in eliminating improper secular terms that were entirely due to the imperfections of the method of solution.

In the first half of the nineteenth century, Poisson, Gauss, Leverrier, Hill, and Delaunay extended the work of Lagrange and Laplace. Hansen and Delaunay, in the middle of the nineteenth century, made great advances in the lunar theory and, during this time, Newcomb established that, in the case of planetary theory, the instantaneous elements can be represented by purely periodic functions of time which formally satisfy the differential equations of motion.

In the latter half of the nineteenth century, Hill developed a new lunar theory based on new concepts and mathematical methods. The fundamental concept was to use a non-Keplerian orbit as the first approximate solution. During this time, Linstedt and Gylden extended the work of Newcomb concerning representation by periodic functions. However, it was the ingenious work of Poincaré that completely revolutionized classical orbit theory. Poincaré was the first to successfully investigate the aspects of series convergence and error of approximation. For a more detailed account of classical orbit theory, reference should be made to Moulton (Ref. 936) or Brouwer and Clemence (Ref. 205), from which the information above has been extracted.

The first significant investigation of satellite theory (differing from classical satellite theory) occurred in the very first part of the twentieth century when Brown extended Hill's classical lunar theory to a high degree of numerical accuracy, culminating with a lunar theory that is still used today (Refs. 212 and 213). In addition, Brown succeeded in developing a planetary theory based on Hamiltonian mechanics (canonical transformations) (Ref. 214) quite similar to the planetary theory developed independently by von Zeipel (Ref. 1357) at about the same time. Additional studies were conducted by Brown concerning both planetary and satellite theory (Refs. 206–211). Because of the success of the Hill–Brown lunar theory, the investigations in the next few years were primarily concerned with planetary theory; for example, Brouwer developed a method of integrating the equations of general planetary theory in rectangular coordinates which is free from most of the disadvantages of the classical methods (Ref. 203). Clemence applied Brouwer's theory shortly afterwards (Ref. 283). Additional investigations into planetary theory were made by both Brouwer (Refs. 199 and 200) and Clemence (Refs. 282, 284, and 285).

Interest in satellite theory was again aroused by the launching of the first artificial satellite on Oct. 4, 1957. During the last decade, a tremendous number of investigations have been concerned with satellite theory. An attempt to indicate the order of these investigations will not be made here. However, Table 1 presents a list of investigators, with their works, who were either one of the first or one of the consistent investigators of the past decade. The works of the remaining investigators may be found in the Bibliography. In Table 1, the work of the investigator or investigators has been classified according to nine categories so that if a particular aspect of satellite theory is of interest, references concerning that aspect can be more readily found. For an investigator with multiple publications, the earliest reference number corresponds to the latest date of investigation. (This procedure has also been followed in the Bibliography.) Finally, more recent investigations into planetary theory have been conducted by Brouwer, Clemence, Duncombe, Danby, Musen, and a number of investigators at the Jet Propulsion Laboratory[3] (Refs. 181, 196, and 393; 276, 278–281; 382–388; 324–331; and 943, 945, 946, 948, 950, 957, and 973).

Several textbooks often referenced, which have not been mentioned previously, include those by Tisserand, Charlier, Moulton, Subbotin, Whittaker, Smart, Goldstein, Finlay-Freundlich, Sterne, Duboshin, and Baker (Refs. 1250; 250; 935 and 936, 1231–1233; 1320; 1172; 494; 426; 1204; 379–381; and 86 and 87). The textbook by Herrick (Ref. 564) and the textbook by Brouwer and Clemence (Ref. 205), both previously referenced, have been found to be exceptionally informative.

B. Methods of Solution

It is well known that if a spherical body moves under the influence of a much more massive spherical body, its path of motion will describe a conic section. The massive spherical body will be located at the principal focus and will act as a point mass. This motion is commonly called central-force-field motion or Keplerian motion (after Kepler, whose three laws concern this type of motion).

However, if the mass distribution of either principal body is not spherical, if there is a resisting medium surrounding the massive body, if there are other celestial bodies attracting or any other way affecting the two principal bodies, or if there is any force acting upon the two principal bodies other than that of the mutual attractions of the two spheres, the path of motion of the less massive body will not describe an exact conic section and the theory of motion of this body in the noncentral force field is then called perturbation theory. The deviations from a reference orbit (not necessarily a conic section) in force, acceleration, velocity, or position are called perturbations; that is, the term perturbations may be used (1) for forces that have not been included in the force field defining the reference orbit, (2) for differences between the components of the total acceleration of a celestial body and those it would have in the reference orbit, and (3) for corrections to the velocity or position components determined from the reference orbit. In some instances in astrodynamics, it is permissible to neglect perturbations, but in the majority of cases the perturbations have to be taken into account. In the case of satellite theory, it is impossible to describe the orbit adequately without considering them.

It is quite common to classify perturbations according to their period and their order. With regard to the period, the perturbations are divided into secular, long-period, short-period, and Poisson or mixed perturbations. (The mixed perturbations are called Poisson perturbations because they were first encountered by Poisson in the discussion of the variations of the major axes of the planetary orbits.) The perturbations that change proportionally to a power of the independent variable are called secular perturbations. The perturbations that are periodic in the independent variable are called periodic perturbations and are either long-period or short-period, depending on whether the period is relatively large or small, respectively. Sometimes perturbations arise that are the products of a power of the independent variable and periodic perturbations. These are called Poisson perturbations. With regard to the order, the perturbations are classified according to their size relative to a small parameter of the problem. (Section II-A-1 presents a more detailed discussion.)

Perturbation theory is of such importance in astrodynamics, and is at the same time so difficult, that efforts of many investigators have been directed toward it. In general, a complete and rigorous integration of the differential equations of motion yielding closed analytical expressions is manifestly impossible in orbit theory. As a result, these investigators have turned to the processes of approximation. It is customary in astrodynamics to distinguish between two classes of perturbation methods

[3]Specific citations may be found in *Publications of the Jet Propulsion Laboratory*, Bibliography 39- series, published annually by the Jet Propulsion Laboratory, Pasadena, Calif.

Table 1. References on satellite theory

Investigators	Motion of a satellite in a gravitational field	Critical eccen-tricity and inclination	Aerodynamic effects	Electro-magnetic effects	Low-thrust effects	Lunar and solar effects	Relativ-istic effects	Solar radiation pressure effects	Reviews and miscellaneous topics
1. Aoki		54, 55							56
2. Beard and Johnson				103, 104					
3. Blitzer	131–135, 137–139, 141–145	136				140			
4. Brouwer	180, 193, 195, 197, 202		183, 190–192					184	182, 185–189, 194, 198
5. Bryant, R. W.								222, 223	
6. Chebotarev	252, 254, 256–259	253, 255							
7. Cook, G. E.	303, 304, 307, 310, 311	306	305, 308, 309, 315–318			312		313	314
8. Diliberto	360–364								
9. Duboshin	370, 374–378								369, 371–373
10. Flandro					434				
11. Garfinkel	453–457, 459–462	458							463
12. Groves	514		515			513			
13. Herget	553, 554								552
14. Herrick	558	560							556, 557, 559, 561–563
15. Hori	587–589, 592	590, 591						586	
16. Izsak	609, 611, 615	613, 614	616						612
17. Jacchia			617–621			622		623	
18. Jastrow				628, 629				627	
19. Kaula						658			654–657
20. King-Hele	692, 694–697		677, 684–691, 693, 698						676, 678–683
21. Kovalevsky	738–750								
22. Kozai	752, 755, 756, 758, 760, 767–769	762, 763				753, 754, 766		757, 761	759, 764, 765
23. Krause	775						773, 774		
24. Lubowe	841, 842, 846	845	844						843
25. Merson	896–899								895
26. Musen	944, 953–955, 959–961, 965, 966, 969–971	956, 958				952, 962– 964		968	943, 947, 949, 951, 967, 972, 974
27. Newton	989	988	987, 990						
28. O'Keefe	1006, 1007								1004, 1005
29. Roberson	1086, 1087		1084, 1085						
30. Sterne	1198, 1199, 1201–1203		1197, 1200						
31. Struble	1218, 1219, 1222								1217, 1220, 1221, 1223
32. Vinti	1280–1282, 1284–1289, 1291, 1293, 1294		1292		1279				1283, 1290

called "special perturbations" and " general perturbations" (both classes consisting of processes of approximation). However, a combination of special and general perturbations is also possible and will be considered herein as a third class of perturbation methods. Figure 2 presents a schematic diagram of perturbation theory showing the three classes of perturbation methods, their differences, and their similarities.

1. Special perturbations. The class of perturbation methods known as special perturbations consists of methods in which the accelerations of the disturbed body are integrated using numerical analysis (numerical integration or mechanical quadratures). As a result, special perturbations methods generate a particular (or special) orbit for a particular (or special) disturbed body given particular (or special) initial conditions. The motion of the disturbed body is determined from point to point in discrete time intervals; that is, a continuous determination of the perturbed orbit is made by means of a step-by-step process. Before the perturbed orbit can be determined at the nth point, it must be determined at the $n - 1$ point.

The methods of special perturbations are normally classified according to the formulation of the equations to be integrated, there being three possibilities as shown in Fig. 2. The first possibility is known as "Cowell's method" and consists of the numerical integration of the total accelerations in terms of coordinates (usually rectangular) with no reference to an osculating or reference orbit. The second possibility is known as "Encke's method" and consists of the integration of perturbative accelerations in terms of coordinates. The perturbative accelerations are the differences in acceleration between those of the actual orbit and a reference orbit. The third possibility is known as the "variation-of-parameters method" and consists of the integration of perturbative variations in terms of parameters such as a, e, i, ω, Ω, M or n, $e \cos \omega$, $e \sin \omega$, i, Ω, M. The parameters characterize an osculating orbit which is a progressively changing reference orbit (not necessarily Keplerian) that yields the actual position and velocity at any given point or instant of time. Since these parameters are changing from instant to instant, they are called instantaneous parameters. (These three methods are also discussed in the Glossary.)

In each of the three basic methods of special perturbations, there are many processes for the numerical integration of the total or perturbative accelerations of the disturbed body, two of which, the Runge–Kutta and the second-sum processes, are indicated in Fig. 2 and are described in the Glossary. The Runge–Kutta process is often used because it features a variable integration step size, whereas the second-sum process has been used for many years in celestial mechanics because of its simplicity. Some processes are simpler to initiate than others, some are more accurate, some require less computing time or computer memory capacity, some permit variable step size (integration interval), etc. It should be recognized that the numerical integration process is not theoretically dictated by the method of special perturbations being used. However, in practice, it may turn out that one numerical integration process is better suited than another for a particular method of special perturbations.

The advent of high-speed electronic computers has eliminated one of the prime difficulties in the methods of special perturbations: the overwhelming computational labor. However, special perturbations methods still have the great drawback of accumulation of error. The source of this error is twofold, consisting of truncation error and round-off or rounding error. Truncation error is the difference between the exact solution of the difference equations which approximate the differential equations and the exact solution of the differential equations themselves. Round-off or rounding error is the difference between the computed and the exact solutions of the difference equations. The buildup of round-off and truncation errors in numerical integration is unavoidable. The influence of truncation error is usually prominent when a large value of step size is used. The influence of round-off error is usually prominent when a large number of steps have been taken, usually occurring when a small value of step size is used. (Because of the round-off errors that are encountered at each step and propagate with each step, a random walk away from the exact solution occurs and there is no tendency for these errors to average out.) Hence, the smaller the step size taken, the less the truncation error, but (for a given total interval of integration) then more steps are required, and thus more round-off error occurs.

Another difficulty that may arise in a special perturbations method is the phenomenon of instability; that is, the occurrence of a solution of the difference equations that is unstable (unbounded) even though the solution of the differential equations is inherently stable (bounded). (Instability is discussed further in the Glossary.) In the Runge–Kutta and second-sum processes, instability will not occur, provided that the step size chosen is sufficiently small (Ref. 292).

The methods of special perturbations have the extremely desirable characteristic of being immediately applicable

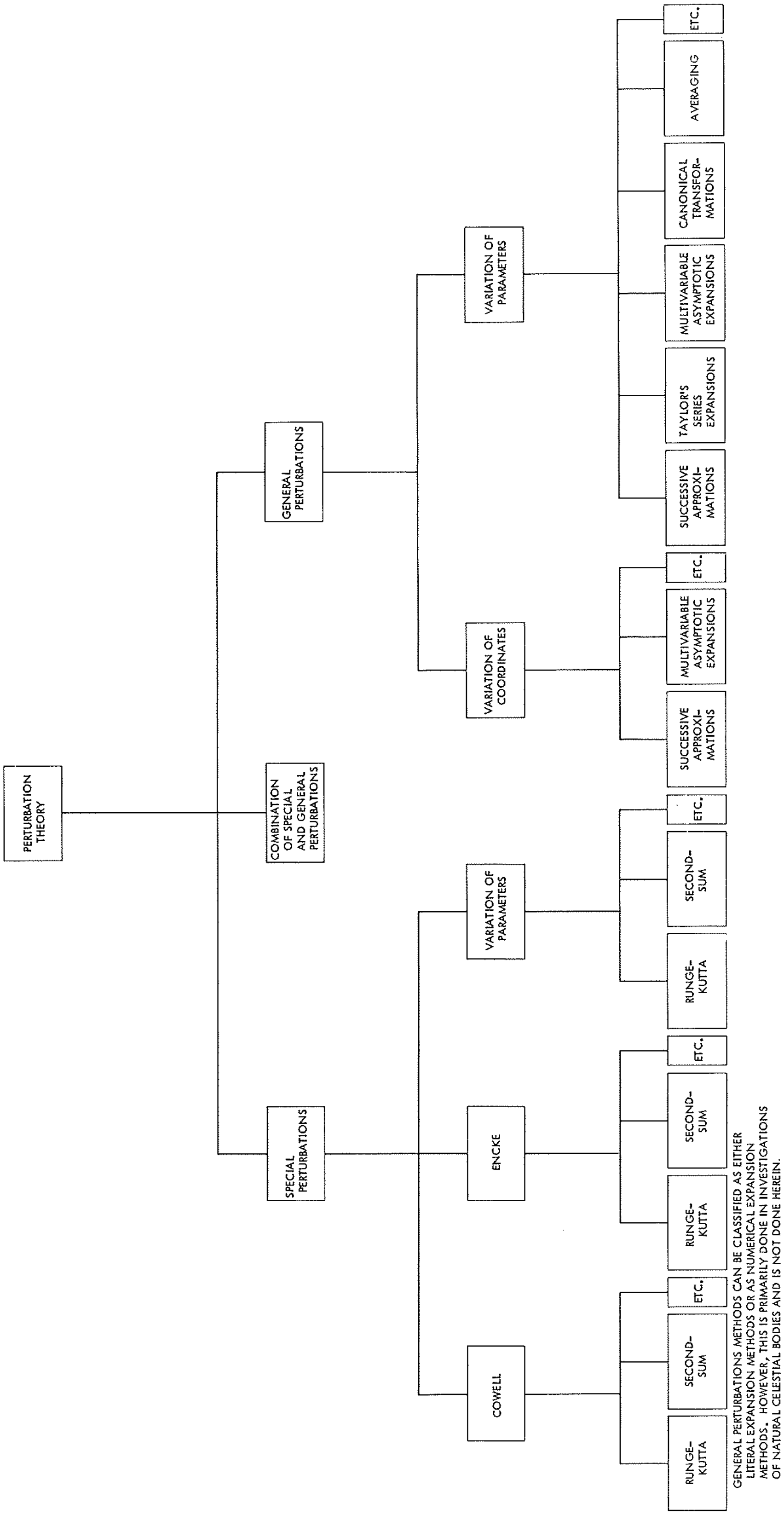

Fig 2. Approximate methods of perturbation theory

to any set of initial conditions (kind of orbit) and to any force field. On the other hand, these methods have the undesirable characteristic of requiring the computation of the coordinates of the disturbed body or the elements of its orbit at all the intermediate epochs prior to their computation at the epoch of interest (a sufficient number of intermediate epochs must be taken for an acceptable numerical accuracy). Furthermore, since these methods are purely numerical, the "general theorems" regarding the stability of the system (boundedness of the motion as the time approaches infinity) can not be obtained (Ref. 936, p. 429).

The methods of special perturbations are ideally suited for calculating orbits having only a limited duration, such as lunar and interplanetary trajectories. These methods can also be used for calculating the motion of any body in the solar system for a few revolutions around its primary with sufficient numerical accuracy.

However, because of the large number of integration steps and the fact that the accumulated error increases with the number of steps, the methods of special perturbations are not well suited for the long-term prediction of the motion of a satellite or a planet. Furthermore, even if the numerical accuracy criterion is not stringent, as in preliminary orbit selection, these methods are still not suitable for long-term orbit prediction because of the large amount of computing time required. Attempts at reducing the amount of computing time have been made by investigators; for example, Taratynova developed a process that consists of numerically integrating only a few revolutions in detail and, for the most part, taking the integration interval equal to the period of revolution (Refs. 861 and 1242–1244).

2. General perturbations. The class of perturbation methods known as general perturbations consists of analytical methods in which the accelerations (usually the perturbative accelerations) are expanded into infinite series and integrated term by term. In practice, only a finite number of the terms in the infinite series is actually used. These methods give the solution of the differential equations of motion in the form of symbolic formulas which express the sought-for quantities as explicit functions of (1) the independent variable (usually time), (2) the constants of the problem, and (3) the constants of integration, which are determined by the initial conditions of the problem. Thus, general perturbations methods yield solutions to orbit problems that are generally applicable to various disturbed bodies, given various values of the constants and initial conditions.

A general perturbations method can be specialized to a particular disturbed body by introducing numerical values for the constants and initial conditions in place of their symbols. This method would then give the solution of the differential equations of motion in the form of symbolic formulas which express the sought-for quantities as explicit functions of the independent variable containing numerical coefficients. Specialization of this sort is usually made in investigations of natural celestial bodies such as the moon or a planet in an effort to minimize the complexity of the investigation, for instance, in Brown's lunar theory (Refs. 212 and 213) in which numerical values are substituted for some of the orbital elements, or in Hill's theory of Jupiter (Refs. 573, 574, and 578), in which numerical values are substituted for all of the orbital elements, leaving time alone symbolized.

Hence, general perturbations methods can be classified as either literal expansion methods or as numerical expansion methods. However, this classification will not be made herein, since the latter is simply a special case of the former. Finally, the motion of the disturbed body in any general perturbations method is determined from the initial point to any point of interest in one step, without determination of the motion at intermediate epochs.

The methods of general perturbations are normally classified according to the formulation of the equations to be integrated, there being two possibilities, as shown in Fig. 2. The first possibility is known as the "variation-of-coordinates method" and consists of the analytical integration of the accelerations (usually perturbative) in terms of coordinates (for example, x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$ or r, α, δ, $\dot{r}$, $\dot{\alpha}$, $\dot{\delta}$). The second possibility is known as the "variation-of-parameters method" and consists of the analytical integration of perturbative variations in terms of parameters (for example, a, e, i, ω, Ω, M or n, $e \cos \omega$, $e \sin \omega$, i, Ω, L).

In each of the two basic methods of general perturbations, there are many processes for the analytical integration of the total or perturbative variations of the disturbed body. Several of these processes—successive approximations, multivariable asymptotic expansions, Taylor's series expansions, canonical transformations, and averaging—are indicated in Fig. 2 and are described in the Glossary and later sections of this report.

The prime difficulty of general perturbations methods is the overwhelming analytical labor required by the astrodynamicist in the development of the symbolic formulas of the methods. However, recent advances in computer programming technology indicate that high-speed electronic computers may be able to execute all the necessary analysis associated with the development of most general perturbations methods and thereby eliminate this difficulty (Refs. 155–158, 216, 669, 1058, 1059, 1126, 1154–1156, 1251, and 1252). Most general perturbations methods consist of sequences of operations that can be computerized, such as expansions of functions in multiple trigonometric series, multiplication of trigonometric or power series, term-by-term integration and differentiation of trigonometric or power series, solution of algebraic systems of linear equations, etc. (Refs. 351, 476, 543, 606–608, 1090, 1168, 1297, and 1309).

The methods of general perturbations have the extremely desirable characteristic of permitting the computation of the coordinates of the disturbed body or the elements of its orbit at any particular time after the initial time without computation at intermediate epochs. Thus, it is possible to obtain the sought-for quantities at some point in the first orbit and then immediately at some point in any subsequent orbit. Since these methods are analytical, the general character of perturbations can be investigated (for example, the effect, order, and interaction of individual perturbations can be obtained) and general theorems regarding the stability of a system can be obtained.

Furthermore, general perturbations methods permit a clear interpretation of the sources of the perturbations in orbit determination; for example, the discovery of Neptune by Adams and Leverrier (independently) was made through the analysis of the motion of Uranus (Ref. 806) and, more recently, the discovery of the Earth's "pear" shape by O'Keefe and Eckels was made through the analysis of long-period terms in orbital eccentricity (Ref. 1005).

On the other hand, most general perturbations methods have the undesirable characteristic of being applicable only to certain force fields; for example, the method which uses averaging requires periodic perturbations. In addition, these methods require that a large number of terms of the infinite series be retained in order to achieve great accuracy.

The methods of general perturbations are ideally suited for the prediction of orbits extending over many periods, such as artificial and natural satellites. These methods permit the development of the simplest and most economical computing algorithms for long-term orbit prediction which, with the aid of high-speed electronic computers, can produce the required solution quickly and accurately.

3. Combination of special and general perturbations. The difficulties and characteristics of the methods of special and general perturbations have been presented in Sections I-B-1 and I-B-2, respectively, and it should be clear that the choice of which of the two classes of methods of solution is better suited for a particular motion depends upon the motion itself. In some cases of quasi-periodic motion, it is probable that the best means of determining the motion will be based on some combination of special and general perturbations. Perhaps the nonconservative perturbations could be numerically integrated and the conservative perturbations could be analytically integrated, or perhaps the nonperiodic perturbations could be numerically integrated and the periodic perturbations could be analytically integrated, etc. In this way, advantage is taken of the repetitive character of the motion.

C. Method of General Perturbations

In order to obtain the most precise and efficient satellite theory described in Section I-A, it should be clear from the discussions in Sections I-B-1 and I-B-2 that a general perturbations method must be used rather than a special perturbations method (it is assumed that the perturbative forces are not of the type requiring a combination of special and general perturbations). Hence, the subsequent sections will be concerned with general perturbations methods only.

1. Classification of theories. Satellite theory, from the mathematical point of view, can be thought of as the application of perturbation theory to the set of ordinary nonlinear differential equations that determine the motion of a satellite. Hence, a satellite theory can be described according to the method of perturbation theory used; for example, a particular satellite theory is the theory of general perturbations using the variation of parameters with multivariable asymptotic expansions (one branch of the diagram in Fig. 2). It should be pointed out that within each class of solutions to the motion of a satellite, there are many factors that can differentiate one solution from another, such as the order of approximation of the solution (see Section I-C-4), the perturbations included, etc. However, these factors are a matter of numerical accuracy and completeness rather than differences between satellite theories. As a result, there are probably no more than ten or twenty different satellite theories. For a classification of solutions to the motion of a satellite, see Ref. 67 (since this reference uses Clemence's interpretation of a satellite theory, solutions to the motion of a satellite are called satellite theories).

It should be pointed out that a satellite theory is considered as fully developed to a particular order if sufficient expressions are given to obtain the position and velocity of the satellite at any time t to that particular order.

2. Selection criteria. It has been stated in previous sections that there are several possible satellite theories; therefore, the question of a suitable satellite theory arises. Unfortunately, there have been very few comparisons of satellite theories appearing in Refs. 1–1370, and thus this question is extremely difficult to answer. As a result, two satellite theories were selected on the basis of the criteria listed below. The theories are compared in detail in subsequent sections.

Criteria for selection of a satellite theory were as follows:

(1) A satellite theory should not introduce artificial secular or Poisson terms into the solution, such as those encountered in classical orbit theory where the independent variable (usually time) was removed from the trigonometric functions (usually sine and cosine). In this manner, the expressions for the sought-for quantities represent the required solution over a very large interval of time.

(2) A satellite theory should not introduce small divisors or mathematical singularities into the solution for particular initial conditions such as those associated with eccentricities that are 0, small, or near 1, with inclinations that are 0, small, near or equal to 180 deg, or near or equal to the critical 63.4 deg, etc. Since the choice of coordinates or parameters is directly related to avoiding small divisors or mathematical singularities, a complete freedom of choice of coordinates or parameters to be used in a satellite theory is desirable. It should be pointed out that the use of a nonsingular set of coordinates or parameters often complicates the formulation of the theory.

(3) A satellite theory should not restrict the initial conditions of the satellite to those associated with orbits having a particular range of eccentricities. For example, several satellite theories restrict the initial conditions of the satellite to those associated with orbits of moderate eccentricity; otherwise, the infinite series of the theory converge very slowly. It is very desirable to have a satellite theory that yields a solution applicable to initial conditions associated with eccentricities in the entire range $0 \leqq e < 1$. The choice of the correct independent variable eliminates slowly converging infinite series and permits initial conditions associated with eccentricities in the range $0 \leqq e < 1$.

(4) A satellite theory should not restrict the perturbative forces acting upon the satellite; for example, a satellite theory should be able to yield the solution to the motion of a satellite being perturbed by conservative forces, nonconservative forces, or a combination of conservative and nonconservative forces. (In the case of a satellite in an orbit around Mars, the effects of oblateness, atmospheric drag, and solar gravitation are all important.)

(5) Finally, a satellite theory should yield a precise and efficient solution, should be easily adaptable to various satellites and force fields, and should be unexcelled in yielding an insight into the physical nature of the motion (Ref. 657).

3. Selected theories. On the basis of the criteria specified above, two satellite theories were selected for comparison from the various satellite theories appearing in Refs. 1–1370. These theories are (1) the theory of general perturbations using the variation of coordinates with two variable asymptotic expansions (Section V-C) and (2) the theory of general perturbations using the variation of parameters with Taylor's series expansions (Section V-D). Both theories are represented in Fig. 2.

Two additional theories described in the glossary and represented in Fig. 2 are the theory of general perturbations using the variation of parameters with canonical transformations and the theory of general perturbations using the variation of parameters with averaging. Although both theories are quite elegant, neither satisfies all of the selection criteria specified in Section I-C-2.

The theory of general perturbations using the variation of parameters with canonical transformations was first conceived by the French astronomer Delaunay, who applied it to the solar perturbations of the moon (Ref. 343). Since then, the theory has been refined by the investigations of Hill, Poincaré, von Zeipel, Brown, and Brouwer–Hori (Refs. 578; 1056; 1357; 214; and 190–192). Using this theory, the investigation by Brouwer and Hori attempted to obtain the motion of a satellite being perturbed by asphericity and drag. They succeeded in establishing certain coupling effects between oblateness and drag perturbations which are not accessible by treating the oblateness and drag perturbations independently. Unfortunately, slowly convergent series appeared, and a device to overcome this lack of convergence was of limited effectiveness only. Finally, the various series that develop in this theory are based upon certain constants or mean parameters which must be determined by an initialization procedure. If these mean parameters are not determined precisely, secular error growths occur (Ref. 67). Additional information concerning this satellite theory may be found in Refs. 88, 90, 183, 188–192, 194, 195, 197, 205, 214, 343, 459, 487, 570, 578, 590, 591, 657, 664, 740, 760, 885, 1056, 1264, and 1357.

The theory of general perturbations using the variation of parameters with averaging was rigorously developed for the first time by the Russian mathematicians Krylov and Bogoliubov in the study of nonlinear oscillations (Ref. 776) (the concept of averaging was used in classical orbit theory but without special emphasis on mathematical rigor). Since then, the theory has been generalized by Bogoliubov and Mitropolsky (Ref. 150). Investigations by Lass and Lorell, Lass and Solloway, Lorell, Kyner, Morrison, and Musen have been concerned with determining the motion of a satellite using this theory (Refs. 798; 797; 825, 826, 828, 829, and 831; 782 and 783; 929–931; and 946 and 947). They have shown that this satellite theory yields excellent first-order solutions to the motion of a satellite under the action of a single perturbation. Unfortunately, when higher-order solutions are desired to the motion of a satellite under the action of more than a single perturbation, the theory becomes extremely cumbersome. However, a theory very similar to the generalized theory of averaging, but much simpler, exists—the theory of multivariable asymptotic expansions, to be discussed and compared subsequently. (Morrison has compared these two theories in Ref. 930, substantiating the relative simplicity of the latter theory.) Additional information concerning the generalized theory of averaging may be found in Refs. 39, 150, 261, 506, 524, 776, 781–783, 797, 798, 825, 826, 828, 829, 831, 832, 919, 921, 922, 929–931, 946, 947, 1218, 1219, 1224, and 1326.

It is interesting to note that the theories of multivariable asymptotic expansions, canonical transformations, and averaging have been compared with each other by Kevorkian, Kyner, and Morrison in Refs. 664, 782, and 930, respectively. Their results indicate that the three theories yield equivalent solutions to the slightly nonlinear oscillations of an autonomous system with one degree of freedom and subject to small damping. This does not mean that the three theories are equivalent or that they yield equivalent solutions to the motion of a satellite under all circumstances. For example, the theory of canonical transformations requires the use of canonical variables, whereas the other two theories do not. As a result, mathematical singularities may arise in the solution obtained using the theory of canonical transformations, whereas a judicious choice of noncanonical variables in the other two theories may prevent the occurrence of these singularities in the solutions.

4. Order of an approximate solution. The concept of order in satellite theory has been used differently by various investigators and, as a result, requires clarification. There appear to be three distinct interpretations of "order of an approximate solution."

The classical interpretation arises during the analytical integration of the derivatives of the parameters of a disturbed body when the process of successive approximations is used. In this interpretation, order is used to specify the approximation number. For example, the equations of motion are of the form

$$q_j = q_{j0} + \int_{t_0}^{t} \dot{q}_j\,(q_1, q_2, \cdots, q_6, \tau)\, d\tau$$

for $j = 1, 2, \cdots, 6$, and the first approximation or first-order solution is obtained by replacing the q_k of the integrands by the $\bar{q}_k$, which are constants or possibly the sum of constants and secular terms; that is,

$$q_j^{(1)} = q_{j0} + \int_{t_0}^{t} \dot{q}_j\,(\bar{q}_1, \bar{q}_2, \cdots, \bar{q}_6, \tau) d\tau$$

Through the use of the first-order solution, the second approximation or second-order solution is obtained by replacing the q_k of the integrands by the $q_j^{(1)}$; that is,

$$q_j^{(2)} = q_{j0} + \int_{t_0}^{t} \dot{q}_j\,(q_1^{(1)}, q_2^{(1)}, \cdots, q_6^{(1)}, \tau)\, d\tau$$

The third approximation or third-order solution is obtained by replacing the q_k of the integrands by the $q_j^{(2)}$, and so forth. The end result would be infinite series representing the q_j. The correspondence between order and approximation number arises when the $\bar{q}_k$ are taken as constants, for then the $q_j^{(1)}$ contain terms of order ϵ (perturbative parameter), the $q_j^{(2)}$ contain terms of order ϵ^2, etc. However, there is no guarantee that the $q_j^{(1)}$ contain all the terms of order ϵ, the $q_j^{(2)}$ contain all the terms of order ϵ^2, etc., so that the correspondence is actually between apparent order and approximation number. As a result, the classical interpretation of order in satellite theory will be referred to herein as apparent order.

The second interpretation of order of an approximate solution arises during the analytical integration of the equations of motion using one of the processes other than successive approximations indicated in Section I-B-2. In this interpretation, order is used to specify the highest order of terms completely determined in the solution. For example, a first-order solution using this interpretation would imply that "all" of the first-order terms in the expressions for q_j have been determined. This interpretation of order of a solution is used herein.

The third interpretation of order of an approximate solution is essentially the same as the second interpretation except that this interpretation requires that all perturbative forces that produce terms of the same order in the expressions for q_j be included in the force field. For example, Brouwer has shown in his solution to the motion of an artificial satellite (Ref. 195) that terms with coefficients J_2 and J_4/J_2 appear in the first-order solution. Thus, although J_4 may be considered to be of order J_2^2, it is necessary to include the perturbative force due to J_4 in the force field in order to obtain a first-order theory (long-term prediction). The difference between these interpretations is that the second is purely mathematical, whereas the third is physical as well.

Investigators who use Clemence's interpretation of a satellite theory (Section I-A-1) refer to "order of a solution" as "order of a theory" and thus their terminology is a "first-order theory" rather than a "first-order solution," etc.

5. Comparison criteria. An indication of the quality of a satellite theory can be obtained by investigating (1) the numerical accuracy of the resultant solution (assuming the satellite theory has been used optimally) to a particular force field, (2) the adaptability of the theory to various satellites and force fields, and (3) the simplicity of both the mathematical operations involved and the form of the solution. These criteria are used to compare one satellite theory with another, assuming that the selection criteria specified in Section I-C-2, with the exception of the last, are satisfied by both theories.

The principal test of a satellite theory herein is that it yields the position and velocity (or equivalent parameters) of a satellite to a sufficiently high accuracy even after many revolutions about the central mass. The only means of deciding whether a given theory yields a sufficiently accurate solution is to compare the resultant solution with a standard of comparison of much higher accuracy than the desired accuracy. Unfortunately, obtaining a standard of comparison with this characteristic is a difficult task. There are four possible approaches in the selection of a standard of comparison:

(1) The most common approach is to choose a standard of comparison determined by a special perturbations method. This approach is unacceptable herein because of the buildup of truncation and round-off error after many revolutions of the satellite about the central mass.

(2) The second approach is to determine a standard of comparison by making the most precise observations of a particular satellite, develop the solution to the motion of that satellite using the satellite theory, and then compare the results. This approach is also unacceptable herein because the "real world" is extremely difficult to simulate in a general perturbations theory; that is, all of the perturbations acting upon an actual satellite can not be successfully incorporated into the satellite theory.

(3) The third approach is to choose another satellite theory to determine the standard of comparison to the required accuracy. This approach is unacceptable because there is no satellite theory as yet that yields a sufficiently accurate solution even after many revolutions of the satellite about the central mass.

(4) The fourth approach, and the one which is adopted herein, is to restrict the force field and the initial conditions of the satellite so that there is an exact analytic solution to the motion of the satellite that can be used as the standard of comparison. Of course, the solution to the motion of the satellite using the satellite theory must be determined accordingly. As a result, there is no guarantee that the accuracy as determined by a study of this type will exist in the solution to the unrestricted motion of a satellite determined using the satellite theory.

The adaptability of a satellite theory to various satellites and force fields is highly desirable. Although the subsequent sections used several satellite theories to obtain the solution to the motion of an equatorial satellite being perturbed only by asphericity of the central mass (second harmonic only), in an actual application to a satellite in an orbit about Mars, the motion of a satellite being perturbed at least by the asphericity of Mars, atmospheric drag, and the gravitational attraction of the sun is required and, as a result, the theories must be adaptable to this motion.

The simplicity of the mathematical operations involved as well as the form of the solution govern the speed with which a solution may be obtained using a high-speed electronic computer. The speed of computation is extremely important in the problem of orbit selection, where the totality of possible orbits must be analyzed economically (the slower the speed, the larger the cost). In addition, the simplicity of the form of the solution determines the ease with which the effects of certain perturbations may be analyzed, an important characteristic in the problems of orbit determination and the determination of physical constants. An indication of the simplicity of one solution with respect to another can be obtained by comparing the amount of information that must be stored in the memory of the computer for a transition from one prediction point to another (core storage requirements) and, of course, the speed of computation per prediction point.

6. Solution procedure. The procedure in obtaining the solution to the motion of a satellite using a particular satellite theory (general perturbations) consists of the steps given in Table 2. The satellite theory selected will define the reference orbit or the solution of the principal part of the equations of motion, the first- and higher-order solutions, and the constants of integration during step 6 of Table 2. Furthermore, the accuracy of the solution (assuming the satellite theory satisfies the selection criteria specified in Section I-C-2) will be strongly dependent upon how well step 1 is performed; that is, how successful the astrodynamicist is in defining the mathematical model, including the physical constants, with sufficient accuracy.

Table 2. Solution procedure

Step No.	Procedure
1.	Mathematically represent and include all of the important physical forces acting on the satellite in the mathematical model of the problem (this includes selecting values for the respective physical constants that appear).
2.	Select the appropriate satellite theory (for example, the theory of general perturbations using the variation of coordinates with multivariable asymptotic expansions).
3.	Select the parameters or coordinates to be represented.
4.	Select the independent variable to be used.
5.	Develop the equations of motion in terms of the selected parameters or coordinates and independent variable.
6.	Develop or integrate the equations of motion according to the satellite theory and obtain the parameters or coordinates of the satellite as functions of the initial conditions and the independent variable.
7.	Develop equations relating the position and velocity of the satellite to the selected parameters or coordinates.

D. Summary of Contents

This report is arranged in seven principal sections. The first section consists of the statement of the problem, the historical background of the problem, the methods of solution of the problem, and many definitions and concepts concerning satellite theory. The second section presents the mathematical foundation for a theory of general perturbations using the variation of coordinates or parameters with multivariable asymptotic expansions, and the third section presents the mathematical foundation for a theory of general perturbations using the variation of parameters first with Taylor's series expansions and then with multivariable asymptotic expansions. Neither of these two sections considers the two extremely difficult questions of the convergence of the infinite series and the analytical estimation of the error committed by replacing the infinite series by finite numbers of their first terms. In general, the series which appear need not be convergent in the sense in which the mathematician defines this term. The terms of these series can first decrease rapidly and then start increasing, since the accuracy desired by the astrodynamicist usually permits him to stop with the first several terms of the series long before these terms have ceased to decrease. The divergence of these series is inconvenient only if they are intended for giving an arbitrarily close approximation or if they are intended for rigorously establishing certain results, such as the stability of the motion under study. Furthermore, some of the mathematical processes which are used in developing the theories are largely formal. While mathematical rigor is desirable when it can be attained, the use of formal processes is justified whenever the results, not obtainable otherwise, are useful for the prediction of physical phenomena.

In Section IV, five perturbation theories are applied to the elementary problem of the damped linear harmonic oscillator in order to illustrate the mathematical processes involved and the advantages and disadvantages of each theory. This problem was chosen as an example because of its simplicity and the fact that an exact analytic solution is obtainable. From the results of this section, the theory of the variation of coordinates with two variable asymptotic expansions and the theory of the variation of parameters with Taylor's series expansions are determined to be more desirable than the other theories.

In Section V, the two more desirable perturbation theories determined in Section IV are applied to the equations of motion of a satellite to obtain specific solutions. In studying the numerical accuracy of approximate solutions to problems without exact analytic solutions, the difficult question of the standard of comparison arises. As a result, it was decided to start the study of approximate solutions of satellite motion on a simple case which has an exact analytic solution that can be used as the standard of comparison—the case where the initial conditions and the perturbative force yield the motion of an equatorial satellite about an oblate body (second harmonic only)—and work up to the more difficult cases, such as the motion of a satellite being perturbed by the asphericity of the central mass, atmospheric drag, and the gravitational attraction of the sun "simultaneously." The objective of this section is to analyze the satellite theories rather than the resulting motion, since the motion of an equatorial satellite about an oblate body has received enough attention already (Refs. 23, 49, 50, 71, 110, 112, 132, 136, 163, 169, 172, 446, 517, 566, 597, 702, 747, 931, 961, 1035, 1038, 1135, 1136, 1286, and 1302). For purposes of additional comparison, the theory of special perturbations using the Cowell formulation with a fourth-order Runge–Kutta process is used to obtain the solution to this example.

After the analysis of a simple case of satellite motion in Section V, Section VI considers the more realistic mathematical model of the satellite problem; that is, the case where the satellite is perturbed by the asphericity of the central mass (second harmonic only), atmospheric drag, and the gravitational attraction of the sun, simultaneously, using the better of the two theories analyzed in Section V. Although Section VI is largely qualitative, it does outline the next important step in the investigation of satellite theory in considerable detail. In addition, the solution to this particular force field should provide an excellent orbit prediction program that can be used to help place an artificial satellite in an orbit about Mars or Venus.

The last section presents a summary of results and recommendations for further study. Finally, the bibliography presents probably the most complete list of references on satellite theory ever compiled.

II. Theory of Multivariable Asymptotic Expansions

The basic concept of the theory of multivariable asymptotic expansions—the use of a distorted time scale in order to remove artificial secular or Poisson terms (terms introduced by the method of solution)—was first used by Lindstedt in his studies of planetary theory (Refs. 818 and 819). Later, Poincaré provided the mathematical foundation for the use of this concept (Ref. 1056). More recently, Cole and Kevorkian generalized the work of Lindstedt and Poincaré into the theory of two-variable asymptotic expansions (Refs. 289 and 668). It appears that in solving for the motion of a satellite being perturbed by more than one perturbation, the work of Cole and Kevorkian must also be generalized into the theory of multivariable asymptotic expansions. Additional investigations into the use of multivariable (primarily two-variable) asymptotic expansions may be found in Refs. 290, 394–398, 664–667, 785, 930, 1162, and 1274.

A. Mathematical Concepts

Although the mathematical foundation of the theory of multivariable asymptotic expansions is not yet perfect in the eyes of a pure mathematician, the approximate solutions obtained by this theory are very useful in practical applications and so warrant its use. (The theory of multivariable asymptotic expansions is extremely similar to the theory of averaging as stated in Section I, and since the mathematical foundation of the theory of averaging is rigorously given by Bogoliubov and Mitropolsky in Ref. 150, the rigorous mathematical foundation of the theory of multivariable asymptotic expansions can be assumed as given there—implicitly—as well.)

1. Order of a function. The mathematical measure of the magnitude of a function depending upon a vanishingly small parameter ϵ in a perturbation solution is called the "order of the function." This concept provides a basis for determining the degree of approximation in a perturbation solution. The order of a function is indicated by the symbol O and is used as follows:

$$x(t;\epsilon) = O(\nu(\epsilon)) \qquad \text{uniformly in } t \text{ as } \epsilon \to 0$$

if

$$\lim_{\epsilon \to 0} \left| \frac{x(t;\epsilon)}{\nu(\epsilon)} \right| < \infty \qquad \text{for all } t \text{ in the range of } t,$$

that is, if in comparing $x(t;\epsilon)$ with some gauge function $\nu(\epsilon)$ (whose limiting behavior is well known), the ratio $x(t;\epsilon)/\nu(\epsilon)$ remains bounded as $\epsilon \to 0$ for all t in the range of t. It should be pointed out that other order symbols exist having different meanings; for example, the symbol o (Refs. 407 and 1274). However, only the order symbol O is used in this report. Furthermore, the gauge functions are always chosen herein to yield the sharpest degree of approximation; for example, $\epsilon \sin t = O(\epsilon)$ would be used rather than $\epsilon \sin t = O(1)$.

The rules for algebraic operations with order symbols are given in Ref. 407; two of these rules used extensively in this report are as follows:

(1) The order of a sum or difference is that of the dominant term; for example, $O(\epsilon) \pm O(\epsilon^2) = O(\epsilon)$.

(2) The order of a product or ratio is the product or ratio of the orders.

In general, order symbols may be integrated but not differentiated with respect to ϵ or an independent variable. However, order symbols are assumed to be differentiable with respect to the independent variable herein.

2. ***Asymptotic sequences and expansions.*** The sequence of functions $\nu_j(\epsilon)$ where $j = 0, 1, 2, \cdots$ is called an asymptotic sequence for $\epsilon \to 0$ if, for each j,

$$\lim_{\epsilon \to 0} \frac{\nu_{j+1}(\epsilon)}{\nu_j(\epsilon)} = 0$$

A very useful asymptotic sequence, because of its simplicity and familiar properties, is $\nu_j(\epsilon) = \epsilon^j$, where $j = 0, 1, 2, \cdots$. Other examples of useful asymptotic sequences may be found in Refs. 290, 407, and 1274.

The series

$$\sum_{j=0}^{J} \nu_j(\epsilon)\, x^{(j)}(t)$$

is called an asymptotic expansion to J terms of $x(t;\epsilon)$ as $\epsilon \to 0$ if

$$x(t;\epsilon) = \sum_{j=0}^{J} \nu_j(\epsilon)\, x^{(j)}(t) + O\left(\nu_{J+1}(\epsilon)\right) \qquad \text{as } \epsilon \to 0 \tag{1}$$

where $\nu_j(\epsilon)$ is an asymptotic sequence as $\epsilon \to 0$. The asymptotic expansion in Eq. (1) is called uniformly valid if the remainder in this equation is of $O\left(\nu_{J+1}(\epsilon)\right)$ uniformly in t (that is, for all t in the range of t).

In general, infinite asymptotic expansions may either converge for some range of ϵ or diverge for all ϵ. The present investigation does not require that the asymptotic expansions be convergent, since the mathematical convergence depends upon the behavior of terms of indefinitely high order, which are usually never determined in practical applications. In general, the astrodynamicist calculates only the first few terms and hopes that they rapidly approach the true solution.

The important characteristic of an asymptotic expansion is that the error is of the order of the first neglected term and therefore tends more rapidly to zero the smaller the size of ϵ. For a fixed value of ϵ, the error can be decreased by adding terms; but if the series is divergent, a point is eventually reached beyond which additional terms increase the error. However, the values of ϵ which appear in astrodynamics are so small that sufficient numerical accuracy is usually obtained, in the case of diverging series, before the point of increasing error is reached.

It is clear from Eq. (1) that the error committed in approximating $x(t;\epsilon)$ by an asymptotic expansion of J terms is of $O\left(\nu_{J+1}(\epsilon)\right)$. Unfortunately, knowing the mathematical order of the error committed does not yield the actual numerical error committed, because no account is kept of constants of proportionality; that is, if the

$$\text{error} = O\left(\nu_{J+1}(\epsilon)\right)$$

then the

$$\text{error} = C\, \nu_{J+1}(\epsilon)$$

where C is finite but otherwise unknown. Theoretically, C can have any finite value (1; 10,000; 1,000,000; etc.). However, in practical applications, the proper choice of normalization constants and ϵ usually yields constants of proportionality near unity ($0 < C < 10$) so that an estimate of the actual numerical error can be obtained.

The series

$$\sum_{j=0}^{J} \nu_j(\epsilon)\, x^{(j)}(\bar{t}, \tilde{t}, \cdots)$$

is called a multivariable asymptotic expansion to J terms of $x(t;\epsilon)$ as $\epsilon \to 0$ if

$$\begin{aligned} x(t;\epsilon) &= X(\bar{t}, \tilde{t}, \cdots; \epsilon) \\ &= \sum_{j=0}^{J} \nu_j(\epsilon)\, x^{(j)}(\bar{t}, \tilde{t}, \cdots) + O\left(\nu_{J+1}(\epsilon)\right) \\ &\qquad \text{as } \epsilon \to 0 \end{aligned} \tag{2}$$

where $\nu_j(\epsilon)$ is an asymptotic sequence as $\epsilon \to 0$ and the variables $\bar{t}, \tilde{t}, \cdots$ are functions of ϵ multiplied linearly by time (for example, $\bar{t} = t\,[1 + \epsilon^2 \tau_2 + \epsilon^3 \tau_3 + O(\epsilon^4)]$ and $\tilde{t} = t\,\epsilon\,[1 + \epsilon \tilde{\tau}_1 + \epsilon^2 \tilde{\tau}_2 + O(\epsilon^3)]$ where $\tau_2, \tau_3, \cdots$ and $\tilde{\tau}_1, \tilde{\tau}_2, \cdots$ are constants). The variables $\bar{t}, \tilde{t}, \cdots$ are either of the fast variable type $\bar{t}$ or of the slow variable type $\tilde{t}$ and are formally treated as being distinct variables. The multivariable asymptotic expansion in Eq. (2) is called uniformly valid if the remainder in this equation is of $O(\nu_{J+1}(\epsilon))$ uniformly in $\bar{t}, \tilde{t}, \cdots$; that is, for all t in the range of t.

3. Uniformity conditions. Two concepts are necessary for the success of the multivariable asymptotic expansion technique; they are the first and second uniformity conditions, discussed below. These correspond to what some investigators call boundedness conditions, a terminology that is misleading.

The first uniformity condition states that an approximate solution, in the form of a multivariable asymptotic expansion, to a differential equation containing a small parameter ϵ can not contain secular or Poisson terms in the fast variable (or variables) if the solution to the same differential equation with $\epsilon = 0$ does not contain secular or Poisson terms in the natural independent variable. Briefly, if the solution of the system with $\epsilon = 0$ is bounded, then the solution of the system with $\epsilon \neq 0$ will not contain secular or Poisson terms in the fast variable (or variables). However, this does not imply that the solution of the system with $\epsilon \neq 0$ must be bounded, since secular and Poisson terms in the slow variable (or variables) are allowed; that is, growth or decay is possible but it must be slow. In this respect, to call this first concept a boundedness condition is misleading.

It should be noted that the first uniformity condition can not be applied to the solution of a differential equation that is unbounded for $\epsilon = 0$. For example, the time in the motion of a satellite about an oblate body can be determined in terms of angle variables, in which case the time will be proportional to the fast angle variable (time is an unbounded function).

The second uniformity condition is a consequence of the definition of a uniformly valid multivariable asymptotic expansion given in Section II-A-2. It states that the

$$\lim_{\epsilon \to 0} \frac{\nu_{j+1}(\epsilon)\, x^{(j+1)}(\bar{t}, \tilde{t}, \cdots)}{\nu_j(\epsilon)\, x^{(j)}(\bar{t}, \tilde{t}, \cdots)} = 0 \tag{3}$$

for any j and all t in the range of t. The condition (3) implies that given any $d > 0$, there exists an ϵ_d such that

$$\left| \frac{\nu_{j+1}(\epsilon_d)\, x^{(j+1)}(\bar{t}_d, \tilde{t}_d, \cdots)}{\nu_j(\epsilon_d)\, x^{(j)}(\bar{t}_d, \tilde{t}_d, \cdots)} \right| < d \tag{4}$$

for any t_d in the range of t and any j. Now, if the ratio $x^{(j+1)}/x^{(j)}$ in (4) contains a secular or Poisson term in any of the slow variables (like $\tilde{t}_d$), assuming the secular and Poisson terms in the fast variables (like $\bar{t}_d$) have been removed by the first uniformity condition, then

$$\left| \frac{\nu_{j+1}(\epsilon_d)\, x^{(j+1)}(\bar{t}_d, \tilde{t}_d, \cdots)}{\nu_j(\epsilon_d)\, x^{(j)}(\bar{t}_d, \tilde{t}_d, \cdots)} \right| > d$$

for some t_d regardless of how small ϵ_d is taken to be, since there exists a t_d such that

$$\frac{\nu_{j+1}(\epsilon_d)}{\nu_j(\epsilon_d)} \tilde{t}_d$$

or

$$\frac{\nu_{j+1}(\epsilon_d)}{\nu_j(\epsilon_d)} \tilde{t}_d \cos \bar{t}_d$$

etc., is of $O(1)$. Thus, the ratio $x^{(j+1)}/x^{(j)}$ can not contain secular or Poisson terms in the slow variables. However, this does not mean that the $x^{(j)}(\bar{t}, \tilde{t}, \cdots)$ can not have secular or Poisson terms in the slow variables; for example, if

$$x^{(1)}(\bar{t}, \tilde{t}, \cdots) = \tilde{t} \cos \bar{t}$$

and

$$x^{(0)}(\bar{t}, \tilde{t}, \cdots) = \tilde{t}$$

then

$$\frac{x^{(1)}(\bar{t}, \tilde{t}, \cdots)}{x^{(0)}(\bar{t}, \tilde{t}, \cdots)} = \cos \bar{t}$$

Hence, $x^{(j+1)}(\bar{t}, \tilde{t}, \cdots)$ can never become as large as $x^{(j)}(\bar{t}, \tilde{t}, \cdots)$ for any j, but $x^{(j)}(\bar{t}, \tilde{t}, \cdots)$ can become large. In this respect, to call this second concept a boundedness condition is misleading.

Note that even if the first uniformity condition can not be applied (that is, if the solution to the unperturbed differential equation is unbounded) the second uniformity condition may still be used to eliminate nonuniform terms. As a result, the concepts of uniformity as used herein appear to be more flexible than those used by previous investigators (boundedness conditions), since even the unbounded function, time, in the motion of a satellite about an oblate body can be handled nicely (see Section V-C).

4. Initially and uniformly valid expansions. The multivariable asymptotic expansion given in Eq. (2) is called initially valid if the remainder in that equation is of $O(\nu_{J+1}(\epsilon))$ uniformly in $\bar{t}, \tilde{t}, \cdots$ only over the initial portions of their ranges; that is, for all t in the range $0 \leqq t < t_1$ where t_1 is finite (the entire range of t is assumed to be

$0 \leqq t < \infty$). It is always possible to find an initially valid multivariable asymptotic expansion but it may not always be possible to find a uniformly valid multivariable asymptotic expansion.

B. Development of Solution

The motion of a satellite about a central mass is governed by three second-order ordinary nonlinear differential equations characterized by the presence of small accelerations active for a long time; that is,

$$\frac{d^2\mathbf{r}}{dt^2} + \mu_p \frac{\mathbf{r}}{r^3} - \mathbf{\ddot{r}}^{\cdot} = 0 \tag{5}$$

where $\mathbf{r}$ is the radius vector from the dynamical center (usually the center of the central mass) to the satellite, μ_p is the mass function, and $\mathbf{\ddot{r}}^{\cdot}$ represents the perturbative accelerations, which act upon the satellite and are small relative to $(d^2\mathbf{r})/(dt^2)$ and $\mu_p(\mathbf{r}/r^3)$. The initial conditions

$$\mathbf{r}(0) \text{ and } \frac{d\mathbf{r}}{dt}(0)$$

as well as the physical constants such as μ_p, J_2, etc., are known (orbit prediction problem).

Most satellite theories yielding solutions in terms of the small parameters measuring the perturbative accelerations and one time variable are not uniformly valid for the complete time interval $0 \leqq t < \infty$ due to the cumulative action of the small accelerations. However, a satellite theory that uses asymptotic expansions in terms of several time variables appears to provide a means of obtaining uniformly valid solutions for the complete time interval $0 \leqq t < \infty$.

The physical significance of various time variables (for example, the fast time variable based upon the period of unperturbed motion, the slow time variable based upon the decay time, etc.) will become clear from the discussion of the damped linear harmonic oscillator in Section IV. The classical problem of the damped linear harmonic oscillator has been chosen because of its simplicity, because its exact analytic solution is obtainable, and because of its similarity to the satellite problem (with damping by drag). The various time variables have physical significance since they are based upon dimensional physical constants that enter into the differential equations of motion. Furthermore, each of the small parameters of the problem can be expressed as a ratio of two of the significant time scales of the problem.

The "customary" development of the solution to the equations in (5) begins with a dimensional analysis to determine the normalization constants, perturbative parameters, fast time scales, and slow time scales. The differential equations of motion are then normalized accordingly. Next, multivariable asymptotic expansions of the form shown in Eq. (2) are assumed to approximate the solution uniformly in time; that is (using vector notation),

$$\begin{aligned}\mathbf{r}(t;\epsilon) &= \mathbf{R}(\bar{t}, \tilde{t}, \cdots; \epsilon) \\ &= \sum_{j=0}^{J} \nu_j(\epsilon)\, \mathbf{r}^{(j)}(\bar{t}, \tilde{t}, \cdots) + O(\nu_{J+1}(\epsilon))\end{aligned} \tag{6}$$

where ϵ is the perturbative parameter. From Eqs. (6), the first and second derivatives of $\mathbf{r}(t;\epsilon)$ are obtained by formal differentiation (it is not, in general, permissible to differentiate asymptotic expansions):

$$\begin{aligned}\frac{d\mathbf{r}}{dt}(t;\epsilon) &= \frac{d\mathbf{R}}{dt}(\bar{t}, \tilde{t}, \cdots; \epsilon) \\ &= \sum_{j=0}^{J} \nu_j(\epsilon) \left[\frac{\partial \mathbf{r}^{(j)}}{\partial \bar{t}}(\bar{t}, \tilde{t}, \cdots) \frac{d\bar{t}}{dt}(\epsilon) \right. \\ &\qquad \left. + \frac{\partial \mathbf{r}^{(j)}}{\partial \tilde{t}}(\bar{t}, \tilde{t}, \cdots) \frac{d\tilde{t}}{dt}(\epsilon) + \cdots \right] \\ &\qquad + O(\nu'_{J+1}(\epsilon))\end{aligned} \tag{7}$$

and

$$\begin{aligned}\frac{d^2\mathbf{r}}{dt^2}(t;\epsilon) &= \frac{d^2\mathbf{R}}{dt^2}(\bar{t}, \tilde{t}, \cdots; \epsilon) \\ &= \sum_{j=0}^{J} \nu_j(\epsilon) \left\{ \frac{\partial^2 \mathbf{r}^{(j)}}{\partial \bar{t}^2}(\bar{t}, \tilde{t}, \cdots) \left[\frac{d\bar{t}}{dt}(\epsilon) \right]^2 \right. \\ &\qquad + 2 \frac{\partial^2 \mathbf{r}^{(j)}}{\partial \bar{t}\, \partial \tilde{t}}(\bar{t}, \tilde{t}, \cdots) \frac{d\bar{t}}{dt}(\epsilon) \frac{d\tilde{t}}{dt}(\epsilon) \\ &\qquad \left. + \frac{\partial^2 \mathbf{r}^{(j)}}{\partial \tilde{t}^2}(\bar{t}, \tilde{t}, \cdots) \left[\frac{d\tilde{t}}{dt}(\epsilon) \right]^2 + \cdots \right\} \\ &\qquad + O(\nu_{J+1}(\epsilon))\end{aligned} \tag{8}$$

where

$$\frac{\partial^2 \mathbf{r}^{(j)}}{\partial \bar{t}\, \partial \tilde{t}}(\bar{t}, \tilde{t}, \cdots) = \frac{\partial^2 \mathbf{r}^{(j)}}{\partial \tilde{t}\, \partial \bar{t}}(\bar{t}, \tilde{t}, \cdots), \qquad \bar{t}, \tilde{t}, \cdots$$

are treated as distinct variables, and the functional dependencies of $\bar{t}, \tilde{t}, \cdots$ upon ϵ and t have been previously chosen (however, the proportionality constants have not been determined as yet). Equations (6), (7), and (8) are introduced into Eqs. (5) (properly normalized), after which terms of like powers of ϵ are combined. Since the expansions must hold (at least in an asymptotic sense) for arbitrary values of the perturbative parameter ϵ, the coefficients of the powers of ϵ must separately equal zero. Setting these coefficients equal to zero yields second-order partial differential equations in the $\mathbf{r}^{(j)}$ which, because of their special form, can be solved in sequence ($\mathbf{r}^{(0)} \to \mathbf{r}^{(1)} \to \mathbf{r}^{(2)} \to$ etc.) by using the theory of ordinary differential equations. The initial conditions $\mathbf{r}^{(j)}(0, 0, \cdots)$ and $(\partial \mathbf{r}^{(j)}/\partial \bar{t})(0, 0, \cdots)$ necessary for the complete solution of the partial differential equations are obtained by evaluating Eqs. (6) and (7) at $t = 0$ (then $\bar{t} = 0, \tilde{t} = 0, \cdots$), by using the known initial conditions $\mathbf{r}(0; \epsilon)$ and $(d\mathbf{r}/dt)(0; \epsilon)$, by combining terms of like powers of ϵ, and by setting the coefficients of the powers of ϵ separately equal to zero. During the sequential solution of the partial differential equations, secular or Poisson terms usually arise in the $\mathbf{r}^{(j)}$ which are eliminated by using the uniformity conditions discussed in Section II-A-3 and the arbitrary constants of proportionality arising from the $\bar{t}, \tilde{t}, \cdots$. Thus, the flexibility introduced into the satellite theory by using several time variables is necessary in order that a uniformly valid solution can be obtained. The $\mathbf{r}^{(j)}$ and the fully determined time variables are now introduced into Eqs. (6), yielding the required solution.

It should be pointed out that the natural independent variable, t, in Eqs. (5) can be transformed into an independent angle variable and the concepts previously discussed would still be applicable (in this case, there would be distinct fast and slow angle variables).

III. Theory of Variation of Parameters

The basic concept of the theory of the variation of parameters (also called the variation of elements or the variation of constants)—the use of an osculating orbit (or trajectory)—was first used in the middle of the eighteenth century by Euler in his studies of the mutual perturbations of the planets Jupiter and Saturn (memoirs by Euler). In this work, Euler did not consider all the orbital elements as being variable simultaneously, and thus the complete development of this theory was given for the first time by Lagrange in 1782 during his investigation of comets moving in elliptical orbits (Ref. 786). Since then, the theory has been used extensively by astronomers and astrodynamicists. Nearly all the textbooks on celestial mechanics or astrodynamics discuss this theory; for example, detailed discussions may be found in Moulton (Ref. 936), Herrick (Ref. 564), or Brouwer and Clemence (Ref. 205) (the discussions in this section follow those of Moulton and Herrick).

A. Principal Concepts

Three principal concepts arise in the theory of the variation of parameters: an osculating orbit, perturbative differentiation, and instantaneous parameters.

1. Osculating orbit. Consider the motion of a spherical body of mass m relative to a much more massive spherical body of mass m_p so that the only force acting upon the smaller mass is the force of attraction of the central mass m_p. According to the results of the two-body problem, the path of motion of the smaller body will describe a conic section C_0 whose elements are uniquely determined from the initial conditions $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$ as shown in Fig. 3. Suppose that when the smaller body arrives at the position $\mathbf{r}_1$, it becomes subject to an instantaneous impulse of intensity I_1 as shown. The position $\mathbf{r}_1$ and the velocity $\dot{\mathbf{r}}_1$ (after the impulse) determine a new conic section C_1 in which the smaller body will move until it is again disturbed by some external source. Suppose that when the smaller body arrives at the position $\mathbf{r}_2$, it becomes subject to another instantaneous impulse of intensity I_2 as shown. The position $\mathbf{r}_2$ and the velocity $\dot{\mathbf{r}}_2$ (after the impulse) determine a new conic section C_2 in which the smaller body will move until it is again disturbed by some external source. Assuming that additional impulses occur at subsequent positions, the smaller body will be moving in conic sections which change from time to time as a result of the disturbing impulses. Finally, suppose that the instantaneous impulses become very small and that the intervals of time between them become shorter and shorter so that, in the limit, the impulses become a continually disturbing force and the path of motion becomes a continually changing conic section. This continually changing conic section is called the osculating orbit in this report.

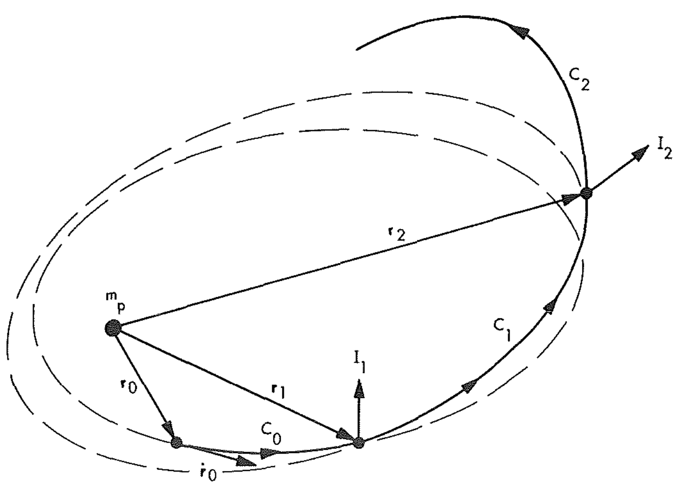

Fig. 3. Osculating orbit of mass m

According to this concept, the smaller body is always moving in a conic section, but in one that changes at each instant. The osculating orbit is tangent to the path of motion at every point and the velocity associated with the osculating orbit at the point of tangency is the same as the actual velocity of the smaller mass at that point. The parameters (or elements) of the osculating orbit are called instantaneous or osculating parameters (or elements). The osculating orbit may be visualized as rolling along the path of motion of the smaller body (varying in the parameters that describe the size, shape, and orientation of the conic section) in such a way that the actual position and velocity of the smaller body can be calculated from the instantaneous values of the parameters of the osculating orbit at each instant by simple Keplerian formulas (two-body formulas). If, at an instant, all disturbing forces are removed (instant of osculation), the smaller body would move in a conic section characterized by the instantaneous parameters at that time.

The perturbations are the differences between the Keplerian parameters of the osculating orbit at the starting epoch (the classical reference or intermediate orbit) and those at the instant of osculation and may be obtained by integrating the variations of the parameters for the given interval of time. It should be pointed out that the reference (or intermediate) orbit does not have to be the osculating orbit at the starting epoch but may be some kind of a continually changing approximation. In this case, the perturbations would be the differences between the parameters of this approximation and the parameters of the osculating orbit at the instant of osculation.

2. ***Perturbative differentiation.*** Consider the variation (time-derivative) of any function $f(p_1, p_2, \cdots, p_j, t)$ of the type that may arise in the theory of the variation of parameters, as the sum of two parts; that is,

$$\frac{df}{dt}(p_1, p_2, \cdots, p_j, t) = \dot{f} + f^{\grave{}} \tag{9}$$

where $p_1, p_2, \cdots, p_j$ are the instantaneous parameters of the osculating orbit which vary only because of the disturbing forces ($j \leq 6$) and where all the variables have been normalized using the proper length and time scales. The function $\dot{f}$ (f-dot) is the Keplerian or two-body variation that remains at the instant of osculation if all the disturbing forces are suddenly removed, and the function $f^{\grave{}}$ (f-grave) is the perturbative variation caused by the disturbing forces. Accordingly, since

$$\frac{df}{dt}(p_1, p_2, \cdots, p_j, t) = \frac{\partial f}{\partial t} + \sum_{j=1}^{J} \frac{\partial f}{\partial p_j}\frac{dp_j}{dt}$$

Eq. (9) yields

$$\dot{f} = \frac{\partial f}{\partial t} \text{ and } f^{\grave{}} = \sum_{j=1}^{J} \frac{\partial f}{\partial p_j}\frac{dp_j}{dt} \tag{10}$$

There are three types of variations which arise in this theory, namely,

$$\frac{df}{dt} = \dot{f} \text{ where } f^{\grave{}} = 0, \tag{11}$$

$$\frac{df}{dt} = f^{\grave{}} \text{ where } \dot{f} = 0, \tag{12}$$

or

$$\frac{df}{dt} = \dot{f} + f^{\grave{}} \text{ where both parts exists.} \tag{13}$$

Since the velocity associated with the osculating orbit at the point of tangency is the same as the actual velocity of the disturbed body at that point,

$$\frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} \text{ and } \mathbf{r}^{\grave{}} = 0 \tag{14}$$

so that the components of $d\mathbf{r}/dt$ in an inertial coordinate system are of the first type, Eq. (11). Variations of the second type, Eq. (12), arise for parameters that would be constant in Keplerian motion such as a, e, i, ω, and Ω so that the variations are due only to the disturbing forces, that is,

$$\frac{da}{dt} = a^{\grave{}}, \frac{de}{dt} = e^{\grave{}}, \text{etc.} \tag{15}$$

where

$$\dot{a} = \dot{e} = \cdots = 0$$

Variations of the third type, Eq. (13), arise for parameters that are referred to perturbed reference directions such as v, M, E and for all accelerations; that is,

$$\left.\begin{aligned} &\frac{dv}{dt} = \dot{v} + v^{\grave{}}, \frac{dM}{dt} = \dot{M} + M^{\grave{}}, \text{etc.} \\ \text{and} \\ &\frac{d^2x}{dt^2} = \ddot{x} + \dot{x}^{\grave{}}, \frac{d^2y}{dt^2} = \ddot{y} + \dot{y}^{\grave{}}, \text{etc.} \end{aligned}\right\} \tag{16}$$

Two techniques are available for obtaining the perturbative part of the variation of a function f (the Keplerian part is either known or can be easily obtained). The first technique consists of developing the total variation of the function, df/dt, and then removing the Keplerian part; that is,

$$f^{\backprime} = \frac{df}{dt} - \dot{f}$$

The second technique consists of using perturbative differentiation as developed by Herrick (Ref. 564). Perturbative differentiation consists of taking the grave derivative of a given relationship in which only the variations due to the disturbing forces are considered and in which Eq. (14) is used. For example, consider the vis viva integral (Ref. 564) in normalized variables,

$$\dot{s}^2 = \left(\frac{2}{r} - \frac{1}{a}\right)$$

Using perturbative differentiation, one obtains

$$2\dot{s}\dot{s}^{\backprime} = \left(-\frac{2r^{\backprime}}{r^2} + \frac{a^{\backprime}}{a^2}\right)$$

or

$$\dot{s}^{\backprime} = \frac{1}{2\dot{s}}\frac{a^{\backprime}}{a^2} \tag{17}$$

since $r^{\backprime} = 0$ from Eq. (14). Henceforth, perturbative differentiation will be used in obtaining the perturbative part of the variation of a function.

3. ***Instantaneous parameters.*** In general, six instantaneous parameters are required to describe the osculating orbit completely; for example, a, e, i, ω, Ω, and M. Using perturbative differentiation, the following perturbative variations of the illustrative parameters may be obtained (Ref. 564) in normalized coordinates and parameters:

$$a^{\backprime} = 2\frac{a^2}{p}\left[\frac{r\dot{r}^{\backprime}}{p^{1/2}}\left(e\frac{p}{r}\sin v\right) + \frac{r^2\dot{v}^{\backprime}}{p^{1/2}}\left(\frac{p}{r}\right)^2\right] \tag{18}$$

$$e^{\backprime} = \frac{r\dot{r}^{\backprime}}{p^{1/2}}\left(\frac{p}{r}\sin v\right) + \frac{r^2\dot{v}^{\backprime}}{p^{1/2}}\left[\left(\frac{p}{r}+1\right)\cos v + e\right] \tag{19}$$

$$i^{\backprime} = \frac{r^2\dot{b}^{\backprime}}{p^{1/2}}\cos u \tag{20}$$

$$\omega^{\backprime} = -\Omega^{\backprime}\cos i - \frac{r\dot{r}^{\backprime}}{e(p)^{1/2}}\left(\frac{p}{r}\cos v\right) + \frac{r^2\dot{v}^{\backprime}}{e(p)^{1/2}}\left(\frac{p}{r}+1\right)\sin v \tag{21}$$

$$\Omega^{\backprime} = \frac{r^2\dot{b}^{\backprime}}{p^{1/2}}\frac{\sin u}{\sin i} \tag{22}$$

and

$$M^{\backprime} = -(1-e^2)^{1/2}\left(\omega^{\backprime} + \Omega^{\backprime}\cos i + \frac{2r\dot{r}^{\backprime}}{p^{1/2}}\right) \tag{23}$$

where

$$\dot{\mathbf{r}}^{\backprime} = \dot{r}^{\backprime}\mathbf{U} + r\dot{v}^{\backprime}\mathbf{V} + r\dot{b}^{\backprime}\mathbf{W} \tag{24}$$

The Keplerian variations of these parameters are

$$\left.\begin{aligned} &\dot{a} = \dot{e} = \dot{i} = \dot{\omega} = \dot{\Omega} = 0 \\ \text{and}\quad &\dot{M} = n = \frac{1}{a^{3/2}} \end{aligned}\right\} \tag{25}$$

The instantaneous parameters are obtained by proper integration of their variations; that is,

$$\left.\begin{aligned} a &= a_0 + \int_0^t a^{\backprime}\,d\tau \\ e &= e_0 + \int_0^t e^{\backprime}\,d\tau \\ i &= i_0 + \int_0^t i^{\backprime}\,d\tau \\ \omega &= \omega_0 + \int_0^t \omega^{\backprime}\,d\tau \\ \Omega &= \Omega_0 + \int_0^t \Omega^{\backprime}\,d\tau \\ \text{and}\quad M &= M_{00} + \int_0^t a^{-3/2}\,d\tau + \int_0^t M^{\backprime}\,d\tau \end{aligned}\right\} \tag{26}$$

(It should be noted that Eqs. (26) are intended as an illustration and are not intended to represent the best set of integral equations that may be used.) In the case of Keplerian motion, the elements a, e, i, ω, Ω, and M_0 have specific geometrical significance (they describe the size, shape, and orientation of the orbit) but in the case of perturbed motion, the corresponding instantaneous elements lose their geometrical significance and are to be regarded merely as parameters in terms of which the theory is developed (in general, it becomes impossible to define elements of the actual orbit).

B. Development of Solution

The differential equations of motion of a satellite about a central mass are, from Section II-B,

$$\frac{d^2\mathbf{r}}{dt^2} + \mu_p \frac{\mathbf{r}}{r^3} - \dot{\mathbf{r}}^{\backprime} = 0$$

where the initial conditions

$$\mathbf{r}(0) \text{ and } \frac{d\mathbf{r}}{dt}(0)$$

and the physical constants are known. As in Section II-B, the development of the solution begins with a dimensional analysis to determine the normalization constants, perturbative parameters, and possibly the fast and slow time scales. The differential equations of motion are then normalized accordingly and transformed into six first-order differential equations. The next step consists of selecting the reference (or intermediate) orbit and the remaining steps in determining the perturbations of various orders for this reference motion.

The reference orbit, or reference motion, can be chosen in such a manner that the most important perturbing forces are included (at least partially) in the differential equations describing the reference motion. Then, the nature of the motion is more or less defined and only small corrections remain to be introduced subsequently; that is, the departure of the actual motion from the reference motion chosen would be small. It should be pointed out that the expressions describing the reference motion must be reasonably simple so that they can be easily manipulated. Reference orbits are classified either as dynamical reference orbits (closed-form solvable orbits of given potential functions) or as geometrical reference orbits (orbits that are defined by specific parameters and their variations but that do not necessarily satisfy any potential function).

At this point of the discussion, it is necessary to differentiate between the development of the solution with Taylor's series expansions and the development of the solution with multivariable asymptotic expansions.

1. With Taylor's series expansions. In this development, the three second-order ordinary nonlinear differential equations given above are transformed into the six standard first-order ordinary differential equations (highly coupled) in terms of the instantaneous parameters of the osculating orbit; for example, Eqs. (18–23) and (25) (the transformation process is illustrated during the analysis of the damped linear harmonic oscillator in Section IV). The radial ($\dot{r}^{\backprime}$), orthogonal (or circumferential) ($r\dot{v}^{\backprime}$), and normal ($r\dot{b}^{\backprime}$) components of the perturbative acceleration $\dot{\mathbf{r}}^{\backprime}$ are known (assuming the mathematical model of the satellite motion has been defined) and can be expressed in terms of the instantaneous parameters. Furthermore, the functions $p/r, (p/r) \sin v, (p/r)^2$, etc., appearing in Eqs. (18–23) can be expressed in terms of the instantaneous parameters (Ref. 564) so that the variations $a^{\backprime}$, $e^{\backprime}$, $i^{\backprime}$, $\omega^{\backprime}$, $\Omega^{\backprime}$ and $M^{\backprime}$ can be expressed entirely in terms of a, e, i, ω, Ω, and M; for example,

$$a^{\backprime} = a^{\backprime}(a, e, i, \omega, \Omega, M)$$

Introducing these expressions for the variations into Eqs. (26) yields six integral equations in terms of the six unknown osculating parameters a, e, i, ω, Ω, and M, which cannot, in general, be solved in closed form. As a result, a process of approximation must be used.

The process of approximation is based upon the development of a reference orbit and associated reference parameters that approximate the actual instantaneous orbit and associated actual instantaneous parameters sufficiently well so that Taylor's series expansions may be used to obtain the actual instantaneous parameters from the reference parameters. Let the state of the actual orbit at some arbitrary time t be represented by the state vector $\mathbf{q}(a, e, i, \omega, \Omega, M)$. Then, from Fig. 4,

$$\mathbf{q}(a, e, i, \omega, \Omega, M) = \bar{\mathbf{q}}(\bar{a}, \bar{e}, \bar{i}, \bar{\omega}, \bar{\Omega}, \bar{M}) + \Delta\mathbf{q}(\bar{a}, \bar{e}, \bar{i}, \bar{\omega}, \bar{\Omega}, \bar{M}) \qquad (27)$$

where $\bar{\mathbf{q}}$ is the reference state vector at the time t and $\Delta\mathbf{q}$ is the difference between the actual state vector and the reference state vector at time t. The $\Delta\mathbf{q}$ is obtained from Taylor's series expansions about $\bar{\mathbf{q}}$ using the derivative (known in terms of a, e, i, ω, Ω, M)

$$\frac{d\mathbf{q}}{dt} = \dot{\mathbf{q}} + \mathbf{q}^{\backprime}$$

where $\dot{\mathbf{q}}$ represents the Keplerian variation of the state vector, Eq. (25), and $\mathbf{q}^{\backprime}$ represents the perturbative variation of the state vector, Eqs. (18–23) after transformation into expressions in a, e, i, ω, Ω, and M. In order to insure that $\Delta\mathbf{q}$ be small, the reference state vector $\bar{\mathbf{q}}$ must contain all the secular and zero-order periodic terms (as

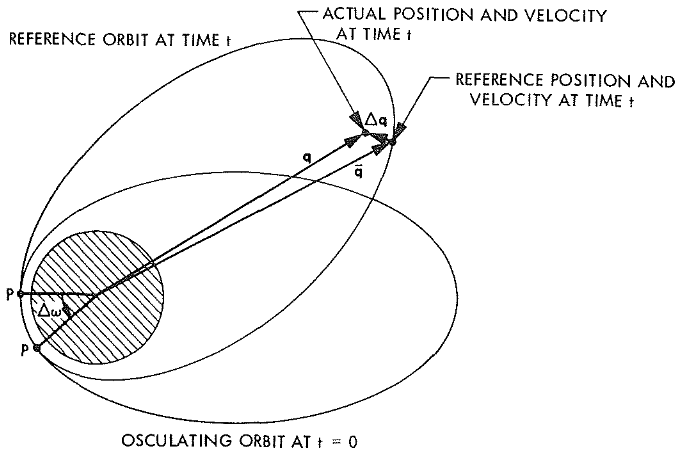

Fig. 4. Reference orbit

well as resonance terms) that may arise. Hence, the first-order perturbations would contain first-order periodic terms, the second-order perturbations would contain second-order periodic terms, etc. Through the use of these concepts, each of the instantaneous parameters is obtained in the form of an infinite series of terms of increasing order in the perturbative parameter ϵ; for example,

$$a(t;\epsilon) = \bar{a}(t;\epsilon) + a^{(1)}(t;\epsilon) + a^{(2)}(t;\epsilon) + O(\epsilon^3)$$

The position and velocity is then obtained by using simple Keplerian formulas; for example,

$$M(t;\epsilon) = E(t;\epsilon) - e(t;\epsilon)\sin E(t;\epsilon)$$

and

$$r(t;\epsilon) = a(t;\epsilon)[1 - e(t;\epsilon)\cos E(t;\epsilon)]$$

These concepts and this development are illustrated in detail during the analysis of the damped linear harmonic oscillator in Section IV-F.

2. ***With multivariable asymptotic expansions.*** In this development, a special functional transformation is chosen in such a manner that the three second-order ordinary nonlinear differential equations are transformed into six first-order ordinary differential equations (highly coupled) in terms of the instantaneous parameters of the osculating orbit but having a special form. The six first-order ordinary differential equations are of the form where fast time variables (excluding the natural independent variable of time) and slow time variables may be used as indicated by the fast and slow time scales arising during the dimensional analysis of the original differential equations of motion. Next, multivariable asymptotic expansions of the form shown in Eq. (2) are assumed to approximate the solution uniformly in time; that is (using the concept of the state vector),

$$\begin{aligned} \mathbf{q}(t;\epsilon) &= \mathbf{Q}(\bar{t}, \tilde{t}, \cdots; \epsilon) \\ &= \sum_{j=0}^{J} \nu_j(\epsilon)\, \mathbf{q}^{(j)}(\bar{t}, \tilde{t}, \cdots) \\ &\quad + O(\nu_{J+1}(\epsilon)) \end{aligned} \quad (28)$$

The subsequent development follows that of Section II-B. In this manner, the instantaneous parameters are obtained in the form of asymptotic expansions that can be introduced into the expressions relating the instantaneous parameters to the original coordinates to obtain the position and velocity of the satellite in the form of asymptotic expansions. These concepts and this development are illustrated in detail during the analysis of the damped linear harmonic oscillator in Section IV-G.

It should be pointed out that the natural independent variable t in the original differential equations of motion, Eqs. (5), can be transformed into an independent angle variable and the concepts discussed in this and the previous sections would still be applicable.

IV. Motion of a Damped Linear Harmonic Oscillator

The mathematical basis for a perturbation theory is often obscured by the number of variables and the complicated expressions that must be used in obtaining an approximate solution to a problem. Many of the essential features of a theory can be illustrated by analyzing a simple example that is not subject to the complexities of many variables and involved expressions. Such an example is that of the damped linear harmonic oscillator. The physical nature of this problem is quite simple and its exact analytic solution is easily obtained (a desirable feature if numerical and behavioristic comparisons are to be made).

A. Mathematical Model

Consider the motion of a body of mass m restrained by a linear spring and damped by a force proportional to the speed as shown in Fig. 5. The symbols C_s, C_d, and C_f represent the spring constant, damping coefficient, and friction-of-rolling coefficient (assumed to be zero or included in C_d), respectively.

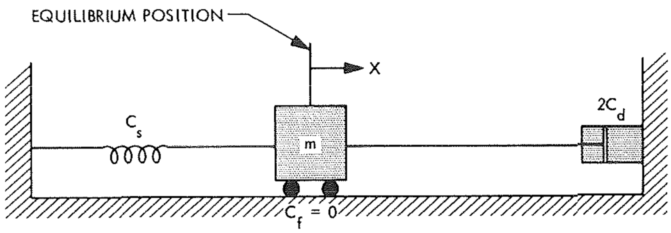

Fig. 5. Damped linear harmonic oscillator

The displacement of the mass is designated as x and is measured from the position of equilibrium of the mass (positive to the right), and the independent variable is chosen as the time t. Since the damping is assumed to be small, the motion is one of damped harmonic oscillations and is therefore bounded. Hence, the differential equation of motion and the selected initial conditions are

$$m\frac{d^2x}{dt^2} + C_s x + 2C_d \frac{dx}{dt} = 0 \tag{29}$$

and

$$x(0) = x_0, \qquad \frac{dx}{dt}(0) = 0 \tag{30}$$

where

$$0 \leqq t < \infty \tag{31}$$

The first step in solving this system of differential equations is to normalize the variables by means of appropriate normalization constants for length and time (the length and time scales). The appropriate constants are determined by introducing normalized variables x^* and t^* into Eq. (29) and requiring that all coefficients be unity except for the coefficient of dx^*/dt^*, which should be a function of a small parameter ϵ.

Let

$$x^* = \frac{x}{L_1} \text{ and } t^* = \frac{t}{T_1} \tag{32}$$

where L_1 and T_1 represent the length and time scales, respectively. Then

$$\left.\begin{aligned} &\frac{dt^*}{dt} = \frac{1}{T_1}, \quad \frac{dx}{dt} = L_1 \frac{dx^*}{dt^*}\frac{dt^*}{dt} = \frac{L_1}{T_1}\frac{dx^*}{dt^*} \\ &\text{and} \\ &\frac{d^2x}{dt^2} = \frac{L_1}{T_1}\frac{d^2x^*}{dt^{*2}}\frac{dt^*}{dt} = \frac{L_1}{T_1^2}\frac{d^2x^*}{dt^{*2}} \end{aligned}\right\} \tag{33}$$

Introducing Eqs. (32) and (33) into Eq. (29) yields

$$\frac{mL_1}{T_1^2}\frac{d^2x^*}{dt^{*2}} + C_s L_1 x^* + 2\frac{C_d L_1}{T_1}\frac{dx^*}{dt^*} = 0$$

or

$$\frac{d^2x^*}{dt^{*2}} + \frac{C_s T_1^2}{m} x^* + 2\frac{C_d T_1}{m}\frac{dx^*}{dt^*} = 0 \tag{34}$$

Equation (34) shows that L_1 is arbitrary at this point,

$$\frac{C_s T_1^2}{m} = 1, \text{ and } \epsilon = \frac{C_d T_1}{m} \tag{35}$$

from which

$$T_1 = \left(\frac{m}{C_s}\right)^{1/2} \text{ and } \epsilon = \frac{C_d}{(C_s m)^{1/2}} \tag{36}$$

The small parameter ϵ has an interesting physical interpretation, which can be seen by rewriting the second equation in (35) as

$$\epsilon = \frac{T_1}{m/C_d} \tag{37}$$

The dimensions of the mass m and the damping coefficient C_d are (using the brackets around the symbol to indicate "the dimension of")

$$[m] = \text{mass and } [C_d] = \frac{\text{mass}}{\text{time}}$$

so that a second characteristic time scale appears, namely,

$$T_2 = \frac{m}{C_d} \tag{38}$$

Hence, the small parameter ϵ becomes, using Eqs. (37) and (38)

$$\epsilon = \frac{T_1}{T_2} \tag{39}$$

the ratio of two characteristic time scales. The physical significance of these two time scales is clear. The time scale T_1 is a measure of the period of the oscillatory behavior of the system produced by the spring (characteristic time for the primary oscillations), while the time scale T_2 is a measure of the period after which the cumulative effects of damping become important (characteristic damping time). The ratio ϵ is a measure of the relative importance of the spring and damping forces and is small as required since the damping is assumed small; that is,

$$C_d << (C_s m)^{1/2} \text{ and thus } \epsilon << 1 \tag{40}$$

as can be seen from the second equation in (36). Furthermore, (39) and (40) show that

$$T_2 >> T_1 \tag{41}$$

so that T_1 characterizes the fast variation of $x(t;\epsilon)$ and T_2 characterizes the slow variation of $x(t;\epsilon)$. Consequently, T_1 is the fast time scale and T_2 is the slow time scale. The existence of these two time scales in the physical context of the problem is a fundamental feature of the method of multivariable asymptotic expansions and will be discussed in more detail in Section IV-D.

Finally, the length scale is chosen as the initial position x_0 since this is the only dimensional constant appearing in the mathematical model with a dimension of length; that is,

$$L_1 \triangleq x_0 \tag{42}$$

Introducing Eqs. (35) into Eq. (34) and using L_1 and T_1 from Eqs. (42) and (36), respectively, in Eqs. (30) and (31) yields the following differential equation of motion and initial conditions in normalized variables:

$$\frac{d^2x^*}{dt^{*2}} + x^* + 2\epsilon\frac{dx^*}{dt^*} = 0 \tag{43}$$

and

$$x^*(0) = 1, \frac{dx^*}{dt^*}(0) = 0 \tag{44}$$

where

$$0 \leqq t^* < \infty \tag{45}$$

Rather than use the cumbersome star notation in the following analysis, let x^* and t^* be replaced by the original x and t with the understanding that now the units of x and t are initial positions and periods of undamped oscillation, respectively, as may be seen from Eqs. (32), (36), and (42). Thus, the normalization procedure may be thought of as simply a change of units of the dependent and independent variables. Accordingly, Eqs. (43), (44), and (45) become

$$\frac{d^2x}{dt^2} + x + 2\epsilon\frac{dx}{dt} = 0 \tag{46}$$

and

$$x(0) = 1, \frac{dx}{dt}(0) = 0 \tag{47}$$

where

$$0 \leqq t < \infty \tag{48}$$

and where the units of x are initial positions x_0 and the units of t are periods of undamped oscillation,

$$T_1 = \left(\frac{m}{C_s}\right)^{1/2}$$

B. Exact Solution

The exact solution of Eq. (46), subject to Eqs. (47) and (48), is easily obtained using the theory of ordinary differential equations. Rewriting Eq. (46) yields

$$\frac{d^2x}{dt^2} + 2\epsilon\frac{dx}{dt} + x = 0$$

so that the auxiliary equation is

$$w^2 + 2\epsilon w + 1 = 0$$

which implies

$$w = \frac{-2\epsilon \pm (4\epsilon^2 - 4)^{1/2}}{2}$$

or

$$w = -\epsilon \pm i(1-\epsilon^2)^{1/2}$$

since $\epsilon << 1$. Hence, the general solution is

$$x(t;\epsilon) = C_1 \exp\{[-\epsilon + i(1-\epsilon^2)^{1/2}]t\} + C_2 \exp\{[-\epsilon - i(1-\epsilon^2)^{1/2}]t\}$$

or

$$x(t;\epsilon) = e^{-\epsilon t}\left[C_3 \cos(1-\epsilon^2)^{1/2} t + C_4 \sin(1-\epsilon^2)^{1/2} t\right] \tag{49}$$

Evaluating Eq. (49) at $t = 0$ and using the first equation in (47) yields

$$C_3 = 1 \tag{50}$$

Differentiating Eq. (49) with respect to t once, evaluating at $t = 0$, and using the second equation in (47) yields

$$C_4 = \frac{\epsilon}{(1-\epsilon^2)^{1/2}} \tag{51}$$

Finally, with the introduction of Eqs. (50) and (51) into Eq. (49), the general solution becomes

$$x(t;\epsilon) = e^{-\epsilon t}\left[\cos(1-\epsilon^2)^{1/2} t + \frac{\epsilon}{(1-\epsilon^2)^{1/2}} \sin(1-\epsilon^2)^{1/2} t\right] \tag{52}$$

where $0 \leqq t < \infty$. The speed dx/dt is not determined explicitly here nor in the following sections but is easily obtained by differentiating $x(t;\epsilon)$ with respect to t.

C. Solution Using Variation of Coordinates With a One-Variable Asymptotic Expansion

Consider the classical perturbation theory, used to obtain approximate solutions to highly nonlinear problems in celestial mechanics, which consists of assuming a solution to Eq. (46) in the form of a one-variable asymptotic expansion; that is,

$$x(t;\epsilon) = x^{(0)}(t) + \epsilon x^{(1)}(t) + O(\epsilon^2) \tag{53}$$

and sequentially solving for the functions of time $x^{(0)}(t)$, $x^{(1)}(t)$, etc. Note that $x^{(0)}(t)$ can be thought of as the reference motion, and $\epsilon x^{(1)}(t)$, $\epsilon^2 x^{(2)}(t)$, etc., can be thought of as the perturbations to the reference motion. Also, note that the reference motion is simply the unperturbed motion of the system; that is, the motion of the system with $\epsilon = 0$.

Differentiating Eq. (53) with respect to t yields

$$\frac{dx}{dt} = \frac{dx^{(0)}}{dt} + \epsilon \frac{dx^{(1)}}{dt} + O(\epsilon^2) \tag{54}$$

and

$$\frac{d^2x}{dt^2} = \frac{d^2x^{(0)}}{dt^2} + \epsilon \frac{d^2x^{(1)}}{dt^2} + O(\epsilon^2) \tag{55}$$

Introducing Eqs. (53), (54), and (55) into Eq. (46) and combining like powers of ϵ, one obtains

$$\left(\frac{d^2x^{(0)}}{dt^2} + x^{(0)}\right) + \left(\frac{d^2x^{(1)}}{dt^2} + x^{(1)} + 2\frac{dx^{(0)}}{dt}\right)\epsilon + O(\epsilon^2) = 0 \tag{56}$$

Since the expansion (53) must hold (at least in an asymptotic sense) for arbitrary values of the perturbative parameter ϵ, the coefficients of the powers of ϵ in Eq. (56) must separately equal zero. Thus,

$$\frac{d^2x^{(0)}}{dt^2} + x^{(0)} = 0 \tag{57}$$

$$\frac{d^2x^{(1)}}{dt^2} + x^{(1)} = -2\frac{dx^{(0)}}{dt} \tag{58}$$

and so forth. The initial conditions necessary for the complete solution of these differential equations are obtained by evaluating Eqs. (53) and (54) at $t = 0$, by using the initial conditions (47), by combining terms of like powers of ϵ, and by setting the coefficients of the powers of ϵ separately equal to zero. Thus,

$$1 = x^{(0)}(0) + \epsilon x^{(1)}(0) + O(\epsilon^2)$$

and

$$0 = \frac{dx^0}{dt}(0) + \epsilon \frac{dx^{(1)}}{dt}(0) + O(\epsilon^2)$$

so that

$$x^{(0)}(0) = 1, \qquad \frac{dx^{(0)}}{dt}(0) = 0 \tag{59}$$

$$x^{(1)}(0) = 0, \qquad \frac{dx^{(1)}}{dt}(0) = 0 \tag{60}$$

and so forth.

The solution to Eq. (57) is

$$x^{(0)}(t) = B_0 \cos t + C_0 \sin t \tag{61}$$

and, by differentiation,

$$\frac{dx^{(0)}}{dt} = -B_0 \sin t + C_0 \cos t \tag{62}$$

Evaluating Eqs. (61) and (62) at $t = 0$ and using Eqs. (59), one obtains

$$B_0 = 1 \text{ and } C_0 = 0 \tag{63}$$

so that Eq. (61) becomes

$$x^{(0)}(t) = \cos t \tag{64}$$

Thus, the reference motion is simple harmonic motion. Introducing Eqs. (63) into Eq. (62) and the results into Eq. (58) yields

$$\frac{d^2x^{(1)}}{dt^2} + x^{(1)} = 2\sin t \tag{65}$$

The solution to the corresponding homogeneous differential equation is

$$x_H^{(1)} = B_1 \cos t + C_1 \sin t$$

and a particular solution, from (A-4) in Appendix A, is

$x_P^{(1)} = -t\cos t$ so that

$$x^{(1)}(t) = B_1 \cos t + C_1 \sin t - t\cos t \tag{66}$$

Differentiating Eq. (66) and combining terms yields

$$\frac{dx^{(1)}}{dt} = (t - B_1)\sin t + (C_1 - 1)\cos t \tag{67}$$

Evaluating Eqs. (66) and (67) at $t = 0$ and using Eqs. (60), one obtains

$$B_1 = 0 \text{ and } C_1 = 1$$

so that Eq. (66) becomes

$$x^{(1)}(t) = \sin t - t\cos t \tag{68}$$

Similarly, the remaining terms of the asymptotic expansion (53) may be obtained. Introducing Eqs. (64) and (68) into Eq. (53) yields

$$x(t;\epsilon) = \cos t + \epsilon(\sin t - t\cos t) + O(\epsilon^2) \tag{69}$$

Equation (69) shows that the asymptotic expansion of the form (53) to $O(\epsilon)$ contains a short-period perturbation, $\epsilon \sin t$, and a Poisson perturbation, $-\epsilon t\cos t$. The Poisson perturbation implies that $x(t;\epsilon)$ decreases indefinitely with the time (both $|\cos t| \leq 1$ and $|\sin t| \leq 1$). However, since $x(t;\epsilon)$ is known to be bounded from Section IV-A, the Poisson perturbation is not basic to the nature of the problem but only to the perturbation theory used. The expansion (69) is only an approximate expression used for $x(t;\epsilon)$, which is useful for a limited time only (initially valid). It appears that there are higher-order Poisson perturbations which, when included, would sum to a bounded function just as

$$e^{-\epsilon t}\cos(1-\epsilon^2)^{1/2} t = \cos t - \epsilon t\cos t + O(\epsilon^2)$$

is a function whose numerical value does not exceed unity although a consideration of its expansion in the form of Eq. (53) to $O(\epsilon)$ would lead to the conclusion that it decreases indefinitely with the time t. In order to show that this is the case, consider the series expansion of the exact solution given in Eq. (52) for $\epsilon << 1$ and $0 \leq t \leq t_1$; that is, using Taylor's series (about $t = 0$),

$$x(t;\epsilon) = [1 - \epsilon t + O(\epsilon^2)]\{\cos[1 - O(\epsilon^2)]t + [\epsilon + O(\epsilon^2)]\sin[1 - O(\epsilon^3)]t\}$$

or

$$x(t;\epsilon) = \cos t + \epsilon(\sin t - t\cos t) + O(\epsilon^2) \tag{70}$$

The expansion given in Eq. (70) is the initially valid expansion of Eq. (52) to $O(\epsilon)$; that is, (70) approximates (52) sufficiently well only over a finite time interval $0 \leq t \leq t_1$ since the representations of the functions $e^{-\epsilon t}$, $\cos(1-\epsilon^2)^{1/2} t$, and $\sin(1-\epsilon^2)^{1/2} t$ by finite numbers of terms of Taylor's series (about $t = 0$) are not accurate for large times. Comparing Eqs. (69) and (70), it is clear that Eq. (69) is an initially valid asymptotic expansion of Eq. 52). Furthermore, the nonuniform representation of the term $e^{-\epsilon t}\cos(1-\epsilon^2)^{1/2} t$ in Eq. (52) introduces the Poisson perturbation, $-\epsilon t\cos t$, into Eqs. (69) and (70) (a representation of a term is said to be uniformly valid if the representation is valid for any time t in the entire range of t). The appearance of a secular or Poisson perturbation in an approximate representation of a bounded function implies a nonuniform representation of some term in that function since all secular and Poisson perturbations are unbounded.

In general, an asymptotic expansion of the form of Eq. (53) leads to an initially valid approximation of the solution. Since the objective of this investigation is to obtain a satellite theory that will predict the position and velocity of a satellite to a sufficiently high accuracy even after many revolutions about the central mass ($t >> 0$), the perturbation theory described in this section is unacceptable (Laplace and Lagrange realized the inadequacies of this perturbation theory during their studies of the perturbations of the planets; see Section I-A-3). An acceptable perturbation theory is one which yields a uniformly valid approximation of a solution; that is, the approximation does not contain nonuniform representations of terms in the exact solution.

D. Solution Using Variation of Coordinates With a Two-Variable Asymptotic Expansion

In this section, the perturbation theory of the previous section is modified in such a manner that an acceptable perturbation theory is obtained.

1. Selection of coordinates and independent variable. The natural coordinates $x(t;\epsilon)$ and dx/dt are chosen as the coordinates, and the natural independent variable t is chosen as the independent variable.

2. Equation of motion. The differential equation of motion and initial conditions, for the coordinates and independent variable chosen (properly normalized), are given by Eqs. (46), (47), and (48); that is,

$$\frac{d^2x}{dt^2} + x + 2\epsilon\frac{dx}{dt} = 0$$

and

$$x(0) = 1, \qquad \frac{dx}{dt}(0) = 0$$

where

$$0 \leqq t < \infty$$

3. Development of solution. Consider the perturbation theory which consists of (1) assuming a solution to Eq. (46) in the form of a two-variable asymptotic expansion, that is,

$$x(t;\epsilon) = X(\bar{t},\tilde{t};\epsilon) \triangleq x^{(0)}(\bar{t},\tilde{t}) + \epsilon x^{(1)}(\bar{t},\tilde{t}) + \epsilon^2 x^{(2)}(\bar{t},\tilde{t}) + O(\epsilon^3) \tag{71}$$

where

$$\bar{t} \triangleq t[1 + \epsilon^2\tau_2 + O(\epsilon^3)] \tag{72}$$

$$\tilde{t} \triangleq \epsilon t \tag{73}$$

and the $\tau_2, \tau_3, \cdots$ are undetermined constants, and (2) sequentially solving for the functions $x^{(0)}(\bar{t},\tilde{t})$, $x^{(1)}(\bar{t},\tilde{t})$, etc., using the concepts discussed in Section II-A. Note that $x^{(0)}(\bar{t},\tilde{t})$ can be thought of as the reference motion and $\epsilon x^{(1)}(\bar{t},\tilde{t})$, $\epsilon^2 x^{(2)}(\bar{t},\tilde{t})$, etc., can be thought of as the perturbations to the reference motion. Also, note that the reference motion is not necessarily the unperturbed motion of the system, since both $\bar{t}$ and $\tilde{t}$ reflect the effects of the perturbative parameter ϵ.

As mentioned in Section II-A-2, the convergence of the asymptotic expansion (71) is of no interest in this investigation. The important objective is that the approximation of the solution by a finite number of terms of an expansion illustrate all the essential features of the motion and provide a close numerical approximation to the exact result for all time t in the range of t, $0 \leqq t < \infty$, assuming the parameter ϵ is sufficiently small (the previous perturbation theory failed in this respect).

The need for "two" distinct time variables $\bar{t}$ and $\tilde{t}$ was indicated in Section IV-A by the occurrence of the two independent time scales, T_1 and T_2, in the physical context of the problem. The time variable $\bar{t}$ is a fast time variable normalized by using the fast time scale T_1, as may be seen from Eq. (72), by returning to natural time units and using the second equation in (32),

$$\bar{t} = t^*[1 + O(\epsilon^2)] = \frac{t}{T_1}[1 + O(\epsilon^2)]$$

The time variable $\tilde{t}$ is a slow time variable normalized by using the slow time scale T_2, as may be seen from Eq. (73), by returning to natural time units and using $t^* = t/T_1$ from Eqs. (32) and Eq. (39),

$$\tilde{t} = \epsilon t^* = \frac{T_1}{T_2}\frac{t}{T_1} = \frac{t}{T_2}$$

The distinct variable $\bar{t}$ is the variable close to t that is introduced to account for the frequency shift appearing in Eq. (52) due to the generalized damping, as is done in the Lindstedt–Poincaré theory for periodic solutions. The distinct variable $\tilde{t}$ is the slowly changing variable that is introduced to account for the slow modulation of the oscillations appearing in Eq. (52); that is, the slow decay.

For conciseness, the following notation is adopted:

$$\left.\begin{aligned} x_1^{(j)} &= \frac{\partial x^{(j)}}{\partial \bar{t}},\; x_2^{(j)} = \frac{\partial x^{(j)}}{\partial \tilde{t}},\; x_{11}^{(j)} = \frac{\partial^2 x^{(j)}}{\partial \bar{t}^2} \\ x_{12}^{(j)} &= \frac{\partial^2 x^{(j)}}{\partial \bar{t}\,\partial \tilde{t}} = x_{21}^{(j)},\; x_{22}^{(j)} = \frac{\partial^2 x^{(j)}}{\partial \tilde{t}^2} \end{aligned}\right\} \tag{74}$$

where

$$j = 0, 1, 2, \cdots$$

Differentiating Eq. (71) with respect to the natural independent variable t and using the notation in Eqs. (74) yields

$$\frac{dx}{dt} = \left(x_1^{(0)} \frac{d\bar{t}}{dt} + x_2^{(0)} \frac{d\tilde{t}}{dt} \right) + \epsilon \left(x_1^{(1)} \frac{d\bar{t}}{dt} + x_2^{(1)} \frac{d\tilde{t}}{dt} \right)$$
$$+ \epsilon^2 \left(x_1^{(2)} \frac{d\bar{t}}{dt} + x_2^{(2)} \frac{d\tilde{t}}{dt} \right) + O(\epsilon^3) \tag{75}$$

Differentiating Eqs. (72) and (73) with respect to t yields

$$\frac{d\bar{t}}{dt} = 1 + \epsilon^2 \tau_2 + O(\epsilon^3) \text{ and } \frac{d\tilde{t}}{dt} = \epsilon \tag{76}$$

Introducing Eqs. (76) into Eq. (75) and combining like powers of ϵ, one obtains

$$\frac{dx}{dt} = x_1^{(0)} + \epsilon \left(x_2^{(0)} + x_1^{(1)} \right)$$
$$+ \epsilon^2 \left(\tau_2 x_1^{(0)} + x_2^{(1)} + x_1^{(2)} \right) + O(\epsilon^3) \tag{77}$$

Differentiating Eq. (77) with respect to the natural independent variable t and using the notation in Eqs. (74), one obtains

$$\frac{d^2x}{dt^2} = \left(x_{11}^{(0)} \frac{d\bar{t}}{dt} + x_{12}^{(0)} \frac{d\tilde{t}}{dt} \right)$$
$$+ \epsilon \left(x_{21}^{(0)} \frac{d\bar{t}}{dt} + x_{22}^{(0)} \frac{d\tilde{t}}{dt} + x_{11}^{(1)} \frac{d\bar{t}}{dt} + x_{12}^{(1)} \frac{d\tilde{t}}{dt} \right)$$
$$+ \epsilon^2 \left[\tau_2 \left(x_{11}^{(0)} \frac{d\bar{t}}{dt} + x_{12}^{(0)} \frac{d\tilde{t}}{dt} \right) \right.$$
$$\left. + x_{21}^{(1)} \frac{d\bar{t}}{dt} + x_{22}^{(1)} \frac{d\tilde{t}}{dt} + x_{11}^{(2)} \frac{d\bar{t}}{dt} + x_{12}^{(2)} \frac{d\tilde{t}}{dt} \right] + O(\epsilon^3) \tag{78}$$

Introducing Eqs. (76) into Eq. (78) and combining like powers of ϵ, one obtains

$$\frac{d^2x}{dt^2} = x_{11}^{(0)} + \epsilon \left(2x_{12}^{(0)} + x_{11}^{(1)} \right)$$
$$+ \epsilon^2 \left(2\tau_2 x_{11}^{(0)} + x_{22}^{(0)} + 2x_{12}^{(1)} + x_{11}^{(2)} \right) + O(\epsilon^3) \tag{79}$$

Introducing Eqs. (71), (77), and (79) into the differential equation of motion (Eq. 46), and combining like powers of ϵ yields

$$\left(x_{11}^{(0)} + x^{(0)} \right) + \epsilon \left(x_{11}^{(1)} + x^{(1)} + 2x_1^{(0)} + 2x_{12}^{(0)} \right)$$
$$+ \epsilon^2 \left(x_{11}^{(2)} + x^{(2)} + 2x_{12}^{(1)} + 2x_1^{(1)} + 2\tau_2 x_{11}^{(0)} + x_{22}^{(0)} + 2x_2^{(0)} \right)$$
$$+ O(\epsilon^3) = 0 \tag{80}$$

Since the expansion (80) must hold (at least in an asymptotic sense) for arbitrary values of the perturbative parameter ϵ, the coefficients of the powers of ϵ in (80) must separately equal zero. Thus,

$$x_{11}^{(0)} + x^{(0)} = 0 \tag{81}$$

$$x_{11}^{(1)} + x^{(1)} = -2x_{12}^{(0)} - 2x_1^{(0)} \tag{82}$$

$$x_{11}^{(2)} + x^{(2)} = -2x_{12}^{(1)} - 2x_1^{(1)} - x_{22}^{(0)} - 2\tau_2 x_{11}^{(0)} - 2x_2^{(0)} \tag{83}$$

and so forth. The initial conditions necessary for the complete solution of these partial differential equations are obtained by evaluating Eqs. (71) and (77) at $t = 0$, by using the initial conditions (Eqs. 47), by combining terms of like powers of ϵ, and by setting the coefficients of the powers of ϵ separately equal to zero. Thus, since $\bar{t} = 0$ and $\tilde{t} = 0$ when $t = 0$ from Eqs. (72) and (73),

$$1 = x^{(0)}(0,0) + \epsilon x^{(1)}(0,0) + \epsilon^2 x^{(2)}(0,0) + O(\epsilon^3)$$

and

$$0 = x_1^{(0)}(0,0) + \epsilon \left[x_2^{(0)}(0,0) + x_1^{(1)}(0,0) \right]$$
$$+ \epsilon^2 \left[\tau_2 x_1^{(0)}(0,0) + x_2^{(1)}(0,0) + x_1^{(2)}(0,0) \right] + O(\epsilon^3)$$

so that

$$x^{(0)}(0,0) = 1, \qquad x_1^{(0)}(0,0) = 0 \tag{84}$$

$$x^{(1)}(0,0) = 0, \qquad x_1^{(1)}(0,0) = -x_2^{(0)}(0,0) \tag{85}$$

$$x^{(2)}(0,0) = 0, \qquad x_1^{(2)}(0,0) = -x_2^{(1)}(0,0) - \tau_2 x_1^{(0)}(0,0) \tag{86}$$

and so forth.

The general solution to Eq. (81),

$$\frac{\partial^2 x^{(0)}}{\partial \bar{t}^2} + x^{(0)} = 0$$

is

$$x^{(0)}(\bar{t},\tilde{t}) = B^{(0)}(\tilde{t})\cos\bar{t} + C^{(0)}(\tilde{t})\sin\bar{t} \tag{87}$$

In solving the partial differential equation (Eq. 81), $\tilde{t}$ can be thought of as a parameter rather than an independent variable since the partial derivative with respect to $\tilde{t}$ does not appear. Hence, the theory of ordinary differential equations may be used, remembering that any constants which ordinarily would appear must be functions of the parameter $\tilde{t}$. This phenomenon is an essential characteristic of the theory of multivariable asymptotic expansions.

Taking the partial derivative of Eq. (87) with respect to $\bar{t}$, one obtains

$$x_1^{(0)} = -B^{(0)}(\tilde{t})\sin\bar{t} + C^{(0)}(\tilde{t})\cos\bar{t} \tag{88}$$

Evaluating Eqs. (87) and (88) at $t = 0$ ($\bar{t} = 0$ and $\tilde{t} = 0$) and using Eqs. (84),

$$B^{(0)}(0) = 1 \text{ and } C^{(0)}(0) = 0 \tag{89}$$

At this point, another essential characteristic of the theory of multivariable asymptotic expansions arises. All of the known conditions on $x^{(0)}(\bar{t},\tilde{t})$ and, hence, on $B^{(0)}(\tilde{t})$ and $C^{(0)}(\tilde{t})$ have been applied without a unique determination of these functions. This attribute is exactly what is required in order to prevent the occurrence of secular or Poisson perturbations in the approximation to the solution, as will be seen during the solution of Eq. (82).

Introducing Eq. (88) and its partial derivative with respect to $\bar{t}$ into Eq. (82) yields

$$x_{11}^{(1)} + x^{(1)} = -2\left[\frac{dC^{(0)}}{d\tilde{t}}\cos\bar{t} - \frac{dB^{(0)}}{d\tilde{t}}\sin\bar{t}\right] - 2\left[C^{(0)}\cos\bar{t} - B^{(0)}\sin\bar{t}\right]$$

or

$$x_{11}^{(1)} + x^{(1)} = -2\left[\frac{dC^{(0)}}{d\tilde{t}} + C^{(0)}\right]\cos\bar{t} + 2\left[\frac{dB^{(0)}}{d\tilde{t}} + B^{(0)}\right]\sin\bar{t} \tag{90}$$

Treating $\tilde{t}$ as a parameter in the same manner as before, the solution one obtains to the corresponding homogeneous partial differential equation is

$$x_H^{(1)}(\bar{t},\tilde{t}) = B^{(1)}(\tilde{t})\cos\bar{t} + C^{(1)}(\tilde{t})\sin\bar{t} \tag{91}$$

and a particular solution, from (A-3) and (A-4) in Appendix A, is

$$x_P^{(1)}(\bar{t},\tilde{t}) = -\left[\frac{dC^{(0)}}{d\tilde{t}} + C^{(0)}\right]\bar{t}\sin\bar{t} - \left[\frac{dB^{(0)}}{d\tilde{t}} + B^{(0)}\right]\bar{t}\cos\bar{t} \tag{92}$$

so that

$$x^{(1)}(\bar{t},\tilde{t}) = B^{(1)}(\tilde{t})\cos\bar{t} + C^{(1)}(\tilde{t})\sin\bar{t} - \left[\frac{dC^{(0)}}{d\tilde{t}} + C^{(0)}\right]\bar{t}\sin\bar{t} - \left[\frac{dB^{(0)}}{d\tilde{t}} + B^{(0)}\right]\bar{t}\cos\bar{t} \tag{93}$$

Now, the solution to the differential equation (46) with $\epsilon = 0$ and subject to the initial conditions (47) is $\cos t$ (periodic in the natural independent variable t) as shown in Section IV-C, so that from the first uniformity condition, as discussed in Section II-A-3, no secular or Poisson perturbations in the fast time variable $\bar{t}$ are allowed. Thus, the first uniformity condition requires that

$$\frac{dB^{(0)}}{d\tilde{t}} + B^{(0)} = 0 \tag{94}$$

and

$$\frac{dC^{(0)}}{d\tilde{t}} + C^{(0)} = 0 \tag{95}$$

Note that secular or Poisson perturbations may arise in the slow time variable $\tilde{t}$, indicating a slow decay because of the damping.

The solutions to Eqs. (94) and (95) are

$$B^{(0)}(\tilde{t}) = B_0 e^{-\tilde{t}} \text{ and } C^{(0)}(\tilde{t}) = C_0 e^{-\tilde{t}} \tag{96}$$

Evaluating Eqs. (96) at $\tilde{t} = 0$ and using Eqs. (89) yields

$$B_0 = 1 \text{ and } C_0 = 0$$

so that Eqs. (96) become

$$B^{(0)}(\tilde{t}) = e^{-\tilde{t}} \text{ and } C^{(0)}(\tilde{t}) = 0 \tag{97}$$

Introducing Eqs. (97) into Eq. (87) yields the uniformly valid first approximation to the solution (reference motion):

$$x^{(0)}(\bar{t},\tilde{t}) = e^{-\tilde{t}}\cos\bar{t} \tag{98}$$

Thus, the reference motion is not the unperturbed motion, $\cos t$, as is the case in Section IV-C.

It should be noted that the uniformly valid first approximation, $x^{(0)}(\bar{t}, \tilde{t})$, is completely determined by applying certain logic (first uniformity condition) to the next highest term of the expansion; that is, $x^{(1)}(\bar{t}, \tilde{t})$. Similarly, the uniformly valid jth approximation is completely determined by applying certain logic (first and second uniformity conditions) to the $j+1$ term of the expansion. Furthermore, although none of the τ_j in the definition of $\bar{t}$ (Eq. 72) have been obtained as yet, these constants are determined as additional terms of the asymptotic expansion (Eq. 71) are obtained.

From Eqs. (93), (94), and (95),

$$x^{(1)}(\bar{t}, \tilde{t}) = B^{(1)}(\tilde{t}) \cos \bar{t} + C^{(1)}(\tilde{t}) \sin \bar{t} \tag{99}$$

and, by partial differentiation with respect to $\bar{t}$,

$$x_1^{(1)} = -B^{(1)}(\tilde{t}) \sin \bar{t} + C^{(1)}(\tilde{t}) \cos \bar{t} \tag{100}$$

Consider the partial differential equation (83) from which the functions $B^{(1)}(\tilde{t})$ and $C^{(1)}(\tilde{t})$ as well as the undetermined constant τ_2 are determined. Taking the required partial derivatives of Eqs. (98) and (100) with respect to $\bar{t}$ and $\tilde{t}$, one obtains

$$x_{11}^{(0)} = -e^{-\tilde{t}} \cos \bar{t} \tag{101}$$

$$x_2^{(0)} = -e^{-\tilde{t}} \cos \bar{t} \tag{102}$$

$$x_{22}^{(0)} = e^{-\tilde{t}} \cos \bar{t} \tag{103}$$

and

$$x_{12}^{(1)} = -\frac{dB^{(1)}}{d\tilde{t}} \sin \bar{t} + \frac{dC^{(1)}}{d\tilde{t}} \cos \bar{t} \tag{104}$$

Introducing Eqs. (100–104) into Eq. (83) yields

$$x_{11}^{(2)} + x^{(2)} = -2\left[\frac{dC^{(1)}}{d\tilde{t}} + C^{(1)} - \left(\tau_2 + \frac{1}{2}\right)e^{-\tilde{t}}\right] \cos \bar{t} + 2\left[\frac{dB^{(1)}}{d\tilde{t}} + B^{(1)}\right] \sin \bar{t} \tag{105}$$

Solving Eq. (105) for $x^{(2)}(\bar{t}, \tilde{t})$ and applying the first uniformity condition results in the two equations yields

$$\frac{dB^{(1)}}{d\tilde{t}} + B^{(1)} = 0$$

and

$$\frac{dC^{(1)}}{d\tilde{t}} + C^{(1)} = \left(\tau_2 + \frac{1}{2}\right)e^{-\tilde{t}}$$

whose solutions are, using Solution (A-1) in Appendix A,

$$\left.\begin{aligned} B^{(1)}(\tilde{t}) &= B_1 e^{-\tilde{t}} \\ \text{and} \quad C^{(1)}(\tilde{t}) &= C_1 e^{-\tilde{t}} + \left(\tau_2 + \frac{1}{2}\right)\tilde{t} e^{-\tilde{t}} \end{aligned}\right\} \tag{106}$$

Evaluating Eqs. (99) and (100) at $t = 0$ ($\bar{t} = 0$ and $\tilde{t} = 0$) and using Eqs. (85) and (102), one obtains

$$B^{(1)}(0) = 0 \text{ and } C^{(1)}(0) = 1 \tag{107}$$

Evaluating Eqs. (106) at $\tilde{t} = 0$ and using Eqs. (107) yields

$$B_1 = 0 \text{ and } C_1 = 1$$

so that Eqs. (106) become

$$\left.\begin{aligned} B^{(1)}(\tilde{t}) &= 0 \\ \text{and} \quad C^{(1)}(\tilde{t}) &= \left[1 + \left(\tau_2 + \frac{1}{2}\right)\tilde{t}\right] e^{-\tilde{t}} \end{aligned}\right\} \tag{108}$$

Introducing Eqs. (108) into Eq. (99) yields

$$x^{(1)}(\bar{t}, \tilde{t}) = \left[1 + \left(\tau_2 + \frac{1}{2}\right)\tilde{t}\right] e^{-\tilde{t}} \sin \bar{t} \tag{109}$$

At this point, another essential characteristic of the theory of multivariable asymptotic expansions arises. All of the known conditions on $x^{(1)}(\bar{t}, \tilde{t})$ have been applied (without the determination of the constant τ_2) except the second uniformity condition as discussed in Section II-A-3. The second uniformity condition requires that

$$\lim_{\epsilon \to 0} \frac{\epsilon x^{(1)}(\bar{t}, \tilde{t})}{x^{(0)}(\bar{t}, \tilde{t})} = 0 \text{ for all } t \text{ in } 0 \leq t < \infty$$

so that, from Eqs. (98) and (109),

$$\lim_{\epsilon \to 0} \frac{\epsilon \left[1 + \left(\tau_2 + \frac{1}{2}\right)\tilde{t}\right] e^{-\tilde{t}} \sin \bar{t}}{e^{-\tilde{t}} \cos \bar{t}} = 0$$

$$\text{for all } t \text{ in } 0 \leqq t < \infty \tag{110}$$

Equation (110) requires that

$$\tau_2 = -\frac{1}{2} \tag{111}$$

so that the coefficient of the term $\tilde{t} e^{-\tilde{t}} \sin \bar{t}$ is zero, and a uniformly valid perturbation $\epsilon x^{(1)}(\bar{t}, \tilde{t})$ is obtained (see Section II-A-3). From Eqs. (109) and (111),

$$x^{(1)}(\bar{t}, \tilde{t}) = e^{-\tilde{t}} \sin \bar{t} \tag{112}$$

where, from Eqs. (72) and (111),

$$\bar{t} = t\left[1 - \frac{1}{2}\epsilon^2 + O(\epsilon^3)\right] \tag{113}$$

Similarly, the remaining constants $\tau_3, \tau_4, \cdots$ in $\bar{t}$ and perturbations $\epsilon^2 x^{(2)}(\bar{t}, \tilde{t})$, $\epsilon^3 x^{(3)}(\bar{t}, \tilde{t})$, $\cdots$ in the asymptotic expansion (71) may be obtained.

The use of two distinct time variables, $\bar{t}$ and $\tilde{t}$, introduces a degree of flexibility into the theory of solution in such a manner that improper secular or Poisson perturbations can be eliminated. The application of the first uniformity condition removes improper secular or Poisson perturbations in the fast variable $\bar{t}$, and the application of the second uniformity condition removes improper secular or Poisson perturbations in the slow variable $\tilde{t}$.

The uniformly valid asymptotic expansion to $O(\epsilon)$ is, from Eqs. (71), (98), and (112),

$$X(\bar{t}, \tilde{t}; \epsilon) = e^{-\tilde{t}} \cos \bar{t} + \epsilon e^{-\tilde{t}} \sin \bar{t} + O(\epsilon^2)$$

or, using Eqs. (73) and (113),

$$x(t;\epsilon) = e^{-\epsilon t}\left\{\cos\left[1 - \frac{1}{2}\epsilon^2 + O(\epsilon^3)\right]t + \epsilon \sin\left[1 - \frac{1}{2}\epsilon^2 + O(\epsilon^3)\right]t\right\} + O(\epsilon^2) \tag{114}$$

A comparison of the uniformly valid asymptotic expansion to $O(\epsilon)$ (Eq. 114), with the exact solution (Eq. 52), shows a remarkable similarity. The expansion (114) exhibits the proper bounded motion, exponential decay, and frequency shift. Furthermore, the reference motion

$$x^{(0)}(t;\epsilon) = e^{-\epsilon t} \cos\left[1 - \frac{1}{2}\epsilon^2 + O(\epsilon^3)\right]t \tag{115}$$

is much closer to the actual motion than the unperturbed motion, $\cos t$. As a result, the perturbation theory described in this section is acceptable.

E. Solution Using Variation of Parameters With Successive Approximations

Consider the classical perturbation theory which consists of (1) defining an osculating trajectory, (2) transforming Eq. (46) into differential equations in terms of the instantaneous parameters of the osculating trajectory, and (3) integrating these differential equations by using successive approximations.

1. Selection of parameters and independent variable. From Eq. (46),

$$\frac{d^2x}{dt^2} + x = -2\epsilon \frac{dx}{dt} \tag{116}$$

Assume that the mass m is always moving in a sinusoidal trajectory (the counterpart of the conic section in Section III-A-1), but in one that changes at each instant. The sinusoidal trajectory is tangent to the path of motion at every point, as shown in Fig. 6, and the velocity associated with this osculating trajectory at the point of tangency is the same as the actual velocity of the mass m at that point. Thus,

$$\frac{dx}{dt} = \dot{x} \text{ and } x' = 0 \tag{117}$$

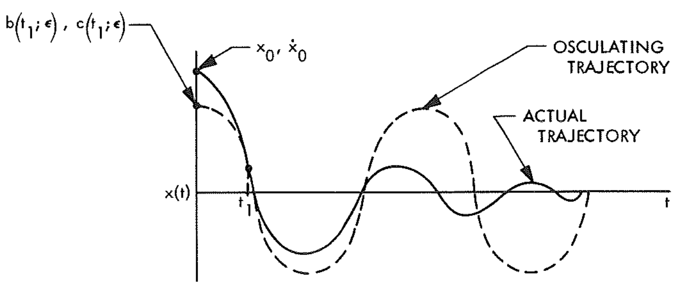

Fig. 6. Osculating trajectory at time t_1

where $\dot{x}$ is the unperturbed variation and $x^{\backprime}$ is the perturbative variation (see Section III-A-2). In addition (from the same section),

$$\frac{d^2x}{dt^2} = \ddot{x} + \dot{x}^{\backprime} \qquad (118)$$

Introducing Eqs. (117) and (118) into Eq. (116) yields

$$\ddot{x} + x = 0 \qquad (119)$$

and

$$\dot{x}^{\backprime} = -2\epsilon\dot{x} \qquad (120)$$

The solution to Eq. (119) is

$$x(t) = b\cos t + c\sin t \qquad (121)$$

and, by differentiation,

$$\dot{x}(t) = -b\sin t + c\cos t \qquad (122)$$

where b and c are the constants of integration in the case of unperturbed motion. In the terminology of astrodynamics, b and c are the parameters (or elements) of the sinusoidal trajectory of the mass m.

The theory of the variation of parameters consists in permitting the parameters b and c (previously constant) to vary with the time t; that is,

$$x(t;\epsilon) = b(t;\epsilon)\cos t + c(t;\epsilon)\sin t \qquad (123)$$

and

$$\dot{x}(t;\epsilon) = -b(t;\epsilon)\sin t + c(t;\epsilon)\cos t \qquad (124)$$

in such a manner that Eqs. (123) and (124) will satisfy the differential equation of perturbed motion (Eq. 116) for all values of time.

The parameters $b(t;\epsilon)$ and $c(t;\epsilon)$ are the instantaneous or osculating parameters that describe the osculating trajectory. From a purely mathematical point of view, Eqs. (123) and (124) represent a special functional transformation from the original dependent variables $x(t;\epsilon)$ and $\dot{x}(t;\epsilon)$ to new dependent variables $b(t;\epsilon)$ and $c(t;\epsilon)$ which vary slowly with time. Note that if Eqs. (121) and (122) are evaluated at $t = 0$,

$$b = x(0) \stackrel{\Delta}{=} x_0 \text{ and } c = \dot{x}(0) \stackrel{\Delta}{=} \dot{x}_0$$

so that their osculating counterparts are

$$b(t;\epsilon) = x_0(t;\epsilon) \text{ and } c(t;\epsilon) = \dot{x}_0(t;\epsilon)$$

that is, the osculating parameters are the instantaneous initial conditions of the osculating trajectory. If, at an instant t_1, the disturbing force is removed ($\epsilon = 0$), the mass m would move in a sinusoidal trajectory of the form

$$x(t;\epsilon) = b(t_1;\epsilon)\cos t + c(t_1;\epsilon)\sin t$$

where $b(t_1;\epsilon)$ and $c(t_1;\epsilon)$ are the osculating parameters (initial conditions) at t_1, as shown in Fig. 6.

The functions $b(t;\epsilon)$ and $c(t;\epsilon)$ and the variable t are chosen as the osculating parameters and independent variable, respectively. It should be pointed out that other selections are possible.

2. ***Equations of motion.*** Equations (123) and (124) are linear in $b(t;\epsilon)$ and $c(t;\epsilon)$ and can be solved for these parameters because the determinant of their coefficients is unity. Hence,

$$\left.\begin{aligned} b(t;\epsilon) &= x(t;\epsilon)\cos t - \dot{x}(t;\epsilon)\sin t \\ &\text{and} \\ c(t;\epsilon) &= x(t;\epsilon)\sin t + \dot{x}(t;\epsilon)\cos t \end{aligned}\right\} \qquad (125)$$

Furthermore, from Eqs. (47), (117), and (125),

$$\left.\begin{aligned} b_0 &\stackrel{\Delta}{=} b(0;\epsilon) = x(0;\epsilon) = 1 \\ &\text{and} \\ c_0 &\stackrel{\Delta}{=} c(0;\epsilon) = \dot{x}(0;\epsilon) = 0 \end{aligned}\right\} \qquad (126)$$

The variations of the parameters $b(t;\epsilon)$ and $c(t;\epsilon)$ can be expressed as the sum of two parts; that is,

$$\frac{db}{dt} = \dot{b} + b^{\backprime} \text{ and } \frac{dc}{dt} = \dot{c} + c^{\backprime} \qquad (127)$$

where $\dot{b}$ and $\dot{c}$ are the sinusoidal (unperturbed) variations of $b(t;\epsilon)$ and $c(t;\epsilon)$ that remain at the instant of osculation if the disturbing force is suddenly removed and $b^{\backprime}$ and $c^{\backprime}$ are the perturbative variations caused by the disturbing force. Both b and c are constants in the case of unperturbed motion according to Eq. (121) so that

$$\dot{b} = \dot{c} = 0 \qquad (128)$$

This result can be substantiated by taking the dot derivative of Eq. (125) and then using Eq. (119) as follows:

$$\dot{b} = -x \sin t + \dot{x} \cos t - \dot{x} \cos t - \ddot{x} \sin t$$

and

$$\dot{c} = x \cos t + \dot{x} \sin t - \dot{x} \sin t + \ddot{x} \cos t$$

or

$$\dot{b} = -(\ddot{x} + x) \sin t = 0$$

and

$$\dot{c} = (\ddot{x} + x) \cos t = 0$$

In addition, using the concept of perturbative differentiation from Section III-A-2, Eqs. (125) yield

$$b^{\backprime} = -\dot{x}^{\backprime} \sin t \text{ and } c^{\backprime} = \dot{x}^{\backprime} \cos t \tag{129}$$

where $x^{\backprime} = 0$ from Eq. (117) and $t^{\backprime} = 0$ (the independent variable t is unaffected by the disturbing force). Introducing Eq. (120) into Eq. (129) and using Eq. (124),

$$b^{\backprime} = 2\epsilon\dot{x} \sin t = 2\epsilon\left[-b(t;\epsilon)\sin^2 t + c(t;\epsilon)\sin t \cos t\right]$$

and

$$c^{\backprime} = -2\epsilon\dot{x} \cos t = -2\epsilon\left[-b(t;\epsilon)\sin t \cos t + c(t;\epsilon)\cos^2 t\right]$$

or, using trigonometric formulas,

$$\left.\begin{aligned} b^{\backprime} &= -\epsilon\left[b(t;\epsilon) - b(t;\epsilon)\cos 2t - c(t;\epsilon)\sin 2t\right] \\ &\text{and} \\ c^{\backprime} &= -\epsilon\left[c(t;\epsilon) + c(t;\epsilon)\cos 2t - b(t;\epsilon)\sin 2t\right] \end{aligned}\right\} \tag{130}$$

Introducing Eqs. (128) and (130) into Eqs. (127) yields the differential equations of motion in terms of the osculating parameters, or, in integral form,

$$\left.\begin{aligned} b &= b_0 + \int_0^t b^{\backprime}\, d\tau \\ &\text{and} \\ c &= c_0 + \int_0^t c^{\backprime}\, d\tau \end{aligned}\right\} \tag{131}$$

where $b^{\backprime}$ and $c^{\backprime}$ are given by Eq. (130).

It should be noted that the procedure in solving the integral equations (Eqs. 131) is just as difficult as the one in solving the original differential equation (Eq. 46) because of the coupling of the integral equations. The objective of the special functional transformation (associated with the osculating trajectory) from $x(t;\epsilon)$ and $\dot{x}(t;\epsilon)$ to $b(t;\epsilon)$ and $c(t;\epsilon)$ is not to simplify the procedure of solution but rather to develop the equations of motion so that series expansions can be effectively used. The appearance of ϵ as a multiplier in Eqs. (130) shows that $b^{\backprime}$ and $c^{\backprime}$ are very small (as is the case in all perturbative variations) so that $b(t;\epsilon)$ and $c(t;\epsilon)$ change very slowly, a desirable characteristic when using series expansions. In general, the total variations of the parameters are separated into variations of unperturbed motion which can be analytically integrated in closed form and perturbative variations which must be determined by using series expansions.

3. ***Analytical integration of equations of motion.*** The process of successive approximations consists of approximating the perturbative variations, $b^{\backprime}$ and $c^{\backprime}$, by successively more accurate integrable expressions which, upon analytic integration, yield successively higher-order solutions. For example, a classical first-order solution is obtained by analytical integrations of Eqs. (131), using $b^{\backprime}$ and $c^{\backprime}$ from Eqs. (130) but with $b(t;\epsilon)$ and $c(t;\epsilon)$ approximated by b_0 and c_0, respectively (in this manner, the reference motion is the unperturbed motion of the system, $\cos t$). Having a first-order solution (which yields better approximations of the functions $b(t;\epsilon)$ and $c(t;\epsilon)$ then b_0 and c_0), a second-order solution is obtained by analytical integrations of Eqs. (131) using $b^{\backprime}$ and $c^{\backprime}$ from Eqs. (130) but now with $b(t;\epsilon)$ and $c(t;\epsilon)$ replaced by their respective first-order approximations, and so forth. It should be noted that the term "order" used in this section refers to "apparent order" as discussed in Section I-C-4.

Consider the first-order solution to Eqs. (131) in the manner just described. From Eqs. (130) with $b(t;\epsilon)$ and $c(t;\epsilon)$ approximated by b_0 and c_0, respectively,

$$\left.\begin{aligned} b^{\backprime} &= -\epsilon(b_0 - b_0\cos 2t - c_0 \sin 2t) + \cdots \\ &\text{and} \\ c^{\backprime} &= -\epsilon(c_0 + c_0\cos 2t - b_0 \sin 2t) + \cdots \end{aligned}\right\} \tag{132}$$

Introducing Eqs. (132) into Eqs. (131), one obtains

$$b(t;\epsilon) = b_0 - \epsilon\int_0^t (b_0 - b_0\cos 2\tau - c_0\sin 2\tau)\, d\tau + \cdots$$

and

$$c(t;\epsilon) = c_0 - \epsilon \int_0^t (c_0 + c_0 \cos 2\tau - b_0 \sin 2\tau)\, d\tau + \cdots$$

so that, upon integration, the first-order approximations of $b(t;\epsilon)$ and $c(t;\epsilon)$ are

$$\left.\begin{aligned} b(t;\epsilon) &= \left(b_0 + \frac{1}{2}\epsilon c_0\right) \\ &\quad - \epsilon\left(b_0 t + \frac{1}{2} c_0 \cos 2t - \frac{1}{2} b_0 \sin 2t\right) + \cdots \\ &\text{and} \\ c(t;\epsilon) &= \left(c_0 + \frac{1}{2}\epsilon b_0\right) \\ &\quad - \epsilon\left(c_0 t + \frac{1}{2} b_0 \cos 2t + \frac{1}{2} c_0 \sin 2t\right) + \cdots \end{aligned}\right\} \tag{133}$$

Introducing Eqs. (130), with $b(t;\epsilon)$ and $c(t;\epsilon)$ replaced by their respective first-order approximations from Eqs. (133), into Eqs. (131) and integrating yields the second-order approximations of $b(t;\epsilon)$ and $c(t;\epsilon)$, and so forth.

From Eqs. (126) and (133),

$$\left.\begin{aligned} b(t;\epsilon) &= 1 - \epsilon\left(t - \frac{1}{2}\sin 2t\right) + \cdots \\ &\text{and} \\ c(t;\epsilon) &= \frac{1}{2}\epsilon(1 - \cos 2t) + \cdots \end{aligned}\right\} \tag{134}$$

Note that at $t = 0$, Eqs. (134) become

$$b_0 = 1 + \cdots \qquad \text{and} \qquad c_0 = 0 + \cdots$$

which satisfy Eqs. (126) to the order considered. Introducing Eqs. (134) into Eq. (123) yields

$$x(t;\epsilon) = \left[1 - \epsilon\left(t - \frac{1}{2}\sin 2t\right)\right]\cos t + \left[\frac{1}{2}\epsilon(1 - \cos 2t)\right]\sin t + \cdots$$

or, using trigonometric formulas,

$$x(t;\epsilon) = \cos t + \epsilon(\sin t - t\cos t) + \cdots \tag{135}$$

Thus, the first-order approximation in Eq. (135) obtained using the variation of parameters with successive approximations is only initially valid (because of the Poisson perturbation) just as the first-order approximation obtained in Section IV-C using the variation of coordinates with a one-variable asymptotic expansion is initially valid. As a result, the perturbation theory described in this section is unacceptable (see the discussion at the end of Section IV-C).

Since Eq. (135) is an initially valid approximation to $O(\epsilon)$, Eqs. (134) and (135) become

$$\left.\begin{aligned} b(t;\epsilon) &= 1 - \epsilon\left(t - \frac{1}{2}\sin 2t\right) + O(\epsilon^2) \\ c(t;\epsilon) &= \frac{1}{2}\epsilon(1 - \cos 2t) + O(\epsilon^2) \\ &\text{and} \\ x(t;\epsilon) &= \cos t + \epsilon(\sin t - t\cos t) + O(\epsilon^2) \end{aligned}\right\} \tag{136}$$

F. Solution Using Variation of Parameters With Taylor's Series Expansions

The perturbation theory in the previous section yields only an initially valid approximation of the solution because of the process used for the analytical integration of the perturbative variations and not because of the formulation of the equations to be integrated. As a result, a process for the analytical integration of the perturbative variations is developed in this section in such a manner that a more accurate perturbation theory is obtained.

1. Selection of parameters and independent variable. The functions $b(t;\epsilon)$ and $c(t;\epsilon)$ and the variable t are chosen as the osculating parameters and independent variable, respectively, just as in the previous section.

2. Equations of motion. The equations of motion in terms of the selected osculating parameters and independent variable, in integral form, are given by Eqs. (130) and (131); that is,

$$b(t;\epsilon) = b_0 + \int_0^t b^{\backprime}\, d\tau$$

and

$$c(t;\epsilon) = c_0 + \int_0^t c^{\backprime}\, d\tau$$

where

$$b^{\backprime} = -\epsilon\,[b(t;\epsilon) - b(t;\epsilon)\cos 2t - c(t;\epsilon)\sin 2t]$$

and

$$c^{\backprime} = -\epsilon\,[c\,(t;\epsilon) + c\,(t;\epsilon)\cos 2t - b\,(t;\epsilon)\sin 2t]$$

The corresponding initial conditions are given by Eqs. (126) as

$$b_0 = 1 \qquad \text{and} \qquad c_0 = 0$$

and the range of t is given by (48) as

$$0 \leqq t < \infty$$

3. Analytical integration of equations of motion. The process of Taylor's series expansions is based upon the development of a reference motion and associated reference parameters, $\bar{b}\,(t;\epsilon)$ and $\bar{c}\,(t;\epsilon)$, which approximate the actual motion and associated instantaneous parameters, $b\,(t;\epsilon)$ and $c\,(t;\epsilon)$, sufficiently well so that Taylor's series expansions may be used to obtain the instantaneous parameters from the reference parameters (see Section III-B-1). The actual motion is then obtained from Eq. (123); that is,

$$x\,(t;\epsilon) = b\,(t;\epsilon)\cos t + c\,(t;\epsilon)\sin t$$

Let

$$\left.\begin{aligned} &b\,(t;\epsilon) = \bar{b}\,(t;\epsilon) + b^{(1)}\,(t;\epsilon) + b^{(2)}\,(t;\epsilon) + O\,(\epsilon^3) \\ &\text{and} \\ &\qquad b \rightarrow c \end{aligned}\right\} \qquad (137)$$

The superscript within a parenthesis indicates the order of the perturbation with respect to the perturbative parameter ϵ; for example,

$$b^{(1)}\,(t;\epsilon) = O\,(\epsilon)\,, \qquad b^{(2)}\,(t;\epsilon) = O\,(\epsilon^2)\,, \qquad \text{etc.}$$

and the notation $b \rightarrow c$ implies that there is a corresponding equation for c which can be obtained from the equation in b by replacing b with c. Note that the reference parameters, $\bar{b}\,(t;\epsilon)$ and $\bar{c}\,(t;\epsilon)$, depend upon ϵ and therefore the reference motion is not the unperturbed motion, $\cos t$. In addition, let

$$b^{\backprime} = (b^{\backprime})_s + (b^{\backprime})_{sp} + (b^{\backprime})_{\ell p} \qquad \text{and} \qquad b \rightarrow c \qquad (138)$$

where the subscripts s, sp, and ℓp indicate those portions of $b^{\backprime}$ or $c^{\backprime}$ which yield secular, short-period, and long-period terms, respectively, upon integration. Poisson terms will be classified as either short-period terms or long-period terms depending upon whether their periodic portion is short-periodic or long-periodic, respectively. Finally, let the short-period and long-period variations be expanded in terms of powers of ϵ so that Eqs. (138) become

$$\left.\begin{aligned} &b^{\backprime} = (b^{\backprime})_s + (b^{\backprime})^{(0)}_{sp} + (b^{\backprime})^{(1)}_{sp} + (b^{\backprime})^{(2)}_{sp} \\ &\qquad + (b^{\backprime})^{(0)}_{\ell p} + (b^{\backprime})^{(1)}_{\ell p} + (b^{\backprime})^{(2)}_{\ell p} + O\,(\epsilon^3) \\ &\text{and} \\ &\qquad b \rightarrow c \end{aligned}\right\} \qquad (139)$$

Introducing Eqs. (137) and (139) into the equations of motion in integral form (Eqs. 131),

$$\left.\begin{aligned} &\bar{b}\,(t;\epsilon) + b^{(1)}\,(t;\epsilon) + b^{(2)}\,(t;\epsilon) + O\,(\epsilon^3) = \\ &b_0 + \int_0^t [(b^{\backprime})_s + (b^{\backprime})^{(0)}_{sp} + (b^{\backprime})^{(0)}_{\ell p} + (b^{\backprime})^{(1)}_{\ell p}]\,d\tau \\ &\quad + \int_0^t [(b^{\backprime})^{(1)}_{sp} + (b^{\backprime})^{(2)}_{\ell p}]\,d\tau \\ &\quad + \int_0^t [(b^{\backprime})^{(2)}_{sp} + (b^{\backprime})^{(3)}_{\ell p}]\,d\tau + O\,(\epsilon^3) \\ &\text{and} \\ &\qquad b \rightarrow c \end{aligned}\right\} \qquad (140)$$

These equations are the basis for choosing the reference parameters, $\bar{b}\,(t;\epsilon)$ and $\bar{c}\,(t;\epsilon)$, and corresponding perturbations. The reference parameters are chosen in such a manner that they are good approximations of the instantaneous parameters, $b\,(t;\epsilon)$ and $c\,(t;\epsilon)$, and yet reasonably simple so that they can be easily manipulated. Thus, the constant terms, secular terms, and terms of order one in Eqs. (140) are defined as composing the reference parameters (in this manner, the reference motion is of the geometrical type) and the perturbations are defined by equating the remaining terms of the same order in Eqs. (140); that is,

$$\left.\begin{aligned} &\bar{b}\,(t;\epsilon) = \\ &\quad b_0 + \int_0^t [(b^{\backprime})_s + (b^{\backprime})^{(0)}_{sp} + (b^{\backprime})^{(0)}_{\ell p} + (b^{\backprime})^{(1)}_{\ell p}]\,d\tau \\ &\qquad - \int^0 [(b^{\backprime})^{(1)}_{sp} + (b^{\backprime})^{(2)}_{\ell p}]\,d\tau + O\,(\epsilon^2) \\ &b^{(1)}\,(t;\epsilon) = \int^t [(b^{\backprime})^{(1)}_{sp} + (b^{\backprime})^{(2)}_{\ell p}]\,d\tau\,, \qquad \text{etc.} \\ &\text{and} \\ &\qquad b \rightarrow c \end{aligned}\right\} \qquad (141)$$

where some of the definite integrals have been separated into functions of t and constants (see "integral" in the Glossary) and where past experience has shown that the long-period variations decrease by one order upon integration and so have been properly accounted for. Note that

$$\left.\begin{aligned} &\int^{t} [\ ]\,d\tau \Rightarrow \text{integrate integrand and evaluate at } \tau = t \text{ only} \\ &\text{and} \\ &\int^{0} [\ ]\,d\tau \Rightarrow \text{integrate integrand and evaluate at } \tau = 0 \text{ only (yielding a constant)} \end{aligned}\right\} \quad (142)$$

In order to obtain the required secular, short-period, and long-period terms in Eqs. (141), $b^{\backprime}$ and $c^{\backprime}$ as given in Eqs. (130) must be expanded in Taylor's series (assumed to be uniformly convergent in the range $0 \leqq t < \infty$) about the point represented by $\bar{b}(t;\epsilon)$ and $\bar{c}(t;\epsilon)$ at time t. Since the perturbative variations, $b^{\backprime}$ and $c^{\backprime}$, are explicit functions of only b and c (ϵ and t are assumed to be fixed) as shown in Eqs. (130),

$$\left.\begin{aligned} b^{\backprime} &= \overline{b^{\backprime}} + \overline{\frac{\partial b^{\backprime}}{\partial b}}\Delta b + \overline{\frac{\partial b^{\backprime}}{\partial c}}\Delta c + \frac{1}{2}\overline{\frac{\partial^2 b^{\backprime}}{\partial b^2}}(\Delta b)^2 \\ &\quad + \frac{1}{2}\overline{\frac{\partial^2 b^{\backprime}}{\partial c^2}}(\Delta c)^2 + \overline{\frac{\partial^2 b^{\backprime}}{\partial b \partial c}}\Delta b \Delta c + \cdots \\ &\text{and} \\ &b^{\backprime} \rightarrow c^{\backprime} \end{aligned}\right\} \quad (143)$$

where the bar over a partial derivative implies that the partial derivative is evaluated using $\bar{b}$ and $\bar{c}$ in place of b and c respectively. From Eqs. (137),

$$\left.\begin{aligned} &\Delta b = b - \bar{b} = b^{(1)} + b^{(2)} + O(\epsilon^3) \\ &\text{and} \\ &b \rightarrow c \end{aligned}\right\} \quad (144)$$

Note that because of the transformation from rapidly changing dependent variables x and $\dot{x}$ to slowly changing dependent variables b and c and because of the careful selection of the reference parameters $\bar{b}$ and $\bar{c}$, the quantities Δb and Δc are very small, allowing the rapid convergence of the two Taylor's series in Eqs. (143).

From Eqs. (130),

$$\left.\begin{aligned} &\frac{\partial b^{\backprime}}{\partial b} = -\epsilon(1 - \cos 2t), \qquad \frac{\partial b^{\backprime}}{\partial c} = \epsilon \sin 2t \\ &\frac{\partial c^{\backprime}}{\partial b} = \epsilon \sin 2t, \qquad \frac{\partial c^{\backprime}}{\partial c} = -\epsilon(1 + \cos 2t) \\ &\text{and} \\ &\frac{\partial^2 b^{\backprime}}{\partial b^2} = \frac{\partial^2 b^{\backprime}}{\partial c^2} = \frac{\partial^2 b^{\backprime}}{\partial b \partial c} = \cdots \\ &\quad = \frac{\partial^2 c^{\backprime}}{\partial b^2} = \frac{\partial^2 c^{\backprime}}{\partial c^2} = \frac{\partial^2 c^{\backprime}}{\partial b \partial c} = \cdots = 0 \end{aligned}\right\} \quad (145)$$

Introducing the reference parameters into Eqs. (130) and (145) yields the $\overline{b^{\backprime}}$, $\overline{c^{\backprime}}$, and the partial derivatives required in Eqs. (143); that is,

$$\left.\begin{aligned} &\overline{b^{\backprime}} = -\epsilon(\bar{b} - \bar{b}\cos 2t - \bar{c}\sin 2t) \\ &\overline{\frac{\partial b^{\backprime}}{\partial b}} = -\epsilon(1 - \cos 2t), \qquad \overline{\frac{\partial b^{\backprime}}{\partial c}} = \epsilon \sin 2t \\ &\overline{c^{\backprime}} = -\epsilon(\bar{c} + \bar{c}\cos 2t - \bar{b}\sin 2t) \\ &\overline{\frac{\partial c^{\backprime}}{\partial b}} = \epsilon \sin 2t, \qquad \overline{\frac{\partial c^{\backprime}}{\partial c}} = -\epsilon(1 + \cos 2t) \\ &\text{and} \\ &\overline{\frac{\partial^2 b^{\backprime}}{\partial b^2}} = \overline{\frac{\partial^2 b^{\backprime}}{\partial c^2}} = \overline{\frac{\partial^2 b^{\backprime}}{\partial b \partial c}} = \cdots \\ &\quad = \overline{\frac{\partial^2 c^{\backprime}}{\partial b^2}} = \overline{\frac{\partial^2 c^{\backprime}}{\partial c^2}} = \overline{\frac{\partial^2 c^{\backprime}}{\partial b \partial c}} = \cdots = 0 \end{aligned}\right\} \quad (146)$$

Introducing Eqs. (144) and (146) into Eqs. (143) yields

$$\left.\begin{aligned} b^{\backprime} &= -\epsilon(\bar{b} - \bar{b}\cos 2t - \bar{c}\sin 2t) \\ &\quad - \epsilon(1 - \cos 2t)\,[b^{(1)} + b^{(2)} + O(\epsilon^3)] \\ &\quad + \epsilon(\sin 2t)\,[c^{(1)} + c^{(2)} + O(\epsilon^3)] \\ &\text{and} \\ c^{\backprime} &= -\epsilon(\bar{c} + \bar{c}\cos 2t - \bar{b}\sin 2t) \\ &\quad + \epsilon(\sin 2t)\,[b^{(1)} + b^{(2)} + O(\epsilon^3)] \\ &\quad - \epsilon(1 + \cos 2t)\,[c^{(1)} + c^{(2)} + O(\epsilon^3)] \end{aligned}\right\} \quad (147)$$

Let $b' = (b')^{(0)} + (b')^{(1)} + (b')^{(2)} + O(\epsilon^3)$ and $b \to c$ in Eqs. (147) so that by equating terms of like powers of ϵ and assuming that $\bar{b}$ and $\bar{c}$ are of order one,

$$(b')^{(0)} = (b')_s^{(0)} + (b')_{sp}^{(0)} + (b')_{lp}^{(0)} = 0 \tag{148}$$

$$\begin{aligned}(b')^{(1)} &= (b')_s^{(1)} + (b')_{sp}^{(1)} + (b')_{lp}^{(1)} \\ &= -\epsilon(\bar{b} - \bar{b}\cos 2t - \bar{c}\sin 2t)\end{aligned} \tag{149}$$

$$\begin{aligned}(b')^{(2)} &= (b')_s^{(2)} + (b')_{sp}^{(2)} + (b')_{lp}^{(2)} \\ &= -\epsilon(b^{(1)} - b^{(1)}\cos 2t - c^{(1)}\sin 2t)\end{aligned} \tag{150}$$

and so forth, and

$$(c')^{(0)} = (c')_s^{(0)} + (c')_{sp}^{(0)} + (c')_{lp}^{(0)} = 0 \tag{151}$$

$$\begin{aligned}(c')^{(1)} &= (c')_s^{(1)} + (c')_{sp}^{(1)} + (c')_{lp}^{(1)} \\ &= -\epsilon(\bar{c} + \bar{c}\cos 2t - \bar{b}\sin 2t)\end{aligned} \tag{152}$$

$$\begin{aligned}(c')^{(2)} &= (c')_s^{(2)} + (c')_{sp}^{(2)} + (c')_{lp}^{(2)} \\ &= -\epsilon(c^{(1)} + c^{(1)}\cos 2t - b^{(1)}\sin 2t)\end{aligned} \tag{153}$$

and so forth.

If the functional dependence of the parameters $\bar{b}$, $\bar{c}$, $b^{(1)}$, $c^{(1)}$, $\cdots$ upon t were known, the secular, short-period, and long-period terms of various orders appearing in Eqs. (141) could be determined from Eqs. (148)–(153). However, these functional forms are not known as yet. Note that the perturbative variations of order j are expressed in terms of the parameters of order $j-1$ and therefore if the reference parameters can be determined, the first-order perturbative variations can be obtained, from which the first-order parameters can be obtained, and so forth.

From Eqs. (141),

$$\left.\begin{aligned}&\bar{b}(t;\epsilon) = \\ &\bar{b}_0 + \int_0^t [(b')_s + (b')_{sp}^{(0)} + (b')_{lp}^{(0)} + (b')_{lp}^{(1)}]\,d\tau \\ &\text{and} \\ &\qquad b \to c\end{aligned}\right\} \tag{154}$$

where

$$\left.\begin{aligned}\bar{b}_0 &\triangleq b_0 - \int^0 [(b')_{sp}^{(1)} + (b')_{lp}^{(2)}]\,d\tau + O(\epsilon^2) \\ &= b_0 - (b^{(1)})_0 + O(\epsilon^2) \\ &\text{and} \\ &\qquad b \to c\end{aligned}\right\} \tag{155}$$

Note that both $\bar{b}_0$ and $\bar{c}_0$ are constants to be determined later.

Since $x(t;\epsilon)$ is bounded (see Section IV-A), it follows from Eq. (123) that both $b(t;\epsilon)$ and $c(t;\epsilon)$ must be bounded and therefore

$$(b')_s = (c')_s = 0 \tag{156}$$

that is, $b(t;\epsilon)$ and $c(t;\epsilon)$ cannot contain unbounded (secular) terms (remember $(b')_s$ and $(c')_s$ are terms that yield secular terms upon integration). Furthermore, Eqs. (148) and (151) imply that

$$(b')_{sp}^{(0)} = (b')_{lp}^{(0)} = (c')_{sp}^{(0)} = (c')_{lp}^{(0)} = 0 \tag{157}$$

since secular, short-period, and long-period terms are of different character. Introducing Eqs. (156) and (157) into Eqs. (154) yields

$$\left.\begin{aligned}&\bar{b}(t;\epsilon) = \bar{b}_0 + \int_0^t (b')_{lp}^{(1)}\,d\tau \\ &\text{and} \\ &\bar{c}(t;\epsilon) = \bar{c}_0 + \int_0^t (c')_{lp}^{(1)}\,d\tau\end{aligned}\right\} \tag{158}$$

It is interesting to note that if

$$\int_0^t (b')_{lp}^{(1)}\,d\tau \qquad \text{and} \qquad \int_0^t (c')_{lp}^{(1)}\,d\tau$$

were not included in the definitions of $\bar{b}(t;\epsilon)$ and $\bar{c}(t;\epsilon)$, respectively, then

$$\bar{b}(t;\epsilon) = \bar{b}_0 \qquad \text{and} \qquad \bar{c}(t;\epsilon) = \bar{c}_0$$

both of which are constants. However, if $\bar{b}(t;\epsilon)$ and $\bar{c}(t;\epsilon)$ are constants, then Eqs. (149) and (152) show that

$$(b')_s^{(1)} = -\epsilon\bar{b}_0 \qquad \text{and} \qquad (c')_s^{(1)} = -\epsilon\bar{c}_0$$

contradicting Eqs. (156), which show that secular terms of all orders are zero (in general, $\bar{b}_0$ and $\bar{c}_0$ are not simultaneously zero).

Remembering that the integral of a long-period function usually decreases by one order of magnitude, let

$$\left.\begin{aligned} \int_0^t (b')^{(1)}_{lp}\, d\tau &= b^{(0)}_{lp}(t;\epsilon) + B_0 \\ \text{and} \\ \int_0^t (c')^{(1)}_{lp}\, d\tau &= c^{(0)}_{lp}(t;\epsilon) + C_0 \end{aligned}\right\} \quad (159)$$

so that Eqs. (158) become

$$\left.\begin{aligned} \bar{b}(t;\epsilon) &= \bar{b}_0 + b^{(0)}_{lp}(t;\epsilon) + B_0 \\ \text{and} \\ \bar{c}(t;\epsilon) &= \bar{c}_0 + c^{(0)}_{lp}(t;\epsilon) + C_0 \end{aligned}\right\} \quad (160)$$

Introducing Eqs. (160) into Eqs. (149) and (152), using the results that the product of a constant or long-period function with a short-period function yields a short-period function, using $(b')^{(1)}_s = (c')^{(1)}_s = 0$ from Eqs. (156), and equating secular, short-period, and long-period terms, respectively, one obtains

$$B_0 = -\bar{b}_0, \qquad C_0 = -\bar{c}_0 \quad (161)$$

$$(b')^{(1)}_{sp} = \epsilon\,(b^{(0)}_{lp}\cos 2t + c^{(0)}_{lp}\sin 2t) \quad (162)$$

$$(c')^{(1)}_{sp} = -\epsilon\,(c^{(0)}_{lp}\cos 2t - b^{(0)}_{lp}\sin 2t) \quad (163)$$

$$\left.\begin{aligned} (b')^{(1)}_{lp} &= -\epsilon b^{(0)}_{lp} \\ \text{and} \\ (c')^{(1)}_{lp} &= -\epsilon c^{(0)}_{lp} \end{aligned}\right\} \quad (164)$$

Introducing Eqs. (161) and (164) into Eqs. (159) yields

$$-\epsilon\int_0^t b^{(0)}_{lp}\, d\tau = b^{(0)}_{lp} - \bar{b}_0$$

and

$$-\epsilon\int_0^t c^{(0)}_{lp}\, d\tau = c^{(0)}_{lp} - \bar{c}_0$$

whose solutions are (differentiating with respect to t and solving the first-order differential equations)

$$\left.\begin{aligned} b^{(0)}_{lp}(t;\epsilon) &= \bar{b}_0 e^{-\epsilon t} \\ \text{and} \\ c^{(0)}_{lp}(t;\epsilon) &= \bar{c}_0 e^{-\epsilon t} \end{aligned}\right\} \quad (165)$$

respectively. Hence, from Eqs. (160), (161), and (165),

$$\left.\begin{aligned} \bar{b}(t;\epsilon) &= \bar{b}_0 e^{-\epsilon t} \\ \text{and} \\ \bar{c}(t;\epsilon) &= \bar{c}_0 e^{-\epsilon t} \end{aligned}\right\} \quad (166)$$

These equations determine the reference parameters and reference motion. It follows from Eqs. (162), (163), and (165) that

$$(b')^{(1)}_{sp} = \epsilon\,(\bar{b}_0 e^{-\epsilon t}\cos 2t + \bar{c}_0 e^{-\epsilon t}\sin 2t) \quad (167)$$

and

$$(c')^{(1)}_{sp} = -\epsilon\,(\bar{c}_0 e^{-\epsilon t}\cos 2t - \bar{b}_0 e^{-\epsilon t}\sin 2t) \quad (168)$$

The next step is to determine the first-order perturbations, $b^{(1)}(t;\epsilon)$ and $c^{(1)}(t;\epsilon)$. Introducing Eqs. (167) and (168) into the expressions for $b^{(1)}(t;\epsilon)$ and $c^{(1)}(t;\epsilon)$ from Eqs. (141), respectively, and letting

$$\left.\begin{aligned} b^{(1)}_{lp} &= \int^t (b')^{(2)}_{lp}\, d\tau \\ \text{and} \\ c^{(1)}_{lp} &= \int^t (c')^{(2)}_{lp}\, d\tau \end{aligned}\right\} \quad (169)$$

yields

$$\left.\begin{aligned} b^{(1)} &= \epsilon\int^t (\bar{b}_0 e^{-\epsilon\tau}\cos 2\tau + \bar{c}_0 e^{-\epsilon\tau}\sin 2\tau)\, d\tau + b^{(1)}_{lp} \\ \text{and} \\ c^{(1)} &= -\epsilon\int^t (\bar{c}_0 e^{-\epsilon\tau}\cos 2\tau - \bar{b}_0 e^{-\epsilon\tau}\sin 2\tau)\, d\tau + c^{(1)}_{lp} \end{aligned}\right\} \quad (170)$$

From Ref. 1372 (p. 196),

$$\int^t e^{-\epsilon\tau}\cos 2\tau\, d\tau = \frac{1}{(4+\epsilon^2)}\, e^{-\epsilon t}(-\epsilon\cos 2t + 2\sin 2t)$$

and

$$\int^t e^{-\epsilon\tau}\sin 2\tau\, d\tau = \frac{1}{(4+\epsilon^2)}\, e^{-\epsilon t}(-\epsilon\sin 2t - 2\cos 2t)$$

so that Eqs. (170) become

$$\left.\begin{aligned}b^{(1)}(t;\epsilon) &= \frac{\epsilon}{(4+\epsilon^2)}e^{-\epsilon t}[-(2\bar{c}_0+\epsilon\bar{b}_0)\cos 2t \\ &\quad + (2\bar{b}_0-\epsilon\bar{c}_0)\sin 2t] + b_{lp}^{(1)}(t;\epsilon) \\ \text{and} \\ c^{(1)}(t;\epsilon) &= -\frac{\epsilon}{(4+\epsilon^2)}e^{-\epsilon t}[(2\bar{b}_0-\epsilon\bar{c}_0)\cos 2t \\ &\quad + (2\bar{c}_0+\epsilon\bar{b}_0)\sin 2t] + c_{lp}^{(1)}(t;\epsilon)\end{aligned}\right\}\quad(171)$$

Introducing Eqs. (171) into Eqs. (150) and (153), using the result that the product of a constant or long-period function with a short-period function yields a short-period function, and equating secular, short-period, and long-period terms, respectively, one obtains

$$\left.\begin{aligned}&(b')_s^{(2)} = (c')_s^{(2)} = 0, \text{ in accordance with Eqs. (156)} \\ &(b')_{sp}^{(2)} = \frac{\epsilon^2}{(4+\epsilon^2)}e^{-\epsilon t}[(2\bar{c}_0+\epsilon\bar{b}_0)\cos 2t \\ &\qquad - (2\bar{b}_0-\epsilon\bar{c}_0)\sin 2t] \\ &\qquad + \epsilon b_{lp}^{(1)}\cos 2t + \epsilon c_{lp}^{(1)}\sin 2t \\ &(c')_{sp}^{(2)} = \frac{\epsilon^2}{(4+\epsilon^2)}e^{-\epsilon t}[(2\bar{b}_0-\epsilon\bar{c}_0)\cos 2t \\ &\qquad + (2\bar{c}_0+\epsilon\bar{b}_0)\sin 2t] \\ &\qquad - \epsilon c_{lp}^{(1)}\cos 2t + \epsilon b_{lp}^{(1)}\sin 2t\end{aligned}\right\}\quad(172)$$

$$\left.\begin{aligned}(b')_{lp}^{(2)} &= -\frac{\epsilon^2}{(4+\epsilon^2)}(2\bar{c}_0+\epsilon\bar{b}_0)e^{-\epsilon t} - \epsilon b_{lp}^{(1)} \\ \text{and} \\ (c')_{lp}^{(2)} &= \frac{\epsilon^2}{(4+\epsilon^2)}(2\bar{b}_0-\epsilon\bar{c}_0)e^{-\epsilon t} - \epsilon c_{lp}^{(1)}\end{aligned}\right\}\quad(173)$$

Introducing Eqs. (173) into Eqs. (169) yields

$$b_{lp}^{(1)} = -\frac{\epsilon^2}{(4+\epsilon^2)}(2\bar{c}_0+\epsilon\bar{b}_0)\int^t e^{-\epsilon\tau}d\tau - \epsilon\int^t b_{lp}^{(1)}d\tau$$

and

$$c_{lp}^{(1)} = \frac{\epsilon^2}{(4+\epsilon^2)}(2\bar{b}_0-\epsilon\bar{c}_0)\int^t e^{-\epsilon\tau}d\tau - \epsilon\int^t c_{lp}^{(1)}d\tau$$

whose solutions are (differentiating with respect to t and solving the first-order differential equations with the help of Solution (A-1) in Appendix A)

$$\left.\begin{aligned}b_{lp}^{(1)}(t;\epsilon) &= -\frac{\epsilon}{(4+\epsilon^2)}e^{-\epsilon t}[B_1 \\ &\quad + (2\bar{c}_0+\epsilon\bar{b}_0)(1+\epsilon t)] \\ \text{and} \\ c_{lp}^{(1)}(t;\epsilon) &= -\frac{\epsilon}{(4+\epsilon^2)}e^{-\epsilon t}[C_1 \\ &\quad - (2\bar{b}_0-\epsilon\bar{c}_0)(1+\epsilon t)]\end{aligned}\right\}\quad(174)$$

where B_1 and C_1 are constants of integration. Since there are no conditions specified for Eqs. (174) (the initial conditions, b_0 and c_0, are used in defining $\bar{b}_0$ and $\bar{c}_0$), the constants B_1 and C_1 are completely arbitrary and can be defined so as to simplify the expressions for $b_{lp}^{(1)}(t;\epsilon)$ and $c_{lp}^{(1)}(t;\epsilon)$. Let

$$B_1 = -(2\bar{c}_0+\bar{b}_0)$$

and

$$C_1 = (2\bar{b}_0-\bar{c}_0)$$

so that Eqs. (174) become

$$\left.\begin{aligned}b_{lp}^{(1)}(t;\epsilon) &= -\frac{\epsilon}{(4+\epsilon^2)}e^{-\epsilon t}(2\bar{c}_0+\epsilon\bar{b}_0)(\epsilon t) \\ \text{and} \\ c_{lp}^{(1)}(t;\epsilon) &= \frac{\epsilon}{(4+\epsilon^2)}e^{-\epsilon t}(2\bar{b}_0-\epsilon\bar{c}_0)(\epsilon t)\end{aligned}\right\}\quad(175)$$

Introducing Eqs. (175) into (171) yields the first-order perturbations

$$\left.\begin{aligned}&b^{(1)}(t;\epsilon) = \\ &\quad -\frac{\epsilon}{(4+\epsilon^2)}e^{-\epsilon t}[2\bar{c}_0+\epsilon\bar{b}_0)(\epsilon t+\cos 2t) \\ &\quad - (2\bar{b}_0-\epsilon\bar{c}_0)\sin 2t] \\ &\text{and} \\ &c^{(1)}(t;\epsilon) = \\ &\quad -\frac{\epsilon}{(4+\epsilon^2)}e^{-\epsilon t}[-(2\bar{b}_0-\epsilon\bar{c}_0)(\epsilon t-\cos 2t) \\ &\quad + (2\bar{c}_0+\epsilon\bar{b}_0)\sin 2t]\end{aligned}\right\}\quad(176)$$

Similarly, the remaining perturbations

$$b^{(2)}(t;\epsilon), b^{(3)}(t;\epsilon), \cdots, c^{(2)}(t;\epsilon), c^{(3)}(t;\epsilon), \cdots$$

in Eqs. (137) may be obtained.

At this point, the initial values of the reference parameters, $\bar{b}_0$ and $\bar{c}_0$, can be determined to $O(\epsilon)$. From Eqs. (155),

$$\bar{b}_0 = b_0 - (b^{(1)})_0 + O(\epsilon^2)$$

and

$$\bar{c}_0 = c_0 - (c^{(1)})_0 + O(\epsilon^2)$$

which become, using Eqs. (176) evaluated at $t = 0$,

$$\left.\begin{aligned} \bar{b}_0 &= b_0 + \frac{\epsilon}{(4+\epsilon^2)}(2\bar{c}_0 + \epsilon\bar{b}_0) + O(\epsilon^2) \\ &\text{and} \\ \bar{c}_0 &= c_0 + \frac{\epsilon}{(4+\epsilon^2)}(2\bar{b}_0 - \epsilon\bar{c}_0) + O(\epsilon^2) \end{aligned}\right\} \qquad (177)$$

Introducing the initial conditions, $b_0 = 1$ and $c_0 = 0$ from Eqs. (126), into Eqs. (177) yields two simultaneous equations in the two unknowns $\bar{b}_0$ and $\bar{c}_0$ whose solutions are

$$\left.\begin{aligned} \bar{b}_0 &= \left(1 + \frac{1}{2}\epsilon^2\right) + O(\epsilon^2) \\ &\text{and} \\ \bar{c}_0 &= \frac{1}{2}\epsilon + O(\epsilon^2) \end{aligned}\right\} \qquad (178)$$

Thus, introducing Eqs. (178) into Eqs. (166) and (176) and the results into Eqs. (137), one obtains

$$\left.\begin{aligned} b(t;\epsilon) &= e^{-\epsilon t}\left\{1 + \frac{1}{2}\epsilon\left[\epsilon(1-\epsilon t) - \epsilon\cos 2t + \sin 2t\right]\right\} \\ &\quad + O(\epsilon^2) \\ &\text{and} \\ c(t;\epsilon) &= e^{-\epsilon t}\left\{\frac{1}{2}\epsilon\left[(1+\epsilon t) - \cos 2t - \epsilon\sin 2t\right]\right\} \\ &\quad + O(\epsilon^2) \end{aligned}\right\} \qquad (179)$$

Note that at $t = 0$, Eqs. (179) become

$$b(0;\epsilon) = b_0 = 1 + O(\epsilon^2) \qquad \text{and} \qquad c(0;\epsilon) = c_0 = 0 + O(\epsilon^2)$$

in agreement with Eqs. (126) to the number of terms considered. Introducing Eqs. (179) into Eq. (123) yields

$$\begin{aligned} x(t;\epsilon) = e^{-\epsilon t}\Bigg[&\left(1 + \frac{1}{2}\epsilon^2 - \frac{1}{2}\epsilon^3 t\right)\cos t + \frac{1}{2}\epsilon(1+\epsilon t)\sin t - \frac{1}{2}\epsilon^2(\cos 2t\cos t + \sin 2t\sin t) \\ &- \frac{1}{2}\epsilon(\cos 2t\sin t - \sin 2t\cos t)\Bigg] + O(\epsilon^2) \end{aligned}$$

or, using trigonometric formulas,

$$x(t;\epsilon) = e^{-\epsilon t}\left[\left(1 - \frac{1}{2}\epsilon^3 t\right)\cos t + \epsilon\left(1 + \frac{1}{2}\epsilon t\right)\sin t\right] + O(\epsilon^2) \qquad (180)$$

Thus, the approximation obtained in Eq. (180) using the variation of parameters with Taylor's series expansions is only initially valid (because of the Poisson terms), just as the approximation obtained in the previous section using the variation of parameters with successive approximations is initially valid. However, the initially valid approximation in Eq. (180) has a larger range of validity than that of the previous section.

Consider the expansion of the exact solution in Eq. (52) using Taylor's series (about $t = 0$) for $\epsilon << 1$ and $0 \leqq t < t_2$ as follows:

$$x(t;\epsilon) = e^{-\epsilon t}\left\{\cos\left[1 - \frac{1}{2}\epsilon^2 + O(\epsilon^4)\right]t + \epsilon\left[1 + \frac{1}{2}\epsilon^2 + O(\epsilon^4)\right]\sin\left[1 - \frac{1}{2}\epsilon^2 + O(\epsilon^4)\right]t\right\}$$

$$= e^{-\epsilon t}\left(\cos t \cos\left[\frac{1}{2}\epsilon^2 + O(\epsilon^4)\right]t + \sin t \sin\left[\frac{1}{2}\epsilon^2 + O(\epsilon^4)\right]t\right.$$

$$\left. + \epsilon\left[1 + \frac{1}{2}\epsilon^2 + O(\epsilon^4)\right]\left\{\sin t \cos\left[\frac{1}{2}\epsilon^2 + O(\epsilon^4)\right]t - \cos t \sin\left[\frac{1}{2}\epsilon^2 + O(\epsilon^4)\right]t\right\}\right)$$

or

$$x(t;\epsilon) = e^{-\epsilon t}\left[\left(1 - \frac{1}{2}\epsilon^3 t\right)\cos t + \epsilon\left(1 + \frac{1}{2}\epsilon t + \frac{1}{2}\epsilon^2\right)\sin t\right] + O(\epsilon^4) \qquad (181)$$

A comparison of the approximation in Eq. (180) with the expansion of the exact solution in Eq. (181) shows that the approximation in Eq. (180) exhibits (1) the proper bounded motion since

$$\lim_{t\to\infty}\left(\frac{t\cos t}{e^{\epsilon t}}\right) = 0$$

and

$$\lim_{t\to\infty}\left(\frac{t\sin t}{e^{\epsilon t}}\right) = 0$$

(2) the proper exponential decay, and (3) the effects of the frequency shift. Furthermore, although Eq. (180) has been developed to first order explicitly, $b(t;\epsilon) = \bar{b}(t;\epsilon) + b^{(1)}(t;\epsilon) + O(\epsilon^2)$ and $b \to c$, the approximation is accurate to nearly third order implicitly (the $1/2\,\epsilon^3 \sin t$ term is missing). Thus, Eq. (180) becomes

$$x(t;\epsilon) = e^{-\epsilon t}\left[\left(1 - \frac{1}{2}\epsilon^3 t\right)\cos t + \epsilon\left(1 + \frac{1}{2}\epsilon t\right)\sin t\right] + O(\epsilon^3) \qquad (182)$$

Finally, the reference motion

$$\bar{x}(t;\epsilon) = \bar{b}(t;\epsilon)\cos t + \bar{c}(t;\epsilon)\sin t$$

or

$$\bar{x}(t;\epsilon) = e^{-\epsilon t}\left[\bar{b}_0\cos t + \bar{c}_0\sin t\right] \qquad (183)$$

obtained from Eqs. (123), (137), and (166) is much closer to the actual motion than the unperturbed motion, $\cos t$.

Although the process for the analytical integration of the perturbative variations in this section is superior to that of the previous section, the resulting approximation is still only initially valid. Thus, the perturbation theory described in this section is unacceptable. However, this perturbation theory can be modified so that a uniformly valid approximation could be obtained. The modification consists of introducing a distorted time scale,

$$\bar{t} = t\left[1 + \epsilon\tau_1 + \epsilon^2\tau_2 + O(\epsilon^3)\right],$$

into the process of integration to account for the frequency shift as is done in the Lindstedt–Poincaré theory for periodic solutions (Ref. 1056, Vol. II). The resulting perturbation theory would then be acceptable.

G. Solution Using Variation of Parameters With Two-Variable Asymptotic Expansions

Rather than modify the perturbation theory of the previous section into an acceptable perturbation theory by introducing a distorted time scale into the process of integration, it is desirable to proceed one step further and use two distorted time scales; that is, develop a process for the analytical integration of the perturbative variations using two-variable asymptotic expansions.

1. Selection of parameters and independent variable. Suppose the functions $b(t;\epsilon)$ and $c(t;\epsilon)$ and the variable t from Section IV-E-1 are chosen as the osculating parameters and independent variable, respectively. Then, the differential equations of motion and the corresponding initial conditions are, from Eqs. (126), (127), (128), and (130),

$$\left.\begin{aligned} \frac{db}{dt} &= -\epsilon\,[b(t;\epsilon) - b(t;\epsilon)\cos 2t - c(t;\epsilon)\sin 2t] \\ \frac{dc}{dt} &= -\epsilon\,[c(t;\epsilon) + c(t;\epsilon)\cos 2t - b(t;\epsilon)\sin 2t] \\ b(0;\epsilon) &= 1 \\ \text{and}\qquad c(0;\epsilon) &= 0 \end{aligned}\right\} \tag{184}$$

The standard procedure in solving this system of equations using two-variable asymptotic expansions is to assume

$$\bar{t} = t, \qquad \tilde{t} = \epsilon t \tag{185}$$

$$\left.\begin{aligned} b(t;\epsilon) &= B(\bar{t},\tilde{t};\epsilon) \\ &= b^{(0)}(\bar{t},\tilde{t}) + \epsilon b^{(1)}(\bar{t},\tilde{t}) + \epsilon^2 b^{(2)}(\bar{t},\tilde{t}) + O(\epsilon^3) \\ \text{and} \\ c(t;\epsilon) &= C(\bar{t},\tilde{t};\epsilon) \\ &= c^{(0)}(\bar{t},\tilde{t}) + \epsilon c^{(1)}(\bar{t},\tilde{t}) + \epsilon^2 c^{(2)}(\bar{t},\tilde{t}) + O(\epsilon^3) \end{aligned}\right\} \tag{186}$$

and then follow a procedure similar to that of Section IV-D-3. Note that the natural independent variable t is chosen as the fast time variable $\bar{t}$ since it appears explicitly in the differential equations of motion in Eqs. (184). Unfortunately, this procedure does not yield a uniformly valid approximation to $x(t;\epsilon)$. Furthermore, all the standard artifices to obtain uniformly valid approximations fail, such as letting

$$\bar{t} = t \qquad \text{and} \qquad \tilde{t} = \epsilon t\,[1 + \epsilon\tau_1 + O(\epsilon^2)]$$

or introducing additional terms to the expansions (186) of the form

$$\frac{B^{(0)}(\tilde{t})}{\epsilon} \text{ and } \frac{C^{(0)}(\tilde{t})}{\epsilon}$$

The difficulty here is easily identified, since a uniformly valid approximation to $x(t;\epsilon)$ has been obtained in Section IV-D, using

$$\bar{t} = t\,[1 + \epsilon^2\tau_2 + O(\epsilon^3)] \qquad \text{and} \qquad \tilde{t} = \epsilon t \tag{187}$$

The difficulty is that the proper form for $\bar{t}$ given in Eqs. (187) cannot be used with Eqs. (184) because of the explicit appearance of the time t in the trigonometric functions of Eqs. (184). As a result, the differential equations of motion in (184) must be modified so that the proper form for $\bar{t}$ can be used. Now, these equations were obtained in Section IV-E by permitting the parameters b and c to vary with the time t in such a manner that Eqs. (123) and (124) would satisfy the differential equation of perturbed motion, Eq. (116), for all the values of the time in the range $0 \leqq t < \infty$. The proper modification to these equations results if, in addition to permitting the parameters b and c to vary with the time t, the frequency of oscillation is permitted to vary from the unperturbed case; that is, the argument t of the trigonometric functions in Eqs. (123) and (124) is replaced by $\bar{t}$. In this manner, Eqs. (123) and (124) become

$$x(t;\epsilon) = \tilde{b}(t;\epsilon)\cos\bar{t} + \tilde{c}(t;\epsilon)\sin\bar{t} \tag{188}$$

and

$$\frac{dx}{dt}(t;\epsilon) = -\tilde{b}(t;\epsilon)\sin\bar{t} + \tilde{c}(t;\epsilon)\cos\bar{t} \tag{189}$$

where

$$\bar{t} = t\,[1 + \epsilon^2\tau_2 + O(\epsilon^3)] \tag{190}$$

It should be noted that if $\bar{t} \neq t$, then from Eqs. (123) and (188),

$$\begin{aligned} x(t;\epsilon) &= b(t;\epsilon)\cos t + c(t;\epsilon)\sin t \\ &= \tilde{b}(t;\epsilon)\cos\bar{t} + \tilde{c}(t;\epsilon)\sin\bar{t} \end{aligned}$$

or, using $t = (\bar{t} - t) + t$ and trigonometric formulas,

$$\left.\begin{aligned} b(t;\epsilon) &= \tilde{b}(t;\epsilon)\cos(\bar{t} - t) + \tilde{c}(t;\epsilon)\sin(\bar{t} - t) \\ \text{and} \\ c(t;\epsilon) &= -\tilde{b}(t;\epsilon)\sin(\bar{t} - t) + \tilde{c}(t;\epsilon)\cos(\bar{t} - t) \end{aligned}\right\} \tag{191}$$

Thus, the instantaneous parameters $\tilde{b}(t;\epsilon)$ and $\tilde{c}(t;\epsilon)$ are not the same as the instantaneous parameters of Sections IV-E and IV-F.

The functions $\tilde{b}(t;\epsilon)$ and $\tilde{c}(t;\epsilon)$ and the variable t are chosen as the instantaneous parameters and independent variable, respectively.

2. ***Equations of motion.*** Equations (188) and (189) are linear in $\tilde{b}(t;\epsilon)$ and $\tilde{c}(t;\epsilon)$ and can be solved for these parameters because the determinant of their coefficients is not identically zero. Hence,

$$\left.\begin{aligned} \tilde{b}(t;\epsilon) &= x(t;\epsilon)\cos\bar{t} - \frac{dx}{dt}(t;\epsilon)\sin\bar{t} \\ &\text{and} \\ \tilde{c}(t;\epsilon) &= x(t;\epsilon)\sin\bar{t} + \frac{dx}{dt}(t;\epsilon)\cos\bar{t} \end{aligned}\right\} \tag{192}$$

Furthermore, from Eqs. (47) and from Eqs. (190) and (192) evaluated at $t = 0$,

$$\left.\begin{aligned} \tilde{b}(0;\epsilon) &= x(0;\epsilon) = 1 \\ &\text{and} \\ \tilde{c}(0;\epsilon) &= \frac{dx}{dt}(0;\epsilon) = 0 \end{aligned}\right\} \tag{193}$$

Differentiating Eqs. (192) with respect to the time t, one obtains

$$\frac{d\tilde{b}}{dt} = -\left(x\sin\bar{t} + \frac{dx}{dt}\cos\bar{t}\right)\frac{d\bar{t}}{dt} + \frac{dx}{dt}\cos\bar{t} - \frac{d^2x}{dt^2}\sin\bar{t}$$

and

$$\frac{d\tilde{c}}{dt} = \left(x\cos\bar{t} - \frac{dx}{dt}\sin\bar{t}\right)\frac{d\bar{t}}{dt} + \frac{dx}{dt}\sin\bar{t} + \frac{d^2x}{dt^2}\cos\bar{t}$$

or, using Eq. (46),

$$\left.\begin{aligned} \frac{d\tilde{b}}{dt} &= -\left(x\sin\bar{t} + \frac{dx}{dt}\cos\bar{t}\right)\frac{d\bar{t}}{dt} \\ &\quad + \left(x\sin\bar{t} + \frac{dx}{dt}\cos\bar{t}\right) + 2\epsilon\frac{dx}{dt}\sin\bar{t} \\ &\text{and} \\ \frac{d\tilde{c}}{dt} &= \left(x\cos\bar{t} - \frac{dx}{dt}\sin\bar{t}\right)\frac{d\bar{t}}{dt} \\ &\quad - \left(x\cos\bar{t} - \frac{dx}{dt}\sin\bar{t}\right) - 2\epsilon\frac{dx}{dt}\cos\bar{t} \end{aligned}\right\} \tag{194}$$

Introducing Eqs. (189) and (192) into Eqs. (194) and using trigonometric formulas yields the differential equations of motion in terms of the instantaneous parameters $\tilde{b}(t;\epsilon)$ and $\tilde{c}(t;\epsilon)$; that is,

$$\left.\begin{aligned} \frac{d\tilde{b}}{dt} &= \tilde{c}(t;\epsilon)\left(1 - \frac{d\bar{t}}{dt}\right) \\ &\quad - \epsilon\,[\tilde{b}(t;\epsilon) - \tilde{b}(t;\epsilon)\cos 2\bar{t} - \tilde{c}(t;\epsilon)\sin 2\bar{t}] \\ &\text{and} \\ \frac{d\tilde{c}}{dt} &= -\tilde{b}(t;\epsilon)\left(1 - \frac{d\bar{t}}{dt}\right) \\ &\quad - \epsilon\,[\tilde{c}(t;\epsilon) + \tilde{c}(t;\epsilon)\cos 2\bar{t} - \tilde{b}(t;\epsilon)\sin 2\bar{t}] \end{aligned}\right\} \tag{195}$$

Note that if $\bar{t} = t$ as in Sections IV-E and IV-F, then Eqs. (193) and (195) become the same as Eqs. (184), as they should, since

$$\frac{d\bar{t}}{dt} = 1$$

and, from Eqs. (191),

$$\tilde{b}(t;\epsilon) = b(t;\epsilon) \qquad \text{and} \qquad \tilde{c}(t;\epsilon) = c(t;\epsilon)$$

An alternate development of Eqs. (195) consists of using concepts analogous to those of perturbative differentiation (Section III-A-2); that is,

$$\frac{d\tilde{b}}{dt} = \dot{\tilde{b}} + \grave{\tilde{b}} \qquad \text{and} \qquad \frac{d\tilde{c}}{dt} = \dot{\tilde{c}} + \grave{\tilde{c}} \tag{196}$$

where the dot and grave derivatives are formally analogous to the unperturbed variation and perturbative variation of the parameter, respectively.

Taking the dot derivatives of Eqs. (192) (remembering that $dx/dt = \dot{x}$ since $\grave{x} = 0$) and using Eq. (119) yields

$$\dot{\tilde{b}} = -x(\sin\bar{t})\frac{d\bar{t}}{dt} - \dot{x}(\cos\bar{t})\frac{d\bar{t}}{dt} + \dot{x}\cos\bar{t} - \ddot{x}\sin\bar{t}$$

and

$$\dot{\tilde{c}} = x(\cos\bar{t})\frac{d\bar{t}}{dt} - \dot{x}(\sin\bar{t})\frac{d\bar{t}}{dt} + \dot{x}\sin\bar{t} + \ddot{x}\cos\bar{t}$$

or

$$\left.\begin{aligned}\dot{\tilde{b}} &= (x\sin\bar{t} + \dot{x}\cos\bar{t})\left(1 - \frac{d\bar{t}}{dt}\right)\\ \text{and}\\ \dot{\tilde{c}} &= -(x\cos\bar{t} - \dot{x}\sin\bar{t})\left(1 - \frac{d\bar{t}}{dt}\right)\end{aligned}\right\} \quad (197)$$

Introducing Eqs. (192) into Eqs. (197), one obtains

$$\dot{\tilde{b}} = \tilde{c}(t;\epsilon)\left(1 - \frac{d\bar{t}}{dt}\right) \quad \text{and} \quad \dot{\tilde{c}} = -\tilde{b}(t;\epsilon)\left(1 - \frac{d\bar{t}}{dt}\right) \quad (198)$$

Note that if $\bar{t} = t$, then

$$\dot{\tilde{b}} = \dot{b} = 0 \quad \text{and} \quad \dot{\tilde{c}} = \dot{c} = 0$$

as they should.

Taking the perturbative derivatives of Eqs. (192) and using Eq. (120) one obtains

$$\left.\begin{aligned}\tilde{b}' &= -\dot{x}'\sin\bar{t} = 2\epsilon\dot{x}\sin\bar{t}\\ \text{and}\\ \tilde{c}' &= \dot{x}'\cos\bar{t} = -2\epsilon\dot{x}\cos\bar{t}\end{aligned}\right\} \quad (199)$$

where, from Eq. (190),

$$\bar{t}' = t'\,[1 + \epsilon^2\tau_2 + O(\epsilon^3)] = 0$$

since $t' = 0$. Introducing Eq. (189) into Eqs. (199) $(dx/dt = \dot{x})$ and using trigonometric formulas yields

$$\left.\begin{aligned}\tilde{b}' &= -\epsilon\,[\tilde{b}(t;\epsilon) - \tilde{b}(t;\epsilon)\cos 2\bar{t} - \tilde{c}(t;\epsilon)\sin 2\bar{t}]\\ \text{and}\\ \tilde{c}' &= -\epsilon\,[\tilde{c}(t;\epsilon) + \tilde{c}(t;\epsilon)\cos 2\bar{t} - \tilde{b}(t;\epsilon)\sin 2\bar{t}]\end{aligned}\right\} \quad (200)$$

Finally, Eqs. (195) follow from Eqs. (196), (198), and (200).

3. ***Development of solution.*** The process of two-variable asymptotic expansions associated with Eqs. (195) consists of assuming

$$\bar{t} = t\,[1 + \epsilon^2\tau_2 + O(\epsilon^3)]\,, \qquad \tilde{t} = \epsilon t \quad (201)$$

$$\left.\begin{aligned}\tilde{b}(t;\epsilon) &= \tilde{B}(\bar{t},\tilde{t};\epsilon)\\ &= \tilde{b}^{(0)}(\bar{t},\tilde{t}) + \epsilon\tilde{b}^{(1)}(\bar{t},\tilde{t}) + \epsilon^2\tilde{b}^{(2)}(\bar{t},\tilde{t})\\ &\quad + O(\epsilon^3)\\ \text{and}\\ \tilde{c}(t;\epsilon) &= \tilde{C}(\bar{t},\tilde{t};\epsilon)\\ &= \tilde{c}^{(0)}(\bar{t},\tilde{t}) + \epsilon\tilde{c}^{(1)}(\bar{t},\tilde{t}) + \epsilon^2\tilde{c}^{(2)}(\bar{t},\tilde{t})\\ &\quad + O(\epsilon^3)\end{aligned}\right\} \quad (202)$$

and then following a procedure similar to that of Section IV-D-3. The actual motion is then obtained from Eq. (188); that is,

$$x(t;\epsilon) = \tilde{b}(t;\epsilon)\cos\bar{t} + \tilde{c}(t;\epsilon)\sin\bar{t}$$

Note that $x^{(0)}(\bar{t},\tilde{t})$ can be thought of as the reference motion where

$$x^{(0)}(\bar{t},\tilde{t}) = \tilde{b}^{(0)}(\bar{t},\tilde{t})\cos\bar{t} + \tilde{c}^{(0)}(\bar{t},\tilde{t})\sin\bar{t} \quad (203)$$

and $\epsilon x^{(1)}(\bar{t},\tilde{t})$, $\epsilon^2 x^{(2)}(\bar{t},\tilde{t})$, etc. can be thought of as the perturbations to the reference motion where

$$x^{(1)}(\bar{t},\tilde{t}) = \tilde{b}^{(1)}(\bar{t},\tilde{t})\cos\bar{t} + \tilde{c}^{(1)}(\bar{t},\tilde{t})\sin\bar{t}$$

$$x^{(2)}(\bar{t},\tilde{t}) = \tilde{b}^{(2)}(\bar{t},\tilde{t})\cos\bar{t} + \tilde{c}^{(2)}(\bar{t},\tilde{t})\sin\bar{t}$$

and so forth.

For conciseness, the following notation is adopted:

$$\left.\begin{aligned}&\tilde{b}^{(j)}_1 = \frac{\partial\tilde{b}^{(j)}}{\partial\bar{t}}, \quad \tilde{b}^{(j)}_2 = \frac{\partial\tilde{b}^{(j)}}{\partial\tilde{t}}\\ &\text{and}\\ &\tilde{c}^{(j)}_1 = \frac{\partial\tilde{c}^{(j)}}{\partial\bar{t}}, \quad \tilde{c}^{(j)}_2 = \frac{\partial\tilde{c}^{j}}{\partial\tilde{t}}\end{aligned}\right\} \quad (204)$$

where $j = 0, 1, 2$, etc. Differentiating Eqs. (201) with respect to the time t yields

$$\frac{d\bar{t}}{dt} = 1 + \epsilon^2\tau_2 + O(\epsilon^3) \quad \text{and} \quad \frac{d\tilde{t}}{dt} = \epsilon \quad (205)$$

and differentiating Eqs. (202) with respect to the time t yields, using Eqs. (205),

$$
\left.
\begin{aligned}
\frac{d\tilde{b}}{dt} &= \tilde{b}_1^{(0)} + \epsilon\,(\tilde{b}_1^{(1)} + \tilde{b}_2^{(0)}) \\
&\quad + \epsilon^2\,(\tilde{b}_1^{(2)} + \tilde{b}_2^{(1)} + \tau_2 \tilde{b}_1^{(0)}) + O\,(\epsilon^3) \\
&\text{and} \\
\frac{d\tilde{c}}{dt} &= \tilde{c}_1^{(0)} + \epsilon\,(\tilde{c}_1^{(1)} + \tilde{c}_2^{(0)}) \\
&\quad + \epsilon^2\,(\tilde{c}_1^{(2)} + \tilde{c}_2^{(1)} + \tau_2 \tilde{c}_1^{(0)}) + O\,(\epsilon^3)
\end{aligned}
\right\} \qquad (206)
$$

Introducing Eqs. (202), the first of Eqs. (205), and Eqs. (206) into Eqs. (195) and combining like powers of ϵ yields

$$
\begin{aligned}
&\tilde{b}_1^{(0)} + \epsilon\,[\tilde{b}_1^{(1)} + \tilde{b}_2^{(0)} + \tilde{b}^{(0)}\,(1 - \cos 2\bar{t}) - \tilde{c}^{(0)} \sin 2\bar{t}] \\
&\quad + \epsilon^2\,[\tilde{b}_1^{(2)} + \tilde{b}_2^{(1)} + \tau_2 \tilde{b}_1^{(0)} + \tau_2 \tilde{c}^{(0)} \\
&\quad + \tilde{b}^{(1)}\,(1 - \cos 2\bar{t}) - \tilde{c}^{(1)} \sin 2\bar{t}] \\
&\quad + O\,(\epsilon^3) = 0
\end{aligned} \qquad (207a)
$$

and

$$
\begin{aligned}
&\tilde{c}_1^{(0)} + \epsilon\,[\tilde{c}_1^{(1)} + \tilde{c}_2^{(0)} + \tilde{c}^{(0)}\,(1 + \cos 2\bar{t}) - \tilde{b}^{(0)} \sin 2\bar{t}] \\
&\quad + \epsilon^2\,[\tilde{c}_1^{(2)} + \tilde{c}_2^{(1)} + \tau_2 \tilde{c}_1^{(0)} - \tau_2 \tilde{b}^{(0)} \\
&\quad + \tilde{c}^{(1)}\,(1 + \cos 2\bar{t}) - \tilde{b}^{(1)} \sin 2\bar{t}] \\
&\quad + O\,(\epsilon^3) = 0
\end{aligned} \qquad (207b)
$$

Since the expansions in Eqs. (207) must hold (at least in an asymptotic sense) for arbitrary values of the perturbative parameter ϵ, the coefficients of the powers of ϵ in Eqs. (207) must separately equal zero. Thus,

$$\tilde{b}_1^{(0)} = 0 \qquad (208)$$

$$\tilde{b}_1^{(1)} = -\tilde{b}^{(0)}\,(1 - \cos 2\bar{t}) + \tilde{c}^{(0)} \sin 2\bar{t} - \tilde{b}_2^{(0)} \qquad (209)$$

$$
\begin{aligned}
\tilde{b}_1^{(2)} &= -\tau_2 \tilde{c}^{(0)} - \tau_2 \tilde{b}_1^{(0)} - \tilde{b}^{(1)}\,(1 - \cos 2\bar{t}) \\
&\quad + \tilde{c}^{(1)} \sin 2\bar{t} - \tilde{b}_2^{(1)}
\end{aligned} \qquad (210)
$$

$$\tilde{c}_1^{(0)} = 0 \qquad (211)$$

$$\tilde{c}_1^{(1)} = -\tilde{c}^{(0)}\,(1 + \cos 2\bar{t}) + \tilde{b}^{(0)} \sin 2\bar{t} - \tilde{c}_2^{(0)} \qquad (212)$$

$$
\begin{aligned}
\tilde{c}_1^{(2)} &= \tau_2 \tilde{b}^{(0)} - \tau_2 \tilde{c}_1^{(0)} - \tilde{c}^{(1)}\,(1 + \cos 2\bar{t}) \\
&\quad + \tilde{b}^{(1)} \sin 2\bar{t} - \tilde{c}_2^{(1)}
\end{aligned} \qquad (213)
$$

and so forth. The initial conditions necessary for the complete solution of these partial differential equations are obtained by evaluating Eqs. (202) at $t = 0$, by using the initial conditions (193), by combining terms of like powers of ϵ, and by setting the coefficients of the powers of ϵ separately equal to zero. Thus, since $\bar{t} = \tilde{t} = 0$ when $t = 0$ from Eqs. (201),

$$\tilde{b}^{(0)}\,(0,0) = 1, \qquad \tilde{c}^{(0)}\,(0,0) = 0 \qquad (214)$$

$$\tilde{b}^{(1)}\,(0,0) = \tilde{c}^{(1)}\,(0,0) = 0 \qquad (215)$$

$$\tilde{b}^{(2)}\,(0,0) = \tilde{c}^{(2)}\,(0,0) = 0 \qquad (216)$$

and so forth.

The general solutions to Eqs. (208) and (211), $\partial\tilde{b}^{(0)}/\partial\bar{t} = 0$ and $\partial\tilde{c}^{(0)}/\partial\bar{t} = 0$, are

$$\tilde{b}^{(0)}\,(\bar{t},\tilde{t}) = B^{(0)}\,(\tilde{t}) \qquad \text{and} \qquad \tilde{c}^{(0)}\,(\bar{t},\tilde{t}) = C^{(0)}\,(\tilde{t}) \qquad (217)$$

respectively (the theory of ordinary differential equations may be used here, remembering that any constants that would ordinarily appear must be functions of the parameter $\tilde{t}$ as discussed in Section IV-D-3). Evaluating Eqs. (217) at $t = 0$ ($\bar{t} = 0$ and $\tilde{t} = 0$) and using Eqs. (214), one obtains

$$B^{(0)}\,(0) = 1 \qquad \text{and} \qquad C^{(0)}\,(0) = 0 \qquad (218)$$

Note that all of the known conditions on $\tilde{b}^{(0)}\,(\bar{t},\tilde{t})$ and $\tilde{c}^{(0)}\,(\bar{t},\tilde{t})$ have been applied without a unique determination of the functions $B^{(0)}\,(\tilde{t})$ and $C^{(0)}\,(\tilde{t})$. This is exactly what is required in order to prevent the occurrence of secular or Poisson perturbations in the approximation to the solution as will be seen during the solution of Eqs. (209) and (212). Introducing Eqs. (217) and the partial derivatives of Eqs. (217) with respect to $\tilde{t}$ into Eqs. (209) and (212) yields

$$
\left.
\begin{aligned}
\tilde{b}_1^{(1)} &= -\left(\frac{dB^{(0)}}{d\tilde{t}} + B^{(0)}\right) + B^{(0)} \cos 2\bar{t} \\
&\quad + C^{(0)} \sin 2\bar{t} \\
&\text{and} \\
\tilde{c}_1^{(1)} &= -\left(\frac{dC^{(0)}}{d\tilde{t}} + C^{(0)}\right) - C^{(0)} \cos 2\bar{t} \\
&\quad + B^{(0)} \sin 2\bar{t}
\end{aligned}
\right\} \qquad (219)
$$

Treating $\tilde{t}$ as a parameter in the same manner as before, one obtains the general solutions to Eqs. (219):

$$\left.\begin{aligned}\tilde{b}^{(1)}(\bar{t},\tilde{t}) &= B^{(1)}(\tilde{t}) - \left(\frac{dB^{(0)}}{d\tilde{t}} + B^{(0)}\right)\bar{t} \\ &\quad + \frac{1}{2}B^{(0)}\sin 2\bar{t} - \frac{1}{2}C^{(0)}\cos 2\bar{t} \\ \text{and} \\ \tilde{c}^{(1)}(\bar{t},\tilde{t}) &= C^{(1)}(\tilde{t}) - \left(\frac{dC^{(0)}}{d\tilde{t}} + C^{(0)}\right)\bar{t} \\ &\quad - \frac{1}{2}C^{(0)}\sin 2\bar{t} - \frac{1}{2}B^{(0)}\cos 2\bar{t}\end{aligned}\right\} \tag{220}$$

Now, the unperturbed motion is bounded (as shown in Section IV-C) so that from the first uniformity condition as discussed in Section II-A-3, no secular or Poisson perturbations in the fast time variable $\bar{t}$ are allowed. Thus, the first uniformity condition requires that

$$\frac{dB^{(0)}}{d\tilde{t}} + B^{(0)} = 0 \qquad \text{and} \qquad \frac{dC^{(0)}}{d\tilde{t}} + C^{(0)} = 0 \tag{221}$$

whose solutions are

$$B^{(0)}(\tilde{t}) = B_0 e^{-\tilde{t}} \qquad \text{and} \qquad C^{(0)}(\tilde{t}) = C_0 e^{-\tilde{t}} \tag{222}$$

respectively. Evaluating Eqs. (222) at $\tilde{t} = 0$ and using Eqs. (218), one obtains

$$B_0 = 1 \qquad \text{and} \qquad C_0 = 0$$

so that Eqs. (222) become

$$B^{(0)}(\tilde{t}) = e^{-\tilde{t}} \qquad \text{and} \qquad C^{(0)}(\tilde{t}) = 0 \tag{223}$$

Introducing Eqs. (223) into Eqs. (217) yields the uniformly valid first approximations

$$\tilde{b}^{(0)}(\bar{t},\tilde{t}) = e^{-\tilde{t}} \qquad \text{and} \qquad \tilde{c}^{(0)}(\bar{t},\tilde{t}) = 0 \tag{224}$$

Note that the resulting reference motion, from Eqs. (203) and (224),

$$x^{(0)}(\bar{t},\tilde{t}) = e^{-\tilde{t}}\cos\bar{t} \tag{225}$$

is not the unperturbed motion, $\cos t$.

From Eqs. (220), (221), and (223),

$$\left.\begin{aligned}\tilde{b}^{(1)}(\bar{t},\tilde{t}) &= B^{(1)}(\tilde{t}) + \frac{1}{2}e^{-\tilde{t}}\sin 2\bar{t} \\ \text{and} \\ \tilde{c}^{(1)}(\bar{t},\tilde{t}) &= C^{(1)}(\tilde{t}) - \frac{1}{2}e^{-\tilde{t}}\cos 2\bar{t}\end{aligned}\right\} \tag{226}$$

and, by partial differentiation with respect to $\tilde{t}$,

$$\left.\begin{aligned}\tilde{b}^{(1)}_2 &= \frac{dB^{(1)}}{d\tilde{t}} - \frac{1}{2}e^{-\tilde{t}}\sin 2\bar{t} \\ \text{and} \\ \tilde{c}^{(1)}_2 &= \frac{dC^{(1)}}{d\tilde{t}} + \frac{1}{2}e^{-\tilde{t}}\cos 2\bar{t}\end{aligned}\right\} \tag{227}$$

Consider the partial differential equations (210) and (213) from which the functions $B^{(1)}(\tilde{t})$ and $C^{(1)}(\tilde{t})$, as well as the undetermined constant τ_2, are determined. Taking the partial derivatives of Eqs. (224) with respect to $\bar{t}$, one obtains

$$\tilde{b}^{(0)}_1 = 0 \qquad \text{and} \qquad \tilde{c}^{(0)}_1 = 0 \tag{228}$$

Introducing Eqs. (224), (226), (227), and (228) into (210) and (213) yields

$$\left.\begin{aligned}\tilde{b}^{(2)}_1 &= -\left(\frac{dB^{(1)}}{d\tilde{t}} + B^{(1)}\right) + B^{(1)}\cos 2\bar{t} \\ &\quad + C^{(1)}\sin 2\bar{t} \\ \text{and} \\ \tilde{c}^{(2)}_1 &= -\left[\frac{dC^{(1)}}{d\tilde{t}} + C^{(1)} - \left(\tau_2 + \frac{1}{2}\right)e^{-\tilde{t}}\right] \\ &\quad - C^{(1)}\cos 2\bar{t} + B^{(1)}\sin 2\bar{t}\end{aligned}\right\} \tag{229}$$

Solving Eqs. (229) for $\tilde{b}^{(2)}(\bar{t},\tilde{t})$ and $\tilde{c}^{(2)}(\bar{t},\tilde{t})$ and applying the first uniformity condition results in the two equations

$$\frac{dB^{(1)}}{d\tilde{t}} + B^{(1)} = 0$$

and

$$\frac{dC^{(1)}}{d\tilde{t}} + C^{(1)} = \left(\tau_2 + \frac{1}{2}\right)e^{-\tilde{t}}$$

whose solutions are, using Solution (A-1) in Appendix A,

$$
\left.\begin{aligned}
&B^{(1)}(\tilde{t}) = B_1 e^{-\tilde{t}} \\
\text{and} \\
&C^{(1)}(\tilde{t}) = \left[C_1 + \left(\tau_2 + \frac{1}{2}\right)\tilde{t}\right] e^{-\tilde{t}}
\end{aligned}\right\} \tag{230}
$$

Evaluating Eqs. (226) at $t = 0$ ($\bar{t} = 0$ and $\tilde{t} = 0$) and using Eqs. (215) yields

$$
B^{(1)}(0) = 0 \qquad \text{and} \qquad C^{(1)}(0) = \frac{1}{2} \tag{231}
$$

Evaluating Eqs. (230) at $\tilde{t} = 0$ and using Eqs. (231), one obtains

$$
B_1 = 0 \qquad \text{and} \qquad C_1 = \frac{1}{2}
$$

so that Eqs. (230) become

$$
\left.\begin{aligned}
&B^{(1)}(\tilde{t}) = 0 \\
\text{and} \\
&C^{(1)}(\tilde{t}) = \left[\frac{1}{2} + \left(\tau_2 + \frac{1}{2}\right)\tilde{t}\right] e^{-\tilde{t}}
\end{aligned}\right\} \tag{232}
$$

Introducing Eqs. (232) into Eqs. (226) yields

$$
b^{(1)}(\bar{t}, \tilde{t}) = \frac{1}{2} e^{-\tilde{t}} \sin 2\bar{t} \tag{233}
$$

and

$$
c^{(1)}(\bar{t}, \tilde{t}) = \frac{1}{2} e^{-\tilde{t}}(1 - \cos 2\bar{t}) + \left(\tau_2 + \frac{1}{2}\right)\tilde{t} e^{-\tilde{t}} \tag{234}
$$

At this point, all of the known conditions on $b^{(1)}(\bar{t}, \tilde{t})$ and $c^{(1)}(\bar{t}, \tilde{t})$ have been applied (without the determination of the constant τ_2) except the second uniformity condition as discussed in Section II-A-3. The second uniformity condition requires that

$$
\lim_{\epsilon \to 0} \frac{\epsilon \tilde{c}^{(1)}(\bar{t}, \tilde{t})}{\tilde{c}^{(0)}(\bar{t}, \tilde{t})} = 0 \text{ for all } t \text{ in } 0 \leqq t < \infty \tag{235}
$$

Since Eq. (235) must be valid for all possible initial conditions, using the second of Eqs. (217), (222), (226), and (230),

$$
\lim_{\epsilon \to 0} \frac{\epsilon \left\{\left[C_1 + \left(\tau_2 + \frac{1}{2}\right)\tilde{t}\right] e^{-\tilde{t}} - \frac{1}{2} e^{-\tilde{t}} \cos 2\bar{t}\right\}}{C_0 e^{-\tilde{t}}} = 0 \tag{236}
$$

for all t in $0 \leqq t < \infty$. Equation (236) requires that

$$
\tau_2 = -\frac{1}{2} \tag{237}
$$

in order that the coefficient of the term $\tilde{t} e^{-\tilde{t}}$ is zero and a uniformly valid approximation is obtained (see Section II-A-3). From Eqs. (234) and (237),

$$
\tilde{c}^{(1)}(\bar{t}, \tilde{t}) = \frac{1}{2} e^{-\tilde{t}}(1 - \cos 2\bar{t}) \tag{238}
$$

where, from the first of Eqs. (201) and from Eq. (237),

$$
\bar{t} = t\left[1 - \frac{1}{2}\epsilon^2 + O(\epsilon^3)\right] \tag{239}
$$

Similarly, the remaining constants $\tau_3, \tau_4, \cdots$ in $\bar{t}$ and the perturbations in the asymptotic expansions in Eqs. (202) may be obtained. Note that the use of two distinct time variables, $\bar{t}$ and $\tilde{t}$, introduces a degree of flexibility into the theory of solution in such a manner that improper secular or Poisson perturbations can be eliminated just as in Section IV-D.

The uniformly valid asymptotic expansions to $O(\epsilon)$ are, from Eqs. (202), (224), (233), and (238),

$$
\tilde{B}(\bar{t}, \tilde{t}; \epsilon) = e^{-\tilde{t}} + \frac{1}{2}\epsilon e^{-\tilde{t}} \sin 2\bar{t} + O(\epsilon^2)
$$

and

$$
\tilde{C}(\bar{t}, \tilde{t}; \epsilon) = \frac{1}{2}\epsilon e^{-\tilde{t}}(1 - \cos 2\bar{t}) + O(\epsilon^2)
$$

or, using the second of Eqs. (201) and Eq. (239),

$$\left.\begin{aligned}\tilde{b}(t;\epsilon) &= e^{-\epsilon t}\left\{1+\frac{1}{2}\epsilon\sin 2\left[1-\frac{1}{2}\epsilon^2+O(\epsilon^3)\right]t\right\} \\ &\quad + O(\epsilon^2) \\ \text{and} \\ \tilde{c}(t;\epsilon) &= \frac{1}{2}\epsilon e^{-\epsilon t}\left\{1-\cos 2\left[1-\frac{1}{2}\epsilon^2+O(\epsilon^3)\right]t\right\} \\ &\quad + O(\epsilon^2)\end{aligned}\right\} \quad (240)$$

Finally, introducing Eqs. (239) and (240) into Eq. (188) and using trigonometric formulas,

$$x(t;\epsilon) = e^{-\epsilon t}\left\{\cos\left[1-\frac{1}{2}\epsilon^2+O(\epsilon^3)\right]t + \epsilon\sin\left[1-\frac{1}{2}\epsilon^2+O(\epsilon^3)\right]t\right\} + O(\epsilon^2) \quad (241)$$

A comparison of the uniformly valid asymptotic expansion to $O(\epsilon)$, Eq. (241), with the result of Section IV-D, Eq. (114), shows that both results are the same. Hence, the perturbation theory described in this section is acceptable (see the discussion at the end of Section IV-D-3).

H. Comparison of Approximate Solutions

The five perturbation theories described in Sections IV-C–IV-G are now considered with respect to numerical accuracy and range of validity. Table 3 presents the exact solution and the approximate solutions obtained through the use of these perturbation theories. In reviewing Table 3, it is apparent that only three different approximate solutions to $x(t;\epsilon)$ have been obtained; that is,

$$\left.\begin{aligned} &X_{A1}(t;\epsilon) = \cos t + \epsilon(\sin t - t\cos t) + O(\epsilon^2) \\ &\text{where } 0 \leqq t < t_1, \\ &X_{A2}(t;\epsilon) = e^{-\epsilon t}\left[\cos\left(1-\frac{1}{2}\epsilon^2+O(\epsilon^3)\right)t + \epsilon\sin\left(1-\frac{1}{2}\epsilon^2+O(\epsilon^3)\right)t\right] + O(\epsilon^2) \\ &\text{where } 0 \leqq t < \infty, \text{ and} \\ &X_{A3}(t;\epsilon) = e^{-\epsilon t}\left[\left(1-\frac{1}{2}\epsilon^3 t\right)\cos t + \epsilon\left(1+\frac{1}{2}\epsilon t\right)\sin t\right] + O(\epsilon^3) \\ &\text{where } 0 \leqq t < t_2. \end{aligned}\right\} \quad (242)$$

Hence, the following comparison is concerned with the numerical accuracy and the range of validity of $X_{A1}(t;\epsilon)$, $X_{A2}(t;\epsilon)$, and $X_{A3}(t;\epsilon)$ where the standard of comparison is the exact solution; that is,

$$X_E(t;\epsilon) = e^{-\epsilon t}\left[\cos(1-\epsilon^2)^{1/2}t + \frac{\epsilon}{(1-\epsilon^2)^{1/2}}\sin(1-\epsilon^2)^{1/2}t\right] \quad (243)$$

where $0 \leqq t < \infty$.

It should be noted that $X_{A1}(t;\epsilon)$ and $X_{A3}(t;\epsilon)$ are initially valid approximations, whereas $X_{A2}(t;\epsilon)$ is a uniformly valid approximation (see Sections IV-C–IV-G).

Consider the numerical accuracy and the range of validity of the three approximations given in Eqs. (242) for $\epsilon = 0.001$, 0.01, and 0.1 (increasing ϵ corresponds to increasing the damping or decreasing the strength of the spring; see Eqs. 36). Figure 7 presents the exact position of the oscillator versus the time, as given in Eq. (243), for each of the three epsilons. (The data for this figure as well as for Figs. 8–16 have been attained using an IBM 1620 electronic computer.) Figures 8–10, 11–13, and 14–16 present the position and position error of the oscillator versus time for $\epsilon = 0.001$, 0.01, and 0.1, respectively. The position error is the difference between the exact position as given in Eq. (243) and the approximate position as given in Eqs. (242). Note that the position, position error, and the time in these figures are given in normalized form (see Section IV-A).

Consider Fig. 7. The three values of epsilon, $\epsilon = 0.001$, 0.01, and 0.1, clearly correspond to a slightly, moderately, and highly damped oscillator, respectively.

Table 3. Exact and approximate solutions to the problem of the damped linear harmonic oscillator

Method of solution	*Solution*
1. Exact (Section IV-B)	$x(t;\epsilon) = e^{-\epsilon t}\left[\cos(1-\epsilon^2)^{1/2} t + \frac{\epsilon}{(1-\epsilon^2)^{1/2}} \sin(1-\epsilon^2)^{1/2} t\right]$ where $0 \leqq t < \infty$
2. Variation of coordinates with a one-variable asymptotic expansion (Section IV-C)	$x(t;\epsilon) = \cos t + \epsilon(\sin t - t\cos t) + O(\epsilon^2)$ where $0 \leqq t < t_1$
3. Variation of coordinates with a two-variable asymptotic expansion (Section IV-D)	$x(t;\epsilon) = e^{-\epsilon t}\left\{\cos\left[1 - \frac{1}{2}\epsilon^2 + O(\epsilon^3)\right]t + \epsilon \sin\left[1 - \frac{1}{2}\epsilon^2 + O(\epsilon^3)\right]t\right\} + O(\epsilon^2)$ where $0 \leqq t < \infty$
4. Variation of parameters with successive approximations (Section IV-E)	$x(t;\epsilon) = \cos t + \epsilon(\sin t - t\cos t) + O(\epsilon^2)$ where $b(t;\epsilon) = 1 - \epsilon\left(t - \frac{1}{2}\sin 2t\right) + O(\epsilon^2)$ $c(t;\epsilon) = \frac{1}{2}\epsilon(1-\cos 2t) + O(\epsilon^2)$ and $0 \leqq t < t_1$
5. Variation of parameters with Taylor's series expansions (Section IV-F)	$x(t;\epsilon) = e^{-\epsilon t}\left[\left(1 - \frac{1}{2}\epsilon^3 t\right)\cos t + \epsilon\left(1 + \frac{1}{2}\epsilon t\right)\sin t\right] + O(\epsilon^3)$ where $b(t;\epsilon) = e^{-\epsilon t}\left\{1 + \frac{1}{2}\epsilon[\epsilon(1-\epsilon t) - \epsilon\cos 2t + \sin 2t]\right\} + O(\epsilon^2)$ $c(t;\epsilon) = \frac{1}{2}\epsilon e^{-\epsilon t}[(1+\epsilon t) - \cos 2t - \epsilon\sin 2t] + O(\epsilon^2)$ and $0 \leqq t < t_2 \quad (t_2 > t_1)$

Table 3 (contd)

Method of solution	*Solution*
6. Variation of parameters with two-variable asymptotic expansions (Section IV-G)	$x(t;\epsilon) = e^{-\epsilon t}\left\{\cos\left[1-\frac{1}{2}\epsilon^2 + O(\epsilon^3)\right]t + \epsilon\sin\left[1-\frac{1}{2}\epsilon^2+O(\epsilon^3)\right]t\right\} + O(\epsilon^2)$ where $\widetilde{b}(t;\epsilon) = e^{-\epsilon t}\left\{1+\frac{1}{2}\epsilon\sin 2\left[1-\frac{1}{2}\epsilon^2+O(\epsilon^3)\right]t\right\} + O(\epsilon^2)$ $\widetilde{c}(t;\epsilon) = \frac{1}{2}\epsilon e^{-\epsilon t}\left\{1-\cos 2\left[1-\frac{1}{2}\epsilon^2+O(\epsilon^3)\right]t\right\} + O(\epsilon^2)$ and $0 \leqq t < \infty$

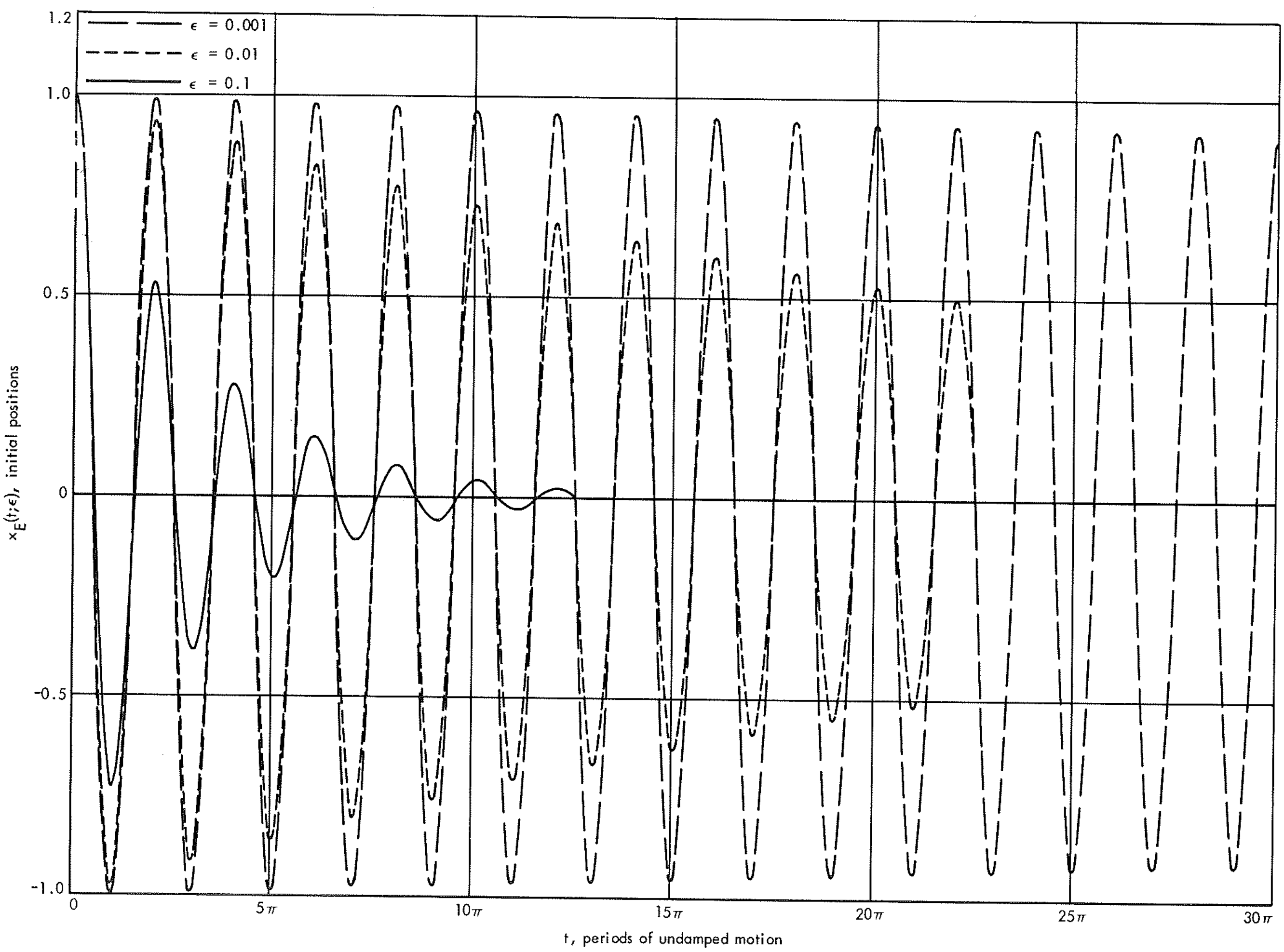

Fig. 7. Exact position of oscillator versus time (ϵ = 0.001, 0.01, 0.1)

Consider Figs. 8–10 ($\epsilon = 0.001$). Although Fig. 8 implies that all three approximations are very close to the exact solution, Figs. 9 and 10 show that the approximation $X_{A2}(t;\epsilon)$ is superior to the two remaining approximations, $X_{A1}(t;\epsilon)$ and $X_{A3}(t;\epsilon)$. The maximum error of $X_{A2}(t;\epsilon)$ is approximately 5.0×10^{-10} (well within the expected error of approximately $\epsilon^2 = 1.0 \times 10^{-6}$ (see the second of Eqs. 242) and occurs at $t = \pi/2$. Furthermore, the relative maximum errors of $X_{A2}(t;\epsilon)$ decrease as time increases, and no secular error growth exists (a highly desirable characteristic). Thus, the range of validity of $X_{A2}(t;\epsilon)$ is $0 \leqq t < \infty$ as expected (uniformly valid). In contrast, the range of validity of $X_{A1}(t;\epsilon)$ is approximately $0 \leqq t < 2\pi$ ($t_1 \cong 2\pi$), whereas the range of validity of $X_{A3}(t;\epsilon)$ cannot be attained (because of insufficient data) but is clearly much larger than that of $X_{A1}(t;\epsilon)$ as expected; that is, $0 \leqq t < t_2$ where $t_2 >> t_1$. It should be noted that the range of validity is assumed to be the range of time during which the actual error is less than or equal to the size of the first neglected term of the approximate solution.

Consider Figs. 11–13 ($\epsilon = 0.01$). Figure 11 clearly shows the inferiority of the approximation $X_{A1}(t;\epsilon)$ and implies that the two remaining approximations, $X_{A2}(t;\epsilon)$ and $X_{A3}(t;\epsilon)$, are very close to the exact solution. However, Fig. 13 shows that the approximation $X_{A2}(t;\epsilon)$ is superior to the approximation $X_{A3}(t;\epsilon)$. The maximum error of $X_{A2}(t;\epsilon)$ is approximately 5.0×10^{-7} (well within the expected error of approximately $\epsilon^2 = 1.0 \times 10^{-4}$) and occurs at $t = \pi/2$. Furthermore, no secular error growth exists, and thus the range of validity of $X_{A2}(t;\epsilon)$ is $0 \leqq t < \infty$ as expected. In contrast, the range of validity of $X_{A1}(t;\epsilon)$ is approximately $0 \leqq t < 3\pi/2$ ($t_1 \cong 3\pi/2$), whereas the range of validity of $X_{A3}(t;\epsilon)$ cannot be attained (because of insufficient data) but is clearly much larger than that of $X_{A1}(t;\epsilon)$ as expected.

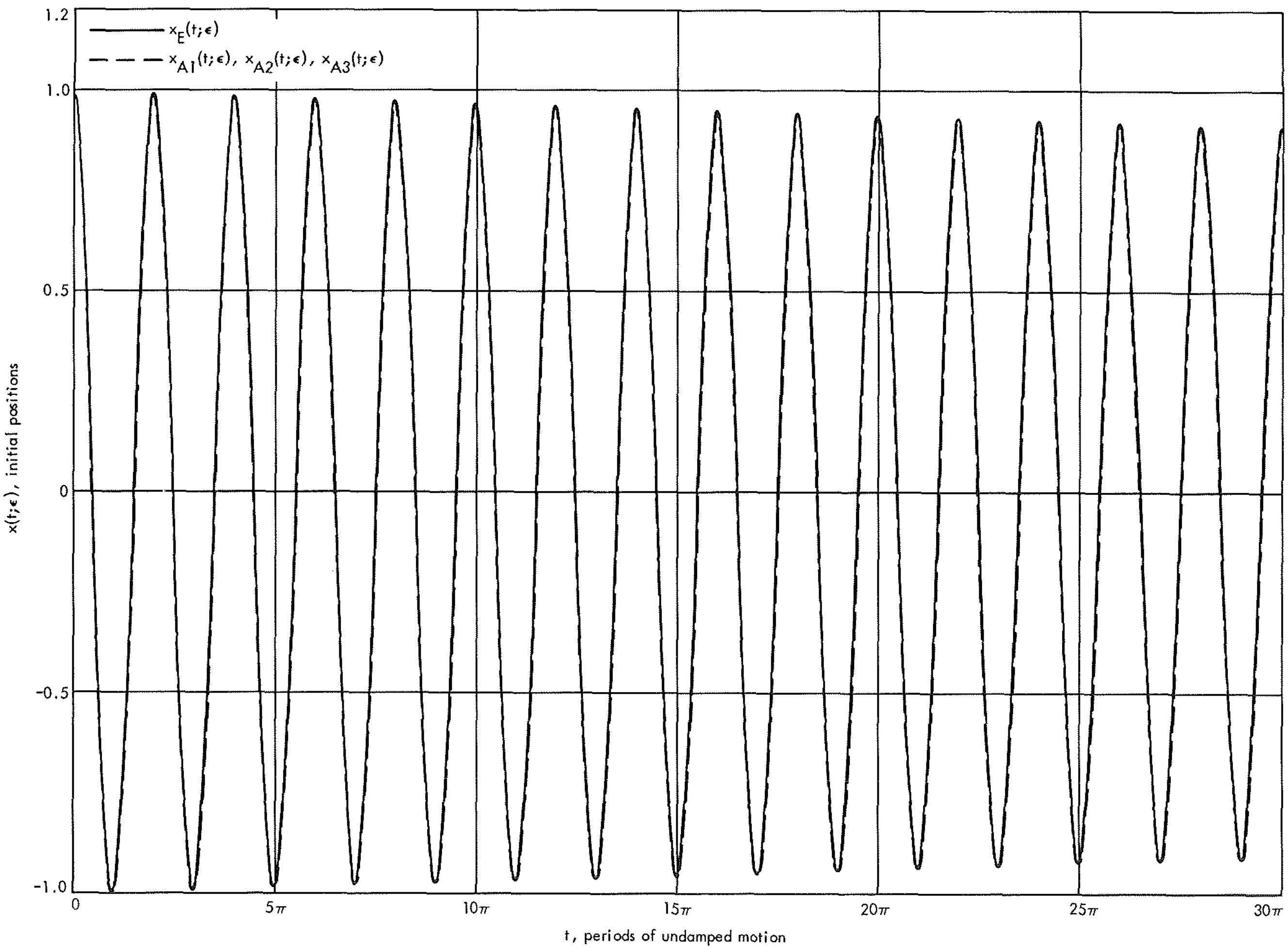

Fig. 8. Exact and approximate positions of oscillator versus time ($\epsilon = 0.001$)

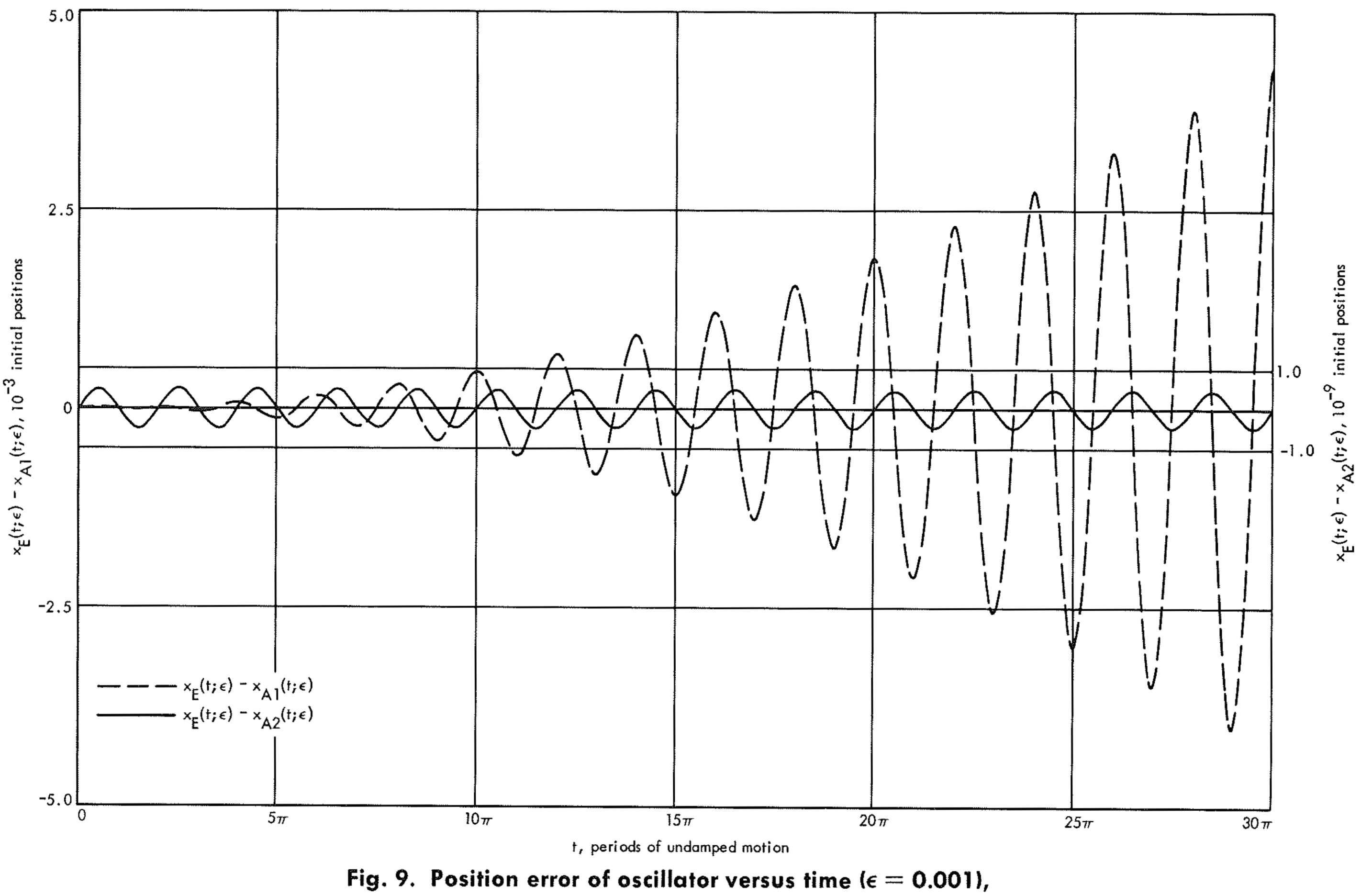

Fig. 9. Position error of oscillator versus time ($\epsilon = 0.001$), $x_E(t;\epsilon) - x_{A1}(t;\epsilon)$, $x_E(t;\epsilon) - x_{A2}(t;\epsilon)$

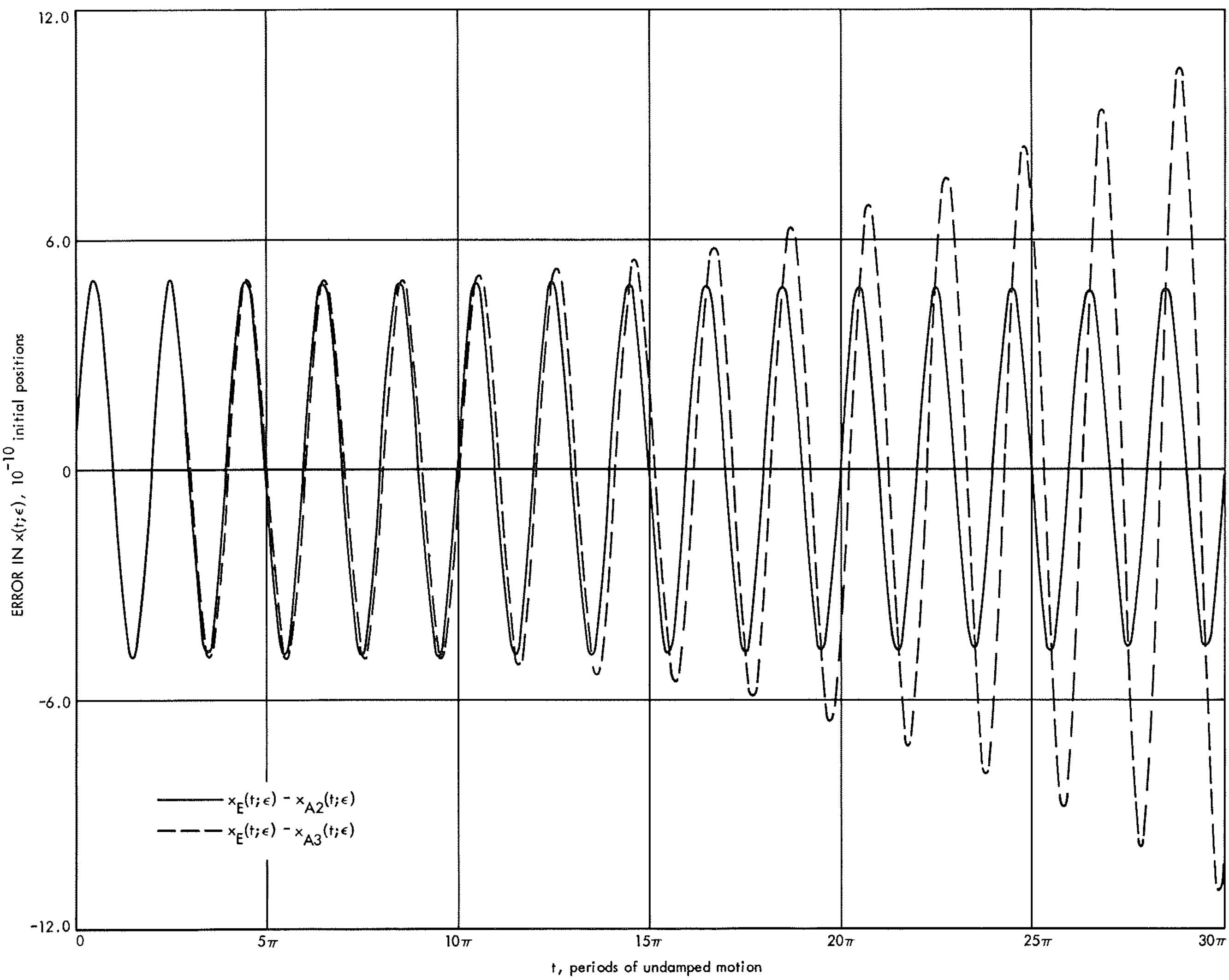

Fig. 10. Position error of oscillator versus time ($\epsilon = 0.001$), $x_E(t;\epsilon) - x_{A2}(t;\epsilon)$, $x_E(t;\epsilon) - x_{A3}(t;\epsilon)$

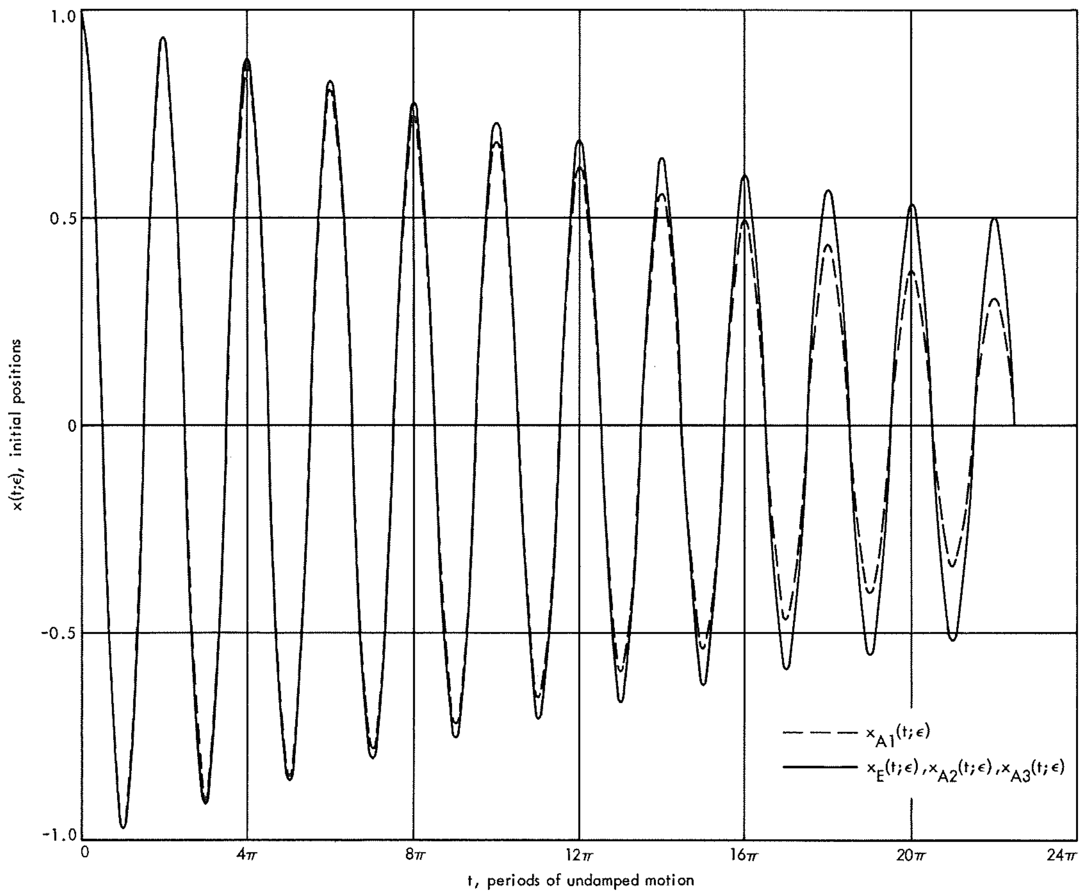

Fig. 11. Exact and approximate positions of oscillator versus time ($\epsilon = 0.01$)

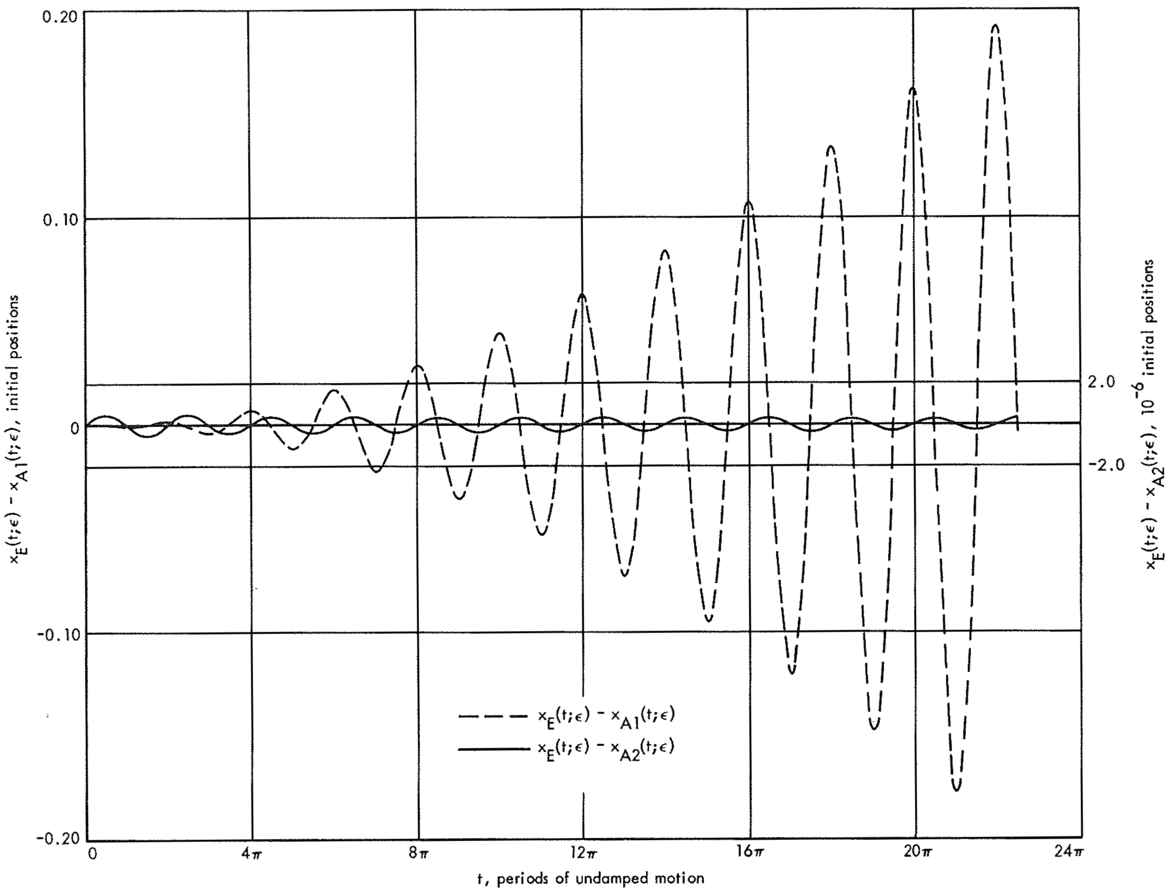

Fig. 12. Position error of oscillator versus time ($\epsilon = 0.01$), $x_E(t;\epsilon) - x_{A1}(t;\epsilon)$, $x_E(t;\epsilon) - x_{A2}(t;\epsilon)$

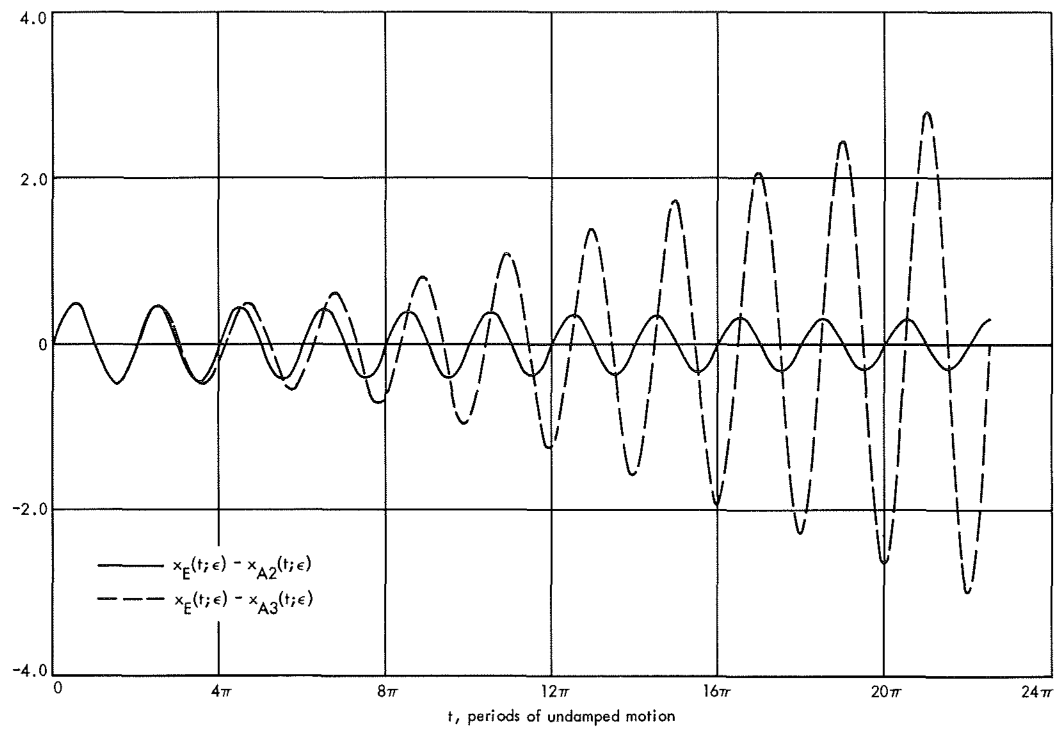

Fig. 13. Position error of oscillator versus time ($\epsilon = 0.01$), $x_E(t;\epsilon) - x_{A2}(t;\epsilon)$, $x_E(t;\epsilon) - x_{A3}(t;\epsilon)$

Consider Figs. 14–16 ($\epsilon = 0.1$). Figure 14 unmistakably shows the inferiority of the approximation $X_{A1}(t;\epsilon)$ and implies that the two remaining approximations, $X_{A2}(t;\epsilon)$ and $X_{A3}(t;\epsilon)$, are very close to the exact solution. However, Fig. 16 shows that the approximation $X_{A2}(t;\epsilon)$ is superior to the approximation $X_{A3}(t;\epsilon)$. The maximum error of $X_{A2}(t;\epsilon)$ is approximately 4.5×10^{-4} (well within the expected error of approximately $\epsilon^2 = 1.0 \times 10^{-2}$) and occurs at $t = \pi/2$. Furthermore, no secular error growth exists and thus the range of validity of $X_{A2}(t;\epsilon)$ is $0 \leqq t < \infty$ as expected. In contrast, the range of validity of $X_{A1}(t;\epsilon)$ is approximately $0 \leqq t < \pi$ ($t_1 \cong \pi$), whereas the range of validity of $X_{A3}(t;\epsilon)$ is $0 \leqq t < \infty$ (although $X_{A3}(t;\epsilon)$ contains Poisson terms, the exponential decay overcomes the secular error growth for this large value of ϵ and yields a uniformly valid approximation).

It is evident from the preceding discussion that the uniformly valid approximation $X_{A2}(t;\epsilon)$ is superior to the remaining two approximations, $X_{A1}(t;\epsilon)$ and $X_{A3}(t;\epsilon)$, regardless of the value of ϵ, as expected. Furthermore, from Eqs. (242) and Table 3, it is apparent that only two of the five perturbation theories considered in Sections IV-C–IV-G yield the approximation $X_{A2}(t;\epsilon)$: namely, the variation of coordinates with a two-variable asymptotic expansion and the variation of parameters with two-variable asymptotic expansions. Hence, the application of either of these two perturbation theories (generalized) to the set of differential equations that govern the motion of a satellite appears desirable. The decision as to which of these two theories is more desirable depends upon the complexity of the set of differential equations that govern the motion of the satellite. If this set of differential equations is relatively simple, as in the motion of an equatorial satellite about an oblate body, it is just as desirable to use the variation of coordinates with two-variable asymptotic expansions as it is to use the variation of parameters with two-variable asymptotic expansions. However, if this set of differential equations is complex, as in the general motion of a satellite acted upon by several disturbing forces, it is more desirable to use the variation of parameters with two-variable asymptotic expansions. In general, this theory involves the solution of first-order partial differential equations, whereas the variation of coordinates with two-variable asymptotic expansions involves the solution of second-order partial differential equations, which are usually more difficult to solve (see Sections IV-D and IV-G).

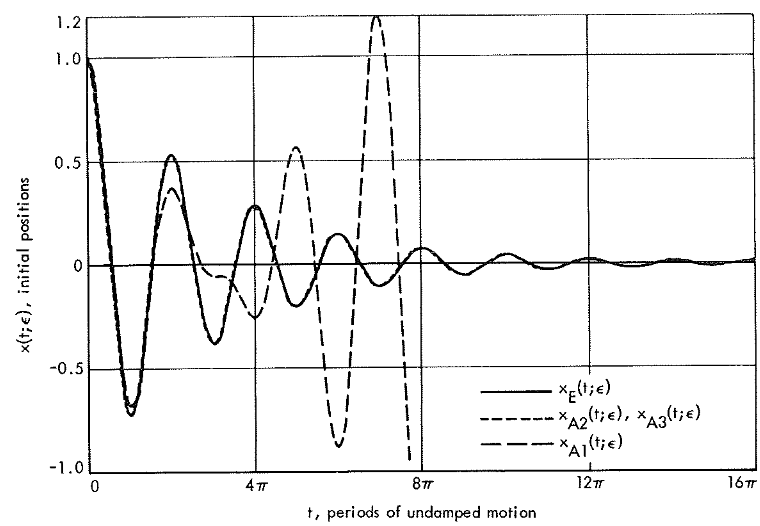

Fig. 14. Exact and approximate positions of oscillator versus time ($\epsilon = 0.1$)

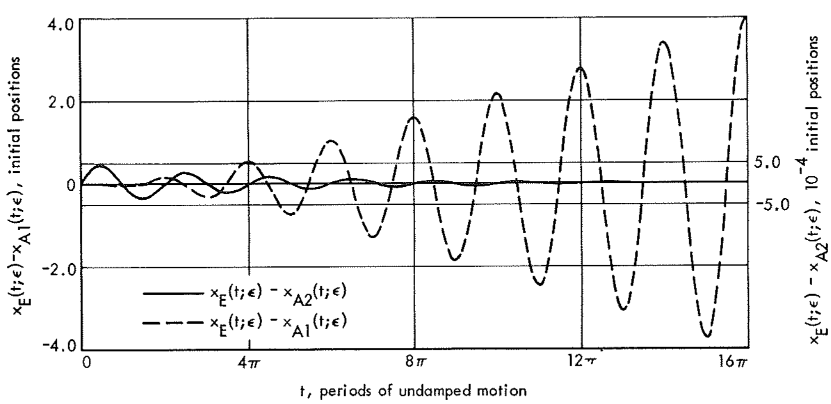

Fig. 15. Position error of oscillator versus time ($\epsilon = 0.1$), $x_E(t;\epsilon) - x_{A1}(t;\epsilon)$, $x_E(t;\epsilon) - x_{A2}(t;\epsilon)$

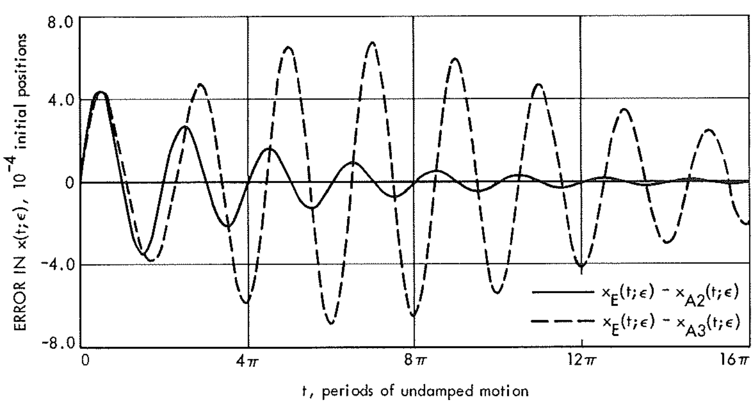

Fig. 16. Position error of oscillator versus time ($\epsilon = 0.1$), $x_E(t;\epsilon) - x_{A2}(t;\epsilon)$, $x_E(t;\epsilon) - x_{A3}(t;\epsilon)$

V. Motion of a Satellite in an Equatorial Orbit About an Oblate Body

The principal test of a satellite theory herein is that it yields the position and velocity (or equivalent parameters) of a satellite to a sufficiently high accuracy even after many revolutions about the central mass. The only means of deciding whether a given theory yields a sufficiently accurate solution is to compare the resultant solution with a standard of much higher accuracy than the desired accuracy. Unfortunately, obtaining a standard of comparison with this characteristic is a difficult task. As a result, it was decided to start the study of satellite theories by restricting the force field and the initial conditions of the satellite in such a way that there is an exact analytic solution to the motion of the satellite that can be used as the standard of comparison. In particular, the force field and the initial conditions of the satellite are chosen to yield the motion of the satellite in an equatorial orbit about an oblate body (second harmonic only).

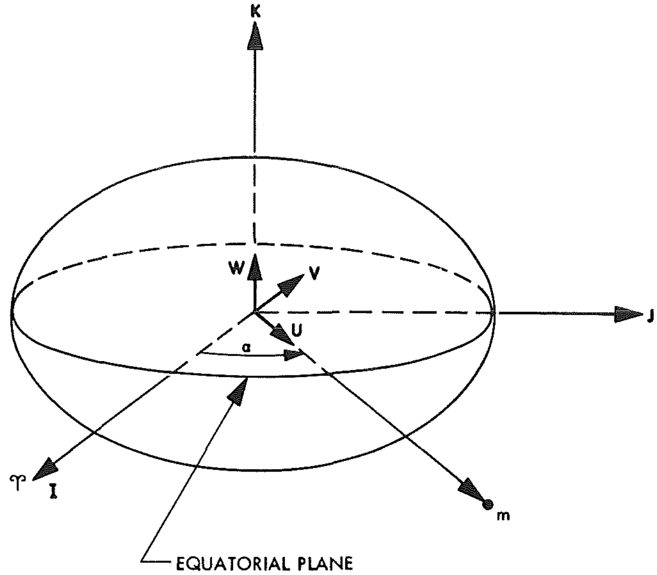

Fig. 17. Motion of a satellite of mass *m* about an axially symmetric oblate body

Two satellite theories are investigated in this section. The two satellite theories consist of the application of (1) the theory of the variation of coordinates with multivariable asymptotic expansions and (2) the theory of the variation of parameters with Taylor's series expansions, respectively, to the differential equations governing the restricted motion of the satellite. These satellite theories are more desirable than the other satellite theories since they are based upon the more desirable perturbation theories from Section IV. The variation of coordinates with multivariable asymptotic expansions yields a uniformly valid approximate solution (the variation of parameters with multivariable asymptotic expansions is equally desirable, yielding the same approximate solution; see Section IV-H), whereas the variation of parameters with Taylor's series expansions yields an approximate solution that is initially valid but one that has a large range of validity (the resulting numerical accuracy may be sufficient over a large enough range of time for practical applications).

A. Mathematical Model

Consider the motion of a satellite of mass m about an axially symmetric oblate body, as shown in Fig. 17. The initial conditions of the satellite and the oblateness of the body are assumed to have values that yield quasi-periodic motion in the equatorial plane of the body (the existence of quasi-periodic motion has been established by several investigators; for example, see Ref. 172). The potential function (also called force function) of the oblate body is assumed to be of the form

$$\Phi = \frac{\mu_p}{r}\left[1 + \frac{1}{2} J_2 \left(\frac{a_p}{r}\right)^2\right] \tag{244}$$

where μ_p is the mass function ($\mu_p = k^2 (m_p + m)$, m_p = mass of body), a_p is the equatorial radius of the body, J_2 is the coefficient of the second harmonic, and r is the radius distance from the dynamical center (in this case, the center of the body). It should be noted that the potential function as given in Eq. (244) implies symmetry in the northern and southern hemispheres of the body (Ref. 564).

The inertial basis or frame of reference, **I**, **J**, **K**, is chosen with origin at the center of mass of the oblate body, with the principal direction **I** directed towards the vernal equinox (indicated by ♈), and with the equatorial plane of the oblate body as its principal plane.

The differential equation of motion of the satellite, in vector form and using the concepts of Section III, is

$$\frac{d^2\mathbf{r}}{dt^2} + \mu_p \frac{\mathbf{r}}{r^3} - \dot{\mathbf{r}}' = 0 \tag{245}$$

where

$$\dot{\mathbf{r}}' = \dot{r}'\mathbf{U} + r\dot{\alpha}'\mathbf{V} \tag{246}$$

and α is the right ascension of the satellite. The perturbative vector $\dot{\mathbf{r}}'$ due to the oblateness of the body can also be written

$$\dot{\mathbf{r}}' = \frac{\partial R}{\partial r}\mathbf{U} + \frac{1}{r}\frac{\partial R}{\partial \alpha}\mathbf{V} \tag{247}$$

where the perturbative function R is, from Eq. (244),

$$R = \frac{1}{2}\mu_p J_2 a_p^2 \left(\frac{1}{r^3}\right) \tag{248}$$

Thus, from Eqs. (246), (247), and (248),

$$\dot{\mathbf{r}}' = \left[-\frac{3}{2}\mu_p J_2 a_p^2 \left(\frac{1}{r^4}\right)\right]\mathbf{U} \tag{249}$$

where

$$\dot{r}_\alpha' = \frac{1}{r}\frac{\partial R}{\partial \alpha} = 0$$

since R is independent of α. From Fig. 17,

$$\mathbf{r} = r\mathbf{U} \tag{250}$$

$$\mathbf{U} = \cos\alpha\mathbf{I} + \sin\alpha\mathbf{J}$$

and

$$\mathbf{V} = -\sin\alpha\mathbf{I} + \cos\alpha\mathbf{J}$$

so that, by differentiation,

$$\frac{d^2\mathbf{r}}{dt^2} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\alpha}{dt}\right)^2\right]\mathbf{U} + \left[\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\alpha}{dt}\right)\right]\mathbf{V} \tag{251}$$

Introducing Eqs. (249), (250), and (251) into Eq. (245) yields

$$\left\{\frac{d^2r}{dt^2} - r\left(\frac{d\alpha}{dt}\right)^2 + \frac{\mu_p}{r^2}\left[1 + \frac{3}{2}J_2\left(\frac{a_p}{r}\right)^2\right]\right\}\mathbf{U} + \left[\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\alpha}{dt}\right)\right]\mathbf{V} = 0$$

or, since $\mathbf{U}$ and $\mathbf{V}$ are orthogonal vectors,

$$\left.\begin{aligned} &\frac{d^2r}{dt^2} - r\left(\frac{d\alpha}{dt}\right)^2 + \frac{\mu_p}{r^2}\left[1 + \frac{3}{2}J_2\left(\frac{a_p}{r}\right)^2\right] = 0 \\ &\text{and} \\ &\frac{d}{dt}\left(r^2\frac{d\alpha}{dt}\right) = 0 \end{aligned}\right\} \tag{252}$$

Furthermore, the initial conditions are

$$\left.\begin{aligned} r(t_0) &= r_0, & \frac{dr}{dt}(t_0) &= \dot{r}_0 \\ \alpha(t_0) &= \alpha_0, & \frac{d\alpha}{dt}(t_0) &= \dot{\alpha}_0 \end{aligned}\right\} \tag{253}$$

where, if $J_2 = 0$, these initial conditions would yield bounded (periodic) motion; that is, the satellite's orbit would be circular or elliptical. Finally, the range of time of interest herein is

$$0 \leq t < \infty \tag{254}$$

Following the procedure of Section IV-A, the differential equations of motion of the satellite are normalized by letting

$$r^* = \frac{r}{L_1}, \qquad t^* = \frac{t - t_0}{T_1}, \qquad \alpha^* = \alpha - \alpha_0 \tag{255}$$

Note that the right ascension α need not be normalized as long as its units are radians. Introducing Eqs. (255) into Eqs. (252) yields

$$\left.\begin{aligned} &\frac{d^2r^*}{dt^{*2}} - r^*\left(\frac{d\alpha^*}{dt^*}\right)^2 \\ &\qquad + \left(\frac{\mu_p T_1^2}{L_1^3}\right)\frac{1}{r^{*2}}\left[1 + \frac{3}{2}J_2\left(\frac{a_p}{L_1}\right)^2\frac{1}{r^{*2}}\right] = 0 \\ &\text{and} \\ &\frac{d}{dt^*}\left(r^{*2}\frac{d\alpha^*}{dt^*}\right) = 0 \end{aligned}\right\} \tag{256}$$

The normalization constants for the length and the time (the length and time scales) L_1 and T_1 and the perturbative parameter ϵ^* are now determined, as in Section IV-A, by setting all of the coefficients in Eqs. (256) equal to unity except for the coefficient of the perturbing term, which is set equal to the perturbative parameter ϵ^*. Hence,

$$\left.\begin{aligned} \frac{\mu_p T_1^2}{L_1^3} &= 1 \\ \text{and} \qquad & \\ \epsilon^* &= \frac{3}{2} J_2 \left(\frac{a_p}{L_1}\right)^2 \end{aligned}\right\} \tag{257}$$

Now, Eqs. (257) are composed of two equations in the three unknowns L_1, T_1, and ϵ^* and thus a unique determination of these unknowns is not possible. However, there appear to be only three attractive choices for the length scale L_1 in the physical context of the problem. These are the equatorial radius of the body a_p, the initial radius distance of the satellite r_0, and the semimajor axis of the unperturbed orbit a_0 (for this orbit, $J_2 = 0$ and the initial conditions are those given in Eqs. 253). The value of a_0 is obtained from the initial conditions (253) and the vis viva integral; that is, from Ref. 564,

$$\dot{s}_0^2 = \mu_p \left(\frac{2}{r_0} - \frac{1}{a_0}\right)$$

where

$$\dot{s}_0^2 = \dot{r}_0^2 + r_0^2 \dot{\alpha}_0^2$$

so that

$$a_0 = \left[\frac{2}{r_0} - \frac{1}{\mu_p}(\dot{r}_0^2 + r_0^2\dot{\alpha}_0^2)\right]^{-1} \tag{258}$$

Consider the three possibilities $L_1 = a_p$, r_0, and a_0, respectively. Using Eqs. (257) and Kepler's third law as modified by Newton (Ref. 564), $P = 2\pi a^{3/2}/\mu_p^{1/2}$, the respective possibilities for T_1 and ϵ^* are obtained as presented in Table 4.

In reviewing Table 4, it is clear that the use of a_0 as the length scale is most characteristic of the perturbed motion since a first approximation to the perturbed motion is simply the unperturbed motion. Thus, the normalization constants are chosen to be

$$L_1 = a_0 \text{ and } T_1 = \left(\frac{a_0^3}{\mu_p}\right)^{1/2} \tag{259}$$

and, as a result,

$$\epsilon^* = \frac{3}{2} J_2 \left(\frac{a_p}{a_0}\right)^2 \tag{260}$$

In Section IV-A, it was shown that the perturbative parameter is the ratio of two characteristic time scales; that is,

$$\epsilon^* = \frac{T_1}{T_2}$$

and thus a second characteristic time scale for this motion may be defined as

$$T_2 = \frac{T_1}{\epsilon^*} \tag{261}$$

Introducing the second of Eqs. (259) and Eq. (260) into Eq. (261) yields

$$T_2 = \frac{2}{3}\frac{1}{J_2}\left(\frac{a_0}{a_p}\right)^2 \left(\frac{a_0^3}{\mu_p}\right)^{1/2} \tag{262}$$

The physical significance of these two time scales is clear. The time scale T_1 is a measure of the period of the primary motion of the satellite produced by the inverse square portion of the force field of the body (characteristic time for the primary motion), while the time scale T_2 is a measure of the period after which the cumulative

Table 4. Normalization constants for the equatorial satellite

L_1	T_1		ϵ^*
a_p	$(a_p^3/\mu_p)^{1/2}$	$1/2\pi$ times the period of a satellite just grazing the surface of the body in a circular orbit	$\frac{3}{2} J_2$
r_0	$(r_0^3/\mu_p)^{1/2}$	$1/2\pi$ times the period of a satellite in a circular orbit whose semimajor axis is equal to the initial radius distance	$\frac{3}{2} J_2 (a_p/r_0)^2$
a_0	$(a_0^3/\mu_p)^{1/2}$	$1/2\pi$ times the period of a satellite in the unperturbed orbit whose initial conditions are given in Eqs. (253)	$\frac{3}{2} J_2 (a_p/a_0)^2$

effects of the second harmonic become important (characteristic perturbing time). The ratio ϵ^* is a measure of the relative importance of the primary and perturbing forces. In fact, ϵ^* is the ratio of the perturbing force to the primary force at a radius distance equal to the semimajor axis of the unperturbed motion a_0. (The initial conditions of this motion are those given in Eqs. 253.) It should be noted that a_0 is, in a sense, a mean radius distance of the satellite since

$$a_0 = \frac{q_0 + r_{A0}}{2}$$

where q_0 = initial perifocal distance and r_{A0} = initial apofocal distance as shown in Fig. 18. The fact that ϵ^* is the ratio of the perturbing force to the primary force at $r = a_0$ is substantiated by using the first of Eqs. (252); that is,

$$\epsilon^* = \left[\frac{\text{perturbing force}}{\text{primary force}}\right]_{r=a_0}$$

or

$$\epsilon^* = \left[\frac{(m\mu_p/r^2)\left(\frac{3}{2}J_2\right)(a_p/r)^2}{m\mu_p/r^2}\right]_{r=a_0} = \frac{3}{2}J_2\left(\frac{a_p}{a_0}\right)^2$$

in agreement with Eq. (260) (the satellite mass m is introduced to obtain force from acceleration).

In general, the value of J_2 is small ($J_2 = 0.001975$ for Mars) and, since $a_0 \geqq a_p$ for a realistic satellite orbit, it is clear from Eq. (260) that $\epsilon^* << 1$. Furthermore, since $\epsilon^* << 1$, it is clear from Eq. (261) that $T_2 >> T_1$, and thus T_1 characterizes the fast (short-period) variations of the satellite and T_2 characterizes the slow (long-period) variations of the satellite. The existence of these two time scales in the physical context of this problem implies that the theory of multivariable asymptotic expansions is ideally suited for obtaining an approximate solution to the motion of the satellite.

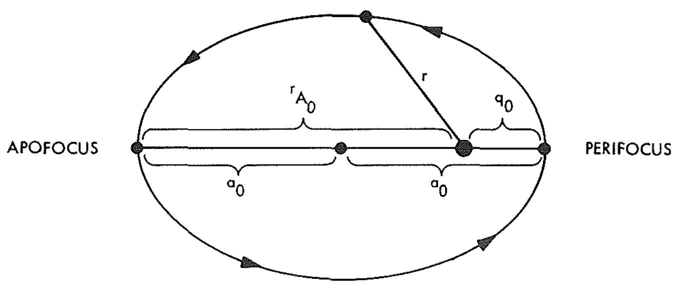

Fig. 18. Mean radius distance of a satellite

Introducing Eqs. (257) into Eqs. (256) and using Eqs. (255) and (259) in Eqs. (253) and (254) yields the following differential equations of motion and initial conditions in normalized variables:

$$\left.\begin{aligned} \frac{d^2r^*}{dt^{*2}} - r^*\left(\frac{d\alpha^*}{dt^*}\right)^2 + \frac{1}{r^{*2}}\left[1 + \epsilon^*\left(\frac{1}{r^{*2}}\right)\right] = 0 \\ \frac{d}{dt^*}\left(r^{*2}\frac{d\alpha^*}{dt^*}\right) = 0 \end{aligned}\right\} \quad (263a)$$

and

$$\left.\begin{aligned} r^*(0) = \frac{r_0}{a_0}, \qquad \frac{dr^*}{dt^*}(0) = \frac{\dot{r}_0}{(\mu_p/a_0)^{1/2}} \\ \alpha^*(0) = 0, \qquad \frac{d\alpha^*}{dt^*}(0) = \left(\frac{a_0^3}{\mu_p}\right)^{1/2}\dot{\alpha}_0 \\ \text{where} \\ 0 \leqq t^* < \infty \end{aligned}\right\} \quad (263b)$$

Note that the quantity $(\mu_p/a_0)^{1/2}$ is the circular-satellite speed at $r = a_0$ of a satellite in the unperturbed orbit whose initial conditions are those of Eqs. (253), as may be seen from the vis viva integral

$$(\dot{s})_{r=a_0} = \left[\mu_p\left(\frac{2}{r} - \frac{1}{a_0}\right)\right]^{1/2}_{r=a_0} = (\mu_p/a_0)^{1/2}$$

Since a_0 can be thought of as the mean distance of the satellite, it follows that $(\mu_p/a_0)^{1/2}$ can be thought of as the mean speed of the satellite; that is,

$$\bar{\dot{s}} = (\mu_p/a_0)^{1/2} \quad (264)$$

Following the procedure of Section IV-A, the cumbersome star notation is eliminated so that Eqs. (263) become, setting $\epsilon^* = h^4\epsilon$ as well,

$$\frac{d^2r}{dt^2} - r\left(\frac{d\alpha}{dt}\right)^2 + \frac{1}{r^2}\left(1 + \epsilon\frac{h^4}{r^2}\right) = 0 \quad (265)$$

$$\frac{d}{dt}\left(r^2\frac{d\alpha}{dt}\right) = 0 \quad (266)$$

and

$$r(0) = r_0, \qquad \frac{dr}{dt}(0) = \dot{r}_0 \tag{267}$$

$$\alpha(0) = 0, \qquad \frac{d\alpha}{dt}(0) = \dot{\alpha}_0 \tag{268}$$

where

$$0 \leqq t < \infty \tag{269}$$

and where the units of the variables and initial conditions are now as follows:

$$\left.\begin{array}{l}\text{[distances]} = \text{semimajor axes of the unperturbed motion } a_0,\\ \text{[speeds]} = \text{circular-satellite speeds of the unperturbed motion at } r = a_0, \bar{\dot{s}},\\ \text{[angular speeds]} = \text{radians per } 1/2\pi \text{ periods of the unperturbed motion, rad}/(P_0/2\pi),\\ \text{and}\\ \text{[time]} = 1/2\pi \text{ periods of the unperturbed motion, } P_0/2\pi.\end{array}\right\} \tag{270}$$

Note that, using Eq. (260),

$$\epsilon = \frac{\epsilon^*}{h^4} = \frac{3}{2} J_2 \left(\frac{a_p}{h^2}\right)^2 \tag{271}$$

where the units of a_p are now semimajor axes of the unperturbed motion, a_0. The quantity h is the normalized angular momentum per unit mass and is introduced for the purpose of simplification, as will be seen in the next section.

B. Exact Solution

The differential equations that govern the motion of an equatorial satellite, given in Eqs. (265) and (266), are clearly nonlinear. Nevertheless, an exact analytical solution is attainable in terms of familiar functions: namely, elliptic integrals (or elliptic functions).

1. Selection of coordinates and independent variable. In order to obtain the exact analytical solution to the motion of the satellite, Eqs. (265–269) are transformed from the natural dependent variables r and α and the natural independent variable t to the reference dependent variables $\eta = h^2/r$ and t and the reference independent variable α. Thus the reference coordinates are $\eta(\alpha; \epsilon)$, $t(\alpha; \epsilon)$, $(d\eta/d\alpha)(\alpha; \epsilon)$, and $(dt/d\alpha)(\alpha; \epsilon)$, and the reference independent variable is α. From these, the selected coordinates $\alpha(r; \epsilon)$, $t(r; \epsilon)$, $(dr/dt)(r; \epsilon)$, and $(d\alpha/dt)(r; \epsilon)$ are determined where the selected independent variable is r.

2. Equations of motion. The transformation of Eqs. (265–269) into the reference variables begins by integrating Eq. (266) to yield

$$r^2 \frac{d\alpha}{dt} = h \tag{272}$$

where, through the use of $r(0) = r_0$ and $(d\alpha/dt)(0) = \dot{\alpha}_0$ from Eqs. (267) and (268),

$$h = r_0^2 \dot{\alpha}_0 \tag{273}$$

The quantity h, the normalized angular momentum of the satellite, is constant (conservation of angular momentum).

Let

$$r = \frac{h^2}{\eta} \tag{274}$$

so that Eq. (272) becomes (r is assumed to be greater than zero)

$$\frac{d\alpha}{dt} = \frac{\eta^2}{h^3} \tag{275}$$

Differentiating Eq. (274) with respect to t and using Eq. (275) yields

$$\frac{dr}{dt} = -\frac{h^2}{\eta^2}\frac{d\eta}{d\alpha}\frac{d\alpha}{dt} = -\frac{1}{h}\frac{d\eta}{d\alpha} \tag{276}$$

and

$$\frac{d^2 r}{dt^2} = -\frac{1}{h}\frac{d^2\eta}{d\alpha^2}\frac{d\alpha}{dt} = -\frac{\eta^2}{h^4}\frac{d^2\eta}{d\alpha^2} \tag{277}$$

Introducing Eqs. (274), (275), and (277) into Eq. (265) yields

$$\frac{d^2\eta}{d\alpha^2} + \eta - 1 - \epsilon\eta^2 = 0$$

Thus, the differential equations of motion and initial conditions in terms of the reference variables are, from

Eq. (275), the previous equation, and from Eqs. (274–276) evaluated at $\alpha = 0$,

$$\frac{d^2\eta}{d\alpha^2} + \eta - 1 - \epsilon\eta^2 = 0 \tag{278}$$

$$\frac{dt}{d\alpha} = \frac{h^3}{\eta^2} \tag{279}$$

and

$$\eta(0) = \frac{h^2}{r_0}, \qquad \frac{d\eta}{d\alpha}(0) = -h\dot{r}_0 \tag{280}$$

$$t(0) = 0, \qquad \frac{dt}{d\alpha}(0) = \frac{1}{\dot{\alpha}_0} \tag{281}$$

where

$$0 \leqq \alpha < \infty \tag{282}$$

3. Relativistic theory of gravitation. The system of equations developed in the preceding section governs the motion of a satellite in an equatorial orbit about an oblate body. The intent of this section is to show that the same system of equations governs relativistic motion of a point mass moving around a central point mass so massive as to dominate the system (the motion of a satellite around a planet and the motion of a planet around the sun can be approximated by such a motion).

According to the theory of general relativity (Ref. 1373), a motion of this type is governed by the ordinary geodesics of the static Schwarzschild metric, which in simplified form is

$$(ds)^2 = -\frac{1}{\gamma}(dr)^2 - r^2(d\alpha)^2 + \gamma c^2 (dt)^2$$

where

$$\delta \equiv 0 \qquad \text{and} \qquad \gamma = 1 - 2\frac{\mu_p}{c^2}\frac{1}{r}$$

and where the quantities c and μ_p are the speed of light and the mass function, respectively. The quantities r, α, and δ ($\delta \equiv 0$) are the position coordinates of the orbiting point mass and are analogous to r, α, and δ in the Newtonian theory of gravitation (preceding section), whereas the quantities t and s are the coordinate time and proper time of the orbiting point mass, respectively. The proper time is analogous to the time t in the Newtonian theory of gravitation.

The differential equations of motion that result from the metric above are

$$\frac{d^2r}{ds^2} - r\left(\frac{d\alpha}{ds}\right)^2 + \frac{\mu_p}{c^2}\frac{1}{r^2}\left(1 + 3\frac{h_0^2}{c^2}\frac{1}{r^2}\right) = 0$$

$$r^2\frac{d\alpha}{ds} = \frac{h_0}{c}$$

$$\frac{dt}{ds} = \frac{\kappa}{\gamma c}$$

subject to the initial conditions

$$r(t_0) = r_0, \qquad \frac{dr}{dt}(t_0) = \dot{r}_0$$

$$\alpha(t_0) = \alpha_0, \qquad \frac{d\alpha}{dt}(t_0) = \dot{\alpha}_0$$

$$s(t_0) = s_0, \qquad \frac{ds}{dt}(t_0) = \frac{\gamma_0 c}{\kappa}$$

where

$$\gamma_0 = 1 - 2\frac{\mu_p}{c^2}\frac{1}{r_0}$$

and

$$0 \leqq t < \infty$$

The quantities h_0 and κ are constants of integration and are given by

$$\kappa = \gamma_0\left[\gamma_0 - \frac{1}{c^2}\frac{\dot{r}_0^2}{\gamma_0} - \frac{1}{c^2}\frac{h^2}{r_0^2}\right]^{-1/2}$$

and

$$h_0 = \frac{\kappa}{\gamma_0}h$$

where

$$h = r_0^2\dot{\alpha}_0$$

Note that the quantity h is the angular momentum per unit mass in the Newtonian theory of gravitation and that h_0 is analogous to h since both γ_0 and κ are dimensionless constants.

Following the procedure of Section V-A, the preceding equations are normalized by letting

$$r^* = \frac{r}{a_0}, \qquad \alpha^* = \alpha - \alpha_0$$

$$t^* = \frac{t - t_0}{\dfrac{\kappa}{\gamma_0}\left(\dfrac{a_0^3}{\mu_p}\right)^{1/2}}, \qquad s^* = \frac{s - s_0}{c\left(\dfrac{a_0^3}{\mu_p}\right)^{1/2}}$$

$$\epsilon^* h_0^{*4} = 3\frac{h_0^2}{c^2 a_0^2}, \qquad h_0^* = \frac{h_0}{(\mu_p a_0)^{1/2}}$$

where

$$a_0 = \left[\frac{2}{r_0} - \frac{1}{\mu_p}(\dot{r}_0^2 + r_0^2\dot{\alpha}_0^2)\right]^{-1}$$

to obtain

$$\frac{d^2 r^*}{ds^{*2}} - r^*\left(\frac{d\alpha^*}{ds^*}\right)^2 + \frac{1}{r^{*2}}\left(1 + \epsilon^*\frac{h_0^{*4}}{r^{*2}}\right) = 0$$

$$r^{*2}\frac{d\alpha^*}{ds^*} = h_0^*$$

$$\frac{dt^*}{ds^*} = \frac{\gamma_0}{\gamma}$$

subject to the initial conditions

$$r^*(0) = \frac{r_0}{a_0}, \qquad \frac{dr^*}{dt^*}(0) = \frac{\dot{r}_0}{\dfrac{\gamma_0}{\kappa}\left(\dfrac{\mu_p}{a_0}\right)^{1/2}}$$

$$\alpha^*(0) = 0, \qquad \frac{d\alpha^*}{dt^*}(0) = \frac{\kappa}{\gamma_0}\left(\frac{a_0^3}{\mu_p}\right)^{1/2}\dot{\alpha}_0$$

$$s^*(0) = 0, \qquad \frac{ds^*}{dt^*}(0) = 1$$

where

$$0 \leq t^* < \infty$$

Furthermore,

$$h^* = \frac{h}{\dfrac{\gamma_0}{\kappa}(\mu_p a_0)^{1/2}}$$

and

$$\epsilon^* = 3\frac{\mu_p^2}{c^2 h_0^2}$$

Now, with the elimination of the cumbersome star notation, these equations become

$$\frac{d^2 r}{ds^2} - r\left(\frac{d\alpha}{ds}\right)^2 + \frac{1}{r^2}\left(1 + \epsilon\frac{h_0^2}{r^2}\right) = 0$$

$$r^2\frac{d\alpha}{ds} = h_0$$

$$\frac{dt}{ds} = \frac{\gamma_0}{\gamma}$$

subject to the initial conditions

$$r(0) = r_0, \qquad \frac{dr}{dt}(0) = \dot{r}_0$$

$$\alpha(0) = 0, \qquad \frac{d\alpha}{dt}(0) = \dot{\alpha}_0$$

$$s(0) = 0, \qquad \frac{ds}{dt}(0) = 1$$

where

$$0 \leq t < \infty$$

and where the units of the variables and initial conditions are now as follows:

[distances] = semimajor axes of the unperturbed motion a_0,

[speeds] = γ_0/κ circular-satellite speeds of the unperturbed motion at $r = a_0, (\gamma_0/\kappa)\,\bar{s}$,

[angular speeds] = radians per $(\kappa/\gamma_0)(1/2\pi)$ periods of the unperturbed motion, rad$/(\kappa/\gamma_0)(P_0/2\pi)$,

[coordinate time] = $(\kappa/\gamma_0)(1/2\pi)$ periods of the unperturbed motion, $(\kappa/\gamma_0)(P_0/2\pi)$,

and

[proper time] = $c/2\pi$ periods of the unperturbed motion, $cP_0/2\pi$.

Note that in normalized variables and initial conditions

$$\gamma = 1 - \frac{1}{c^2}\frac{1}{r}, \qquad \gamma_0 = 1 - \frac{1}{c^2}\frac{1}{r_0}$$

$$\kappa = \left(\gamma_0 + \frac{1}{c^2}\dot{r}_0^2 + \frac{\gamma_0}{c^2}\frac{h^2}{r_0^2}\right)^{1/2}$$

$$h_0 = h = r_0^2\dot{\alpha}_0$$

and

$$\epsilon = 3\frac{1}{c^2 h_0^2}$$

where the speed of light c is now expressed in units of circular-satellite speeds of the unperturbed motion at $r = a_0$ (this is equivalent to setting

$$c^* = \frac{c}{\left(\frac{\mu_p}{a_0}\right)^{1/2}}$$

and then dropping the star notation).

Finally, letting $r = h^2/\eta$, using $h_0 = h$, and using $d\alpha/ds = h/r^2 = \eta^2/h^3$, one obtains the differential equations of motion and initial conditions

$$\frac{d^2\eta}{d\alpha^2} + \eta - 1 - \epsilon\eta^2 = 0$$

$$\frac{ds}{d\alpha} = \frac{h^3}{\eta^2}$$

$$\frac{dt}{d\alpha} = \frac{\gamma_0 h^3}{\gamma\eta^2}$$

and

$$\eta(0) = \frac{h^2}{r_0}, \qquad \frac{d\eta}{d\alpha}(0) = -h\dot{r}_0$$

$$s(0) = 0, \qquad \frac{ds}{d\alpha}(0) = \frac{1}{\dot{\alpha}_0}$$

$$t(0) = 0, \qquad \frac{dt}{d\alpha}(0) = \frac{1}{\dot{\alpha}_0}$$

where

$$\gamma = 1 - \frac{1}{c^2 h^2}\eta$$

and

$$0 \leq \alpha < \infty$$

In comparing the preceding differential equations of motion and initial conditions with those given in Eqs. (278–281), it is apparent that the first two differential equations of relativistic motion and their respective initial conditions are the same as Eqs. (278–281) if the proper time s is interpreted as being analogous to the Newtonian time t. Hence, the solution to Eqs. (278–281) represents (1) the relativistic motion of a point mass moving around a central point mass so massive as to dominate the system and (2) the motion of a satellite in an equatorial orbit about an oblate body. Note that in the case of the relativistic motion, the solution to the differential equation for the coordinate time t must be obtained in addition to the solution to Eqs. (278–281).

4. Analytical integration of equations of motion. The quasilinear differential equation given in Eq. (278) can be solved by using the substitution

$$\xi = \frac{d\eta}{d\alpha} \tag{283}$$

Differentiating Eq. (283) with respect to α yields

$$\frac{d^2\eta}{d\alpha^2} = \frac{d\xi}{d\eta}\frac{d\eta}{d\alpha} = \xi\frac{d\xi}{d\eta}$$

and thus Eq. (278) becomes

$$\xi\frac{d\xi}{d\eta} + \eta - 1 - \epsilon\eta^2 = 0$$

or

$$\xi d\xi = (\epsilon\eta^2 - \eta + 1)\, d\eta \tag{284}$$

Integrating Eq. (284) yields

$$\xi^2 = \frac{2}{3}\epsilon\eta^3 - \eta^2 + 2\eta + E_T \tag{285}$$

where E_T is a constant of integration. Evaluating Eq. (285) at $\alpha = 0$ and using Eqs. (280) and (283), one obtains

$$E_T = 2h^2\left[\frac{1}{2}\left(\dot{r}_0^2 + \frac{h^2}{r_0^2}\right) - \left(\frac{1}{r_0} + \frac{1}{3}\epsilon\frac{h^4}{r_0^3}\right)\right] \tag{286}$$

Note that, from Eqs. (272), (274), (276), and (283),

$$\xi^2 + \eta^2 = h^2\left(\frac{dr}{dt}\right)^2 + \frac{h^4}{r^2} = h^2\left[\left(\frac{dr}{dt}\right) + \left(r\frac{d\alpha}{dt}\right)^2\right]$$

or

$$\xi^2 + \eta^2 = (2h^2)\,(\text{normalized kinetic energy})$$

and that, from Eqs. (271) and (274),

$$2\left(\eta + \frac{1}{3}\epsilon\eta^3\right) = 2h^2\frac{1}{r}\left[1 + \frac{1}{2}J_2\left(\frac{a_p}{r}\right)^2\right]$$

or

$$2\left(\eta + \frac{1}{3}\,\epsilon\eta^3\right) = -(2h^2)\,(\text{normalized potential energy})$$

so that Eq. (285) can be rewritten as

$$\text{normalized kinetic energy} + \text{normalized potential energy} = \frac{1}{2}\frac{E_T}{h^2}$$

that is,

$$\text{normalized total energy} = \frac{1}{2}\frac{E_T}{h^2}$$

Thus, since the quantity h is constant as shown in Eq. (273), the normalized total energy of the satellite is constant (conservation of total energy). Note that the quantity E_T can be thought of as a normalized total energy as well.

Introducing Eq. (283) into Eq. (285) yields

$$\frac{d\eta}{d\alpha} = \pm\left(\frac{2}{3}\,\epsilon\eta^3 - \eta^2 + 2\eta + E_T\right)^{1/2} \tag{287}$$

from which

$$\alpha = \pm\int_{\eta_0}^{\eta} \frac{d\zeta}{\left(\frac{2}{3}\,\epsilon\zeta^3 - \zeta^2 + 2\zeta + E_T\right)^{1/2}} \tag{288}$$

where, from the first of Eqs. (280),

$$\eta_0 \overset{\Delta}{=} \eta(0) = \frac{h^2}{r_0} \tag{289}$$

In addition, from Eqs. (279) and (287),

$$\frac{dt}{d\eta} = \frac{dt}{d\alpha}\frac{d\alpha}{d\eta} = \pm\frac{h^3}{\eta^2}\,\frac{1}{\left(\frac{2}{3}\,\epsilon\eta^3 - \eta^2 + 2\eta + E_T\right)^{1/2}}$$

and therefore, using the first of Eqs. (280) and (281),

$$t = \pm h^3\int_{\eta_0}^{\eta} \frac{d\zeta}{\zeta^2\left(\frac{2}{3}\,\epsilon\zeta^3 - \zeta^2 + 2\zeta + E_T\right)^{1/2}} \tag{290}$$

The proper signs in Eqs. (287), (288), and (290) depend upon the behavior of $\eta = h^2/r$; that is, upon whether r is decreasing or increasing, as is shown in Appendix B.

The integrals in Eqs. (288) and (290) are expressed in terms of Legendre normal elliptic integrals of the first, second, and third kinds in Appendix B. There exist complete and detailed tables of these normal elliptic integrals, and their properties have been discussed exhaustively in the literature (for example, see Refs. 1374 and 1375). Elliptic integrals are basically as easy to use as circular functions. However, since they do not arise in the sciences as often as circular functions, they are not as familiar to most investigators.

A summary of the integrals in Eqs. (288) and (290) in terms of Legendre normal elliptic integrals, as well as pertinent constants and relationships, is given in Tables B-1 and B-2 in Appendix B. The functions $F(\hat{\phi}\backslash\hat{m})$, $E(\hat{\phi}\backslash\hat{m})$, $\Pi(\hat{n};\hat{\phi}\backslash\hat{m})$ are the Legendre normal elliptic integrals of the first, second, and third kinds, respectively. It should be noted that since $\hat{\phi}_0$, $\hat{n}$, and $\hat{m}$ are constants determined by the initial conditions, the quantities $F(\hat{\phi}_0\backslash\hat{m})$, $E(\hat{\phi}_0\backslash\hat{m})$, and $\Pi(\hat{n};\hat{\phi}_0\backslash\hat{m})$ are also constants. As a result, the right ascension α of the satellite consists of constants and a Legendre normal elliptic integral of the first kind, whereas the time t associated with the position of the satellite consists of constants, powers of r, and Legendre normal elliptic integrals of the first, second, and third kinds. Thus, $\alpha(r;\epsilon)$ and $t(r;\epsilon)$ are developed where $r = h^2/\eta$ is the independent variable. In order to determine actual numerical data, values of the elliptic integrals for various values of r must be obtained from tables of Legendre normal elliptic integrals of the first, second, and third kinds or from numerical determinations on a digital computer using elliptic integral computer programs. The latter are used in this investigation. For an insight into the tables of elliptic integrals, see Ref. 1375.

Note that the reference coordinates are $\eta(\alpha;\epsilon)$ and $t(\alpha;\epsilon)$, where α is the reference independent variable, but that Eqs. (288) and (290) yield $\alpha(r;\epsilon)$ and $t(r;\epsilon)$ where r is the independent variable instead. Actually, Eq. (288) can be expressed in terms of a Jacobian elliptic function in such a manner that $\eta(\alpha;\epsilon)$ is obtained (Refs. 112 and 172). Then, introducing $\eta(\alpha;\epsilon)$ into (279) and integrating (a difficult task), $t(\alpha;\epsilon)$ can be obtained. Thus, although $\alpha(r;\epsilon)$ and $t(r;\epsilon)$ are developed herein, it may be assumed that $\eta(\alpha;\epsilon)$ and $t(\alpha;\epsilon)$ can be obtained as well.

Suppose noncircular quasi-periodic motion exists ($h^4 > 6J_2a_p^2$ in normalized quantities; see Ref. 172) so that $r_{\min}$ and $r_{\max}$ both exist, and suppose the initial position of the satellite occurs after or at perifocal passage but before apofocal passage (see Fig. 18) so that $\dot{r}_0 > 0$ or

$$\dot{r}_0 = 0 \qquad \text{with} \qquad \frac{d^2 r}{dt^2}(0) > 0$$

then the equations in the first column of Table B-1, as well as

$$\frac{d\alpha}{dt} = \frac{h}{r^2}$$

represent the motion of the satellite until apofocal passage is reached ($r = r_{max}$). Actual numerical data are obtained by entering increasing values of r into these equations until r_{max} is reached. The number of data points is arbitrary and is not determined by an accuracy criterion, as in the case of the numerical integration of the differential equations of motion of the satellite. This is the essential advantage of representing the motion of the satellite by elliptic integrals.

Once the satellite attains apofocal passage, a different set of equations represents the motion until perifocal passage is reached ($r = r_{min}$). These equations are those in the last column of Table B-1 but with the initial point, represented by $\overset{\Delta}{\phi}_0$, replaced by the apofocus, represented by $\overset{\Delta}{\phi}_A$. From Tables B-1 and B-2,

$$\sin^2 \overset{\Delta}{\phi}_A = -\frac{\dfrac{h^2}{r_{max}} + \nu}{\overset{\Delta}{m}\left(\sigma_1^2 - \dfrac{h^2}{r_{max}} - \nu\right)}, \qquad \frac{h^2}{r_{max}} = \frac{1}{2\epsilon} - \Lambda_{max}$$

and

$$\nu = -\left(\frac{1}{2\epsilon} - \Lambda_{min}\right)$$

so that

$$\overset{\Delta}{\phi}_A = \sin^{-1}\left[\frac{\Lambda_{max} - \Lambda_{min}}{\overset{\Delta}{m}\left(\sigma_1^2 + \Lambda_{max} - \Lambda_{min}\right)}\right]^{1/2} \tag{291}$$

Thus,

$$\alpha = \alpha_{0A} + \left(\frac{6}{\epsilon\left(\sigma_1^2 + \sigma_3^2\right)}\right)^{1/2}\left[F\left(\overset{\Delta}{\phi}_A \backslash \overset{\Delta}{m}\right) - F\left(\overset{\Delta}{\phi} \backslash \overset{\Delta}{m}\right)\right] \tag{292}$$

$$\begin{aligned} t = t_{0A} + \frac{h^3}{E_T}\Bigg\{ & \left[\left(\frac{2}{3}\epsilon h^2\right)\left(\frac{1}{r_{max}}\right) - 1 + \left(\frac{2}{h^2}\right) r_{max} + \left(\frac{E_T}{h^4}\right) r_{max}^2\right]^{1/2} \\ & - \left[\left(\frac{2}{3}\epsilon h^2\right)\left(\frac{1}{r}\right) - 1 + \left(\frac{2}{h^2}\right) r + \left(\frac{E_T}{h^4}\right) r^2\right]^{1/2} \\ & - \sigma_1^2\left(\frac{2\epsilon}{3\left(\sigma_1^2 + \sigma_3^2\right)}\right)^{1/2}\left[\Pi\left(\overset{\Delta}{m}; \overset{\Delta}{\phi}_A \backslash \overset{\Delta}{m}\right) - \Pi\left(\overset{\Delta}{m}; \overset{\Delta}{\phi} \backslash \overset{\Delta}{m}\right)\right] \\ & + \left[\left(\sigma_1^2 - \nu\right)\left(\frac{2\epsilon}{3\left(\sigma_1^2 + \sigma_3^2\right)}\right)^{1/2} + \frac{\overset{\Delta}{m}}{\nu \overset{\Delta}{n}}\left(\frac{6}{\epsilon\left(\sigma_1^2 + \sigma_3^2\right)}\right)^{1/2}\right]\left[F\left(\overset{\Delta}{\phi}_A \backslash \overset{\Delta}{m}\right) - F\left(\overset{\Delta}{\phi} \backslash \overset{\Delta}{m}\right)\right] \\ & + \frac{1}{\nu}\left(1 - \frac{\overset{\Delta}{m}}{\overset{\Delta}{n}}\right)\left(\frac{6}{\epsilon\left(\sigma_1^2 + \sigma_3^2\right)}\right)^{1/2}\left[\Pi\left(\overset{\Delta}{n}; \overset{\Delta}{\phi}_A \backslash \overset{\Delta}{m}\right) - \Pi\left(\overset{\Delta}{n}; \overset{\Delta}{\phi} \backslash \overset{\Delta}{m}\right)\right]\Bigg\} \end{aligned} \tag{293}$$

and

$$\left.\begin{aligned}\frac{dr}{dt} &= -\frac{1}{h}\left[\left(\frac{2}{3}\epsilon h^6\right)\left(\frac{1}{r^3}\right) - (h^4)\left(\frac{1}{r^2}\right) + (2h^2)\left(\frac{1}{r}\right) + E_T\right]^{1/2} \\ \frac{d\alpha}{dt} &= \frac{h}{r^2}\end{aligned}\right\} \qquad (294)$$

The remaining pertinent expressions are given in the last column of Table B-1 and in Table B-2. The quantity α_{0A} appearing in Eq. (292) and the quantity t_{0A} appearing in Eq. (293) are the differences in the right ascension and the time, respectively, between the initial point and the apofocus as determined from the equations for α and t in the first column of Table B-1 with $\hat{\phi}$ replaced by $\hat{\phi}_A$. Actual numerical data are obtained by entering decreasing values of r in Eqs. (292–294) and the associated expressions in Tables B-1 and B-2 until $r_{\min}$ is reached.

Once the satellite attains perifocal passage, a different set of equations represents the motion until apofocal passage is once again reached. These equations are those in the first column of Table B-1 but with the initial point, represented by $\hat{\phi}_0$, replaced by the perifocus, represented by $\hat{\phi}_P$. From Tables B-1 and B-2,

$$\sin\hat{\phi}_P = \frac{1}{\sigma_2}\left(\frac{h^2}{r_{\min}} + \nu\right)^{1/2}, \qquad \frac{h^2}{r_{\min}} = \frac{1}{2\epsilon} - \Lambda_{\min} \quad \text{and} \quad \nu = -\left(\frac{1}{2\epsilon} - \Lambda_{\max}\right)$$

so that

$$\hat{\phi}_P = \sin^{-1}\left[\frac{1}{\sigma_2}(\Lambda_{\max} - \Lambda_{\min})^{1/2}\right] \qquad (295)$$

Thus,

$$\alpha = \alpha_{0A} + \alpha_{AP} + \frac{1}{\sigma_1}\left(\frac{6}{\epsilon}\right)^{1/2}\left[F\left(\hat{\phi}_P\backslash\hat{m}\right) - F\left(\hat{\phi}\backslash\hat{m}\right)\right] \qquad (296)$$

$$t = t_{0A} + t_{AP} + \frac{h^3}{E_T}\left\{-\left[\left(\frac{2}{3}\epsilon h^2\right)\left(\frac{1}{r_{\min}}\right) - 1 + \left(\frac{2}{h^2}\right)r_{\min} + \left(\frac{E_T}{h^4}\right)r_{\min}^2\right]^{1/2}\right.$$

$$+ \left[\left(\frac{2}{3}\epsilon h^2\right)\left(\frac{1}{r}\right) - 1 + \left(\frac{2}{h^2}\right)r + \left(\frac{E_T}{h^4}\right)r^2\right]^{1/2} - \sigma_1\left(\frac{2\epsilon}{3}\right)^{1/2}\left[E\left(\hat{\phi}_P\backslash\hat{m}\right) - E\left(\hat{\phi}\backslash\hat{m}\right)\right]$$

$$\left. + \left(\frac{\sigma_1^2 - \nu}{\sigma_1}\right)\left(\frac{2\epsilon}{3}\right)^{1/2}\left[F\left(\hat{\phi}_P\backslash\hat{m}\right) - F\left(\hat{\phi}\backslash\hat{m}\right)\right] + \frac{1}{\sigma_1\nu}\left(\frac{6}{\epsilon}\right)^{1/2}\left[\Pi\left(\hat{n};\hat{\phi}_P\backslash\hat{m}\right) - \Pi\left(\hat{n};\hat{\phi}\backslash\hat{m}\right)\right]\right\} \qquad (297)$$

and

$$\left.\begin{aligned}\frac{dr}{dt} &= \frac{1}{h}\left[\left(\frac{2}{3}\epsilon h^6\right)\left(\frac{1}{r^3}\right) - (h^4)\left(\frac{1}{r^2}\right) + (2h^2)\left(\frac{1}{r}\right) + E_T\right]^{1/2} \\ \frac{d\alpha}{dt} &= \frac{h}{r^2}\end{aligned}\right\} \qquad (298)$$

The remaining pertinent expressions are given in the first column of Table B-1 and in Table B-2. The quantity α_{AP} appearing in Eq. (296) and the quantity t_{AP} appearing in Eq. (297) are the differences in the right ascension and the time, respectively, between the apofocus and the perifocus as determined from the equations for α and t in the last column of Table B-1 with $\hat{\phi}_0$ replaced by $\hat{\phi}_A$ and $\hat{\phi}$ replaced by zero. Actual numerical data are obtained by entering increasing values of r into Eqs. (296–298) and associated expressions in Tables B-1 and B-2 until r_{max} is reached.

At this point, the three sets of equations describing the motion of the satellite for a partial revolution (from the initial position to the apofocus) and one complete revolution (from the apofocus to the perifocus to the apofocus) have been described. From these sets of equations, the position and velocity of the satellite after several revolutions can be obtained; that is,

$$\left.\begin{aligned}\alpha_j &= (j-1)(\alpha_{AP} + \alpha_{PA}) + \alpha \\ &\text{and} \\ t_j &= (j-1)(t_{AP} + t_{PA}) + t\end{aligned}\right\} \qquad (299)$$

where

j is the revolution number measured from the first apofocal passage;

α_j and t_j are the right ascension and the time of the satellite, respectively, in the jth revolution;

α_{AP} and t_{AP} are the differences in the right ascension and the time of the satellite, respectively, between the apofocus and the perifocus (previously determined);

α_{PA} and t_{PA} are the differences in the right ascension and the time of the satellite, respectively, between the perifocus and the apofocus as determined from the equations for α and t in the first column of Table B-1 with $\hat{\phi}_0$ replaced by $\hat{\phi}_P$ and $\hat{\phi}$ replaced by zero;

and where the expressions for α, t, dr/dt, and $d\alpha/dt$ are given in Eqs. (292–294) for $dr/dt < 0$ and in Eqs. (296–298) for $dr/dt > 0$.

It should be noted that in one complete revolution, the change in the right ascension of the satellite does not equal 2π radians, $(\alpha_{AP} + \alpha_{PA}) \neq 2\pi$, as in the case of Keplerian motion ($\epsilon = 0$). Furthermore, it is clear from Eqs. (299) that the position and velocity of the satellite in any revolution (other than the first) can be determined from the data of the first complete revolution and the constants α_{0A}, α_{AP}, α_{PA}, t_{0A}, t_{AP}, t_{PA}, and j. Finally, each data point determined from the elliptic integrals is independent of the error of the previous data point, and thus error propagation does not occur as in the case of the numerical integration of the differential equations of motion of the satellite. Consequently, the numerical accuracy of the position and the velocity of the satellite after many revolutions can be controlled by the astrodynamicist. This is the principal reason for restricting the force field and the initial conditions of the satellite. Numerical comparisons in Section V-E-1 between approximate solutions and the exact solution to the motion of the satellite are free from any uncertainty in the data of the exact solution.

In the case where the initial position of the satellite occurs after or at apofocal passage but before perifocal passage (see Fig. 18) so that $\dot{r}_0 < 0$ or

$$\dot{r}_0 = 0 \quad \text{with} \quad \frac{d^2r}{dt^2}(0) < 0$$

the previous discussion must be modified with respect to the order of the sets of equations representing the motion of the satellite. The equations in the last column of Table B-1, as well as

$$\frac{d\alpha}{dt} = \frac{h}{r^2}$$

represent the motion of the satellite from the initial position to the perifocus. From the perifocus to the apofocus, Eqs. (295–298), with $(\alpha_{0A} + \alpha_{AP})$ and $(t_{0A} + t_{0P})$ replaced by α_{0P} and t_{0P}, respectively, represent the motion of the satellite. The quantities α_{0P} and t_{0P} are the differences in the right ascension and the time of the satellite, respectively, between the initial position and the perifocus as determined from the equations for α and t in the last column of Table B-1, with $\hat{\phi}$ replaced by zero. From the apofocus to the perifocus (second perifocal passage), Eqs. (291–294), with α_{0A} and t_{0A} replaced by $(\alpha_{0P} + \alpha_{PA})$ and $(t_{0P} + t_{PA})$, respectively, represent the motion of the satellite. The quantities α_{PA} and t_{PA} are the differences in the right ascension and the time of the satellite, respectively, between the perifocus and the apofocus, as determined from the equations for α and t in the first column of Table B-1, with $\hat{\phi}_0$ replaced by $\hat{\phi}_P$ and $\hat{\phi}$ replaced by zero.

At this point, the three sets of equations describing the motion of the satellite for a partial revolution (from the initial position to the perifocus) and one complete revolution (from the perifocus to the apofocus to the perifocus) have been described. From these sets of equations, the position and velocity of the satellite after several revolutions can be obtained; that is,

$$\left.\begin{aligned} \alpha_j &= (j-1)(\alpha_{PA} + \alpha_{AP}) + \alpha \\ &\text{and} \\ t_j &= (j-1)(t_{PA} + t_{AP}) + t \end{aligned}\right\} \tag{300}$$

where

j is the revolution number measured from the first perifocal passage;

α_j and t_j are the right ascension and the time of the satellite, respectively, in the jth revolution;

α_{PA} and t_{PA} are the differences in the right ascension and the time of the satellite, respectively, between the perifocus and the apofocus (previously determined);

α_{AP} and t_{AP} are the differences in the right ascension and the time of the satellite, respectively, between the apofocus and the perifocus, as determined from the equations for α and t in the last column of Table B-1, with $\hat{\phi}_0$ replaced by $\hat{\phi}_A$ and $\hat{\phi}$ replaced by zero;

and where the expressions for α, t, dr/dt, and $d\alpha/dt$ are given for $dr/dt < 0$ in Eqs. (292–294) with α_{0A} and t_{0A} replaced by $(\alpha_{0P} + \alpha_{PA})$ and $(t_{0P} + t_{PA})$, respectively, and for $dr/dt > 0$ in Eqs. (296–298) with $(\alpha_{0A} + \alpha_{AP})$ and $(t_{0A} + t_{0P})$ replaced by α_{0P} and t_{0P}, respectively. Again, the position and velocity of the satellite in any revolution (other than the first) can be determined from the data of the first complete revolution and certain constants (in this case, α_{0P}, α_{PA}, α_{AP}, t_{0P}, t_{PA}, t_{AP}, and j).

If circular motion exists so that $r_{\min} = r_{\max}$, then Eqs. (B-21) from Appendix B represent the motion of the satellite; that is,

$$r = q_0, \qquad \frac{dr}{dt} = 0 \tag{301}$$

$$\alpha = \left(\frac{h}{q_0^2}\right)t, \qquad \frac{d\alpha}{dt} = \frac{h}{q_0^2} \tag{302}$$

The initial conditions for this case are, from Eqs. (B-20) in Appendix B,

$$\left.\begin{aligned} &r(0) = r_0 = q_0, \quad \frac{dr}{dt}(0) = \dot{r}_0 = 0 \quad \alpha(0) = 0, \\ &\frac{d\alpha}{dt}(0) = \dot{\alpha}_0 = \frac{1}{q_0^{3/2}}\left[1 + \frac{3}{2}J_2\left(\frac{a_p}{q_0}\right)^2\right]^{1/2} \end{aligned}\right\} \tag{303}$$

Note that the expression for $\dot{\alpha}_0$ is obtained by using Eqs. (B-20) in Appendix B in conjunction with Eq. (273) and with Eq. (B-17) in Appendix B; that is,

$$\dot{\alpha}_0 = \frac{h}{r_0^2} = \frac{1}{r_0^{3/2}}(1 + e_0)^{1/2} \tag{304}$$

Since $r_{\min} = r_{\max}$, a complete revolution for this case is defined by a change of 2π radians in the right ascension of the satellite.

Finally, it should be pointed out that Tables B-1 and B-2 present the equations representing the motion of a spacecraft in an escape trajectory as well (second and third columns in Table B-1). However, since the present investigation concerns only quasi-periodic motion, a detailed discussion of the escape trajectory is not presented.

In addition to r, α, dr/dt, $d\alpha/dt$, and t, several additional quantities are useful in satellite theory. These are the total speed of the satellite ds/dt and the osculating parameters of the satellite's orbit, a, e, q, v, and ω (see the Glossary). From Ref. 564 and Section III,

$$\frac{ds}{dt} = \left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\alpha}{dt}\right)^2\right]^{1/2} \tag{305}$$

where

$$\frac{ds}{dt} = \dot{s}, \qquad \frac{dr}{dt} = \dot{r} \tag{306}$$

and

$$\frac{d\alpha}{dt} = \dot{\alpha} = \dot{v} = \frac{p^{1/2}}{r^2} = \frac{h}{r^2} \tag{307}$$

Note that $s' = r' = \alpha' = 0$ but that $v' \neq 0$. From the vis viva integral (Ref. 564) in normalized form,

$$\left(\frac{ds}{dt}\right)^2 = \frac{2}{r} - \frac{1}{a}$$

so that

$$a = \left[\frac{2}{r} - \left(\frac{ds}{dt}\right)^2\right]^{-1} \tag{308}$$

From Ref. 564,

$$p = h^2 = a(1 - e^2)$$

and thus

$$e = \left(1 - \frac{h^2}{a}\right)^{1/2} \tag{309}$$

Also,

$$q = a(1 - e) \tag{310}$$

From Ref. 564 in normalized forms,

$$\frac{dr}{dt} = \frac{1}{h} e \sin v \qquad \text{and} \qquad p = h^2 = r(1 + e\cos v)$$

so that

$$v = \tan^{-1}\left[\frac{h\dfrac{dr}{dt}}{\dfrac{h^2}{r} - 1}\right] \tag{311}$$

where

$$0 < v < \pi \qquad \text{if} \qquad \frac{dr}{dt} > 0$$

$$\pi < v < 2\pi \qquad \text{if} \qquad \frac{dr}{dt} > 0$$

$$v = 0 \qquad \text{if} \qquad \frac{dr}{dt} = 0 \qquad \text{and} \qquad \frac{d^2r}{dt^2} > 0$$

and

$$v = \pi \qquad \text{if} \qquad \frac{dr}{dt} = 0 \qquad \text{and} \qquad \frac{d^2r}{dt^2} < 0$$

Finally, since the line of nodes is physically nonexistent for an equatorial orbit, the longitude of the ascending node can be defined in a suitable manner; for example,

$$\Omega \stackrel{\Delta}{=} 0 \tag{312}$$

so that

$$\ell = \Omega + \omega + v = \omega + v = u \tag{313}$$

and, since $\alpha = \ell$ in this case,

$$\omega = \alpha - v \tag{314}$$

where $0 \leq \omega \leq 2\pi$. It should be noted that if Ω is not defined equal to zero, then ω is replaced by $\varpi = \Omega + \omega$ so that

$$\varpi = \alpha - v$$

that is, the longitude of perifocus is used in place of the argument of perifocus. Equations (312–314) are used herein.

C. Solution Using Variation of Coordinates With Two-Variable Asymptotic Expansions

There are two perturbation theories that use multivariable asymptotic expansions. Either of these perturbation theories can be used to find the solution for the motion of an equatorial satellite about an oblate body. The first theory, the variation of coordinates with multivariable asymptotic expansions, and the second theory, the variation of parameters with multivariable asymptotic expansions, are discussed in Sections IV-D and IV-G, respectively, with regard to the damped linear harmonic oscillator. It is shown in Section IV-H that these theories are very similar, yielding the same approximate solution, but that the variation of parameters is more desirable whenever the differential equations of motion are complex. Since the differential equations of motion for the equatorial satellite about an oblate body are simple, there appears to be no advantage of one theory over the other theory. Consequently, the variation of coordinates with multivariable asymptotic expansions is arbitrarily chosen in this section and applied to the differential equations of motion of an equatorial satellite about an oblate body.

1. *Selection of coordinates and independent variable.* It is desirable to select $\eta(\alpha;\epsilon)$, $t(\alpha;\epsilon)$, $d\eta/d\alpha(\alpha;\epsilon)$, and $dt/d\alpha(\alpha;\epsilon)$ as the coordinates and α as the independent variable, in place of the natural coordinates $r(t;\epsilon)$, $\alpha(t;\epsilon)$, $dr/dt(t;\epsilon)$, and $d\alpha/dt(t;\epsilon)$ and the natural independent variable t.

2. *Equations of motion.* The differential equations of motion and initial conditions in terms of the selected variables are given by Eqs. (278–282); that is,

$$\frac{d^2\eta}{d\alpha^2} + \eta - 1 - \epsilon\eta^2 = 0$$

$$\frac{dt}{d\alpha} = \frac{h^3}{\eta^2}$$

and

$$\eta(0) = \frac{h^2}{r_0}, \frac{d\eta}{d\alpha}(0) = -h\dot{r}_0$$

$$t(0) = 0, \frac{dt}{d\alpha}(0) = \frac{1}{\dot{\alpha}_0}$$

where

$$0 \leqq \alpha < \infty$$

Through the use of Eqs. (273) and (289), the initial conditions can be rewritten as

$$\eta(0) = \eta_0, \frac{d\eta}{d\alpha}(0) = \xi_0 \tag{315}$$

$$t(0) = 0, \frac{dt}{d\alpha}(0) = \frac{h^3}{\eta_0^2} \tag{316}$$

where

$$\eta_0 \triangleq \frac{h^2}{r_0}, \qquad \xi_0 \triangleq -h\dot{r}_0 \tag{317}$$

and

$$p = h^2 = r_0^4\dot{\alpha}_0^2 \tag{318}$$

3. *Development of solution.* The approximate solution is assumed to be of the form

$$\left.\begin{aligned} \eta(\alpha;\epsilon) = H(\bar{\alpha},\tilde{\alpha};\epsilon) &\triangleq h_0(\epsilon)\,\eta^{(0)}(\bar{\alpha},\tilde{\alpha}) \\ &+ h_1(\epsilon)\,\eta^{(1)}(\bar{\alpha},\tilde{\alpha}) \\ &+ h_2(\epsilon)\,\eta^{(2)}(\bar{\alpha},\tilde{\alpha}) \\ &+ O(h_3(\epsilon)) \\ \text{and} \\ t(\alpha;\epsilon) = T(\bar{\alpha},\tilde{\alpha};\epsilon) &\triangleq \tau_0(\epsilon)\,t^{(0)}(\bar{\alpha},\tilde{\alpha}) \\ &+ \tau_1(\epsilon)\,t^{(1)}(\bar{\alpha},\tilde{\alpha}) \\ &+ \tau_2(\epsilon)\,t^{(2)}(\bar{\alpha},\tilde{\alpha}) \\ &+ O(\tau_3(\epsilon)) \end{aligned}\right\} \tag{319}$$

where the sequences of functions $h_j(\epsilon)$ and $\tau_j(\epsilon)$, $j = 0, 1, 2, \cdots$, are asymptotic sequences (see Section II-A-2); where the fast variable $\bar{\alpha}$ is dependent upon α in a manner that expresses a frequency shift; that is,

$$\bar{\alpha} \triangleq \alpha\left[1 + a_1(\epsilon)\,\alpha_1 + a_2(\epsilon)\,\alpha_2 + O(a_3(\epsilon))\right] \tag{320}$$

and where the slow variable is given by

$$\tilde{\alpha} \triangleq A(\epsilon)\,\alpha \tag{321}$$

The sequence of functions $a_j(\epsilon)$, $j = 1, 2, \cdots$, is also an asymptotic sequence, and the α_j, $j = 1, 2, \cdots$, are undetermined constants. The need for two distinct angle variables $\bar{\alpha}$ and $\tilde{\alpha}$ is indicated in Section V-A by the occurrence of the two independent time scales T_1 and T_2 in the physical context of the problem (see the discussion of two time scales with regard to the damped linear harmonic oscillator in Sections IV-A and IV-D-3).

Unfortunately, the correct functional forms of $h_j(\epsilon)$, $\tau_j(\epsilon)$, $a_j(\epsilon)$, and $A(\epsilon)$ are not easily determined and, consequently, a certain amount of experience and foresight is required by the astrodynamicist in the selection of these functions. Let

$$h_j(\epsilon) = \tau_j(\epsilon) = \epsilon^j \tag{322}$$

where $j = 0, 1, 2, \cdots$, and let

$$a_j(\epsilon) = \epsilon^j \qquad \text{and} \qquad A(\epsilon) = \epsilon \tag{323}$$

where $j = 2, 3, \cdots$ and $a_1(\epsilon) = 0$. Then Eqs. (319), (320), and (321) become

$$\eta(\alpha;\epsilon) = H(\bar{\alpha},\tilde{\alpha};\epsilon) = \eta^{(0)}(\bar{\alpha},\tilde{\alpha}) + \epsilon\eta^{(1)}(\bar{\alpha},\tilde{\alpha}) + \epsilon^2\eta^{(2)}(\bar{\alpha},\tilde{\alpha}) + \epsilon^3\eta^{(3)}(\bar{\alpha},\tilde{\alpha}) + O(\epsilon^4) \qquad (324)$$

and

$$t(\alpha;\epsilon) = T(\bar{\alpha},\tilde{\alpha};\epsilon) = t^{(0)}(\bar{\alpha},\tilde{\alpha}) + \epsilon t^{(1)}(\bar{\alpha},\tilde{\alpha}) + \epsilon^2 t^{(2)}(\bar{\alpha},\tilde{\alpha}) + \epsilon^3 t^{(3)}(\bar{\alpha},\tilde{\alpha}) + O(\epsilon^4) \qquad (325)$$

where

$$\bar{\alpha} = \alpha[1 + \epsilon^2\alpha_2 + \epsilon^3\alpha_3 + O(\epsilon^4)] \qquad (326)$$

and

$$\tilde{\alpha} = \epsilon\alpha \qquad (327)$$

Note that $\eta^{(0)}(\bar{\alpha},\tilde{\alpha})$ and $t^{(0)}(\bar{\alpha},\tilde{\alpha})$ can be thought of as representing the reference motion of the satellite, and that $\epsilon\eta^{(1)}(\bar{\alpha},\tilde{\alpha})$, $\epsilon^2\eta^{(2)}(\bar{\alpha},\tilde{\alpha})$, etc., and $\epsilon t^{(1)}(\bar{\alpha},\tilde{\alpha})$, $\epsilon^2 t^{(1)}(\bar{\alpha},\tilde{\alpha})$, etc. can be thought of as the perturbations to the reference motion. Furthermore, note that the reference motion is not necessarily the unperturbed motion of the system, since both $\bar{\alpha}$ and $\tilde{\alpha}$ reflect the effects of the perturbative parameter ϵ.

In accordance with Section II-A-2, the convergence of the asymptotic expansions (324) and (325) is of no interest in this investigation. The important objective is that the approximation of the solution by the two expansions, each consisting of a finite number of terms, illustrates all the essential features of the motion and provides a close numerical approximation to the exact result for all values of α in the range of α, $0 \leq \alpha < \infty$, assuming the parameter ϵ is sufficiently small.

For conciseness, the following notation is adopted (similar to that of Section IV-D-3):

$$\left.\begin{aligned} &\eta_1^{(j)} = \frac{\partial\eta^{(j)}}{\partial\bar{\alpha}}, \quad \eta_2^{(j)} = \frac{\partial\eta^{(j)}}{\partial\tilde{\alpha}}, \quad \eta_{11}^{(j)} = \frac{\partial^2\eta^{(j)}}{\partial\bar{\alpha}^2} \\ &\eta_{12}^{(j)} = \frac{\partial^2\eta^{(j)}}{\partial\bar{\alpha}\partial\tilde{\alpha}} = \eta_{21}^{(j)}, \quad \eta_{22}^{(j)} = \frac{\partial^2\eta^{(j)}}{\partial\tilde{\alpha}^2} \end{aligned}\right\} \qquad (328)$$

$$t_1^{(j)} = \frac{\partial t^{(j)}}{\partial\bar{\alpha}} \quad \text{and} \quad t_2^{(j)} = \frac{\partial t^{(j)}}{\partial\tilde{\alpha}} \qquad (329)$$

where

$$j = 0, 1, 2, \cdots$$

Consider the development of the expansion (324) using the first of the differential equations of motion in Section V-C-2 (Eq. 278). Differentiating Eq. (324) with respect to the right ascension α and using the notation in Eqs. (328), one obtains

$$\begin{aligned} \frac{d\eta}{d\alpha} &= \left(\eta_1^{(0)}\frac{d\bar{\alpha}}{d\alpha} + \eta_2^{(0)}\frac{d\tilde{\alpha}}{d\alpha}\right) + \epsilon\left(\eta_1^{(1)}\frac{d\bar{\alpha}}{d\alpha} + \eta_2^{(1)}\frac{d\tilde{\alpha}}{d\alpha}\right) \\ &+ \epsilon^2\left(\eta_1^{(2)}\frac{d\bar{\alpha}}{d\alpha} + \eta_2^{(2)}\frac{d\tilde{\alpha}}{d\alpha}\right) + \epsilon^3\left(\eta_1^{(3)}\frac{d\bar{\alpha}}{d\alpha} + \eta_2^{(3)}\frac{d\tilde{\alpha}}{d\alpha}\right) + O(\epsilon^4) \end{aligned} \qquad (330)$$

Differentiating Eqs. (326) and (327) with respect to α yields

$$\left.\begin{aligned} &\frac{d\bar{\alpha}}{d\alpha} = 1 + \epsilon^2\alpha_2 + \epsilon^3\alpha_3 + O(\epsilon^4) \\ &\text{and} \\ &\frac{d\tilde{\alpha}}{d\alpha} = \epsilon \end{aligned}\right\} \qquad (331)$$

Introducing Eqs. (331) into Eq. (330), and combining like powers of ϵ, one obtains

$$\begin{aligned} \frac{d\eta}{d\alpha} &= \eta_1^{(0)} + \epsilon(\eta_2^{(0)} + \eta_1^{(1)}) + \epsilon^2(\alpha_2\eta_1^{(0)} + \eta_2^{(1)} + \eta_1^{(2)}) \\ &+ \epsilon^3(\alpha_3\eta_1^{(0)} + \alpha_2\eta_1^{(1)} + \eta_2^{(2)} + \eta_1^{(3)}) + O(\epsilon^4) \end{aligned} \qquad (332)$$

Differentiating Eq. (332) with respect to the right ascension α, using Eqs. (328) and (331), and combining like powers of ϵ, one obtains

$$\frac{d^2\eta}{d\alpha^2} = \left[\frac{\partial}{\partial\bar{\alpha}}\left(\frac{d\eta}{d\alpha}\right)\right]\frac{d\bar{\alpha}}{d\alpha} + \left[\frac{\partial}{\partial\tilde{\alpha}}\left(\frac{d\eta}{d\alpha}\right)\right]\frac{d\tilde{\alpha}}{d\alpha}$$

or

$$\begin{aligned} \frac{d^2\eta}{d\alpha^2} &= \eta_{11}^{(0)} + \epsilon(2\eta_{12}^{(0)} + \eta_{11}^{(1)}) \\ &+ \epsilon^2(2\alpha_2\eta_{11}^{(0)} + \eta_{22}^{(0)} + 2\eta_{12}^{(1)} + \eta_{11}^{(2)}) \\ &+ \epsilon^3(2\alpha_3\eta_{11}^{(0)} + 2\alpha_2\eta_{12}^{(0)} + 2\alpha_2\eta_{11}^{(1)} \\ &+ \eta_{22}^{(1)} + 2\eta_{12}^{(2)} + \eta_{11}^{(3)}) + O(\epsilon^4) \end{aligned} \qquad (333)$$

In addition, from Eq. (324),

$$\eta^2 = (\eta^{(0)})^2 + \epsilon(2\eta^{(0)}\eta^{(1)}) + \epsilon^2[(\eta^{(1)})^2 + 2\eta^{(0)}\eta^{(2)}] + \epsilon^3(2\eta^{(1)}\eta^{(2)} + 2\eta^{(0)}\eta^{(3)}) + O(\epsilon^4) \tag{334}$$

Introducing Eqs. (324), (333), and (334) into the differential equation of motion for $\eta(\alpha;\epsilon)$, Eq. (278), and combining like powers of ϵ, one obtains

$$\begin{aligned}(\eta^{(0)}_{11} + \eta^{(0)} - 1) &+ \epsilon[\eta^{(1)}_{11} + \eta^{(1)} - (\eta^{(0)})^2 + 2\eta^{(0)}_{12}] \\ &+ \epsilon^2[\eta^{(2)}_{11} + \eta^{(2)} - 2\eta^{(0)}\eta^{(1)} + 2\eta^{(1)}_{12} + 2\alpha_2\eta^{(0)}_{11} + \eta^{(0)}_{22}] \\ &+ \epsilon^3[\eta^{(3)}_{11} + \eta^{(3)} - 2\eta^{(0)}\eta^{(2)} + 2\eta^{(2)}_{12} - (\eta^{(1)})^2 \\ &+ 2\alpha_2\eta^{(1)}_{11} + \eta^{(1)}_{22} + 2\alpha_3\eta^{(0)}_{11} + 2\alpha_2\eta^{(0)}_{12}] \\ &+ O(\epsilon^4) = 0\end{aligned} \tag{335}$$

Since the expansion (335) must hold (at least in an asymptotic sense) for arbitrary values of the perturbative parameter ϵ, the coefficients of the powers of ϵ in Eq. (335) must separately equal zero. Thus,

$$\eta^{(0)}_{11} + \eta^{(0)} = 1 \tag{336}$$

$$\eta^{(1)}_{11} + \eta^{(1)} = (\eta^{(0)})^2 - 2\eta^{(0)}_{12} \tag{337}$$

$$\eta^{(2)}_{11} + \eta^{(2)} = 2\eta^{(0)}\eta^{(1)} - 2\eta^{(1)}_{12} - 2\alpha_2\eta^{(0)}_{11} - \eta^{(0)}_{22} \tag{338}$$

$$\begin{aligned}\eta^{(3)}_{11} + \eta^{(3)} = 2\eta^{(0)}\eta^{(2)} &- 2\eta^{(2)}_{12} + (\eta^{(1)})^2 - 2\alpha_2\eta^{(1)}_{11} \\ &- \eta^{(1)}_{22} - 2\alpha_3\eta^{(0)}_{11} - 2\alpha_2\eta^{(0)}_{12}\end{aligned} \tag{339}$$

and so forth. The initial conditions necessary for the complete solution of these partial differential equations are obtained by evaluating Eqs. (324) and (332) at $\alpha = 0$, by using the initial conditions (315), by combining terms of like powers of ϵ, and by setting the coefficients of the powers of ϵ separately equal to zero. Thus, since $\bar{\alpha} = 0$ and $\tilde{\alpha} = 0$ when $\alpha = 0$ from Eqs. (326) and (327),

$$0 = [\eta^{(0)}(0,0) - \eta_0] + \epsilon\eta^{(1)}(0,0) + \epsilon^2\eta^{(2)}(0,0) + \epsilon^3\eta^{(3)}(0,0) + O(\epsilon^4)$$

and

$$\begin{aligned}0 = [\eta^{(0)}_1(0,0) - \xi_0] &+ \epsilon[\eta^{(0)}_2(0,0) + \eta^{(1)}_1(0,0)] \\ &+ \epsilon^2[\alpha_2\eta^{(0)}_1(0,0) + \eta^{(1)}_2(0,0) + \eta^{(2)}_1(0,0)] \\ &+ \epsilon^3[\alpha_3\eta^{(0)}_1(0,0) + \alpha_2\eta^{(1)}_1(0,0) \\ &+ \eta^{(2)}_2(0,0) + \eta^{(3)}_1(0,0)] + O(\epsilon^4)\end{aligned}$$

so that

$$\left.\begin{aligned}\eta^{(0)}(0,0) &= \eta_0 \\ \eta^{(0)}_1(0,0) &= \xi_0\end{aligned}\right\} \tag{340}$$

$$\left.\begin{aligned}\eta^{(1)}(0,0) &= 0 \\ \eta^{(1)}_1(0,0) &= -\eta^{(0)}_2(0,0)\end{aligned}\right\} \tag{341}$$

$$\left.\begin{aligned}\eta^{(2)}(0,0) &= 0 \\ \eta^{(2)}_1(0,0) &= -\alpha_2\eta^{(0)}_1(0,0) - \eta^{(1)}_2(0,0)\end{aligned}\right\} \tag{342}$$

$$\left.\begin{aligned}\eta^{(3)}(0,0) &= 0 \\ \eta^{(3)}_1(0,0) &= -\alpha_3\eta^{(0)}_1(0,0) - \alpha_2\eta^{(1)}_1(0,0) \\ &\quad - \eta^{(2)}_2(0,0)\end{aligned}\right\} \tag{343}$$

and so forth.

The general solution to Eq. (336), through the use of the procedure in Section IV-D-3 and Solution (A-2) in Appendix A, is

$$\eta^{(0)}(\bar{\alpha},\tilde{\alpha}) = \tilde{a}^{(0)}_N(\tilde{\alpha})\cos\bar{\alpha} + \tilde{a}^{(0)}_M(\tilde{\alpha})\sin\bar{\alpha} + 1 \tag{344}$$

Let

$$\tilde{a}^{(0)}_N(\tilde{\alpha}) \triangleq \tilde{e}^{(0)}(\tilde{\alpha})\cos\tilde{\omega}^{(0)}(\tilde{\alpha})$$

and

$$\tilde{a}^{(0)}_M(\tilde{\alpha}) \triangleq \tilde{e}^{(0)}(\tilde{\alpha})\sin\tilde{\omega}^{(0)}(\tilde{\alpha}) \tag{345}$$

so that Eq. (344) becomes

$$\eta^{(0)}(\bar{\alpha},\tilde{\alpha}) = 1 + \tilde{e}^{(0)}(\tilde{\alpha})\cos[\bar{\alpha} - \tilde{\omega}^{(0)}(\tilde{\alpha})] \tag{346}$$

Note that, through the use of Eqs. (274) and (318), Eq. (346) can be written as

$$p = r^{(0)}(\bar{\alpha},\tilde{\alpha})\{1 + \tilde{e}^{(0)}(\tilde{\alpha})\cos[\bar{\alpha} - \tilde{\omega}^{(0)}(\tilde{\alpha})]\} \tag{347}$$

which is similar to the equation of perturbed motion; that is,

$$p = r(\alpha;\epsilon)\{1 + e(\alpha;\epsilon)\cos[\alpha - \omega(\alpha;\epsilon)]\} \tag{348}$$

where $v(\alpha;\epsilon) = [\alpha - \omega(\alpha;\epsilon)]$. At first glance, Eqs. (347) and (348) imply that

$$r(\alpha;\epsilon) = r^{(0)}(\bar{\alpha},\tilde{\alpha}) + \epsilon r^{(1)}(\bar{\alpha},\tilde{\alpha}) + O(\epsilon^2) \tag{349}$$

$$e(\alpha;\epsilon) = \tilde{e}^{(0)}(\tilde{\alpha}) + \epsilon\tilde{e}^{(1)}(\tilde{\alpha}) + O(\epsilon^2) \tag{350}$$

and

$$\omega(\alpha;\epsilon) = \widetilde{\omega}^{(0)}(\widetilde{\alpha}) + \epsilon\widetilde{\omega}^{(1)}(\widetilde{\alpha}) + O(\epsilon^2) \tag{351}$$

However, a careful scrutiny of Eqs. (347) and (348) shows that Eqs. (350) and (351) are incorrect, inasmuch as Eq. (347) contains the fast variable $\overline{\alpha}$ explicitly, whereas Eq. (348) contains the fast variable α explicitly. The proper implication from these equations is

$$e(\alpha;\epsilon)\cos[\alpha - \omega(\alpha;\epsilon)] = \widetilde{e}^{(0)}(\widetilde{\alpha})\cos[\overline{\alpha} - \widetilde{\omega}^{(0)}(\widetilde{\alpha})] + O(\epsilon)$$

Although $\widetilde{e}^{(0)}(\widetilde{\alpha})$ and $\widetilde{\omega}^{(0)}(\widetilde{\alpha})$ are not the zero-order approximations of the instantaneous parameters $e(\alpha;\epsilon)$ and $\omega(\alpha;\epsilon)$, respectively, it is still proper to consider them as zero-order approximations of an eccentricity and an argument of perifocus, respectively, as may be seen by considering the multivariable plane where $\overline{\alpha}$ is the fast variable and the physical plane where α is the fast variable. Figure 19 shows sketches of Eq. (347) and of Eq. (348) to $O(\epsilon)$ where

$$r(\alpha;\epsilon) = r^{(0)}(\alpha;\epsilon) + O(\epsilon)$$

$$e(\alpha;\epsilon) = e^{(0)}(\alpha;\epsilon) + O(\epsilon)$$

and

$$\omega(\alpha;\epsilon) = \omega^{(0)}(\alpha;\epsilon) + O(\epsilon)$$

Now, $\widetilde{e}^{(0)}(\widetilde{\alpha})$ and $\widetilde{\omega}^{(0)}(\widetilde{\alpha})$ are simply the zero-order approximations of the instantaneous eccentricity and argument of perifocus, respectively, in the multivariable plane where $\overline{\alpha}$ is the fast variable, whereas $e^{(0)}(\alpha;\epsilon)$ and $\omega^{(0)}(\alpha;\epsilon)$ are the zero-order approximations of the instantaneous eccentricity and argument of perifocus, respectively, in the physical plane where α is the fast variable. In order to distinguish between these two types of instantaneous parameters, those parameters such as $\widetilde{e}(\alpha;\epsilon)$ and $\widetilde{\omega}(\alpha;\epsilon)$ defined in the multivariable plane will be called instantaneous multivariable parameters, whereas those parameters such as $e(\alpha;\epsilon)$ and $\omega(\alpha;\epsilon)$ defined in the physical plane will be called simply instantaneous parameters as usual (see the related discussion of the functions $b(t;\epsilon)$ and $c(t;\epsilon)$ and the functions $\widetilde{b}(t;\epsilon)$ and $\widetilde{c}(t;\epsilon)$ in Section IV-G-1).

Taking the partial derivative of Eq. (346) with respect to $\overline{\alpha}$ yields

$$\eta_1^{(0)}(\overline{\alpha},\widetilde{\alpha}) = -\widetilde{e}^{(0)}(\widetilde{\alpha})\sin[\overline{\alpha} - \widetilde{\omega}^{(0)}(\widetilde{\alpha})] \tag{352}$$

Evaluating Eqs. (346) and (352) at $\alpha = 0$ ($\overline{\alpha} = 0$ and $\widetilde{\alpha} = 0$) and using Eqs. (340), one obtains

$$\left.\begin{aligned} \eta_0 &= 1 + \widetilde{e}^{(0)}(0)\cos\widetilde{\omega}^{(0)}(0) \\ &\text{and} \\ \xi_0 &= \widetilde{e}^{(0)}(0)\sin\widetilde{\omega}^{(0)}(0) \end{aligned}\right\} \tag{353}$$

from which

$$\widetilde{e}^{(0)}(0) = [(\eta_0 - 1)^2 + \xi_0^2]^{1/2} \overset{\Delta}{=} e_0 \tag{354}$$

and

$$\widetilde{\omega}^{(0)}(0) = \tan^{-1}\left[\frac{\xi_0}{(\eta_0 - 1)}\right] \overset{\Delta}{=} \omega_0 \tag{355}$$

Note that for the initial conditions $\eta_0 = 1$ and $\xi_0 = 0$, $e_0 = 0$ and ω_0 is undefined. However, for these initial conditions, Eqs. (317) and (318) yield

$$\dot{\alpha}_0 = \frac{1}{r_0^{3/2}} \qquad \text{and} \qquad \dot{r}_0 = 0 \tag{356}$$

and from Eq. (265) evaluated at $t = 0$ ($\alpha = 0$) and Eqs. (356),

$$\frac{d^2r}{dt^2}(0) = -\frac{\epsilon}{r_0^2} \tag{357}$$

Now, $d^2r/dt^2(0) < 0$ since $\epsilon > 0$ and $r_0^2 > 0$. Hence, from Eqs. (311) evaluated at $t = 0$ ($\alpha = 0$), for $\dot{r}_0 = 0$ and $d^2r/dt^2(0) < 0$, one obtains $v_0 = \pi$, which corresponds to $\omega_0 = -\pi$ from Eq. (314) evaluated at $t = 0$ ($\alpha = 0$) or $\omega_0 = \pi$, since $0 \leq \omega_0 \leq 2\pi$. Therefore, for the initial conditions $\eta_0 = 1$ and $\xi_0 = 0$,

$$e_0 = 0 \qquad \text{and} \qquad \omega_0 = \pi \tag{358}$$

Taking the partial derivative of Eq. (352) with respect to $\widetilde{\alpha}$ yields

$$\eta_{12}^{(0)}(\overline{\alpha},\widetilde{\alpha}) = -\frac{d\widetilde{e}^{(0)}}{d\widetilde{\alpha}}\sin[\overline{\alpha} - \widetilde{\omega}^{(0)}(\widetilde{\alpha})] + \widetilde{e}^{(0)}(\widetilde{\alpha})\frac{d\widetilde{\omega}^{(0)}}{d\widetilde{\alpha}}\cos[\overline{\alpha} - \widetilde{\omega}^{(0)}(\widetilde{\alpha})] \tag{359}$$

and, from Eq. (346) and trigonometric formulas,

$$[\eta^{(0)}(\overline{\alpha},\widetilde{\alpha})]^2 = 1 + \frac{1}{2}[\widetilde{e}^{(0)}(\widetilde{\alpha})]^2 + 2\widetilde{e}^{(0)}(\widetilde{\alpha})\cos[\overline{\alpha} - \widetilde{\omega}^{(0)}(\widetilde{\alpha})] + \frac{1}{2}[\widetilde{e}^{(0)}(\widetilde{\alpha})]^2\cos 2[\overline{\alpha} - \widetilde{\omega}^{(0)}(\widetilde{\alpha})] \tag{360}$$

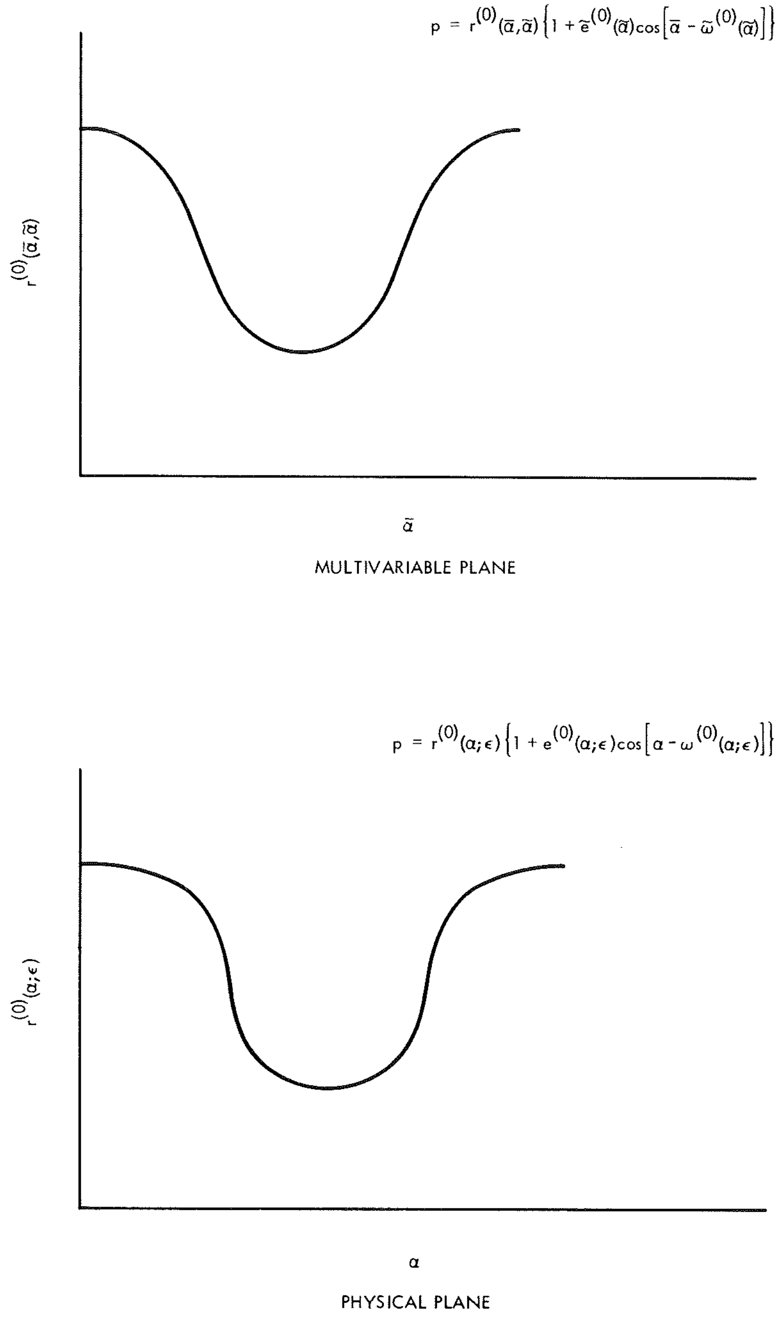

Fig. 19. Instantaneous parameters in the multivariable and physical planes

Introducing Eqs. (359) and (360) into Eq. (337) yields

$$\eta_{11}^{(1)} + \eta^{(1)} = 1 + \frac{1}{2}(\tilde{e}^{(0)})^2 + 2\tilde{e}^{(0)}\left(1 - \frac{d\tilde{\omega}^{(0)}}{d\tilde{\alpha}}\right)\cos(\bar{\alpha} - \tilde{\omega}^{(0)}) + 2\frac{d\tilde{e}^{(0)}}{d\tilde{\alpha}}\sin(\bar{\alpha} - \tilde{\omega}^{(0)}) + \frac{1}{2}(\tilde{e}^{(0)})^2\cos 2(\bar{\alpha} - \tilde{\omega}^{(0)}) \tag{361}$$

Note that the dependence of a function on a variable or variables is not indicated explicitly in Eq. (361). Whenever the dependence of a function upon a variable or variables has been established or is obvious subsequently, the dependence is not usually indicated explicitly.

The general solution to Eq. (361), through the use of the procedure in Section IV-D-3 and Solutions (A-2)–(A-5) in Appendix A, is

$$\eta^{(1)}(\bar{\alpha},\tilde{\alpha}) = \tilde{a}_N^{(1)}(\tilde{\alpha})\cos\bar{\alpha} + \tilde{a}_M^{(1)}(\tilde{\alpha})\sin\bar{\alpha} + 1 + \frac{1}{2}(\tilde{e}^{(0)})^2 + \tilde{e}^{(0)}\left(1 - \frac{d\tilde{\omega}^{(0)}}{d\tilde{\alpha}}\right)\bar{\alpha}\sin(\bar{\alpha} - \tilde{\omega}^{(0)}) - \frac{d\tilde{e}^{(0)}}{d\tilde{\alpha}}\bar{\alpha}\cos(\bar{\alpha} - \tilde{\omega}^{(0)}) - \frac{1}{6}(\tilde{e}^{(0)})^2\cos 2(\bar{\alpha} - \tilde{\omega}^{(0)}) \tag{362}$$

Let

$$\tilde{a}_N^{(1)}(\tilde{\alpha}) = \tilde{e}^{(1)}(\tilde{\alpha})\cos\tilde{\omega}^{(1)}(\tilde{\alpha})$$

and

$$\tilde{a}_M^{(1)}(\tilde{\alpha}) = \tilde{e}^{(1)}(\tilde{\alpha})\sin\tilde{\omega}^{(1)}(\tilde{\alpha}) \tag{363}$$

Now, the solution to the differential equation (278) with $\epsilon = 0$ and subject to the initial conditions given by Eqs. (280) and (281) is simply bounded Keplerian motion (the initial conditions are restricted so that quasi-periodic motion exists; see Section V-A) so that from the first uniformity condition as discussed in Section II-A-3, no secular or Poisson perturbations in the fast angle variable $\bar{\alpha}$ are allowed. Thus, the first uniformity condition requires that

$$\tilde{e}^{(0)}\left(1 - \frac{d\tilde{\omega}^{(0)}}{d\tilde{\alpha}}\right)\sin\tilde{\omega}^{(0)} + \frac{d\tilde{e}^{(0)}}{d\tilde{\alpha}}\cos\tilde{\omega}^{(0)} = 0 \tag{365}$$

and

$$\tilde{e}^{(0)}\left(1 - \frac{d\tilde{\omega}^{(0)}}{d\tilde{\alpha}}\right)\cos\tilde{\omega}^{(0)} - \frac{d\tilde{e}^{(0)}}{d\tilde{\alpha}}\sin\tilde{\omega}^{(0)} = 0 \tag{366}$$

Multiplying Eqs. (365) and (366) by $\sin\tilde{\omega}^{(0)}$ and $\cos\tilde{\omega}^{(0)}$, respectively, and adding the results, one obtains

$$\tilde{e}^{(0)}\left(1 - \frac{d\tilde{\omega}^{(0)}}{d\tilde{\alpha}}\right) = 0 \tag{367}$$

whereas by multiplying Eqs. (365) and (366) by $\cos\tilde{\omega}^{(0)}$ and $\sin\tilde{\omega}^{(0)}$, respectively, and subtracting the results, one obtains

$$\frac{d\tilde{e}^{(0)}}{d\tilde{\alpha}} = 0 \tag{368}$$

It follows from Eqs. (354) and (368) that

$$\tilde{e}^{(0)}(\tilde{\alpha}) = e_0 \tag{369}$$

and from Eqs. (355), (367), and (369), since e_0 is not necessarily zero, that

$$\tilde{\omega}^{(0)}(\tilde{\alpha}) = \omega_0 + \tilde{\alpha} \tag{370}$$

so that Eq. (362) becomes, through the use of trigonometric formulas,

$$\eta^{(1)}(\bar{\alpha},\tilde{\alpha}) = 1 + \frac{1}{2}(\tilde{e}^{(0)})^2 + \tilde{e}^{(1)}\cos(\bar{\alpha} - \tilde{\omega}^{(1)}) - \frac{1}{6}(\tilde{e}^{(0)})^2\cos 2(\bar{\alpha} - \tilde{\omega}^{(0)}) - \left[\tilde{e}^{(0)}\left(1 - \frac{d\tilde{\omega}^{(0)}}{d\tilde{\alpha}}\right)\sin\tilde{\omega}^{(0)} + \frac{d\tilde{e}^{(0)}}{d\tilde{\alpha}}\cos\tilde{\omega}^{(0)}\right]\bar{\alpha}\cos\bar{\alpha} + \left[\tilde{e}^{(0)}\left(1 - \frac{d\tilde{\omega}^{(0)}}{d\tilde{\alpha}}\right)\cos\tilde{\omega}^{(0)} - \frac{d\tilde{e}^{(0)}}{d\tilde{\alpha}}\sin\tilde{\omega}^{(0)}\right]\bar{\alpha}\sin\bar{\alpha} \tag{364}$$

Introducing Eqs. (369) and (370) into Eq. (346) yields the uniformly valid first approximation to the motion of the satellite (reference motion):

$$\eta^{(0)}(\overline{\alpha},\widetilde{\alpha}) = 1 + e_0 \cos(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) \qquad (371)$$

Thus, the reference motion is not the Keplerian motion

$$\eta(\alpha) = 1 + e_0 \cos(\alpha - \omega_0)$$

where

$$\eta(\alpha) = \frac{p}{r(\alpha)} \qquad \text{and} \qquad v = \alpha - \omega_0$$

From Eqs. (364), (365), (366), (369), and (370),

$$\eta^{(1)}(\overline{\alpha},\widetilde{\alpha}) = \left(1 + \frac{1}{2}e_0^2\right) + \widetilde{e}^{(1)} \cos(\overline{\alpha} - \widetilde{\omega}^{(1)})$$
$$- \frac{1}{6} e_0^2 \cos 2(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) \qquad (372)$$

Consider the partial differential equation (338), from which the functions $\widetilde{e}^{(1)}(\widetilde{\alpha})$ and $\widetilde{\omega}^{(1)}(\widetilde{\alpha})$, as well as the undetermined constant α_2 (see Eq. 326), are determined. From Eqs. (371) and (372) and from trigonometric formulas,

$$\eta^{(0)}\eta^{(1)} = \left(1 + \frac{1}{2}e_0^2\right) + e_0\left(1 + \frac{5}{12}e_0^2\right)\cos(\overline{\alpha} - \omega_0 - \widetilde{\alpha})$$
$$- \frac{1}{6} e_0^2 \cos 2(\overline{\alpha} - \omega_0 - \widetilde{\alpha})$$
$$- \frac{1}{12} e_0^3 \cos 3(\overline{\alpha} - \omega_0 - \widetilde{\alpha})$$
$$+ \widetilde{e}^{(1)} \cos(\overline{\alpha} - \widetilde{\omega}^{(1)})$$
$$+ \frac{1}{2} e_0 \widetilde{e}^{(1)} \cos(\omega_0 + \widetilde{\alpha} - \widetilde{\omega}^{(1)})$$
$$+ \frac{1}{2} e_0 \widetilde{e}^{(1)} \cos(2\overline{\alpha} - \omega_0 - \widetilde{\alpha} - \widetilde{\omega}^{(1)}) \qquad (373)$$

Taking the required partial derivatives of Eqs. (371) and (372) with respect to $\overline{\alpha}$ and $\widetilde{\alpha}$ yields

$$\eta_{11}^{(0)} = -e_0 \cos(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) \qquad (374)$$

$$\eta_{22}^{(0)} = -e_0 \cos(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) \qquad (375)$$

and

$$\eta_{12}^{(1)} = -\frac{d\widetilde{e}^{(1)}}{d\widetilde{\alpha}} \sin(\overline{\alpha} - \widetilde{\omega}^{(1)}) + \widetilde{e}^{(1)} \frac{d\widetilde{\omega}^{(1)}}{d\widetilde{\alpha}} \cos(\overline{\alpha} - \widetilde{\omega}^{(1)})$$
$$- \frac{2}{3} e_0^2 \cos 2(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) \qquad (376)$$

where

$$\eta_2^{(0)} = e_0 \sin(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) \qquad (377)$$

and

$$\eta_1^{(1)} = -\widetilde{e}^{(1)} \sin(\overline{\alpha} - \widetilde{\omega}^{(1)}) + \frac{1}{3} e_0^2 \sin 2(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) \qquad (378)$$

Evaluating Eqs. (372), (377), and (378) at $\alpha = 0$ ($\overline{\alpha} = 0$ and $\widetilde{\alpha} = 0$) and using Eq. (341), one obtains

$$0 = \left(1 + \frac{1}{2} e_0^2\right) + \widetilde{e}^{(1)}(0) \cos[-\widetilde{\omega}^{(1)}(0)]$$
$$- \frac{1}{6} e_0^2 \cos 2(-\omega_0)$$

and

$$-\widetilde{e}^{(1)}(0) \sin[-\widetilde{\omega}^{(1)}(0)]$$
$$+ \frac{1}{3} e_0^2 \sin 2(-\omega_0) = -e_0 \sin(-\omega_0)$$

from which

$$\left.\begin{aligned} \widetilde{e}^{(1)}(0) &= \left\{\left[\frac{1}{6} e_0^2 \cos 2\omega_0 - \left(1 + \frac{1}{2} e_0^2\right)\right]^2 + \left[\frac{1}{3} e_0^2 \sin 2\omega_0 + e_0 \sin \omega_0\right]^2\right\}^{1/2} \triangleq e_1 \\ &\text{and} \\ \widetilde{\omega}^{(1)}(0) &= \tan^{-1}\left[\frac{\frac{1}{3} e_0^2 \sin 2\omega_0 + e_0 \sin \omega_0}{\frac{1}{6} e_0^2 \cos 2\omega_0 - \left(1 + \frac{1}{2} e_0^2\right)}\right] \triangleq \omega_1 \end{aligned}\right\} \qquad (379)$$

Introducing Eqs. (373–376) into Eq. (338) yields

$$
\begin{aligned}
\eta_{11}^{(2)} + \eta^{(2)} = 2\left(1 + \frac{1}{2} e_0^2\right) \\
&+ 2e_0\left(\frac{3}{2} + \frac{5}{12} e_0^2 + \alpha_2\right)\cos(\bar{\alpha} - \omega_0 - \tilde{\alpha}) \\
&+ e_0^2 \cos 2(\bar{\alpha} - \omega_0 - \tilde{\alpha}) \\
&- \frac{1}{6} e_0^3 \cos 3(\bar{\alpha} - \omega_0 - \tilde{\alpha}) \\
&+ 2\tilde{e}^{(1)}\left(1 - \frac{d\tilde{\omega}^{(1)}}{d\tilde{\alpha}}\right)\cos(\bar{\alpha} - \tilde{\omega}^{(1)}) \\
&+ 2\frac{d\tilde{e}^{(1)}}{d\tilde{\alpha}} \sin(\bar{\alpha} - \tilde{\omega}^{(1)}) \\
&+ e_0 \tilde{e}^{(1)} \cos(\omega_0 + \tilde{\alpha} - \tilde{\omega}^{(1)}) \\
&+ e_0 \tilde{e}^{(1)} \cos(2\bar{\alpha} - \omega_0 - \tilde{\alpha} - \tilde{\omega}^{(1)})
\end{aligned}
\tag{380}
$$

The general solution to Eq. (380), if one uses Solutions (A-2)–(A-5) in Appendix A, lets

$$
\left.
\begin{aligned}
\tilde{a}_N^{(2)}(\tilde{\alpha}) &= \tilde{e}^{(2)}(\tilde{\alpha}) \cos \tilde{\omega}^{(2)}(\tilde{\alpha}) \\
&\text{and} \\
\tilde{a}_M^{(2)}(\tilde{\alpha}) &= \tilde{e}^{(2)}(\tilde{\alpha}) \sin \tilde{\omega}^{(2)}(\tilde{\alpha})
\end{aligned}
\right\}
\tag{381}
$$

From the first uniformity condition, as discussed in Section II-A-3, the coefficients of the Poisson terms must be set equal to zero; that is,

$$
\left.
\begin{aligned}
& e_0\left(\frac{3}{2} + \frac{5}{12} e_0^2 + \alpha_2\right)\sin(\omega_0 + \tilde{\alpha}) \\
& + \tilde{e}^{(1)}\left(1 - \frac{d\tilde{\omega}^{(1)}}{d\tilde{\alpha}}\right)\sin\tilde{\omega}^{(1)} + \frac{d\tilde{e}^{(1)}}{d\tilde{\alpha}}\cos\tilde{\omega}^{(1)} = 0 \\
& \text{and} \\
& e_0\left(\frac{3}{2} + \frac{5}{12} e_0^2 + \alpha_2\right)\cos(\omega_0 + \tilde{\alpha}) \\
& + \tilde{e}^{(1)}\left(1 - \frac{d\tilde{\omega}^{(1)}}{d\tilde{\alpha}}\right)\cos\tilde{\omega}^{(1)} - \frac{d\tilde{e}^{(1)}}{d\tilde{\alpha}}\sin\tilde{\omega}^{(1)} = 0
\end{aligned}
\right\}
\tag{383}
$$

Equations (383) can be solved more easily if they are transformed back to $\tilde{a}_N^{(1)}(\tilde{\alpha})$ and $\tilde{a}_M^{(1)}(\tilde{\alpha})$. Taking the derivatives of Eqs. (363) with respect to $\tilde{\alpha}$ yields

$$
\left.
\begin{aligned}
\frac{d\tilde{a}_N^{(1)}}{d\tilde{\alpha}} &= \frac{d\tilde{e}^{(1)}}{d\tilde{\alpha}}\cos\tilde{\omega}^{(1)} - \tilde{e}^{(1)}\frac{d\tilde{\omega}^{(1)}}{d\tilde{\alpha}}\sin\tilde{\omega}^{(1)} \\
&\text{and} \\
\frac{d\tilde{a}_M^{(1)}}{d\tilde{\alpha}} &= \frac{d\tilde{e}^{(1)}}{d\tilde{\alpha}}\sin\tilde{\omega}^{(1)} + \tilde{e}^{(1)}\frac{d\tilde{\omega}^{(1)}}{d\tilde{\alpha}}\cos\tilde{\omega}^{(1)}
\end{aligned}
\right\}
\tag{384}
$$

---

and uses trigonometric formulas, is

$$
\begin{aligned}
\eta^{(2)}(\bar{\alpha}, \tilde{\alpha}) = {} & 2\left(1 + \frac{1}{2} e_0^2\right) + e_0 \tilde{e}^{(1)}\cos(\omega_0 + \tilde{\alpha} - \tilde{\omega}^{(1)}) + \tilde{e}^{(2)}\cos(\bar{\alpha} - \tilde{\omega}^{(2)}) - \frac{1}{3} e_0^2 \cos 2(\bar{\alpha} - \omega_0 - \tilde{\alpha}) \\
& + \frac{1}{48} e_0^3 \cos 3(\bar{\alpha} - \omega_0 - \tilde{\alpha}) - \frac{1}{3} e_0 \tilde{e}^{(1)} \cos(2\bar{\alpha} - \omega_0 - \tilde{\alpha} - \tilde{\omega}^{(1)}) \\
& - \left[e_0\left(\frac{3}{2} + \frac{5}{12} e_0^2 + \alpha_2\right)\sin(\omega_0 + \tilde{\alpha}) + \tilde{e}^{(1)}\left(1 - \frac{d\tilde{\omega}^{(1)}}{d\tilde{\alpha}}\right)\sin\tilde{\omega}^{(1)} + \frac{d\tilde{e}^{(1)}}{d\tilde{\alpha}}\cos\tilde{\omega}^{(1)}\right]\bar{\alpha}\cos\bar{\alpha} \\
& + \left[e_0\left(\frac{3}{2} + \frac{5}{12} e_0^2 + \alpha_2\right)\cos(\omega_0 + \tilde{\alpha}) + \tilde{e}^{(1)}\left(1 - \frac{d\tilde{\omega}^{(1)}}{d\tilde{\alpha}}\right)\cos\tilde{\omega}^{(1)} - \frac{d\tilde{e}^{(1)}}{d\tilde{\alpha}}\sin\tilde{\omega}^{(1)}\right]\bar{\alpha}\sin\bar{\alpha}
\end{aligned}
\tag{382}
$$

Introducing Eqs. (363) and (384) into Eq. (383) yields

$$\frac{d\tilde{a}_N^{(1)}}{d\tilde{\alpha}} = -\tilde{a}_M^{(1)} - e_0\left(\frac{3}{2} + \frac{5}{12}e_0^2 + \alpha_2\right)\sin(\omega_0 + \tilde{\alpha}) \tag{385}$$

and

$$\frac{d\tilde{a}_M^{(1)}}{d\tilde{\alpha}} = \tilde{a}_N^{(1)} + e_0\left(\frac{3}{2} + \frac{5}{12}e_0^2 + \alpha_2\right)\cos(\omega_0 + \tilde{\alpha}) \tag{386}$$

Differentiating Eqs. (385) and (386) with respect to $\tilde{\alpha}$ yields

$$\frac{d^2\tilde{a}_N^{(1)}}{d\tilde{\alpha}^2} = -\frac{d\tilde{a}_M^{(1)}}{d\tilde{\alpha}} - e_0\left(\frac{3}{2} + \frac{5}{12}e_0^2 + \alpha_2\right)\cos(\omega_0 + \tilde{\alpha}) \tag{387}$$

and

$$\frac{d^2\tilde{a}_M^{(1)}}{d\tilde{\alpha}^2} = \frac{d\tilde{a}_N^{(1)}}{d\tilde{\alpha}} - e_0\left(\frac{3}{2} + \frac{5}{12}e_0^2 + \alpha_2\right)\sin(\omega_0 + \tilde{\alpha}) \tag{388}$$

Introducing Eq. (386) into Eq. (387) and Eq. (385) into Eq. (388) yields the required equations

$$\left.\begin{aligned} &\frac{d^2\tilde{a}_N^{(1)}}{d\tilde{\alpha}^2} + \tilde{a}_N^{(1)} = -2e_0\left(\frac{3}{2} + \frac{5}{12}e_0^2 + \alpha_2\right)\cos(\omega_0 + \tilde{\alpha}) \\ &\text{and} \\ &\frac{d^2\tilde{a}_M^{(1)}}{d\tilde{\alpha}^2} + \tilde{a}_M^{(1)} = -2e_0\left(\frac{3}{2} + \frac{5}{12}e_0^2 + \alpha_2\right)\sin(\omega_0 + \tilde{\alpha}) \end{aligned}\right\} \tag{389}$$

The solutions to Eqs. (389), through the use of Solutions (A-3) and (A-4) in Appendix A, are

$$\tilde{a}_N^{(1)}(\tilde{\alpha}) = C_1\cos\tilde{\alpha} + C_2\sin\tilde{\alpha} - e_0\left(\frac{3}{2} + \frac{5}{12}e_0^2 + \alpha_2\right)\tilde{\alpha}\sin(\omega_0 + \tilde{\alpha}) \tag{390}$$

and

$$\tilde{a}_M^{(1)}(\tilde{\alpha}) = C_3\cos\tilde{\alpha} + C_4\sin\tilde{\alpha} + e_0\left(\frac{3}{2} + \frac{5}{12}e_0^2 + \alpha_2\right)\tilde{\alpha}\cos(\omega_0 + \tilde{\alpha}) \tag{391}$$

where C_1, C_2, C_3, and C_4 are constants determined from the initial conditions.

Differentiating Eq. (390) with respect to $\tilde{\alpha}$ yields

$$\frac{d\tilde{a}_N^{(1)}}{d\tilde{\alpha}} = -C_1\sin\tilde{\alpha} + C_2\cos\tilde{\alpha} - e_0\left(\frac{3}{2} + \frac{5}{12}e_0^2 + \alpha_2\right)[\sin(\omega_0 + \tilde{\alpha}) + \tilde{\alpha}\cos(\omega_0 + \tilde{\alpha})] \tag{392}$$

Introducing Eqs. (391) and (392) into Eq. (385) yields

$$(C_2 + C_3)\cos\tilde{\alpha} - (C_1 - C_4)\sin\tilde{\alpha} = 0$$

so that, since $\tilde{\alpha}$ is not a constant,

$$C_2 = -C_3 \qquad \text{and} \qquad C_1 = C_4 \tag{393}$$

Introducing Eqs. (393) into Eqs. (390) and (391) yields

$$\left.\begin{aligned} &\tilde{a}_N^{(1)}(\tilde{\alpha}) = C_1\cos\tilde{\alpha} - C_3\sin\tilde{\alpha} - e_0\left(\frac{3}{2} + \frac{5}{12}e_0^2 + \alpha_2\right)\tilde{\alpha}\sin(\omega_0 + \tilde{\alpha}) \\ &\text{and} \\ &\tilde{a}_M^{(1)}(\tilde{\alpha}) = C_3\cos\tilde{\alpha} + C_1\sin\tilde{\alpha} + e_0\left(\frac{3}{2} + \frac{5}{12}e_0^2 + \alpha_2\right)\tilde{\alpha}\cos(\omega_0 + \tilde{\alpha}) \end{aligned}\right\} \tag{394}$$

Now, evaluating Eqs. (363) at $\tilde{\alpha} = 0$ and using Eqs. (379), one obtains

$$\left.\begin{aligned} &\tilde{a}_N^{(1)}(0) = e_1\cos\omega_1 \\ \text{and} \quad &\tilde{a}_M^{(1)}(0) = e_1\sin\omega_1 \end{aligned}\right\} \tag{395}$$

Thus, evaluating Eqs. (394) at $\tilde{\alpha} = 0$ and using Eqs. (395), one obtains

$$e_1 \cos \omega_1 = C_1 \qquad \text{and} \qquad e_1 \sin \omega_1 = C_3 \tag{396}$$

It follows from Eqs. (394) and (396) and from trigonometric formulas that

$$\left.\begin{aligned} \tilde{a}_N^{(1)}(\tilde{\alpha}) &= e_1 \cos(\omega_1 + \tilde{\alpha}) - e_0 \left(\frac{3}{2} + \frac{5}{12} e_0^2 + \alpha_2\right) \tilde{\alpha} \sin(\omega_0 + \tilde{\alpha}) \\ &\text{and} \\ \tilde{a}_M^{(1)}(\tilde{\alpha}) &= e_1 \sin(\omega_1 + \tilde{\alpha}) + e_0 \left(\frac{3}{2} + \frac{5}{12} e_0^2 + \alpha_2\right) \tilde{\alpha} \cos(\omega_0 + \tilde{\alpha}) \end{aligned}\right\} \tag{397}$$

Finally, from trigonometric formulas and from Eqs. (363) and (397),

$$\begin{aligned} \tilde{e}^{(1)} \cos(\bar{\alpha} - \tilde{\omega}^{(1)}) &= \tilde{e}^{(1)} \cos\bar{\alpha} \cos\tilde{\omega}^{(1)} + \tilde{e}^{(1)} \sin\bar{\alpha} \sin\tilde{\omega}^{(1)} = \tilde{a}_N^{(1)} \cos\bar{\alpha} + \tilde{a}_M^{(1)} \sin\bar{\alpha} \\ &= e_1 \cos\bar{\alpha} \cos(\omega_1 + \tilde{\alpha}) - e_0 \left(\frac{3}{2} + \frac{5}{12} e_0^2 + \alpha_2\right) \tilde{\alpha} \cos\bar{\alpha} \sin(\omega_0 + \tilde{\alpha}) \\ &\quad + e_1 \sin\bar{\alpha} \sin(\omega_1 + \tilde{\alpha}) + e_0 \left(\frac{3}{2} + \frac{5}{12} e_0^2 + \alpha_2\right) \tilde{\alpha} \sin\bar{\alpha} \cos(\omega_0 + \tilde{\alpha}) \end{aligned}$$

or

$$\tilde{e}^{(1)} \cos(\bar{\alpha} - \tilde{\omega}^{(1)}) = e_1 \cos(\bar{\alpha} - \omega_1 - \tilde{\alpha}) + e_0 \left(\frac{3}{2} + \frac{5}{12} e_0^2 + \alpha_2\right) \tilde{\alpha} \sin(\bar{\alpha} - \omega_0 - \tilde{\alpha}) \tag{398}$$

Introducing Eq. (398) into Eq. (372) yields

$$\begin{aligned} \eta^{(1)}(\bar{\alpha}, \tilde{\alpha}) &= \left(1 + \frac{1}{2} e_0^2\right) + e_1 \cos(\bar{\alpha} - \omega_1 - \tilde{\alpha}) - \frac{1}{6} e_0^2 \cos 2(\bar{\alpha} - \omega_0 - \tilde{\alpha}) \\ &\quad + e_0 \left(\frac{3}{2} + \frac{5}{12} e_0^2 + \alpha_2\right) \tilde{\alpha} \sin(\bar{\alpha} - \omega_0 - \tilde{\alpha}) \end{aligned} \tag{399}$$

At this point, all of the known conditions on $\eta^{(1)}(\bar{\alpha}, \tilde{\alpha})$ have been applied (without the determination of the constant α_2) except the second uniformity condition as discussed in Section II-A-3. The second uniformity condition requires that

$$\lim_{\epsilon \to 0} \frac{\epsilon \eta^{(1)}(\bar{\alpha}, \tilde{\alpha})}{\eta^{(0)}(\bar{\alpha}, \tilde{\alpha})} = 0 \qquad \text{for all } \alpha \text{ in } 0 \leqq \alpha < \infty$$

so that, from Eqs. (371) and (399),

$$\lim_{\epsilon \to 0} \left\{ \frac{\epsilon\left[\left(1+\frac{1}{2}e_0^2\right)+e_1\cos(\overline{\alpha}-\omega_1-\widetilde{\alpha})-\frac{1}{6}e_0^2\cos 2(\overline{\alpha}-\omega_0-\widetilde{\alpha})\right]}{1+e_0\cos(\overline{\alpha}-\omega_0-\widetilde{\alpha})} + \frac{\epsilon\left[e_0\left(\frac{3}{2}+\frac{5}{12}e_0^2+\alpha_2\right)\widetilde{\alpha}\sin(\overline{\alpha}-\omega_0-\widetilde{\alpha})\right]}{1+e_0\cos(\overline{\alpha}-\omega_0-\widetilde{\alpha})} \right\} = 0 \tag{400}$$

Equation (400) requires that

$$\alpha_2 = -\frac{1}{12}(18+5e_0^2) \tag{401}$$

As a result, the coefficient of the term $\widetilde{\alpha}\sin(\overline{\alpha}-\omega_0-\widetilde{\alpha})$ is zero, and a uniformly valid perturbation $\epsilon\eta^{(1)}(\overline{\alpha},\widetilde{\alpha})$ is obtained (see Section II-A-3). Consequently, from Eqs. (363), (397), and (401),

$$\widetilde{e}^{(1)}(\widetilde{\alpha})\cos\widetilde{\omega}^{(1)}(\widetilde{\alpha}) = e_1\cos(\omega_1+\widetilde{\alpha})$$

and

$$\widetilde{e}^{(1)}(\widetilde{\alpha})\sin\widetilde{\omega}^{(1)}(\widetilde{\alpha}) = e_1\sin(\omega_1+\widetilde{\alpha})$$

so that

$$\widetilde{e}^{(1)}(\widetilde{\alpha}) = e_1 \qquad \text{and} \qquad \widetilde{\omega}^{(1)}(\widetilde{\alpha}) = \omega_1+\widetilde{\alpha} \tag{402}$$

Furthermore, introducing Eq. (402) into Eq. (372) yields

$$\eta^{(1)}(\overline{\alpha},\widetilde{\alpha}) = \left(1+\frac{1}{2}e_0^2\right)+e_1\cos(\overline{\alpha}-\omega_1-\widetilde{\alpha})-\frac{1}{6}e_0^2\cos 2(\overline{\alpha}-\omega_0-\widetilde{\alpha}) \tag{403}$$

and introducing Eqs. (383) and (402) into Eq. (382) yields

$$\eta^{(2)}(\overline{\alpha},\widetilde{\alpha}) = 2\left(1+\frac{1}{2}e_0^2\right)+e_0e_1\cos(\omega_0-\omega_1)+\widetilde{e}^{(2)}\cos(\overline{\alpha}-\widetilde{\omega}^{(2)})-\frac{1}{3}e_0^2\cos 2(\overline{\alpha}-\omega_0-\widetilde{\alpha})$$
$$+\frac{1}{48}e_0^3\cos 3(\overline{\alpha}-\omega_0-\widetilde{\alpha})-\frac{1}{3}e_0e_1\cos(2\overline{\alpha}-\omega-\omega_1-2\widetilde{\alpha}) \tag{404}$$

where, from Eqs. (326) and (401),

$$\overline{\alpha} = \alpha\left[1-\frac{1}{12}\epsilon^2(18+5e_0^2)+\epsilon^3\alpha_3+O(\epsilon^4)\right] \tag{405}$$

Note that if α_2 did not appear in Eq. (399), the nonuniform term could not be eliminated and the asymptotic expansion being developed would not be uniformly valid. The domain of validity would be

$$0 \leqq \alpha < O\left(\frac{1}{\epsilon^2}\right) \tag{406}$$

As a result, if the fast and slow angle variables are chosen as

$$\overline{\alpha} = \alpha \qquad \text{and} \qquad \widetilde{\alpha} = \epsilon\alpha \tag{407}$$

that is, if the constant α_2 is not introduced into the development by means of $\overline{\alpha}$, the resulting asymptotic expansion to $O(\epsilon^2)$ would not be uniformly valid.

Consider the partial differential equation for $\eta^{(3)}(\bar{\alpha}\tilde{\alpha})$ (Eq. 339) from which the functions $\tilde{e}^{(2)}(\tilde{\alpha})$ and $\tilde{\omega}^{(2)}(\tilde{\alpha})$, as well as the undetermined constant α_3 (see Eq. 326), are determined. From Eqs. (371) and (404) and from trigonometric formulas,

$$\begin{aligned}
\eta^{(0)}\eta^{(2)} = &\left[2\left(1+\frac{1}{2}e_0^2\right)+e_0 e_1\cos(\omega_0-\omega_1)\right]+\frac{1}{2}e_0\tilde{e}^{(2)}\cos(\omega_0+\tilde{\alpha}-\tilde{\omega}^{(2)}) \\
&+e_0\left[2\left(1+\frac{5}{12}e_0^2\right)+e_0 e_1\cos(\omega_0-\omega_1)\right]\cos(\bar{\alpha}-\omega_0-\tilde{\alpha}) \\
&-\frac{1}{6}e_0^2 e_1\cos(\bar{\alpha}-\omega_1-\tilde{\alpha})+\tilde{e}^{(2)}\cos(\bar{\alpha}-\tilde{\omega}^{(2)})-\frac{1}{3}e_0^2\left(1-\frac{1}{32}e_0^2\right)\cos 2(\bar{\alpha}-\omega_0-\tilde{\alpha}) \\
&-\frac{1}{3}e_0 e_1\cos(2\bar{\alpha}-\omega_0-\omega_1-2\tilde{\alpha})+\frac{1}{2}e_0\tilde{e}^{(2)}\cos(2\bar{\alpha}-\omega_0-\tilde{\alpha}-\tilde{\omega}^{(2)}) \\
&-\frac{7}{48}e_0^3\cos 3(\bar{\alpha}-\omega_0-\tilde{\alpha})-\frac{1}{6}e_0^2 e_1\cos(3\bar{\alpha}-2\omega_0-\omega_1-3\tilde{\alpha})+\frac{1}{96}e_0^4\cos 4(\bar{\alpha}-\omega_0-\tilde{\alpha})
\end{aligned} \tag{408}$$

and from Eq. (403) and trigonometric formulas,

$$\begin{aligned}
(\eta^{(1)})^2 = &\left[\left(1+\frac{1}{2}e_0^2\right)^2+\frac{1}{72}e_0^4+\frac{1}{2}e_1^2\right]+2e_1\left(1+\frac{1}{2}e_0^2\right)\cos(\bar{\alpha}-\omega_1-\tilde{\alpha}) \\
&-\frac{1}{6}e_0^2 e_1\cos(\bar{\alpha}-2\omega_0+\omega_1-\tilde{\alpha})-\frac{1}{3}e_0^2\left(1+\frac{1}{2}e_0^2\right)\cos 2(\bar{\alpha}-\omega_0-\tilde{\alpha}) \\
&+\frac{1}{2}e_1^2\cos 2(\bar{\alpha}-\omega_1-\tilde{\alpha})-\frac{1}{6}e_0^2 e_1\cos(3\bar{\alpha}-2\omega_0-\omega_1-3\tilde{\alpha})+\frac{1}{72}e_0^4\cos 4(\bar{\alpha}-\omega_0-\tilde{\alpha})
\end{aligned} \tag{409}$$

Taking the required partial derivatives of Eqs. (377), (403), and (404) with respect to $\bar{\alpha}$ and $\tilde{\alpha}$ yields

$$\eta_{12}^{(0)} = e_0\cos(\bar{\alpha}-\omega_0-\tilde{\alpha}) \tag{410}$$

$$\eta_{11}^{(1)} = -e_1\cos(\bar{\alpha}-\omega_1-\tilde{\alpha})+\frac{2}{3}e_0^2\cos 2(\bar{\alpha}-\omega_0-\tilde{\alpha}) \tag{411}$$

$$\eta_{22}^{(1)} = -e_1\cos(\bar{\alpha}-\omega_1-\tilde{\alpha})+\frac{2}{3}e_0^2\cos 2(\bar{\alpha}-\omega_0-\tilde{\alpha}) \tag{412}$$

and

$$\begin{aligned}
\eta_{12}^{(2)} = &-\frac{d\tilde{e}^{(2)}}{d\tilde{\alpha}}\sin(\bar{\alpha}-\tilde{\omega}^{(2)})+\tilde{e}^{(2)}\frac{d\tilde{\omega}^{(2)}}{d\tilde{\alpha}}\cos(\bar{\alpha}-\tilde{\omega}^{(2)})-\frac{4}{3}e_0^2\cos 2(\bar{\alpha}-\omega_0-\tilde{\alpha}) \\
&+\frac{3}{16}e_0^3\cos 3(\bar{\alpha}-\omega_0-\tilde{\alpha})-\frac{4}{3}e_0 e_1\cos(2\bar{\alpha}-\omega_0-\omega_1-2\tilde{\alpha})
\end{aligned} \tag{413}$$

where

$$\eta_1^{(0)} = -e_0 \sin(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) \tag{414}$$

$$\eta_1^{(1)} = -e_1 \sin(\overline{\alpha} - \omega_1 - \widetilde{\alpha}) + \frac{1}{3} e_0^2 \sin 2(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) \tag{415}$$

$$\eta_2^{(1)} = e_1 \sin(\overline{\alpha} - \omega_1 - \widetilde{\alpha}) - \frac{1}{3} e_0^2 \sin 2(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) \tag{416}$$

and

$$\eta_1^{(2)} = -\widetilde{e}^{(2)} \sin(\overline{\alpha} - \widetilde{\omega}^{(2)}) + \frac{2}{3} e_0^2 \sin 2(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) - \frac{1}{16} e_0^3 \sin 3(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) + \frac{2}{3} e_0 e_1 \sin(2\overline{\alpha} - \omega_0 - \omega_1 - 2\widetilde{\alpha}) \tag{417}$$

Evaluating Eqs. (404), (414), (416), and (417) at $\alpha = 0$ ($\overline{\alpha} = 0$ and $\widetilde{\alpha} = 0$) and using Eqs. (342) and (401), one obtains

$$0 = 2\left(1 + \frac{1}{2} e_0^2\right) + e_0 e_1 \cos(\omega_0 - \omega_1) + \widetilde{e}^{(2)}(0) \cos[-\widetilde{\omega}^{(2)}(0)] - \frac{1}{3} e_0^2 \cos 2(-\omega_0) + \frac{1}{48} e_0^3 \cos 3(-\omega_0) - \frac{1}{3} e_0 e_1 \cos(-\omega_0 - \omega_1)$$

and

$$-\widetilde{e}^{(2)}(0) \sin[-\widetilde{\omega}^{(2)}(0)] + \frac{2}{3} e_0^2 \sin 2(-\omega_0) - \frac{1}{16} e_0^3 \sin 3(-\omega_0) + \frac{2}{3} e_0 e_1 \sin(-\omega_0 - \omega_1) = -\frac{1}{12} e_0 (18 + 5e_0^2) \sin(-\omega_0) - e_1 \sin(-\omega_1) + \frac{1}{3} e_0^2 \sin 2(-\omega_0)$$

from which

$$\left.\begin{aligned} \widetilde{e}^{(2)}(0) &= \left\{\left[-2\left(1 + \frac{1}{2} e_0^2\right) + \frac{1}{3} e_0 e_1 \cos(\omega_0 + \omega_1) - e_0 e_1 \cos(\omega_0 - \omega_1) + \frac{1}{3} e_0^2 \cos 2\omega_0 - \frac{1}{48} e_0^3 \cos 3\omega_0\right]^2 \right. \\ &\quad \left. + \left[\frac{2}{3} e_0 e_1 \sin(\omega_0 + \omega_1) + \frac{1}{12} e_0 (18 + 5e_0^2) \sin\omega_0 + \frac{1}{3} e_0^2 \sin 2\omega_0 - \frac{1}{16} e_0^3 \sin 3\omega_0 + e_1 \sin\omega_1\right]^2\right\}^{1/2} \\ &\stackrel{\Delta}{=} e_2 \\ &\text{and} \\ \widetilde{\omega}^{(2)}(0) &= \tan^{-1}\left\{\frac{\frac{2}{3} e_0 e_1 \sin(\omega_0 + \omega_1) + \frac{1}{12} e_0 (18 + 5e_0^2) \sin\omega_0 + \frac{1}{3} e_0^2 \sin 2\omega_0 - \frac{1}{16} e_0^3 \sin 3\omega_0 + e_1 \sin\omega_1}{-2\left(1 + \frac{1}{2} e_0^2\right) + \frac{1}{3} e_0 e_1 \cos(\omega_0 + \omega_1) - e_0 e_1 \cos(\omega_0 - \omega_1) + \frac{1}{3} e_0^2 \cos 2\omega_0 - \frac{1}{48} e_0^3 \cos 3\omega_0}\right\} \\ &\stackrel{\Delta}{=} \omega_2 \end{aligned}\right\} \tag{418}$$

Introducing Eqs. (374), (401), and (408–413) into Eq. (339) yields

$$
\begin{aligned}
\eta_{11}^{(3)} + \eta^{(3)} = {} & 4\left(1+\frac{1}{2}e_0^2\right) + \left(1+\frac{1}{2}e_0^2\right)^2 + \frac{1}{72}e_0^2 + \frac{1}{2}e_1^2 + 2e_0\,e_1\cos(\omega_0-\omega_1) + e_0\,\tilde{e}^{(2)}\cos(\omega_0+\tilde{\alpha}-\tilde{\omega}^{(2)}) \\
& + 2e_0\left[\frac{1}{2}\left(7+\frac{5}{2}e_0^2\right) + e_0\,e_1\cos(\omega_0-\omega_1) + \alpha_3\right]\cos(\bar{\alpha}-\omega_0-\tilde{\alpha}) \\
& - \frac{1}{6}e_0^2\,e_1\cos(\bar{\alpha}-\omega_1-\tilde{\alpha}) - \frac{1}{6}e_0^2\,e_1\cos(\bar{\alpha}-2\omega_0+\omega_1-\tilde{\alpha}) \\
& + 2\tilde{e}^{(2)}\left(1-\frac{d\tilde{\omega}^{(2)}}{d\tilde{\alpha}}\right)\cos(\bar{\alpha}-\tilde{\omega}^{(2)}) + 2\frac{d\tilde{e}^{(2)}}{d\tilde{\alpha}}\sin(\bar{\alpha}-\tilde{\omega}^{(2)}) \\
& + \frac{1}{3}e_0^2\left(9+\frac{59}{48}e_0^2\right)\cos 2(\bar{\alpha}-\omega_0-\tilde{\alpha}) + \frac{1}{2}e_1^2\cos 2(\bar{\alpha}-\omega_1-\tilde{\alpha}) + 2e_0\,e_1\cos(2\bar{\alpha}-\omega_0-\omega_1-2\tilde{\alpha}) \\
& + e_0\,\tilde{e}^{(2)}\cos(2\bar{\alpha}-\omega_0-\tilde{\alpha}-\tilde{\omega}^{(2)}) - \frac{2}{3}e_0^3\cos 3(\bar{\alpha}-\omega_0-\tilde{\alpha}) - \frac{1}{2}e_0^2\,e_1\cos(3\bar{\alpha}-2\omega_0-\omega_1-3\tilde{\alpha}) \\
& + \frac{5}{144}e_0^4\cos 4(\bar{\alpha}-\omega_0-\tilde{\alpha})
\end{aligned}
\tag{419}
$$

The general solution to Eq. (419), using Solutions (A-2) through (A-5) in Appendix A, letting

and

$$
\left.
\begin{aligned}
\tilde{a}_N^{(3)}(\tilde{\alpha}) &= \tilde{e}^{(3)}(\tilde{\alpha})\cos\tilde{\omega}^{(3)}(\tilde{\alpha}) \\
\tilde{a}_M^{(3)}(\tilde{\alpha}) &= \tilde{e}^{(3)}(\tilde{\alpha})\sin\tilde{\omega}^{(3)}(\tilde{\alpha})
\end{aligned}
\right\}
\tag{420}
$$

and using trigonometric formulas, is

$$
\begin{aligned}
\eta^{(3)}(\bar{\alpha},\tilde{\alpha}) = {} & 4\left(1+\frac{1}{2}e_0^2\right) + \left(1+\frac{1}{2}e_0^2\right)^2 + \frac{1}{72}e_0^4 + \frac{1}{2}e_1^2 + 2e_0\,e_1\cos(\omega_0-\omega_1) + e_0\tilde{e}^{(2)}\cos(\omega_0+\tilde{\alpha}-\tilde{\omega}^{(2)}) \\
& + \tilde{e}^{(3)}\cos(\bar{\alpha}-\tilde{\omega}^{(3)}) - \frac{1}{9}e_0^2\left(9+\frac{59}{48}e_0^2\right)\cos 2(\bar{\alpha}-\omega_0-\tilde{\alpha}) - \frac{1}{6}e_1^2\cos 2(\bar{\alpha}-\omega_1-\tilde{\alpha}) \\
& - \frac{2}{3}e_0\,e_1\cos(2\bar{\alpha}-\omega_0-\omega_1-2\tilde{\alpha}) - \frac{1}{3}e_0\,\tilde{e}^{(2)}\cos(2\bar{\alpha}-\omega_0-\tilde{\alpha}-\tilde{\omega}^{(2)}) \\
& + \frac{1}{12}e_0^3\cos 3(\bar{\alpha}-\omega_0-\tilde{\alpha}) + \frac{1}{16}e_0^2\,e_1\cos(3\bar{\alpha}-2\omega_0-\omega_1-3\tilde{\alpha}) - \frac{1}{432}e_0^4\cos 4(\bar{\alpha}-\omega_0-\tilde{\alpha}) \\
& - \left\{e_0\left[\frac{1}{2}\left(7+\frac{5}{2}e_0^2\right) + e_0\,e_1\cos(\omega_0-\omega_1) + \alpha_3\right]\sin(\omega_0+\tilde{\alpha}) - \frac{1}{12}e_0^2\,e_1\sin(\omega_1+\tilde{\alpha})\right. \\
& \left. - \frac{1}{12}e_0^2\,e_1\sin(2\omega_0-\omega_1+\tilde{\alpha}) + \tilde{e}^{(2)}\left(1-\frac{d\tilde{\omega}^{(2)}}{d\tilde{\alpha}}\right)\sin\tilde{\omega}^{(2)} + \frac{d\tilde{e}^{(2)}}{d\tilde{\alpha}}\cos\tilde{\omega}^{(2)}\right\}\bar{\alpha}\cos\bar{\alpha} \\
& + \left\{e_0\left[\frac{1}{2}\left(7+\frac{5}{2}e_0^2\right) + e_0\,e_1\cos(\omega_0-\omega_1) + \alpha_3\right]\cos(\omega_0+\tilde{\alpha}) - \frac{1}{12}e_0^2\,e_1\cos(\omega_1+\tilde{\alpha})\right. \\
& \left. - \frac{1}{12}e_0^2\,e_1\cos(2\omega_0-\omega_1+\tilde{\alpha}) + \tilde{e}^{(2)}\left(1-\frac{d\tilde{\omega}^{(2)}}{d\tilde{\alpha}}\right)\cos\tilde{\omega}^{(2)} - \frac{d\tilde{e}^{(2)}}{d\tilde{\alpha}}\sin\tilde{\omega}^{(2)}\right\}\bar{\alpha}\sin\bar{\alpha}
\end{aligned}
\tag{421}
$$

From the first uniformity condition, as discussed in Section II-A-3, the coefficients of the Poisson terms must be set equal to zero; that is,

$$\left.\begin{aligned} &C_0 \sin(\omega_0 + \tilde{\alpha}) - \frac{1}{12} e_0^2 e_1 \sin(\omega_1 + \tilde{\alpha}) - \frac{1}{12} e_0^2 e_1 \sin(2\omega_0 - \omega_1 + \tilde{\alpha}) + \tilde{e}^{(2)} \left(1 - \frac{d\tilde{\omega}^{(2)}}{d\tilde{\alpha}}\right) \sin \tilde{\omega}^{(2)} + \frac{d\tilde{e}^{(2)}}{d\tilde{\alpha}} \cos \tilde{\omega}^{(2)} = 0 \\ &\text{and} \\ &C_0 \cos(\omega_0 + \tilde{\alpha}) - \frac{1}{12} e_0^2 e_1 \cos(\omega_1 + \tilde{\alpha}) - \frac{1}{12} e_0^2 e_1 \cos(2\omega_0 - \omega_1 + \tilde{\alpha}) + \tilde{e}^{(2)} \left(1 - \frac{d\tilde{\omega}^{(2)}}{d\tilde{\alpha}}\right) \cos \tilde{\omega}^{(2)} - \frac{d\tilde{e}^{(2)}}{d\tilde{\alpha}} \sin \tilde{\omega}^{(2)} = 0 \end{aligned}\right\} \quad (422)$$

where

$$C_0 = e_0 \left[\frac{1}{2}\left(7 + \frac{5}{2} e_0^2\right) + e_0 e_1 \cos(\omega_0 - \omega_1) + \alpha_3\right] \quad (423)$$

Equations (422) can be solved more easily if they are transformed back to $\tilde{a}_N^{(2)}(\tilde{\alpha})$ and $\tilde{a}_M^{(2)}(\tilde{\alpha})$. Taking the derivatives of Eqs. (381) with respect to $\tilde{\alpha}$, we obtain

$$\frac{d\tilde{a}_N^{(2)}}{d\tilde{\alpha}} = \frac{d\tilde{e}^{(2)}}{d\tilde{\alpha}} \cos \tilde{\omega}^{(2)} - \tilde{e}^{(2)} \frac{d\tilde{\omega}^{(2)}}{d\tilde{\alpha}} \sin \tilde{\omega}^{(2)} \qquad \text{and} \qquad \frac{d\tilde{a}_M^{(2)}}{d\tilde{\alpha}} = \frac{d\tilde{e}^{(2)}}{d\tilde{\alpha}} \sin \tilde{\omega}^{(2)} + \tilde{e}^{(2)} \frac{d\tilde{\omega}^{(2)}}{d\tilde{\alpha}} \cos \tilde{\omega}^{(2)} \quad (424)$$

Introducing Eqs. (381) and (424) into Eqs. (422) yields

$$\frac{d\tilde{a}_N^{(2)}}{d\tilde{\alpha}} = -\tilde{a}_M^{(2)} - C_0 \sin(\omega_0 + \tilde{\alpha}) + \frac{1}{12} e_0^2 e_1 \sin(\omega_1 + \tilde{\alpha}) + \frac{1}{12} e_0^2 e_1 \sin(2\omega_0 - \omega_1 + \tilde{\alpha}) \quad (425)$$

and

$$\frac{d\tilde{a}_M^{(2)}}{d\tilde{\alpha}} = \tilde{a}_N^{(2)} + C_0 \cos(\omega_0 + \tilde{\alpha}) - \frac{1}{12} e_0^2 e_1 \cos(\omega_1 + \tilde{\alpha}) - \frac{1}{12} e_0^2 e_1 \cos(2\omega_0 - \omega_1 + \tilde{\alpha}) \quad (426)$$

Differentiating Eqs. (425) and (426) with respect to $\tilde{\alpha}$ yields

$$\frac{d^2 \tilde{a}_N^{(2)}}{d\tilde{\alpha}^2} = -\frac{d\tilde{a}_M^{(2)}}{d\tilde{\alpha}} - C_0 \cos(\omega_0 + \tilde{\alpha}) + \frac{1}{12} e_0^2 e_1 \cos(\omega_1 + \tilde{\alpha}) + \frac{1}{12} e_0^2 e_1 \cos(2\omega_0 - \omega_1 + \tilde{\alpha}) \quad (427)$$

and

$$\frac{d^2 \tilde{a}_M^{(2)}}{d\tilde{\alpha}^2} = \frac{d\tilde{a}_N^{(2)}}{d\tilde{\alpha}} - C_0 \sin(\omega_0 + \tilde{\alpha}) + \frac{1}{12} e_0^2 e_1 \sin(\omega_1 + \tilde{\alpha}) + \frac{1}{12} e_0^2 e_1 \sin(2\omega_0 - \omega_1 + \tilde{\alpha}) \quad (428)$$

Introducing Eq. (426) into Eq. (427) and Eq. (425) into Eq. (428) yields the required equations

$$\left.\begin{aligned} &\frac{d^2 \tilde{a}_N^{(2)}}{d\tilde{\alpha}^2} + \tilde{a}_N^{(2)} = 2C_0 \cos(\omega_0 + \tilde{\alpha}) + \frac{1}{6} e_0^2 e_1 \cos(\omega_1 + \tilde{\alpha}) + \frac{1}{6} e_0^2 e_1 \cos(2\omega_0 - \omega_1 + \tilde{\alpha}) \\ &\text{and} \\ &\frac{d^2 \tilde{a}_M^{(2)}}{d\tilde{\alpha}^2} + \tilde{a}_M^{(2)} = -2C_0 \sin(\omega_0 + \tilde{\alpha}) + \frac{1}{6} e_0^2 e_1 \sin(\omega_1 + \tilde{\alpha}) + \frac{1}{6} e_0^2 e_1 \sin(2\omega_0 - \omega_1 + \tilde{\alpha}) \end{aligned}\right\} \quad (429)$$

Through the use of Solutions (A-3) and (A-4) in Appendix A, the solutions to Eqs. (429) are

$$\tilde{a}_N^{(2)}(\tilde{\alpha}) = C_5 \cos\tilde{\alpha} + C_6 \sin\tilde{\alpha} - C_0 \tilde{\alpha} \sin(\omega_0 + \tilde{\alpha}) + \frac{1}{12} e_0^2 e_1 \tilde{\alpha} \sin(\omega_1 + \tilde{\alpha}) + \frac{1}{12} e_0^2 e_1 \tilde{\alpha} \sin(2\omega_0 - \omega_1 + \tilde{\alpha}) \tag{430}$$

and

$$\tilde{a}_M^{(2)}(\tilde{\alpha}) = C_7 \cos\tilde{\alpha} + C_8 \sin\tilde{\alpha} + C_0 \tilde{\alpha} \cos(\omega_0 + \tilde{\alpha}) - \frac{1}{12} e_0^2 e_1 \tilde{\alpha} \cos(\omega_1 + \tilde{\alpha}) - \frac{1}{12} e_0^2 e_1 \tilde{\alpha} \cos(2\omega_0 - \omega_1 + \tilde{\alpha}) \tag{431}$$

where C_5, C_6, C_7, and C_8 are constants determined from the initial conditions.

Differentiating Eq. (430) with respect to $\tilde{\alpha}$ yields

$$\frac{d\tilde{a}_N^{(2)}}{d\tilde{\alpha}} = -C_5 \sin\tilde{\alpha} + C_6 \cos\tilde{\alpha} - C_0 \sin(\omega_0 + \tilde{\alpha}) - C_0 \tilde{\alpha} \cos(\omega_0 + \tilde{\alpha}) + \frac{1}{12} e_0^2 e_1 \sin(\omega_1 + \tilde{\alpha}) + \frac{1}{12} e_0^2 e_1 \tilde{\alpha} \cos(\omega_1 + \tilde{\alpha})$$
$$+ \frac{1}{12} e_0^2 e_1 \sin(2\omega_0 - \omega_1 + \tilde{\alpha}) + \frac{1}{12} e_0^2 e_1 \tilde{\alpha} \cos(2\omega_0 - \omega_1 + \tilde{\alpha}) \tag{432}$$

Introducing Eqs. (431) and (432) into Eq. (425) yields

$$(C_6 + C_7)\cos\tilde{\alpha} - (C_5 - C_8)\sin\tilde{\alpha} = 0$$

so that, since $\tilde{\alpha}$ is not a constant,

$$C_6 = -C_7 \qquad \text{and} \qquad C_5 = C_8 \tag{433}$$

Introducing Eqs. (433) into Eqs. (430) and (431) yields

$$\left.\begin{aligned} \tilde{a}_N^{(2)}(\tilde{\alpha}) &= C_5 \cos\tilde{\alpha} - C_7 \sin\tilde{\alpha} - C_0 \tilde{\alpha} \sin(\omega_0 + \tilde{\alpha}) + \frac{1}{12} e_0^2 e_1 \tilde{\alpha} \sin(\omega_1 + \tilde{\alpha}) + \frac{1}{12} e_0^2 e_1 \tilde{\alpha} \sin(2\omega_0 - \omega_1 + \tilde{\alpha}) \\ &\text{and} \\ \tilde{a}_M^{(2)}(\tilde{\alpha}) &= C_7 \cos\tilde{\alpha} + C_5 \sin\tilde{\alpha} + C_0 \tilde{\alpha} \cos(\omega_0 + \tilde{\alpha}) - \frac{1}{12} e_0^2 e_1 \tilde{\alpha} \cos(\omega_1 + \tilde{\alpha}) - \frac{1}{12} e_0^2 e_1 \tilde{\alpha} \cos(2\omega_0 - \omega_1 + \tilde{\alpha}) \end{aligned}\right\} \tag{434}$$

Now, evaluating Eqs. (381) at $\tilde{\alpha} = 0$ and using Eqs. (418), one obtains

$$\tilde{a}_N^{(2)}(0) = e_2 \cos\omega_2 \qquad \text{and} \qquad \tilde{a}_M^{(2)}(0) = e_2 \sin\omega_2 \tag{435}$$

Thus, evaluating Eqs. (434) at $\tilde{\alpha} = 0$ and using Eqs. (435), one obtains

$$e_2 \cos\omega_2 = C_5 \qquad \text{and} \qquad e_2 \sin\omega_2 = C_7 \tag{436}$$

It follows from Eqs. (434) and (436) and trigonometric formulas that

$$\left.\begin{aligned} \tilde{a}_N^{(2)}(\tilde{\alpha}) &= e_2 \cos(\omega_2 + \tilde{\alpha}) - C_0 \tilde{\alpha} \sin(\omega_0 + \tilde{\alpha}) + \frac{1}{12} e_0^2 e_1 \tilde{\alpha} \sin(\omega_1 + \tilde{\alpha}) + \frac{1}{12} e_0^2 e_1 \tilde{\alpha} \sin(2\omega_0 - \omega_1 + \tilde{\alpha}) \\ &\text{and} \\ \tilde{a}_M^{(2)}(\tilde{\alpha}) &= e_2 \sin(\omega_2 + \tilde{\alpha}) + C_0 \tilde{\alpha} \cos(\omega_0 + \tilde{\alpha}) - \frac{1}{12} e_0^2 e_1 \tilde{\alpha} \cos(\omega_1 + \tilde{\alpha}) - \frac{1}{12} e_0^2 e_1 \tilde{\alpha} \cos(2\omega_0 - \omega_1 + \tilde{\alpha}) \end{aligned}\right\} \tag{437}$$

Finally, from trigonometric formulas and from Eqs. (381) and (437),

$$\tilde{e}^{(2)} \cos(\bar{\alpha} - \tilde{\omega}^{(2)}) = \tilde{e}^{(2)} \cos\bar{\alpha} \cos\tilde{\omega}^{(2)} + \tilde{e}^{(2)} \sin\bar{\alpha} \sin\tilde{\omega}^{(2)} = \tilde{a}_N^{(2)} \cos\bar{\alpha} + \tilde{a}_M^{(2)} \sin\bar{\alpha}$$

$$= e_2 \cos(\bar{\alpha} - \omega_2 - \tilde{\alpha}) + C_0 \tilde{\alpha} \sin(\bar{\alpha} - \omega_0 - \tilde{\alpha}) - \frac{1}{12} e_0^2 e_1 \tilde{\alpha} [\sin(\bar{\alpha} - \omega_1 - \tilde{\alpha}) + \sin(\bar{\alpha} - 2\omega_0 + \omega_1 - \tilde{\alpha})]$$

$$= e_2 \cos(\bar{\alpha} - \omega_2 - \tilde{\alpha}) + \left[C_0 - \frac{1}{6} e_0^2 e_1 \cos(\omega_0 - \omega_1)\right] \tilde{\alpha} \sin(\bar{\alpha} - \omega_0 - \tilde{\alpha})$$

or, through the use of Eq. (423),

$$\tilde{e}^{(2)} \cos(\bar{\alpha} - \tilde{\omega}^{(2)}) = e_2 \cos(\bar{\alpha} - \omega_2 - \tilde{\alpha}) + \frac{1}{2} e_0 \left[\left(7 + \frac{5}{2} e_0^2\right) + \frac{5}{3} e_0 e_1 \cos(\omega_0 - \omega_1) + 2\alpha_3\right] \tilde{\alpha} \sin(\bar{\alpha} - \omega_0 - \tilde{\alpha}) \tag{438}$$

Introducing Eq. (438) into Eq. (404) yields

$$\eta^{(2)}(\bar{\alpha}, \tilde{\alpha}) = \left[2\left(1 + \frac{1}{2} e_0^2\right) + e_0 e_1 \cos(\omega_0 - \omega_1)\right] + e_2 \cos(\bar{\alpha} - \omega_2 - \tilde{\alpha}) - \frac{1}{3} e_0^2 \cos 2(\bar{\alpha} - \omega_0 - \tilde{\alpha})$$
$$+ \frac{1}{48} e_0^3 \cos 3(\bar{\alpha} - \omega_0 - \tilde{\alpha}) - \frac{1}{3} e_0 e_1 \cos(2\bar{\alpha} - \omega_0 - \omega_1 - 2\tilde{\alpha})$$
$$+ \frac{1}{2} e_0 \left[\left(7 + \frac{5}{2} e_0^2\right) + \frac{5}{3} e_0 e_1 \cos(\omega_0 - \omega_1) + 2\alpha_3\right] \tilde{\alpha} \sin(\bar{\alpha} - \omega_0 - \tilde{\alpha}) \tag{439}$$

At this point, all of the known conditions on $\eta^{(2)}(\bar{\alpha}, \tilde{\alpha})$ have been applied (without the determination of the constant α_3) except the second uniformity condition as discussed in Section II-A-3. The second uniformity condition requires that

$$\lim_{\epsilon \to 0} \frac{\epsilon^2 \eta^{(2)}(\bar{\alpha}, \tilde{\alpha})}{\eta^{(1)}(\bar{\alpha}, \tilde{\alpha})} = \lim_{\epsilon \to 0} \frac{\epsilon \eta^{(2)}(\bar{\alpha}, \tilde{\alpha})}{\eta^{(1)}(\bar{\alpha}, \tilde{\alpha})} = 0 \qquad \text{for all } \alpha \text{ in } 0 \leqq \alpha < \infty$$

so that, from Eqs. (403) and (439),

$$\lim_{\epsilon \to 0} \left\{ \frac{\epsilon\left[2\left(1 + \frac{1}{2} e_0^2\right) + e_0 e_1 \cos(\omega_0 - \omega_1) + e_2 \cos(\bar{\alpha} - \omega_2 - \tilde{\alpha})\right] + \epsilon\left[-\frac{1}{3} e_0^2 \cos 2(\bar{\alpha} - \omega_0 - \tilde{\alpha}) + \frac{1}{48} e_0^3 \cos 3(\bar{\alpha} - \omega_0 - \tilde{\alpha})\right]}{\left(1 + \frac{1}{2} e_0^2\right) + e_1 \cos(\bar{\alpha} - \omega_1 - \tilde{\alpha}) - \frac{1}{6} e_0^2 \cos 2(\bar{\alpha} - \omega_0 - \tilde{\alpha})} \right.$$
$$\left. + \frac{\epsilon\left[-\frac{1}{3} e_0 e_1 \cos(2\bar{\alpha} - \omega_0 - \omega_1 - 2\tilde{\alpha})\right] + \epsilon\left(\frac{1}{2} e_0\right)\left[\left(7 + \frac{5}{2} e_0^2\right) + \frac{5}{3} e_0 e_1 \cos(\omega_0 - \omega_1) + 2\alpha_3\right] \tilde{\alpha} \sin(\bar{\alpha} - \omega_0 - \tilde{\alpha})}{\left(1 + \frac{1}{2} e_0^2\right) + e_1 \cos(\bar{\alpha} - \omega_1 - \tilde{\alpha}) - \frac{1}{6} e_0^2 \cos 2(\bar{\alpha} - \omega_0 - \tilde{\alpha})} \right\}$$
$$= 0 \tag{440}$$

Equation (440) requires that

$$\alpha_3 = -\frac{1}{2}\left[\left(7 + \frac{5}{2}e_0^2\right) + \frac{5}{3}e_0 e_1 \cos(\omega_0 - \omega_1)\right] \qquad (441)$$

so that the coefficient of the term $\tilde{\alpha}\sin(\bar{\alpha} - \omega_0 - \tilde{\alpha})$ is zero, and a uniformly valid perturbation $\epsilon^2 \eta^{(2)}(\bar{\alpha},\tilde{\alpha})$ is obtained (see Section II-A-3). Consequently, from Eqs. (381), (423), (437), and (441) and from trigonometric formulas,

$$\tilde{e}^{(2)}(\tilde{\alpha}) \cos \tilde{\omega}^{(2)}(\tilde{\alpha}) = e_2 \cos(\omega_2 + \tilde{\alpha})$$

and

$$\tilde{e}^{(2)}(\tilde{\alpha}) \sin \tilde{\omega}^{(2)}(\tilde{\alpha}) = e_2 \sin(\omega_2 + \tilde{\alpha})$$

and therefore

$$\tilde{e}^{(2)}(\tilde{\alpha}) = e_2 \qquad \text{and} \qquad \tilde{\omega}^{(2)}(\tilde{\alpha}) = \omega_2 + \tilde{\alpha} \qquad (442)$$

Furthermore, introducing Eqs. (442) into Eq. (404), one obtains

$$\begin{aligned}\eta^{(2)}(\bar{\alpha},\tilde{\alpha}) = {} & 2\left(1 + \frac{1}{2}e_0^2\right) + e_0 e_1 \cos(\omega_0 - \omega_1) \\ & + e_2 \cos(\bar{\alpha} - \omega_2 - \tilde{\alpha}) \\ & - \frac{1}{3}e_0^2 \cos 2(\bar{\alpha} - \omega_0 - \tilde{\alpha}) \\ & + \frac{1}{48}e_0^3 \cos 3(\bar{\alpha} - \omega_0 - \tilde{\alpha}) \\ & - \frac{1}{3}e_0 e_1 \cos(2\bar{\alpha} - \omega_0 - \omega_1 - 2\tilde{\alpha}) \qquad (443)\end{aligned}$$

and introducing Eqs. (422) and (442) into Eq. (421), one obtains

$$\begin{aligned}\eta^{(3)}(\bar{\alpha},\tilde{\alpha}) = {} & 4\left(1 + \frac{1}{2}e_0^2\right) + \left(1 + \frac{1}{2}e_0^2\right)^2 + \frac{1}{72}e_0^4 + \frac{1}{2}e_1^2 + 2e_0 e_1 \cos(\omega_0 - \omega_1) \\ & + e_0 e_2 \cos(\omega_0 - \omega_2) + \tilde{e}^{(3)} \cos(\bar{\alpha} - \tilde{\omega}^{(3)}) - \frac{1}{9}e_0^2\left(9 + \frac{59}{48}e_0^2\right) \cos 2(\bar{\alpha} - \omega_0 - \tilde{\alpha}) \\ & - \frac{1}{6}e_1^2 \cos 2(\bar{\alpha} - \omega_1 - \tilde{\alpha}) - \frac{2}{3}e_0 e_1 \cos(2\bar{\alpha} - \omega_0 - \omega_1 - 2\tilde{\alpha}) \\ & - \frac{1}{3}e_0 e_2 \cos(2\bar{\alpha} - \omega_0 - \omega_2 - 2\tilde{\alpha}) + \frac{1}{12}e_0^3 \cos 3(\bar{\alpha} - \omega_0 - \tilde{\alpha}) \\ & + \frac{1}{16}e_0^2 e_1 \cos(3\bar{\alpha} - 2\omega_0 - \omega_1 - 3\tilde{\alpha}) - \frac{1}{432}e_0^4 \cos 4(\bar{\alpha} - \omega_0 - \tilde{\alpha}) \qquad (444)\end{aligned}$$

where, from Eqs. (405) and (441),

$$\begin{aligned}\bar{\alpha} = \alpha \Bigg\{ 1 & - \frac{1}{12}\epsilon^2(18 + 5e_0^2) \\ & - \frac{1}{2}\epsilon^3\left[\left(7 + \frac{5}{2}e_0^2\right) + \frac{5}{3}e_0 e_1 \cos(\omega_0 - \omega_1)\right] \\ & + O(\epsilon^4)\Bigg\} \qquad (445)\end{aligned}$$

Similarly, the remaining constants $\alpha_4, \alpha_5, \cdots$ in $\bar{\alpha}$ and perturbations $\epsilon^3 \eta^{(3)}(\bar{\alpha},\tilde{\alpha})$, $\epsilon^4 \eta^{(4)}(\bar{\alpha},\tilde{\alpha})$, $\cdots$ in the asymptotic expansion (324) may be obtained.

Thus far, sufficient information is available to describe the orbit of the satellite to an accuracy of $O(\epsilon^2)$. However, additional information is necessary to describe the position of the satellite in its orbit: namely, the asymptotic expansion for time. For orbits inclined with respect to the equatorial plane, the differential equations of motion can not be uncoupled (unlike Eqs. 278 and 279) and thus the asymptotic expansions for $\eta(\alpha;\epsilon)$ and $t(\alpha;\epsilon)$ must be developed simultaneously.

Consider the development of the expansion (325) using the second of the differential equations of motion in Section V-C-2 (Eq. 279). Differentiating Eq. (325) with respect to the right ascension α, using the notation in Eq. (329), using Eqs. (331), and combining like powers of ϵ, one obtains

$$\begin{aligned}\frac{dt}{d\alpha} = {} & t_1^{(0)} + \epsilon(t_2^{(0)} + t_1^{(1)}) + \epsilon^2(\alpha_2 t_1^{(0)} + t_2^{(1)} + t_1^{(2)}) \\ & + \epsilon^3(\alpha_3 t_1^{(0)} + \alpha_2 t_1^{(1)} + t_2^{(2)} + t_1^{(3)}) + O(\epsilon^4) \qquad (446)\end{aligned}$$

Introducing Eqs. (334) and (446) into the differential equation of motion for $t(\alpha;\epsilon)$, Eq. (279), and combining like powers of ϵ, one obtains

$$
\begin{aligned}
&[(\eta^{(0)})^2 t_1^{(0)} - h^3] + \epsilon\,[(\eta^{(0)})^2 (t_2^{(0)} + t_1^{(1)}) + 2\eta^{(0)}\eta^{(1)} t_1^{(0)}] \\
&\quad + \epsilon^2 \{(\eta^{(0)})^2 (\alpha_2 t_1^{(0)} + t_2^{(1)} + t_1^{(2)}) + 2\eta^{(0)}\eta^{(1)} (t_2^{(0)} + t_1^{(1)}) + [(\eta^{(1)})^2 + 2\eta^{(0)}\eta^{(2)}]\, t_1^{(0)}\} \\
&\quad + \epsilon^3 \{(\eta^{(0)})^2 (\alpha_3 t_1^{(0)} + \alpha_2 t_1^{(1)} + t_2^{(2)} + t_1^{(3)}) + 2\eta^{(0)}\eta^{(1)} (\alpha_2 t_1^{(0)} + t_2^{(1)} + t_1^{(2)}) \\
&\qquad + [(\eta^{(1)})^2 + 2\eta^{(0)}\eta^{(2)}]\,(t_2^{(0)} + t_1^{(1)}) + (2\eta^{(0)}\eta^{(3)} + 2\eta^{(1)}\eta^{(2)})\, t_1^{(0)}\} + O(\epsilon^4) = 0
\end{aligned}
\tag{447}
$$

Since the expansion (447) must hold (at least in an asymptotic sense) for arbitrary values of the perturbative parameter ϵ, the coefficients of the powers of ϵ in Eq. (447) must separately equal zero. Thus,

$$t_1^{(0)} = \frac{h^3}{(\eta^{(0)})^2} \tag{448}$$

$$t_1^{(1)} = -\frac{2\eta^{(1)}}{\eta^{(0)}} t_1^{(0)} - t_2^{(0)} \tag{449}$$

$$t_1^{(2)} = -\frac{2\eta^{(1)}}{\eta^{(0)}} (t_2^{(0)} + t_1^{(1)}) - \left[\left(\frac{\eta^{(1)}}{\eta^{(0)}}\right)^2 + \frac{2\eta^{(2)}}{\eta^{(0)}} + \alpha_2\right] t_1^{(0)} - t_2^{(1)} \tag{450}$$

$$
\begin{aligned}
t_1^{(3)} = &-\frac{2\eta^{(1)}}{\eta^{(0)}} (\alpha_2 t_1^{(0)} + t_2^{(1)} + t_1^{(2)}) - \left[\left(\frac{\eta^{(1)}}{\eta^{(0)}}\right)^2 + \frac{2\eta^{(2)}}{\eta^{(0)}}\right] t_2^{(0)} \\
&- \left[\left(\frac{\eta^{(1)}}{\eta^{(0)}}\right)^2 + \frac{2\eta^{(2)}}{\eta^{(0)}} + \alpha_2\right] t_1^{(1)} - \left[\frac{2\eta^{(3)}}{\eta^{(0)}} + \frac{2\eta^{(1)}\eta^{(2)}}{(\eta^{(0)})^2} + \alpha_3\right] t_1^{(0)} - t_2^{(2)}
\end{aligned}
\tag{451}
$$

and so forth. The initial conditions necessary for the complete solution of these partial differential equations are obtained by evaluating Eqs. (325) and (446) at $\alpha = 0$, by using the initial conditions (316), by combining terms of like powers of ϵ, and by setting the coefficients of the powers of ϵ separately equal to zero. Thus, since $\overline{\alpha} = 0$ and $\widetilde{\alpha} = 0$ when $\alpha = 0$ from Eqs. (326) and (327),

$$0 = t^{(0)}(0,0) + \epsilon\, t^{(1)}(0,0) + \epsilon^2 t^{(2)}(0,0) + \epsilon^3 t^{(3)}(0,0) + O(\epsilon^4)$$

and

$$
\begin{aligned}
0 = &\left[t_1^{(0)}(0,0) - \frac{h^3}{\eta_0^2}\right] + \epsilon\,[t_2^{(0)}(0,0) + t_1^{(1)}(0,0)] + \epsilon^2 [\alpha_2 t_1^{(0)}(0,0) + t_2^{(1)}(0,0) + t_1^{(2)}(0,0)] \\
&+ \epsilon^3 [\alpha_3 t_1^{(0)}(0,0) + \alpha_2 t_1^{(1)}(0,0) + t_2^{(2)}(0,0) + t_1^{(3)}(0,0)] + O(\epsilon^4)
\end{aligned}
$$

so that

$$t^{(0)}(0,0) = 0, \qquad t_1^{(0)}(0,0) = \frac{h^3}{\eta_0^2} \tag{452}$$

$$t^{(1)}(0,0) = 0, \qquad t_1^{(1)}(0,0) = -t_2^{(0)}(0,0) \tag{453}$$

$$t^{(2)}(0,0) = 0, \qquad t_1^{(2)}(0,0) = -\alpha_2 t_1^{(0)}(0,0) - t_2^{(1)}(0,0) \tag{454}$$

$$t^{(3)}(0,0) = 0, \qquad t_1^{(3)}(0,0) = -\alpha_3 t_1^{(0)}(0,0) - \alpha_2 t_1^{(1)}(0,0) - t_2^{(2)}(0,0) \tag{455}$$

and so forth. Note that the initial conditions for the partial derivatives will be satisfied automatically and are given only for the purpose of checking results.

Introducing Eq. (371) into Eq. (448) yields

$$\frac{\partial t^{(0)}}{\partial \bar{\alpha}} = \frac{h^3}{[1 + e_0 \cos(\bar{\alpha} - \omega_0 - \tilde{\alpha})]^2}$$

from which

$$t^{(0)}(\bar{\alpha}, \tilde{\alpha}) = h^3 \int \frac{d(\bar{\alpha} - \omega_0 - \tilde{\alpha})}{[1 + e_0 \cos(\bar{\alpha} - \omega_0 - \tilde{\alpha})]^2} + \tilde{T}^{(0)}(\tilde{\alpha}) \tag{456}$$

where the constant of integration is replaced by a function of the slow variable $\tilde{\alpha}$ and where $\tilde{\alpha}$ behaves as a constant in the integral since $\bar{\alpha}$ and $\tilde{\alpha}$ are distinct variables and, consequently, the integration can be performed with respect to $(\bar{\alpha} - \omega_0 - \tilde{\alpha})$ rather than $\bar{\alpha}$. It should be noted that if a partial differential equation consists only of derivatives with respect to one independent variable, its solution may be obtained by treating the partial differential equation as if it were an ordinary differential equation and then replacing the constants of integration that would normally arise by functions of the other independent variables.

At this point, it is desirable to introduce the instantaneous multivariable true anomaly and eccentric anomaly

$$\bar{v}(\bar{\alpha}, \tilde{\alpha}) = \bar{\alpha} - \omega_0 - \tilde{\alpha} \tag{457}$$

and

$$\tan\frac{1}{2}\bar{v}(\bar{\alpha}, \tilde{\alpha}) = \left(\frac{1 + e_0}{1 - e_0}\right)^{1/2} \tan\frac{1}{2}\bar{E}(\bar{\alpha}, \tilde{\alpha}) \tag{458}$$

respectively. It should be noted that if $\kappa_1 \pi \leqq \bar{v}(\bar{\alpha}, \tilde{\alpha}) \leqq \kappa_2 \pi$, then $\kappa_1 \pi \leqq \bar{E}(\bar{\alpha}, \tilde{\alpha}) \leqq \kappa_2 \pi$ where $\kappa_1 < \kappa_2$ and both are integers. It follows from Eq. (458) and trigonometric formulas that

$$\cos \bar{E}(\bar{\alpha}, \tilde{\alpha}) = \frac{\cos \bar{v}(\bar{\alpha}, \tilde{\alpha}) + e_0}{1 + e_0 \cos \bar{v}(\bar{\alpha}, \tilde{\alpha})} \tag{459}$$

$$\sin \bar{E}(\bar{\alpha}, \tilde{\alpha}) = \frac{(1 - e_0^2)^{1/2} \sin \bar{v}(\bar{\alpha}, \tilde{\alpha})}{1 + e_0 \cos \bar{v}(\bar{\alpha}, \tilde{\alpha})} \tag{460}$$

and

$$\frac{1}{1 + e_0 \cos \bar{v}(\bar{\alpha}, \tilde{\alpha})} = \frac{1}{1 - e_0^2}[1 - e_0 \cos \bar{E}(\bar{\alpha}, \tilde{\alpha})] \tag{461}$$

Taking the partial derivative of Eq. (461) with respect to $\bar{\alpha}$ yields

$$\frac{(e_0 \sin \bar{v})\dfrac{\partial \bar{v}}{\partial \bar{\alpha}}}{(1 + e_0 \cos \bar{v})^2} = \frac{(e_0 \sin \bar{E})\dfrac{\partial \bar{E}}{\partial \bar{\alpha}}}{1 - e_0^2}$$

or, using Eqs. (460) and (461),

$$\frac{\partial \bar{E}}{\partial \bar{\alpha}} = \frac{1 - e_0 \cos \bar{E}}{(1 - e_0^2)^{1/2}} \frac{\partial \bar{v}}{\partial \bar{\alpha}} \tag{462}$$

Similarly,

$$\frac{\partial \bar{E}}{\partial \tilde{\alpha}} = \frac{1 - e_0 \cos \bar{E}}{(1 - e_0^2)^{1/2}} \frac{\partial \bar{v}}{\partial \tilde{\alpha}} \tag{463}$$

Taking the partial derivatives of Eq. (457) with respect to $\bar{\alpha}$ and $\tilde{\alpha}$, respectively, yields

$$\frac{\partial \bar{v}}{\partial \bar{\alpha}} = 1 \tag{464}$$

and

$$\frac{\partial \bar{v}}{\partial \tilde{\alpha}} = -1 \tag{465}$$

Introducing Eq. (464) into Eq. (462) and Eq. (465) into Eq. (463) yields

$$\left.\begin{aligned} \frac{\partial \bar{E}}{\partial \bar{\alpha}} &= \frac{1 - e_0 \cos \bar{E}}{(1 - e_0^2)^{1/2}} \\ \text{and} \qquad \frac{\partial \bar{E}}{\partial \tilde{\alpha}} &= -\frac{1 - e_0 \cos \bar{E}}{(1 - e_0^2)^{1/2}} \end{aligned}\right\} \tag{466}$$

respectively, so that

$$\frac{\partial \bar{E}}{\partial \bar{\alpha}} = -\frac{\partial \bar{E}}{\partial \tilde{\alpha}} \tag{467}$$

Now, introducing Eq. (457) into Eq. (456) and using Eq. (C-8) in Appendix C (with $\bar{e} = e_0$), one obtains

$$t^{(0)}(\bar{\alpha}, \tilde{\alpha}) = \frac{h^3}{(1 - e_0^2)^{3/2}}[\bar{E}(\bar{\alpha}, \tilde{\alpha}) - e_0 \sin \bar{E}(\bar{\alpha}, \tilde{\alpha})] + \tilde{T}^{(0)}(\tilde{\alpha}) \tag{468}$$

Equation (468) can be rewritten in an interesting form. From the normalized equations (see Section V-A)

$$p = h^2 = 1 - e_0^2$$

it follows that

$$h^3 = (1 - e_0^2)^{3/2} \tag{469}$$

Introducing Eq. (469) into Eq. (468) yields

$$t^{(0)}(\bar{\alpha}, \tilde{\alpha}) = \bar{E}(\bar{\alpha}, \tilde{\alpha}) - e_0 \sin \bar{E}(\bar{\alpha}, \tilde{\alpha}) + \tilde{T}^{(0)}(\tilde{\alpha}) \tag{470}$$

Furthermore, from

$$\bar{M}^{(0)}(\bar{\alpha}, \tilde{\alpha}) = t^{(0)}(\bar{\alpha}, \tilde{\alpha}) - \tilde{T}^{(0)}(\tilde{\alpha}) \tag{471}$$

where $\bar{M}^{(0)}(\bar{\alpha}, \tilde{\alpha})$ is the zero-order approximation of the instantaneous multivariable mean anomaly,

$$\bar{M}^{(0)}(\bar{\alpha}, \tilde{\alpha}) = \bar{E}(\bar{\alpha}, \tilde{\alpha}) - e_0 \sin \bar{E}(\bar{\alpha}, \tilde{\alpha}) \tag{472}$$

which shows that Eq. (468) is equivalent to a zero-order multivariable Kepler equation.

It follows from Eqs. (457) and (458) evaluated at $\alpha = 0$ ($\bar{\alpha} = 0$ and $\tilde{\alpha} = 0$), respectively, that

$$\bar{E}(0,0) = -2 \tan^{-1}\left[\left(\frac{1 - e_0}{1 + e_0}\right)^{1/2} \tan(\omega_0/2)\right] \tag{473}$$

and from the first of Eqs. (452), as well as Eq. (470) evaluated at $\alpha = 0$ ($\bar{\alpha} = 0$ and $\tilde{\alpha} = 0$), that

$$\tilde{T}^{(0)}(0) = -[\bar{E}(0,0) - e_0 \sin \bar{E}(0,0)] \triangleq T_0 \tag{474}$$

Taking the partial derivatives of Eq. (470) with respect to $\bar{\alpha}$ and $\tilde{\alpha}$ yields

$$\left.\begin{aligned} t_1^{(0)} &= (1 - e_0 \cos \bar{E}) \frac{\partial \bar{E}}{\partial \bar{\alpha}} \\ &\text{and} \\ t_2^{(0)} &= (1 - e_0 \cos \bar{E}) \frac{\partial \bar{E}}{\partial \tilde{\alpha}} + \frac{d\tilde{T}^{(0)}}{d\tilde{\alpha}} \end{aligned}\right\} \tag{475}$$

from which, using Eq. (467),

$$t_2^{(0)} = -t_1^{(0)} + \frac{d\tilde{T}^{(0)}}{d\tilde{\alpha}} \tag{476}$$

Introducing Eq. (476) into Eq. (449) yields

$$t_1^{(1)} = \left(1 - \frac{2\eta^{(1)}}{\eta^{(0)}}\right) t_1^{(0)} - \frac{d\tilde{T}^{(0)}}{d\tilde{\alpha}}$$

or, using Eq. (448),

$$t_1^{(1)} = \frac{h^3}{(\eta^{(0)})^2} - \frac{2h^3 \eta^{(1)}}{(\eta^{(0)})^3} - \frac{d\tilde{T}^{(0)}}{d\tilde{\alpha}} \tag{477}$$

Introducing Eqs. (371) and (403) into Eq. (477) yields a partial differential equation in terms of $\bar{\alpha}$ and $\tilde{\alpha}$ but with no partial derivatives in $\tilde{\alpha}$, so that, by integration,

$$\begin{aligned} t^{(1)}(\bar{\alpha}, \tilde{\alpha}) = h^3 \Bigg\{ & \int \frac{d(\bar{\alpha} - \omega_0 - \tilde{\alpha})}{[1 + e_0 \cos(\bar{\alpha} - \omega_0 - \tilde{\alpha})]^2} - 2\left(1 + \frac{1}{3} e_0^2\right) \int \frac{d(\bar{\alpha} - \omega_0 - \tilde{\alpha})}{[1 + e_0 \cos(\bar{\alpha} - \omega_0 - \tilde{\alpha})]^3} \\ & - 2e_1 \cos(\omega_0 - \omega_1) \int \frac{\cos(\bar{\alpha} - \omega_0 - \tilde{\alpha})\, d(\bar{\alpha} - \omega_0 - \tilde{\alpha})}{[1 + e_0 \cos(\bar{\alpha} - \omega_0 - \tilde{\alpha})]^3} + 2e_1 \sin(\omega_0 - \omega_1) \int \frac{\sin(\bar{\alpha} - \omega_0 - \tilde{\alpha})\, d(\bar{\alpha} - \omega_0 - \tilde{\alpha})}{[1 + e_0 \cos(\bar{\alpha} - \omega_0 - \tilde{\alpha})]^3} \\ & - \frac{2}{3} e_0^2 \int \frac{\sin^2(\bar{\alpha} - \omega_0 - \tilde{\alpha})\, d(\bar{\alpha} - \omega_0 - \tilde{\alpha})}{[1 + e_0 \cos(\bar{\alpha} - \omega_0 - \tilde{\alpha})]^3} \Bigg\} - (\bar{\alpha} - \omega_0 - \tilde{\alpha}) \frac{d\tilde{T}^{(0)}}{d\tilde{\alpha}} + \tilde{T}^{(1)}(\tilde{\alpha}) \end{aligned} \tag{478}$$

where

$$\left.\begin{aligned} \cos(\bar{\alpha} - \omega_1 - \tilde{\alpha}) &= \cos(\bar{\alpha} - \omega_0 - \tilde{\alpha}) \cos(\omega_0 - \omega_1) - \sin(\bar{\alpha} - \omega_0 - \tilde{\alpha}) \sin(\omega_0 - \omega_1) \\ &\text{and} \\ \cos 2(\bar{\alpha} - \omega_0 - \tilde{\alpha}) &= 1 - 2 \sin^2(\bar{\alpha} - \omega_0 - \tilde{\alpha}) \end{aligned}\right\} \tag{479}$$

and where all the constants and functions of the slow variable $\tilde{\alpha}$ resulting from the individual integrals are included in $\tilde{T}^{(1)}(\tilde{\alpha})$. Now, introducing Eq. (457) into Eq. (478) and using Eqs. (C-8), (C-9), (C-11), (C-16), and (C-18) in Appendix C (with $\bar{e} = e_0$) as well as Eq. (469), one obtains

$$t^{(1)}(\bar{\alpha},\tilde{\alpha}) = \tau_1 \bar{E} + \tau_2 \cos \bar{E} + \tau_3 \sin \bar{E} + \tau_4 \cos 2\bar{E} + \tau_5 \sin 2\bar{E} - \frac{d\tilde{T}^{(0)}}{d\tilde{\alpha}} \bar{v} + \tilde{T}^{(1)} \tag{480}$$

where

$$\left.\begin{aligned}
\tau_1 &\triangleq -\frac{1}{1-e_0^2}\left[1 + 3e_0^2 - 3e_0 e_1 \cos(\omega_0 - \omega_1)\right] \\
\tau_2 &\triangleq -\frac{2e_1}{(1-e_0^2)^{1/2}} \sin(\omega_0 - \omega_1) \\
\tau_3 &\triangleq \frac{1}{1-e_0^2}\left[3e_0\left(1 + \frac{7}{9}e_0^2\right) - 2(1+e_0^2)\, e_1 \cos(\omega_0 - \omega_1)\right] \\
\tau_4 &\triangleq \frac{1}{2}\frac{e_0 e_1}{(1-e_0^2)^{1/2}} \sin(\omega_0 - \omega_1) \\
\text{and} \qquad & \\
\tau_5 &\triangleq -\frac{e_0}{1-e_0^2}\left[\frac{1}{3} e_0 (1+e_0^2) - \frac{1}{2} e_1 \cos(\omega_0 - \omega_1)\right]
\end{aligned}\right\} \tag{481}$$

It follows from Eqs. (457) and (458) that both $\bar{v}(\bar{\alpha},\tilde{\alpha})$ and $\bar{E}(\bar{\alpha},\tilde{\alpha})$ are unbounded functions of $\bar{\alpha}$, and since $\bar{E}(\bar{\alpha},\tilde{\alpha})$ appears secularly in $t^{(0)}(\bar{\alpha},\tilde{\alpha})$, (Eq. 470), and in $t^{(1)}(\bar{\alpha},\tilde{\alpha})$, (Eq. 480), both of these functions (and therefore $t(\alpha;\epsilon)$) are unbounded. Consequently, the first uniformity condition (see Section II-A-3) can not be applied to $t(\alpha;\epsilon)$. However, from the second uniformity condition (see Section II-A-3),

$$\lim_{\epsilon \to 0} \frac{\epsilon t^{(1)}(\bar{\alpha},\tilde{\alpha})}{t^{(0)}(\bar{\alpha},\tilde{\alpha})} = 0 \qquad \text{for all } \alpha \text{ in } 0 \leqq \alpha < \infty$$

so that, from Eqs. (470) and (480),

$$\lim_{\epsilon \to 0}\left[\frac{\epsilon(\tau_2 \cos \bar{E} + \tau_3 \sin \bar{E} + \tau_4 \cos 2\bar{E} + \tau_5 \sin 2\bar{E})}{\bar{E} - e_0 \sin \bar{E} + \tilde{T}^{(0)}} + \frac{\epsilon\left(\tau_1 \bar{E} - \dfrac{d\tilde{T}^{(0)}}{d\tilde{\alpha}}\bar{v} + \tilde{T}^{(1)}\right)}{\bar{E} - e_0 \sin \bar{E} + \tilde{T}^{(0)}}\right] = 0 \tag{482}$$

Equation (482) requires that

$$\frac{d\tilde{T}^{(0)}}{d\tilde{\alpha}} = 0 \tag{483}$$

as shown in Appendix D, so that the coefficient of the term $\bar{v}$ is zero and a uniformly valid perturbation $\epsilon t^{(1)}(\bar{\alpha},\tilde{\alpha})$ is obtained. Consequently, from Eqs. (474) and (483),

$$\tilde{T}^{(0)}(\tilde{\alpha}) = T_0 \tag{484}$$

Furthermore, introducing Eq. (484) into Eq. (470), one obtains

$$t^{(0)}(\bar{\alpha},\tilde{\alpha}) = \bar{E} - e_0 \sin \bar{E} + T_0 \tag{485}$$

and introducing Eq. (483) into Eq. (480) results in

$$\begin{aligned} t^{(1)}(\bar{\alpha},\tilde{\alpha}) = {} & \tau_1 \bar{E} + \tau_2 \cos \bar{E} + \tau_3 \sin \bar{E} + \tau_4 \cos 2\bar{E} \\ & + \tau_5 \sin 2\bar{E} + \tilde{T}^{(1)} \end{aligned} \tag{486}$$

Note that if the initial conditions of the satellite correspond to the case where $e_0 = 0$ and $\omega_0 = \pi$, (Eqs. 358), then Eqs. (457) and (458) yield

$$\overline{E} = \overline{v} = \overline{\alpha} - \omega_0 - \widetilde{\alpha}$$

and Eq. (486) becomes, using Eqs. (481) with $e_0 = 0$ and $\omega_0 = \pi$ and with $e_1 = 1$ and $\omega_1 = \pi$ (from Eqs. 379, with $e_0 = 0$ and $\omega_0 = \pi$),

$$t^{(1)}(\overline{\alpha},\widetilde{\alpha}) = -(\overline{\alpha} - \pi - \widetilde{\alpha}) - 2\sin(\overline{\alpha} - \pi - \widetilde{\alpha}) + \widetilde{T}^{(1)} \tag{487}$$

Now, if the argument of consistency used in Ref. 665 (for example, see p. 268) is applied to Eq. (480), that is, if $\overline{\alpha}$ is not allowed to appear as a power, the correct result (Eq. 483) is obtained. However, for $e_0 = 0$ and $\omega_0 = \pi$, $t^{(1)}(\overline{\alpha},\widetilde{\alpha})$ is given by Eq. (487), which shows that $t^{(1)}(\overline{\alpha},\widetilde{\alpha})$ actually contains $\overline{\alpha}$ to a power, namely, one. As a result, the argument of consistency can not be applied rigorously to an unbounded function like $t(\alpha;\epsilon)$ since the nonexistence of $\overline{\alpha}$ to a power is never known *a priori*. For this reason, the second uniformity condition is preferred herein rather than the argument of consistency.

Consider the partial differential equation for $t^{(2)}(\overline{\alpha},\widetilde{\alpha})$ (Eq. 450) from which the function $\widetilde{T}^{(1)}(\widetilde{\alpha})$ is determined.

Taking the partial derivatives of Eq. (486) with respect to $\overline{\alpha}$ and $\widetilde{\alpha}$, respectively, yields

$$t_1^{(1)} = (\tau_1 - \tau_2\sin\overline{E} + \tau_3\cos\overline{E} - 2\tau_4\sin 2\overline{E} + 2\tau_5\cos 2\overline{E})\frac{\partial\overline{E}}{\partial\overline{\alpha}} \tag{488}$$

and

$$t_2^{(1)} = (\tau_1 - \tau_2\sin\overline{E} + \tau_3\cos\overline{E} - 2\tau_4\sin 2\overline{E} + 2\tau_5\cos 2\overline{E})\frac{\partial\overline{E}}{\partial\widetilde{\alpha}} + \frac{d\widetilde{T}^{(1)}}{d\widetilde{\alpha}} \tag{489}$$

so that, introducing Eq. (467) into Eq. (488) and the results into Eq. (489), one obtains

$$t_2^{(1)} = -t_1^{(1)} + \frac{d\widetilde{T}^{(1)}}{d\widetilde{\alpha}} \tag{490}$$

Also, from Eqs. (476) and (483),

$$t_2^{(0)} = -t_1^{(0)} \tag{491}$$

and from Eqs. (491) and (449),

$$t_1^{(1)} = \left(1 - \frac{2\eta^{(1)}}{\eta^{(0)}}\right)t_1^{(0)} \tag{492}$$

Introducing Eqs. (490), (491), and (492) into Eq. (450) yields

$$t_1^{(2)} = \left[3\left(\frac{\eta^{(1)}}{\eta^{(0)}}\right)^2 + \frac{2(\eta^{(1)} + \eta^{(2)})}{\eta^{(0)}} + (1 - \alpha_2)\right]t_1^{(0)} - \frac{d\widetilde{T}^{(1)}}{d\widetilde{\alpha}}$$

or, using Eq. (448),

$$t_1^{(2)} = h^3\left[\frac{(1-\alpha_2)}{(\eta^{(0)})^2} - 2\frac{(\eta^{(1)} + \eta^{(2)})}{(\eta^{(0)})^3} + 3\frac{(\eta^{(1)})^2}{(\eta^{(0)})^4}\right] - \frac{d\widetilde{T}^{(1)}}{d\widetilde{\alpha}} \tag{493}$$

Now, from Eqs. (403) and (443),

$$\eta^{(1)} + \eta^{(2)} = 3\left(1 + \frac{1}{2}e_0^2\right) + e_0 e_1\cos(\omega_0 - \omega_1) + e_1\cos(\overline{\alpha} - \omega_1 - \widetilde{\alpha}) + e_2\cos(\overline{\alpha} - \omega_2 - \widetilde{\alpha})$$
$$- \frac{1}{2}e_0^2\cos 2(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) + \frac{1}{48}e_0^3\cos 3(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) - \frac{1}{3}e_0 e_1\cos(2\overline{\alpha} - \omega_0 - \omega_1 - 2\widetilde{\alpha})$$

or, using Eqs. (479) as well as similar expressions,

$$\eta^{(1)} + \eta^{(2)} = -\frac{1}{2}[c_0 + c_1\cos(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) + c_2\sin(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) + c_3\sin(\overline{\alpha} - \omega_0 - \widetilde{\alpha})\cos(\overline{\alpha} - \omega_0 - \widetilde{\alpha})$$
$$+ c_4\sin^2(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) + c_5\cos^3(\overline{\alpha} - \omega_0 - \widetilde{\alpha})] \tag{494}$$

where

$$
\left.
\begin{aligned}
c_0 &\triangleq -2\left[(3+e_0^2)+\frac{2}{3}e_0e_1\cos(\omega_0-\omega_1)\right]\\
c_1 &\triangleq 2\left[\frac{1}{16}e_0^3-e_1\cos(\omega_0-\omega_1)-e_2\cos(\omega_0-\omega_2)\right]\\
c_2 &\triangleq 2\left[e_1\sin(\omega_0-\omega_1)+e_2\sin(\omega_0-\omega_2)\right]\\
c_3 &\triangleq -\frac{4}{3}e_0e_1\sin(\omega_0-\omega_1)\\
c_4 &\triangleq -2e_0\left[e_0+\frac{2}{3}e_1\cos(\omega_0-\omega_1)\right]
\end{aligned}
\right\}
\tag{495}
$$

and

$$
c_5 \triangleq -\frac{1}{6}e_0^3
$$

From Eqs. (409) and (479), as well as similar expressions,

$$
\begin{aligned}
(\eta^{(1)})^2 = \frac{1}{3}\,[c_6 &+ c_7\cos(\overline{\alpha}-\omega_0-\widetilde{\alpha})+c_8\sin(\overline{\alpha}-\omega_0-\widetilde{\alpha})+c_9\sin(\overline{\alpha}-\omega_0-\widetilde{\alpha})\cos(\overline{\alpha}-\omega_0-\widetilde{\alpha})\\
&+c_{10}\sin^2(\overline{\alpha}-\omega_0-\widetilde{\alpha})+c_{11}\cos^3(\overline{\alpha}-\omega_0-\widetilde{\alpha})+c_{12}\sin^3(\overline{\alpha}-\omega_0-\widetilde{\alpha})+c_{13}\cos^4(\overline{\alpha}-\omega_0-\widetilde{\alpha})]
\end{aligned}
\tag{496}
$$

where

$$
\left.
\begin{aligned}
c_6 &\triangleq 3\left[\left(1+\frac{2}{3}e_0^2+\frac{1}{2}e_1^2\right)+\frac{1}{2}e_1^2\cos 2(\omega_0-\omega_1)\right]\\
c_7 &\triangleq 3\left[2e_1\left(1+\frac{2}{3}e_0^2\right)\cos(\omega_0-\omega_1)\right]\\
c_8 &\triangleq -3\left[2e_1\left(1+\frac{1}{3}e_0^2\right)\sin(\omega_0-\omega_1)\right]\\
c_9 &\triangleq -3e_1^2\sin 2(\omega_0-\omega_1)\\
c_{10} &\triangleq 3\left[\frac{2}{3}e_0^2\left(1+\frac{2}{3}e_0^2\right)-e_1^2\cos 2(\omega_0-\omega_1)\right]\\
c_{11} &\triangleq -2e_0^2e_1\cos(\omega_0-\omega_1)\\
c_{12} &\triangleq -2e_0^2e_1\sin(\omega_0-\omega_1)
\end{aligned}
\right\}
\tag{497}
$$

and

$$
c_{13} \triangleq \frac{1}{3}e_0^4
$$

Evaluating Eq. (486) at $\alpha = 0$ ($\overline{\alpha}=0$ and $\widetilde{\alpha}=0$) and using the first of Eqs. (453) yields

$$
\widetilde{T}^{(1)}(0) = -\left[\tau_1\overline{E}(0,0)+\tau_2\cos\overline{E}(0,0)+\tau_3\sin\overline{E}(0,0)+\tau_4\cos 2\overline{E}(0,0)+\tau_5\sin 2\overline{E}(0,0)\right] \triangleq T_1 \tag{498}
$$

where $\overline{E}(0,0)$ is given by Eq. (473).

Introducing Eqs. (371), (494), and (496) into Eq. (493) yields a partial differential equation in terms of $\bar{\alpha}$ and $\tilde{\alpha}$ but with no partial derivatives in $\tilde{\alpha}$, so that, by integration,

$$\begin{aligned}
t^{(2)}(\bar{\alpha},\tilde{\alpha}) = h^3 \Bigg\{ & (1-\alpha_2)\int \frac{d(\bar{\alpha}-\omega_0-\tilde{\alpha})}{[1+e_0\cos(\bar{\alpha}-\omega_0-\tilde{\alpha})]^2} + c_0 \int \frac{d(\bar{\alpha}-\omega_0-\tilde{\alpha})}{[1+e_0\cos(\bar{\alpha}-\omega_0-\tilde{\alpha})]^3} \\
& + c_1 \int \frac{\cos(\bar{\alpha}-\omega_0-\tilde{\alpha})\, d(\bar{\alpha}-\omega_0-\tilde{\alpha})}{[1+e_0\cos(\bar{\alpha}-\omega_0-\tilde{\alpha})]^3} + c_2 \int \frac{\sin(\bar{\alpha}-\omega_0-\tilde{\alpha})\, d(\bar{\alpha}-\omega_0-\tilde{\alpha})}{[1+e_0\cos(\bar{\alpha}-\omega_0-\tilde{\alpha})]^3} \\
& + c_3 \int \frac{\sin(\bar{\alpha}-\omega_0-\tilde{\alpha})\cos(\bar{\alpha}-\omega_0-\tilde{\alpha})\, d(\bar{\alpha}-\omega_0-\tilde{\alpha})}{[1+e_0\cos(\bar{\alpha}-\omega_0-\tilde{\alpha})]^3} \\
& + c_4 \int \frac{\sin^2(\bar{\alpha}-\omega_0-\tilde{\alpha})\, d(\bar{\alpha}-\omega_0-\tilde{\alpha})}{[1+e_0\cos(\bar{\alpha}-\omega_0-\tilde{\alpha})]^3} + c_5 \int \frac{\cos^3(\bar{\alpha}-\omega_0-\tilde{\alpha})\, d(\bar{\alpha}-\omega_0-\tilde{\alpha})}{[1+e_0\cos(\bar{\alpha}-\omega_0-\tilde{\alpha})]^3} \\
& + c_6 \int \frac{d(\bar{\alpha}-\omega_0-\tilde{\alpha})}{[1+e_0\cos(\bar{\alpha}-\omega_0-\tilde{\alpha})]^4} + c_7 \int \frac{\cos(\bar{\alpha}-\omega_0-\tilde{\alpha})\, d(\bar{\alpha}-\omega_0-\tilde{\alpha})}{[1+e_0\cos(\bar{\alpha}-\omega_0-\tilde{\alpha})]^4} \\
& + c_8 \int \frac{\sin(\bar{\alpha}-\omega_0-\tilde{\alpha})\, d(\bar{\alpha}-\omega_0-\tilde{\alpha})}{[1+e_0\cos(\bar{\alpha}-\omega_0-\tilde{\alpha})]^4} \\
& + c_9 \int \frac{\sin(\bar{\alpha}-\omega_0-\tilde{\alpha})\cos(\bar{\alpha}-\omega_0-\tilde{\alpha})\, d(\bar{\alpha}-\omega_0-\tilde{\alpha})}{[1+e_0\cos(\bar{\alpha}-\omega_0-\tilde{\alpha})]^4} \\
& + c_{10} \int \frac{\sin^2(\bar{\alpha}-\omega_0-\tilde{\alpha})\, d(\bar{\alpha}-\omega_0-\tilde{\alpha})}{[1+e_0\cos(\bar{\alpha}-\omega_0-\tilde{\alpha})]^4} + c_{11} \int \frac{\cos^3(\bar{\alpha}-\omega_0-\tilde{\alpha})\, d(\bar{\alpha}-\omega_0-\tilde{\alpha})}{[1+e_0\cos(\bar{\alpha}-\omega_0-\tilde{\alpha})]^4} \\
& + c_{12} \int \frac{\sin^3(\bar{\alpha}-\omega_0-\tilde{\alpha})\, d(\bar{\alpha}-\omega_0-\tilde{\alpha})}{[1+e_0\cos(\bar{\alpha}-\omega_0-\tilde{\alpha})]^4} + c_{13} \int \frac{\cos^4(\bar{\alpha}-\omega_0-\tilde{\alpha})\, d(\bar{\alpha}-\omega_0-\tilde{\alpha})}{[1+e_0\cos(\bar{\alpha}-\omega_0-\tilde{\alpha})]^4} \Bigg\} \\
& - (\bar{\alpha}-\omega_0-\tilde{\alpha})\frac{d\tilde{T}^{(1)}}{d\tilde{\alpha}} + \tilde{T}^{(2)}(\tilde{\alpha})
\end{aligned} \tag{499}$$

where all the constants and functions of the slow variable $\tilde{\alpha}$ resulting from the individual integrals are included in $\tilde{T}^{(2)}(\tilde{\alpha})$. Now, introducing Eq. (457) into Eq. (499) and using Eqs. (C-8–C-22) in Appendix C (with $\bar{e} = e_0$) as well as Eq. (469), one obtains

$$t^{(2)}(\bar{\alpha},\tilde{\alpha}) = \tau_7 \bar{E} + \tau_8 \cos\bar{E} + \tau_9 \sin\bar{E} + \tau_{10}\cos 2\bar{E} + \tau_{11}\sin 2\bar{E} + \tau_{12}\cos 3\bar{E} + \tau_{13}\sin 3\bar{E} - \left(\frac{d\tilde{T}^{(1)}}{d\tilde{\alpha}} - \frac{h^3}{6}\right)\bar{v} + \tilde{T}^{(2)} \tag{500}$$

where

$$\tau_7 \triangleq \frac{1}{2}\Bigg\{(2-2\alpha_2+c_4) + \frac{1}{1-e_0^2}[c_0(2+e_0^2) - 3c_1 e_0 + c_{10}] + \frac{1}{(1-e_0^2)^2}[c_6(2+3e_0^2) - c_7 e_0(4+e_0^2) - c_{11}e_0(3+2e_0^2) + 2c_{14}]\Bigg\} \tag{501}$$

$$\tau_8 \triangleq -\frac{1}{4}\frac{1}{(1-e_0^2)^{1/2}}\Bigg\{(4c_2 - 4c_3 e_0 + 3c_{12}) + \frac{1}{1-e_0^2}[c_8(4+e_0^2) - 5c_9 e_0]\Bigg\} \tag{502}$$

$$\tau_9 \triangleq -\frac{1}{4}\Bigg\{4(1-\alpha_2)e_0 + \frac{1}{1-e_0^2}[8c_0 e_0 - 4c_1(1+e_0^2) + c_{10}e_0] + \frac{1}{(1-e_0^2)^2}[3c_6 e_0(4+e_0^2) - c_7(4+11e_0^2) - 3c_{11}(1+4e_0^2) - 4c_{15}]\Bigg\} \tag{503}$$

$$\tau_{10} \triangleq \frac{1}{4}\frac{1}{(1-e_0^2)^{1/2}}\left\{(c_2 e_0 - c_3) + \frac{1}{1-e_0^2}\left[2c_8 e_0 - c_9(1+e_0^2)\right]\right\} \tag{504}$$

$$\tau_{11} \triangleq -\frac{1}{4}\left\{c_4 - \frac{1}{(1-e_0^2)}(c_0 e_0^2 - c_1 e_0 - c_{10}) - \frac{1}{(1-e_0^2)^2}\left[3c_6 e_0^2 - c_7 e_0(2+e_0^2) - 3c_{11} e_0 + 4c_{16}\right]\right\} \tag{505}$$

$$\tau_{12} \triangleq \frac{1}{12}\frac{1}{(1-e_0^2)^{1/2}}\left[c_{12} - \frac{1}{1-e_0^2}(c_8 e_0^2 - c_9 e_0)\right] \tag{506}$$

and

$$\tau_{13} \triangleq \frac{1}{12}\frac{1}{1-e_0^2}\left[c_{10} e_0 - \frac{1}{1-e_0^2}(c_6 e_0^3 - c_7 e_0^2 - c_{11} - 12c_{17})\right] \tag{507}$$

Although the integrals with coefficients c_5 and c_{13} in Eq. (499) have double forms individually (one form for $0 < e_0 < 1$ and another form for $e_0 = 0$), their sum has only one form (valid for $0 \leqq e_0 < 1$); that is, through the use of Eqs. (C-13) and (C-15) in Appendix C, as well as c_5 from Eqs. (495) and c_{13} from Eqs. (497),

$$c_5 \int \frac{\cos^3 \bar{v}\, d\bar{v}}{(1+e_0\cos\bar{v})^3} + c_{13}\int \frac{\cos^4 \bar{v}\, d\bar{v}}{(1+e_0\cos\bar{v})^4} = \frac{1}{(1-e_0^2)^{7/2}}(c_{14}\bar{E} + c_{15}\sin\bar{E} + c_{16}\sin 2\bar{E} + c_{17}\sin 3\bar{E}) + \frac{1}{6}\bar{v} \tag{508}$$

for $0 \leqq e_0 < 1$ where

$$\left.\begin{aligned} c_{14} &\triangleq -\frac{1}{12}(2 - 7e_0^2 + 5e_0^4 - 10e_0^6), \qquad & c_{15} &\triangleq -\frac{1}{12}e_0(2 - 5e_0^2 + 18e_0^4) \\ c_{16} &\triangleq -\frac{1}{24}e_0^2(1 - 7e_0^2), \qquad \text{and} & c_{17} &= -\frac{1}{36}e_0^3 \end{aligned}\right\} \tag{509}$$

As before, the first uniformity condition can not be applied to $t(\alpha;\epsilon)$. However, from the second uniformity condition,

$$\lim_{\epsilon\to 0}\frac{\epsilon^2 t^{(2)}(\bar{\alpha},\tilde{\alpha})}{\epsilon t^{(1)}(\bar{\alpha},\tilde{\alpha})} = \lim_{\epsilon\to 0}\frac{\epsilon t^{(2)}(\bar{\alpha},\tilde{\alpha})}{t^{(1)}(\bar{\alpha},\tilde{\alpha})} = 0 \qquad \text{for all } \alpha \text{ in } 0 \leqq \alpha < \infty$$

so that, from Eqs. (486) and (500),

$$\lim_{\epsilon\to 0}\left\{\frac{\epsilon(\tau_8\cos\bar{E} + \tau_9\sin\bar{E} + \tau_{10}\cos 2\bar{E} + \tau_{11}\sin 2\bar{E} + \tau_{12}\cos 3\bar{E} + \tau_{13}\sin 3\bar{E})}{\tau_1\bar{E} + \tau_2\cos\bar{E} + \tau_3\sin\bar{E} + \tau_4\cos 2\bar{E} + \tau_5\sin 2\bar{E} + \tilde{T}^{(1)}} + \frac{\epsilon\left[\tau_7\bar{E} - \left(\dfrac{d\tilde{T}^{(1)}}{d\tilde{\alpha}} - \dfrac{h^3}{6}\right)\bar{v} + \tilde{T}^{(2)}\right]}{\tau_1\bar{E} + \tau_2\cos\bar{E} + \tau_3\sin\bar{E} + \tau_4\cos 2\bar{E} + \tau_5\sin 2\bar{E} + \tilde{T}^{(1)}}\right\} = 0 \tag{510}$$

Following the same procedure as that used in Appendix D, one can reduce Eq. (510) to

$$\lim_{\epsilon \to 0} \epsilon \left[\frac{\left(\frac{d\widetilde{T}^{(1)}}{d\widetilde{\alpha}} - \frac{h^3}{6} \right) \overline{v} - \widetilde{T}^{(2)}}{\overline{E}} \right] = 0$$

which requires that

$$\frac{d\widetilde{T}^{(1)}}{d\widetilde{\alpha}} - \frac{h^3}{6} = \text{a constant} \triangleq 0 \tag{511}$$

and

$$\lim_{\epsilon \to 0} \frac{\epsilon \widetilde{T}^{(2)}}{\overline{E}} = 0 \tag{512}$$

The constant in Eq. (511) can be defined as zero because there are no other conditions on $d\widetilde{T}^{(1)}/d\widetilde{\alpha}$; that is, the constant is arbitrary. Consequently, from Eqs. (469), (498), and (511),

$$\widetilde{T}^{(1)}(\widetilde{\alpha}) = \tau_0 \widetilde{\alpha} + T_1 \tag{513}$$

where

$$\tau_0 \triangleq \frac{1}{6}(1 - e_0^2)^{3/2} \tag{514}$$

Note that $\widetilde{T}^{(1)}(\widetilde{\alpha})$ as determined in Eq. (513) satisfies the required condition (D-7) in Appendix D; that is, $\widetilde{T}^{(1)}(\widetilde{\alpha})$ is a linear function of α since $\widetilde{\alpha} = \epsilon\alpha$. Furthermore, it follows from Eq. (512) that $\widetilde{T}^{(2)}(\widetilde{\alpha})$ must be either a bounded function of $\widetilde{\alpha}$ or, if unbounded, at most a linear function of α since the unbounded part of $\overline{E}(\overline{\alpha}, \widetilde{\alpha})$ is a linear function of α (remember $\widetilde{\alpha} = \epsilon\alpha$ and $\overline{\alpha} = (1 + \epsilon^2 \alpha_2 + \cdots)\alpha$).

Introducing Eq. (513) into Eq. (486) and into Eq. (500) yields, respectively,

$$t^{(1)}(\overline{\alpha}, \widetilde{\alpha}) = \tau_0 \widetilde{\alpha} + \tau_1 \overline{E} + \tau_2 \cos \overline{E} + \tau_3 \sin \overline{E} + \tau_4 \cos 2\overline{E} + \tau_5 \sin 2\overline{E} + T_1 \tag{515}$$

and

$$t^{(2)}(\overline{\alpha}, \widetilde{\alpha}) = \tau_7 \overline{E} + \tau_8 \cos \overline{E} + \tau_9 \sin \overline{E} + \tau_{10} \cos 2\overline{E} + \tau_{11} \sin 2\overline{E} + \tau_{12} \cos 3\overline{E} + \tau_{13} \sin 3\overline{E} + \widetilde{T}^{(2)} \tag{516}$$

Note that $\widetilde{T}^{(2)}(\widetilde{\alpha})$ can not be determined completely without developing third-order expressions and, as a result, is not completely determined herein. However, from the first of Eqs. (454) and from Eq. (516),

$$\begin{aligned} \widetilde{T}^{(2)}(0) = & -[\tau_7 \overline{E}(0,0) + \tau_8 \cos \overline{E}(0,0) + \tau_9 \sin \overline{E}(0,0) \\ & + \tau_{10} \cos 2\overline{E}(0,0) + \tau_{11} \sin 2\overline{E}(0,0) \\ & + \tau_{12} \cos 3\overline{E}(0,0) + \tau_{13} \sin 3\overline{E}(0,0)] \\ & \triangleq T_2 \end{aligned} \tag{517}$$

where $\overline{E}(0,0)$ is given by Eq. (473) and thus

$$\widetilde{T}^{(2)}(\widetilde{\alpha}) = T_2 + \tau_6 \widetilde{T}_u^{(2)}(\widetilde{\alpha}) \tag{518}$$

where $\widetilde{T}_u^{(2)}(\widetilde{\alpha})$ represents the undetermined portion of $\widetilde{T}^{(2)}(\widetilde{\alpha})$. Now

$$\widetilde{T}_u^{(2)}(\widetilde{\alpha}) \to 0 \text{ as } \alpha \to 0$$

and since

$$\widetilde{T}_u^{(2)}(\widetilde{\alpha}) << \overline{E}(\overline{\alpha}, \widetilde{\alpha}) \text{ for } \alpha > 0$$

the quantity $\widetilde{T}_u^{(2)}(\widetilde{\alpha})$ contributes very little to $t^{(2)}(\overline{\alpha}, \widetilde{\alpha})$. Thus, Eq. (516) can be replaced by

$$t^{(2)}(\overline{\alpha}, \widetilde{\alpha}) \cong \tau_7 \overline{E} + \tau_8 \cos \overline{E} + \tau_9 \sin \overline{E} + \tau_{10} \cos 2\overline{E} + \tau_{11} \sin 2\overline{E} + \tau_{12} \cos 3\overline{E} + \tau_{13} \sin 3\overline{E} + T_2 \tag{519}$$

At this point of the development, the uniformly valid asymptotic expansions to $O(\epsilon^2)$ are determined for $\eta(\alpha; \epsilon)$ and for $t(\alpha; \epsilon)$. In addition to these, the quantities $(d\eta/d\alpha)(\alpha; \epsilon)$ and $(dt/d\alpha)(\alpha; \epsilon)$ are required to completely describe the motion of the satellite (see Section V-C-1).

As in Section V-B-3, let

$$\xi(\alpha; \epsilon) = \frac{d\eta}{d\alpha}(\alpha; \epsilon) \tag{520}$$

In addition, let

$$\xi(\alpha; \epsilon) = \Xi(\overline{\alpha}, \widetilde{\alpha}; \epsilon) \triangleq \xi^{(0)}(\overline{\alpha}, \widetilde{\alpha}) + \epsilon\xi^{(1)}(\overline{\alpha}, \widetilde{\alpha}) + \epsilon^2\xi^{(2)}(\overline{\alpha}, \widetilde{\alpha}) + \epsilon^3\xi^{(3)}(\overline{\alpha}, \widetilde{\alpha}) + O(\epsilon^4) \tag{521}$$

Introducing Eqs. (332) and (521) into Eq. (520), combining like powers of ϵ, and setting the coefficients of the powers of ϵ equal to zero (since the expansion must hold for arbitrary values of the perturbative parameter ϵ) yields

$$\xi^{(0)} = \eta_1^{(0)} \tag{522}$$

$$\xi^{(1)} = \eta_2^{(0)} + \eta_1^{(1)} \tag{523}$$

$$\xi^{(2)} = \alpha_2 \eta_1^{(0)} + \eta_2^{(1)} + \eta_1^{(2)} \tag{524}$$

and so forth. Now, from Eqs. (414) and (522),

$$\xi^{(0)}(\overline{\alpha}, \widetilde{\alpha}) = -e_0 \sin(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) \tag{525}$$

from Eqs. (377), (415), and (523),

$$\xi^{(1)}(\overline{\alpha}, \widetilde{\alpha}) = e_0 \sin(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) + \frac{1}{3} e_0^2 \sin 2(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) - e_1 \sin(\overline{\alpha} - \omega_1 - \widetilde{\alpha}) \tag{526}$$

and from Eqs. (414), (416), (417), (442), and (524),

$$\begin{aligned} \xi^{(2)}(\overline{\alpha}, \widetilde{\alpha}) = & -\alpha_2 e_0 \sin(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) \\ & + \frac{1}{3} e_0^2 \sin 2(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) \\ & - \frac{1}{16} e_0^3 \sin 3(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) \\ & + e_1 \sin(\overline{\alpha} - \omega_1 - \widetilde{\alpha}) \\ & + \frac{2}{3} e_0 e_1 \sin(2\overline{\alpha} - \omega_0 - \omega_1 - 2\widetilde{\alpha}) \\ & - e_2 \sin(\overline{\alpha} - \omega_2 - \widetilde{\alpha}) \end{aligned} \tag{527}$$

so that the uniformly valid asymptotic expansion to $O(\epsilon^2)$ is determined for $(d\eta/d\alpha)(\alpha;\epsilon) = \xi(\alpha;\epsilon)$. Also, from Eq. (279),

$$\frac{dt}{d\alpha}(\alpha;\epsilon) = \frac{h^3}{\eta^2(\alpha;\epsilon)} \tag{528}$$

where $\eta(\alpha;\epsilon)$ is given by Eqs. (324), (371), (403), and (443) to $O(\epsilon^2)$.

Finally, from the selected coordinates $\eta(\alpha;\epsilon)$, $t(\alpha;\epsilon)$, $(d\eta/d\alpha)(\alpha;\epsilon)$, and $(dt/d\alpha)(\alpha;\epsilon)$, the desirable quantities $r(\alpha;\epsilon)$, $(dr/dt)(\alpha;\epsilon)$, $(d\alpha/dt)(\alpha;\epsilon)$, $(ds/dt)(\alpha;\epsilon)$, $a(\alpha;\epsilon)$, $e(\alpha;\epsilon)$, $q(\alpha;\epsilon)$, $v(\alpha;\epsilon)$, and $\omega(\alpha;\epsilon)$ can be obtained. Thus, from Eq. (274),

$$r(\alpha;\epsilon) = \frac{h^2}{\eta(\alpha;\epsilon)} \tag{529}$$

from Eqs. (276) and (520),

$$\frac{dr}{dt}(\alpha;\epsilon) = -\frac{1}{h}\frac{d\eta}{d\alpha}(\alpha;\epsilon) = -\frac{1}{h}\xi(\alpha;\epsilon) \tag{530}$$

from Eq. (305),

$$\frac{ds}{dt}(\alpha;\epsilon) = \left\{\left[\frac{dr}{dt}(\alpha;\epsilon)\right]^2 + \left[r(\alpha;\epsilon)\frac{d\alpha}{dt}(\alpha;\epsilon)\right]^2\right\}^{1/2} \tag{531}$$

from Eq. (308),

$$a(\alpha;\epsilon) = \left\{\frac{2}{r(\alpha;\epsilon)} - \left[\frac{ds}{dt}(\alpha;\epsilon)\right]^2\right\}^{-1} \tag{532}$$

from Eq. (309),

$$e(\alpha;\epsilon) = \left[1 - \frac{h^2}{a(\alpha;\epsilon)}\right]^{1/2} \tag{533}$$

from Eq. (310),

$$q(\alpha;\epsilon) = a(\alpha;\epsilon)[1 - e(\alpha;\epsilon)] \tag{534}$$

from Eq. (311),

$$v(\alpha;\epsilon) = \tan^{-1}\left[\frac{h\dfrac{dr}{dt}(\alpha;\epsilon)}{\dfrac{h^2}{r(\alpha;\epsilon)} - 1}\right] \tag{535}$$

where

$$0 < v(\alpha;\epsilon) < \pi \qquad \text{if} \qquad \frac{dr}{dt}(\alpha;\epsilon) > 0$$

$$\pi < v(\alpha;\epsilon) < 2\pi \qquad \text{if} \qquad \frac{dr}{dt}(\alpha;\epsilon) < 0$$

$$v = 0 \qquad \text{if} \qquad \frac{dr}{dt}(\alpha;\epsilon) = 0 \qquad \text{and} \qquad \frac{d^2r}{dt^2}(\alpha;\epsilon) > 0$$

$$v = \pi \qquad \text{if} \qquad \frac{dr}{dt}(\alpha;\epsilon) = 0 \qquad \text{and} \qquad \frac{d^2r}{dt^2}(\alpha;\epsilon) < 0$$

and where, through the use of Eqs. (277) and (278),

$$\frac{d^2r}{dt^2}(\alpha;\epsilon) = -\frac{1}{h^4}[\eta(\alpha;\epsilon)]^2\{1 - \eta(\alpha;\epsilon) + \epsilon[\eta(\alpha;\epsilon)]^2\}$$

and, from Eq. (314),

$$\omega(\alpha;\epsilon) = \alpha - v(\alpha;\epsilon) \tag{536}$$

where $0 \leqq \omega \leqq 2\pi$.

In conclusion, the pertinent relationships of this section are summarized in Tables 5, 6, and 7.

Table 5. Summary of constants arising in the two-variable asymptotic expansions representing the motion of the equatorial satellite (Part 1)

$$h = r_0^2 \dot{\alpha}_0, \qquad \eta_0 = h^2/r_0, \qquad \xi_0 = -h\dot{r}_0$$

$$e_0 = [(\eta_0 - 1)^2 + \xi_0^2]^{1/2}, \qquad \omega_0 = \begin{cases} \tan^{-1}[\xi_0/(\eta_0 - 1)] & \text{for all } \xi_0 \text{ and } \eta_0 \text{ except } \xi_0 = 0 \text{ with } \eta_0 = 1 \\ \pi & \text{for } \xi_0 = 0 \text{ with } \eta_0 = 1 \end{cases}$$

$$e_1 = \left\{ \left[\frac{1}{6} e_0^2 \cos 2\omega_0 - \left(1 + \frac{1}{2} e_0^2\right) \right]^2 + \left[\frac{1}{3} e_0^2 \sin 2\omega_0 + e_0 \sin \omega_0 \right]^2 \right\}^{1/2}$$

$$\omega_1 = \tan^{-1} \left\{ \left[\frac{1}{3} e_0^2 \sin 2\omega_0 + e_0 \sin \omega_0 \right] \left[\frac{1}{6} e_0^2 \cos 2\omega_0 - \left(1 + \frac{1}{2} e_0^2\right) \right]^{-1} \right\}$$

$$e_2 = \left\{ \left[-2\left(1 + \frac{1}{2} e_0^2\right) + \frac{1}{3} e_0 e_1 \cos(\omega_0 + \omega_1) - e_0 e_1 \cos(\omega_0 - \omega_1) + \frac{1}{3} e_0^2 \cos 2\omega_0 - \frac{1}{48} e_0^3 \cos 3\omega_0 \right]^2 \right.$$
$$\left. + \left[\frac{2}{3} e_0 e_1 \sin(\omega_0 + \omega_1) + \frac{1}{12} e_0 (18 + 5e_0^2) \sin \omega_0 + \frac{1}{3} e_0^2 \sin 2\omega_0 - \frac{1}{16} e_0^3 \sin 3\omega_0 + e_1 \sin \omega_1 \right]^2 \right\}^{1/2}$$

$$\omega_2 = \tan^{-1} \left\{ \left[\frac{2}{3} e_0 e_1 \sin(\omega_0 + \omega_1) + \frac{1}{12} e_0 (18 + 5e_0^2) \sin \omega_0 + \frac{1}{3} e_0^2 \sin 2\omega_0 - \frac{1}{16} e_0^3 \sin 3\omega_0 \right. \right.$$
$$\left. \left. + e_1 \sin \omega_1 \right] \left[-2\left(1 + \frac{1}{2} e_0^2\right) + \frac{1}{3} e_0 e_1 \cos(\omega_0 + \omega_1) - e_0 e_1 \cos(\omega_0 - \omega_1) + \frac{1}{3} e_0^2 \cos 2\omega_0 - \frac{1}{48} e_0^3 \cos 3\omega_0 \right]^{-1} \right\}$$

$$\alpha_2 = \frac{1}{12}(18 + 5e_0^2), \qquad \alpha_3 = -\frac{1}{2}\left[\left(7 + \frac{5}{2} e_0^2\right) + \frac{5}{3} e_0 e_1 \cos(\omega_0 - \omega_1) \right]$$

$$\overline{E}(0,0) = -2 \tan^{-1} \left[\left(\frac{1 - e_0}{1 + e_0} \right)^{1/2} \tan \frac{\omega_0}{2} \right] \qquad \begin{cases} \text{if } \kappa_1 \pi \leqq \omega_0 \leqq \kappa_2 \pi, \text{ then } \kappa_1 \pi \leqq \overline{E}(0,0) \leqq \kappa_2 \pi \\ \text{where } \kappa_1 < \kappa_2 \text{ and both are integers} \end{cases}$$

$$c_0 = -2\left[(3 + e_0^2) + \frac{2}{3} e_0 e_1 \cos(\omega_0 - \omega_1) \right], \qquad c_1 = 2\left[\frac{1}{16} e_0^3 - e_1 \cos(\omega_0 - \omega_1) - e_2 \cos(\omega_0 - \omega_2) \right]$$

$$c_2 = 2\left[e_1 \sin(\omega_0 - \omega_1) + e_2 \sin(\omega_0 - \omega_2) \right], \qquad c_3 = -\frac{4}{3} e_0 e_1 \sin(\omega_0 - \omega_1)$$

$$c_4 = -2e_0 \left[e_0 + \frac{2}{3} e_1 \cos(\omega_0 - \omega_1) \right], \qquad c_6 = 3\left[\left(1 + \frac{2}{3} e_0^2 + \frac{1}{2} e_1^2\right) + \frac{1}{2} e_1^2 \cos 2(\omega_0 - \omega_1) \right]$$

$$c_7 = 3\left[2e_1 \left(1 + \frac{2}{3} e_0^2\right) \cos(\omega_0 - \omega_1) \right], \qquad c_8 = -3\left[2e_1 \left(1 + \frac{1}{3} e_0^2\right) \sin(\omega_0 - \omega_1) \right]$$

$$c_9 = -3 e_1^2 \sin 2(\omega_0 - \omega_1), \qquad c_{10} = 3\left[\frac{2}{3} e_0^2 \left(1 + \frac{2}{3} e_0^2\right) - e_1^2 \cos 2(\omega_0 - \omega_1) \right]$$

$$c_{11} = -2e_0^2 e_1 \cos(\omega_0 - \omega_1), \qquad c_{12} = -2e_0^2 e_1 \sin(\omega_0 - \omega_1)$$

$$c_{14} = -\frac{1}{12}(2 - 7e_0^2 + 5e_0^4 - 10e_0^6), \qquad c_{15} = -\frac{1}{12} e_0 (2 - 5e_0^2 + 18e_0^4)$$

$$c_{16} = -\frac{1}{24} e_0^2 (1 - 7e_0^2), \qquad c_{17} = -\frac{1}{36} e_0^3$$

Table 6. Summary of constants arising in the two-variable asymptotic expansions representing the motion of the equatorial satellite (Part 2)

$$\tau_0 = \frac{1}{6}(1-e_0^2)^{3/2}, \qquad \tau_1 = -\frac{1}{1-e_0^2}[1+3e_0^2-3e_0e_1\cos(\omega_0-\omega_1)], \qquad \tau_2 = -\frac{2e_1}{(1-e_0^2)^{1/2}}\sin(\omega_0-\omega_1)$$

$$\tau_3 = \frac{1}{1-e_0^2}\left[3e_0\left(1+\frac{7}{9}e_0^2\right)-2(1+e_0^2)\,e_1\cos(\omega_0-\omega_1)\right], \qquad \tau_4 = \frac{1}{2}\frac{e_0e_1}{(1-e_0^2)^{1/2}}\sin(\omega_0-\omega_1)$$

$$\tau_5 = -\frac{e_0}{1-e_0^2}\left[\frac{1}{3}e_0(1+e_0^2)-\frac{1}{2}e_1\cos(\omega_0-\omega_1)\right]$$

$$\tau_7 = \frac{1}{2}\left\{(2-2\alpha_2+c_4)+\frac{1}{1-e_0^2}[c_0(2+e_0^2)-3c_1e_0+c_{10}] + \frac{1}{(1-e_0^2)^2}[c_6(2+3e_0^2)-c_7e_0(4+e_0^2)-c_{11}e_0(3+2e_0^2)+2c_{14}]\right\}$$

$$\tau_8 = -\frac{1}{4}\frac{1}{(1-e_0^2)^{1/2}}\left\{(4c_2-4c_3e_0+3c_{12})+\frac{1}{1-e_0^2}[c_8(4+e_0^2)-5c_9e_0]\right\}$$

$$\tau_9 = -\frac{1}{4}\left\{4(1-\alpha_2)\,e_0+\frac{1}{1-e_0^2}[8c_0e_0-4c_1(1+e_0^2)+c_{10}e_0] + \frac{1}{(1-e_0^2)^2}[3c_6e_0(4+e_0^2)-c_7(4+11e_0^2)-3c_{11}(1+4e_0^2)-4c_{15}]\right\}$$

$$\tau_{10} = \frac{1}{4}\frac{1}{(1-e_0^2)^{1/2}}\left\{(c_2e_0-c_3)+\frac{1}{1-e_0^2}[2c_8e_0-c_9(1+e_0^2)]\right\}$$

$$\tau_{11} = -\frac{1}{4}\left\{c_4-\frac{1}{1-e_0^2}(c_0e_0^2-c_1e_0-c_{10})-\frac{1}{(1-e_0^2)^2}[3c_6e_0^2-c_7e_0(2+e_0^2)-3c_{11}e_0+4c_{16}]\right\}$$

$$\tau_{12} = \frac{1}{12}\frac{1}{(1-e_0^2)^{1/2}}\left[c_{12}-\frac{1}{(1-e_0^2)}(c_8e_0^2-c_9e_0)\right]$$

$$\tau_{13} = \frac{1}{12}\frac{1}{1-e_0^2}\left[c_{10}e_0-\frac{1}{1-e_0^2}(c_6e_0^3-c_7e_0^2-c_{11}-12c_{17})\right]$$

$$T_0 = -[\bar{E}(0,0)-e_0\sin\bar{E}(0,0)]$$

$$T_1 = -[\tau_1\bar{E}(0,0)+\tau_2\cos\bar{E}(0,0)+\tau_3\sin\bar{E}(0,0)+\tau_4\cos 2\bar{E}(0,0)+\tau_5\sin 2\bar{E}(0,0)]$$

$$T_2 = -[\tau_7\bar{E}(0,0)+\tau_8\cos\bar{E}(0,0)+\tau_9\sin\bar{E}(0,0)+\tau_{10}\cos 2\bar{E}(0,0)+\tau_{11}\sin 2\bar{E}(0,0) + \tau_{12}\cos 3\bar{E}(0,0)+\tau_{13}\sin 3\bar{E}(0,0)]$$

Table 7. Summary of the two-variable asymptotic expansions representing the motion of the equatorial satellite

$$\overline{\alpha} = \alpha\left(1 + \epsilon^2\alpha_2 + \epsilon^3\alpha_3 + O(\epsilon^4)\right), \qquad \widetilde{\alpha} = \epsilon\alpha$$

$$\eta^{(0)} = 1 + e_0 \cos(\overline{\alpha} - \omega_0 - \widetilde{\alpha})$$

$$\eta^{(1)} = \left(1 + \frac{1}{2}e_0^2\right) + e_1 \cos(\overline{\alpha} - \omega_1 - \widetilde{\alpha}) - \frac{1}{6}e_0^2 \cos 2(\overline{\alpha} - \omega_0 - \widetilde{\alpha})$$

$$\eta^{(2)} = 2\left(1 + \frac{1}{2}e_0^2\right) + e_0e_1\cos(\omega_0 - \omega_1) + e_2\cos(\overline{\alpha} - \omega_2 - \widetilde{\alpha}) - \frac{1}{3}e_0^2\cos 2(\overline{\alpha} - \omega_0 - \widetilde{\alpha})$$
$$+ \frac{1}{48}e_0^3\cos 3(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) - \frac{1}{3}e_0e_1\cos(2\overline{\alpha} - \omega_0 - \omega_1 - 2\widetilde{\alpha})$$

$$\overline{E} = 2\tan^{-1}\left[\left(\frac{1-e_0}{1+e_0}\right)^{1/2}\tan\frac{1}{2}(\overline{\alpha} - \omega_0 - \widetilde{\alpha})\right] \quad \begin{cases}\text{if } \kappa_1\pi \leqq (\overline{\alpha} - \omega_0 - \widetilde{\alpha}) \leqq \kappa_2\pi, \text{ then } \kappa_1\pi \leqq \overline{E} \leqq \kappa_2\pi \\ \text{where } \kappa_1 < \kappa_2 \text{ and both are integers}\end{cases}$$

$$t^{(0)} = \overline{E} - e_0\sin\overline{E} + T_0$$

$$t^{(1)} = \tau_0\widetilde{\alpha} + \tau_1\overline{E} + \tau_2\cos\overline{E} + \tau_3\sin\overline{E} + \tau_4\cos 2\overline{E} + \tau_5\sin 2\overline{E} + T_1$$

$$t^{(2)} \cong \tau_7\overline{E} + \tau_8\cos\overline{E} + \tau_9\sin\overline{E} + \tau_{10}\cos 2\overline{E} + \tau_{11}\sin 2\overline{E} + \tau_{12}\cos 3\overline{E} + \tau_{13}\sin 3\overline{E} + T_2$$

$$\xi^{(0)} = -e_0\sin(\overline{\alpha} - \omega_0 - \widetilde{\alpha}), \qquad \xi^{(1)} = e_0\sin(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) + \frac{1}{3}e_0^2\sin 2(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) - e_1\sin(\overline{\alpha} - \omega_1 - \widetilde{\alpha})$$

$$\xi^{(2)} = -\alpha_2e_0\sin(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) + \frac{1}{3}e_0^2\sin 2(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) - \frac{1}{16}e_0^3\sin 3(\overline{\alpha} - \omega_0 - \widetilde{\alpha}) + e_1\sin(\overline{\alpha} - \omega_1 - \widetilde{\alpha})$$
$$+ \frac{2}{3}e_0e_1\sin(2\overline{\alpha} - \omega_0 - \omega_1 - 2\widetilde{\alpha}) - e_2\sin(\overline{\alpha} - \omega_2 - \widetilde{\alpha})$$

$$\eta = \eta^{(0)} + \epsilon\eta^{(1)} + \epsilon^2\eta^{(2)} + O(\epsilon^3), \qquad \xi = \xi^{(0)} + \epsilon\xi^{(1)} + \epsilon^2\xi^{(2)} + O(\epsilon^3)$$

$$t = t^{(0)} + \epsilon t^{(1)} + \epsilon^2 t^{(2)} + O(\epsilon^3), \qquad r = h^2/\eta$$

$$\frac{dr}{dt} = -\frac{\xi}{h}, \qquad \frac{d^2r}{dt^2} = -\frac{\eta^2}{h^4}(1 - \eta + \epsilon\eta^2), \qquad \frac{d\alpha}{dt} = \frac{\eta^2}{h^3}, \qquad \frac{ds}{dt} = \left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\alpha}{dt}\right)^2\right]^{1/2}$$

$$a = \left[\frac{2}{r} - \left(\frac{ds}{dt}\right)^2\right]^{-1}, \qquad e = \left(1 - \frac{h^2}{a}\right)^{1/2}, \qquad q = a(1 - e)$$

$$v = \tan^{-1}\left[\frac{h\dfrac{dr}{dt}}{\dfrac{h^2}{r} - 1}\right] \quad \text{where} \quad \begin{cases} 0 < v < \pi & \text{if } \dfrac{dr}{dt} > 0 \\ \pi < v < 2\pi & \text{if } \dfrac{dr}{dt} < 0 \\ v = 0 & \text{if } \dfrac{dr}{dt} = 0 \text{ and } \dfrac{d^2r}{dt^2} > 0 \\ v = \pi & \text{if } \dfrac{dr}{dt} = 0 \text{ and } \dfrac{d^2r}{dt^2} < 0 \end{cases}$$

$$\omega = \alpha - v \quad \text{where} \quad 0 \leqq \omega \leqq 2\pi$$

D. Solution Using Variation of Parameters With Taylor's Series Expansions

Consider the satellite theory that consists of the variation of parameters with Taylor's series expansions applied to the differential equations of motion of an equatorial satellite about an oblate body. The principal concepts of this theory are given in Section III. In addition, the variation of parameters with Taylor's series expansions is used to obtain the solution to the damped linear harmonic oscillator in Section IV-F (the essential features of this perturbation theory are given there). Note that this theory yields an initially valid approximate solution but one that has a large range of validity.

1. Selection of parameters and independent variable. In general, six instantaneous parameters are required to describe the osculating orbit completely: for example, a, e, i, ω, Ω, and M (see Fig. 20). However, in the case of an equatorial satellite, $i \equiv 0$ and Ω is physically indeterminant, as may be seen from Figs. 17 and 20. A suitable definition for the longitude of the ascending mode is $\Omega \overset{\Delta}{=} 0$ (an often-used definition for Ω is the limit of the expression for the Ω for all inclinations as $i \rightarrow 0$). Furthermore, as is shown in Section V-B (Eq. 272), the angular momentum of the satellite (for the restricted motion) is constant. As a result of these factors, only three instantaneous parameters are required to describe the osculating orbit of the equatorial satellite about an oblate body: for example, e, ω, and M.

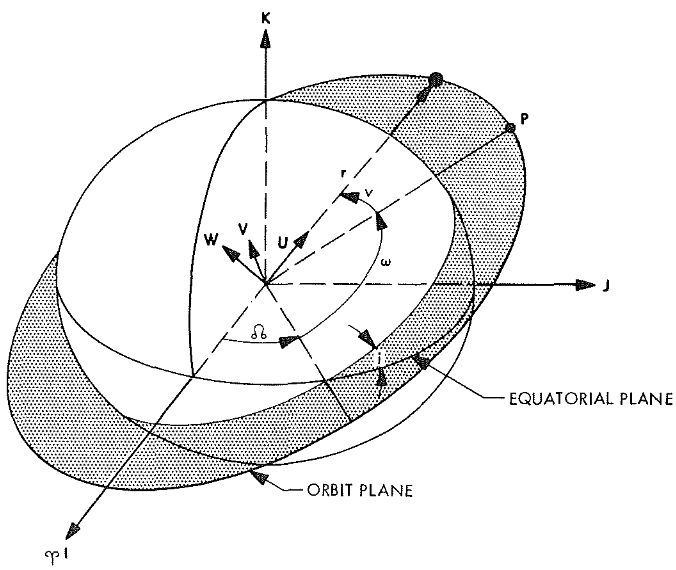

Fig. 20. Osculating orbit plane

In order to avoid small devisors and mathematical singularities in the solution due to small and zero eccentricities, respectively, the classical parameters e, ω, and M are replaced by $a_N = e \cos \omega$, $a_M = e \sin \omega$, and M (also, the use of a_N and a_M instead of e and ω introduces certain symmetries into the development of the solution; this choice enables mathematical checks to be made as the development proceeds and, consequently, permits the elimination of algebraic errors more easily).

Finally, in order to avoid slowly converging infinite series for particular values of eccentricity, the natural independent variable t is replaced by the right ascension α.

Thus, it is desirable to select $a_N(\alpha; \epsilon)$, $a_M(\alpha; \epsilon)$, and $t(\alpha; \epsilon)$ as the parameters and α as the independent variable in place of the classical parameters $e(t; \epsilon)$, $\omega(t; \epsilon)$, and $M(t; \epsilon)$ and the natural independent variable t.

2. Equations of motion. The differential equations of motion for the selected instantaneous parameters and the selected independent variable are, using Eqs. (E-16) from Appendix E,

$$\left.\begin{aligned} \frac{da_N}{d\alpha} &= \frac{\dfrac{da_N}{dt}}{\dfrac{d\alpha}{dt}} = \frac{r^2}{p^{1/2}} \grave{a}_N \\[2ex] \frac{da_M}{d\alpha} &= \frac{\dfrac{da_M}{dt}}{\dfrac{d\alpha}{dt}} = \frac{r^2}{p^{1/2}} \grave{a}_M \end{aligned}\right\} \qquad (537)$$

and

$$\frac{dt}{d\alpha} = \frac{r^2}{p^{1/2}} \qquad (538)$$

where $\dot{a}_N = \dot{a}_M = 0$ since $\dot{e} = \dot{\omega} = 0$ and where $\grave{a}_N$ and $\grave{a}_M$ are as given in Appendix E. Introducing Eqs. (E-18) into Eqs. (537) yields

$$\left.\begin{aligned} \frac{da_N}{d\alpha} &= -\epsilon\left[\left(\frac{p}{r}\right)^2 \sin\alpha\right] \\ \text{and} \qquad \frac{da_M}{d\alpha} &= \epsilon\left[\left(\frac{p}{r}\right)^2 \cos\alpha\right] \end{aligned}\right\} \qquad (539)$$

Next, the functions $r^2/p^{1/2}$, $(p/r)^2 \sin\alpha$, and $(p/r)^2 \cos\alpha$ are expressed in terms of the selected instantaneous parameters. From Eq. (348) and trigonometric formulas,

$$\frac{p}{r} = 1 + a_N \cos\alpha + a_M \sin\alpha \tag{540}$$

It follows from Eq. (540) and trigonometric formulas that

$$\left(\frac{p}{r}\right)^2 = 1 + \frac{1}{2}(a_N^2 + a_M^2) + 2a_N \cos\alpha + 2a_M \sin\alpha + \frac{1}{2}(a_N^2 - a_M^2)\cos 2\alpha + a_N a_M \sin 2\alpha \tag{541}$$

and that

$$\left(\frac{p}{r}\right)^3 = 1 + \frac{3}{2}(a_N^2 + a_M^2) + \frac{3}{4}a_N(a_N^2 + a_M^2 + 4)\cos\alpha + \frac{3}{4}a_M(a_N^2 + a_M^2 + 4)\sin\alpha + \frac{3}{2}(a_N^2 - a_M^2)\cos 2\alpha + 3a_N a_M \sin 2\alpha + \frac{1}{4}a_N(a_N^2 - 3a_M^2)\cos 3\alpha + \frac{1}{4}a_M(3a_M^2 - a_N^2)\sin 3\alpha \tag{542}$$

Consider the partial derivatives of $(p/r)^j$ with respect to a_N and a_M (see Ref. 564); that is,

$$\left.\begin{aligned} \frac{\partial}{\partial a_N}\left(\frac{p}{r}\right)^j &= j\left(\frac{p}{r}\right)^{j-1}\frac{\partial}{\partial a_N}\left(\frac{p}{r}\right) \\ \text{and} \qquad \frac{\partial}{\partial a_M}\left(\frac{p}{r}\right)^j &= j\left(\frac{p}{r}\right)^{j-1}\frac{\partial}{\partial a_M}\left(\frac{p}{r}\right) \end{aligned}\right\} \tag{543}$$

From Eq. (540),

$$\frac{\partial}{\partial a_N}\left(\frac{p}{r}\right) = \cos\alpha \qquad \text{and} \qquad \frac{\partial}{\partial a_M}\left(\frac{p}{r}\right) = \sin\alpha$$

so that Eqs. (543) become

$$\left.\begin{aligned} \frac{\partial}{\partial a_N}\left(\frac{p}{r}\right)^j &= j\left(\frac{p}{r}\right)^{j-1}\cos\alpha \\ \text{and} \qquad \frac{\partial}{\partial a_M}\left(\frac{p}{r}\right)^j &= j\left(\frac{p}{r}\right)^{j-1}\sin\alpha \end{aligned}\right\} \tag{544}$$

Thus, using Eqs. (544) with $j = 3$, one obtains

$$\left(\frac{p}{r}\right)^2 \cos\alpha = \frac{1}{3}\frac{\partial}{\partial a_N}\left(\frac{p}{r}\right)^3$$

and

$$\left(\frac{p}{r}\right)^2 \sin\alpha = \frac{1}{3}\frac{\partial}{\partial a_M}\left(\frac{p}{r}\right)^3$$

or, from Eq. (542),

$$\left(\frac{p}{r}\right)^2 \cos\alpha = a_N + \left(1 + \frac{3}{4}a_N^2 + \frac{1}{4}a_M^2\right)\cos\alpha + \frac{1}{2}a_N a_M \sin\alpha + a_N \cos 2\alpha + a_M \sin 2\alpha + \frac{1}{4}(a_N^2 - a_M^2)\cos 3\alpha + \frac{1}{2}a_N a_M \sin 3\alpha \tag{545}$$

and

$$\left(\frac{p}{r}\right)^2 \sin\alpha = a_M + \frac{1}{2}a_N a_M \cos\alpha + \left(1 + \frac{1}{4}a_N^2 + \frac{3}{4}a_M^2\right)\sin\alpha - a_M \cos 2\alpha + a_N \sin 2\alpha - \frac{1}{2}a_N a_M \cos 3\alpha + \frac{1}{4}(a_N^2 - a_M^2)\sin 3\alpha \tag{546}$$

Introducing Eqs. (545) and (546) into Eqs. (539) and Eq. (541) into Eq. (538) yields the differential equations of motion in terms of the selected osculating parameters and independent variable

$$\frac{da_N}{d\alpha} = -\epsilon\left[a_M + \frac{1}{2}a_N a_M \cos\alpha + \left(1 + \frac{1}{4}a_N^2 + \frac{3}{4}a_M^2\right)\sin\alpha - a_M \cos 2\alpha + a_N \sin 2\alpha - \frac{1}{2}a_N a_M \cos 3\alpha + \frac{1}{4}(a_N^2 - a_M^2)\sin 3\alpha\right] \tag{547}$$

$$\frac{da_M}{d\alpha} = \epsilon\left[a_N + \left(1 + \frac{3}{4}a_N^2 + \frac{1}{4}a_M^2\right)\cos\alpha + \frac{1}{2}a_N a_M \sin\alpha + a_N\cos 2\alpha + a_M \sin 2\alpha + \frac{1}{4}(a_N^2 - a_M^2)\cos 3\alpha + \frac{1}{2}a_N a_M \sin 3\alpha\right] \quad (548)$$

and

$$\frac{dt}{d\alpha} = \frac{p^{3/2}}{(1 + a_N\cos\alpha + a_M\sin\alpha)^2} \quad (549)$$

The initial conditions and the range of the independent variable for these differential equations of motion are

$$\left.\begin{aligned} a_N(0) &\stackrel{\Delta}{=} a_{N0} \\ a_M(0) &\stackrel{\Delta}{=} a_{M0} \\ t(0) &= 0 \end{aligned}\right\} \quad (550)$$

and

$$0 \leq \alpha < \infty \quad (551)$$

respectively. It should be noted that

$$\left.\begin{aligned} a_{N0} &= e_0 \cos\omega_0 \\ \text{and} \qquad a_{M0} &= e_0 \sin\omega_0 \end{aligned}\right\} \quad (552)$$

where, at $\alpha = 0$,

$$p = h^2 = r_0^4 \dot{\alpha}_0^2, \qquad v_0 = -\omega_0$$

$$e_0 = \left[\left(\frac{p}{r_0} - 1\right)^2 + p\dot{r}_0^2\right]^{1/2}$$

$$\omega_0 = \begin{cases} \tan^{-1}\left[\dfrac{-p^{1/2}\dot{r}_0}{\left(\dfrac{p}{r_0} - 1\right)}\right] & \text{for all } \dot{r}_0 \text{ and } \dfrac{p}{r_0} \text{ except } \dot{r}_0 = 0 \text{ with } \dfrac{p}{r_0} = 1 \\ \pi & \text{for } \dot{r}_0 = 0 \text{ with } \dfrac{p}{r_0} = 1 \end{cases}$$

Finally, it follows from Eqs. (547–552) that

$$\left.\begin{aligned} a_N &= a_{N0} + \int_0^\alpha \frac{da_N}{d\alpha}(\alpha'; \epsilon)\,d\alpha' \\ a_M &= a_{M0} + \int_0^\alpha \frac{da_M}{d\alpha}(\alpha'; \epsilon)\,d\alpha' \\ \text{and} \qquad t &= \int_0^\alpha \frac{dt}{d\alpha}(\alpha'; \epsilon)\,d\alpha' \end{aligned}\right\} \quad (553)$$

where $i \equiv 0$, $\Omega \stackrel{\Delta}{=} 0$, and $p = h^2$ and is constant.

3. *Analytical integration of equations of motion.* The process of Taylor's series expansions is based upon the development of a reference motion and associated reference parameters, $\bar{a}_N(\alpha; \epsilon)$, $\bar{a}_M(\alpha; \epsilon)$, and $\bar{t}(\alpha; \epsilon)$, which approximate the actual motion and associated instantaneous parameters, $a_N(\alpha; \epsilon)$, $a_M(\alpha; \epsilon)$, and $t(\alpha; \epsilon)$, sufficiently well so that Taylor's series expansions may be used to obtain the instantaneous parameters from the reference parameters (see Sections III-B-1 and IV-F-3).

Following the procedure of Section IV-F-3, let

$$\left.\begin{aligned} a_N(\alpha; \epsilon) &= \bar{a}_N(\alpha; \epsilon) + a_N^{(1)}(\alpha; \epsilon) + a_N^{(2)}(\alpha; \epsilon) + O(\epsilon^3) \\ \text{and} \qquad & a_N \to a_M \to t \end{aligned}\right\} \quad (554)$$

and let

$$\left.\begin{aligned} \frac{da_N}{d\alpha} &= \left(\frac{da_N}{d\alpha}\right)_s + \left(\frac{da_N}{d\alpha}\right)_{sp}^{(0)} + \left(\frac{da_N}{d\alpha}\right)_{sp}^{(1)} + \left(\frac{da_N}{d\alpha}\right)_{sp}^{(2)} \\ &\quad + \left(\frac{da_N}{d\alpha}\right)_{tp}^{(0)} + \left(\frac{da_N}{d\alpha}\right)_{tp}^{(1)} + \left(\frac{da_N}{d\alpha}\right)_{tp}^{(2)} + O(\epsilon^3) \\ \text{and} \qquad & a_N \to a_M \to t \end{aligned}\right\} \quad (555)$$

The notation used here corresponds to that of Section IV-F-3. Introducing Eqs. (554) and (555) into the equations of motion in integral form (Eqs. 553) and following the procedure of Section IV-F-3, one obtains

$$\left.\begin{aligned}
\bar{a}_N(\alpha;\epsilon) &= a_{N0} + \int_0^\alpha \left[\left(\frac{da_N}{d\alpha}\right)_s + \left(\frac{da_N}{d\alpha}\right)_{sp}^{(0)} + \left(\frac{da_N}{d\alpha}\right)_{tp}^{(0)} + \left(\frac{da_N}{d\alpha}\right)_{tp}^{(1)}\right] d\alpha' \\
&\quad - \int^0 \left[\left(\frac{da_N}{d\alpha}\right)_{sp}^{(1)} + \left(\frac{da_N}{d\alpha}\right)_{tp}^{(2)}\right] d\alpha' \\
&\quad - \int^0 \left[\left(\frac{da_N}{d\alpha}\right)_{sp}^{(2)} + \left(\frac{da_N}{d\alpha}\right)_{tp}^{(3)}\right] d\alpha' + O(\epsilon^3) \\
a_N^{(1)}(\alpha;\epsilon) &= \int^\alpha \left[\left(\frac{da_N}{d\alpha}\right)_{sp}^{(1)} + \left(\frac{da_N}{d\alpha}\right)_{tp}^{(2)}\right] d\alpha' \\
a_N^{(2)}(\alpha;\epsilon) &= \int^\alpha \left[\left(\frac{da_N}{d\alpha}\right)_{sp}^{(2)} + \left(\frac{da_N}{d\alpha}\right)_{tp}^{(3)}\right] d\alpha', \qquad \text{etc.}
\end{aligned}\right\} \tag{556}$$

and

$$a_N \rightarrow a_M \rightarrow t$$

Consider the determination of the reference (mean) parameters $\bar{a}_N(\alpha;\epsilon)$ and $\bar{a}_M(\alpha;\epsilon)$ (the parameter $\bar{t}(\alpha;\epsilon)$ is determined later). From Eqs. (556),

$$\bar{a}_N(\alpha;\epsilon) = \bar{a}_{N0} + \int_0^\alpha \left[\left(\frac{da_N}{d\alpha}\right)_s + \left(\frac{da_N}{d\alpha}\right)_{sp}^{(0)} + \left(\frac{da_N}{d\alpha}\right)_{tp}^{(0)} + \left(\frac{da_N}{d\alpha}\right)_{tp}^{(1)}\right] d\alpha' \tag{557}$$

and

$$a_N \rightarrow a_M$$

where

$$\left.\begin{aligned}
\bar{a}_{N0} &\triangleq a_{N0} - \int^0 \left[\left(\frac{da_N}{d\alpha}\right)_{sp}^{(1)} + \left(\frac{da_N}{d\alpha}\right)_{sp}^{(2)} + \left(\frac{da_N}{d\alpha}\right)_{tp}^{(2)} + \left(\frac{da_N}{d\alpha}\right)_{tp}^{(3)}\right] d\alpha' + O(\epsilon^3) \\
&= a_{N0} - (a_N^{(1)})_0 - (a_N^{(2)})_0 + O(\epsilon^3)
\end{aligned}\right\} \tag{558}$$

and

$$a_N \rightarrow a_M$$

Note that $\bar{a}_{N0}$ and $\bar{a}_{M0}$ are constants to be determined later.

Since $r(t;\epsilon)$ is bounded (the initial conditions are restricted so that quasi-periodic motion exists), it follows from

$$a_N = e\cos\omega \qquad \text{and} \qquad a_M = e\sin\omega \tag{559}$$

and from $e < 1$, $|\cos\omega| \leq 1$, and $|\sin\omega| \leq 1$ that

$$|a_N| \leq 1 \qquad \text{and} \qquad |a_M| \leq 1$$

that is, a_N and a_M are bounded. Consequently,

$$\left(\frac{da_N}{d\alpha}\right)_s = \left(\frac{da_M}{d\alpha}\right)_s = 0 \tag{560}$$

Furthermore, Eqs. (F-7) and (F-10) in Appendix F imply that

$$\left(\frac{da_N}{d\alpha}\right)_{sp}^{(0)} = \left(\frac{da_N}{d\alpha}\right)_{tp}^{(0)} = \left(\frac{da_M}{d\alpha}\right)_{sp}^{(0)} = \left(\frac{da_M}{d\alpha}\right)_{tp}^{(0)} = 0 \tag{561}$$

since secular, short-period, and long-period terms are of different character. Introducing Eqs. (560) and (561) into Eqs. (557) yields

$$\bar{a}_N(\alpha;\epsilon) = \bar{a}_{N0} + \int_0^\alpha \left(\frac{da_N}{d\alpha}\right)_{tp}^{(1)} d\alpha' \tag{562}$$

and

$$\bar{a}_M(\alpha;\epsilon) = \bar{a}_{M0} + \int_0^\alpha \left(\frac{da_M}{d\alpha}\right)_{tp}^{(1)} d\alpha'$$

Remembering that the integral of a long-period function usually decreases by one order of magnitude, one can let

$$\left.\begin{aligned}\int_0^\alpha \left(\frac{da_N}{d\alpha}\right)_{lp}^{(1)} d\alpha' &= (a_N)_{lp}^{(0)}(\alpha;\epsilon) + A_{N0}\\ &\text{and}\\ \int_0^\alpha \left(\frac{da_M}{d\alpha}\right)_{lp}^{(1)} d\alpha' &= (a_M)_{lp}^{(0)}(\alpha;\epsilon) + A_{M0}\end{aligned}\right\} \quad (563)$$

so that Eqs. (562) become

$$\left.\begin{aligned}\overline{a}_N(\alpha;\epsilon) &= \overline{a}_{N0} + (a_N)_{lp}^{(0)} + A_{N0}\\ &\text{and}\\ \overline{a}_M(\alpha;\epsilon) &= \overline{a}_{M0} + (a_M)_{lp}^{(0)} + A_{M0}\end{aligned}\right\} \quad (564)$$

Introducing Eqs. (564) into Eqs. (F-8) and (F-11) in Appendix F, using the results that the product of a constant or long-period function with a short-period function yields a short-period function, using

$$\left(\frac{da_N}{d\alpha}\right)_s^{(1)} = \left(\frac{da_M}{d\alpha}\right)_s^{(1)} = 0$$

from Eqs. (560), and equating secular, short-period, and long-period terms, respectively, one obtains

$$A_{N0} = -\overline{a}_{N0}, \qquad A_{M0} = -\overline{a}_{M0} \quad (565)$$

$$\begin{aligned}\left(\frac{da_N}{d\alpha}\right)_{sp}^{(1)} = -\epsilon\bigg(&\frac{1}{2}(a_N)_{lp}^{(0)}(a_M)_{lp}^{(0)}\cos\alpha\\ &+\left\{1+\frac{1}{4}[(a_N)_{lp}^{(0)}]^2+\frac{3}{4}[(a_M)_{lp}^{(0)}]^2\right\}\sin\alpha\\ &-(a_M)_{lp}^{(0)}\cos 2\alpha + (a_N)_{lp}^{(0)}\sin 2\alpha\\ &-\frac{1}{2}(a_N)_{lp}^{(0)}(a_M)_{lp}^{(0)}\cos 3\alpha\\ &+\frac{1}{4}\{[(a_N)_{lp}^{(0)}]^2-[(a_M)_{lp}^{(0)}]^2\}\sin 3\alpha\bigg)\end{aligned} \quad (566)$$

$$\begin{aligned}\left(\frac{da_M}{d\alpha}\right)_{sp}^{(1)} = \epsilon\bigg(&\left\{1+\frac{3}{4}[(a_N)_{lp}^{(0)}]^2+\frac{1}{4}[(a_M)_{lp}^{(0)}]^2\right\}\cos\alpha\\ &+\frac{1}{2}(a_N)_{lp}^{(0)}(a_M)_{lp}^{(0)}\sin\alpha\\ &+(a_N)_{lp}^{(0)}\cos 2\alpha + (a_M)_{lp}^{(0)}\sin 2\alpha\\ &+\frac{1}{4}\{[(a_N)_{lp}^{(0)}]^2-[(a_M)_{lp}^{(0)}]^2\}\cos 3\alpha\\ &+\frac{1}{2}(a_N)_{lp}^{(0)}(a_M)_{lp}^{(0)}\sin 3\alpha\bigg)\end{aligned} \quad (567)$$

$$\left.\begin{aligned}\left(\frac{da_N}{d\alpha}\right)_{lp}^{(1)} &= -\epsilon(a_M)_{lp}^{(0)}\\ &\text{and}\\ \left(\frac{da_M}{d\alpha}\right)_{lp}^{(1)} &= \epsilon(a_N)_{lp}^{(0)}\end{aligned}\right\} \quad (568)$$

Now, introducing Eqs. (565) and (568) into Eqs. (563) yields

$$-\epsilon\int_0^\alpha (a_M)_{lp}^{(0)} d\alpha' = (a_N)_{lp}^{(0)} - \overline{a}_{N0}$$

and

$$\epsilon\int_0^\alpha (a_N)_{lp}^{(0)} d\alpha' = (a_M)_{lp}^{(0)} - \overline{a}_{M0}$$

whose solutions are

$$\left.\begin{aligned}(a_N)_{lp}^{(0)} &= \overline{a}_{N0}\cos\epsilon\alpha - \overline{a}_{M0}\sin\epsilon\alpha\\ &\text{and}\\ (a_M)_{lp}^{(0)} &= \overline{a}_{M0}\cos\epsilon\alpha + \overline{a}_{N0}\sin\epsilon\alpha\end{aligned}\right\} \quad (569)$$

respectively. Hence, from Eqs. (564), (565), and (569),

$$\left.\begin{aligned}\overline{a}_N &= \overline{a}_{N0}\cos\epsilon\alpha - \overline{a}_{M0}\sin\epsilon\alpha\\ &\text{and}\\ \overline{a}_M &= \overline{a}_{M0}\cos\epsilon\alpha + \overline{a}_{N0}\sin\epsilon\alpha\end{aligned}\right\} \quad (570)$$

Let

$$\overline{a}_{N0} \overset{\Delta}{=} \overline{e}\cos\overline{\omega}_0 \qquad \text{and} \qquad \overline{a}_{M0} \overset{\Delta}{=} \overline{e}\sin\overline{\omega}_0 \quad (571)$$

so that Eqs. (570) become, using trigonometric formulas,

$$\overline{a}_N = \overline{e}\cos(\overline{\omega}_0 + \epsilon\alpha) \qquad \text{and} \qquad \overline{a}_M = \overline{e}\sin(\overline{\omega}_0 + \epsilon\alpha) \quad (572)$$

Finally, let

$$\bar{\omega} \triangleq \bar{\omega}_0 + \epsilon\alpha \tag{573}$$

so that Eqs. (572) become

$$\bar{a}_N = \bar{e}\cos\bar{\omega} \qquad \text{and} \qquad \bar{a}_M = \bar{e}\sin\bar{\omega} \tag{574}$$

In this manner, $\bar{a}_N$ and $\bar{a}_M$ are analogous to a_N and a_M. These equations and the equation for $\bar{t}$, to be obtained later, determine the reference parameters and reference motion.

The next step is to determine the first-order perturbations, $a_N^{(1)}(\alpha;\epsilon)$ and $a_M^{(1)}(\alpha;\epsilon)$. Following the procedure of Section IV-F-2, using the equations for $a_N^{(1)}(\alpha;\epsilon)$ and $a_M^{(1)}(\alpha;\epsilon)$ from Eqs. (556), using Eqs. (569), (571), (573), and (574), using Eqs. (F-9) and (F-12) in Appendix F, and using trigonometric formulas, one obtains

$$\left.\begin{aligned} a_N^{(1)} = \frac{1}{4}\epsilon\,[&-4C_5\bar{e}(\epsilon\alpha)\sin\bar{\omega} + 2(2+\bar{e}^2)\cos\alpha \\ &- C_1\bar{e}^2\cos(\alpha-2\bar{\omega}) + 4C_2\bar{e}\cos(2\alpha-\bar{\omega}) \\ &+ C_3\bar{e}^2\cos(3\alpha-2\bar{\omega})] \\ \text{and}& \\ a_M^{(1)} = \frac{1}{4}\epsilon\,[&4C_5\bar{e}(\epsilon\alpha)\cos\bar{\omega} + 2(2+\bar{e}^2)\sin\alpha \\ &+ C_1\bar{e}^2\sin(\omega-2\bar{\omega}) + 4C_2\bar{e}\sin(2\alpha-\bar{\omega}) \\ &+ C_3\bar{e}^2\sin(3\alpha-2\bar{\omega})] \end{aligned}\right\} \tag{575}$$

where

$$\left.\begin{aligned} &C_1 \triangleq \frac{1}{1-2\epsilon}, \qquad C_2 \triangleq \frac{1}{2-\epsilon}, \qquad C_3 \triangleq \frac{1}{3-2\epsilon} \\ &C_4 \triangleq 5 - 16\epsilon + 8\epsilon^2 \\ &C_5 \triangleq C_2(3-\epsilon) + \frac{1}{4}C_1C_3C_4\bar{e}^2 \end{aligned}\right\} \tag{576}$$

and where

$$\left.\begin{aligned} (a_N)_{tp}^{(1)} &= -\epsilon C_5\bar{e}(\epsilon\alpha)\sin\bar{\omega} \\ \text{and} \quad (a_M)_{tp}^{(1)} &= \epsilon C_5\bar{e}(\epsilon\alpha)\cos\bar{\omega} \end{aligned}\right\} \tag{577}$$

It should be noted that an excessive amount of tedious algebra is required in attaining $a_N^{(1)}$ and $a_M^{(1)}$ in Eqs. (575). Furthermore, the appearance of the Poisson terms, (Eqs. 577), in Eqs. (575) shows that this solution is not uniformly valid.

In a similar manner, the second-order perturbations, $a_N^{(2)}(\alpha;\epsilon)$ and $a_M^{(2)}(\alpha;\epsilon)$, are obtained using the equations for $a_N^{(2)}(\alpha;\epsilon)$ and $a_M^{(2)}(\alpha;\epsilon)$ from Eqs. (556), using Eqs. (573–575), and (577), using Eqs. (F-9) and (F-12) in Appendix F, and using trigonometric formulas:

$$\begin{aligned} a_N^{(2)} = -\frac{1}{4}\epsilon^2\{&2C_5[C_1\bar{e}^2(\epsilon\alpha)\sin(\alpha-2\bar{\omega}) \\ &- 2C_2\bar{e}(\epsilon\alpha)\sin(2\alpha-\bar{\omega}) \\ &- C_3\bar{e}^2(\epsilon\alpha)\sin(3\alpha-2\bar{\omega})] - 4C_9\cos\alpha \\ &+ 2C_1^2C_{10}\bar{e}^2\cos(\alpha-2\bar{\omega}) \\ &- C_1C_3C_6\bar{e}^3\cos(2\alpha-3\bar{\omega}) \\ &- 8C_5C_2^2\bar{e}\cos(2\alpha-\bar{\omega}) \\ &- 2C_3^2C_{11}\bar{e}^2\cos(3\alpha-2\bar{\omega}) \\ &+ C_1C_3C_7\bar{e}^3\cos(4\alpha-3\bar{\omega})\} + (a_N)_{tp}^{(2)} \end{aligned} \tag{578}$$

and

$$\begin{aligned} a_M^{(2)} = -\frac{1}{4}\epsilon^2\{&2C_5[C_1\bar{e}^2(\epsilon\alpha)\cos(\alpha-2\bar{\omega}) \\ &+ 2C_2\bar{e}(\epsilon\alpha)\cos(2\alpha-\bar{\omega}) \\ &+ C_3\bar{e}^2(\epsilon\alpha)\cos(3\alpha-2\bar{\omega})] - 4C_9\sin\alpha \\ &- 2C_1^2C_{10}\bar{e}^2\sin(\alpha-2\bar{\omega}) \\ &+ C_1C_3C_6\bar{e}^3\sin(2\alpha-3\bar{\omega}) \\ &- 8C_5C_2^2\bar{e}\sin(2\alpha-\bar{\omega}) \\ &- 2C_3^2C_{11}\bar{e}^2\sin(3\alpha-2\bar{\omega}) \\ &+ C_1C_3C_7\bar{e}^3\sin(4\alpha-3\bar{\omega})\} + (a_M)_{tp}^{(2)} \end{aligned} \tag{579}$$

where

$$\left.\begin{aligned} &C_6 \triangleq \frac{1}{2-3\epsilon}, \qquad C_7 \triangleq \frac{1}{4-3\epsilon} \\ &C_8 \triangleq 1 - 7\epsilon + 4\epsilon^2 \\ &C_9 \triangleq 2 + C_2(3-\epsilon)\bar{e}^2 \\ &C_{10} \triangleq C_5\epsilon + C_2C_3C_8 \\ &C_{11} \triangleq C_5\epsilon + C_1C_2C_8 \end{aligned}\right\} \tag{580}$$

and where

$$(a_N)_{tp}^{(2)} = \int^{\alpha}\left(\frac{da_N}{d\alpha}\right)_{tp}^{(3)} d\alpha'$$

and

$$(a_M)_{tp}^{(2)} = \int^{\alpha} \left(\frac{da_M}{d\alpha}\right)_{tp}^{(3)} d\alpha'$$

Note that $(a_N)_{tp}^{(2)}$ and $(a_M)_{tp}^{(2)}$ can not be determined without developing third-order expressions and, as a result, $a_N^{(2)}$ and $a_M^{(2)}$ are not completely determined herein. It should be clear that $(a_N)_{tp}^{(2)}$ and $(a_M)_{tp}^{(2)}$, as well as the remaining perturbations $a_N^{(3)}, a_N^{(4)}, \cdots$ and $a_M^{(3)}, a_M^{(4)}, \cdots$, can be obtained from the higher-order expressions (the use of a high-speed electronic computer is strongly recommended; see Section I-B-2).

The reference (mean) parameter $\bar{t}(\alpha; \epsilon)$ and the perturbations $t^{(1)}(\alpha; \epsilon)$ and $t^{(2)}(\alpha;\epsilon)$ are now considered. Since the time t is an unbounded function of α, the previously used concept of boundedness can not be applied in the determination of these functions. As a result, the expressions for $\bar{t}(\alpha; \epsilon)$, $t^{(1)}(\alpha; \epsilon)$, and $t^{(2)}(\alpha; \epsilon)$ from Eqs. (556) are rewritten as

$$\left.\begin{aligned}
\bar{t}(\alpha; \epsilon) &= \bar{t}_0 + \int_0^{\alpha} \left[\left(\frac{dt}{d\alpha}\right)_s^{(0)} + \left(\frac{dt}{d\alpha}\right)_{sp}^{(0)} + \left(\frac{dt}{d\alpha}\right)_{tp}^{(0)}\right] d\alpha' \\
&\quad + \int_0^{\alpha} \left[\left(\frac{dt}{d\alpha}\right)_s^{(1)} + \left(\frac{dt}{d\alpha}\right)_{tp}^{(1)}\right] d\alpha' + \int \left(\frac{dt}{d\alpha}\right)_s^{(2)} + O(\epsilon^3) \\
t^{(1)}(\alpha; \epsilon) &= \int^{\alpha} \left(\frac{dt}{d\alpha}\right)_{sp}^{(1)} d\alpha' + \int^{\alpha} \left(\frac{dt}{d\alpha}\right)_{tp}^{(2)} d\alpha' \\
\text{and} \qquad & \\
t^{(2)}(\alpha; \epsilon) &= \int^{\alpha} \left(\frac{dt}{d\alpha}\right)_{sp}^{(2)} d\alpha' + \int^{\alpha} \left(\frac{dt}{d\alpha}\right)_{tp}^{(3)} d\alpha'
\end{aligned}\right\} \tag{581}$$

where

$$\left.\begin{aligned}
\bar{t}_0 &= -\int^{0} \left[\left(\frac{dt}{d\alpha}\right)_{sp}^{(1)} + \left(\frac{dt}{d\alpha}\right)_{sp}^{(2)} + \left(\frac{dt}{d\alpha}\right)_{tp}^{(2)} + \left(\frac{dt}{d\alpha}\right)_{tp}^{(3)}\right] d\alpha' + O(\epsilon^3) \\
&= -(t^{(1)})_0 - (t^{(2)})_0 + O(\epsilon^3)
\end{aligned}\right\} \tag{582}$$

and where the integrals are to be determined individually rather than the integrands. It should be noted that the short-period terms in $t^{(2)}(\alpha; \epsilon)$ can not be determined completely (nor can the long-period terms) without determining $(a_N)_{tp}^{(2)}$ and $(a_M)_{tp}^{(2)}$ from third-order expressions (the expression for $(dt/d\alpha)_{sp}^{(2)}$ contains $(a_N)_{tp}^{(2)}$ and $(a_M)_{tp}^{(2)}$ multiplied by short-period functions yielding short-period functions; see Eqs. (578) and (579) and Eq. (F-15) in Appendix F). Consequently, $t^{(2)}(\alpha; \epsilon)$ is not determined herein.

Introducing Eqs. (574), (575), (578), and (579) into Eqs. (F-13), (F-14), and (F-15) in Appendix F and using trigonometric formulas, introducing the results into Eqs. (581) and using the integral expressions in Appendix C with $\bar{v}(\alpha; \epsilon) = \alpha - \bar{\omega}(\alpha, \epsilon)$ yields

$$\bar{t}(\alpha; \epsilon) = \tau_1 \bar{E} - \tau_2 \sin \bar{E} + [\tau_3 - \tau_4 (1 - \bar{e} \cos \bar{E})^2] (\epsilon\alpha) + T_0 \tag{583}$$

and

$$\begin{aligned}
t^{(1)}(\alpha; \epsilon) = \epsilon \{&\tau_5 \sin \bar{E} + \tau_6 \sin 2\bar{E} + [\tau_7 (1 - \bar{e} \cos \bar{E})^2 + \tau_8 (\cos \bar{E} - \bar{e})(1 - \bar{e} \cos \bar{E}) \\
&+ \tau_9 (\cos \bar{E} - \bar{e})^2] (1 - \bar{e} \cos \bar{E}) (\epsilon\alpha)\}
\end{aligned} \tag{584}$$

where

$$\bar{v}(\alpha; \epsilon) = \alpha - \bar{\omega} = (1 - \epsilon)\alpha - \bar{\omega}_0 \tag{585}$$

and

$$\overline{E}\,(\alpha;\epsilon) = 2\tan^{-1}\left[\left(\frac{1-\overline{e}}{1+\overline{e}}\right)^{1/2}\tan\frac{1}{2}\,\overline{v}\,(\alpha;\epsilon)\right] \tag{586}$$

where

$$\left.\begin{aligned}
&C_{12} \triangleq \frac{1}{1-\epsilon}\,, \qquad C_{13} \triangleq 5 - 4\epsilon + \epsilon^2\,, \qquad C_{14} \triangleq 13 - 32\epsilon + 16\epsilon^2\,, \qquad C_{15} \triangleq 5 - 2\epsilon^2 \\
&C_{16} \triangleq 7 - 16\epsilon + 8\epsilon^2\,, \qquad C_{17} \triangleq 12 - 22\epsilon + 11\epsilon^2 \\
&C_{18} \triangleq C_{12}\overline{e}\,\frac{(1-e_0^2)^{3/2}}{(1-\overline{e}^2)^{7/2}}\left[\frac{1}{2}(2+\overline{e}^2)(4-\overline{e}^2) - 2C_2(1+\overline{e}^2) - \frac{1}{2}C_1C_3\overline{e}^2(2+\overline{e}^2) - C_5C_{12}\,\epsilon\,(1-\overline{e}^2)\right] \\
&\text{and} \\
&C_{19} \triangleq C_{12}\overline{e}^2\,\frac{(1-e_0^2)^{3/2}}{(1-\overline{e}^2)^{7/2}}\left[\frac{3}{2}(2+\overline{e}^2) - C_2(1+2\overline{e}^2) + \frac{1}{2}C_1C_3(2-5\overline{e}^2) - C_5C_{12}\epsilon\,(1-\overline{e}^2)\right]
\end{aligned}\right\} \tag{587}$$

and where

$$\begin{aligned}
\tau_1 \triangleq \frac{1}{2}C_{12}\,\frac{(1-e_0^2)^{3/2}}{(1-\overline{e}^2)^{5/2}}\Bigg\{&2(1-\overline{e}^2) - \epsilon\left[(2+\overline{e}^2)^2 - 6C_2\overline{e}^2 + 3C_1C_3\overline{e}^4 - 2C_5C_{12}\epsilon\,(1-\overline{e}^2)\right] \\
&- \epsilon^2\Bigg[2C_9(2+\overline{e}^2) + 6C_1^2C_2C_3^2\left(C_8 + \frac{4C_5\epsilon}{C_2C_{12}}\right)\overline{e}^4 - C_1C_3C_6C_7(8 - 20\overline{e}^2 + 15\overline{e}^4) \\
&- \frac{3}{2}\left(2 + C_2^2C_{13}\overline{e}^2 + \frac{1}{8}C_1C_3C_{14}\overline{e}^4\right)\frac{2+3\overline{e}^2}{1-\overline{e}^2} + 6C_2\left(1 + \frac{1}{4}C_1C_3C_4\overline{e}^2\right)\frac{\overline{e}^2(4+\overline{e}^2)}{1-\overline{e}^2} \\
&+ \frac{15}{2}C_1C_3(C_2^2C_{15} + \overline{e}^2)\frac{\overline{e}^4}{1-\overline{e}^2} - 15C_1C_2C_3\frac{\overline{e}^6}{1-\overline{e}^2} + \frac{3}{8}\frac{C_1^2C_3^2}{1-\overline{e}^2}(16 - 56\overline{e}^2 + 70\overline{e}^4 - 35\overline{e}^6) \\
&- 6C_2^2C_5C_{12}(2-\epsilon^2)\overline{e}^2 + 2C_5C_{12}\epsilon\left(1 + \frac{1}{4}C_1C_3C_{16}\overline{e}^2\right)(2+\overline{e}^2) - \frac{1}{2}C_1C_3C_5C_{12}\epsilon\overline{e}^2(2 - \overline{e}^2 + 8\overline{e}^4)\Bigg]\Bigg\}
\end{aligned} \tag{588}$$

$$\left.\begin{aligned}
&\tau_2 \triangleq C_{12}\overline{e}\left(\frac{1-e_0^2}{1-\overline{e}^2}\right)^{3/2}, \qquad \tau_3 \triangleq C_1^2C_3^2C_6C_7C_{17}\epsilon\,(1-e_0^2)^{3/2}, \qquad \tau_4 \triangleq C_5C_{12}\epsilon\,\frac{(1-e_0^2)^{3/2}}{(1-\overline{e}^2)^2}\,, \qquad \tau_5 \triangleq C_{18} - C_{19}\overline{e} \\
&\tau_6 \triangleq \frac{1}{2}(C_{19} - C_{18}\overline{e})\,, \qquad \tau_7 \triangleq 2C_5C_{12}\epsilon\left(1 + \frac{2C_1C_3}{C_{12}^2}\overline{e}^2\right)\frac{(1-e_0^2)^{3/2}}{(1-\overline{e}^2)^3} \\
&\tau_8 \triangleq 2C_2C_5C_{12}\epsilon\overline{e}\,\frac{(1-e_0^2)^{3/2}}{(1-\overline{e}^2)^3}\,, \qquad \tau_9 \triangleq -2C_1C_3C_5C_{12}\epsilon\overline{e}^2\,\frac{(1-e_0^2)^{3/2}}{(1-\overline{e}^2)^3} \\
&\text{and} \\
&T_0 \triangleq \overline{t}_0 - \left[\tau_1\overline{E}\,(0) - C_{12}\overline{e}\left(\frac{1-e_0^2}{1-\overline{e}^2}\right)^{3/2}\sin\overline{E}\,(0)\right]
\end{aligned}\right\} \tag{589}$$

It should be noted that if $\kappa_1\pi \leqq \overline{v}\,(\alpha;\epsilon) \leqq \kappa_2\pi$, then $\kappa_1\pi \leqq \overline{E}\,(\alpha;\epsilon) \leqq \kappa_2\pi$, where $\kappa_1 < \kappa_2$ and both are integers.

At this point, the initial values of the reference parameters can be determined. From Eqs. (558) and (582),

$$\left.\begin{aligned}\bar{a}_{N0} &= a_{N0} - (a_N^{(1)})_0 - (a_N^{(2)})_0 + O(\epsilon^3)\\ \bar{a}_{M0} &= a_{M0} - (a_M^{(1)})_0 - (a_M^{(2)})_0 + O(\epsilon^3)\\ \text{and}\qquad \bar{t}_0 &= -(t^{(1)})_0 + O(\epsilon^2)\end{aligned}\right\} \tag{590}$$

Introducing Eqs. (575), (578), (579), and (584), evaluated at $\alpha = 0$, into Eqs. (590) yields

$$\begin{aligned}\bar{a}_{N0} = a_{N0} &- \frac{1}{4}\epsilon\,[2(2+\bar{e}^2) + 4C_2\bar{e}\cos\bar{\omega}_0 - (C_1 - C_3)\,\bar{e}^2\cos 2\bar{\omega}_0]\\ &- \frac{1}{4}\epsilon^2\,[4C_9 + 8C_5C_2^2\bar{e}\cos\bar{\omega}_0 - 2(C_1^2C_{10} - C_3^2C_{11})\,\bar{e}^2\cos 2\bar{\omega}_0\\ &+ C_1C_3(C_6 - C_7)\bar{e}^3\cos 3\bar{\omega}_0] - (a_N)_{ip}^{(2)}(0) + O(\epsilon^3)\end{aligned} \tag{591}$$

$$\begin{aligned}\bar{a}_{M0} = a_{M0} &+ \frac{1}{4}\epsilon\,[4C_2\bar{e}\sin\bar{\omega}_0 + (C_1 + C_3)\,\bar{e}^2\sin 2\bar{\omega}_0]\\ &+ \frac{1}{4}\epsilon^2\,[8C_5C_2^2\bar{e}\sin\bar{\omega}_0 + 2(C_1^2C_{10} + C_3^2C_{11})\,\bar{e}^2\sin 2\bar{\omega}_0\\ &- C_1C_3(C_6 + C_7)\bar{e}^3\sin 3\bar{\omega}_0] - (a_M)_{ip}^{(2)}(0) + O(\epsilon^3)\end{aligned} \tag{592}$$

and

$$\bar{t}_0 = -\epsilon\,[\tau_5\sin\bar{E}(0) + \tau_6\sin 2\bar{E}(0)] + O(\epsilon^2) \tag{593}$$

It should be noted that a_{N0} and a_{M0} are given by Eqs. (550) and that

$$\bar{a}_{N0} = \bar{e}\cos\bar{\omega}_0 \qquad \text{and} \qquad \bar{a}_{M0} = \bar{e}\sin\bar{\omega}_0$$

from Eqs. (571) so that Eqs. (591) and (592) are two simultaneous equations in the two unknowns $\bar{e}$ and $\bar{\omega}_0$ (the quantities $(a_N)_{ip}^{(2)}(0)$ and $(a_M)_{ip}^{(2)}(0)$ are assumed to be negligible in this determination). The solution to these equations is attained by an iteration process and the use of a high-speed electronic computer. The quantity $\bar{t}_0$ is then obtained from $\bar{e}$, $\bar{\omega}_0$, and Eq. (593).

It is clear from the appearance of the Poisson terms in Eqs. (575), (578), and (579) that an initially valid approximate solution has been obtained. Furthermore, although $a_N(\alpha;\epsilon)$ and $a_M(\alpha;\epsilon)$ have been developed to $O(\epsilon^2)$, except for the second-order long-period terms, the time can be developed to $O(\epsilon)$ only.

Finally, from the selected parameters $a_N(\alpha;\epsilon)$, $a_M(\alpha;\epsilon)$, and $t(\alpha;\epsilon)$, the desirable quantities $e(\alpha;\epsilon)$, $\omega(\alpha;\epsilon)$, $v(\alpha;\epsilon)$, $a(\alpha;\epsilon)$, $q(\alpha;\epsilon)$, $r(\alpha;\epsilon)$, $(dr/dt)(\alpha;\epsilon)$, $(d\alpha/dt)(\alpha;\epsilon)$, and $(ds/dt)(\alpha;\epsilon)$ can be obtained ($\Omega \triangleq 0$ and $i \equiv 0$). Thus, from the definitions of $a_N(\alpha;\epsilon)$ and $a_M(\alpha;\epsilon)$,

$$e(\alpha;\epsilon) = \{[a_N(\alpha;\epsilon)]^2 + [a_M(\alpha;\epsilon)]^2\}^{1/2} \tag{594}$$

and

$$\omega(\alpha;\epsilon) = \begin{cases}\tan^{-1}\left[\dfrac{a_M(\alpha;\epsilon)}{a_N(\alpha;\epsilon)}\right] & \text{for } e(\alpha;\epsilon) \neq 0\\ \pi & \text{for } e(\alpha;\epsilon) = 0\end{cases} \tag{595}$$

where

$$0 \leq \omega \leq 2\pi$$

From Eq. (314),

$$v(\alpha;\epsilon) = \alpha - \omega(\alpha;\epsilon) \qquad \text{where } 0 \leq v \leq 2\pi \tag{596}$$

Table 8. Summary of constants arising in the Taylor's series expansions representing the motion of the equatorial satellite (Part 1)

$$h = r_0^2 \dot{\alpha}_0 , \qquad e_0 = \left[\left(\frac{h^2}{r_0} - 1 \right)^2 + h^2 \dot{r}_0^2 \right]^{1/2}$$

$$\omega_0 = \begin{cases} \tan^{-1} \left(\dfrac{-h\dot{r}_0}{\dfrac{h^2}{r_0} - 1} \right) & \text{for all } \dot{r}_0 \text{ and } \dfrac{h^2}{r_0} \text{ except } \dot{r}_0 = 0 \text{ with } \dfrac{h^2}{r_0} = 1 \\ \pi & \text{for } r_0 = 0 \text{ with } \dfrac{h^2}{r_0} = 1 \end{cases}$$

$$\overline{E}(0) = -2 \tan^{-1} \left[\left(\frac{1 - \overline{e}}{1 + \overline{e}} \right)^{1/2} \tan \frac{\overline{\omega}_0}{2} \right] \qquad \begin{cases} \text{if } \kappa_1 \pi \leqq \overline{\omega}_0 \leqq \kappa_2 \pi, \text{ then } \kappa_1 \pi \leqq \overline{E}(0) \leqq \kappa_2 \pi \\ \text{where } \kappa_1 < \kappa_2 \text{ and both are integers} \end{cases}$$

$$C_1 = \frac{1}{1 - 2\epsilon}, \qquad C_2 = \frac{1}{2 - \epsilon}, \qquad C_3 = \frac{1}{3 - 2\epsilon}, \qquad C_4 = 5 - 16\epsilon + 8\epsilon^2$$

$$C_5 = C_2 (3 - \epsilon) + \frac{1}{4} C_1 C_3 C_4 \overline{e}^2 , \qquad C_6 = \frac{1}{2 - 3\epsilon}, \qquad C_7 = \frac{1}{4 - 3\epsilon}$$

$$C_8 = 1 - 7\epsilon + 4\epsilon^2 , \qquad C_9 = 2 + C_2 (3 - \epsilon) \overline{e}^2 , \qquad C_{10} = C_5 \epsilon + C_2 C_3 C_8$$

$$C_{11} = C_5 \epsilon + C_1 C_2 C_8 , \qquad C_{12} = \frac{1}{1 - \epsilon}, \qquad C_{13} = 5 - 4\epsilon + \epsilon^2 , \qquad C_{14} = 13 - 32\epsilon + 16\epsilon^2$$

$$C_{15} = 5 - 2\epsilon^2 , \qquad C_{16} = 7 - 16\epsilon + 8\epsilon^2 , \qquad C_{17} = 12 - 22\epsilon + 11\epsilon^2$$

$$C_{18} = C_{12} \overline{e} \frac{(1 - e_0^2)^{3/2}}{(1 - \overline{e}^2)^{7/2}} \left[\frac{1}{2} (2 + \overline{e}^2)(4 - \overline{e}^2) - 2C_2 (1 + \overline{e}^2) - \frac{1}{2} C_1 C_3 \overline{e}^2 (2 + \overline{e}^2) - C_5 C_{12} \epsilon (1 - \overline{e}^2) \right]$$

$$C_{19} = C_{12} \overline{e}^2 \frac{(1 - e_0^2)^{3/2}}{(1 - \overline{e}^2)^{7/2}} \left[\frac{3}{2} (2 + \overline{e}^2) - C_2 (1 + 2\overline{e}^2) + \frac{1}{2} C_1 C_3 (2 - 5\overline{e}^2) - C_5 C_{12} \epsilon (1 - \overline{e}^2) \right]$$

$$\left. \begin{aligned} \overline{e} \cos \overline{\omega}_0 \cong {} & e_0 \cos \omega_0 - \frac{1}{4} \epsilon \left[2 (2 + \overline{e}^2) + 4C_2 \overline{e} \cos \overline{\omega}_0 - (C_1 - C_3) \overline{e}^2 \cos 2\overline{\omega}_0 \right] \\ & - \frac{1}{4} \epsilon^2 \left[4C_9 + 8C_5 C_2^2 \overline{e} \cos \overline{\omega}_0 - 2 (C_1^2 C_{10} - C_3^2 C_{11}) \overline{e}^2 \cos 2\overline{\omega}_0 + C_1 C_3 (C_6 - C_7) \overline{e}^3 \cos 3\overline{\omega}_0 \right] \\ \overline{e} \sin \overline{\omega}_0 \cong {} & e_0 \sin \omega_0 + \frac{1}{4} \epsilon \left[4C_2 \overline{e} \sin \overline{\omega}_0 + (C_1 + C_3) \overline{e}^2 \sin 2\overline{\omega}_0 \right] \\ & + \frac{1}{4} \epsilon^2 \left[8C_5 C_2^2 \overline{e} \sin \overline{\omega}_0 + 2 (C_1^2 C_{10} + C_3^2 C_{11}) \overline{e}^2 \sin 2\overline{\omega}_0 - C_1 C_3 (C_6 + C_7) \overline{e}^3 \sin 3\overline{\omega}_0 \right] \end{aligned} \right\} \text{Solve by iteration for } \overline{e} \text{ and } \overline{\omega}_0$$

from Eq. (309),

$$a(\alpha;\epsilon)=\frac{h^2}{1-[e(\alpha;\epsilon)]^2} \tag{597}$$

from Eq. (310),

$$q(\alpha;\epsilon)=a(\alpha;\epsilon)[1-e(\alpha;\epsilon)] \tag{598}$$

from Ref. 564,

$$r(\alpha;\epsilon)=\frac{h^2}{1+e(\alpha;\epsilon)\cos v(\alpha;\epsilon)} \tag{599}$$

and

$$\frac{dr}{dt}(\alpha;\epsilon)=\frac{1}{h}e(\alpha;\epsilon)\sin v(\alpha;\epsilon) \tag{600}$$

from Eqs. (307),

$$\frac{d\alpha}{dt}(\alpha;\epsilon)=\frac{h}{[r(\alpha;\epsilon)]^2} \tag{601}$$

and from Eq. (531),

$$\frac{ds}{dt}(\alpha;\epsilon)=\left\{\left[\frac{dr}{dt}(\alpha;\epsilon)\right]^2+\left[r(\alpha;\epsilon)\frac{d\alpha}{dt}(\alpha;\epsilon)\right]^2\right\}^{1/2} \tag{602}$$

In conclusion, the pertinent relationships of this section are summarized in Tables 8, 9, and 10.

Table 9. Summary of constants arising in the Taylor's series expansions representing the motion of the equatorial satellite (Part 2)

$$\tau_1=\frac{1}{2}C_{12}\frac{(1-e_0^2)^{3/2}}{(1-\bar{e}^2)^{5/2}}\left\{2(1-\bar{e}^2)-\epsilon[(2+\bar{e}^2)^2-6C_2\bar{e}^2+3C_1C_3\bar{e}^4-2C_5C_{12}\,\epsilon(1-\bar{e}^2)]\right.$$

$$-\epsilon^2\left[2C_9(2+\bar{e}^2)+6C_1^2C_2C_3^2\left(C_8+\frac{4C_5\epsilon}{C_2C_{12}}\right)\bar{e}^4-C_1C_3C_6C_7(8-20\bar{e}^2+15\bar{e}^4)\right.$$

$$-\frac{3}{2}\left(2+C_2^2C_{13}\bar{e}^2+\frac{1}{8}C_1C_{13}C_{14}\bar{e}^4\right)\frac{2+3\bar{e}^2}{1-\bar{e}^2}+6C_2\left(1+\frac{1}{4}C_1C_3C_4\bar{e}^2\right)\frac{\bar{e}^2(4+\bar{e}^2)}{1-\bar{e}^2}$$

$$+\frac{15}{2}C_1C_3(C_2^2C_{15}+\bar{e}^2)\frac{\bar{e}^4}{1-\bar{e}^2}-15C_1C_2C_3\frac{\bar{e}^6}{1-\bar{e}^2}+\frac{3}{8}\frac{C_1^2C_3^2}{1-\bar{e}^2}(16-56\bar{e}^2+70\bar{e}^4-35\bar{e}^6)$$

$$\left.\left.-6C_2^2C_5C_{12}(2-\epsilon^2)\bar{e}^2+2C_5C_{12}\epsilon\left(1+\frac{1}{4}C_1C_3C_{16}\bar{e}^2\right)(2+\bar{e}^2)-\frac{1}{2}C_1C_3C_5C_{12}\epsilon\bar{e}^2(2-\bar{e}^2+8\bar{e}^4)\right]\right\}$$

$$\tau_2=C_{12}\bar{e}\left(\frac{1-e_0^2}{1-\bar{e}^2}\right)^{3/2},\qquad \tau_3=C_1^2C_3^2C_6C_7C_{17}\epsilon(1-e_0^2)^{3/2},\qquad \tau_4=C_5C_{12}\epsilon\frac{(1-e_0^2)^{3/2}}{(1-\bar{e}^2)^2}$$

$$\tau_5=C_{18}-C_{19}\bar{e},\qquad \tau_6=\frac{1}{2}(C_{19}-C_{18}\bar{e}),\qquad \tau_7=2C_5C_{12}\epsilon\left(1+\frac{2C_1C_3}{C_{12}^2}\bar{e}^2\right)\frac{(1-e_0^2)^{3/2}}{(1-\bar{e}^2)^3}$$

$$\tau_8=2C_2C_5C_{12}\epsilon e\frac{(1-e_0^2)^{3/2}}{(1-\bar{e}^2)^3},\qquad \tau_9=2C_1C_3C_5C_{12}\epsilon\bar{e}^2\frac{(1-e_0^2)^{3/2}}{(1-\bar{e}^2)^3}$$

$$\bar{t}_0\cong-\epsilon[\tau_5\sin\bar{E}(0)+\tau_6\sin 2\bar{E}(0)]$$

$$T_0=\bar{t}_0-\left[\tau_1\bar{E}(0)-C_{12}\bar{e}\left(\frac{1-e_0^2}{1-\bar{e}^2}\right)^{3/2}\sin\bar{E}(0)\right]$$

Table 10. Summary of the Taylor's series expansions representing the motion of the equatorial satellite

$$\bar{\omega} = \bar{\omega}_0 + \epsilon\alpha\,, \qquad \bar{a}_N = \bar{e}\cos\bar{\omega}\,, \qquad \bar{a}_M = \bar{e}\sin\bar{\omega}$$

$$a_N^{(1)} = \frac{1}{4}\,\epsilon\,[-4C_5\bar{e}\,(\epsilon\alpha)\sin\bar{\omega} + 2\,(2+\bar{e}^2)\cos\alpha - C_1\bar{e}^2\cos(\alpha - 2\bar{\omega})$$
$$+ 4C_2\bar{e}\cos(2\alpha - \bar{\omega}) + C_3\bar{e}^2\cos(3\alpha - 2\bar{\omega})]$$

$$a_M^{(1)} = \frac{1}{4}\,\epsilon\,[4C_5\bar{e}\,(\epsilon\alpha)\cos\bar{\omega} + 2\,(2+\bar{e}^2)\sin\alpha + C_1\bar{e}^2\sin(\alpha - 2\bar{\omega})$$
$$+ 4C_2\bar{e}\sin(2\alpha - \bar{\omega}) + C_3\bar{e}^2\sin(3\alpha - 2\bar{\omega})]$$

$$a_N^{(2)} \cong -\frac{1}{4}\,\epsilon^2\,\{2C_5\,[C_1\bar{e}^2\sin(\alpha - 2\bar{\omega}) - 2C_2\bar{e}\sin(2\alpha - \bar{\omega}) - C_3\bar{e}^2\sin(3\alpha - 2\bar{\omega})]\,(\epsilon\alpha)$$
$$- 4C_9\cos\alpha + 2C_1^2C_{10}\bar{e}^2\cos(\alpha - 2\bar{\omega}) - C_1C_3C_6\bar{e}^3\cos(2\alpha - 3\bar{\omega})$$
$$- 8C_2^2C_5\bar{e}\cos(2\alpha - \bar{\omega}) - 2C_3^2C_{11}\bar{e}^2\cos(3\alpha - 2\bar{\omega}) + C_1C_3C_7\bar{e}^3\cos(4\alpha - 3\bar{\omega})\}$$

$$a_M^{(2)} \cong -\frac{1}{4}\,\epsilon^2\,\{2C_5\,[C_1\bar{e}^2\cos(\alpha - 2\bar{\omega}) + 2C_2\bar{e}\cos(2\alpha - \bar{\omega}) + C_3\bar{e}^2\cos(3\alpha - 2\bar{\omega})]\,(\epsilon\alpha)$$
$$- 4C_9\sin\alpha - 2C_1^2C_{10}\bar{e}^2\sin(\alpha - 2\bar{\omega}) + C_1C_3C_6\bar{e}^3\sin(2\alpha - 3\bar{\omega})$$
$$- 8C_2^2C_5\bar{e}\sin(2\alpha - \bar{\omega}) - 2C_3^2C_{11}\bar{e}^2\sin(3\alpha - 2\bar{\omega}) + C_1C_3C_7\bar{e}^3\sin(4\alpha - 3\bar{\omega})\}$$

$$\bar{E} = 2\tan^{-1}\left[\left(\frac{1-\bar{e}}{1+\bar{e}}\right)^{1/2}\tan\frac{1}{2}(\alpha - \bar{\omega})\right] \quad \begin{cases}\text{if } \kappa_1\pi \leqq (\alpha - \bar{\omega}) \leqq \kappa_2\pi, \text{ then } \kappa_1\pi \leqq \bar{E} \leqq \kappa_2\pi \\ \text{where } \kappa_1 < \kappa_2 \text{ and both are integers}\end{cases}$$

$$\bar{t} = \tau_1\bar{E} - \tau_2\sin\bar{E} + [\tau_3 - \tau_4(1 - \bar{e}\cos\bar{E})^2]\,(\epsilon\alpha) + T_0$$

$$t^{(1)} = \epsilon\,\{\tau_5\sin\bar{E} + \tau_6\sin 2\bar{E} + [\tau_7(1 - \bar{e}\cos\bar{E})^2 + \tau_8(\cos\bar{E} - \bar{e})(1 - \bar{e}\cos\bar{E}) + \tau_9(\cos\bar{E} - \bar{e})^2](1 - \bar{e}\cos\bar{E})\,(\epsilon\alpha)\}$$

$$a_N = \bar{a}_N + a_N^{(1)} + a_N^{(2)} + O(\epsilon^3)\,, \qquad a_M = \bar{a}_M + a_M^{(1)} + a_M^{(2)} + O(\epsilon^3)$$

$$t = \bar{t} + t^{(1)} + O(\epsilon^2)\,, \qquad e = (a_N^2 + a_M^2)^{1/2}$$

$$\omega = \begin{cases}\tan^{-1}\left(\dfrac{a_M}{a_N}\right) & \text{for } e \neq 0 \\ \pi & \text{for } e = 0\end{cases} \qquad \text{where } 0 \leqq \omega \leqq 2\pi$$

$$v = \alpha - \omega \text{ where } 0 \leqq v \leqq 2\pi\,, \qquad a = \frac{h^2}{1-e^2}\,, \qquad q = a(1-e)$$

$$r = \frac{h^2}{1 - e\cos v}\,, \qquad \frac{dr}{dt} = \frac{e}{h}\sin v\,, \qquad \frac{d\alpha}{dt} = \frac{h}{r^2}$$

$$\frac{ds}{dt} = \left[\left(\frac{dr}{dt}\right)^2 + \left(r\,\frac{d\alpha}{dt}\right)^2\right]^{1/2}, \qquad \Omega \stackrel{\Delta}{=} 0 \qquad \text{and} \qquad i \equiv 0$$

E. Comparison of Approximate Solutions

The quality of each of the two satellite theories being investigated can be obtained by examining the numerical accuracy of the resultant approximate solution, the adaptability of the theory to various satellites and force fields, and the simplicity of both the mathematical operations involved and the form of the approximate solution (see Section 1-C-5).

1. Numerical accuracy. Consider the numerical accuracy of the two approximate solutions developed in Section V-C and V-D for the six sets of initial conditions given in Table 11 (these six sets summarize the eleven sets of initial conditions for which orbital data were obtained). The standard of comparison is the exact solution given in Section V-B.

Table 11. Initial conditions for a satellite in the equatorial plane of Mars ($\mu_{♂} = 42829.5$ km^3/s^2, $a_{♂} = 3393.4$ km, $J_{2♂} = 0.001975$)

Set	e_0	q_0, km	v_0, deg	ϵ
1	0	3578	180	0.002665
2	$\frac{3}{2} J_{2♂} (a_{♂}/r_0)^2 =$ 0.002665	3578	0	0.002651
3	0.50	3578	0	0.001184
4	0.95	3578	0	0.000701
5	0.50	8300	0	0.000220
6	0.50	13,393	0	0.000085

Note that the initial true anomaly v_0 of Set 1 is 180 deg. It follows from Eq. (311) that if

$$\left.\begin{aligned} &\frac{dr}{dt}(0) = \dot{r}_0 = 0 \qquad \text{and} \qquad \frac{d^2r}{dt^2}(0) < 0 \\ &\text{then} \\ &\qquad v_0 = 180 \text{ deg} \end{aligned}\right\} \tag{603}$$

Now, from the equations for dr/dt and r on page 73,

$$\frac{dr}{dt}(0) = \dot{r}_0 = 0 \qquad \text{and} \qquad r_0 = h^2 \tag{604}$$

for the value of the initial eccentricity e_0 of zero. From the result $r_0 = h^2$ and Eqs. (265) and (272),

$$\frac{d^2r}{dt^2}(0) = -\frac{\epsilon}{r_0^2} \tag{605}$$

and since $\epsilon > 0$ and $r_0^2 > 0$,

$$\frac{d^2r}{dt^2}(0) < 0 \tag{606}$$

Thus, v_0 must be 180 deg for $e_0 = 0$; that is, the satellite is initially at apofocus for $e_0 = 0$. In addition, because of Eq. (606) and

$$\Delta r = r - r_0 = \left[\frac{dr}{dt}(0)\right](t) + \frac{1}{2}\left[\frac{d^2r}{dt^2}(0)\right](t^2) + \cdots$$

it is clear that $\Delta r \neq 0$, and therefore the satellite orbit is not circular. In contrast, the initial eccentricity of Set 2 is slightly greater than zero and the satellite orbit is circular as shown in Appendix B (see Eq. B-18). For this case, v_0 must be 0 deg. In Sets 3–6, even though all the values of v_0 are permissible with each of the e_0, only the value of $v_0 = 0$ deg is exhibited.

The six sets of initial conditions in Table 11 represent a sufficiently broad spectrum of the possible sets of initial conditions. The dependence of the numerical accuracy of each of the two approximate solutions upon both the initial eccentricity and the initial perifocal distance is exhibited (the dependence of the numerical accuracy upon the initial true anomaly of the satellite was investigated and found to be essentially independent and, consequently, is not exhibited herein).

Figures G-1–G-140 in Appendix G present orbital data for the selected sets of initial conditions. A tabular summary of Appendix G is given in Table 12. The data for these figures were obtained using an IBM 7094 electronic computer in conjunction with "double precision" programs. The special perturbations program consists of the Cowell formulation and a fourth-order Runge–Kutta integration process with a variable step size (see Section I-B-1).

Table 12 shows that the errors in the coordinates and osculating parameters (the error in v is equal to the negative of the error in ω and, consequently, is not exhibited) are exhibited for each of the sets of initial conditions, whereas the actual variations of the coordinates and osculating parameters are exhibited only for those sets of initial conditions that yield elliptical motion, with the exception of the elliptical motion determined by special perturbations (the last column in Table 12). The error in a coordinate or parameter is defined to be the difference between the value of that coordinate or parameter determined in the exact solution and that determined in the approximate solution; for example, the error in t is

Table 12. Synopsis of the parametric comparisons for a satellite in the equatorial plane of Mars

	General perturbations solutions								Special perturbations solutions	
	Circular orbit			Elliptical orbits					Circular orbit	Elliptical orbit
e_0	$\frac{3}{2} J_{2♂} (a_{♂}/r_0)^2 = 0.002665$			0	0.50	0.95	0.50	0.50	$\frac{3}{2} J_{2♂} (a_{♂}/r_0)^2 = 0.002665$	0.95
q_0, km	3578			3578	3578	3578	8300	13,393	3578	3578
v_0, deg	0			180	0	0	0	0	0	0
Revolution	1	5,000	10,000	10,000	10,000	10,000	10,000	10,000	1	1
Error in t	G-1	G-10	G-19	G-28	G-47	G-66	G-85	G-104	G-123	G-132
Error in r	G-2	G-11	G-20	G-29	G-48	G-67	G-86	G-105	G-124	G-133
Error in dr/dt	G-3	G-12	G-21	G-30	G-49	G-68	G-87	G-106	G-125	G-134
Error in $d\alpha/dt$	G-4	G-13	G-22	G-31	G-50	G-69	G-88	G-107	G-126	G-135
Error in ds/dt	G-5	G-14	G-23	G-32	G-51	G-70	G-89	G-108	G-127	G-136
Error in a	G-6	G-15	G-24	G-33	G-52	G-71	G-90	G-109	G-128	G-137
Error in e	G-7	G-16	G-25	G-34	G-53	G-72	G-91	G-110	G-129	G-138
Error in ω	G-8	G-17	G-26	G-35	G-54	G-73	G-92	G-111	G-130	G-139
Error in q	G-9	G-18	G-27	G-36	G-55	G-74	G-93	G-112	G-131	G-140
t	—	—	—	G-37	G-56	G-75	G-94	G-113	—	—
r	—	—	—	G-38	G-57	G-76	G-95	G-114	—	—
dr/dt	—	—	—	G-39	G-58	G-77	G-96	G-115	—	—
$d\alpha/dt$	—	—	—	G-40	G-59	G-78	G-97	G-116	—	—
ds/dt	—	—	—	G-41	G-60	G-79	G-98	G-117	—	—
a	—	—	—	G-42	G-61	G-80	G-99	G-118	—	—
e	—	—	—	G-43	G-62	G-81	G-100	G-119	—	—
ω	—	—	—	G-44	G-63	G-82	G-101	G-120	—	—
v	—	—	—	G-45	G-64	G-83	G-102	G-121	—	—
q	—	—	—	G-46	G-65	G-84	G-103	G-122	—	—

equal to the exact t minus the approximate t. In addition, the independent variable for these figures is the right ascension of the satellite modified to yield portions of a revolution (the largest integral number of apsidal periods of revolution of the satellite is subtracted from the right ascension).

Consider the numerical error in the time t associated with the position of the satellite made by the approximate solutions. Figures G-1, G-10, and G-19 exhibit the numerical error in t during revolutions 1, 5000, and 10,000, respectively, for Set 2 of the initial conditions (circular orbit). It follows from these figures that the multivariable asymptotic expansion predicts the time with greater accuracy after many revolutions than does the Taylor's series expansion (in the first revolution, the errors in t are approximately the same in both expansions). The error in the prediction of the time by either expansion is extremely small; for example, Fig. G-19 shows that the error in the time is less than 3 s in the 10,000th revolution, that is, after approximately 750 days. However, it should be noted that the error in the prediction of the time by both expansions exhibits a secular growth, so that, eventually, the error will become significantly large. This behavior is characteristic in approximating an unbounded function such as the time by a series or an asymptotic expansion. As a result, the order of either solution is dictated by the required accuracy of the time in the largest revolution of interest.

Figures G-19, G-28, G-47, and G-66 exhibit the dependence of the numerical error in t, during the 10,000th revolution, upon the initial eccentricity. The increasing magnitude of the error in t with an increasing e_0 that is shown is partially due to the increasing initial period of revolution of the satellite, so that if the error in t is plotted in normalized form, a significant reduction in the increase of the error with e_0 will be exhibited. Again, the superiority of the multivariable asymptotic expansion of the time after many revolutions is exhibited.

Figures G-47, G-85, and G-104 exhibit the dependence of the numerical error in t, during the 10,000th revolution, upon the initial perifocal distance (if the error in t is plotted in normalized form, a significant reduction in the increase of the error with q_0 will be exhibited). The superiority of the multivariable asymptotic expansion of the time after many revolutions is still exhibited in

Figs. G-47 and G-85 but not in Fig. G-104, where the ϵ is very small (since $\epsilon \propto 1/h^4$ from Eq. (271) and $h^2 \propto q_0$ when e_0 is fixed from $p = h^2 = q_0(1 + e_0)$, $\epsilon \propto 1/q_0^2$, so that for a very large q_0, ϵ is very small; see Table 11). The Taylor's series expansion of the time is just as good as the multivariable asymptotic expansion of the time with respect to numerical accuracy if the ϵ is sufficiently small (this is a consequence of a theorem discussed by Picard in Ref. 1376).

Finally, Figs. G-1 and G-123 exhibit the superiority of the multivariable asymptotic expansion of the time (even in Revolution 1) to a special perturbations determination of the time (not necessarily the best special perturbations solution) in the case of a circular orbit; that is, for Set 2 of the initial conditions in Table 11. A similar result is exhibited by Fig. G-66 (the maximum error shown for the 10,000th revolution is approximately 6×10^5 s, about 0.01 percent, so that the maximum error in the first revolution is approximately 60 s) and by Fig. G-132 for Set 4 of the initial conditions.

Next, consider the numerical error in the radius distance r of the satellite made by the approximate solutions. Figures G-2, G-11, and G-20 exhibit the numerical error in r during Revolutions 1, 5000, and 10,000, respectively, for Set 2 of the initial conditions (circular orbit). It follows from these figures that the multivariable asymptotic expansion predicts the radius distance with greater accuracy after many revolutions than does the Taylor's series expansion. The error in the prediction of the radius distance by either expansion is extremely small; for example, Fig. G-20 shows that the error in the radius distance is less than 1 m in 3,578,000 m in the 10,000th revolution (after 750 days). (Note that $\epsilon^3 a_0 \cong 0.07$ m, so that the resulting error agrees with the expected error.) However, it should be noted that the error in the prediction of the radius distance by the Taylor's series expansion exhibits a Poisson growth (for example, $\epsilon\alpha \cos\alpha$) so that, eventually, the error will become significantly large. In contrast, the error in the prediction of the radius distance by the multivariable asymptotic expansion is strictly periodic (without a secular or Poisson growth). This behavior is characteristic in approximating a bounded function such as the radius distance by a multivariable asymptotic expansion. It should be noted that the apparent phase shift of the error curves in Revolutions 1, 5000, and 10,000 is caused by the definition of a revolution for a circular orbit that is used (a change in the right ascension of the satellite of 2π rad) and does not occur in the error curves for the other sets of initial conditions.

Figures G-20, G-29, G-48, and G-67 exhibit the dependence of the numerical error in r, during the 10,000th revolution, upon the initial eccentricity. The increasing magnitude of the error in r with an increasing e_0 that is shown is partially due to the increasing initial semimajor axis of the satellite, so that if the error in r is plotted in normalized form, a significant reduction in the increase of the error with e_0 will be exhibited. Again, the highly desirable periodic variation of the numerical error made by the multivariable asymptotic expansion of the radius distance is exhibited. It should be noted that the principal cause of the numerical error made by the multivariable asymptotic expansion of r after many revolutions is the approximation of the fast angle variable,

$$\overline{\alpha} = \alpha\,(1 + \epsilon^2\alpha_2 + \epsilon^3\alpha_3 + O\,(\epsilon^4))$$

as may be seen by the phase shift of the approximate r relative to the exact r in Figs. G-38, G-57, and G-76.

Figures G-48, G-86, and G-105 exhibit the dependence of the numerical error in r, during the 10,000th revolution, upon the initial perifocal distance (if the error in r is plotted in normalized form, a significant reduction in the increase of the error with q_0 will be exhibited). The superiority of the multivariable asymptotic expansion of the radius distance after many revolutions is still exhibited in Fig. G-48 but not in Figs. G-86 and G-105, where the values of ϵ are very small (the Taylor's series expansion of the radius distance is just as good as the multivariable asymptotic expansion of the radius distance with respect to numerical accuracy if the ϵ is sufficiently small).

Finally, Figs. G-2 and G-124 exhibit the superiority of the multivariable asymptotic expansion of the radius distance (even in Revolution 1) to a special perturbations determination of the radius distance in the case of a circular orbit; that is, for Set 2 of the initial conditions in Table 11. A similar result is exhibited by Fig. G-67 (the maximum error shown for the 10,000th revolution is approximately 4800 km, about 4 percent, so that the maximum error in the first revolution is approximately 0.5 km) and by Fig. G-133 for Set 4 of the initial conditions.

The remaining unbounded and bounded functions presented in Figs. G-1–G-140 in Appendix G behave similarly to t and r, respectively, and consequently are not discussed explicitly.

The preceding parametric comparisons disclose clearly that the use of the variation of coordinates with multivariable asymptotic expansions yields a numerically more accurate approximate solution after many revolutions of the satellite than does the use of the variation of parameters with Taylor's series expansions.

2. *Adaptability.* Consider the adaptability of each of the two satellite theories under discussion to various satellites and force fields. The first satellite theory, the variation of coordinates with multivariable asymptotic expansions applied to the differential equations of motion of a satellite, is applicable to various satellites and force fields as shown in Section II (it may be necessary to use the variation of parameters in place of the variation of coordinates for some force fields). In general, three multivariable asymptotic expansions must be developed simultaneously, and more than two distinct time or angle variables must be used.

The second satellite theory, the variation of parameters with Taylor's series expansions applied to the differential equations of motion of a satellite, is equally applicable to various satellites and force fields as shown in Section III. In general, the angular momentum is not constant nor is the inclination identically equal to zero, so that expansions for six parameters must be developed simultaneously; for example, $n(\theta;\epsilon)$, $a_N(\theta;\epsilon)$, $a_M(\theta;\epsilon)$, $i(\theta;\epsilon)$, $L(\theta;\epsilon)$, and $t(\theta;\epsilon)$, where θ is an angle measured in the initial orbital plane and from an inertial reference direction ($\theta = \alpha$ for an equatorial orbit).

3. *Simplicity.* Consider the simplicity of the mathematical operations involved in each of the two satellite theories being compared and consider the simplicity both of the required mathematical operations and of the form of each of the respective approximate solutions obtained. It follows from Sections II and V-C that the mathematical operations involved in the variation of coordinates with multivariable asymptotic expansions are straightforward except for, possibly, the solution of the second-order partial differential equations that arise (the use of the variation of parameters with multivariable asymptotic expansions involves the solution of only first-order partial differential equations and is, therefore, more desirable). In contrast, it follows from Sections III and V-D that the mathematical operations involved in the variation of parameters with Taylor's series expansions are straightforward, although an excessive amount of tedious algebra is necessary (avoiding algebraic errors is extremely difficult).

Finally, comparisons of Tables 5, 6, and 7 with Tables 8, 9, and 10, respectively, show that both the mathematical operations involved and the form of each of the two approximate solutions are of equal simplicity. Furthermore, the amount of information that must be stored in the memory of the IBM 7094 for a transition from one prediction point to another and the speed of computation per prediction point for the two approximate solutions are essentially the same (the third approximate solution, the special perturbations solution, required much longer computation times).

VI. Motion of a Satellite After Many Revolutions

In general, in the first-order solution for the motion of a satellite, different disturbing forces cause different perturbations (also called inequalities), although each disturbing force produces a multitude of first-order perturbations. In a higher-order solution, a perturbation may arise from the simultaneous effect of several disturbing forces, producing the effect known as perturbation superposition. This effect can lead to significant variations in the solution. (Most solutions for the motion of a satellite presume the linear independence of the perturbations resulting from the different disturbing forces, an incorrect premise since the differential equations of motion of a satellite are nonlinear.) Furthermore, several disturbing forces can act in resonance so that the resulting variations could be more significant than the sum of the individual variations. For these reasons, the motion of a satellite, even after many revolutions, subject to the simultaneous effect of the three principal disturbing forces is considered in this section.

A. Selection of Mathematical Model

A disturbing or perturbative force (or perturbation) acting on a satellite can be categorized as either a gravitational force or a nongravitational force. The gravitational forces consist of those forces that are due to the asphericity of the central mass, n bodies where $n > 2$ (usually assumed to be point masses but possibly including asphericity effects), and relativity, whereas the nongravitational forces consist of those forces that are due to aerodynamic drag and lift, low thrust, electromagnetism, solar radiation pressure, and meteoritic pressure. (Table 1 presents a partial list of references that discuss the effects of these disturbing forces upon the motion of a satellite.)

The instantaneous effects of the disturbing force (or forces) can be easily summarized by considering the orthogonal components of the force (or sum of forces) in the directions of the tangential unit vector **S**, the normal unit vector **T**, and the binormal unit vector **W** (**S** is in the line of the tangent and is positive in the direction of motion, **T** lies in the orbital plane perpendicular to **S** and is positive in the direction of the central mass, and **W** is perpendicular to the orbital plane and forms a right-handed orthonormal set with **S** and **T**; see Fig. 21). Thus, Table 13 and Fig. 21 summarize the instantaneous effects of the components of the disturbing force upon the osculating parameters of a satellite (Ref. 936). The results are for positive values of the components (for negative components, the variations change sign). It should be noted that $u = v + \omega$ (see Fig. 20).

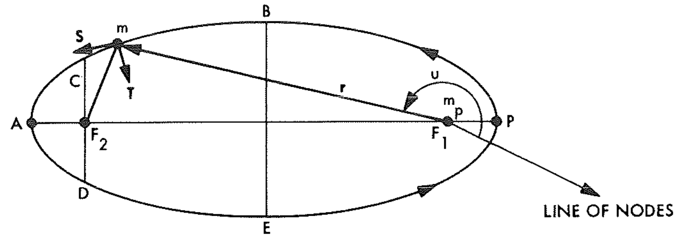

Fig. 21. The instantaneous effects of a disturbing force upon the osculating parameters of a satellite

Unfortunately, the three principal disturbing forces acting upon a satellite can not be selected unless the central mass and the range of radius distances from that central mass are specified. For example, Fig. 22 presents the approximate magnitudes of the larger accelerations acting upon a satellite about the earth (Ref. 510). It follows from this figure that the three principal disturbing forces acting upon a near-earth satellite are those due to the second and fourth harmonics of the aspherical earth and to atmospheric drag, whereas the three principal disturbing forces acting upon a distant-earth satellite are those due to the second harmonic of the aspherical earth and to lunar and

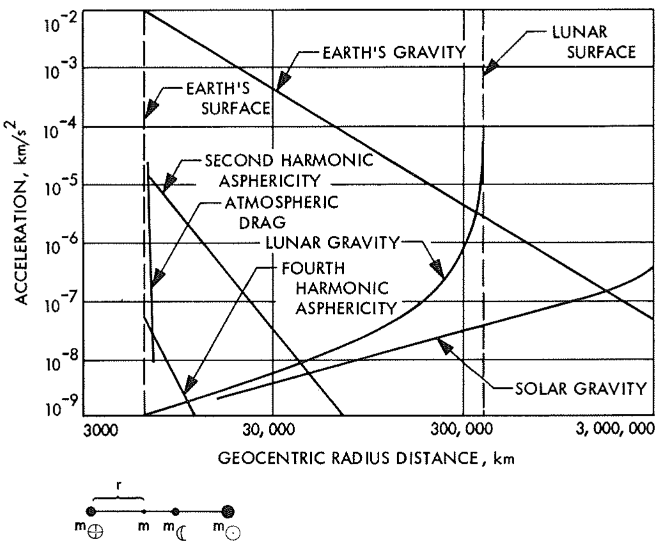

Fig. 22. Accelerations acting upon a satellite about the earth

solar gravity (the satellite is assumed to be of the type with small area-to-mass ratios so that the disturbing force due to solar radiation pressure is relatively small). In contrast, Table 14 presents the approximate magnitudes of the larger accelerations acting upon a satellite about Mars (Ref. 923). It follows from this table that the three principal disturbing forces acting upon a near-Mars satellite are those due to the second harmonic of Mars, atmospheric drag, and solar gravity (the satellite is assumed to have a representative value of the area-to-mass ratio of 0.20 cm^2/gm so that the disturbing force due to solar radiation pressure is relatively small), whereas the three principal disturbing forces acting upon a distant-Mars satellite are those due to the second harmonic of Mars, solar gravity, and solar radiation pressure.

Table 13. The instantaneous effects of a disturbing force upon the osculating parameters of a satellite

Osculating parameter	Effect for indicated direction component of disturbing force		
	S	T	W
a	Always increases	0	0
e	Increases in interval EPB; decreases in interval BAE	Decreases in interval PBA; increases in interval AEP	0
i	0	0	Increases in quadrants 1 and 4 of u; decreases in quadrants 2 and 3 of u
ω	Increases in interval PBA; decreases in interval AEP	Increases in interval DPC; decreases in interval CAD	Opposite of variation in Ω multiplied by cos i
Ω	0	0	Increases in quadrants 1 and 2 of u; decreases in quadrants 3 and 4 of u

Because of the interest in the exploration of Mars and Venus by placing artificial satellites in orbit about these celestial bodies (see Section I-A-2), the present section considers the motion of a satellite having a close perifocal

Table 14. Accelerations acting upon a satellite about Mars

Source	Magnitude of acceleration, km/s²		Direction of acceleration
	At $r = 7400$ km	At $r = 50{,}000$ km	
Martian gravity	79.0×10^{-5}	1.7×10^{-5}	To center of Mars
Second harmonic asphericity of Mars	49.0×10^{-8}	2.3×10^{-10}	Toward center of Mars
Atmospheric drag	2.7×10^{-10}	2.7×10^{-14}	Opposite to velocity vector
Solar gravity	1.6×10^{-10}	1.1×10^{-9}	Toward sun–Mars line
Solar radiation pressure	4.0×10^{-11}	4.0×10^{-11}	Opposite to direction of sun
Jupiter's gravity	—	10^{-13}	—
Earth's gravity	—	10^{-14}	—
Phobos' gravity	—	10^{-14}	—
Deimos' gravity	—	10^{-15}	—

distance and subject, simultaneously, to the three principal disturbing forces due to the second harmonic of the aspherical planet, atmospheric drag, and solar gravity.

Let **I′**, **J′**, **K′** be an inertial basis or frame of reference having a plane parallel to the equatorial plane of the planet as the principal plane; that is, the **I′**–**J′** plane in Fig. 23 (the effect of the precession of the axis of rotation of the planet upon the motion of the satellite is assumed to be negligible; see Ref. 180). Let **I**, **J**, **K**, be a translation

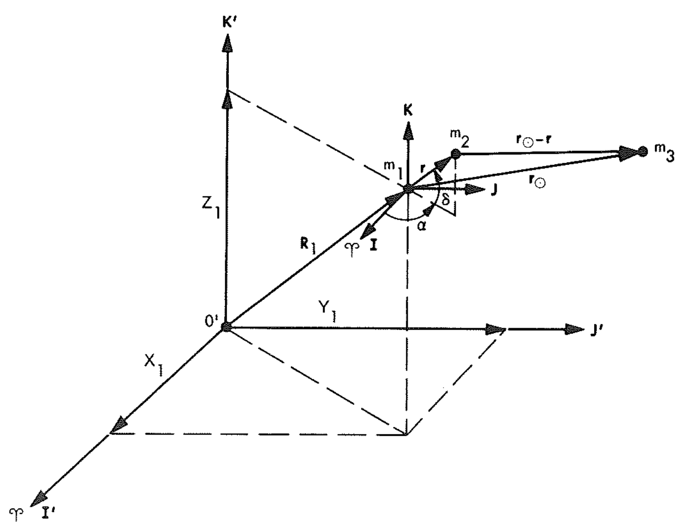

Fig. 23. The inertial basis for the motion of a satellite

of the **I′**, **J′**, **K′** basis from the inertial center 0′ to the center of the moving planet m_1 so that the **I**, **J**, **K** basis differs from the **I′**, **J′**, **K′** basis only in origin. Note that the **I**, **J**, **K** basis is not inertially fixed since its origin is the center of the accelerating planet and is used only to simplify the decomposition of vectors and vector equations. All of the vectors, their components, and the governing dynamical equations are defined relative to the **I′**, **J′**, **K′** basis.

From the law of universal gravitation and Newton's second law of motion,

$$m_i \frac{d^2 \mathbf{R}_i}{dt^2} = k^2 \sum_{j=1}^{n} m_i m_j \frac{\mathbf{R}_{ij}}{R_{ij}^3} + m_i \mathbf{p}_i \qquad (607)$$

where

$i = 1, 2, \cdots, n$, and $j \neq i$

k^2 is the gravitational constant ($k^2 = G$)

m_i and m_j are the masses of the ith and jth bodies, respectively, in the system of n bodies

$\mathbf{R}_i$ is the position vector of the ith body relative to the inertial basis

$\mathbf{p}_i$ represents all other vector accelerations existing in the system of n bodies relative to the inertial basis

and

$\mathbf{R}_{ij} = \mathbf{R}_j - \mathbf{R}_i$, from which $\mathbf{R}_{ij} = -\mathbf{R}_{ji}$ and $R_{ij} = R_{ji}$

Consider the motion of a satellite ($m_2 = m$) about a planet ($m_1 = m_p$) being disturbed by the gravitational attraction of the sun ($m_3 = m_\odot$). From Eqs. (607),

$$\left.\begin{aligned} \frac{d^2 \mathbf{R}_1}{dt^2} &= k^2 \left(m_2 \frac{\mathbf{R}_{12}}{R_{12}^3} + m_3 \frac{\mathbf{R}_{13}}{R_{13}^3} \right) + \mathbf{p}_1 \\ &\text{and} \\ \frac{d^2 \mathbf{R}_2}{dt^2} &= k^2 \left(m_1 \frac{\mathbf{R}_{21}}{R_{21}^3} + m_3 \frac{\mathbf{R}_{23}}{R_{23}^3} \right) + \mathbf{p}_2 \end{aligned}\right\} \qquad (608)$$

Let **r** be the position vector of the satellite with respect to the center of the planet; that is,

$$\mathbf{r} = \mathbf{R}_2 - \mathbf{R}_1 \qquad (609)$$

as shown in Fig. 23. Differentiating Eqs. (609) with respect to the time twice yields

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{R}_2}{dt^2} - \frac{d^2\mathbf{R}_1}{dt^2}$$

and, using Eqs. (608), one obtains

$$\frac{d^2\mathbf{r}}{dt^2} = k^2\left(m_1\frac{\mathbf{R}_{21}}{R_{21}^3} - m_2\frac{\mathbf{R}_{12}}{R_{12}^3}\right) + k^2 m_3\left(\frac{\mathbf{R}_{23}}{R_{23}^3} - \frac{\mathbf{R}_{13}}{R_{13}^3}\right) + \mathbf{p}_2 - \mathbf{p}_1 \qquad (610)$$

Introducing

$$\mathbf{R}_{21} = -\mathbf{R}_{12} \qquad \text{and} \qquad R_{21} = R_{12}$$

into Eqs. (610) yields

$$\frac{d^2\mathbf{r}}{dt^2} = -k^2(m_1 + m_2)\frac{\mathbf{R}_{12}}{R_{12}^3} + k^2 m_3\left(\frac{\mathbf{R}_{23}}{R_{23}^3} - \frac{\mathbf{R}_{13}}{R_{13}^3}\right) + \mathbf{p}_2 - \mathbf{p}_1 \qquad (611)$$

Through the use of

$$\left.\begin{aligned} &\mathbf{r} = \mathbf{R}_{12}, \qquad r = R_{12} \\ &\mathbf{R}_{23} = \mathbf{r}_\odot - \mathbf{r}, \qquad R_{23} = |\mathbf{r}_\odot - \mathbf{r}| \\ &\mathbf{R}_{13} = \mathbf{r}_\odot, \qquad R_{13} = r_\odot \\ &\mu_p = k^2(m_1 + m_2) = k^2(m_p + m) \\ &\text{and} \\ &\mu_\odot = k^2 m_3 = k^2 m_\odot \end{aligned}\right\} \qquad (612)$$

as shown in Fig. 23, Eqs. (611) become

$$\frac{d^2\mathbf{r}}{dt^2} = -\mu_p\frac{\mathbf{r}}{r^3} + \mu_\odot\left(\frac{\mathbf{r}_\odot - \mathbf{r}}{|\mathbf{r}_\odot - \mathbf{r}|^3} - \frac{\mathbf{r}_\odot}{r_\odot^3}\right) + \mathbf{p}_2 - \mathbf{p}_1 \qquad (613)$$

which are the relative motion form of the equations of motion of a satellite.

The unperturbed motion is taken as the classical two-body motion so that Eqs. (613) yield

$$\ddot{\mathbf{r}} = -\mu_p\frac{\mathbf{r}}{r^3} \qquad (614)$$

and

$$\ddot{\mathbf{r}}' = \mu_\odot\left(\frac{\mathbf{r}_\odot - \mathbf{r}}{|\mathbf{r}_\odot - \mathbf{r}|^3} - \frac{\mathbf{r}_\odot}{r_\odot^3}\right) + \mathbf{p}_2 - \mathbf{p}_1 \qquad (615)$$

where

$$\frac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}} + \ddot{\mathbf{r}}' \qquad (616)$$

from Section III-A-2. Two additional perturbing accelerations or perturbations are considered in this section. They are the perturbations due to the asphericity of the planet and the atmosphere surrounding the planet. Hence,

$$\ddot{\mathbf{r}}' = \ddot{\mathbf{r}}'_{J_2} + \ddot{\mathbf{r}}'_D + \ddot{\mathbf{r}}'_\odot \qquad (617)$$

where

$$\mathbf{p}_2 - \mathbf{p}_1 = \ddot{\mathbf{r}}'_{J_2} + \ddot{\mathbf{r}}'_D \qquad (618)$$

and

$$\ddot{\mathbf{r}}'_\odot = \mu_\odot\left(\frac{\mathbf{r}_\odot - \mathbf{r}}{|\mathbf{r}_\odot - \mathbf{r}|^3} - \frac{\mathbf{r}_\odot}{r_\odot^3}\right) \qquad (619)$$

Consider the spherical components of the vector equation of motion in Eqs. (613). It follows from Fig. 24 that

$$\begin{pmatrix}\mathbf{i}\\ \mathbf{j}\\ \mathbf{k}\end{pmatrix} = \begin{bmatrix}\cos\alpha\cos\delta & \sin\alpha\cos\delta & \sin\delta\\ -\sin\alpha & \cos\alpha & 0\\ -\cos\alpha\sin\delta & -\sin\alpha\sin\delta & \cos\delta\end{bmatrix}\begin{pmatrix}\mathbf{I}\\ \mathbf{J}\\ \mathbf{K}\end{pmatrix} \qquad (620)$$

where α and δ are the right ascension and declination of the satellite with respect to the center of the planet. Differentiating Eqs. (620) with respect to the time, noting that

$$\frac{d\mathbf{I}}{dt} = \frac{d\mathbf{J}}{dt} = \frac{d\mathbf{K}}{dt} = 0$$

since **I**, **J**, **K** is simply a translation of the inertial basis **I**′, **J**′, **K**′, and using Eqs. (620) in the results yields

$$\begin{pmatrix}\dfrac{d\mathbf{i}}{dt}\\ \dfrac{d\mathbf{j}}{dt}\\ \dfrac{d\mathbf{k}}{dt}\end{pmatrix} = \begin{bmatrix}0 & \dfrac{d\alpha}{dt}\cos\delta & \dfrac{d\delta}{dt}\\ -\dfrac{d\alpha}{dt}\cos\delta & 0 & \dfrac{d\alpha}{dt}\sin\delta\\ -\dfrac{d\delta}{dt} & -\dfrac{d\alpha}{dt}\sin\delta & 0\end{bmatrix}\begin{pmatrix}\mathbf{i}\\ \mathbf{j}\\ \mathbf{k}\end{pmatrix} \qquad (621)$$

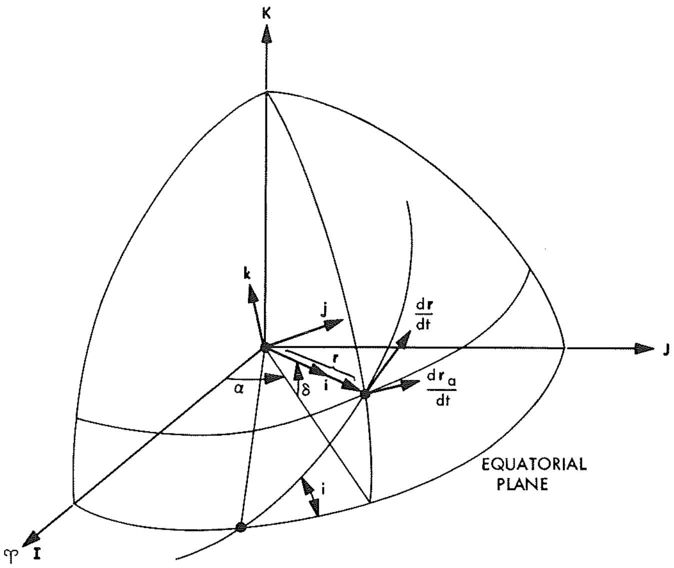

Fig. 24. The moving basis for the motion of a satellite

Now

$$\mathbf{r} = r\mathbf{i} \tag{622}$$

as shown in Fig. 24. Differentiating Eqs. (622) with respect to the time and using Eqs. (621) yields

$$\frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{i} + r\frac{d\alpha}{dt}\cos\delta\,\mathbf{j} + r\frac{d\delta}{dt}\mathbf{k} \tag{623}$$

and

$$\begin{aligned}\frac{d^2\mathbf{r}}{dt^2} = &\left[\frac{d^2 r}{dt^2} - r\left(\frac{d\delta}{dt}\right)^2 - r\left(\frac{d\alpha}{dt}\right)^2\cos^2\delta\right]\mathbf{i} \\ &+ \frac{1}{(r\cos\delta)}\left[\frac{d}{dt}\left(r^2\frac{d\alpha}{dt}\cos^2\delta\right)\right]\mathbf{j} \\ &+ \left[\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\delta}{dt}\right) + r\left(\frac{d\alpha}{dt}\right)^2\sin\delta\cos\delta\right]\mathbf{k}\end{aligned} \tag{624}$$

Equations (624) express the acceleration vector in terms of its spherical components. Introducing Eqs. (614), (622), and (624) into Eqs. (616) and taking the dot product of the result with respect to **i**, **j**, and **k**, respectively, yields

$$\left.\begin{aligned}&\frac{d^2 r}{dt^2} - r\left(\frac{d\delta}{dt}\right)^2 - r\left(\frac{d\alpha}{dt}\right)^2\cos^2\delta = -\frac{\mu_p}{r^2} + \ddot{\mathbf{r}}'\cdot\mathbf{i} \\ &\frac{1}{r\cos\delta}\left[\frac{d}{dt}\left(r^2\frac{d\alpha}{dt}\cos^2\delta\right)\right] = \ddot{\mathbf{r}}'\cdot\mathbf{j} \\ &\text{and} \\ &\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\delta}{dt}\right) + r\left(\frac{d\alpha}{dt}\right)^2\sin\delta\cos\delta = \ddot{\mathbf{r}}'\cdot\mathbf{k}\end{aligned}\right\} \tag{625}$$

Next, consider the perturbation due to the second harmonic of the aspherical planet, $\ddot{\mathbf{r}}'_{J_2}$. The perturbative function due to J_2 of the planet is given in Ref. 564:

$$R = \frac{1}{2}\mu_p J_2 a_p^2\left(\frac{1}{r^3}\right)(1 - 3\sin^2\delta) \tag{626}$$

It should be noted that the planet is assumed to be an axially symmetric oblate body with symmetry in the northern and southern hemispheres. The perturbative acceleration $\ddot{\mathbf{r}}'_{J_2}$ can be written as

$$\ddot{\mathbf{r}}'_{J_2} = \frac{\partial R}{\partial r}\mathbf{i} + \frac{1}{r\cos\delta}\frac{\partial R}{\partial\alpha}\mathbf{j} + \frac{1}{r}\frac{\partial R}{\partial\delta}\mathbf{k}$$

so that, using Eq. (626), one obtains

$$\left.\begin{aligned}&\ddot{\mathbf{r}}'_{J_2}\cdot\mathbf{i} = \frac{\partial R}{\partial r} = -\frac{3}{2}\mu_p J_2 a_p^2\left(\frac{1}{r^4}\right)(1 - 3\sin^2\delta) \\ &\ddot{\mathbf{r}}'_{J_2}\cdot\mathbf{j} = \frac{1}{r\cos\delta}\frac{\partial R}{\partial\alpha} = 0 \\ &\text{and} \\ &\ddot{\mathbf{r}}'_{J_2}\cdot\mathbf{k} = \frac{1}{r}\frac{\partial R}{\partial\delta} = -3\mu_p J_2 a_p^2\left(\frac{1}{r^4}\right)(\sin\delta\cos\delta)\end{aligned}\right\} \tag{627}$$

Consider the drag perturbation due to the atmosphere surrounding the planet, $\ddot{\mathbf{r}}'_D$. The perturbative acceleration $\ddot{\mathbf{r}}'_D$ can be written as (Ref. 657)

$$\ddot{\mathbf{r}}'_D = -\frac{1}{2}\frac{C_D S}{m}\rho(r)\left|\frac{d\mathbf{r}}{dt} - \frac{d\mathbf{r}_a}{dt}\right|\left(\frac{d\mathbf{r}}{dt} - \frac{d\mathbf{r}_a}{dt}\right) \tag{628}$$

where

$\rho(r)$ is the atmospheric density, which varies with r (other functional dependencies are assumed negligible)

$d\mathbf{r}/dt$ is the velocity of the satellite given by Eqs. 623

$d\mathbf{r}_a/dt$ is the velocity of the atmosphere

C_D is the drag coefficient, which depends upon the shape and orientation of the satellite and the manner of reflection of the atmospheric particles

S is a characteristic area of the satellite

and

m is the mass of the satellite

Finally, introducing Eqs. (619), (627), and (628) into Eqs. (617) yields

$$\dot{\mathbf{r}}' = -3\mu_p J_2 a_p^2 \left(\frac{1}{r^4}\right)\left[\frac{1}{2}(1 - 3\sin^2\delta)\,\mathbf{i} + (\sin\delta\cos\delta)\,\mathbf{k}\right] - \frac{1}{2}\frac{C_D S}{m}\rho(r)\left|\frac{d\mathbf{r}}{dt} - \frac{d\mathbf{r}_a}{dt}\right|\left(\frac{d\mathbf{r}}{dt} - \frac{d\mathbf{r}_a}{dt}\right) + \mu_\odot\left(\frac{\mathbf{r}_\odot - \mathbf{r}}{|\mathbf{r}_\odot - \mathbf{r}|^3} - \frac{\mathbf{r}_\odot}{r_\odot^3}\right) \quad (629)$$

from which $\dot{\mathbf{r}}' \cdot \mathbf{i}$, $\dot{\mathbf{r}}' \cdot \mathbf{j}$, and $\dot{\mathbf{r}}' \cdot \mathbf{k}$ can be obtained. These are then introduced into Eqs. (625), yielding the desired differential equations of motion of the satellite. The initial conditions for these equations are

$$\left.\begin{aligned} r(t_0) &= r_0, & \frac{dr}{dt}(t_0) &= \dot{r}_0 \\ \alpha(t_0) &= \alpha_0, & \frac{d\alpha}{dt}(t_0) &= \dot{\alpha}_0 \\ \delta(t_0) &= \delta_0, \quad \text{and} & \frac{d\delta}{dt}(t_0) &= \dot{\delta}_0 \end{aligned}\right\} \quad (630)$$

where, if $J_2 = C_D = \mu_\odot = 0\,(\dot{\mathbf{r}}' = 0)$, these initial conditions would yield bounded (periodic) motion; that is, the satellite's orbit would be circular or elliptical. In addition, the range of time that is of interest in this formulation is

$$t_0 \leqq t \leqq \infty \quad (631)$$

It should be noted that simplifications of both the perturbation due to atmospheric drag and the perturbation due to solar gravity are necessary before an approximate solution to the motion of the satellite can be obtained using a general perturbations method. The necessary simplifications as well as the proper normalization of the differential equations of motion are left as initial steps in the next phase of investigation into the motion of a satellite.

B. Selection of Method of Solution

Due to the highly nonlinear nature of the differential equations of motion given by Eqs. (625) and (629), an exact analytic solution is not possible, so that an approximate method of solution must be used. In accordance with the comparisons in Sections IV-H and V-E, it is concluded that the selection criteria specified in Section I-C-2 are best satisfied by the satellite theory consisting of the application of the variation of parameters with multivariable asymptotic expansions to the set of ordinary nonlinear differential equations governing the motion of the satellite. This satellite theory is described in Section III.

C. Selection of Parameters and Independent Variable

In order to avoid small divisors, mathematical singularities, and slowly converging infinite series, a nonsingular set of instantaneous multivariable parameters (see Section V-C-3) and an angular variable (measured with respect to an inertial direction) are selected as the osculating parameters and independent variable, respectively. The explicit selection of these quantities is left as a step in the next phase of investigation into the motion of a satellite.

D. Equations of Motion

The differential equations of motion for the selected instantaneous multivariable parameters and the selected independent variable are obtained from Eqs. (625) and (629) using the procedure of Section IV-G-2. These equations are first-order ordinary nonlinear differential equations.

E. Development of Solution

The procedure in developing the approximate solution is given in Section III-B. It should be noted that a large amount of analytical labor is required in this section. Not only is this overwhelming analysis time-consuming, but a small error at the beginning of the development of the solution propagates itself through the analysis, vitiating the effort. Consequently, the use of automated manipulation techniques is desirable.

In order for automated manipulation to be useful, it should be necessary to state only the particulars of the problem and of the development of the solution in order to obtain the solution in the desired form. The computer algorithms for many of the basic manipulations which would be comprised in such a hybrid program have already been developed: for example, the ALPAK system developed at Bell Telephone Laboratories (Ref. 216), the FORMAC system developed at International Business Machines, Incorporated (Refs. 155–158, 1251, and 1252), and the FORMAN system under development at Computer Sciences Corporation, Los Angeles, Calif.

VII. Summary

The final section of this report summarizes the conclusions and original contributions and presents recommendations for further study (for a summary of the contents of the entire report, refer to Section I-D).

A. Conclusions

It follows from Section I-B-2 that the methods of general perturbations are ideally suited for the prediction of orbits extending over many periods such as for those for artificial and natural satellites. (In some cases of quasi-periodic motion, it is probable that the best means of determining the motion will be based on some combination of special and general perturbations.)

In Section IV, five theories of general perturbations are applied to the elementary problem of the damped linear harmonic oscillator illustrating the mathematical processes involved and the advantages and disadvantages of each theory. Only two of these theories yield the superior uniformly valid approximation: the variation of coordinates with multivariable asymptotic expansions and the variation of parameters with multivariable asymptotic expansions. The application of either of these theories to the set of differential equations that govern the motion of a satellite constitutes an acceptable satellite theory (see Section I-C-2). The decision as to which of these theories is more desirable depends upon the complexity of the set of differential equations that govern the motion of the satellite. If this set of differential equations is relatively simple, as in the motion of an equatorial satellite about an oblate body, it is just as desirable to use the variation of coordinates with multivariable asymptotic expansions as it is to use the variation of parameters with multivariable asymptotic expansions. However, if this set of differential equations is complex, as in the general motion of a satellite acted upon by several disturbing forces, it is more desirable to use the variation of parameters with multivariable asymptotic expansions (more than two distinct variables are necessary in this case). Finally, a third theory, the variation of parameters with Taylor's series expansions, yields an approximate solution that is initially valid but has a large range of validity, so that the resulting numerical accuracy may be sufficient over a large enough range of time for practical applications.

Two satellite theories are investigated in Section V: the application of the theory of the variation of coordinates with multivariable asymptotic expansions and the theory of the variation of parameters with Taylor's series expansions, respectively, to the differential equations governing the motion of a satellite in an equatorial orbit about an oblate body (second harmonic only). In addition, the exact solution is developed for this restricted satellite motion. In the exact solution, the right ascension of the satellite, $\alpha(r;\epsilon)$, consists of constants and a Legendre normal elliptic integral of the first kind, whereas the time associated with the position of the satellite, $t(r;\epsilon)$, consists of constants, powers of r, and Legendre normal elliptic integrals of the first, second, and third kinds. In the theory of the variation of coordinates with multivariable asymptotic expansions, the inverse of the radius distance of the satellite and the time associated with the position of the satellite are both successfully approximated to $O(\epsilon^2)$ by uniformly valid asymptotic expansions in terms of α. During the development of the asymptotic expansion for the time, the argument of consistency used by Kevorkian (Ref. 665) is inappropriate and is replaced by the second uniformity condition given in Section II-A-3. As expected, the theory of the variation of parameters with Taylor's series expansions yields initially valid expansions and, therefore, is inferior with respect to the numerical accuracy after many revolutions of the satellite about the oblate body (the majority of the satellite theories reviewed in Refs. 1–1370 yield initially valid solutions). In addition, the variation of parameters with Taylor's series expansions, although equally adaptable, is more difficult to apply to general satellite motion than is the variation of coordinates (or parameters) with multivariable asymptotic expansions.

Finally, the recommended satellite theory consists of the application of the variation of parameters with multivariable asymptotic expansions to the set of ordinary nonlinear differential equations governing the motion of a satellite. This theory predicts the position and velocity of a satellite to a sufficiently high accuracy even after many revolutions about the central mass (the remaining selection criteria specified in Section I-C-2 are also satisfied).

B. Original Contributions

There are four primary contributions. First, a systematic study on the prediction of the position and velocity of a satellite after many revolutions about the central mass is presented. Second, a comprehensive discussion of the theory of multivariable asymptotic expansions is provided and the concepts of the uniformity conditions are introduced (see Section II). Prior to this discussion, this theory had been limited primarily to two-variable asymptotic expansions and to the use of boundedness and consistency conditions. Third, the theory of the variation of parameters with multivariable asymptotic expansions is developed and its usefulness demonstrated (see Sections III and IV-G, respectively). Finally, thorough numerical accuracy studies of the uniformly valid asymptotic expansions obtained by using the concepts of multivariable asymptotic expansions are made, and the effectiveness of these concepts is established (see Sections IV-H and V-E-1 as well as Appendix G).

In addition to the primary contributions, there are four secondary contributions. First, a complete set of exact equations for the motion of a satellite in an equatorial orbit about an oblate body (second harmonic only) is obtained; that is, equations are developed which yield both the position and velocity of the satellite after many revolutions (see Section V-B). Prior to this analysis, only the radius distance as a function of an inertial angle had been developed. Second, a complete and uniformly valid set of approximate expressions (to $O(\epsilon^2)$) for the motion of a satellite in an equatorial orbit about an oblate body is presented (see Section V-C). This approximate solution clearly shows the effects of the oblateness of the central mass upon the motion of the satellite, whereas the exact solution does not. Third, the limited usefulness of a solution obtained by using the theory of the variation of parameters with Taylor's series expansions as developed in Sections IV-F and V-D is exhibited (the solution is initially valid). Finally, the bibliography provides one of the largest lists of references on satellite theory.

C. Recommendations for Further Study

The next important step in the investigation of the motion of a satellite after many revolutions is to apply the variation of parameters with multivariable asymptotic expansions to the differential equations of motion of a satellite being perturbed by the asphericity of the central mass (second harmonic only), atmospheric drag, and the gravitational attraction of the sun, simultaneously, as outlined in Section VI of this dissertation (it may be advantageous to first apply this perturbation theory to the differential equations of motion of a satellite being perturbed by only the asphericity of the central mass, allowing all inclinations, and then to the differential equations of motion of a satellite in an equatorial orbit being perturbed by the asphericity of the central mass, atmospheric drag, and the gravitational attraction of the sun, simultaneously). Furthermore, since a large amount of tedious algebra is involved (characteristic of general perturbations methods), a study into the use of a high-speed electronic computer to aid in the development of the solution is desirable.

Eventually, the motion of a satellite acted upon by a more complex force field than those previously discussed will be required. Consequently, the use of the variation of parameters with multivariable asymptotic expansions in conjunction with a special perturbations method will probably be necessary and should be investigated.

Nomenclature

Definitions

Analytic solution. A solution in the form of symbolic formulas which express the sought-for quantities as explicit functions of the independent variable (usually time), the constants of the problem, and the constants of integration which are determined by the initial conditions of the problem.

Apofocus. A point on an orbit farthest from the principal focus or dynamical center. Designated r_A or $r_{\max}$.

Apsidal period of revolution. The time elapsed when an orbiting body completes one revolution from perifocus to perifocus (usually more or less than 360 deg).

Apsis. A point on an orbit where the radius distance is a minimum (perifocus, designated q or $r_{\min}$) or a maximum (apofocus, designated r_A or $r_{\max}$).

Asymptotic expansion of $x(t;\epsilon)$ as $\epsilon \to 0$. The series

$$\sum_{j=0}^{J} \nu_j(\epsilon)\, x^{(j)}(t)$$

where

$$x(t;\epsilon) = \sum_{j=0}^{J} \nu_j(\epsilon)\, x^{(j)}(t) + O(\nu_{J+1}(\epsilon)) \text{ as } \epsilon \to 0$$

and where $\nu_j(\epsilon)$ is an asymptotic sequence as $\epsilon \to 0$. (See Section II-A-2.)

Asymptotic sequence for $\epsilon \to 0$. A sequence of functions $\nu_j(\epsilon)$ where $j = 0, 1, 2, \cdots$ and where

$$\lim_{\epsilon \to 0} \frac{\nu_{j+1}(\epsilon)}{\nu_j(\epsilon)} = 0$$

for each j. (See Section II-A-2.)

Classical lunar theory. The theory of motion of the moon under the gravitational attraction of the earth and the disturbing sun where all three bodies are treated as point masses. Sometimes called the main problem of the lunar theory.

Classical planetary theory. The theory of motion of a planet under the gravitational attraction of the sun and a disturbing planet, where all three bodies are treated as point masses.

Cowell's method of special perturbations. The method in astrodynamics which consists of the direct, step-by-step, numerical integration of the total acceleration (central and perturbative) of a body in motion with no reference to an osculating or other reference orbit. The differential equations of motion, which must be integrated twice to obtain the position of the disturbed body, are (in normalized vector form)

$$\frac{d^2 \mathbf{r}}{dt^2} = -\frac{\mathbf{r}}{r^3} + \ddot{\mathbf{r}}'$$

where $\ddot{\mathbf{r}}'$ is the total perturbative acceleration due to asphericity, drag, and so forth. It should be noted that this method is not strictly a perturbation method since the term perturbation implies that a distinction is made between the principal terms and the perturbation terms in the accelerations. However, in keeping with astronomical tradition, this method is classified as a perturbation method. (See Ref. 564.)

Critical inclination. The perturbations due to the aspherical central mass introduce various terms into the solution of the motion of the satellite, which have the denominator $(1 - 5\cos^2 i)$. This denominator becomes zero at the critical inclination of 63.4 deg and, as a result, mathematical singularities arise in the solution of the motion of the satellite.

Encke's method of special perturbations. The method in astrodynamics which consists of the step-by-step numerical integration of the difference between the total acceleration of a body in motion and the acceleration of the body in an osculating or reference orbit; that is, the acceleration the body would have if it continued to move in the conic section corresponding to the position and velocity at a particular instant called the epoch of osculation (the departures from the osculating orbit are called perturbations and are zero at the epoch of osculation). There are a relatively small number of epochs of osculation in this method as compared with the variation of parameters method of special perturbations, where the reference orbit is continuously changing. The differential equations of motion, which must be integrated twice to obtain the perturbation in position, are (in normalized vector form)

$$\frac{d^2 \boldsymbol{\rho}}{dt^2} = -\frac{\mathbf{r}}{r^3} + \frac{\mathbf{r}_e}{r_e^3} + \ddot{\mathbf{r}}'$$

where $\ddot{\mathbf{r}}'$ is the total perturbative acceleration due to asphericity, drag, and so forth and where $\boldsymbol{\rho}$ is the difference between the actual position vector and that of the osculating orbit; that is, $\boldsymbol{\rho} = \mathbf{r} - \mathbf{r}_e$. (See Ref. 564.)

Epoch. An instant of time for which the initial conditions, elements, parameters, or constants of an orbit or trajectory are specified. Designated t_0.

Equatorial satellite. A satellite whose orbital plane coincides with the equatorial plane of the central mass.

General perturbations. The class of perturbation methods that consists of analytical methods in which the accelerations (usually the perturbative accelerations) are expanded into infinite series and integrated term by term.

Harmonics. A mathematical harmonic progression of terms that represent the dynamical differences between a reference spheroid and a sphere. The zonal harmonics represent latitudinal differences, the sectoral harmonics represent longitudinal differences, and the tesseral harmonics represent both latitudinal and longitudinal differences.

Initially valid asymptotic expansion of $x(t;\epsilon)$ as $\epsilon \rightarrow 0$. The asymptotic expansion of $x(t;\epsilon)$ as $\epsilon \rightarrow 0$ where the remainder is of $O(\nu_{J+1}(\epsilon))$ only in $0 \leqq t < t_1$ and where t_1 is finite (the entire range of t is assumed to be $0 \leqq t < \infty$). (See Section II-A-2).

Instability in numerical integration. The occurrence of an approximate solution calculated by a finite difference method that is unstable (unbounded) even though the solution of the differential equation is inherently stable (bounded). This phenomenon usually occurs when the difference equation used is of higher order than the differential equation, for it then has more independent solutions than the differential equation, and among them there may be increasing solutions even when the differential equation possesses only decreasing solutions. (See Refs. 292, 357, and 358.)

Instantaneous or osculating parameters (or elements). The continually changing parameters (or elements) of the osculating orbit.

Integral. In general,

$$\int_0^t Q(q_1, q_2, \cdots, \tau)\, d\tau = q(q_1, q_2, \cdots, t) - q(q_1(0), q_2(0), \cdots, 0)$$

where $q_1, q_2, \cdots$ are functions of t. Let

$$q(q_1, q_2, \cdots, t) \triangleq \int^t Q(q_1, q_2, \cdots, \tau)\, d\tau$$

and

$$q(q_1(0), q_2(0), \cdots, 0) \triangleq \int^0 Q(q_1, q_2, \cdots, \tau)\, d\tau$$

so that

$$\int_0^t Q(q_1, q_2, \cdots, \tau)\, d\tau = \int^t Q(q_1, q_2, \cdots, \tau)\, d\tau - \int^0 Q(q_1, q_2, \cdots, \tau)\, d\tau$$

In this manner,

$$\int^t Q(q_1, q_2, \cdots, \tau)\, d\tau \Rightarrow \text{integrate integrand and evaluate at } \tau = t$$

and

$$\int^0 Q(q_1, q_2, \cdots, \tau)\, d\tau \Rightarrow \text{integrate integrand and evaluate at } \tau = 0 \text{ (yielding a constant).}$$

Long-period perturbation. A perturbation that is periodic in the independent variable and that has a large period; for example, $\cos \epsilon\alpha$.

Multivariable asymptotic expansion of $x(t;\epsilon)$ as $\epsilon \rightarrow 0$. The series

$$\sum_{j=0}^{J} \nu_j(\epsilon)\, x^{(j)}(\bar{t}, \tilde{t}, \cdots)$$

where

$$x(t;\epsilon) = X(\bar{t}, \tilde{t}, \cdots; \epsilon) = \sum_{j=0}^{J} \nu_j(\epsilon)\, x^{(j)}(\bar{t}, \tilde{t}, \cdots) + O\left(\nu_{J+1}(\epsilon)\right) \text{ as } \epsilon \rightarrow 0$$

and where $\nu_j(\epsilon)$ is an asymptotic sequence as $\epsilon \rightarrow 0$ and the variables $\bar{t}, \tilde{t}, \cdots$ are functions of ϵ multiplied linearly by t. The series is uniformly valid if the remainder is of $O\left(\nu_{J+1}(\epsilon)\right)$ for all t in the range of t. (See Sections II-A-2 and II-A-4.)

Normalization of variables. The process of dividing each variable of a set of equations by an appropriate scale, for example, $x^* = x/L_1$ and $t^* = t/T_1$ where L_1 is a length scale and T_1 is a time scale.

Order of a function. The mathematical measure of the magnitude of a function depending upon a vanishingly small parameter ϵ in a perturbation solution. (See Section II-A-1.)

Order of an approximate solution. The highest order of terms completely determined in the solution; for example, a first-order solution implies that "all" of the first-order terms in the solution have been determined.

Osculating orbit. A continually changing conic section that is tangent to the path of motion at every point and that yields the same velocity at the point of tangency as the actual velocity at that point.

Perifocus. A point on an orbit closest to the principal focus or dynamical center. Designated q or r_{min}.

Perturbation. A deviation from a reference orbit (not necessarily a conic section) in force, acceleration, velocity, or position.

Perturbation theory. The application of a special method of solution to a set of differential equations of motion containing a small parameter (or parameters). In astrodynamics, the theory of motion of a body in a noncentral force field; that is, a central force field plus perturbative forces.

Perturbative differentiation. The process of differentiation in which only the variations due to the disturbing forces are considered and in which $\mathbf{r}' = 0$ (see *Variation*).

Planetary theory. The theory of motion of a point mass moving around a central point mass so massive as to dominate the system but under the disturbing influence of other point masses, relativity, and so forth. In general, the point mass moves in a nearly circular orbit that lies in a slightly varying plane.

Poisson or mixed perturbation. A perturbation that is a product of a power of the independent variable and a periodic perturbation; for example, $\alpha \cos \alpha$ or $\alpha \cos \epsilon\alpha$.

Potential or force function. A function Φ whose partial derivative with respect to a coordinate or parameter yields the force component associated with that coordinate or parameter; for example,

$$F_r = m\frac{\partial \Phi}{\partial r}$$

In general, this function is the negative of the potential energy per unit mass, for example, the potential function for Keplerian motion is

$$\Phi = \frac{\mu_p}{r}$$

Range of validity. The range of the independent variable during which the actual error of an approximate solution is less than or equal to the magnitude of the first neglected term of the approximate solution.

Reference or intermediate orbit. An approximate orbit to which perturbations are added to obtain the actual or perturbed orbit.

Round-off or rounding error in numerical integration. The difference between the computed and the exact solutions of the difference equations. The influence of round-off error is usually prominent when a large number of steps have been taken, usually occurring when a small value of step size is used. (See Refs. 292, 357, and 358.)

Runge–Kutta numerical integration process. A self-starting process for the integration of ordinary differential equations using numerical analysis. For example, the general fourth-order integration formulas for a second-order differential equation in the variable r are

$$r_{i+1} = r_i + \frac{1}{6}h(\ell_1 + 2\ell_2 + 2\ell_3 + \ell_4)$$

and

$$\dot{r}_{i+1} = \dot{r}_i + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4)$$

where h is the interval of integration (step size) and where the ℓ_j and k_j, $j = 1, 2, 3, 4$, depend upon h, r_i, and $\dot{r}_i$. (See Ref. 564.)

Satellite theory. The application of a perturbation theory to the set of ordinary nonlinear differential equations that determine the motion of a satellite. More specifically, the theory of motion of a point mass moving around a central mass (not necessarily a point mass) so massive as to dominate the system but under the disturbing influence of other point masses, asphericity of the central mass, atmospheric resistance, and so forth. In general, the point mass moves in an osculating elliptical orbit that lies in a moderately varying plane.

Second-sum or Gauss–Jackson numerical integration process. A process for the integration of ordinary differential equations using numerical analysis (it should be noted that the starting values of the sum and difference table must be supplied). For example, the general integration formula for a second-order differential equation in the variable r is

$$r_i = h^2\left(\Sigma^2 \ddot{r}_i + \frac{1}{12}\ddot{r}_i - \frac{1}{240}\delta^2 \ddot{r}_i + \cdots\right)$$

where h is the interval of integration (step size), $\Sigma^2 \ddot{r}_i$ is the second sum of $\ddot{r}_i$, and $\delta^2 \ddot{r}_i$ is the second difference of $\ddot{r}_i$. (See Ref. 564.)

Secular perturbation. A perturbation that is proportional to a power of the independent variable; for example, $C\alpha$.

Short-period perturbation. A perturbation that is periodic in the independent variable and that has a short period; for example, $\cos \alpha$.

Special perturbations. The class of perturbation methods that consists of methods in which the accelerations are integrated using numerical analysis (numerical integration or mechanical quadratures).

State vector. A set of position and velocity components or osculating parameters that completely describe the dynamical state of a moving object at a time t; for example, the set $r, \alpha, \delta, \dot{r}, \dot{\alpha}, \dot{\delta}$ or the set $a, e, i, \omega, \Omega, M$.

Theory of general perturbations using the variation of parameters with averaging. In this theory, the orbit of a satellite is defined by six parameters of elliptic motion expressed as symbols and thought of as continuously varying because of the perturbations that they undergo. Thus, the equations of motion will be of the form

$$\frac{dq_j}{dt} = \epsilon Q_j (q_1, q_2, \cdots, q_6, t), \qquad j = 1, 2, \cdots, 6$$

where ϵ is a small parameter, $\epsilon << 1$. A typical set of q_j is the set $a, e, i, \omega, \Omega, M$. Introducing the shorthand notation $Q_j(q, t)$ in place of

$$Q_j (q_1, q_2, \cdots, q_6, t)$$

(a similar notation is used for other functions that arise), the independent variable t is such that the $Q_j(q, t)$ are periodic in t, of period τ, so that

$$Q_j (q, t + \tau) = Q_j (q, t) \text{ for all } j$$

Also, $Q_j(q, t)$ can be represented by

$$Q_j (q, t) = Q_j^{(0)} (q, t) + \epsilon\, Q_j^{(1)} (q, t) + \cdots$$

Because the first derivatives dq_j/dt are proportional to the small parameter ϵ, it is plausible to consider the q_j as slowly varying quantities. Hence, each q_j can be represented by the sum of a smoothly varying term $\overline{q}_j$ (secular or long-period terms) and small vibrational terms (short-period terms). Thus,

$$q_j = \overline{q}_j + \epsilon \eta_j^{(1)} (\overline{q}, t) + \epsilon^2 \eta_j^{(2)} (\overline{q}, t) + \cdots$$

where the $\eta_j^{(k)} (\overline{q}, t)$ have period τ in t and the smoothly varying quantities $\overline{q}_j$ satisfy the averaged equations

$$\frac{d\overline{q}_j}{dt} = \epsilon \xi_j^{(1)} (\overline{q}) + \epsilon^2 \xi_j^{(2)} (\overline{q}) + \cdots$$

It can be shown that

$$\xi_j^{(1)} (\overline{q}) = \frac{1}{\tau} \int_0^\tau Q_j^{(0)} (\overline{q}, t)\, dt \triangleq \overline{Q}_j^{(1)} (\overline{q})$$

and

$$\eta_j^{(1)} (\overline{q}, t) = \int [Q_j^{(0)} (q, t) - \overline{Q}_j^{(1)} (\overline{q})]\, dt \triangleq Q_{jp}^{(1)} (\overline{q}, t)$$

where the $\overline{q}_j$ are held constant in evaluating the integrals and the constants of integration associated with the indefinite integrals are set equal to zero. Furthermore,

$$\xi_j^{(2)} (\overline{q}) = \frac{1}{\tau} \int_0^\tau \left[Q_j^{(1)} (\overline{q}, t) + \frac{\partial Q_j^{(0)} (\overline{q}, t)}{\partial \overline{q}_k} Q_{kp}^{(1)} (\overline{q}, t) \right] dt \triangleq \overline{Q}_j^{(2)} (\overline{q})$$

and

$$\eta_j^{(2)} (\overline{q}, t) = \int \left[Q_j^{(1)} (\overline{q}, t) + \frac{\partial Q_j^{(0)} (\overline{q}, t)}{\partial \overline{q}_k} Q_{kp}^{(1)} (\overline{q}, t) - \overline{Q}_j^{(2)} (\overline{q}) - \frac{\partial Q_{jp}^{(1)} (\overline{q}, t)}{\partial \overline{q}_k} \overline{Q}_k^{(1)} (\overline{q}) \right] dt \triangleq Q_{jp}^{(2)} (\overline{q}, t)$$

where the $\overline{q}_j$ are held constant in evaluating the integrals, the constants of integration associated with the indefinite integrals are set equal to zero, and the convention of summing over repeated indices is used. Similarly, higher-order terms can be obtained. Finally,

$$\frac{d\overline{q}_j}{dt} = \epsilon \overline{Q}_j^{(1)} (\overline{q}) + \epsilon^2 \overline{Q}_j^{(2)} (\overline{q}) + \cdots$$

from which the $\overline{q}_j$ are obtained and

$$q_j = \overline{q}_j + \epsilon Q_{jp}^{(1)} (\overline{q}, t) + \epsilon^2 Q_{jp}^{(2)} (\overline{q}, t) + \cdots$$

(See Refs. 150 and 828.)

Theory of general perturbations using the variation of parameters with canonical transformations. In this theory, the orbit of a satellite is initially defined by six parameters of elliptic motion expressed as symbols, thought of as continuously varying because of the perturbations that they undergo, and chosen in such a way that they have the canonical form; that is, the parameters consist of three pairs, and, for any pair, the derivative of one with respect to the time is equal to the partial derivative of a function F with respect to the other, while the derivative of the other with respect to the time is equal to the negative partial of F with respect to the one. For example, for a conservative force field,

$$\frac{dL_j}{dt} = \frac{\partial F}{\partial \ell_j}, \qquad \frac{d\ell_j}{dt} = -\frac{\partial F}{\partial L_j}, \qquad j = 1, 2, 3$$

(for a nonconservative force field, these equations take the form

$$\frac{dL_j}{dt} = \frac{\partial F}{\partial \ell_j} + P_j, \qquad \frac{d\ell_j}{dt} = -\frac{\partial F}{\partial L_j} - Q_j, \qquad j = 1, 2, 3$$

where P_j and Q_j represent the nonconservative forces). The function F, called the Hamiltonian, is developed in an infinite series in terms of the six parameters. By a succession of transformations of variables, in which the canonical form of the equations is preserved, the Hamiltonian is made to take a simpler and simpler form, until in the end the continuously varying parameters are expressed explicitly as functions of six constants and the time. These constants must be determined by an initialization procedure. The contribution made by von Zeipel consisted of introducing a determining function to accomplish the equivalent of a Delaunay transformation. In this manner, a whole class of terms can be eliminated from the Hamiltonian simultaneously in a single transformation rather than just one periodic term as in a typical Delaunay transformation. (See Ref. 205.)

Truncation error in numerical integration. The difference between the exact solution of the difference equations that approximate the differential equations and the exact solution of the differential equations themselves. The influence of truncation error is usually prominent when a large value of step size is used. (See Refs. 292, 357, and 358.)

Uniformly valid asymptotic expansion of $x(t;\epsilon)$ as $\epsilon \rightarrow 0$. The asymptotic expansion of $x(t;\epsilon)$ as $\epsilon \rightarrow 0$ where the remainder is of $O(\nu_{J+1}(\epsilon))$ uniformly in t (that is, for all t in the range of t). (See Section II-A-2.)

Variation or derivative. The rate of change of a coordinate or parameter with respect to an independent variable, for example, time or right ascension. The variation of a function f is composed of two parts, $\dot{f}$ and $f^{`}$. The function $\dot{f}$ is the Keplerian or two-body variation that remains at the instant of osculation if all the disturbing forces are suddenly removed, and the function $f^{`}$ is the perturbative variation caused by the disturbing forces. (See Ref. 564.)

Variation of coordinates method of general perturbations. The method in astrodynamics which consists of the analytical integration of the accelerations (usually perturbative) in terms of the coordinates of the disturbed body (for example, $x, y, z, \dot{x}, \dot{y}, \dot{z}$ or $r, \alpha, \delta, \dot{r}, \dot{\alpha}, \dot{\delta}$).

Variation of parameters method of general perturbations. The method in astrodynamics which consists of the analytical integration of perturbative variations in terms of parameters of the disturbed body (for example, $a, e, i, \omega, \Omega, M$ or n, a_N, a_M, i, ℓ, L).

Variation of parameters method of special perturbations. The method in astrodynamics which consists of the step-by-step numerical integration of the perturbative variations of the osculating parameters of the disturbed body. As an example, a complete set of integral equations that must be integrated is (in normalized form)

$$n = n_0 + \int_{t_0}^{t} n^{`}\, d\tau, \qquad a_M = a_{M0} + \int_{t_0}^{t} a_M^{`}\, d\tau$$

$$a_N = a_{N0} + \int_{t_0}^{t} a_N^{`}\, d\tau, \qquad i = i_0 + \int_{t_0}^{t} i^{`}\, d\tau$$

$$\Omega = \Omega_0 + \int_{t_0}^{t} \Omega^{`}\, d\tau$$

and

$$M = M_{00} + n_0 (t - t_0) + \iint_{t_0}^{t} n^{`}\, d\tau^2 + \int_{t_0}^{t} M^{`}\, d\tau$$

It should be noted that the reference orbit varies gradually in such a way that it always yields exactly the same position and velocity as those associated with the actual path; that is, $p(t)$, $e(t)$, and $v(t)$ vary in such a way that

$$p(t) = r(t)\,[1 + e(t) \cos v(t)]$$

is always satisfied. Hence, the varying reference orbit is always "osculating" and the constant elements of the two-body problem become varying parameters defining the varying orbit.

English Symbols

a the semimajor axis of an orbit (a mean distance)

a_M the quantity $e \sin \omega$

$\tilde{a}_M^{(0)}, \tilde{a}_M^{(1)}, \cdots$ a set of functions that depend upon a slow variable and that arise in the terms of an expansion (these functions are defined explicitly in Section V-C-3)

a_N the quantity $e \cos \omega$

$\tilde{a}_N^{(0)}, \tilde{a}_N^{(1)}, \cdots$ a set of functions that depend upon a slow variable and that arise in the terms of an expansion (these functions are defined explicitly in Section V-C-3)

a_p the equatorial radius of a planet (or oblate body)

$B_0, B_1, \cdots$ a set of constants that arise in the terms of an expansion (these constants are defined explicitly as they arise in the sections)

$B^{(0)}, B^{(1)}, \cdots$ a set of functions that depend upon a slow variable and that arise in the terms of an expansion (these functions are defined explicitly as they arise in the sections)

b an angle measured along a great circle normal to the orbital plane or a parameter for the damped linear harmonic oscillator (see Section IV-E-1)

$b^{(0)}, b^{(1)}, \cdots$ a set of functions that arise in an expansion (these functions are defined explicitly as they arise in the sections)

$\tilde{b}$ a slowly varying parameter for the damped linear harmonic oscillator (see Section IV-G-1)

$\tilde{b}^{(0)}, \tilde{b}^{(1)}, \cdots$ a set of slowly varying functions that arise in an expansion (these functions are defined explicitly in Section IV-G-3)

$C_0, C_1, \cdots$ a set of constants that arise in the terms of an expansion (these constants are defined explicitly as they arise in the sections)

$C^{(0)}, C^{(1)}, \cdots$ a set of functions that depend upon a slow variable and that arise in the terms of an expansion (these functions are defined explicitly as they arise in the sections)

C_D the damping coefficient or the drag coefficient

C_f the friction of rolling coefficient

C_s the spring constant

c a parameter for the damped linear harmonic oscillator (see Section IV-E-1)

$c_0, c_1, \cdots$ a set of constants that arise in the terms of an expansion (these constants are defined explicity in Section V-C-3)

$c^{(0)}, c^{(1)}, \cdots$ a set of functions that arise in an expansion (these functions are defined explicitly as they arise in the sections)

$\tilde{c}$ a slowly varying parameter for the damped linear harmonic oscillator (see Section IV-G-1)

$\tilde{c}^{(0)}, \tilde{c}^{(1)}, \cdots$ a set of slowly varying functions that arise in an expansion (these functions are defined explicitly in Section IV-G-3)

E the eccentric anomaly

$E(\overset{\Delta}{\phi} \backslash \overset{\Delta}{m})$ the Legendre normal elliptic integral of the second kind; that is,

$$E(\overset{\Delta}{\phi} \backslash \overset{\Delta}{m}) = \int_0^{\overset{\Delta}{\phi}} (1 - \overset{\Delta}{m} \sin^2 \overset{\Delta}{\theta})^{1/2} \, d\overset{\Delta}{\theta}$$

E_T the total energy per unit mass of the satellite

$\bar{E}$ the multivariable eccentric anomaly or the reference (mean) instantaneous eccentric anomaly

e the eccentricity of an orbit (should not be confused with the exponential; for example, e^θ)

$e_0, e_1, \cdots$ a set of constants that arise in the terms of an expansion (these constants are defined explicitly in Section V-C-3)

$\bar{e}$ the initial eccentricity e_0 or the reference (mean) instantaneous eccentricity of an orbit

$\tilde{e}$ the instantaneous multivariable eccentricity of an orbit

$\tilde{e}^{(0)}, \tilde{e}^{(1)}, \cdots$ a set of functions that depend upon a slow variable and that arise in the terms of an expansion (these functions are defined explicitly in Section V-C-3)

F the force or the negative of the Hamiltonian function ($F = \Phi -$ kinetic energy per unit mass)

$F(\overset{\Delta}{\phi} \backslash \overset{\Delta}{m})$ the Legendre normal elliptic integral of the first kind; that is,

$$F(\overset{\Delta}{\phi} \backslash \overset{\Delta}{m}) = \int_0^{\overset{\Delta}{\phi}} (1 - \overset{\Delta}{m} \sin^2 \overset{\Delta}{\theta})^{-1/2} d\overset{\Delta}{\theta}$$

f an arbitrary function

G the universal constant of gravitation (equivalent to k^2)

h the angular momentum (equivalent to $p^{1/2}$) or the interval of integration (step size)

I′, J′, K′ an inertial basis with an arbitrary fixed origin O' having a plane parallel to the equatorial plane of the central mass as the principal plane and the direction of the vernal equinox ♈ as the principal direction (**I′** is the unit vector in the direction of the vernal equinox, **J′** forms a right-handed orthonormal set with **I′** and **K′**, and **K′** is the unit vector in the direction of the north celestial pole)

I, J, K a translation of the **I′**, **J′**, **K′** basis from the inertial center O' to the center of the moving central mass (the **I**, **J**, **K** basis differs from the **I′**, **J′**, **K′** basis only in origin)

i the inclination of an orbital plane to the principal plane

i, j, k a rotating basis with the center of the moving central mass as the origin, the plane **i–j** as the principal plane, and the unit vector **i** in the direction of the radius vector of the satellite as the principal direction (**j** is the unit vector perpendicular to **i** and lying in the principal plane of the **I**, **J**, **K** basis in the general direction of the motion of the satellite and **k** forms a right-handed orthonormal set with **i** and **j**)

J_2 the coefficient of the second zonal harmonic in the gravitational potential of a spheroid of revolution

J′ see **I′**, **J′**, **K′**

J see **I**, **J**, **K**

j a summation index or the revolution number

j see **i**, **j**, **k**

K′ see **I′**, **J′**, **K′**

K see **I**, **J**, **K**

k a gravitational constant ($k^2 = G$)

k see **i**, **j**, **k**

L the mean longitude ($L = M + \omega + \Omega$)

L_1 the length scale

ℓ the true longitude ($\ell = v + \omega + \Omega$)

M the mean anomaly ($M = M_0 + n(t - t_0) = E - e \sin E$)

m the mass of the object whose motion is under study

m_i, m_j the masses of the ith and jth bodies, respectively

$m_\odot, m_\oplus, \cdots$ the masses of the sun, earth, $\cdots$

m_p the mass of the planet or central body

$\overset{\Delta}{m}$ the parameter in the Legendre normal elliptic integrals of the first, second, and third kinds

n the mean angular motion

$\overset{\Delta}{n}$ the characteristic in the Legendre normal elliptic integral of the third kind

$P(\sigma)$ a polynomial in σ (see Appendix B)

P, Q, W a basis with the center of the moving central mass as the origin, the orbital plane as the principal plane, and the direction of the perifocus as the principal direction (**P** is the unit vector in the direction of the perifocus, **Q** forms a right-handed orthonormal set with **P** and **W**, and **W** is the unit vector perpendicular to the orbital plane)

p the semilatus rectum or parameter of an orbit ($p = h^2$)

p_j the jth instantaneous parameter of the osculating orbit (usually $j = 1, 2, \cdots, 6$)

$\mathbf{p}_i$ the perturbative acceleration acting upon the ith body

Q_j the jth generalized function

Q see **P, Q, W**

q the perifocal distance of an orbit ($q = a(1 - e)$ and $q = r_{\min}$)

q_j the jth generalized coordinate

R the perturbative function

$\mathbf{R}_i$ the position vector of the ith body relative to the inertial basis $\mathbf{I}', \mathbf{J}', \mathbf{K}'$ ($R_i = |\mathbf{R}_i|$)

$\mathbf{R}_{ij}$ the position vector of the jth body relative to the ith body expressed with respect to the inertial basis $\mathbf{I}', \mathbf{J}', \mathbf{K}'$ ($\mathbf{R}_{ij} = \mathbf{R}_j - \mathbf{R}_i$ and $R_{ij} = |\mathbf{R}_{ij}|$)

r_A the apofocal distance of an orbit ($r_A = a(1 + e)$ and $r_A = r_{\max}$)

$\mathbf{r}$ the radius vector of the object under study with respect to the dynamical center (center of the central mass)

$d\mathbf{r}_a/dt$ the velocity of the atmosphere of a planet

S a characteristic area of a satellite

S, T, W a basis with the center of the object under study as the origin, the orbital plane as the principal plane, and the direction of the tangent in the direction of motion as the principal direction (**S** is the unit vector in the line of the tangent and positive in the direction of motion, **T** is the unit vector lying in the orbital plane perpendicular to **S** and positive in the direction of the central mass, and **W** is perpendicular to the orbital plane and forms a right-handed orthonormal set with **S** and **T**).

s the magnitude of the velocity vector of the object under study (note that $ds/dt = \dot{s}$ since $d\mathbf{r}/dt = \dot{\mathbf{r}}$ and $\mathbf{r}^{\backprime} = 0$)

T the time of perifocal passage

$T_0, T_1, \cdots$ a set of constants which arise in the terms of an expansion (these constants are defined explicitly as they arise in the sections)

T_1 the fast time scale

T_2 the slow time scale

$\widetilde{T}^{(0)}, \widetilde{T}^{(1)}, \cdots$ a set of functions that depend upon a slow variable and that arise in the terms of an expansion (these functions are defined explicitly in Section V-C-3)

T see **S, T, W**

t the time

$t^{(0)}, t^{(1)}, \cdots$ a set of functions that arise in an expansion (these functions are defined explicitly as they arise in the sections)

$\bar{t}$ the fast time variable ($\bar{t} = t(1 + \epsilon^2\tau_2 + O(\epsilon^3))$)

$\tilde{t}$ the slow time variable ($\tilde{t} = \epsilon t$)

U, V, W a basis with the center of the moving central mass as the origin, the orbital plane as the principal plane, and the direction to the object under study as the principal direction (**U** is the unit vector in the direction of the object under study, **V** forms a right-handed orthonormal set with **U** and **W**, and **W** is the unit vector perpendicular to the orbital plane)

u the argument of latitude ($u = v + \omega$)

V see **U, V, W**

v the true anomaly

$\bar{v}$ the multivariable true anomaly or the reference (mean) instantaneous true anomaly

$\dot{v}$ the angular velocity

$$\left(\dot{v} = \frac{p^{1/2}}{r^2} = \frac{h}{r^2} \text{ and } \frac{dv}{dt} = \dot{v} + v^{\backprime}\right)$$

W see **P, Q, W**; **S, T, W**; and **U, V, W**

X_i, Y_i, Z_i the coordinates of the ith body relative to the inertial basis $\mathbf{I}', \mathbf{J}', \mathbf{K}'$

x, y, z the coordinates of the object under study relative to the basis $\mathbf{I}, \mathbf{J}, \mathbf{K}$

$x^{(0)}, x^{(1)}, \cdots$ a set of functions that arise in an expansion (these functions are defined explicitly as they arise in the sections)

Greek Symbols

α the right ascension of the object under study

$\alpha_2, \alpha_3, \cdots$ a set of constants that arise in the fast angle variable $\overline{\alpha}$ (these constants are defined explicitly in Section V-C-3).

$\overline{\alpha}$ the fast angle variable ($\overline{\alpha} = \alpha (1 + \epsilon^2 \alpha_2 + O(\epsilon^3))$)

$\widetilde{\alpha}$ the slow angle variable ($\widetilde{\alpha} = \epsilon\alpha$)

$\Gamma_1, \Gamma_2, \Gamma_3$ a set of transformation constants explicitly defined in Appendix B

γ a transformation constant explicitly defined in Appendix B

Δ a small increment

δ the declination of the object under study or a difference in numerical analysis ($\delta, \delta^2, \cdots$)

ϵ the perturbative parameter (necessarily small)

ζ the integration variable associated with $\eta = h^2/r$

η the quantity h^2/r

$\eta^{(0)}, \eta^{(1)}, \cdots$ a set of functions that arise in an expansion (these functions are defined explicitly in Section V-C-3)

Θ a function of θ (see Appendix A)

θ an angle measured in the initial orbital plane from an inertial reference direction to the radius vector of the object under study or an arbitrary variable (see Appendix A)

$\overset{\Delta}{\theta}$ the integration variable associated with $\overset{\Delta}{\phi}$ in the Legendre normal elliptic integrals

κ_i a set of increasing integers; that is, $\kappa_1 < \kappa_2 < \cdots$

Λ a transformation variable for the Legendre normal elliptic integrals

$\Lambda_1, \Lambda_2, \Lambda_3$ a set of transformation constants for the Legendre normal elliptic integrals (these constants are defined explicitly in Appendix B)

μ the mass function (in the two-body problem, $\mu = k^2 (m_1 + m_2)$)

$\mu_\odot, \mu_\oplus, \cdots$ the mass functions of the sun ($\mu_\odot = k^2 m_\odot$), earth ($\mu_\oplus = k^2 m_\oplus$), $\cdots$ acting as disturbing bodies

μ_p the mass function of the central body or planet ($\mu_p = k^2 (m_p + m)$)

ν a gauge function or a transformation variable for the Legendre normal elliptic integrals

ν_j a sequence of functions (these functions are defined explicitly as they arise in the sections) or a set of transformation constants for the Legendre normal elliptic integrals (these constants are defined explicitly in Appendix B)

ξ the quantity $d\eta/d\alpha$

$\xi^{(0)}, \xi^{(1)}, \cdots$ a set of functions that arise in an expansion (these functions are defined explicitly in Section V-C-3)

$\Pi(\overset{\Delta}{n}; \overset{\Delta}{\phi} \backslash \overset{\Delta}{m})$ the Legendre normal elliptic integral of the third kind; that is,

$$\Pi(\overset{\Delta}{n}; \overset{\Delta}{\phi} \backslash \overset{\Delta}{m}) = \int_0^{\overset{\Delta}{\phi}} (1 - \overset{\Delta}{n} \sin^2 \overset{\Delta}{\theta})^{-1} (1 - \overset{\Delta}{m} \sin^2 \overset{\Delta}{\theta})^{-1/2} d\overset{\Delta}{\theta}$$

π the ratio of the circumference of a circle to its diameter ($\pi \cong 3.1416$)

ϖ the longitude of perifocus ($\varpi = \omega + \Omega$; the symbol ϖ is script pi and should not be confused with $\widetilde{\omega}$ or π, both of which have different meanings)

ρ the atmospheric density of a planet or the magnitude of the vector $\boldsymbol{\rho}$

$\boldsymbol{\rho}$ the difference in position between the actual orbit and a reference orbit in Encke's method of special perturbations ($\rho = |\boldsymbol{\rho}|$)

Σ a sum in numerical analysis ($\Sigma, \Sigma^2, \cdots$)

σ a transformation variable for the Legendre normal elliptic integrals ($\sigma^2 = \zeta + \nu$)

$\sigma_1, \sigma_2, \sigma_3$ a set of transformation constants for the Legendre normal elliptic integrals (these constants are defined explicitly in Appendix B)

τ the integration variable associated with the time

$\tau_0, \tau_1, \cdots$ a set of constants that arise in the fast time variable (these constants are defined explicitly as they arise in the sections) or a set of constants that arise in the terms of an expansion (these constants are defined explicitly as they arise in the sections)

Φ the potential function

$\overset{\Delta}{\phi}$ the amplitude in the Legendre normal elliptic integrals of the first, second, and third kinds

Ω a forcing function (should not be confused with ☊; see Appendix A)

$\Omega_1, \Omega_2, \Omega_3$ an arbitrary set of constants or functions of a parameter treated as a constant (see Appendix A)

ω the argument of perifocus of an orbit

$\omega_0, \omega_1, \cdots$ a set of constants that arise in the terms of an expansion (these constants are defined explicitly as they arise in the sections)

$\overline{\omega}$ the reference (mean) instantaneous argument of perifocus of an orbit

$\widetilde{\omega}$ the instantaneous multivariable argument of perifocus of an orbit

$\widetilde{\omega}^{(0)}, \widetilde{\omega}^{(1)}, \cdots$ a set of functions that depend upon a slow variable and that arise in the terms of an expansion (these functions are defined explicitly in Section V-C-3)

Special Symbols

A boldface letter denotes a vector

$\bar{\ }$ a bar above a letter denotes a reference (mean) quantity or fast variable

$\tilde{\ }$ a wavy line above a letter denotes a slow variable or a slowly varying function

[ ] brackets around a letter denote "the dimension of"

* a starred letter denotes a normalized variable

$\dot{\ }, \ddot{\ }$ the Keplerian or two-body variations that remain at the instant of osculation if all the disturbing forces are suddenly removed

$\grave{\ }, \grave{\grave{\ }}$ the perturbative variations caused by the disturbing forces

$\frac{d}{d\theta}, \frac{d^2}{d\theta^2}$ the total derivatives with respect to the independent variable θ where

$$\theta = t, \alpha, \cdots \text{ and } \frac{d}{d\theta} = \dot{\ } + \grave{\ }$$

$_1, _2, _{12}, \cdots$ the quantities $\frac{\partial}{\partial \overline{\theta}}, \frac{\partial}{\partial \widetilde{\theta}}, \frac{\partial}{\partial \overline{\theta}\, \partial \widetilde{\theta}}, \cdots,$ respectively, where $\theta = t, \alpha, \cdots$

$^{(0)}, ^{(1)}, \cdots$ the superscripts within parenthesis indicate the order of the function, term, or perturbation

O order of

☊ the longitude of the ascending node of an orbit

♈ vernal equinox

$q_1 \rightarrow q_2, q_3, \cdots$ other equations can be obtained by replacing q_1 by $q_2, q_3, \cdots$

$\int^t (\ )\, d\tau$ implies integrate and evaluate at $\tau = t$

$\int^0 (\ )\, d\tau$ implies integrate and evaluate at $\tau = 0$ (yielding a constant)

$|\ |$ absolute value or magnitude

$\equiv$ identically equal to

$\overset{\Delta}{=}$ equal to by definition

$\doteqdot$	equal to except for an additive constant
$\neq$	not equal to
$\cong$	approximately equal to
$\propto$	proportional to
$<<$	much smaller than
$>, <$	greater than, less than
$\geqq, \leqq$	greater than or equal to, less than or equal to
$\Rightarrow$	implies

Subscripts

A	apofocus or approximation
a	atmosphere
D	drag
e	Encke reference orbit
f	friction
i, j	integers; for example, the ith or jth mass
ℓp	long-period
0	value associated with the epoch, t_0
$\mathbf{p}$	perifocus
p	planet or central mass
s	secular or spring
sp	short-period
$\odot$	sun
$\oplus$	earth
☾	moon
♂	Mars

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Appendix A
Particular Solutions of Differential Equations

During the analysis in Sections IV and V, linear ordinary and partial differential equations are solved by determining the solutions to the corresponding homogeneous differential equations and adding particular solutions to them. These particular solutions are given in this appendix. It should be noted that if a partial differential equation consists only of derivatives with respect to one independent variable, its solution may be obtained by treating the partial differential equation as if it were an ordinary differential equation and then replacing the constants of integration which would normally arise by functions of the other independent variables (these variables may be thought of as parameters treated as constants during the process of solution).

Let Ω_1, Ω_2, and Ω_3 be constants or functions of parameters treated as constants and let $D\,\Theta(\theta)$ represent either an ordinary or partial derivative of $\Theta(\theta)$ with respect to θ. The solutions are given in Table A-1.

Table A-1. Particular solutions of differential equations

Differential equation	$\Omega(\theta)$	*Particular solution*	*Solution number*
$(D+1)\,\Theta(\theta) = \Omega(\theta)$	$\Omega_1 e^{-\theta}$	$\Omega_1 \theta e^{-\theta}$	(A-1)
$(D^2+1)\,\Theta(\theta) = \Omega(\theta)$	Ω_1	Ω_1	(A-2)
	$\Omega_1 \cos(\theta + \Omega_2)$	$\frac{1}{2}\Omega_1 \theta \sin(\theta + \Omega_2)$	(A-3)
	$\Omega_1 \sin(\theta + \Omega_2)$	$-\frac{1}{2}\Omega_1 \theta \cos(\theta + \Omega_2)$	(A-4)
	$\Omega_1 \cos \Omega_3(\theta + \Omega_2)$	$\frac{\Omega_1}{(1-\Omega_3^2)} \cos \Omega_3(\theta + \Omega_2)$	(A-5)
	$\Omega_1 \sin \Omega_3(\theta + \Omega_2)$	$\frac{\Omega_1}{(1-\Omega_3^2)} \sin \Omega_3(\theta + \Omega_2)$	(A-6)

Appendix B
Transformation of Elliptic Integrals Into Legendre Normal Form

Consider the transformation of the integrals in Eqs. (288) and (290), Section V-B-4,

$$\alpha = \pm \int_{\eta_0}^{\eta} \frac{d\zeta}{\left(\frac{2}{3}\epsilon\zeta^3 - \zeta^2 + 2\zeta + E_T\right)^{1/2}} \tag{288}$$

and

$$t = \pm \int_{\eta_0}^{\eta} \frac{d\zeta}{\zeta^2\left(\frac{2}{3}\epsilon\zeta^3 - \zeta^2 + 2\zeta + E_T\right)^{1/2}} \tag{290}$$

into Legendre normal elliptic integrals. Let

$$\zeta = \sigma^2 - \nu \tag{B-1}$$

where ν is a constant to be determined. Introducing Eq. (B-1) into Eq. (288) yields

$$\alpha = \pm \int_{(\eta_0+\nu)^{1/2}}^{(\eta+\nu)^{1/2}} \frac{2\sigma d\sigma}{\left[\frac{2}{3}\epsilon(\sigma^2-\nu)^3 - (\sigma^2-\nu)^2 + 2(\sigma^2-\nu) + E_T\right]^{1/2}} \tag{B-2}$$

Consider the polynomial appearing in the denominator of the integrand in Eq. (B-2), designated $P(\sigma)$. The polynomial can be rewritten as

$$P(\sigma) = \frac{2}{3}\epsilon\sigma^6 - (1+2\epsilon\nu)\sigma^4 + 2(1+\nu+\epsilon\nu^2)\sigma^2 \quad - \frac{2}{3}\epsilon\left(\nu^3 + \frac{3}{2\epsilon}\nu^2 + \frac{3}{\epsilon}\nu - \frac{3E_T}{2\epsilon}\right) \tag{B-3}$$

Choose ν so that

$$\nu^3 + \frac{3}{2\epsilon}\nu^2 + \frac{3}{\epsilon}\nu - \frac{3E_T}{2\epsilon} = 0 \tag{B-4}$$

From the theory of cubic equations (see Ref. 1375, p. 17), let

$$\nu = \Lambda - \frac{1}{2\epsilon} \tag{B-5}$$

so that Eq. (B-4) becomes

$$\Lambda^3 + C_1\Lambda + C_2 = 0 \qquad \text{where} \qquad C_1 = \frac{3}{\epsilon}\left(1 - \frac{1}{4\epsilon}\right) \qquad \text{and} \qquad C_2 = \frac{1}{2\epsilon}\left(\frac{1}{2\epsilon^2} - \frac{3}{\epsilon} - 3E_T\right) \tag{B-6}$$

Then

$$\left.\begin{aligned}
\Lambda_1 &= C_3 + C_4 \\
\Lambda_2 &= -\frac{C_3 + C_4}{2} + \frac{C_3 - C_4}{2}(-3)^{1/2} \\
&\text{and} \\
\Lambda_3 &= -\frac{C_3 + C_4}{2} - \frac{C_3 - C_4}{2}(-3)^{1/2} \\
&\text{where} \\
C_3 &= \left[-\frac{C_2}{2} + \left(\frac{C_2^2}{4} + \frac{C_1^3}{27}\right)^{1/2}\right]^{1/3} \\
&\text{and} \\
C_4 &= \left[\frac{C_2}{2} - \left(\frac{C_2^2}{4} + \frac{C_1^3}{27}\right)^{1/2}\right]^{1/3}
\end{aligned}\right\} \qquad \text{(B-7)}$$

Now, if

$$\left(\frac{C_2^2}{4} + \frac{C_1^3}{27}\right) > 0$$

there will be one real root and two conjugate imaginary roots; if

$$\left(\frac{C_2^2}{4} + \frac{C_1^3}{27}\right) = 0$$

there will be three real roots, of which at least two are equal; and if

$$\left(\frac{C_2^2}{4} + \frac{C_1^3}{27}\right) < 0$$

there will be three real and unequal roots.

The question of which root Λ_j to choose in determining ν can be answered by investigating the maximum and minimum values of the radius distance r. From Eq. (276), Section V-B-2,

$$\frac{d\eta}{d\alpha} = -h\frac{dr}{dt}$$

so that, since $dr/dt = 0$ at an extremum,

$$\frac{d\eta}{d\alpha} = 0 \qquad \text{(B-8)}$$

Introducing Eq. (B-8) into (287) from Section V-B-4 yields

$$\frac{2}{3}\epsilon\eta^3 - \eta^2 + 2\eta + E_T = 0$$

or

$$(-\eta)^3 + \frac{3}{2\epsilon}\eta^2 + \frac{3}{\epsilon}(-\eta) - \frac{3E_T}{2\epsilon} = 0 \qquad \text{(B-9)}$$

Comparing Eqs. (B-4) and (B-9) shows that the roots of Eq. (B-9) are the same as the roots of Eq. (B-4); that is,

$$-\eta_j = \nu_j \qquad \text{where} \qquad j = 1, 2, 3 \qquad \text{(B-10)}$$

Thus, introducing Eq. (B-10) into Eq. (274) from Section V-B-2 and using (B-5) yields

$$r_j = \frac{h^2}{-\nu_j} = \frac{h^2}{\dfrac{1}{2\epsilon} - \Lambda_j} \qquad \text{where} \qquad j = 1, 2, 3 \qquad \text{(B-11)}$$

Equations (B-11) clearly show that there are three theoretical extremums. However, physically, there is at least one extremum (a minimum radius distance, r_{min}) or there are at most two extremums (a maximum and a minimum radius distance, r_{max} and r_{min}). Three physical cases may arise. First, the initial conditions of the satellite may be such that an escape trajectory exists so that only one extremum arises, r_{min}. Second, the initial conditions of the satellite may be such that a circular orbit exists so that two extremums arise but both are the same, $r_{min} = r_{max}$. Third, the initial conditions of the satellite may be such that a periodic trajectory exists so that two different extremums arise, r_{min} and r_{max}.

In all three cases, the radius distance must be real and satisfy

$$r > a_p > 0 \qquad \text{(B-12)}$$

which excludes trajectories that intersect the surface of the oblate body. As a result of inequalities (B-12) and Eqs. (B-11), the root Λ must be real (h and ϵ are real) and satisfy

$$\frac{h^2}{\left(\dfrac{1}{2\epsilon} - \Lambda\right)} > a_p > 0$$

$$\frac{h^2}{a_p} > \left(\frac{1}{2\epsilon} - \Lambda\right) > 0$$

or

$$\frac{1}{2\epsilon} > \Lambda > \left(\frac{1}{2\epsilon} - \frac{h^2}{a_p}\right) \qquad \text{(B-13)}$$

Inequalities (B-13) will help to determine the physically meaningful extremums in each of the three cases discussed above.

One final condition required to determine the physically meaningful extremums concerns the existence of a circular orbit. Suppose that the initial conditions are such that a circular orbit exists at $t = 0$; that is,

$$\frac{dr}{dt}(0) = \dot{r}_0 = 0 \qquad \text{(B-14)}$$

Consider the value of r at a small time later by expanding r in a Taylor series about the point $t = 0$; that is,

$$\Delta r = r - r_0 = \left[\frac{dr}{dt}(0)\right](t) + \frac{1}{2}\left[\frac{d^2r}{dt^2}(0)\right](t^2) + \cdots$$

or, using Eqs. (B-14),

$$\Delta r = \frac{1}{2}\left[\frac{d^2r}{dt^2}(0)\right](t^2) + \cdots \qquad \text{(B-15)}$$

For a circular orbit, $\Delta r = 0$ for all t so that Eq. (B-15) requires

$$\frac{d^2r}{dt^2}(0) = 0$$

and so forth. Thus, with the use of Eqs. (265) and (272) from Sections V-A and V-B-2, respectively, evaluated at $t = 0$ and the first of Eqs. (267),

$$\frac{h^2}{r_0^3} - \frac{1}{r_0^2}\left(1 + \epsilon\frac{h^4}{r_0^2}\right) = 0$$

or

$$\frac{h^2}{r_0} = \left(1 + \epsilon\frac{h^4}{r_0^2}\right) \qquad \text{(B-16)}$$

The equation of an osculating conic is $p = r(1 + e\cos v)$ where $p = h^2$, so that at $t = 0$

$$h^2 = r_0(1 + e_0) \qquad \text{(B-17)}$$

where $v_0 = 0$ since $\dot{r}_0 = 0$ ($v_0 = \pi$ is impossible as is subsequently shown). It follows from Eqs. (B-16) and (B-17) that

$$1 + e_0 = 1 + \epsilon\frac{h^4}{r_0^2}$$

or, using Eqs. (271) from Section V-A,

$$e_0 = \frac{3}{2}J_2\left(\frac{a_p}{r_0}\right)^2 \qquad \text{(B-18)}$$

in order to have a circular orbit for all time. Note that if $(dr/dt)(0) = 0$, then

$$\frac{d^jr}{dt^j}(0) = [f^j(r)]_{t=0}\frac{dr}{dt}(0) = 0$$

$$\text{for} \qquad j = 3, 4, \cdots \qquad \text{(B-19)}$$

since

$$\frac{d^2r}{dt^2} = \frac{h^2}{r^3} - \frac{1}{r^2}\left(1 + \epsilon\frac{h^4}{r^2}\right)$$

from Eq. (265), Section V-A. Furthermore, note that if $v_0 = \pi$ instead of $v_0 = 0$ as in Eq. (B-17), $e_0 = -\epsilon(h^4/r_0^2)$, which is impossible since $e_0 \geqq 0$. Thus, a necessary and sufficient condition for the existence of a circular orbit is that at $t = 0$,

$$r(0) = q_0, \qquad \frac{dr}{dt}(0) = 0, \qquad \text{and} \qquad e_0 = \frac{3}{2}J_2\left(\frac{a_p}{q_0}\right)^2 \qquad \text{(B-20)}$$

Then, using Eq. (272) from Section V-B-2, one obtains

$$r = q_0, \qquad \frac{dr}{dt} = 0, \qquad \alpha = \frac{h}{q_0^2}t$$

and

$$\frac{d\alpha}{dt} = \frac{h}{q_0^2} \qquad \text{(B-21)}$$

where $0 \leqq t < \infty$. Now, because of their simplicity, Eqs. (B-21) are used in the case of a circular orbit rather than Eqs. (288) and (290). Consequently, the possibility of a circular orbit is excluded in determining the proper Λ; that is,

$$r_{\min} \neq r_{\max} \qquad \text{(B-22)}$$

The necessary information for determining the proper root Λ, and thus ν, is now available. Consider the three possibilities for the quantity $[(C_2^2/4) + (C_1^3/27)]$ in Eqs. (B-7):

(1) The possibility that $[(C_2^2/4) + (C_1^3/27)] > 0$. For this possibility, the condition that Λ be real eliminates the two conjugate imaginary roots so that

$$\Lambda_{\min} = C_3 + C_4 \qquad \text{and} \qquad r_{\min} = \frac{h^2}{\dfrac{1}{2\epsilon} - \Lambda_{\min}} \qquad \text{(B-23)}$$

are the only physically meaningful root and extremum, respectively. This possibility corresponds to an escape trajectory (case 1).

(2) The possibility that $[(C_2^2/4) + (C_1^3/27)] = 0$. For this possibility, Eqs. (B-7) become

$$\Lambda_1 = (-4C_2)^{1/3} \qquad \text{and} \qquad \Lambda_2 = \Lambda_3 = \left(\frac{C_2}{2}\right)^{1/3} \tag{B-24}$$

where

$$C_3 = C_4 = \left(-\frac{C_2}{2}\right)^{1/3}$$

Introducing the last of Eqs. (B-6) into Eqs. (B-24) yields

$$\left.\begin{aligned} \Lambda_1 &= \left[\frac{2}{\epsilon}\left(3E_T + \frac{3}{\epsilon} - \frac{1}{2\epsilon^2}\right)\right]^{1/3} \\ &\text{and} \\ \Lambda_2 = \Lambda_3 &= \left[\frac{1}{4\epsilon}\left(\frac{1}{2\epsilon^2} - \frac{3}{\epsilon} - 3E_T\right)\right]^{1/3} \end{aligned}\right\} \tag{B-25}$$

Imposing the condition (B-13) yields

$$\frac{1}{2\epsilon} > \left[\frac{2}{\epsilon}\left(3E_T + \frac{3}{\epsilon} - \frac{1}{2\epsilon^2}\right)\right]^{1/3} > \left(\frac{1}{2\epsilon} - \frac{h^2}{a_p}\right) \tag{B-26}$$

and

$$\frac{1}{2\epsilon} > \left[\frac{1}{4\epsilon}\left(\frac{1}{2\epsilon^2} - \frac{3}{\epsilon} - 3E_T\right)\right]^{1/3} > \left(\frac{1}{2\epsilon} - \frac{h^2}{a_p}\right) \tag{B-27}$$

Suppose inequality (B-26) is not satisfied; then Λ_1 is eliminated as a possible root. This leaves $\Lambda_2 = \Lambda_3$ as the only possible roots, which implies $\Lambda_{min} = \Lambda_{max}$ or, using Eqs. (B-11), $r_{min} = r_{max}$. But this contradicts inequality (B-22) so that inequality (B-26) must be satisfied and Λ_1 is a possible root. Now, if inequality (B-27) is not satisfied, $\Lambda_2 = \Lambda_3$ are not possible roots, so that

$$\left.\begin{aligned} \Lambda_{min} &= \left[\frac{2}{\epsilon}\left(3E_T + \frac{3}{\epsilon} - \frac{1}{2\epsilon^2}\right)\right]^{1/3} \\ &\text{and} \\ r_{min} &= \frac{h^2}{\dfrac{1}{2\epsilon} - \Lambda_{min}} \end{aligned}\right\} \tag{B-28}$$

are the only physically meaningful root and extremum, respectively. These conditions correspond to an escape trajectory (case 1). However, if inequality (B-27) is satisfied, $\Lambda_2 = \Lambda_3$ are possible roots, so that

$$\left.\begin{aligned} &\Lambda_{min} = \text{smaller of } \Lambda_1 \text{ and } \Lambda_2 (= \Lambda_3) \\ &\Lambda_{max} = \text{larger of } \Lambda_1 \text{ and } \Lambda_2 (= \Lambda_3) \\ &r_{min} = \frac{h^2}{\dfrac{1}{2\epsilon} - \Lambda_{min}} \quad \text{and} \quad r_{max} = \frac{h^2}{\dfrac{1}{2\epsilon} - \Lambda_{max}} \end{aligned}\right\} \tag{B-29}$$

are the physically meaningful roots and extremums, respectively. These conditions correspond to a periodic trajectory (case 3).

(3) The possibility that $[(C_2^2/4) + (C_1^3/27)] < 0$. For this possibility, a trigonometric solution for the Λ_j is desirable. From Ref. 1375, p. 17, let

$$\Lambda_j = 2\left(-\frac{C_1}{3}\right)^{1/2} \cos \Gamma_j \qquad \text{for} \qquad j = 1, 2, 3 \tag{B-30}$$

where

$$\Gamma_1 = \frac{\gamma}{3}, \qquad \Gamma_2 = \frac{1}{3}(\gamma + 2\pi), \qquad \Gamma_3 = \frac{1}{3}(\gamma + 4\pi) \tag{B-31}$$

and

$$\cos \gamma = -\frac{C_2}{2\left(\dfrac{-C_1^3}{27}\right)^{1/2}} \tag{B-32}$$

Imposing the condition (B-13) yields

$$\frac{1}{2\epsilon} > 2\left(-\frac{C_1}{3}\right)^{1/2} \cos\Gamma_j > \left(\frac{1}{2\epsilon} - \frac{h^2}{a_p}\right)$$

$$\text{for} \qquad j = 1, 2, 3$$

or, using the second of Eqs. (B-6),

$$\frac{1}{2(1-4\epsilon)^{1/2}} > \cos\Gamma_j > \frac{1 - \dfrac{2\epsilon h^2}{a_p}}{2(1-4\epsilon)^{1/2}}$$

$$\text{for} \qquad j = 1, 2, 3 \tag{B-33}$$

Consider the conditions for which inequalities (B-33) are valid. Both the lower and upper limits of $\cos\Gamma_j$ are near ½ since ϵ is small. Furthermore, since $\epsilon > 0$, it follows that the upper limit of $\cos\Gamma_j$ satisfies

$$\frac{1}{2(1-4\epsilon)^{1/2}} > \frac{1}{2} \tag{B-34}$$

Now, the lower limit of $\cos\Gamma_j$ may be either less than or greater than ½ depending upon h^2/a_p; that is,

$$\frac{1 - \dfrac{2\epsilon h^2}{a_p}}{2(1-4\epsilon)^{1/2}} \gtrless \frac{1}{2}$$

$$\left(1 - 4\frac{\epsilon h^2}{a_p} + 4\frac{\epsilon^2 h^4}{a_p^2}\right) \gtrless (1 - 4\epsilon)$$

or

$$\left\{\frac{h^2}{a_p} - \frac{1}{2\epsilon}[1 + (1-4\epsilon)^{1/2}]\right\}\left\{\frac{h^2}{a_p} - \frac{1}{2\epsilon}[1 - (1-4\epsilon)^{1/2}]\right\} \gtrless 0$$

so that

$$\frac{1 - \dfrac{2\epsilon h^2}{a_p}}{2(1-4\epsilon)^{1/2}} < \frac{1}{2} \quad \text{if both} \quad \frac{h^2}{a_p} > \frac{1}{2\epsilon}[1 - (1-4\epsilon)^{1/2}] \quad \text{and} \quad \frac{h^2}{a_p} < \frac{1}{2\epsilon}[1 + (1-4\epsilon)^{1/2}] \tag{B-35}$$

whereas

$$\frac{1 - \dfrac{2\epsilon h^2}{a_p}}{2(1-4\epsilon)^{1/2}} > \frac{1}{2} \quad \text{if} \quad \frac{h^2}{a_p} < \frac{1}{2\epsilon}[1 - (1-4\epsilon)^{1/2}] \quad \text{or} \quad \frac{h^2}{a_p} > \frac{1}{2\epsilon}[1 + (1-4\epsilon)^{1/2}] \tag{B-36}$$

Figure B-1 shows both sets of conditions.

The fact that the Γ_j differ from each other by exactly 120 deg (see Eqs. B-31) forces at least one of the Γ_j to be invalid in either set of conditions, and, since the Γ_j define the Λ_j through Eq. (B-30), at least one of the Λ_j is invalid. Furthermore, it follows from the geometry shown in Fig. B-1 that for conditions (B-35) there are two valid roots, Λ_{min} and Λ_{max}, and that for conditions (B-36) there is just one valid root, Λ_{min}. Hence, the condition (B-13),

$$\frac{1}{2\epsilon} > \Lambda > \left(\frac{1}{2\epsilon} - \frac{h^2}{a_p}\right)$$

eliminates at least one of the roots Λ_j. For conditions (B-35), the two valid roots, Λ_{min} and Λ_{max}, yield

$$r_{min} = \frac{h^2}{\dfrac{1}{2\epsilon} - \Lambda_{min}} \qquad \text{and} \qquad r_{max} = \frac{h^2}{\dfrac{1}{2\epsilon} - \Lambda_{max}} \tag{B-37}$$

as the only physically meaningful extremums. These conditions correspond to a periodic trajectory (case 3). For conditions (B-36), the valid root, Λ_{min}, yields

$$r_{min} = \frac{h^2}{\dfrac{1}{2\epsilon} - \Lambda_{min}} \tag{B-38}$$

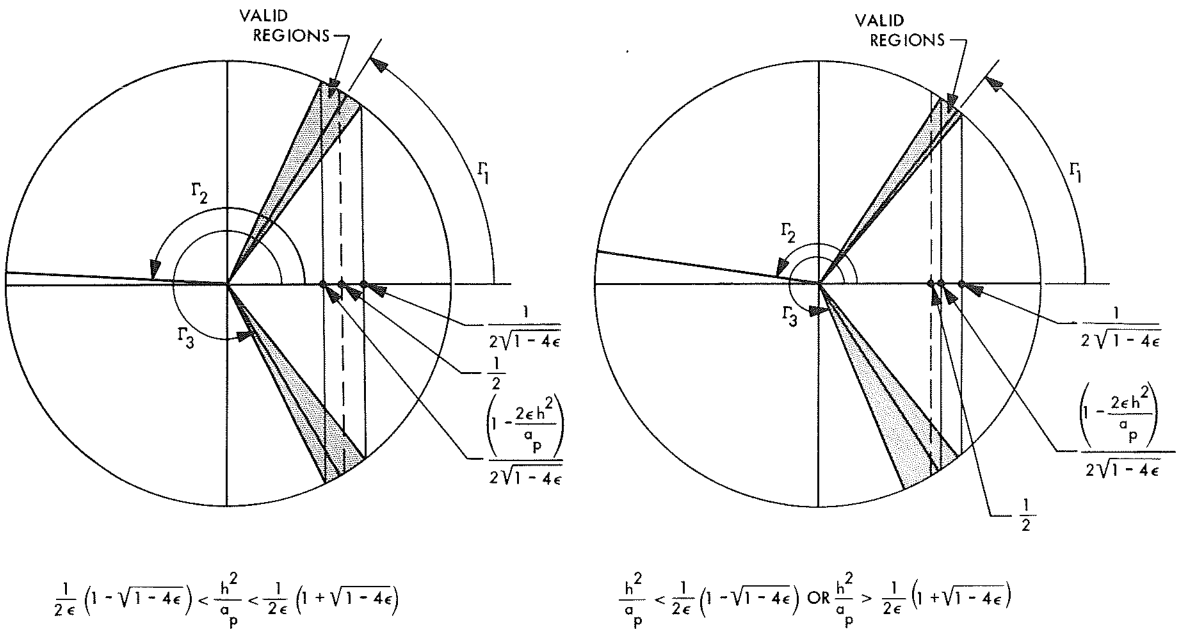


Fig. B-1. Determination of the transformation constants for the elliptic integrals

as the only physically meaningful extremum. These conditions correspond to an escape trajectory (case 1).

For a periodic trajectory, both $r_{\min}$ and $r_{\max}$ exist and are constants so that the motion of the satellite is restricted to an area of the equatorial plane that is bounded by two concentric circles as shown in Fig. B-2.

At this point, it is clear that the quantity ν may be one of two possibilities, using Eq. (B-5),

$$\left.\begin{aligned} \nu &= -\left(\frac{1}{2\epsilon} - \Lambda_{\min}\right) \\ &\text{or} \\ \nu &= -\left(\frac{1}{2\epsilon} - \Lambda_{\max}\right) \end{aligned}\right\} \qquad \text{(B-39)}$$

where both possibilities satisfy Eq. (B-4). The proper choice between these two possibilities is made subsequently.

Since ν satisfies Eq. (B-4), the polynomial (B-3) becomes

$$P(\sigma) = \frac{2}{3}\epsilon\sigma^2\left[\sigma^4 - \frac{3}{2\epsilon}(1+2\epsilon\nu)\,\sigma^2 + \frac{3}{\epsilon}(1+\nu+\epsilon\nu^2)\right]$$

or

$$P(\sigma) = \frac{2}{3}\epsilon\sigma^2\left[(\sigma_1^2-\sigma^2)(\sigma_2^2-\sigma^2)\right] \qquad \text{(B-40)}$$

where

$$\left.\begin{aligned} \sigma_1^2 &= \frac{3}{4\epsilon}\left\{(1+2\epsilon\nu) + \left[(1+2\epsilon\nu)^2 - \frac{16\epsilon}{3}(1+\nu+\epsilon\nu^2)\right]^{1/2}\right\} \\ &\text{and} \\ \sigma_2^2 &= \frac{3}{4\epsilon}\left\{(1+2\epsilon\nu) - \left[(1+2\epsilon\nu)^2 - \frac{16\epsilon}{3}(1+\nu+\epsilon\nu^2)\right]^{1/2}\right\} \end{aligned}\right\} \qquad \text{(B-41)}$$

The form of Eq. (B-40) is dictated by the fact that $\sigma^2 < \sigma_1^2$, since $\sigma_1^2 = O(1/\epsilon)$ and $\sigma^2 = O(1)$ as may be seen from Eqs. (B-1), and (B-5) and inequality (B-13). Introducing Eq. (B-40) into Eq. (B-2) yields the canonical form for the elliptic integral of the first kind:

$$\alpha = \pm\left(\frac{6}{\epsilon}\right)^{1/2}\int_{(\eta_0+\nu)^{1/2}}^{(\eta+\nu)^{1/2}} \frac{d\sigma}{\left[(\sigma_1^2-\sigma^2)(\sigma_2^2-\sigma^2)\right]^{1/2}} \qquad \text{(B-42)}$$

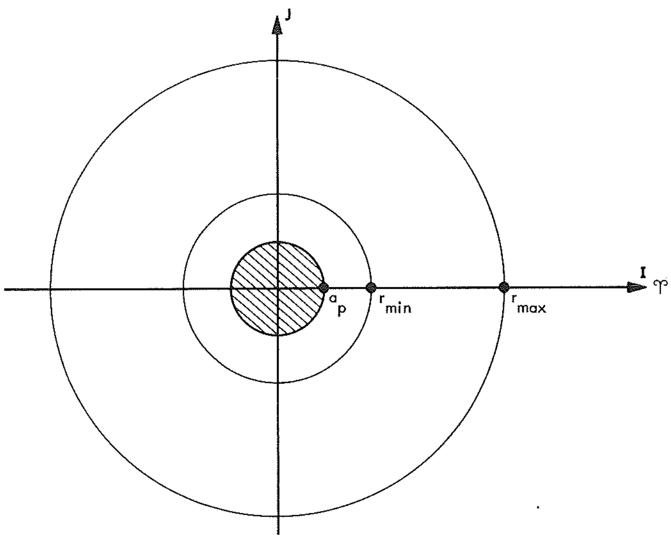


Fig. B-2. Region of motion in the equatorial plane

Next, consider the transformation of the integral (290) into elliptic integrals in canonical form. From a theorem due to Legendre (see Reference 1374, p. 297), the integral (290) can be expressed as a linear combination (with constant coefficients) of an integral of a rational function of ζ and of integrals of the following types:

$$\left.\begin{aligned} I_1 &= \int_{\eta_0}^{\eta} \frac{d\zeta}{\left(\frac{2}{3}\epsilon\zeta^3 - \zeta^2 + 2\zeta + E_T\right)^{1/2}} \\ I_2 &= \frac{\epsilon}{6}\int_{\eta_0}^{\eta} \frac{\zeta d\zeta}{\left(\frac{2}{3}\epsilon\zeta^3 - \zeta^2 + 2\zeta + E_T\right)^{1/2}} \\ I_3 &= \int_{\eta_0}^{\eta} \frac{d\zeta}{\zeta\left(\frac{2}{3}\epsilon\zeta^3 - \zeta^2 + 2\zeta + E_T\right)^{1/2}} \\ &\text{and} \\ I_4 &= \int_{\eta_0}^{\eta} \frac{\zeta d\zeta}{\left(\frac{2}{3}\epsilon\zeta^3 - \zeta^2 + 2\zeta + E_T\right)^{1/2}} = \frac{6}{\epsilon} I_2 \end{aligned}\right\} \quad \text{(B-43)}$$

where I_1, I_2, and I_3 are the elliptic integrals of the first, second, and third kinds, respectively. Consequently, using a recurrence relation from Ref. 1374, p. 299,

$$\frac{\epsilon}{3} I_4 - I_3 - E_T \int_{\eta_0}^{\eta} \frac{d\zeta}{\zeta^2\left(\frac{2}{3}\epsilon\zeta^3 - \zeta^2 + 2\zeta + E_T\right)^{1/2}} = \frac{1}{\zeta}\left(\frac{2}{3}\epsilon\zeta^3 - \zeta^2 + 2\zeta + E_T\right)^{1/2}\Bigg|_{\eta_0}^{\eta} \quad \text{(B-44)}$$

Rewriting Eq. (B-44) and using $I_4 = (6/\epsilon) I_2$ from the last of Eqs. (B-43), one obtains

$$\int_{\eta_0}^{\eta} \frac{d\zeta}{\zeta^2\left(\frac{2}{3}\epsilon\zeta^3 - \zeta^2 + 2\zeta + E_T\right)^{1/2}} = -\frac{1}{E_T\zeta}\left(\frac{2}{3}\epsilon\zeta^3 - \zeta^2 + 2\zeta + E_T\right)^{1/2}\Bigg|_{\eta_0}^{\eta} + \frac{2}{E_T} I_2 - \frac{1}{E_T} I_3 \quad \text{(B-45)}$$

Introducing Eqs. (B-43) into Eq. (B-45) and the results into Eq. (290) (at the beginning of this appendix) yields

$$t = \pm\frac{h^3}{E_T}\left[-\frac{1}{\zeta}\left(\frac{2}{3}\epsilon\zeta^3 - \zeta^2 + 2\zeta + E_T\right)^{1/2}\Bigg|_{\eta_0}^{\eta} + \frac{\epsilon}{3}\int_{\eta_0}^{\eta} \frac{\zeta d\zeta}{\left(\frac{2}{3}\epsilon\zeta^3 - \zeta^2 + 2\zeta + E_T\right)^{1/2}} - \int_{\eta_0}^{\eta} \frac{d\zeta}{\zeta\left(\frac{2}{3}\epsilon\zeta^3 - \zeta^2 + 2\zeta + E_T\right)^{1/2}}\right] \quad \text{(B-46)}$$

Following the procedure used in obtaining the canonical form of α, (Eq. B-42), let $\zeta = \sigma^2 - \nu$ so that

$$t = \pm\frac{h^3}{E_T}\left[\frac{h|\dot{r}_0|}{\eta_0} - \left(\frac{2}{3}\epsilon\eta - 1 + \frac{2}{\eta} + \frac{E_T}{\eta^2}\right)^{1/2} + \left(\frac{2\epsilon}{3}\right)^{1/2}\int_{(\eta_0+\nu)^{1/2}}^{(\eta+\nu)^{1/2}} \frac{(\sigma^2 - \nu)\, d\sigma}{[(\sigma_1^2 - \sigma^2)(\sigma_2^2 - \sigma^2)]^{1/2}} - \left(\frac{6}{\epsilon}\right)^{1/2}\int_{(\eta_0+\nu)^{1/2}}^{(\eta+\nu)^{1/2}} \frac{d\sigma}{(\sigma^2 - \nu)[(\sigma_1^2 - \sigma^2)(\sigma_2^2 - \sigma^2)]^{1/2}}\right] \quad \text{(B-47)}$$

where, using Eqs. (283) and (285) evaluated at $\alpha = 0$ as well as $(d\eta/d\alpha)(0) = -h\dot{r}_0$ from Eqs. (280), all from Section V-B-2,

$$\left(\frac{2}{3}\epsilon\eta_0^3 - \eta_0^2 + 2\eta_0 + E_T\right)^{1/2} = (h^2\dot{r}_0^2)^{1/2} = h|\dot{r}_0|$$

It is now possible to express α and t in terms of Legendre normal elliptic integrals of the first, second, and third kinds. Let (see Ref. 1375)

$$\left.\begin{aligned} &\frac{\sigma}{\sigma_2} = \sin\overset{\Delta}{\theta}, \qquad \frac{(\eta+\nu)^{1/2}}{\sigma_2} = \sin\overset{\Delta}{\phi}, \qquad \frac{(\eta_0+\nu)^{1/2}}{\sigma_2} = \sin\overset{\Delta}{\phi}_0 \\ &\overset{\Delta}{m} = \left(\frac{\sigma_2}{\sigma_1}\right)^2, \qquad \text{and} \qquad \overset{\Delta}{n} = \frac{\sigma_2^2}{\nu} \end{aligned}\right\} \tag{B-48}$$

where

$$\sigma_2^2 \geqq \sigma_1^2$$

as may be seen from Eqs. (B-41). Then

$$\int_{(\eta_0+\nu)^{1/2}}^{(\eta+\nu)^{1/2}} \frac{d\sigma}{[(\sigma_1^2-\sigma^2)(\sigma_2^2-\sigma^2)]^{1/2}} = \int_{(\eta_0+\nu)^{1/2}}^{(\eta+\nu)^{1/2}} \frac{d\sigma}{\sigma_1\sigma_2\left\{\left[1-\left(\frac{\sigma}{\sigma_2}\right)^2\right]\left[1-\left(\frac{\sigma_2}{\sigma_1}\right)^2\left(\frac{\sigma}{\sigma_2}\right)^2\right]\right\}^{1/2}} = \frac{1}{\sigma_1}\int_{\overset{\Delta}{\phi}_0}^{\overset{\Delta}{\phi}} \frac{d\overset{\Delta}{\theta}}{(1-\overset{\Delta}{m}\sin^2\overset{\Delta}{\theta})^{1/2}} \tag{B-49}$$

$$\begin{aligned} \int_{(\eta_0+\nu)^{1/2}}^{(\eta+\nu)^{1/2}} \frac{(\sigma^2-\nu)\,d\sigma}{[(\sigma_1^2-\sigma^2)(\sigma_2^2-\sigma^2)]^{1/2}} &= -\int_{(\eta_0+\nu)^{1/2}}^{(\eta+\nu)^{1/2}} \frac{(\sigma_1^2-\sigma^2)\,d\sigma}{[(\sigma_1^2-\sigma^2)(\sigma_2^2-\sigma^2)]^{1/2}} + (\sigma_1^2-\nu)\int_{(\eta_0+\nu)^{1/2}}^{(\eta+\nu)^{1/2}} \frac{d\sigma}{[(\sigma_1^2-\sigma^2)(\sigma_2^2-\sigma^2)]^{1/2}} \\ &= -\int_{(\eta_0+\nu)^{1/2}}^{(\eta+\nu)^{1/2}} \frac{\sigma_1\left[1-\left(\frac{\sigma_2}{\sigma_1}\right)^2\left(\frac{\sigma}{\sigma_2}\right)^2\right]^{1/2} d\sigma}{\sigma_2\left[1-\left(\frac{\sigma}{\sigma_2}\right)^2\right]^{1/2}} + \frac{\sigma_2^2-\nu}{\sigma_1}\int_{\overset{\Delta}{\phi}_0}^{\overset{\Delta}{\phi}} \frac{d\overset{\Delta}{\theta}}{(1-\overset{\Delta}{m}\sin^2\overset{\Delta}{\theta})^{1/2}} \\ &= -\sigma_1\int_{\overset{\Delta}{\phi}_0}^{\overset{\Delta}{\phi}} (1-\overset{\Delta}{m}\sin^2\overset{\Delta}{\theta})^{1/2}\,d\overset{\Delta}{\theta} + \frac{\sigma_1^2-\nu}{\sigma_1}\int_{\overset{\Delta}{\phi}_0}^{\overset{\Delta}{\phi}} \frac{d\overset{\Delta}{\theta}}{(1-\overset{\Delta}{m}\sin^2\overset{\Delta}{\theta})^{1/2}} \end{aligned} \tag{B-50}$$

and

$$\begin{aligned} \int_{(\eta_0+\nu)^{1/2}}^{(\eta+\nu)^{1/2}} \frac{d\sigma}{(\sigma^2-\nu)[(\sigma_1^2-\sigma^2)(\sigma_2^2-\sigma^2)]^{1/2}} &= -\frac{1}{\sigma_1\nu}\int_{(\eta_0+\nu)^{1/2}}^{(\eta+\nu)^{1/2}} \frac{d\sigma}{\sigma_2\left(1-\frac{1}{\nu}\sigma^2\right)\left\{\left[1-\left(\frac{\sigma}{\sigma_2}\right)^2\right]\left[1-\left(\frac{\sigma_2}{\sigma_1}\right)^2\left(\frac{\sigma}{\sigma_2}\right)^2\right]\right\}^{1/2}} \\ &= -\frac{1}{\sigma_1\nu}\int_{\overset{\Delta}{\phi}_0}^{\overset{\Delta}{\phi}} \frac{d\overset{\Delta}{\theta}}{(1-\overset{\Delta}{n}\sin^2\overset{\Delta}{\theta})(1-\overset{\Delta}{m}\sin^2\overset{\Delta}{\theta})^{1/2}} \end{aligned} \tag{B-51}$$

Introducing Eqs. (B-49), (B-50), and (B-51) into (B-42) and (B-47), respectively yields

$$\alpha = \pm\frac{1}{\sigma_1}\left(\frac{6}{\epsilon}\right)^{1/2}\int_{\overset{\Delta}{\phi}_0}^{\overset{\Delta}{\phi}} \frac{d\overset{\Delta}{\theta}}{(1-\overset{\Delta}{m}\sin^2\overset{\Delta}{\theta})^{1/2}}$$

and

$$t = \pm \frac{h^3}{E_T}\left[\frac{h|\dot{r}_0|}{\eta_0} - \left(\frac{2}{3}\epsilon\eta - 1 + \frac{2}{\eta} + \frac{E_T}{\eta^2}\right)^{1/2} - \sigma_1\left(\frac{2\epsilon}{3}\right)^{1/2}\int_{\overset{\Delta}{\phi}_0}^{\overset{\Delta}{\phi}} (1 - \overset{\Delta}{m}\sin^2\overset{\Delta}{\theta})^{1/2}\, d\overset{\Delta}{\theta}\right.$$

$$\left. + \frac{\sigma_1^2 - \nu}{\sigma_1}\left(\frac{2\epsilon}{3}\right)^{1/2}\int_{\overset{\Delta}{\phi}_0}^{\overset{\Delta}{\phi}} \frac{d\overset{\Delta}{\theta}}{(1 - \overset{\Delta}{m}\sin^2\overset{\Delta}{\theta})^{1/2}} + \frac{1}{\sigma_1\nu}\left(\frac{6}{\epsilon}\right)^{1/2}\int_{\overset{\Delta}{\phi}_0}^{\overset{\Delta}{\phi}} \frac{d\overset{\Delta}{\theta}}{(1 - \overset{\Delta}{n}\sin^2\overset{\Delta}{\theta})(1 - \overset{\Delta}{m}\sin^2\overset{\Delta}{\theta})^{1/2}}\right]$$

or, with the use of the mathematical notation

$$F(\overset{\Delta}{\phi}\backslash\overset{\Delta}{m}) = \int_0^{\overset{\Delta}{\phi}} \frac{d\overset{\Delta}{\theta}}{(1 - \overset{\Delta}{m}\sin^2\overset{\Delta}{\theta})^{1/2}}$$

$$E(\overset{\Delta}{\phi}\backslash\overset{\Delta}{m}) = \int_0^{\overset{\Delta}{\phi}} (1 - \overset{\Delta}{m}\sin^2\overset{\Delta}{\theta})^{1/2}\, d\overset{\Delta}{\theta} \qquad \text{(B-52)}$$

and

$$\Pi(\overset{\Delta}{n};\overset{\Delta}{\phi}\backslash\overset{\Delta}{m}) = \int_0^{\overset{\Delta}{\phi}} \frac{d\overset{\Delta}{\theta}}{(1 - \overset{\Delta}{n}\sin^2\overset{\Delta}{\theta})(1 - \overset{\Delta}{m}\sin^2\overset{\Delta}{\theta})^{1/2}}$$

the quantities α and t become

$$\alpha = \pm\frac{1}{\sigma_1}\left(\frac{6}{\epsilon}\right)^{1/2}[F(\overset{\Delta}{\phi}\backslash\overset{\Delta}{m}) - F(\overset{\Delta}{\phi}_0\backslash\overset{\Delta}{m})]$$

and

$$t = \pm\frac{h^3}{E_T}\left\{\frac{h|\dot{r}_0|}{\eta_0} - \left(\frac{2}{3}\epsilon\eta - 1 + \frac{2}{\eta} + \frac{E_T}{\eta^2}\right)^{1/2} - \sigma_1\left(\frac{2\epsilon}{3}\right)^{1/2}[E(\overset{\Delta}{\phi}\backslash\overset{\Delta}{m}) - E(\overset{\Delta}{\phi}_0\backslash\overset{\Delta}{m})]\right. \qquad \text{(B-53)}$$

$$\left. + \frac{\sigma_1^2 - \nu}{\sigma_1}\left(\frac{2\epsilon}{3}\right)^{1/2}[F(\overset{\Delta}{\phi}\backslash\overset{\Delta}{m}) - F(\overset{\Delta}{\phi}_0\backslash\overset{\Delta}{m})] + \frac{1}{\sigma_1\nu}\left(\frac{6}{\epsilon}\right)^{1/2}[\Pi(\overset{\Delta}{n};\overset{\Delta}{\phi}\backslash\overset{\Delta}{m}) - \Pi(\overset{\Delta}{n};\overset{\Delta}{\phi}_0\backslash\overset{\Delta}{m})]\right\}$$

The question of the proper signs in Eqs. (B-53) can now be considered. From Eqs. (276) and (287), Section V-B-2,

$$\frac{dr}{dt} = \mp\frac{1}{h}\left(\frac{2}{3}\epsilon\eta^3 - \eta^2 + 2\eta + E_T\right)^{1/2}$$

or, with the use of Eq. (274) from Section V-B-2,

$$\frac{dr}{dt} = \mp\frac{1}{h}\left[\left(\frac{2}{3}\epsilon h^6\right)\frac{1}{r^3} - h^4\frac{1}{r^2} + 2h^2\frac{1}{r} + E_T\right]^{1/2} \qquad \text{(B-54)}$$

Equation (B-54) shows that the upper signs in Eqs. (B-53) and (B-54) correspond to a decreasing r, since then $dr/dt<0$, and the lower signs in Eqs. (B-53) and (B-54) correspond to an increasing r, since then $dr/dt > 0$. Hence, the proper signs in Eqs. (B-53) and (B-54) are initially determined by the initial condition $\dot{r}_0$ as follows:

$$\left.\begin{array}{l} \text{if } \dot{r}_0 > 0,\ r \text{ is increasing and the lower signs should be used;} \\ \text{if } \dot{r}_0 < 0,\ r \text{ is decreasing and the upper signs should be used;} \\ \text{if } \dot{r}_0 = 0 \text{ and } d^2r/dt^2\,(0) > 0,\ r \text{ is increasing and the lower signs should be used;} \\ \text{and} \\ \text{if } \dot{r}_0 = 0 \text{ and } d^2r/dt^2\,(0) < 0,\ r \text{ is decreasing and the upper signs should be used.} \end{array}\right\} \quad \text{(B-55)}$$

The latter two statements follow from the Taylor series expansion of r about the point $t = 0$; that is,

$$r = r_0 + \dot{r}_0 t + \frac{1}{2}\frac{d^2r}{dt^2}(0)\,t^2 + \cdots$$

or

$$\Delta r = r - r_0 = \frac{1}{2}\frac{d^2r}{dt^2}(0)\,t^2 + \cdots \qquad \text{for } \dot{r}_0 = 0$$

The case where $\dot{r}_0 = d^2r/dt^2\,(0) = 0$ is excluded from Eqs. (B-55) since then $d^jr/dt^j\,(0) = 0$ for $j = 3, 4, \cdots$ from Eqs. (B-19) and $\Delta r = 0$; that is, the satellite is in a circular orbit so that Eqs. (B-21) are used and not Eqs. (B-53) and (B-54).

At this point of the analysis, the proper choice for ν from Eqs. (B-39) can be determined. It follows from Eq. (B-10) that, at an extremum,

$$\eta + \nu = 0 \qquad \text{(B-56)}$$

In addition, from Eq. (B-1),

$$\eta + \nu = \sigma^2 \geqq 0 \qquad \text{(B-57)}$$

where ζ is the integration variable associated with η; that is, $\zeta = \eta$ in Eq. (B-57). Thus, Eqs. (B-56) and (B-57) show that the proper choice for ν depends upon whether r is increasing or decreasing and whether r_{max} exists.

If r is increasing and r_{max} exists, then $r < r_{max}$ and

$$\nu = -\left(\frac{1}{2\epsilon} - \Lambda_{max}\right)$$

from Eqs. (B-39) so that, with the use of Eq. (274) from Section V-B-2 and Eqs. (B-11),

$$\eta + \nu = \frac{h^2}{r} - \left(\frac{1}{2\epsilon} - \Lambda_{max}\right) = h^2\left(\frac{1}{r} - \frac{1}{r_{max}}\right) \qquad \text{(B-58)}$$

Equation (B-58) shows that

$$\eta + \nu \to 0 \qquad \text{as} \qquad r \to r_{max}$$

$$\eta + \nu = 0 \qquad \text{at} \qquad r = r_{max}$$

and

$$\eta + \nu \geqq 0 \qquad \text{since} \qquad \left(\frac{1}{r} - \frac{1}{r_{max}}\right) \geqq 0$$

for $r \leqq r_{max}$ and since $h^2 > 0$; that is, $\eta + \nu$ satisfies the conditions (B-56) and (B-57).

If r is decreasing, then $r > r_{min}$ and

$$\nu = -\left(\frac{1}{2\epsilon} - \Lambda_{min}\right)$$

from Eqs. (B-39) so that, with the use of Eq. (274) from Section V-B-2 and Eqs. (B-11),

$$\eta + \nu = \frac{h^2}{r} - \left(\frac{1}{2\epsilon} - \Lambda_{min}\right) = h^2\left(\frac{1}{r} - \frac{1}{r_{min}}\right) \qquad \text{(B-59)}$$

Equation (B-59) shows that

$$\eta + \nu \to 0 \qquad \text{as} \qquad r \to r_{min}$$

and

$$\eta + \nu = 0 \qquad \text{at} \qquad r = r_{min}$$

that is, $\eta + \nu$ satisfies the condition (B-56). However,

$$\eta + \nu \leqq 0 \qquad \text{since} \qquad \left(\frac{1}{r} - \frac{1}{r_{\min}}\right) \leqq 0$$

for $r \geqq r_{\min}$ and since $h^2 > 0$; that is, $\eta + \nu$ does not satisfy the condition (B-57). In order to eliminate this contradiction, the transformation (B-1) must be replaced by

$$\zeta = -(\sigma^2 + \nu) \tag{B-60}$$

for r decreasing. Introducing Eq. (B-60) into Eq. (288) (at the beginning of this appendix) and Eq. (B-46) yields

$$\left.\begin{aligned} \alpha &= -\left(\frac{6}{\epsilon}\right)^{1/2} \int_{-(\eta_0+\nu)^{1/2}}^{-(\eta+\nu)^{1/2}} \frac{d\sigma}{[-(\sigma^2+\sigma_1^2)(\sigma^2+\sigma_2^2)]^{1/2}} \\ &\text{and} \\ t &= \frac{h^3}{E_T}\left\{-\frac{h\dot{r}_0}{\eta_0} - \left(\frac{2}{3}\epsilon\eta - 1 + \frac{2}{\eta} + \frac{E_T}{\eta^2}\right)^{1/2} \right. \\ &\quad + \left(\frac{2\epsilon}{3}\right)^{1/2} \int_{-(\eta_0+\nu)^{1/2}}^{-(\eta+\nu)^{1/2}} \frac{(\sigma^2+\nu)\,d\sigma}{[-(\sigma^2+\sigma_1^2)(\sigma^2+\sigma_2^2)]^{1/2}} \\ &\quad \left. - \left(\frac{6}{\epsilon}\right)^{1/2} \int_{-(\eta_0+\nu)^{1/2}}^{-(\eta+\nu)^{1/2}} \frac{d\sigma}{(\sigma^2+\nu)[-(\sigma^2+\sigma_1^2)(\sigma^2+\sigma_2^2)]^{1/2}} \right\} \end{aligned}\right\} \tag{B-61}$$

respectively, where

$$P(\sigma) = -\frac{2}{3}\epsilon\sigma^6 - (1 + 2\epsilon\nu)\sigma^4 - 2(1 + \nu + \epsilon\nu^2)\sigma^2 - \frac{2}{3}\epsilon\left(\nu^3 + \frac{3}{2\epsilon}\nu^2 + \frac{3}{\epsilon}\nu - \frac{3E_T}{2\epsilon}\right)$$

which shows that ν is still determined from Eq. (B-4) and that σ_1^2 and σ_2^2 are still determined from Eq. (B-41) (the discussion following Eq. (B-4) is also applicable here). Now, although Eq. (B-56) is still applicable, Eq. (B-57) is replaced by

$$-(\eta + \nu) = \sigma^2 \geqq 0 \tag{B-62}$$

in accordance with (B-60). If one rewrites Eqs. (B-59),

$$-(\eta + \nu) = h^2\left(\frac{1}{r_{\min}} - \frac{1}{r}\right)$$

which shows that

$$-(\eta + \nu) \to 0 \qquad \text{as} \qquad r \to r_{\min}$$

$$-(\eta + \nu) = 0 \qquad \text{at} \qquad r = r_{\min}$$

and

$$-(\eta + \nu) \geqq 0 \qquad \text{since} \qquad \left(\frac{1}{r_{\min}} - \frac{1}{r}\right) \geqq 0$$

for $r \geqq r_{\min}$ and since $h^2 > 0$; that is, $-(\eta + \nu)$ satisfies the conditions (B-56) and (B-62).

If r is increasing and $r_{\max}$ does not exist (escape trajectory), then

$$\left.\begin{aligned} \nu &= -\left(\frac{1}{2\epsilon} - \Lambda_{\min}\right) \\ \alpha &= \left(\frac{6}{\epsilon}\right)^{1/2} \int_{-(\eta_0+\nu)^{1/2}}^{-(\eta+\nu)^{1/2}} \frac{d\sigma}{[-(\sigma^2+\sigma_1^2)(\sigma^2+\sigma_2^2)]^{1/2}} \\ &\text{and} \\ t &= \frac{h^3}{E_T}\left\{-\frac{h\dot{r}_0}{\eta_0} + \left(\frac{2}{3}\epsilon\eta - 1 + \frac{2}{\eta} + \frac{E_T}{\eta^2}\right)^{1/2} - \left(\frac{2\epsilon}{3}\right)^{1/2} \int_{-(\eta_0+\nu)^{1/2}}^{-(\eta+\nu)^{1/2}} \frac{(\sigma^2+\nu)\,d\sigma}{[-(\sigma^2+\sigma_1^2)(\sigma^2+\sigma_2^2)]^{1/2}} \right. \\ &\quad \left. + \left(\frac{6}{\epsilon}\right)^{1/2} \int_{-(\eta_0+\nu)^{1/2}}^{-(\eta+\nu)^{1/2}} \frac{d\sigma}{(\sigma^2+\nu)[-(\sigma^2+\sigma_1^2)(\sigma^2+\sigma_2^2)]^{1/2}} \right\} \end{aligned}\right\} \tag{B-63}$$

It is now apparent that Eqs. (B-53) are applicable only when r is increasing and r_{max} exists. Furthermore, the proper signs in Eqs. (B-53) are clearly the lower (minus) signs. If r is decreasing or if r is increasing with r_{max} nonexistent, Eqs. (B-61) or Eqs. (B-63), respectively, must be used.

Consider the transformation of the integrals in Eqs. (B-61) and (B-63) into Legendre normal elliptic integrals. Since the elliptic integrals in these equations are real,

$$-(\sigma^2 + \sigma_1^2)(\sigma^2 + \sigma_2^2) > 0$$

for all σ except when $\sigma^2 = -\sigma_1^2$ or $\sigma^2 = -\sigma_2^2$, two isolated points in the range of σ. There are only two ways that this inequality can be satisfied, namely, $(\sigma^2 + \sigma_1^2) < 0$ or $(\sigma^2 + \sigma_2^2) < 0$, which can be rewritten as $-\sigma_1^2 > \sigma^2$ or $-\sigma_2^2 > \sigma^2$, respectively. Furthermore, since $\sigma^2 \geqq 0$ for real elliptic integrals, either $-\sigma_1^2 > 0$ or $-\sigma_2^2 > 0$; that is, either $\sigma_1^2 < 0$ or $\sigma_2^2 < 0$. It is clear from Eqs. (B-41) that $\sigma_1^2 \geqq \sigma_2^2$ so that $\sigma_2^2 < 0$ and $\sigma_1^2 > 0$; otherwise both σ_1^2 and σ_2^2 would be less than zero. Let (see Ref. 1375)

$$\frac{\sigma^2(\sigma_1^2 + \sigma_3^2)}{\sigma_3^2(\sigma_1^2 + \sigma^2)} = \sin^2 \overset{\Delta}{\theta} \tag{B-64}$$

$$-\frac{(\eta + \nu)(\sigma_1^2 + \sigma_3^2)}{\sigma_3^2(\sigma_1^2 - \eta - \nu)} = \sin^2 \overset{\Delta}{\phi} \tag{B-65}$$

$$\frac{-(\eta_0 + \nu)(\sigma_1^2 + \sigma_3^2)}{\sigma_3^2(\sigma_1^2 - \eta_0 - \nu)} = \sin^2 \overset{\Delta}{\phi}_0 \tag{B-66}$$

$$\overset{\Delta}{m} = \frac{\sigma_3^2}{\sigma_1^2 + \sigma_3^2} \tag{B-67}$$

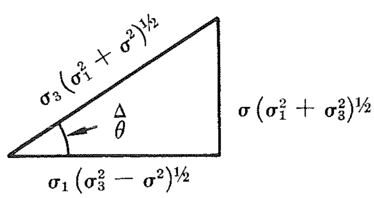
and

$$\overset{\Delta}{n} = \frac{\overset{\Delta}{m}(\nu - \sigma_1^2)}{\nu} \tag{B-68}$$

where $\sigma_3^2 = -\sigma_2^2$ and where σ_1^2 and σ_2^2 are given in Eqs. (B-41). Thus, from Eq. (B-64),

$$d\sigma = \frac{\overset{\Delta}{m}(\sigma_1^2 + \sigma^2)^2}{\sigma_1^2 \sigma} \sin \overset{\Delta}{\theta} \cos \overset{\Delta}{\theta}\, d\overset{\Delta}{\theta} \tag{B-69}$$

from Eq. (B-64) and trigonometry,



$$\sin \overset{\Delta}{\theta} = \frac{\sigma(\sigma_1^2 + \sigma_3^2)^{1/2}}{\sigma_3(\sigma_1^2 + \sigma^2)^{1/2}} \quad \text{and} \quad \cos \overset{\Delta}{\theta} = \frac{\sigma_1}{\sigma_3}\left(\frac{\sigma_3^2 - \sigma^2}{\sigma_1^2 + \sigma^2}\right)^{1/2} \tag{B-70}$$

from Eqs. (B-64) and (B-67),

$$(1 - \overset{\Delta}{m} \sin^2 \overset{\Delta}{\theta}) = 1 - \frac{\sigma^2}{\sigma_1^2 + \sigma^2} = \frac{\sigma_1^2}{\sigma_1^2 + \sigma^2} \tag{B-71}$$

and from Eq. (B-71),

$$\sigma^2 + \nu = \frac{\sigma_1^2}{(1 - \overset{\Delta}{m} \sin^2 \overset{\Delta}{\theta})} - \sigma_1^2 + \nu = \frac{\nu + \overset{\Delta}{m}(\sigma_1^2 - \nu)\sin^2 \overset{\Delta}{\theta}}{(1 - \overset{\Delta}{m} \sin^2 \overset{\Delta}{\theta})} \tag{B-72}$$

Introducing Eqs. (B-64–68) into the integrals in Eqs. (B-61) and (B-63) yields

$$\int_{-(\eta_0+\nu)^{1/2}}^{-(\eta+\nu)^{1/2}} \frac{d\sigma}{[-(\sigma^2 + \sigma_1^2)(\sigma^2 + \sigma_2^2)]^{1/2}} = \int_{-(\eta_0+\nu)^{1/2}}^{-(\eta+\nu)^{1/2}} \frac{d\sigma}{[(\sigma_1^2 + \sigma^2)(\sigma_3^2 - \sigma^2)]^{1/2}}$$

$$= \int_{\overset{\Delta}{\phi}_0}^{\overset{\Delta}{\phi}} \frac{\sin \overset{\Delta}{\theta} \cos \overset{\Delta}{\theta}\, d\overset{\Delta}{\theta}}{\dfrac{\sigma_1^2 \sigma}{\overset{\Delta}{m}(\sigma_1^2 + \sigma^2)^2} [(\sigma_1^2 + \sigma^2)(\sigma_3^2 - \sigma^2)]^{1/2}} = \int_{\overset{\Delta}{\phi}_0}^{\overset{\Delta}{\phi}} \frac{\sin \overset{\Delta}{\theta} \cos \overset{\Delta}{\theta}\, d\overset{\Delta}{\theta}}{\left[\dfrac{\sigma_1^2 \sigma}{\overset{\Delta}{m}(\sigma_1^2 + \sigma^2)}\right]\left[\dfrac{(\sigma_3^2 - \sigma^2)}{(\sigma_1^2 + \sigma^2)}\right]^{1/2}}$$

$$= \int_{\overset{\Delta}{\phi}_0}^{\overset{\Delta}{\phi}} \frac{\sin \overset{\Delta}{\theta} \cos \overset{\Delta}{\theta}\, d\overset{\Delta}{\theta}}{\left[\dfrac{\sigma_1^2}{\overset{\Delta}{m}(\sigma_1^2 + \sigma^2)^{1/2}}\right]\left[\dfrac{\sigma_3}{(\sigma_1^2 + \sigma_3^2)^{1/2}} \sin \overset{\Delta}{\theta}\right]\left(\dfrac{\sigma_3}{\sigma_1} \cos \overset{\Delta}{\theta}\right)} \quad \text{using Eqs. (B-70),}$$

$$= \int_{\overset{\Delta}{\phi}_0}^{\overset{\Delta}{\phi}} \left(\frac{\overset{\Delta}{m}^{1/2}}{\sigma_3}\right)\left[\frac{(\sigma_1^2 + \sigma^2)^{1/2}}{\sigma_1}\right] d\overset{\Delta}{\theta} = \frac{\overset{\Delta}{m}^{1/2}}{\sigma_3} \int_{\overset{\Delta}{\phi}_0}^{\overset{\Delta}{\phi}} \frac{d\overset{\Delta}{\theta}}{(1 - \overset{\Delta}{m} \sin^2 \overset{\Delta}{\theta})^{1/2}} \quad \text{using Eqs. (B-71)}$$

or

$$\int_{-(\eta_0+\nu)^{1/2}}^{-(\eta+\nu)^{1/2}} \frac{d\sigma}{[-(\sigma^2 + \sigma_1^2)(\sigma^2 + \sigma_2^2)]^{1/2}} = \frac{1}{(\sigma_1^2 + \sigma_3^2)^{1/2}} \int_{\overset{\Delta}{\phi}_0}^{\overset{\Delta}{\phi}} \frac{d\overset{\Delta}{\theta}}{(1 - \overset{\Delta}{m} \sin^2 \overset{\Delta}{\theta})^{1/2}} \tag{B-73}$$

$$\int_{-(\eta_0+\nu)^{1/2}}^{-(\eta+\nu)^{1/2}} \frac{(\sigma^2 + \nu)\, d\sigma}{[-(\sigma^2 + \sigma_1^2)(\sigma^2 + \sigma_2^2)]^{1/2}} = \int_{-(\eta_0+\nu)^{1/2}}^{-(\eta+\nu)^{1/2}} \frac{[(\sigma_1^2 + \sigma^2) + (\nu - \sigma_1^2)]\, d\sigma}{[(\sigma_1^2 + \sigma^2)(\sigma_3^2 - \sigma^2)]^{1/2}}$$

or

$$\int_{-(\eta_0+\nu)^{1/2}}^{-(\eta+\nu)^{1/2}} \frac{(\sigma^2 + \nu)\, d\sigma}{[-(\sigma^2 + \sigma_1^2)(\sigma^2 + \sigma_2^2)]^{1/2}} = \frac{\sigma_1^2}{(\sigma_1^2 + \sigma_3^2)^{1/2}} \int_{\overset{\Delta}{\phi}_0}^{\overset{\Delta}{\phi}} \frac{d\overset{\Delta}{\theta}}{(1 - \overset{\Delta}{m} \sin^2 \overset{\Delta}{\theta})^{3/2}} + \frac{\nu - \sigma_1^2}{(\sigma_1^2 + \sigma_3^2)^{1/2}} \int_{\overset{\Delta}{\phi}_0}^{\overset{\Delta}{\phi}} \frac{d\overset{\Delta}{\theta}}{(1 - \overset{\Delta}{m} \sin^2 \overset{\Delta}{\theta})^{1/2}} \tag{B-74}$$

and

$$\int_{-(\eta_0+\nu)^{1/2}}^{-(\eta+\nu)^{1/2}} \frac{d\sigma}{(\sigma^2+\nu)\,[-(\sigma^2+\sigma_1^2)(\sigma^2+\sigma_2^2)]^{1/2}} = \frac{1}{(\sigma_1^2+\sigma_3^2)^{1/2}} \int_{\overset{\Delta}{\phi}_0}^{\overset{\Delta}{\phi}} \frac{(1-\overset{\Delta}{m}\sin^2\overset{\Delta}{\theta})\,d\overset{\Delta}{\theta}}{[\nu+\overset{\Delta}{m}(\sigma_1^2-\nu)\sin^2\overset{\Delta}{\theta}]\,(1-\overset{\Delta}{m}\sin^2\overset{\Delta}{\theta})^{1/2}} \quad \text{using (B-72)},$$

$$= \frac{1}{\nu(\sigma_1^2+\sigma_3^2)^{1/2}} \int_{\overset{\Delta}{\phi}_0}^{\overset{\Delta}{\phi}} \frac{(1-\overset{\Delta}{m}\sin^2\overset{\Delta}{\theta})\,d\overset{\Delta}{\theta}}{(1-\overset{\Delta}{n}\sin^2\overset{\Delta}{\theta})\,(1-\overset{\Delta}{m}\sin^2\overset{\Delta}{\theta})^{1/2}}$$

or

$$\int_{-(\eta_0+\nu)^{1/2}}^{-(\eta+\nu)^{1/2}} \frac{d\sigma}{(\sigma^2+\nu)\,[-(\sigma^2+\sigma_1^2)(\sigma^2+\sigma_2^2)]^{1/2}} = \frac{1}{\nu(\sigma_1^2+\sigma_3^2)^{1/2}} \left[\left(1-\frac{\overset{\Delta}{m}}{\overset{\Delta}{n}}\right) \int_{\overset{\Delta}{\phi}_0}^{\overset{\Delta}{\phi}} \frac{d\overset{\Delta}{\theta}}{(1-\overset{\Delta}{n}\sin^2\overset{\Delta}{\theta})\,(1-\overset{\Delta}{m}\sin^2\overset{\Delta}{\theta})^{1/2}} + \frac{\overset{\Delta}{m}}{\overset{\Delta}{n}} \int_{\overset{\Delta}{\phi}_0}^{\overset{\Delta}{\phi}} \frac{d\overset{\Delta}{\theta}}{(1-\overset{\Delta}{m}\sin^2\overset{\Delta}{\theta})^{1/2}}\right] \tag{B-75}$$

Finally, introducing Eqs. (B-73), (B-74), and (B-75) into Eqs. (B-61) and (B-63), respectively, yields, for decreasing r,

$$\alpha = \left[\frac{6}{\epsilon(\sigma_1^2+\sigma_3^2)}\right]^{1/2} [F(\overset{\Delta}{\phi}_0\backslash\overset{\Delta}{m}) - F(\overset{\Delta}{\phi}\backslash\overset{\Delta}{m})]$$

and

$$t = \frac{h^3}{E_T}\left\{-\frac{h\dot{r}_0}{\eta_0} - \left(\frac{2}{3}\epsilon\eta - 1 + \frac{2}{\eta} + \frac{E_T}{\eta^2}\right)^{1/2} - \sigma_1^2\left[\frac{2\epsilon}{3(\sigma_1^2+\sigma_3^2)}\right]^{1/2} [\Pi(\overset{\Delta}{m};\overset{\Delta}{\phi}_0\backslash\overset{\Delta}{m}) - \Pi(\overset{\Delta}{m};\overset{\Delta}{\phi}\backslash\overset{\Delta}{m})] \right.$$

$$+ \left[(\sigma_1^2-\nu)\left[\frac{2\epsilon}{3(\sigma_1^2+\sigma_3^2)}\right]^{1/2} + \frac{\overset{\Delta}{m}}{\nu\overset{\Delta}{n}}\left[\frac{6}{\epsilon(\sigma_1^2+\sigma_3^2)}\right]^{1/2}\right] [F(\overset{\Delta}{\phi}_0\backslash\overset{\Delta}{m}) - F(\overset{\Delta}{\phi}\backslash\overset{\Delta}{m})]$$

$$\left. + \frac{1}{\nu}\left(1-\frac{\overset{\Delta}{m}}{\overset{\Delta}{n}}\right)\left[\frac{6}{\epsilon(\sigma_1^2+\sigma_3^2)}\right]^{1/2} [\Pi(\overset{\Delta}{n};\overset{\Delta}{\phi}_0\backslash\overset{\Delta}{m}) - \Pi(\overset{\Delta}{n};\overset{\Delta}{\phi}\backslash\overset{\Delta}{m})]\right\} \tag{B-76}$$

and, for increasing r and r_{max} nonexistent,

$$\alpha = \left[\frac{6}{\epsilon(\sigma_1^2 + \sigma_3^2)}\right]^{1/2} [F(\hat{\phi}\backslash\hat{m}) - F(\hat{\phi}_0\backslash\hat{m})]$$

and

$$\left.\begin{aligned} t = \frac{h^3}{E_T}\Bigg\{ & -\frac{h\dot{r}_0}{\eta_0} + \left(\frac{2}{3}\epsilon\eta - 1 + \frac{2}{\eta} + \frac{E_T}{\eta^2}\right)^{1/2} - \sigma_1^2\left[\frac{2\epsilon}{3(\sigma_1^2 + \sigma_3^2)}\right]^{1/2} [\Pi(\hat{m};\hat{\phi}\backslash\hat{m}) - \Pi(\hat{m};\hat{\phi}_0\backslash\hat{m})] \\ & + \left[(\sigma_1^2 - \nu)\left[\frac{2\epsilon}{3(\sigma_1^2 + \sigma_3^2)}\right]^{1/2} + \frac{\hat{m}}{\nu\hat{n}}\left[\frac{6}{\epsilon(\sigma_1^2 + \sigma_3^2)}\right]^{1/2}\right][F(\hat{\phi}\backslash\hat{m}) - F(\hat{\phi}_0\backslash\hat{m})] \\ & + \frac{1}{\nu}\left(1 - \frac{\hat{m}}{\hat{n}}\right)\left[\frac{6}{\epsilon(\sigma_1^2 + \sigma_3^2)}\right]^{1/2} [\Pi(\hat{n};\hat{\phi}\backslash\hat{m}) - \Pi(\hat{n};\hat{\phi}_0\backslash\hat{m})]\Bigg\} \end{aligned}\right\} \qquad \text{(B-77)}$$

where $F(\hat{\phi}\backslash\hat{m})$, $E(\hat{\phi}\backslash\hat{m})$, and $\Pi(\hat{n};\hat{\phi}\backslash\hat{m})$ are given in Eqs. (B-52) and where, from Ref. 1375, p. 600,

$$\Pi(\hat{m};\hat{\phi}\backslash\hat{m}) = \frac{1}{1-\hat{m}} E(\hat{\phi}\backslash\hat{m}) - \frac{\hat{m}}{1-\hat{m}} \frac{\sin 2\hat{\phi}}{2(1 - \hat{m}\sin^2\hat{\phi})^{1/2}} \qquad \text{(B-78)}$$

In conclusion, the pertinent relationships of this appendix are summarized in Tables B-1 and B-2.

Table B-1. Summary of the transformation of elliptic integrals into Legendre normal form

Increasing r and $r_{\max}$ existing	*Increasing r and $r_{\max}$ nonexistent*	*Decreasing r*
$\alpha = \frac{1}{\sigma_1}\left(\frac{6}{\epsilon}\right)^{1/2}[F(\overset{\Delta}{\phi}_0\backslash\overset{\Delta}{m}) - F(\overset{\Delta}{\phi}\backslash\overset{\Delta}{m})]$ $t = \frac{h^3}{E_T}\left\{-\frac{r_0\dot{r}_0}{h} + \left[\left(\frac{2}{3}\epsilon h^2\right)\frac{1}{r} - 1 + \frac{2}{h^2}r + \frac{E_T}{h^4}r^2\right]^{1/2}\right.$ $- \sigma_1\left(\frac{2\epsilon}{3}\right)^{1/2}[E(\overset{\Delta}{\phi}_0\backslash\overset{\Delta}{m}) - E(\overset{\Delta}{\phi}\backslash\overset{\Delta}{m})]$ $+ \frac{\sigma_1^2 - \nu}{\sigma_1}\left(\frac{2\epsilon}{3}\right)^{1/2}[F(\overset{\Delta}{\phi}_0\backslash\overset{\Delta}{m}) - F(\overset{\Delta}{\phi}\backslash\overset{\Delta}{m})]$ $\left. + \frac{1}{\sigma_1\nu}\left(\frac{6}{\epsilon}\right)^{1/2}[\Pi(\overset{\Delta}{n};\overset{\Delta}{\phi}_0\backslash\overset{\Delta}{m}) - \Pi(\overset{\Delta}{n};\overset{\Delta}{\phi}\backslash\overset{\Delta}{m})]\right\}$ $\frac{dr}{dt} = \frac{1}{h}\left[\left(\frac{2}{3}\epsilon h^6\right)\frac{1}{r^3} - h^4\frac{1}{r^2} + 2h^2\frac{1}{r} + E_T\right]^{1/2}$ where $\nu = -\left(\frac{1}{2\epsilon} - \Lambda_{\max}\right)$ $\sigma_1^2 = \frac{3}{4\epsilon}\left\{(1+2\epsilon\nu) + \left[(1+2\epsilon\nu)^2 - \frac{16\epsilon}{3}(1+\nu+\epsilon\nu^2)\right]^{1/2}\right\}$ $\sigma_2^2 = \frac{3}{4\epsilon}\left\{(1+2\epsilon\nu) - \left[(1+2\epsilon\nu)^2 - \frac{16\epsilon}{3}(1+\nu+\epsilon\nu^2)\right]^{1/2}\right\}$ $\overset{\Delta}{m} = \left(\frac{\sigma_2}{\sigma_1}\right)^2, \quad \overset{\Delta}{n} = \frac{\sigma_2^2}{\nu}$ $\sin\overset{\Delta}{\phi} = \frac{1}{\sigma_2}\left(\frac{h^2}{r} + \nu\right)^{1/2}$ and $\sin\overset{\Delta}{\phi}_0 = \frac{1}{\sigma_2}\left(\frac{h^2}{r_0} + \nu\right)^{1/2}$	$\alpha = \left[\frac{6}{\epsilon(\sigma_1^2+\sigma_3^2)}\right]^{1/2}[F(\overset{\Delta}{\phi}\backslash\overset{\Delta}{m}) - F(\overset{\Delta}{\phi}_0\backslash\overset{\Delta}{m})]$ $t = \frac{h^3}{E_T}\left(-\frac{r_0\dot{r}_0}{h} + \left[\left(\frac{2}{3}\epsilon h^2\right)\frac{1}{r} - 1 + \frac{2}{h^2}r + \frac{E_T}{h^4}r^2\right]^{1/2}\right.$ $- \sigma_1^2\left[\frac{2\epsilon}{3(\sigma_1^2+\sigma_3^2)}\right]^{1/2}[\Pi(\overset{\Delta}{m};\overset{\Delta}{\phi}\backslash\overset{\Delta}{m}) - \Pi(\overset{\Delta}{m};\overset{\Delta}{\phi}_0\backslash\overset{\Delta}{m})]$ $+ \left\{(\sigma_1^2 - \nu)\left[\frac{2\epsilon}{3(\sigma_1^2+\sigma_3^2)}\right]^{1/2} + \frac{\overset{\Delta}{m}}{\nu\overset{\Delta}{n}}\left[\frac{6}{\epsilon(\sigma_1^2+\sigma_3^2)}\right]^{1/2}\right\}[F(\overset{\Delta}{\phi}\backslash\overset{\Delta}{m}) - F(\overset{\Delta}{\phi}_0\backslash\overset{\Delta}{m})]$ $\left. + \frac{1}{\nu}\left(1 - \frac{\overset{\Delta}{m}}{\overset{\Delta}{n}}\right)\left[\frac{6}{\epsilon(\sigma_1^2+\sigma_3^2)}\right]^{1/2}[\Pi(\overset{\Delta}{n};\overset{\Delta}{\phi}\backslash\overset{\Delta}{m}) - \Pi(\overset{\Delta}{n};\overset{\Delta}{\phi}_0\backslash\overset{\Delta}{m})]\right)$ $\frac{dr}{dt} = \frac{1}{h}\left[\left(\frac{2}{3}\epsilon h^6\right)\frac{1}{r^3} - h^4\frac{1}{r^2} + 2h^2\frac{1}{r} + E_T\right]^{1/2}$ where $\nu = -\left(\frac{1}{2\epsilon} - \Lambda_{\min}\right)$ $\sigma_1^2 = \frac{3}{4\epsilon}\left\{(1+2\epsilon\nu) + \left[(1+2\epsilon\nu)^2 - \frac{16\epsilon}{3}(1+\nu+\epsilon\nu^2)\right]^{1/2}\right\}$ $\sigma_3^2 = -\sigma_2^2 = -\frac{3}{4\epsilon}\left\{(1+2\epsilon\nu) - \left[(1+2\epsilon\nu)^2 - \frac{16\epsilon}{3}(1+\nu+\epsilon\nu^2)\right]^{1/2}\right\}$ $\overset{\Delta}{m} = \frac{\sigma_3^2}{\sigma_1^2+\sigma_3^2}, \quad \overset{\Delta}{n} = \frac{\overset{\Delta}{m}(\nu - \sigma_1^2)}{\nu}$ $\sin^2\overset{\Delta}{\phi} = -\frac{\frac{h^2}{r} + \nu}{\overset{\Delta}{m}\left(\sigma_1^2 - \frac{h^2}{r} - \nu\right)}$ and $\sin^2\overset{\Delta}{\phi}_0 = -\frac{\frac{h^2}{r_0} + \nu}{\overset{\Delta}{m}\left(\sigma_1^2 - \frac{h^2}{r_0} - \nu\right)}$	$\alpha = \left[\frac{6}{\epsilon(\sigma_1^2+\sigma_3^2)}\right]^{1/2}[F(\overset{\Delta}{\phi}_0\backslash\overset{\Delta}{m}) - F(\overset{\Delta}{\phi}\backslash\overset{\Delta}{m})]$ $t = \frac{h^3}{E_T}\left(-\frac{r_0\dot{r}_0}{h} - \left[\left(\frac{2}{3}\epsilon h^2\right)\frac{1}{r} - 1 + \frac{2}{h^2}r + \frac{E_T}{h^4}r^2\right]^{1/2}\right.$ $- \sigma_1^2\left[\frac{2\epsilon}{3(\sigma_1^2+\sigma_3^2)}\right]^{1/2}[\Pi(\overset{\Delta}{m};\overset{\Delta}{\phi}_0\backslash\overset{\Delta}{m}) - \Pi(\overset{\Delta}{m};\overset{\Delta}{\phi}\backslash\overset{\Delta}{m})]$ $+ \left\{(\sigma_1^2 - \nu)\left[\frac{2\epsilon}{3(\sigma_1^2+\sigma_3^2)}\right]^{1/2} + \frac{\overset{\Delta}{m}}{\nu\overset{\Delta}{n}}\left[\frac{6}{\epsilon(\sigma_1^2+\sigma_3^2)}\right]^{1/2}\right\}[F(\overset{\Delta}{\phi}_0\backslash\overset{\Delta}{m}) - F(\overset{\Delta}{\phi}\backslash\overset{\Delta}{m})]$ $\left. + \frac{1}{\nu}\left(1 - \frac{\overset{\Delta}{m}}{\overset{\Delta}{n}}\right)\left[\frac{6}{\epsilon(\sigma_1^2+\sigma_3^2)}\right]^{1/2}[\Pi(\overset{\Delta}{n};\overset{\Delta}{\phi}_0\backslash\overset{\Delta}{m}) - \Pi(\overset{\Delta}{n};\overset{\Delta}{\phi}\backslash\overset{\Delta}{m})]\right)$ $\frac{dr}{dt} = -\frac{1}{h}\left[\left(\frac{2}{3}\epsilon h^6\right)\frac{1}{r^3} - h^4\frac{1}{r^2} + 2h^2\frac{1}{r} + E_T\right]^{1/2}$ where $\nu = -\left(\frac{1}{2\epsilon} - \Lambda_{\min}\right)$ $\sigma_1^2 = \frac{3}{4\epsilon}\left\{(1+2\epsilon\nu) + \left[(1+2\epsilon\nu)^2 - \frac{16\epsilon}{3}(1+\nu+\epsilon\nu^2)\right]^{1/2}\right\}$ $\sigma_3^2 = -\sigma_2^2 = -\frac{3}{4\epsilon}\left\{(1+2\epsilon\nu) - \left[(1+2\epsilon\nu)^2 - \frac{16\epsilon}{3}(1+\nu+\epsilon\nu^2)\right]^{1/2}\right\}$ $\overset{\Delta}{m} = \frac{\sigma_3^2}{\sigma_1^2+\sigma_3^2}, \quad \overset{\Delta}{n} = \frac{\overset{\Delta}{m}(\nu - \sigma_1^2)}{\nu}$ $\sin^2\overset{\Delta}{\phi} = -\frac{\frac{h^2}{r} + \nu}{\overset{\Delta}{m}\left(\sigma_1^2 - \frac{h^2}{r} - \nu\right)}$ and $\sin^2\overset{\Delta}{\phi}_0 = -\frac{\frac{h^2}{r_0} + \nu}{\overset{\Delta}{m}\left(\sigma_1^2 - \frac{h^2}{r_0} - \nu\right)}$

Note: If $r(0) = q_0$, $\frac{dr}{dt}(0) = 0$, and $e_0 = \frac{3}{2}J_2\left(\frac{a_p}{q_0}\right)^2$, then a circular orbit exists; that is, $r = q_0$, $\frac{dr}{dt} = 0$, $\alpha = \frac{h}{q_0^2}t$, and $\frac{d\alpha}{dt} = \frac{h}{q_0^2}$ where $0 \leqq t < \infty$.

Table B-2. Constants and relationships arising in the transformation of elliptic integrals into Legendre normal form

$$E_T = 2h^2\left[\frac{1}{2}\left(\dot{r}_0^2 + \frac{h^2}{r_0^2}\right) - \left(\frac{1}{r_0} + \frac{1}{3}\epsilon\frac{h^4}{r_0^3}\right)\right] \text{ and } \epsilon = \frac{3}{2}J_2\left(\frac{a_p}{h^2}\right)^2 \text{ where } h = r_0^2\dot{\alpha}_0$$

$$C_1 = \frac{3}{\epsilon}\left(1 - \frac{1}{4\epsilon}\right), C_2 = \frac{1}{2\epsilon}\left(\frac{1}{2\epsilon^2} - \frac{3}{\epsilon} - 3E_T\right), C_3 = \left[-\frac{C_2}{2} + \left(\frac{C_2^2}{4} + \frac{C_1^3}{27}\right)^{1/2}\right]^{1/3}, \text{ and } C_4 = \left[-\frac{C_2}{2} - \left(\frac{C_2^2}{4} + \frac{C_1^3}{27}\right)^{1/2}\right]^{1/3}$$

If $\left(\frac{C_2^2}{4} + \frac{C_1^3}{27}\right) > 0$, then $\Lambda_{min} = C_3 + C_4$ and $r_{min} = \dfrac{h^2}{\frac{1}{2\epsilon} - \Lambda_{min}}$

If $\left(\frac{C_2^2}{4} + \frac{C_1^3}{27}\right) = 0$ and if $\frac{1}{2\epsilon} > \left[\frac{1}{4\epsilon}\left(\frac{1}{2\epsilon^2} - \frac{3}{\epsilon} - 3E_T\right)\right]^{1/3} > \left(\frac{1}{2\epsilon} - \frac{h^2}{a_p}\right)$ is not satisfied,

then $\Lambda_{min} = \left[\frac{2}{\epsilon}\left(3E_T + \frac{3}{\epsilon} - \frac{1}{2\epsilon^2}\right)\right]^{1/3}$ and $r_{min} = \dfrac{h^2}{\frac{1}{2\epsilon} - \Lambda_{min}}$

If $\left(\frac{C_2^2}{4} + \frac{C_1^3}{27}\right) = 0$ and if $\frac{1}{2\epsilon} > \left[\frac{1}{4\epsilon}\left(\frac{1}{2\epsilon^2} - \frac{3}{\epsilon} - 3E_T\right)\right]^{1/3} > \left(\frac{1}{2\epsilon} - \frac{h^2}{a_p}\right)$

then Λ_{min} = smaller of Λ_1 and Λ_2 $(= \Lambda_3)$, Λ_{max} = larger of Λ_1 and Λ_2 $(= \Lambda_3)$

$$r_{min} = \frac{h^2}{\frac{1}{2\epsilon} - \Lambda_{min}}, \text{ and } r_{max} = \frac{h^2}{\frac{1}{2\epsilon} - \Lambda_{max}}$$

where $\Lambda_1 = \left[\frac{2}{\epsilon}\left(3E_T + \frac{3}{\epsilon} - \frac{1}{2\epsilon^2}\right)\right]^{1/3}$ and $\Lambda_2 = \Lambda_3 = \left[\frac{1}{4\epsilon}\left(\frac{1}{2\epsilon^2} - \frac{3}{\epsilon} - 3E_T\right)\right]^{1/3}$

If $\left(\frac{C_2^2}{4} + \frac{C_1^3}{27}\right) < 0$, then $\Lambda_j = 2\left(-\frac{C_1}{3}\right)^{1/2}\cos\Gamma_j$ for $j = 1, 2, 3$

where $\Gamma_1 = \frac{\gamma}{3}$, $\Gamma_2 = \frac{1}{3}(\gamma + 2\pi)$, $\Gamma_3 = \frac{1}{3}(\gamma + 4\pi)$, and $\cos\gamma = -\dfrac{C_2}{2\left(-\frac{C_1^3}{27}\right)^{1/2}}$

If there are two of the Λ_j which satisfy the necessary condition $\frac{1}{2\epsilon} > \Lambda_j > \left(\frac{1}{2\epsilon} - \frac{h^2}{a_p}\right)$ for $j = 1, 2, 3$

then Λ_{min} = smaller of the two valid Λ_j, Λ_{max} = larger of the two valid Λ_j

$$r_{min} = \frac{h^2}{\frac{1}{2\epsilon} - \Lambda_{min}}, \text{ and } r_{max} = \frac{h^2}{\frac{1}{2\epsilon} - \Lambda_{max}}$$

If there is only one Λ_j which satisfies the previous necessary condition,

then Λ_{min} = the valid Λ_j and $r_{min} = \dfrac{h^2}{\frac{1}{2\epsilon} - \Lambda_{min}}$

$$F(\hat{\phi}\backslash\hat{m}) = \int_0^{\hat{\phi}} \frac{d\hat{\theta}}{(1 - \hat{m}\sin^2\hat{\theta})^{1/2}}, \qquad E(\hat{\phi}\backslash\hat{m}) = \int_0^{\hat{\phi}} (1 - \hat{m}\sin^2\hat{\theta})^{1/2}\,d\hat{\theta},$$

$$\Pi(\hat{n};\hat{\phi}\backslash\hat{m}) = \int_0^{\hat{\phi}} \frac{d\hat{\theta}}{(1 - \hat{n}\sin^2\hat{\theta})(1 - \hat{m}\sin^2\hat{\theta})^{1/2}}$$

and

$$\Pi(\hat{m};\hat{\phi}\backslash\hat{m}) = \frac{1}{(1 - \hat{m})}E(\hat{\phi}\backslash\hat{m}) - \frac{\hat{m}}{1 - \hat{m}}\,\frac{\sin^2\hat{\phi}}{2(1 - \hat{m}\sin^2\hat{\phi})^{1/2}}$$

Appendix C

Integrals in the Time Expressions

The following special integrals appear in the development of the time expressions in Section V-C-3 and V-D-3:

$$\left.\begin{aligned}
&\int \frac{d\bar{v}}{(1+\bar{e}\cos\bar{v})^{\kappa}}, \qquad \int \frac{\cos^{\sigma}\bar{v}\,d\bar{v}}{(1+\bar{e}\cos\bar{v})^{k}}, \\
&\int \frac{\sin^{s}\bar{v}\,d\bar{v}}{(1+\bar{e}\cos\bar{v})^{k}}, \qquad \int \frac{\sin\bar{v}\cos\bar{v}\,d\bar{v}}{(1+\bar{e}\cos\bar{v})^{k}}, \\
&\text{and} \qquad \int \frac{\bar{v}\sin\bar{v}\,d\bar{v}}{(1+\bar{e}\cos\bar{v})^{3}}
\end{aligned}\right\} \tag{C-1}$$

where $\kappa = 2, 3, 4$; $k = 3, 4$; $\sigma = 1, 3$ and $s = 1, 2$ for $k = 3$; $\sigma = 1, 3, 4$ and $s = 1, 2, 3$ for $k = 4$.

The parameter $\bar{v}$ is either the multivariable (Section V-C-3) or the mean (Section V-D-3) instantaneous true anomaly, whereas the parameter $\bar{e}$ is either e_0 (Section V-C-3) or the mean instantaneous eccentricity (Section V-D-3). It should be noted that $0 \leqq \bar{e} < 1$ so that the integrands in (C-1) are finite for all possible values of $\bar{v}$; that is, the integrals exist. Furthermore, $\bar{e}$ is a constant in both sections.

In order to evaluate the integrals above, the classical astronomical transformation

$$\tan\frac{1}{2}\bar{v} = \left(\frac{1+\bar{e}}{1-\bar{e}}\right)^{1/2} \tan\frac{1}{2}\bar{E} \tag{C-2}$$

is desirable where the parameter $\bar{E}$ is either the multivariable (Section V-C-3) or the mean (Section V-D-3) instantaneous eccentric anomaly. With the use of Eq. (C-2) and trigonometric formulas, the following identities, which will be useful in evaluating the above integrals, can be derived:

$$\frac{\sin\bar{v}}{1+\bar{e}\cos\bar{v}} = \frac{1}{(1-\bar{e}^2)^{1/2}}\sin\bar{E} \tag{C-3}$$

$$\frac{\cos\bar{v}}{1+\bar{e}\cos\bar{v}} = \frac{1}{1-\bar{e}^2}(\cos\bar{E}-\bar{e}) \tag{C-4}$$

$$\frac{1}{1+\bar{e}\cos\bar{v}} = \frac{1}{1-\bar{e}^2}(1-\bar{e}\cos\bar{E}) \tag{C-5}$$

and, by differentiation,

$$d\bar{v} = \frac{(1-\bar{e}^2)^{1/2}}{1-\bar{e}\cos\bar{E}}\,d\bar{E} \tag{C-6}$$

In addition, from Ref. 1372, p. 148, and Eq. (C-2),

$$\begin{aligned}
\int \frac{d\bar{E}}{1-\bar{e}\cos\bar{E}} &\doteq \frac{2}{(1-\bar{e}^2)^{1/2}}\tan^{-1}\left[\left(\frac{1+\bar{e}}{1-\bar{e}}\right)^{1/2}\tan\frac{1}{2}\bar{E}\right] \\
&\doteq \frac{1}{(1-\bar{e}^2)^{1/2}}\bar{v}
\end{aligned} \tag{C-7}$$

where the constant of integration is omitted so that the symbol $\doteq$ means that the functions on the left and right of this symbol differ by a constant. This procedure is followed throughout the appendix. Consider the integral in (C-1),

$$\int \frac{d\bar{v}}{(1+\bar{e}\cos\bar{v})^2}$$

From Eqs. (C-5) and (C-6),

$$\int \frac{d\bar{v}}{(1+\bar{e}\cos\bar{v})^2} \doteq \int \frac{(1-\bar{e}\cos\bar{E})^2}{(1-\bar{e}^2)^2}\,\frac{(1-\bar{e}^2)^{1/2}}{1-\bar{e}\cos\bar{E}}\,d\bar{E} \doteq \frac{1}{(1-\bar{e}^2)^{3/2}}\int (1-\bar{e}\cos\bar{E})\,d\bar{E}$$

or

$$\int \frac{d\bar{v}}{(1+\bar{e}\cos\bar{v})^2} \doteq \frac{1}{(1-\bar{e}^2)^{3/2}}(\bar{E}-\bar{e}\sin\bar{E}) \tag{C-8}$$

Consider the integral in (C-1),

$$\int \frac{d\bar{v}}{(1+\bar{e}\cos\bar{v})^3}$$

From Eqs. (C-5) and (C-6),

$$\int \frac{d\bar{v}}{(1+\bar{e}\cos\bar{v})^3} \doteq \int \frac{(1-\bar{e}\cos\bar{E})^3}{(1-\bar{e}^2)^3}\,\frac{(1-\bar{e}^2)^{1/2}}{1-\bar{e}\cos\bar{E}}\,d\bar{E} \doteq \frac{1}{(1-\bar{e}^2)^{5/2}}\int (1-\bar{e}\cos\bar{E})^2\,d\bar{E}$$

$$\doteq \frac{1}{(1-\bar{e}^2)^{5/2}}\int (1-2\bar{e}\cos\bar{E}+\bar{e}^2\cos^2\bar{E})\,d\bar{E}$$

$$\doteq \frac{1}{(1-\bar{e}^2)^{5/2}}\int \left[\left(1+\frac{1}{2}\bar{e}^2\right) - 2\bar{e}\cos\bar{E} + \frac{1}{2}\bar{e}^2\cos 2\bar{E}\right] d\bar{E}$$

or

$$\int \frac{d\bar{v}}{(1+\bar{e}\cos\bar{v})^3} \doteq \frac{1}{(1-\bar{e}^2)^{5/2}}\left[\left(1+\frac{1}{2}\bar{e}^2\right)\bar{E} - 2\bar{e}\sin\bar{E} + \frac{1}{4}\bar{e}^2\sin 2\bar{E}\right] \tag{C-9}$$

Consider the integral in (C-1),

$$\int \frac{d\bar{v}}{(1+\bar{e}\cos\bar{v})^4}$$

From Eqs. (C-5) and (C-6),

$$\int \frac{d\bar{v}}{(1+\bar{e}\cos\bar{v})^4} \doteq \int \frac{(1-\bar{e}\cos\bar{E})^4}{(1-\bar{e}^2)^4}\,\frac{(1-\bar{e}^2)^{1/2}}{1-\bar{e}\cos\bar{E}}\,d\bar{E} \doteq \frac{1}{(1-\bar{e}^2)^{7/2}}\int (1-\bar{e}\cos\bar{E})^3\,d\bar{E}$$

$$\doteq \frac{1}{(1-\bar{e}^2)^{7/2}}\int (1-3\bar{e}\cos\bar{E}+3\bar{e}^2\cos^2\bar{E}-\bar{e}^3\cos^3\bar{E})\,d\bar{E}$$

$$\doteq \frac{1}{(1-\bar{e}^2)^{7/2}}\int \left[\left(1+\frac{3}{2}\bar{e}^2\right) - 3\bar{e}\left(1+\frac{1}{4}\bar{e}^2\right)\cos\bar{E} + \frac{3}{2}\bar{e}^2\cos 2\bar{E} - \frac{1}{4}\bar{e}^3\cos 3\bar{E}\right] d\bar{E}$$

or

$$\int \frac{d\bar{v}}{(1+\bar{e}\cos\bar{v})^4} \doteq \frac{1}{(1-\bar{e}^2)^{7/2}}\left[\left(1+\frac{3}{2}\bar{e}^2\right)\bar{E} - 3\bar{e}\left(1+\frac{1}{4}\bar{e}^2\right)\sin\bar{E} + \frac{3}{4}\bar{e}^2\sin 2\bar{E} - \frac{1}{12}\bar{e}^3\sin 3\bar{E}\right] \tag{C-10}$$

Consider the integral in (C-1),

$$\int \frac{\cos\bar{v}\,d\bar{v}}{(1+\bar{e}\cos\bar{v})^3}$$

From Eqs. (C-4), (C-5), and (C-6),

$$\int \frac{\cos\bar{v}\,d\bar{v}}{(1+\bar{e}\cos\bar{v})^3} \doteq \int \frac{\cos\bar{E}-\bar{e}}{1-\bar{e}^2}\,\frac{(1-\bar{e}\cos\bar{E})^2}{(1-\bar{e}^2)^2}\,\frac{(1-\bar{e}^2)^{1/2}}{1-\bar{e}\cos\bar{E}}\,d\bar{E}$$

$$\doteq \frac{1}{(1-\bar{e}^2)^{5/2}}\int (\cos\bar{E}-\bar{e})(1-\bar{e}\cos\bar{E})\,d\bar{E}$$

$$\doteq \frac{1}{(1-\bar{e}^2)^{5/2}}\int \left[-\frac{3}{2}\bar{e} + (1+\bar{e}^2)\cos\bar{E} - \frac{1}{2}\bar{e}\cos 2\bar{E}\right] d\bar{E}$$

or

$$\int \frac{\cos \bar{v}\, d\bar{v}}{(1+\bar{e}\cos \bar{v})^3} = \frac{1}{(1-\bar{e}^2)^{5/2}}\left[-\frac{3}{2}\bar{e}\bar{E} + (1+\bar{e}^2)\sin \bar{E} - \frac{1}{4}\bar{e}\sin 2\bar{E}\right] \tag{C-11}$$

Consider the integral in (C-1),

$$\int \frac{\cos \bar{v}\, d\bar{v}}{(1+\bar{e}\cos \bar{v})^4}$$

From Eqs. (C-4), (C-5), and (C-6),

$$\int \frac{\cos \bar{v}\, d\bar{v}}{(1+\bar{e}\cos \bar{v})^4} = \int \frac{\cos \bar{E} - \bar{e}}{1-\bar{e}^2}\, \frac{(1-\bar{e}\cos \bar{E})^3}{(1-\bar{e}^2)^3}\, \frac{(1-\bar{e}^2)^{1/2}}{1-\bar{e}\cos \bar{E}}\, d\bar{E}$$

$$= \frac{1}{(1-\bar{e}^2)^{7/2}} \int (\cos \bar{E} - \bar{e})(1-\bar{e}\cos \bar{E})^2\, d\bar{E}$$

$$= \frac{1}{(1-\bar{e}^2)^{7/2}} \int \left[-2\bar{e}\left(1+\frac{1}{4}\bar{e}^2\right) + \left(1+\frac{11}{4}\bar{e}^2\right)\cos \bar{E} - \frac{1}{2}\bar{e}(2+\bar{e}^2)\cos 2\bar{E} + \frac{1}{2}\bar{e}^2\cos 3\bar{E}\right] d\bar{E}$$

or

$$\int \frac{\cos \bar{v}\, d\bar{v}}{(1+\bar{e}\cos \bar{v})^4} = \frac{1}{(1-\bar{e}^2)^{7/2}}\left[-2\bar{e}\left(1+\frac{1}{4}\bar{e}^2\right)\bar{E} + \left(1+\frac{11}{4}\bar{e}^2\right)\sin \bar{E}\right.$$

$$\left. - \frac{1}{4}\bar{e}(2+\bar{e}^2)\sin 2\bar{E} + \frac{1}{12}\bar{e}^2\sin 3\bar{E}\right] \tag{C-12}$$

Consider the integral in (C-1),

$$\int \frac{\cos^3 \bar{v}\, d\bar{v}}{(1+\bar{e}\cos \bar{v})^3}$$

From Eqs. (C-4), (C-5), and (C-6), for $0 < \bar{e} < 1$,

$$\int \frac{\cos^3 \bar{v}\, d\bar{v}}{(1+\bar{e}\cos \bar{v})^3} = \int \frac{(\cos \bar{E} - \bar{e})^3}{(1-\bar{e}^2)^3}\, \frac{(1-\bar{e}^2)^{1/2}}{1-\bar{e}\cos \bar{E}}\, d\bar{E} = \frac{1}{(1-\bar{e}^2)^{5/2}} \int \frac{\frac{1}{\bar{e}^3}[(1-\bar{e}^2) - (1-\bar{e}\cos \bar{E})]^3}{1-\bar{e}\cos \bar{E}}\, d\bar{E}$$

or

$$= \frac{1}{\bar{e}^3(1-\bar{e}^2)^{5/2}}\left[\int \frac{(1-\bar{e}^2)^3 - 3(1-\bar{e}^2)^2(1-\bar{e}\cos \bar{E})}{1-\bar{e}\cos \bar{E}}\, d\bar{E}\right.$$

$$\left. + \int \frac{3(1-\bar{e}^2)(1-\bar{e}\cos \bar{E})^2 - (1-\bar{e}\cos \bar{E})^3}{1-\bar{e}\cos \bar{E}}\, d\bar{E}\right]$$

$$= \frac{1}{\bar{e}^3(1-\bar{e}^2)^{5/2}}\left[(1-\bar{e}^2)^3 \int \frac{d\bar{E}}{1-\bar{e}\cos \bar{E}} - 3(1-\bar{e}^2)^2 \int d\bar{E} + 3(1-\bar{e}^2)\int (1-\bar{e}\cos \bar{E})\, d\bar{E}\right.$$

$$\left. - \int (1-\bar{e}\cos \bar{E})^2\, d\bar{E}\right]$$

$$= \frac{1}{\bar{e}^3(1-\bar{e}^2)^{5/2}}\left[(1-\bar{e}^2)^{5/2}\bar{v} - 3(1-\bar{e}^2)^2\bar{E}\right.$$

$$\left. + 3(1-\bar{e}^2)(\bar{E} - \bar{e}\sin \bar{E}) - \left(1+\frac{1}{2}\bar{e}^2\right)\bar{E} + 2\bar{e}\sin \bar{E} - \frac{1}{4}\bar{e}^2\sin 2\bar{E}\right]$$

or

$$\int \frac{\cos^3 \bar{v}\, d\bar{v}}{(1+\bar{e}\cos\bar{v})^3} \doteq \frac{1}{\bar{e}^3}\bar{v} + \frac{1}{\bar{e}^3(1-\bar{e}^2)^{5/2}}\left[-\left(1-\frac{5}{2}\bar{e}^2+3\bar{e}^4\right)\bar{E} - \bar{e}(1-3\bar{e}^2)\sin\bar{E} - \frac{1}{4}\bar{e}^2\sin 2\bar{E}\right]$$

and for $\bar{e}=0$, $\bar{v}=\bar{E}$ from Eq. (C-2), and

$$\int \frac{\cos^3 \bar{v}\, d\bar{v}}{(1+\bar{e}\cos\bar{v})^3} \doteq \int \cos^3\bar{v}\, d\bar{v} \doteq \int \cos^3 \bar{E} d\bar{E} \doteq \frac{1}{4}\int (3\cos\bar{E} + \cos 3\bar{E})\, d\bar{E} \doteq \frac{1}{4}\left(3\sin\bar{E} + \frac{1}{3}\sin 3\bar{E}\right)$$

so that

$$\int \frac{\cos^3 \bar{v}\, d\bar{v}}{(1+\bar{e}\cos\bar{v})^3} \doteq \begin{cases} \left\{\frac{1}{\bar{e}^3}\bar{v} - \frac{1}{\bar{e}^3(1-\bar{e}^2)^{5/2}}\left[\left(1-\frac{5}{2}\bar{e}^2+3\bar{e}^4\right)\bar{E} + \bar{e}(1-3\bar{e}^2)\sin\bar{E} + \frac{1}{4}\bar{e}^2\sin 2\bar{E}\right]\right\} & \text{for } 0<\bar{e}<1 \\ \frac{1}{12}(9\sin\bar{E} + \sin 3\bar{E}) \qquad \text{for } \bar{e}=0 & \end{cases} \tag{C-13}$$

Consider the integral in (C-1),

$$\int \frac{\cos^3 \bar{v}\, d\bar{v}}{(1-\bar{e}\cos\bar{v})^4}$$

From Eqs. (C-4), (C-5), and (C-6),

$$\int \frac{\cos^3 \bar{v}\, d\bar{v}}{(1+\bar{e}\cos\bar{v})^4} \doteq \int \frac{(\cos\bar{E}-\bar{e})^3}{(1-\bar{e}^2)^3}\,\frac{1-\bar{e}\cos\bar{E}}{1-\bar{e}^2}\,\frac{(1-\bar{e}^2)^{1/2}}{1-\bar{e}\cos\bar{E}}\, d\bar{E} = \frac{1}{(1-\bar{e}^2)^{7/2}}\int (\cos\bar{E}-\bar{e})^3\, d\bar{E}$$

$$\doteq \frac{1}{(1-\bar{e}^2)^{7/2}}\int\left[-\bar{e}\left(\frac{3}{2}+\bar{e}^2\right) + 3\left(\frac{1}{4}+\bar{e}^2\right)\cos\bar{E} - \frac{3}{2}\bar{e}\cos 2\bar{E} + \frac{1}{4}\cos 3\bar{E}\right] d\bar{E}$$

or

$$\int \frac{\cos^3 \bar{v}\, d\bar{v}}{(1+\bar{e}\cos\bar{v})^4} \doteq -\frac{1}{(1-\bar{e}^2)^{7/2}}\left[\frac{1}{2}\bar{e}(3+2\bar{e}^2)\bar{E} - \frac{3}{4}(1+4\bar{e}^2)\sin\bar{E} + \frac{3}{4}\bar{e}\sin 2\bar{E} - \frac{1}{12}\sin 3\bar{E}\right] \tag{C-14}$$

Consider the integral in (C-1),

$$\int \frac{\cos^4 \bar{v}\, d\bar{v}}{(1+\bar{e}\cos\bar{v})^4}$$

From Eqs. (C-4), (C-5), and (C-6), for $0<\bar{e}<1$,

$$\int \frac{\cos^4 \bar{v}\, d\bar{v}}{(1+\bar{e}\cos\bar{v})^4} \doteq \int \frac{(\cos\bar{E}-\bar{e})^4}{(1-\bar{e}^2)^4}\,\frac{(1-\bar{e}^2)^{1/2}}{1-\bar{e}\cos\bar{E}}\, d\bar{E} \doteq \frac{1}{(1-\bar{e}^2)^{7/2}}\int \frac{\frac{1}{\bar{e}^4}[(1-\bar{e}^2)-(1-\bar{e}\cos\bar{E})]^4}{1-\bar{e}\cos\bar{E}}\, d\bar{E}$$

$$\doteq \frac{1}{\bar{e}^4(1-\bar{e}^2)^{7/2}}\left[\int \frac{(1-\bar{e}^2)^4 - 4(1-\bar{e}^2)^3(1-\bar{e}\cos\bar{E})}{1-\bar{e}\cos\bar{E}}\, d\bar{E}\right.$$

$$\left. + \int \frac{6(1-\bar{e}^2)^2(1-\bar{e}\cos\bar{E})^2 - 4(1-\bar{e}^2)(1-\bar{e}\cos\bar{E})^3}{1-\bar{e}\cos\bar{E}}\, d\bar{E} + \int \frac{(1-\bar{e}\cos\bar{E})^4}{1-\bar{e}\cos\bar{E}}\, d\bar{E}\right]$$

$$\doteq \frac{1}{\bar{e}^4(1-\bar{e}^2)^{7/2}}\left[(1-\bar{e}^2)^4\int \frac{d\bar{E}}{1-\bar{e}\cos\bar{E}} - 4(1-\bar{e}^2)^3\int d\bar{E} + 6(1-\bar{e}^2)^2\int (1-\bar{e}\cos\bar{E})\, d\bar{E}\right.$$

$$- 4(1-\bar{e}^2)\int(1-\bar{e}\cos\bar{E})^2\,d\bar{E} + \int(1-\bar{e}\cos\bar{E})^3\,d\bar{E}\Bigg]$$

$$\doteq \frac{1}{\bar{e}^4(1-\bar{e}^2)^{7/2}}\Bigg\{(1-\bar{e}^2)^{7/2}\bar{v} - 4(1-\bar{e}^2)^3\bar{E}$$

$$+ 6(1-\bar{e}^2)^2(\bar{E}-\bar{e}\sin\bar{E}) - 4(1-\bar{e}^2)\left[\left(1+\frac{1}{2}\bar{e}^2\right)\bar{E} - 2\bar{e}\sin\bar{E} + \frac{1}{4}\bar{e}^2\sin 2\bar{E}\right]$$

$$+\left(1+\frac{3}{2}\bar{e}^2\right)\bar{E} - 3\bar{e}\left(1+\frac{1}{4}\bar{e}^2\right)\sin\bar{E} + \frac{3}{4}\bar{e}^2\sin 2\bar{E} - \frac{1}{12}\bar{e}^3\sin 3\bar{E}\Bigg\}$$

or

$$\int\frac{\cos^4\bar{v}\,d\bar{v}}{(1+\bar{e}\cos\bar{v})^4} \doteq \frac{1}{\bar{e}^4}\bar{v} + \frac{1}{\bar{e}^4(1-\bar{e}^2)^{7/2}}\Bigg[-\left(1-\frac{7}{2}\bar{e}^2+4\bar{e}^4-4\bar{e}^6\right)\bar{E} - \bar{e}\left(1-\frac{13}{4}\bar{e}^2+6\bar{e}^4\right)\sin\bar{E} - \frac{1}{4}\bar{e}^2(1-4\bar{e}^2)\sin 2\bar{E}$$

$$- \frac{1}{12}\bar{e}^3\sin 3\bar{E}\Bigg]$$

and for $\bar{e}=0$, $\bar{v}=\bar{E}$ from Eq. (C-2), and

$$\int\frac{\cos^4\bar{v}\,d\bar{v}}{(1+\bar{e}\cos\bar{v})^4} \doteq \int\cos^4\bar{v}\,d\bar{v} = \int\cos^4\bar{E}\,d\bar{E} \doteq \frac{1}{8}\int(\cos 4\bar{E} + 4\cos 2\bar{E} + 3)\,d\bar{E} \doteq \frac{1}{8}\left(\frac{1}{4}\sin 4\bar{E} + 2\sin 2\bar{E} + 3\bar{E}\right)$$

so that

$$\int\frac{\cos^4\bar{v}\,d\bar{v}}{(1+\bar{e}\cos\bar{v})^4} \doteq \left\{\begin{array}{ll} \Bigg\{\frac{1}{\bar{e}^4}\bar{v} - \frac{1}{\bar{e}^4(1-\bar{e}^2)^{7/2}}\Bigg[\left(1-\frac{7}{2}\bar{e}^2+4\bar{e}^4-4\bar{e}^6\right)\bar{E} & \\ \quad + \bar{e}\left(1-\frac{13}{4}\bar{e}^2+6\bar{e}^4\right)\sin\bar{E} + \frac{1}{4}\bar{e}^2(1-4\bar{e}^2)\sin 2\bar{E} + \frac{1}{12}\bar{e}^3\sin 3\bar{E}\Bigg]\Bigg\} & \text{for } 0<\bar{e}<1 \\ \frac{1}{32}(12\bar{E} + 8\sin 2\bar{E} + \sin 4\bar{E}) \qquad \text{for } \bar{e}=0 & \end{array}\right\} \tag{C-15}$$

Consider the integral in (C-1),

$$\int\frac{\sin\bar{v}\,d\bar{v}}{(1+\bar{e}\cos\bar{v})^3}$$

From Eqs. (C-3), (C-5), and (C-6),

$$\int\frac{\sin\bar{v}\,d\bar{v}}{(1+\bar{e}\cos\bar{v})^3} \doteq \int\frac{\sin\bar{E}}{(1-\bar{e}^2)^{1/2}}\,\frac{(1-\bar{e}\cos\bar{E})^2}{(1-\bar{e}^2)^2}\,\frac{(1-\bar{e}^2)^{1/2}}{1-\bar{e}\cos\bar{E}}\,d\bar{E}$$

$$\doteq \frac{1}{(1-\bar{e}^2)^2}\int\left(\sin\bar{E} - \frac{1}{2}\bar{e}\sin 2\bar{E}\right)d\bar{E}$$

or

$$\int\frac{\sin\bar{v}\,d\bar{v}}{(1+\bar{e}\cos\bar{v})^3} \doteq \frac{1}{(1-\bar{e}^2)^2}\left(-\cos\bar{E} + \frac{1}{4}\bar{e}\cos 2\bar{E}\right) \tag{C-16}$$

Consider the integral in (C-1),

$$\int \frac{\sin \bar{v}\, d\bar{v}}{(1+\bar{e}\cos\bar{v})^4}$$

From Eqs. (C-3), (C-5), and (C-6),

$$\int \frac{\sin \bar{v}\, d\bar{v}}{(1+\bar{e}\cos\bar{v})^4} = \int \frac{\sin \bar{E}}{(1-\bar{e}^2)^{1/2}} \frac{(1-\bar{e}\cos\bar{E})^3}{(1-\bar{e}^2)^3} \frac{(1-\bar{e}^2)^{1/2}}{1-\bar{e}\cos\bar{E}} d\bar{E}$$

$$= \frac{1}{(1-\bar{e}^2)^3} \int \left[\left(1+\frac{1}{4}\bar{e}^2\right)\sin\bar{E} - \bar{e}\sin 2\bar{E} + \frac{1}{4}\bar{e}^2 \sin 3\bar{E}\right] d\bar{E}$$

or

$$\int \frac{\sin \bar{v}\, d\bar{v}}{(1+\bar{e}\cos\bar{v})^4} = \frac{1}{(1-\bar{e}^2)^3}\left[-\left(1+\frac{1}{4}\bar{e}^2\right)\cos\bar{E} + \frac{1}{2}\bar{e}\cos 2\bar{E} - \frac{1}{12}\bar{e}^2 \cos{}^3\bar{E}\right] \tag{C-17}$$

Consider the integral in (C-1),

$$\int \frac{\sin^2 \bar{v}\, d\bar{v}}{(1+\bar{e}\cos\bar{v})^3}$$

From Eqs. (C-3), (C-5), and (C-6),

$$\int \frac{\sin^2 \bar{v}\, d\bar{v}}{(1+\bar{e}\cos\bar{v})^3} = \int \frac{\sin^2 \bar{E}}{(1-\bar{e}^2)} \frac{(1-\bar{e}\cos\bar{E})}{1-\bar{e}^2} \frac{(1-\bar{e}^2)^{1/2}}{1-\bar{e}\cos\bar{E}} d\bar{E}$$

$$= \frac{1}{(1-\bar{e}^2)^{3/2}} \int \sin^2\bar{E}\, d\bar{E}$$

$$= \frac{1}{2(1-\bar{e}^2)^{3/2}} \int (1-\cos 2\bar{E})\, d\bar{E}$$

or

$$\int \frac{\sin^2 \bar{v}\, d\bar{v}}{(1+\bar{e}\cos\bar{v})^3} = \frac{1}{2(1-\bar{e}^2)^{3/2}}\left(\bar{E} - \frac{1}{2}\sin 2\bar{E}\right) \tag{C-18}$$

Consider the integral in (C-1),

$$\int \frac{\sin^2 \bar{v}\, d\bar{v}}{(1+\bar{e}\cos\bar{v})^4}$$

From Eqs. (C-3), (C-5), and (C-6),

$$\int \frac{\sin^2 \bar{v}\, d\bar{v}}{(1+\bar{e}\cos\bar{v})^4} = \int \frac{\sin^2 \bar{E}}{(1-\bar{e}^2)} \frac{(1-\bar{e}\cos\bar{E})^2}{(1-\bar{e}^2)^2} \frac{(1-\bar{e}^2)^{1/2}}{1-\bar{e}\cos\bar{E}} d\bar{E}$$

$$= \frac{1}{(1-\bar{e}^2)^{5/2}} \int (\sin^2\bar{E})(1-\bar{e}\cos\bar{E})\, d\bar{E}$$

$$= \frac{1}{2(1-\bar{e}^2)^{5/2}} \int \left(1 - \frac{1}{2}\bar{e}\cos\bar{E} - \cos 2\bar{E} + \frac{1}{2}\bar{e}\cos 3\bar{E}\right) d\bar{E}$$

or

$$\int \frac{\sin^2 \bar{v}\, d\bar{v}}{(1+\bar{e}\cos\bar{v})^4} = \frac{1}{2(1-\bar{e}^2)^{5/2}}\left(\bar{E} - \frac{1}{2}\bar{e}\sin\bar{E} - \frac{1}{2}\sin 2\bar{E} + \frac{1}{6}\bar{e}\sin 3\bar{E}\right) \tag{C-19}$$

Consider the integral in (C-1),

$$\int \frac{\sin^3 \bar{v}\, d\bar{v}}{(1 + \bar{e}\cos \bar{v})^4}$$

From Eqs. (C-3), (C-5), and (C-6),

$$\int \frac{\sin^3 \bar{v}\, d\bar{v}}{(1 + \bar{e}\cos \bar{v})^4} \doteq \int \frac{\sin^3 \bar{E}}{(1 - \bar{e}^2)^{3/2}} \frac{1 - \bar{e}\cos \bar{E}}{1 - \bar{e}^2} \frac{(1 - \bar{e}^2)^{1/2}}{1 - \bar{e}\cos \bar{E}}\, d\bar{E}$$

$$\doteq \frac{1}{(1 - \bar{e}^2)^2} \int \sin^3 \bar{E}\, d\bar{E}$$

$$\doteq \frac{1}{4(1 - \bar{e}^2)^2} \int (3 \sin \bar{E} - \sin 3\bar{E})\, d\bar{E}$$

or

$$\int \frac{\sin^3 \bar{v}\, d\bar{v}}{(1 + \bar{e}\cos \bar{v})^4} \doteq \frac{1}{4(1 - \bar{e}^2)^2} \left(-3 \cos \bar{E} + \frac{1}{3} \cos 3\bar{E} \right) \tag{C-20}$$

Consider the integral in (C-1),

$$\int \frac{\sin \bar{v} \cos \bar{v}\, d\bar{v}}{(1 + \bar{e}\cos \bar{v})^3}$$

From Eqs. (C-3–6),

$$\int \frac{\sin \bar{v} \cos \bar{v}\, d\bar{v}}{(1 + \bar{e}\cos \bar{v})^3} \doteq \int \frac{\sin \bar{E}}{(1 - \bar{e}^2)^{1/2}} \frac{\cos \bar{E} - \bar{e}}{1 - \bar{e}^2} \frac{1 - \bar{e}\cos \bar{E}}{1 - \bar{e}^2} \frac{(1 - \bar{e}^2)^{1/2}}{1 - \bar{e}\cos \bar{E}}\, d\bar{E} \doteq \frac{1}{(1 - \bar{e}^2)^2} \int \left(\frac{1}{2} \sin 2\bar{E} - \bar{e} \sin \bar{E} \right) d\bar{E}$$

or

$$\int \frac{\sin \bar{v} \cos \bar{v}\, d\bar{v}}{(1 + \bar{e}\cos \bar{v})^3} \doteq \frac{1}{(1 - \bar{e}^2)^2} \left(\bar{e} \cos \bar{E} - \frac{1}{4} \cos 2\bar{E} \right) \tag{C-21}$$

Consider the integral in (C-1),

$$\int \frac{\sin \bar{v} \cos \bar{v}\, d\bar{v}}{(1 + \bar{e}\cos \bar{v})^4}$$

From Eqs. (C-3–6),

$$\int \frac{\sin \bar{v} \cos \bar{v}\, d\bar{v}}{(1 + \bar{e}\cos \bar{v})^4} \doteq \int \frac{\sin \bar{E}}{(1 - \bar{e}^2)^{1/2}} \frac{\cos \bar{E} - \bar{e}}{1 - \bar{e}^2} \frac{(1 - \bar{e}\cos \bar{E})^2}{(1 - \bar{e}^2)^2} \frac{(1 - \bar{e}^2)^{1/2}}{1 - \bar{e}\cos \bar{E}}\, d\bar{E} \doteq \frac{1}{(1 - \bar{e}^2)^3} \int (\sin \bar{E})(\cos \bar{E} - \bar{e})(1 - \bar{e}\cos \bar{E})\, d\bar{E}$$

$$\doteq \frac{1}{(1 - \bar{e}^2)^3} \int \left[-\frac{5}{4} \bar{e} \sin \bar{E} + \frac{1}{2}(1 + \bar{e}^2) \sin 2\bar{E} - \frac{1}{4} \bar{e} \sin 3\bar{E} \right] d\bar{E}$$

or

$$\int \frac{\sin \bar{v} \cos \bar{v}\, d\bar{v}}{(1 + \bar{e}\cos \bar{v})^4} \doteq \frac{1}{(1 - \bar{e}^2)^3} \left[\frac{5}{4} \bar{e} \cos \bar{E} - \frac{1}{4}(1 + \bar{e}^2) \cos 2\bar{E} + \frac{1}{12} \bar{e} \cos 3\bar{E} \right] \tag{C-22}$$

Finally, consider the integral in (C-1),

$$\int \frac{\bar{v} \sin \bar{v}\, d\bar{v}}{(1 + \bar{e}\cos \bar{v})^3}$$

Since

$$\frac{d}{d\overline{v}} \frac{\overline{v}}{(1+\overline{e}\cos\overline{v})^2} = \frac{1}{(1+\overline{e}\cos\overline{v})^2} + \frac{2\overline{e}\,\overline{v}\sin\overline{v}}{(1+\overline{e}\cos\overline{v})^3}$$

then

$$\frac{\overline{v}\sin\overline{v}}{(1+\overline{e}\cos\overline{v})^3} = \frac{1}{2\overline{e}} \frac{d}{d\overline{v}} \frac{\overline{v}}{(1+\overline{e}\cos\overline{v})^2} - \frac{1}{2\overline{e}} \frac{1}{(1+\overline{e}\cos\overline{v})^2}$$

for $0 < \overline{e} < 1$ and

$$\int \frac{\overline{v}\sin\overline{v}\,d\overline{v}}{(1+\overline{e}\cos\overline{v})^3} \doteq \frac{1}{2\overline{e}} \frac{\overline{v}}{(1+\overline{e}\cos\overline{v})^2} - \frac{1}{2\overline{e}} \int \frac{d\overline{v}}{(1+\overline{e}\cos\overline{v})^2}$$

or, with the use of Eqs. (C-5) and (C-8),

$$\int \frac{\overline{v}\sin\overline{v}\,d\overline{v}}{(1+\overline{e}\cos\overline{v})^3} \doteq \frac{\overline{v}}{2\overline{e}} \frac{(1-\overline{e}\cos\overline{E})^2}{(1-\overline{e}^2)^2} - \frac{1}{2\overline{e}} \frac{(\overline{E}-\overline{e}\sin\overline{E})}{(1-\overline{e}^2)^{3/2}}$$

$$\doteq \frac{1}{2\overline{e}(1-\overline{e}^2)^2}\,\overline{v}(1-\overline{e}\cos\overline{E})^2 - \frac{1}{2\overline{e}(1-\overline{e}^2)^{3/2}}(\overline{E}-\overline{e}\sin\overline{E})$$

and for $\overline{e} = 0$, $\overline{v} = \overline{E}$ from Eq. (C-2), and

$$\int \frac{\overline{v}\sin\overline{v}\,d\overline{v}}{(1+\overline{e}\cos\overline{v})^3} \doteq \int \overline{E}\sin\overline{E}\,d\overline{E} \doteq \sin\overline{E} - \overline{E}\cos\overline{E}$$

so that

$$\int \frac{\overline{v}\sin\overline{v}\,d\overline{v}}{(1+\overline{e}\cos\overline{v})^3} \doteq \left\{ \begin{array}{ll} \left[\dfrac{1}{2\overline{e}(1-\overline{e}^2)^2}\,\overline{v}(1-\overline{e}\cos\overline{E})^2 - \dfrac{1}{2\overline{e}(1-\overline{e}^2)^{3/2}}(\overline{E}-\overline{e}\sin\overline{E})\right] & \text{for } 0<\overline{e}<1 \\ \sin\overline{E} - \overline{E}\cos\overline{E} \qquad \text{for } \overline{e} = 0 & \end{array} \right\} \qquad \text{(C-23)}$$

Appendix D

Determination of the Function $d\widetilde{T}^{(0)}/d\widetilde{\alpha}$

Consider the determination of the function $d\widetilde{T}^{(0)}/d\widetilde{\alpha}$ that arises in Eq. (482) in Section V-C-3. The limit of the first term in Eq. (482) is zero since the numerator is simply ϵ times a bounded function and the denominator is greater than zero as long as $\alpha > 0$. Thus, dividing the numerator and the denominator of the remaining term in Eq. (482) by $\overline{E}(\overline{\alpha},\widetilde{\alpha})$ where $\overline{E}(\overline{\alpha},\widetilde{\alpha}) > 0$ for $\alpha > 0$, one obtains

$$\lim_{\epsilon\to 0}\left[\frac{\epsilon\tau_1}{1-\dfrac{e_0\sin\overline{E}}{\overline{E}}+\dfrac{\widetilde{T}^{(0)}}{\overline{E}}}-\epsilon\frac{\dfrac{\dfrac{d\widetilde{T}^{(0)}}{d\widetilde{\alpha}}\overline{v}-\widetilde{T}^{(1)}}{\overline{E}}}{1-\dfrac{e_0\sin\overline{E}}{\overline{E}}+\dfrac{\widetilde{T}^{(0)}}{\overline{E}}}\right]=0 \tag{D-1}$$

The limit of the first term in Eq. (D-1) is zero since the numerator is just ϵ times a constant and the denominator is greater than zero as long as $\alpha > 0$. Now, $\overline{E}(\overline{\alpha},\widetilde{\alpha}) > e_0\sin\overline{E}(\overline{\alpha},\widetilde{\alpha})$ for all $\alpha > 0$. Also, $\overline{E}(\overline{\alpha},\widetilde{\alpha}) > \widetilde{T}^{(0)}(\widetilde{\alpha})$ for all $\alpha > 0$ since $\overline{E}(\overline{\alpha},\widetilde{\alpha})$ is an unbounded function of $(\overline{\alpha}-\omega_0-\widetilde{\alpha})$, which is a fast-varying argument because of the $\overline{\alpha}$, and the worst $\widetilde{T}^{(0)}(\widetilde{\alpha})$ can be is an unbounded function of the slow variable $\widetilde{\alpha}$. Hence, the denominator of the second term in Eq. (D-1) is greater than zero but finite for all $\alpha > 0$, so that

$$\lim_{\epsilon\to 0}\epsilon\frac{\dfrac{d\widetilde{T}^{(0)}}{d\widetilde{\alpha}}\overline{v}-\widetilde{T}^{(1)}}{\overline{E}}=0 \tag{D-2}$$

Now, from Ref. 205, pp. 62 and 63,

$$\overline{v}=\overline{E}+2\sum_{j=1}^{\infty}\frac{A^j}{j}\sin j\overline{E} \tag{D-3}$$

where

$$A=\frac{1}{e_0}[1-(1-e_0^2)^{1/2}] \qquad \text{or} \qquad e_0=\frac{2A}{1+A^2} \tag{D-4}$$

It should be noted that $\overline{v}$ and $\overline{E}$ are the instantaneous multivariable true anomaly and eccentric anomaly, respectively, whereas Ref. 205 develops the Fourier series expansion of the true anomaly v in terms of the eccentric anomaly E. However, the relationships between $\overline{v}$ and $\overline{E}$ are the same as those between v and E (see Eq. 458 in Section V-C-3) so that Eq. (D-3) follows from its counterpart in Ref. 205.

Introducing Eq. (D-3) into Eq. (D-2) yields

$$\lim_{\epsilon\to 0}\left(\epsilon\frac{d\widetilde{T}^{(0)}}{d\widetilde{\alpha}}+\frac{2\epsilon\dfrac{d\widetilde{T}^{(0)}}{d\widetilde{\alpha}}\displaystyle\sum_{j=1}^{\infty}\frac{A^j}{j}\sin j\overline{E}}{\overline{E}}-\frac{\epsilon\widetilde{T}^{(1)}}{\overline{E}}\right)=0 \tag{D-5}$$

The only way the limit of the first term can be zero for all α is if

$$\frac{d\widetilde{T}^{(0)}}{d\widetilde{\alpha}}=\text{a constant}\stackrel{\Delta}{=}0 \tag{D-6}$$

The constant in Eq. (D-6) can be defined as zero because there are no other conditions on $d\widetilde{T}^{(0)}/d\widetilde{\alpha}$; that is, the constant is arbitrary. With the use of Eq. (D-6), the limit of the second term in Eq. (D-5) is clearly zero since $\overline{E}(\overline{\alpha}, \widetilde{\alpha}) > 0$ for $\alpha > 0$. Thus, Eq. (D-5) becomes

$$\lim_{\epsilon \to 0} \frac{\epsilon \widetilde{T}^{(1)}}{\overline{E}} = 0 \tag{D-7}$$

Although $\widetilde{T}^{(1)}(\widetilde{\alpha})$ must satisfy Eq. (D-7), this equation does not sufficiently determine $\widetilde{T}^{(1)}(\widetilde{\alpha})$. Nevertheless, it appears that $\widetilde{T}^{(1)}(\widetilde{\alpha})$ must be either a bounded function of $\widetilde{\alpha}$ or, if unbounded, at most a linear function of α since the unbounded part of $\overline{E}(\overline{\alpha}, \widetilde{\alpha})$ is a linear function of α (remember $\widetilde{\alpha} = \epsilon\alpha$ and $\overline{\alpha} = (1 + \epsilon^2 \alpha_2 + \cdot \cdot \cdot)\alpha$).

Appendix E

Derivation of the Perturbative Variations

Consider the perturbative variations of the parameters a_N and a_M of Section V-D. It follows from $a_N = e\cos\omega$ and $a_M = e\sin\omega$ that

$$a_N^{\backprime} = e^{\backprime}\cos\omega - e\omega^{\backprime}\sin\omega \tag{E-1}$$

and

$$a_M^{\backprime} = e^{\backprime}\sin\omega + e\omega^{\backprime}\cos\omega \tag{E-2}$$

It follows from Figs. 17 and 20 and the definition

$$\Omega \triangleq 0 \tag{E-3}$$

that

$$i \equiv 0 \tag{E-4}$$

and

$$\alpha = v + \omega \tag{E-5}$$

Consequently, from Eq. (E-5) and the concepts in Section III-A-2,

$$\dot{\alpha} = \dot{v} \tag{E-6}$$

since $\dot{\omega} = 0$. Taking the perturbative derivative (Section III-A-2) of Eq. (E-6) yields

$$\dot{\alpha}^{\backprime} = \dot{v}^{\backprime} \tag{E-7}$$

Furthermore, since the right ascension α is referred to a fixed reference direction,

$$\alpha^{\backprime} = 0 \tag{E-8}$$

so that the perturbative derivative of Eq. (E-5) yields

$$v^{\backprime} = -\omega^{\backprime} \tag{E-9}$$

From the definition of the total derivative in Section III-A-2,

$$\frac{d\alpha}{dt} = \dot{\alpha} + \alpha^{\backprime}$$

or, using Eqs. (E-6) and (E-8),

$$\left.\begin{aligned} \frac{d\alpha}{dt} &= \dot{v} \\ \text{and}\qquad \frac{d^2\alpha}{dt^2} &= \ddot{v} + \dot{v}^{\backprime} \end{aligned}\right\} \tag{E-10}$$

Finally, since $r^{\backprime} = 0$,

$$\left.\begin{aligned} \frac{dr}{dt} &= \dot{r} \\ \text{and}\qquad \frac{d^2r}{dt^2} &= \ddot{r} + \dot{r}^{\backprime} \end{aligned}\right\} \tag{E-11}$$

From Eqs. (19) and (21) in Section III-A-3 and from Eqs. (E-3) and (E-9),

$$e^{\backprime} = \frac{r\dot{r}^{\backprime}}{(p)^{1/2}}\left(\frac{p}{r}\sin v\right) + \frac{r^2\dot{v}^{\backprime}}{(p)^{1/2}}\left[\left(\frac{p}{r} + 1\right)\cos v + e\right] \tag{E-12}$$

and

$$\omega^{\backprime} = -v^{\backprime} = -\frac{r\dot{r}^{\backprime}}{e\,(p)^{1/2}}\left(\frac{p}{r}\cos v\right) + \frac{r^2\dot{v}^{\backprime}}{e\,(p)^{1/2}}\left(\frac{p}{r} + 1\right)\sin v \tag{E-13}$$

where the coordinates and parameters are normalized as in Section V-A; that is, the normalization constants are given by Eqs. (259) in Section V-A. In this manner, the units of the variables in Eqs. (E-12) and (E-13) are given by Eqs. (270) in Section V-A.

It follows from Eqs. (265) and (266) in Section V-A and from Eqs. (E-10) and (E-11) that

$$\left.\begin{aligned} \dot{r}^{\backprime} &= -\epsilon\frac{p^2}{r^4} \\ \text{and}\qquad r\dot{v}^{\backprime} &= 0 \end{aligned}\right\} \tag{E-14}$$

where

$$\ddot{r} - r\dot{v}^2 = -\frac{1}{r^2} \qquad \text{and} \qquad r\ddot{v} + 2\dot{r}\dot{v} = 0 \tag{E-15}$$

and, using Eq. (272) in Section V-B-2 and Eqs. (E-10),

$$p = h^2 = r^4 \left(\frac{d\alpha}{dt}\right)^2 = r^4 \dot{v}^2 \tag{E-16}$$

Introducing Eqs. (E-14) into Eqs. (E-12) and (E-13) yields

$$e^{\backprime} = -\frac{\epsilon}{p^{3/2}} \left(\frac{p}{r}\right)^3 \left(\frac{p}{r} \sin v\right) \qquad \text{and} \qquad e\omega^{\backprime} = \frac{\epsilon}{p^{3/2}} \left(\frac{p}{r}\right)^3 \left(\frac{p}{r} \cos v\right) \tag{E-17}$$

Introducing Eqs. (E-17) into Eqs. (E-1) and (E-2), using trigonometric formulas, and using Eqs. (E-5), one obtains

$$a_N^{\backprime} = -\frac{\epsilon}{p^{3/2}} \left(\frac{p}{r}\right)^3 \left(\frac{p}{r} \sin \alpha\right) \qquad \text{and} \qquad a_M^{\backprime} = \frac{\epsilon}{p^{3/2}} \left(\frac{p}{r}\right)^3 \left(\frac{p}{r} \cos v\right) \tag{E-18}$$

Appendix F
Development of the Secular and Periodic Total Derivatives in Terms of the Reference Parameters

In order to obtain the required secular, short-period, and long-period terms in Eqs. (556), Section V-D-3, the derivatives $da_N/d\alpha$, $da_M/d\alpha$, and $dt/d\alpha$ in Eqs. (547), (548), and (549), respectively (Section V-D-2), must be expanded in Taylor's series (assumed to be uniformly convergent in the range $0 \leqq \alpha < \infty$) about the point represented by $\bar{a}_N(\alpha;\epsilon)$, $\bar{a}_M(\alpha;\epsilon)$, and $\bar{t}(\alpha;\epsilon)$ at right ascension α. Since these derivatives are explicit functions of only a_N and a_M (ϵ and α are assumed to be fixed) as shown in Eqs. (547), (548), and (549),

$$\left.\begin{aligned}\frac{da_N}{d\alpha} &= \overline{\frac{da_N}{d\alpha}} + \overline{\frac{\partial}{\partial a_N}\left(\frac{da_N}{d\alpha}\right)}\Delta a_N + \overline{\frac{\partial}{\partial a_M}\left(\frac{da_N}{d\alpha}\right)}\Delta a_M + \frac{1}{2}\overline{\frac{\partial^2}{\partial a_N^2}\left(\frac{da_N}{d\alpha}\right)}(\Delta a_N)^2 + \frac{1}{2}\overline{\frac{\partial^2}{\partial a_M^2}\left(\frac{da_N}{d\alpha}\right)}(\Delta a_M)^2 \\ &\quad + \overline{\frac{\partial^2}{\partial a_N\,\partial a_M}\left(\frac{da_N}{d\alpha}\right)}\Delta a_N\,\Delta a_M + \cdots \\ \text{and}\quad & \frac{da_N}{d\alpha} \to \frac{da_M}{d\alpha} \to \frac{dt}{d\alpha}\end{aligned}\right\} \quad \text{(F-1)}$$

where the bar over the partial derivative implies that the partial derivative is evaluated using $\bar{a}_N$ and $\bar{a}_M$ in place of a_N and a_M, respectively. From Eqs. (554) in Section V-D-3,

$$\left.\begin{aligned}\Delta a_N &= a_N - \bar{a}_N = a_N^{(1)} + a_N^{(2)} + O(\epsilon^3) \\ \text{and}\quad & a_N \to a_M\end{aligned}\right\} \quad \text{(F-2)}$$

Taking the partial derivatives of Eqs. (547), (548), and (549) in Section V-D-2 with respect to a_N and a_M and evaluating them using $\bar{a}_N$ and $\bar{a}_M$ yields

$$\left.\begin{aligned}\overline{\frac{\partial}{\partial a_N}\left(\frac{da_N}{d\alpha}\right)} &= -\frac{1}{2}\epsilon(\bar{a}_M\cos\alpha + \bar{a}_N\sin\alpha + 2\sin 2\alpha - \bar{a}_M\cos 3\alpha + \bar{a}_N\sin 3\alpha) \\ \overline{\frac{\partial}{\partial a_M}\left(\frac{da_N}{d\alpha}\right)} &= -\frac{1}{2}\epsilon(2 + \bar{a}_N\cos\alpha + 3\bar{a}_M\sin\alpha - 2\cos 2\alpha - \bar{a}_N\cos 3\alpha - \bar{a}_M\sin 3\alpha) \\ \overline{\frac{\partial^2}{\partial a_N^2}\left(\frac{da_N}{d\alpha}\right)} &= \overline{\frac{\partial^2}{\partial a_M^2}\left(\frac{da_N}{d\alpha}\right)} = \overline{\frac{\partial^2}{\partial a_N\,\partial a_M}\left(\frac{da_N}{d\alpha}\right)} = O(\epsilon)\end{aligned}\right\} \quad \text{(F-3)}$$

$$\left.\begin{aligned}\overline{\frac{\partial}{\partial a_N}\left(\frac{da_M}{d\alpha}\right)} &= \frac{1}{2}\epsilon(2 + 3\bar{a}_N\cos\alpha + \bar{a}_M\sin\alpha + 2\cos 2\alpha + \bar{a}_N\cos 3\alpha + \bar{a}_M\sin 3\alpha) \\ \overline{\frac{\partial}{\partial a_M}\left(\frac{da_M}{d\alpha}\right)} &= \frac{1}{2}\epsilon(\bar{a}_M\cos\alpha + \bar{a}_N\sin\alpha + 2\sin 2\alpha - \bar{a}_M\cos 3\alpha + \bar{a}_N\sin 3\alpha) \\ \overline{\frac{\partial^2}{\partial a_N^2}\left(\frac{da_M}{d\alpha}\right)} &= \overline{\frac{\partial^2}{\partial a_M^2}\left(\frac{da_M}{d\alpha}\right)} = \overline{\frac{\partial^2}{\partial a_N\,\partial a_M}\left(\frac{da_M}{d\alpha}\right)} = O(\epsilon)\end{aligned}\right\} \quad \text{(F-4)}$$

$$\left.\begin{aligned}\overline{\frac{\partial}{\partial a_N}\left(\frac{dt}{d\alpha}\right)} &= -\frac{2p^{3/2}\cos\alpha}{(1 + \bar{a}_N\cos\alpha + \bar{a}_M\sin\alpha)^3} \\ \overline{\frac{\partial}{\partial a_M}\left(\frac{dt}{d\alpha}\right)} &= -\frac{2p^{3/2}\sin\alpha}{(1 + \bar{a}_N\cos\alpha + \bar{a}_M\sin\alpha)^3}\end{aligned}\right\} \quad \text{(F-5)}$$

$$
\left.
\begin{aligned}
\overline{\frac{\partial^2}{\partial a_N^2}\left(\frac{dt}{d\alpha}\right)} &= \frac{3p^{3/2}(1+\cos 2\alpha)}{(1+\bar{a}_N\cos\alpha+\bar{a}_M\sin\alpha)^4} \\
\overline{\frac{\partial^2}{\partial a_M^2}\left(\frac{dt}{d\alpha}\right)} &= \frac{3p^{3/2}(1-\cos 2\alpha)}{(1+\bar{a}_N\cos\alpha+\bar{a}_M\sin\alpha)^4} \\
\overline{\frac{\partial^2}{\partial a_N\,\partial a_M}\left(\frac{dt}{d\alpha}\right)} &= \frac{3p^{3/2}\sin 2\alpha}{(1+\bar{a}_N\cos\alpha+\bar{a}_M\sin\alpha)^4}
\end{aligned}
\right\}
\qquad \text{(F-6)}
$$

and so forth. Now, introducing Eqs. (547), (548), and (549) in Section V-D-2 evaluated using $\bar{a}_N$ and $\bar{a}_M$, and Eqs. (F-2–6) into Eq. (F-1), introducing

$$
\frac{da_N}{d\alpha} = \left(\frac{da_N}{d\alpha}\right)^{(0)} + \left(\frac{da_N}{d\alpha}\right)^{(1)} + \left(\frac{da_N}{d\alpha}\right)^{(2)} + O(\epsilon^3)
$$

and

$$
a_N \rightarrow a_M \rightarrow t
$$

and equating terms of like powers of ϵ (assuming the reference parameters are of order one) yields

$$
\left(\frac{da_N}{d\alpha}\right)^{(0)} = \left(\frac{da_N}{d\alpha}\right)_s^{(0)} + \left(\frac{da_N}{d\alpha}\right)_{sp}^{(0)} + \left(\frac{da_N}{d\alpha}\right)_{tp}^{(0)} = 0 \qquad \text{(F-7)}
$$

$$
\begin{aligned}
\left(\frac{da_N}{d\alpha}\right)^{(1)} &= \left(\frac{da_N}{d\alpha}\right)_s^{(1)} + \left(\frac{da_N}{d\alpha}\right)_{sp}^{(1)} + \left(\frac{da_N}{d\alpha}\right)_{tp}^{(1)} \\
&= -\epsilon\left[\bar{a}_M + \frac{1}{2}\bar{a}_N\bar{a}_M\cos\alpha + \left(1+\frac{1}{4}\bar{a}_N^2+\frac{3}{4}\bar{a}_M^2\right)\sin\alpha - \bar{a}_M\cos 2\alpha + \bar{a}_N\sin 2\alpha \right. \\
&\quad \left. -\frac{1}{2}\bar{a}_N\bar{a}_M\cos 3\alpha + \frac{1}{4}(\bar{a}_N^2-\bar{a}_M^2)\sin 3\alpha\right]
\end{aligned}
\qquad \text{(F-8)}
$$

$$
\begin{aligned}
\left(\frac{da_N}{d\alpha}\right)^{(2)} &= \left(\frac{da_N}{d\alpha}\right)_s^{(2)} + \left(\frac{da_N}{d\alpha}\right)_{sp}^{(2)} + \left(\frac{da_N}{d\alpha}\right)_{tp}^{(2)} \\
&= -\frac{1}{2}\epsilon\left[a_N^{(1)}(\bar{a}_M\cos\alpha + \bar{a}_N\sin\alpha + 2\sin 2\alpha - \bar{a}_M\cos 3\alpha + \bar{a}_N\sin 3\alpha)\right. \\
&\quad \left. + a_M^{(1)}(2+\bar{a}_N\cos\alpha + 3\bar{a}_M\sin\alpha - 2\cos 2\alpha - \bar{a}_N\cos 3\alpha - \bar{a}_M\sin 3\alpha)\right]
\end{aligned}
\qquad \text{(F-9)}
$$

and so forth,

$$
\left(\frac{da_M}{d\alpha}\right)^{(0)} = \left(\frac{da_M}{d\alpha}\right)_s^{(0)} + \left(\frac{da_M}{d\alpha}\right)_{sp}^{(0)} + \left(\frac{da_M}{d\alpha}\right)_{tp}^{(0)} = 0 \qquad \text{(F-10)}
$$

$$
\begin{aligned}
\left(\frac{da_M}{d\alpha}\right)^{(1)} &= \left(\frac{da_M}{d\alpha}\right)_s^{(1)} + \left(\frac{da_M}{d\alpha}\right)_{sp}^{(1)} + \left(\frac{da_M}{d\alpha}\right)_{tp}^{(1)} \\
&= \epsilon\left[\bar{a}_N + \left(1+\frac{3}{4}\bar{a}_N^2+\frac{1}{4}\bar{a}_M^2\right)\cos\alpha + \frac{1}{2}\bar{a}_N\bar{a}_M\sin\alpha + \bar{a}_N\cos 2\alpha + \bar{a}_M\sin 2\alpha\right. \\
&\quad \left. + \frac{1}{4}(\bar{a}_N^2-\bar{a}_M^2)\cos 3\alpha + \frac{1}{2}\bar{a}_N\bar{a}_M\sin 3\alpha\right]
\end{aligned}
\qquad \text{(F-11)}
$$

$$\left(\frac{da_M}{d\alpha}\right)^{(2)} = \left(\frac{da_M}{d\alpha}\right)_s^{(2)} + \left(\frac{da_M}{d\alpha}\right)_{sp}^{(2)} + \left(\frac{da_M}{d\alpha}\right)_{lp}^{(2)}$$

$$= \frac{1}{2}\,\epsilon\,[a_N^{(1)}\,(2 + 3\bar{a}_N \cos\alpha + \bar{a}_M \sin\alpha + 2\cos 2\alpha + \bar{a}_N \cos 3\alpha + \bar{a}_M \sin 3\alpha)$$

$$+ a_M^{(1)}\,(\bar{a}_M \cos\alpha + \bar{a}_N \sin\alpha + 2\sin 2\alpha - \bar{a}_M \cos 3\alpha + \bar{a}_N \sin 3\alpha)] \qquad \text{(F-12)}$$

and so forth, and

$$\left(\frac{dt}{d\alpha}\right)^{(0)} = \left(\frac{dt}{d\alpha}\right)_s^{(0)} + \left(\frac{dt}{d\alpha}\right)_{sp}^{(0)} + \left(\frac{dt}{d\alpha}\right)_{lp}^{(0)} = \frac{p^{3/2}}{(1 + \bar{a}_N \cos\alpha + \bar{a}_M \sin\alpha)^2} \qquad \text{(F-13)}$$

$$\left(\frac{dt}{d\alpha}\right)^{(1)} = \left(\frac{dt}{d\alpha}\right)_s^{(1)} + \left(\frac{dt}{d\alpha}\right)_{sp}^{(1)} + \left(\frac{dt}{d\alpha}\right)_{lp}^{(1)} = -\frac{2p^{3/2}}{(1 + \bar{a}_N \cos\alpha + \bar{a}_M \sin\alpha)^3}\,(a_N^{(1)} \cos\alpha + a_M^{(1)} \sin\alpha) \qquad \text{(F-14)}$$

$$\left(\frac{dt}{d\alpha}\right)^{(2)} = \left(\frac{dt}{d\alpha}\right)_s^{(2)} + \left(\frac{dt}{d\alpha}\right)_{sp}^{(2)} + \left(\frac{dt}{d\alpha}\right)_{lp}^{(2)}$$

$$= -\frac{2p^{3/2}}{(1 + \bar{a}_N \cos\alpha + \bar{a}_M \sin\alpha)^3}\,(a_N^{(2)} \cos\alpha + a_M^{(2)} \sin\alpha)$$

$$+ \frac{\frac{3}{2}p^{3/2}}{(1 + \bar{a}_N \cos\alpha + \bar{a}_M \sin\alpha)^4}\,\{[(a_N^{(1)})^2 + (a_M^{(1)})^2] + [(a_N^{(1)})^2 - (a_M^{(1)})^2]\cos 2\alpha + 2a_N^{(1)}a_M^{(1)} \sin 2\alpha\} \qquad \text{(F-15)}$$

and so forth.

If the functional dependence of the parameters $\bar{a}_N, \bar{a}_M, a_N^{(1)}, a_M^{(1)}, \cdots$ upon α were known, the secular, short-period, and long-period terms of various orders required in Eqs. (556), Section V-D-3, could be determined from Eqs. (F-7–15). The functional dependence of these parameters is determined in Section V-D-3.

Appendix G

Plots of Satellite Coordinates and Parameters Vs Revolution Number (Figs. G-1 Through G-140)

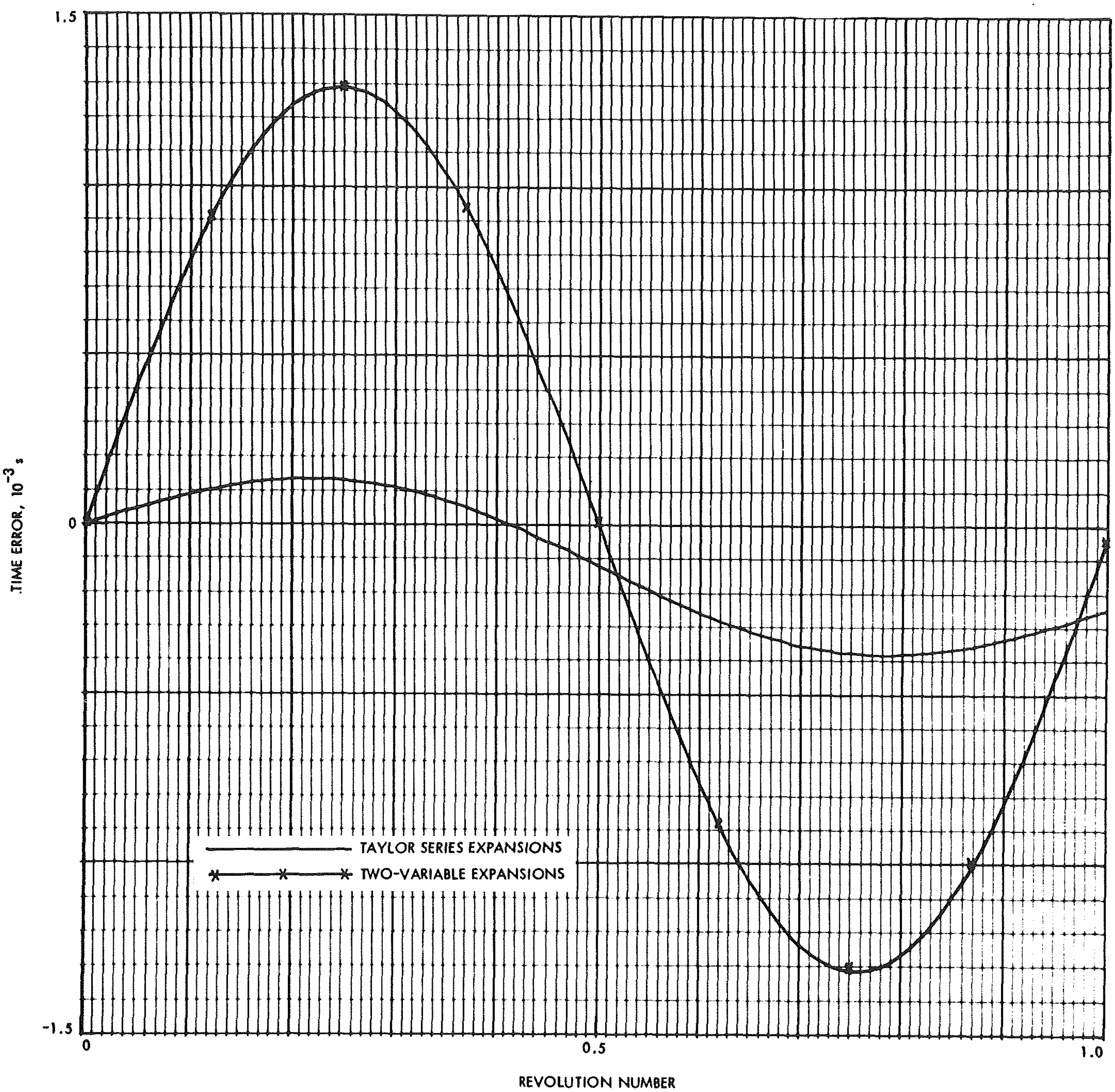


Fig. G-1. Time error in revolution 1, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

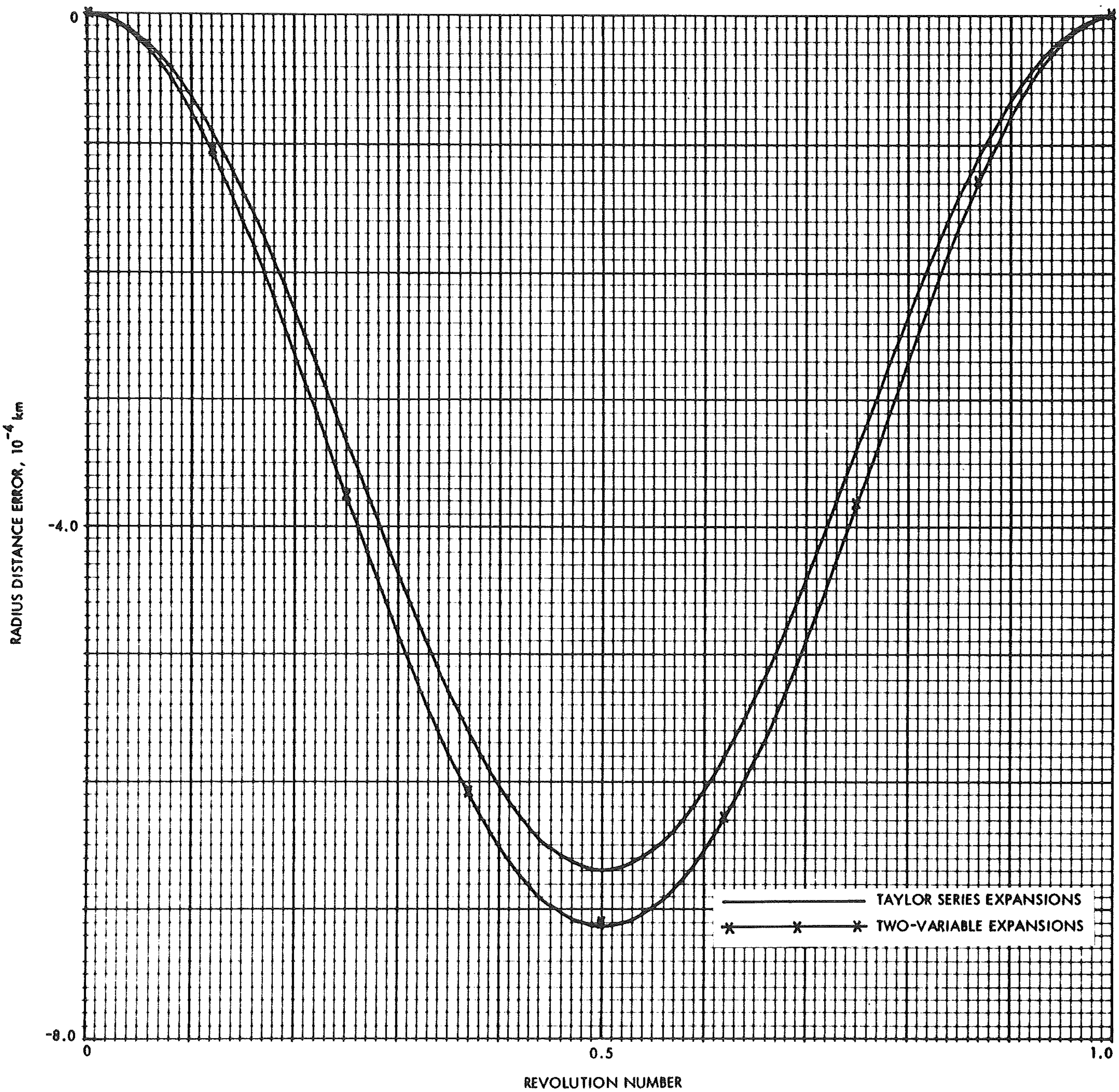


Fig. G-2. Radius distance error in revolution 1, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

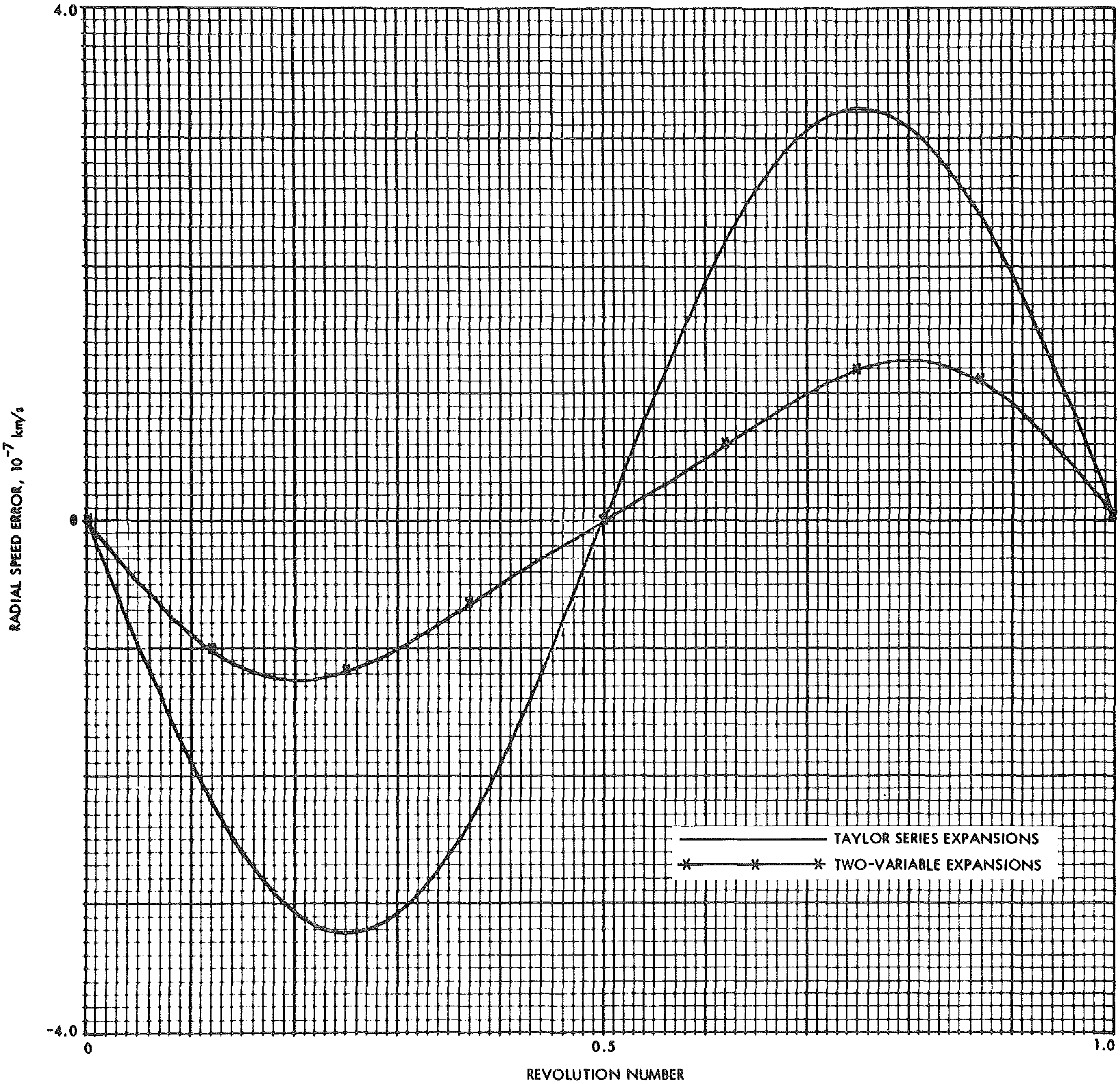


Fig. G-3. Radial speed error in revolution 1, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

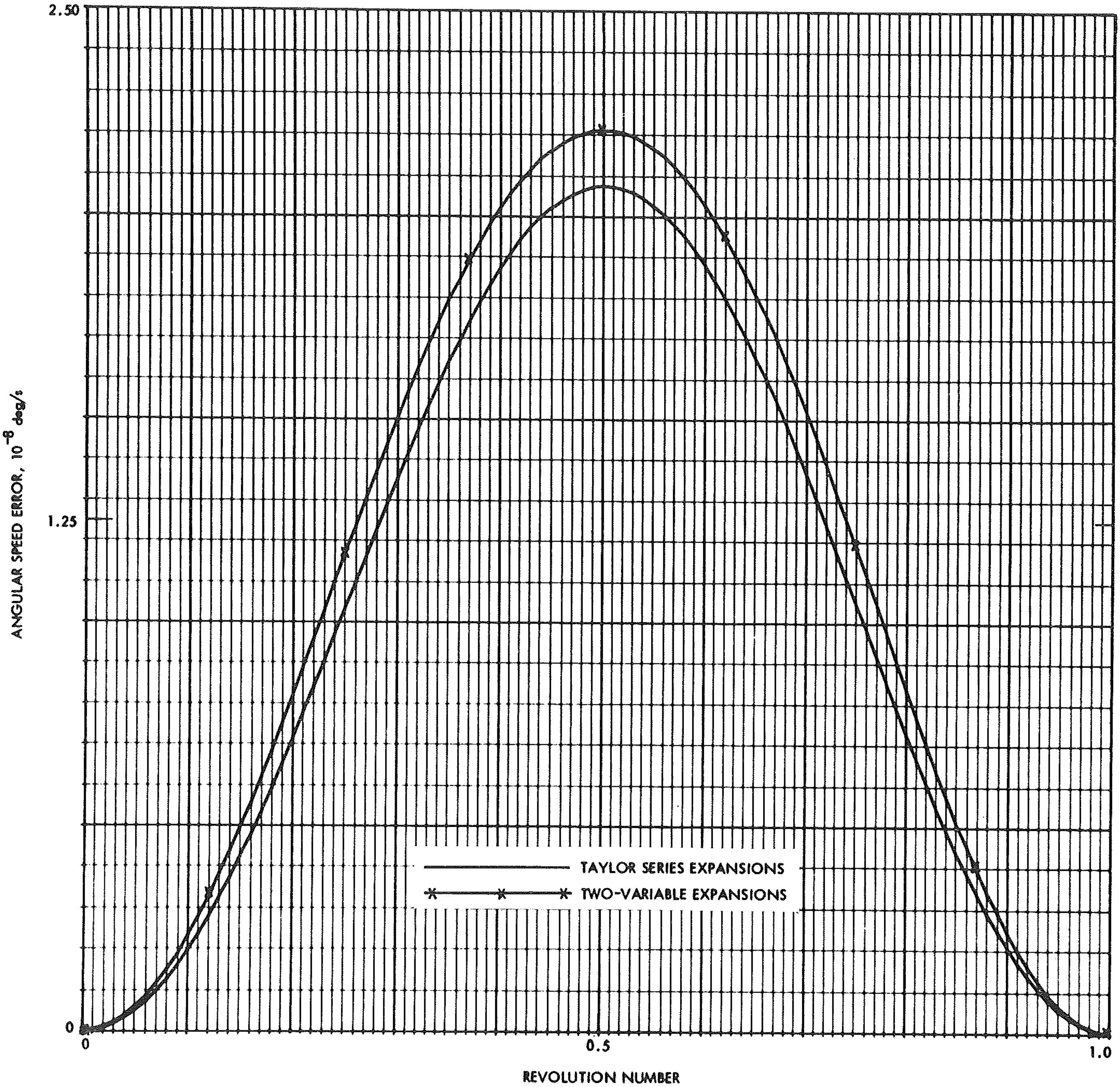


Fig. G-4. Angular speed error in revolution 1, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

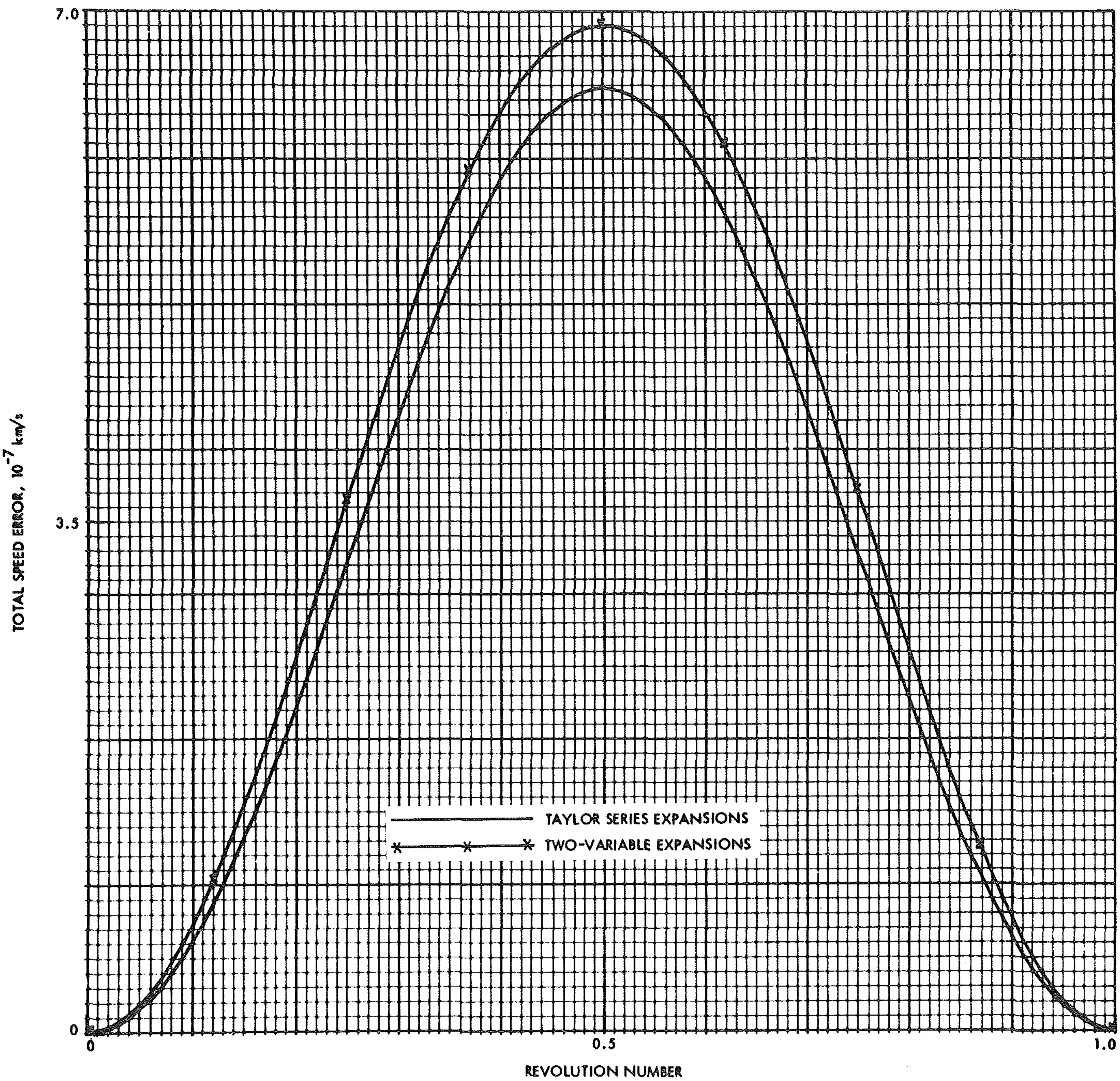


Fig. G-5. Total speed error in revolution 1, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

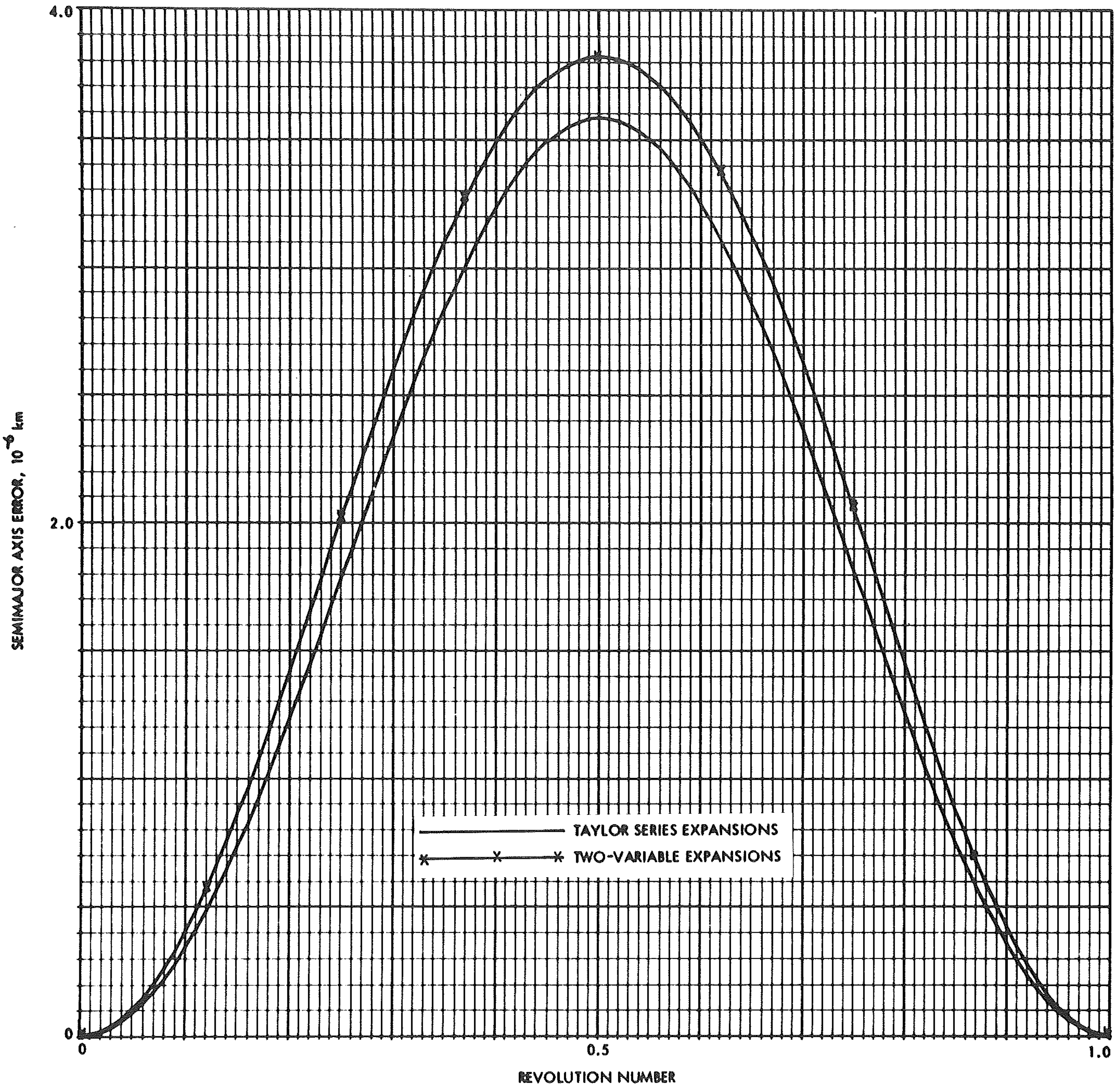


Fig. G-6. Semimajor axis error in revolution 1, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

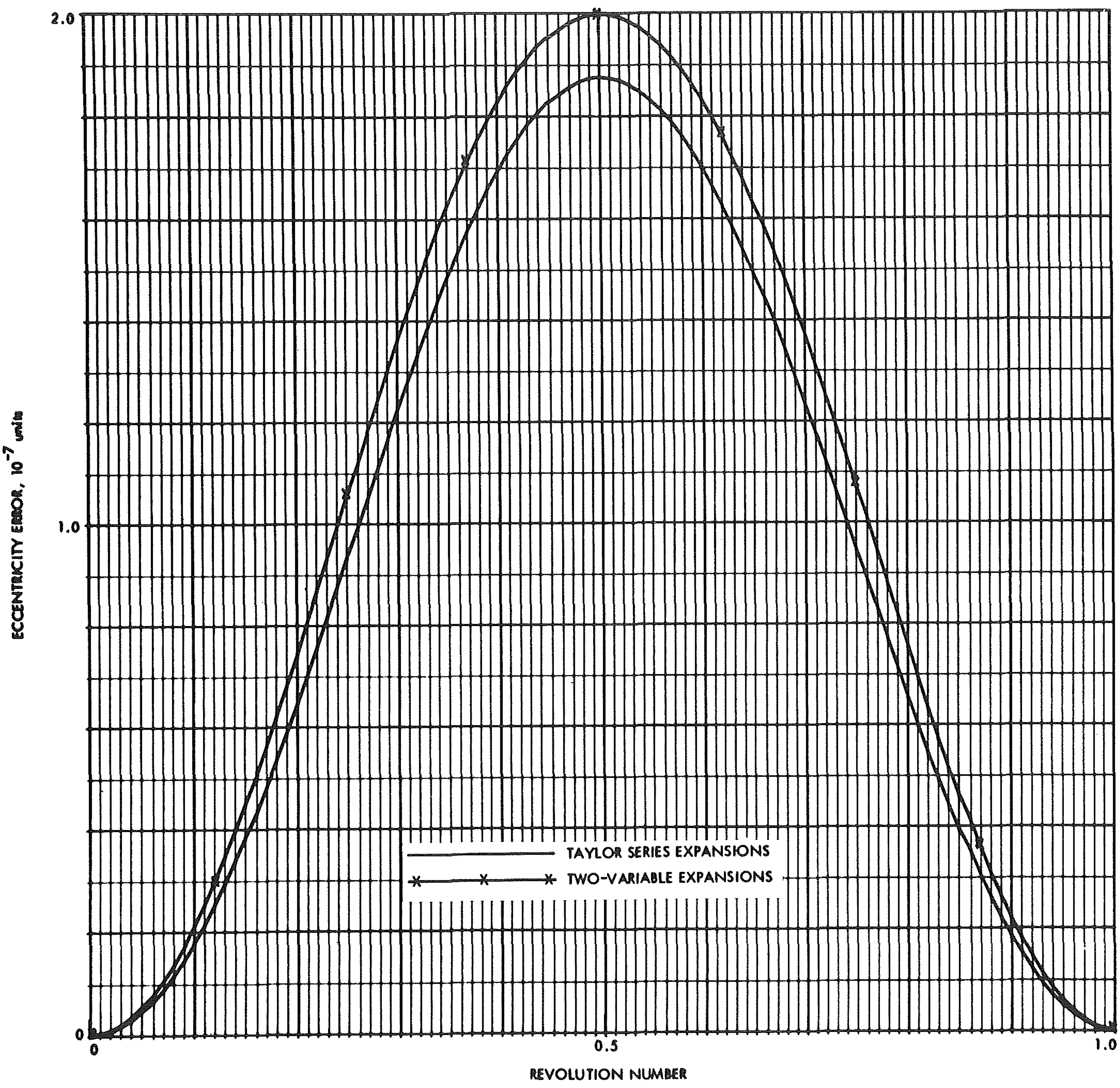


Fig. G-7. Eccentricity error in revolution 1, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

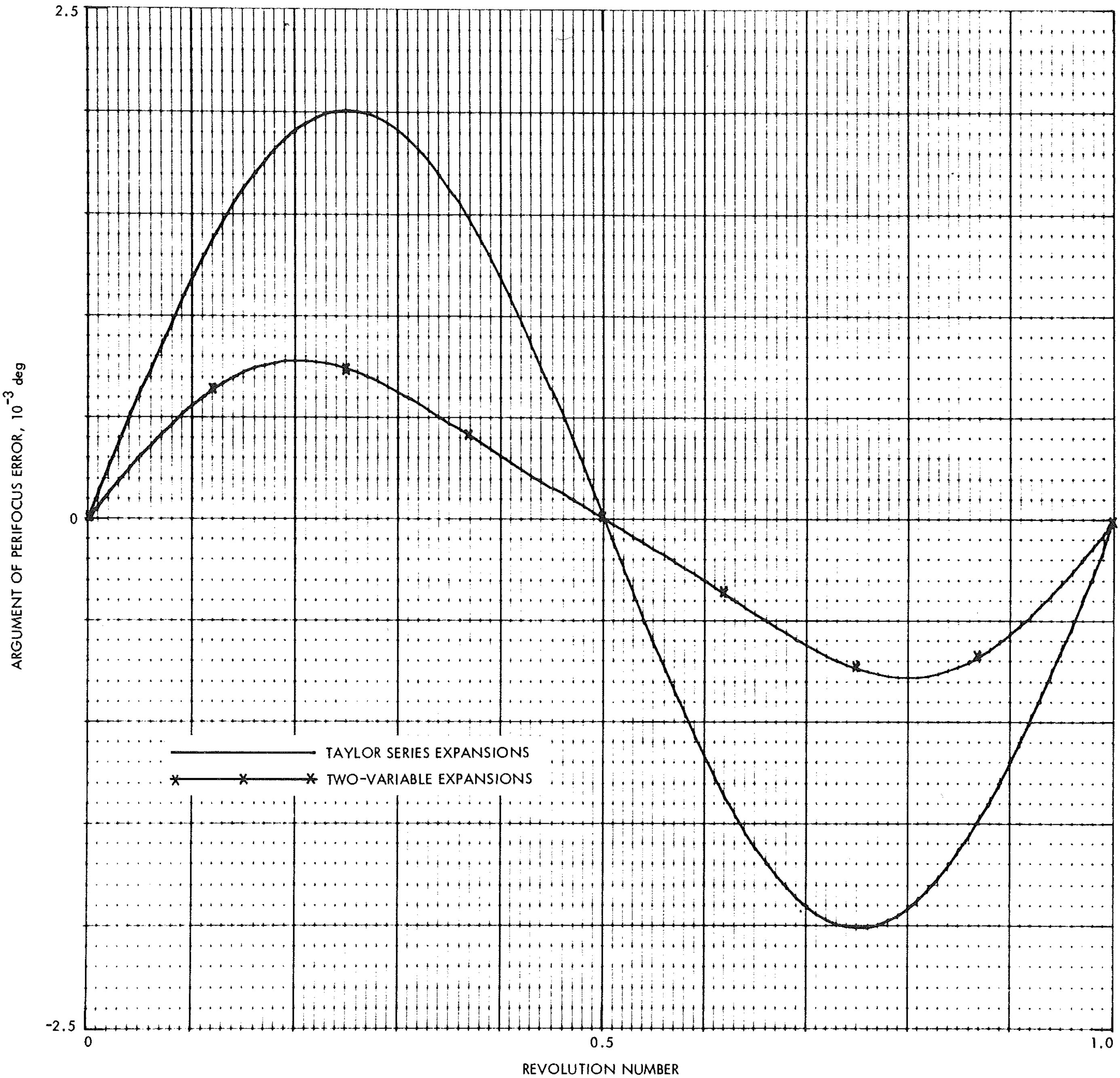


Fig. G-8. Argument of perifocus error in revolution 1, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

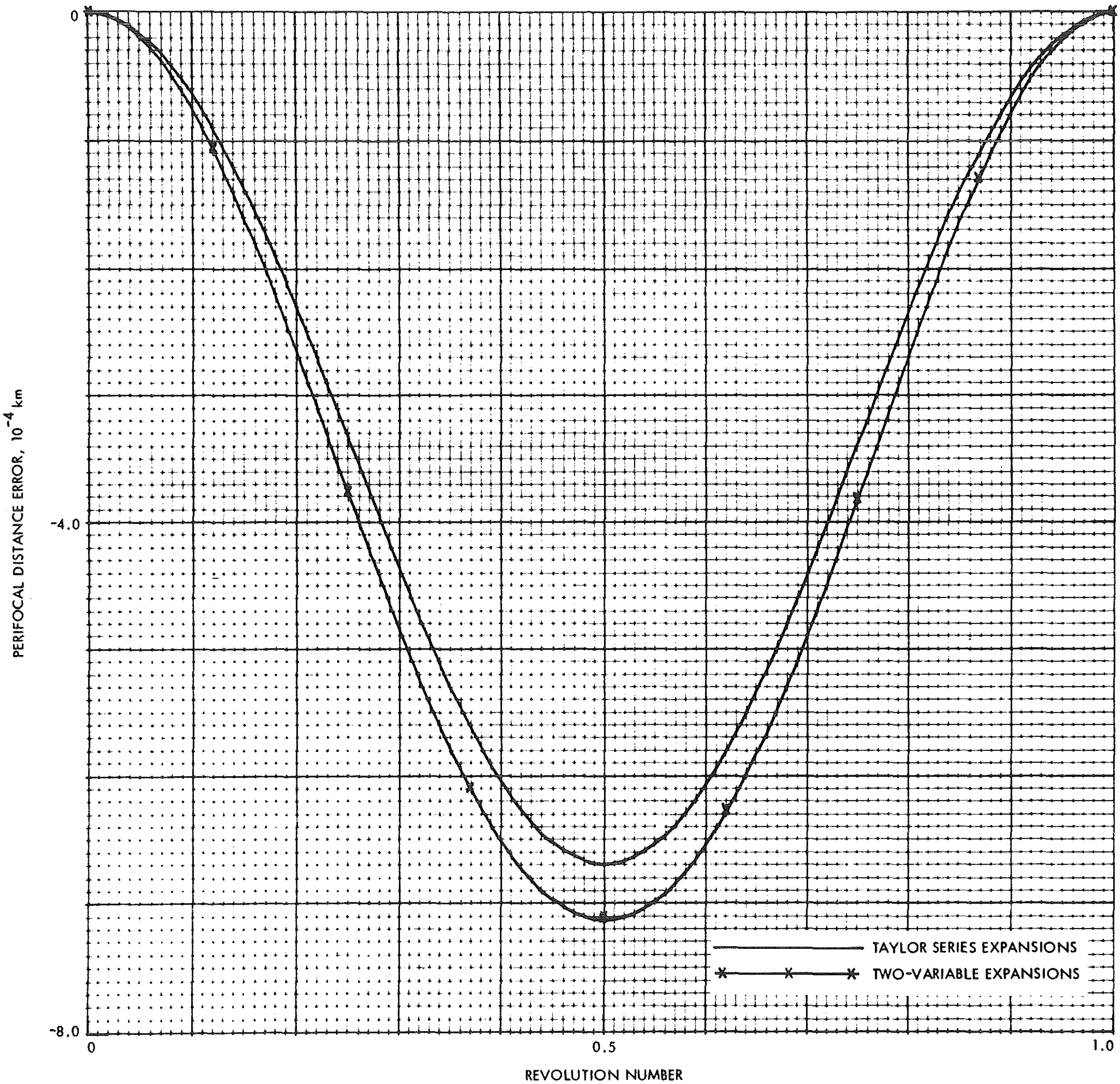


Fig. G-9. Perifocal distance error in revolution 1, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

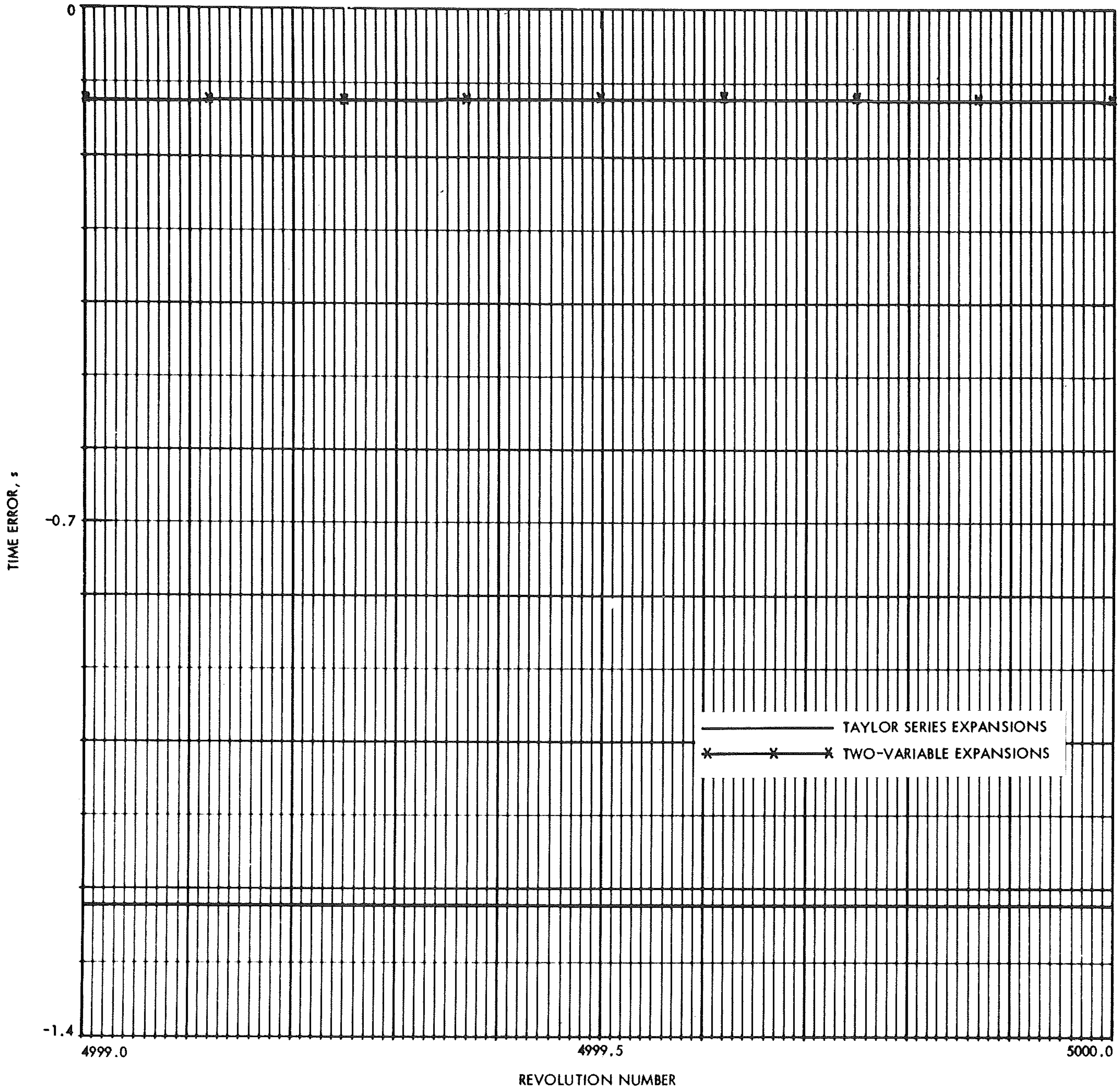


Fig. G-10. Time error in revolution 5000, general perturbations solutions (e_0 = 0.002665, q_0 = 3578 km, v_0 = 0 deg)

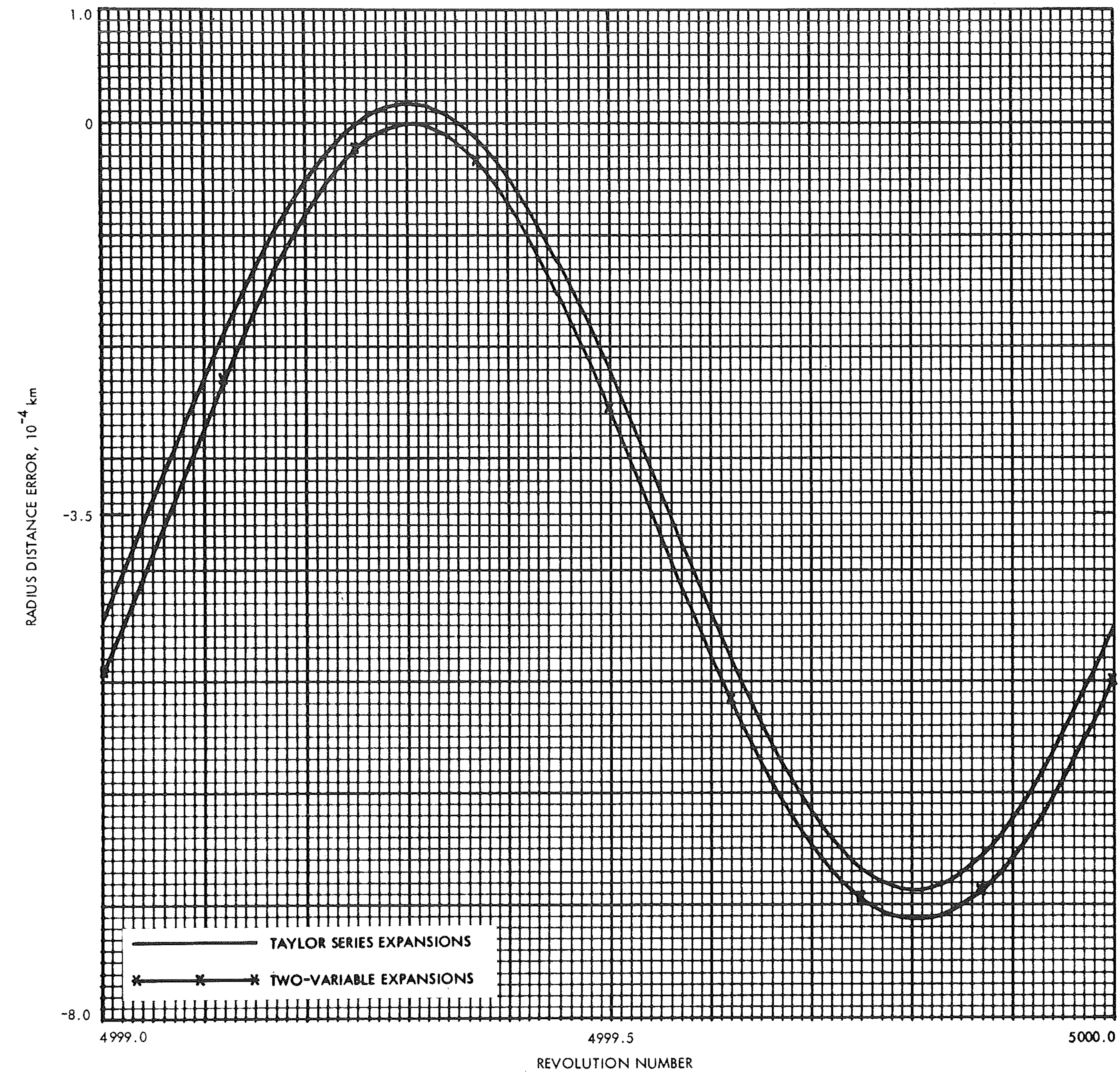


Fig. G-11. Radius distance error in revolution 5000, general perturbations solutions (e_0 = 0.002665, q_0 = 3578 km, v_0 = 0 deg)

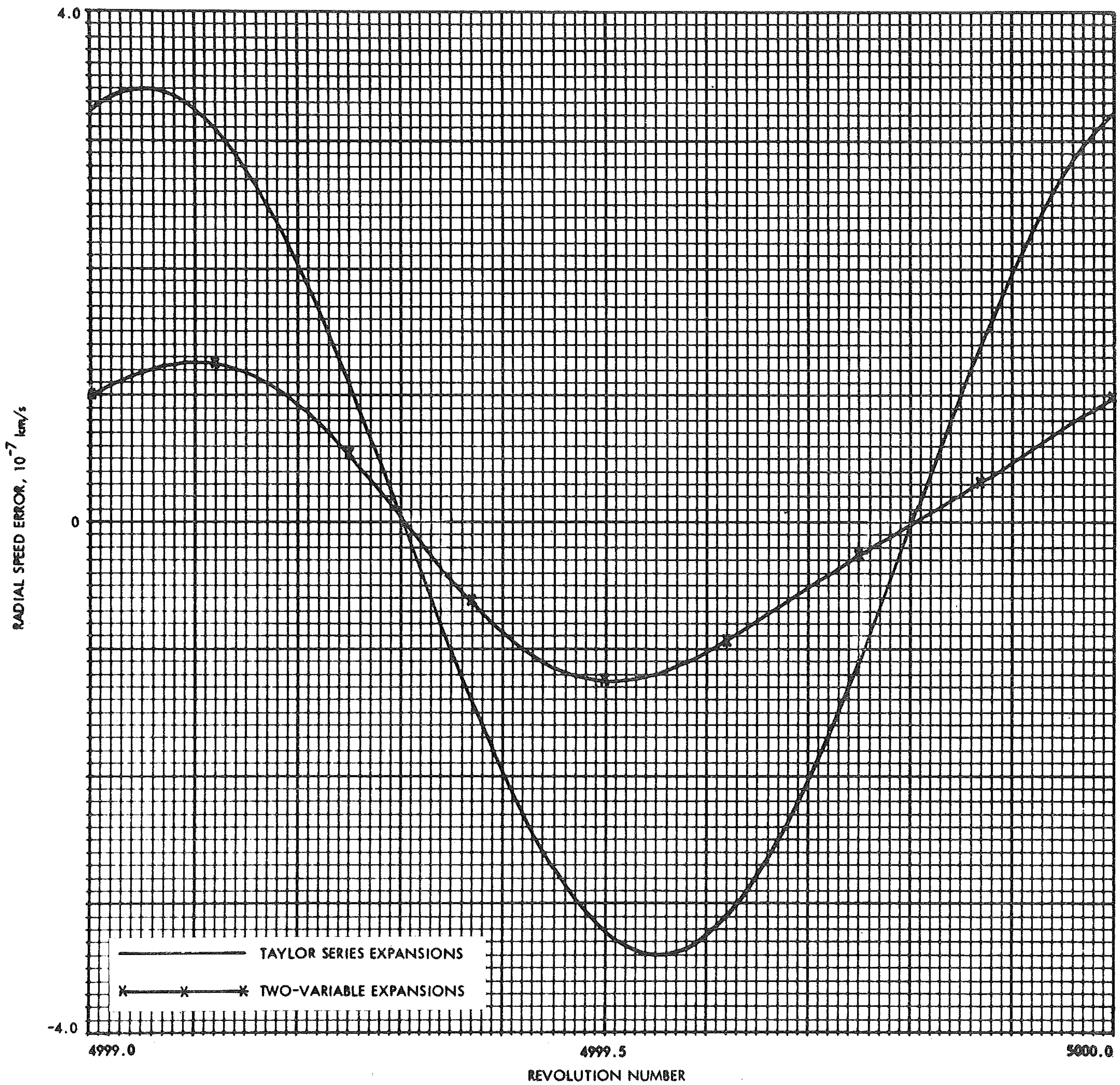


Fig. G-12. Radial speed error in revolution 5000, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

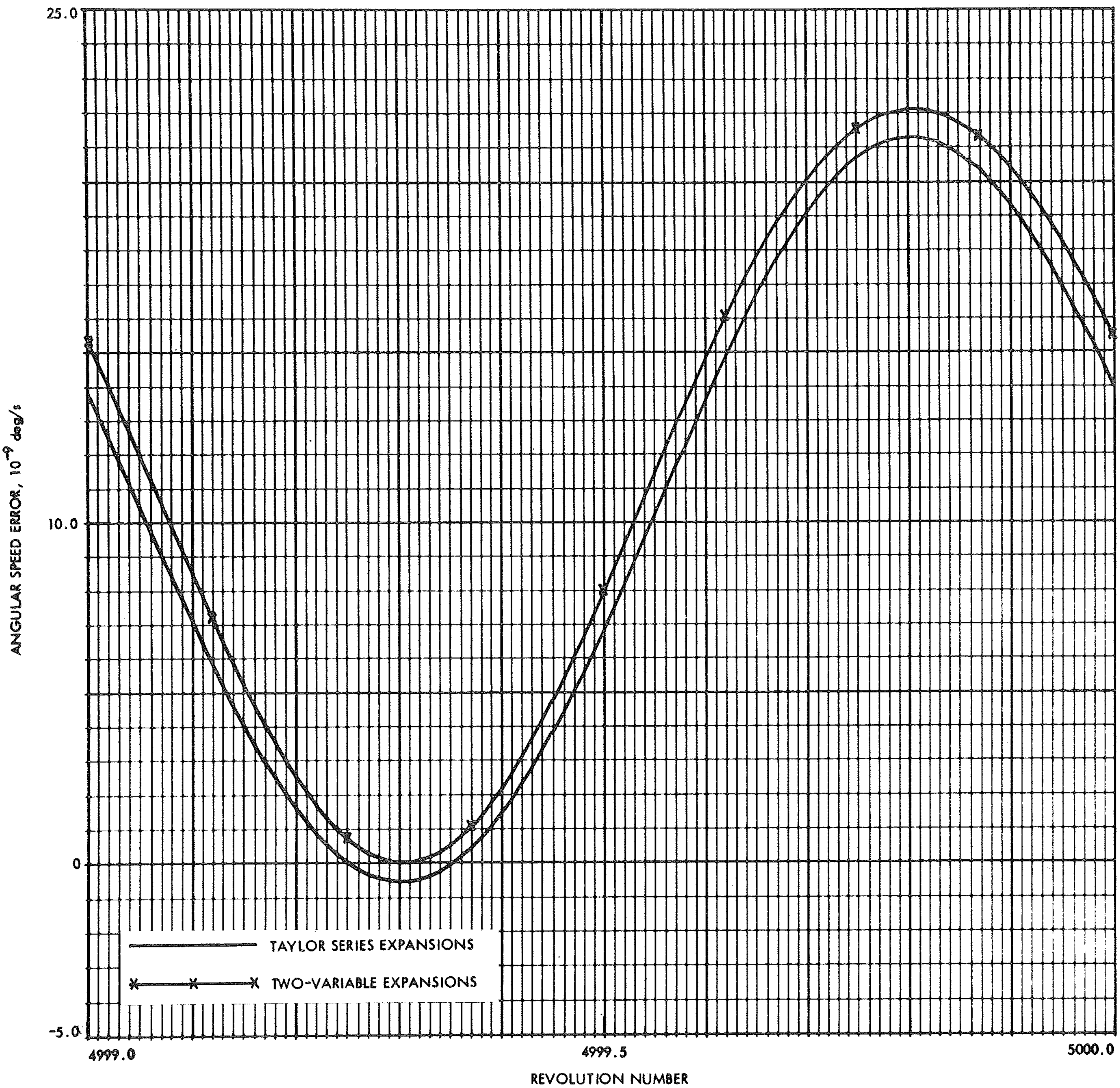


Fig. G-13. Angular speed error in revolution 5000, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

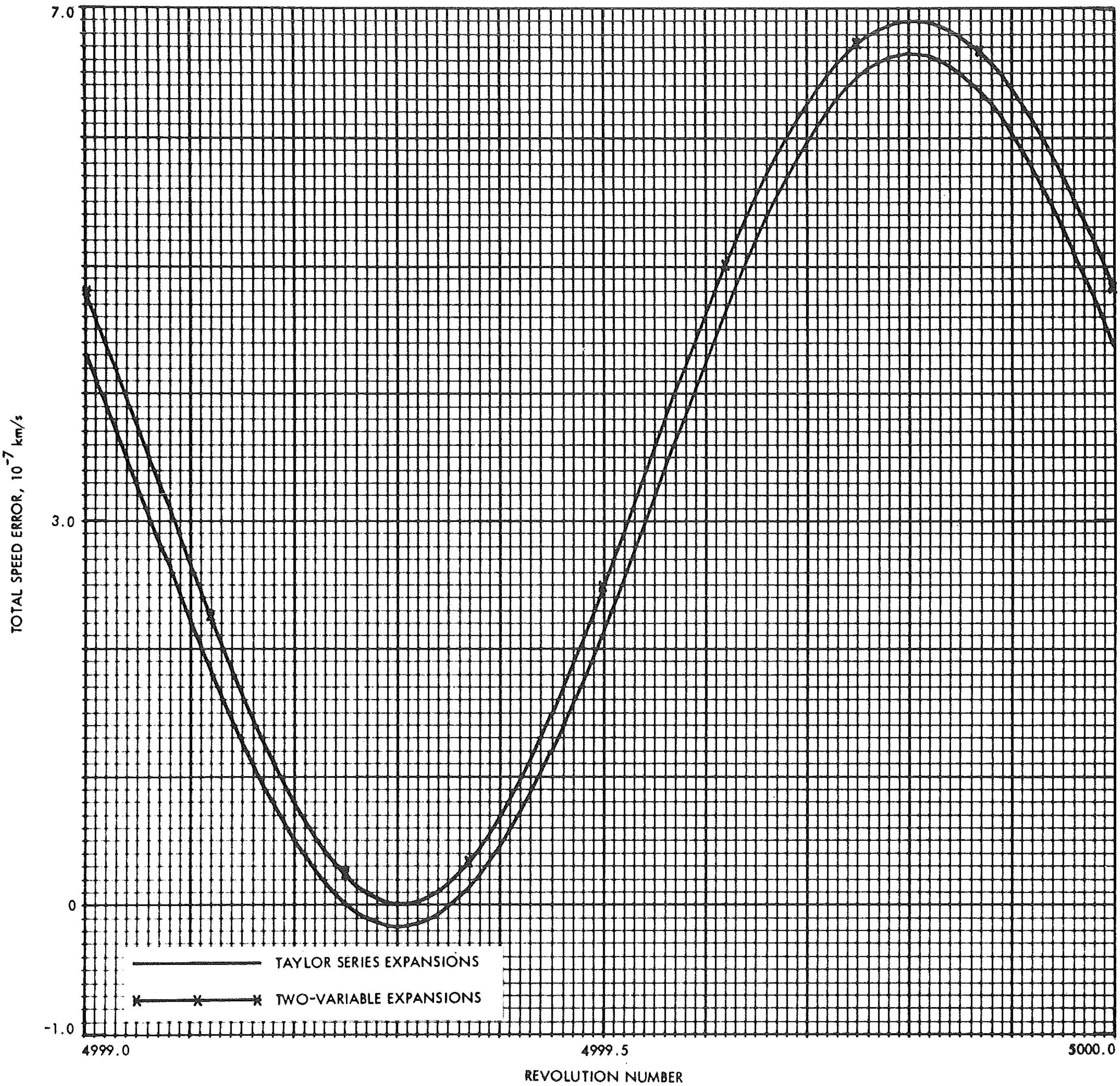


Fig. G-14. Total speed error in revolution 5000, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

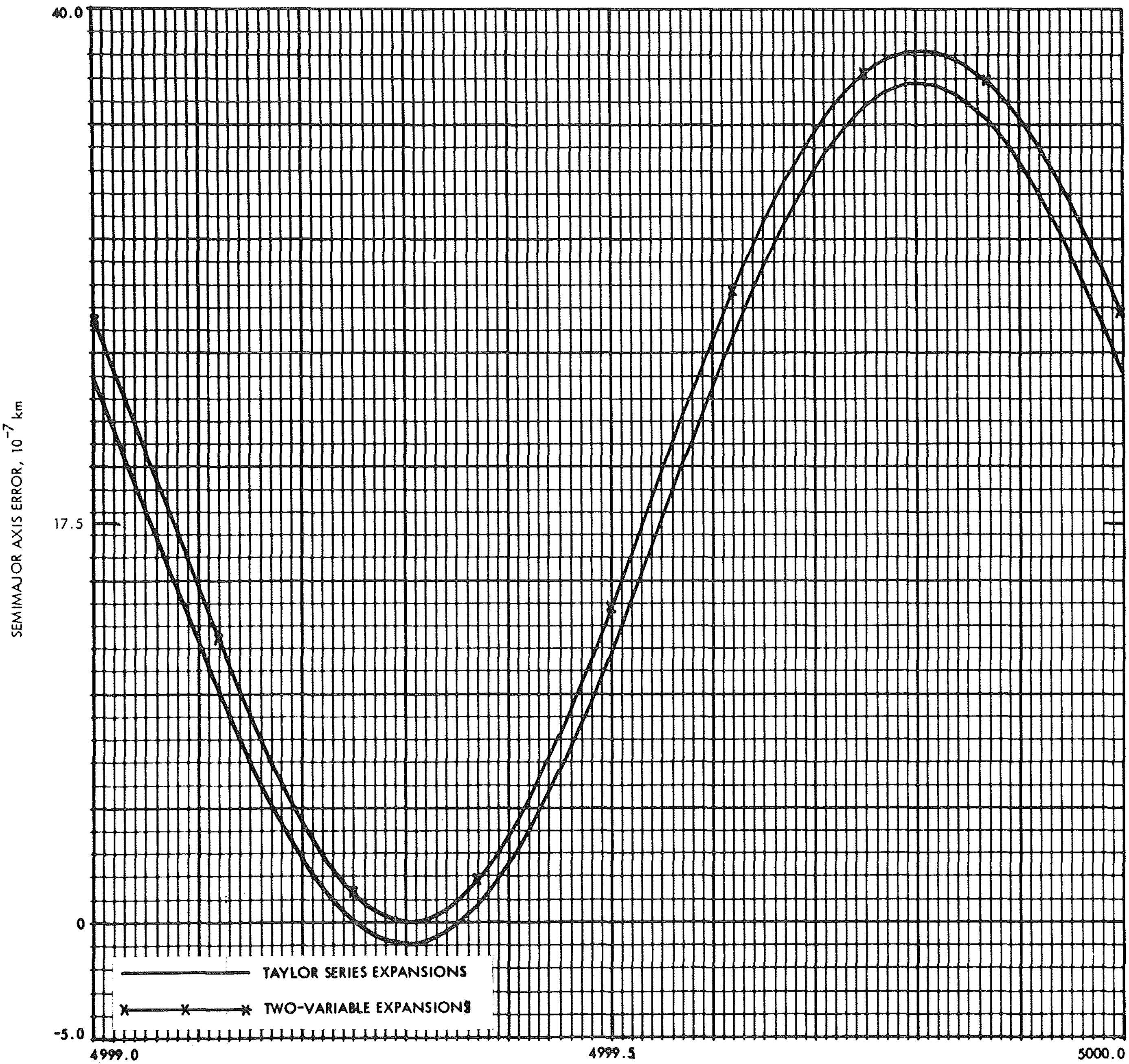


Fig. G-15. Semimajor axis error in revolution 5000, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

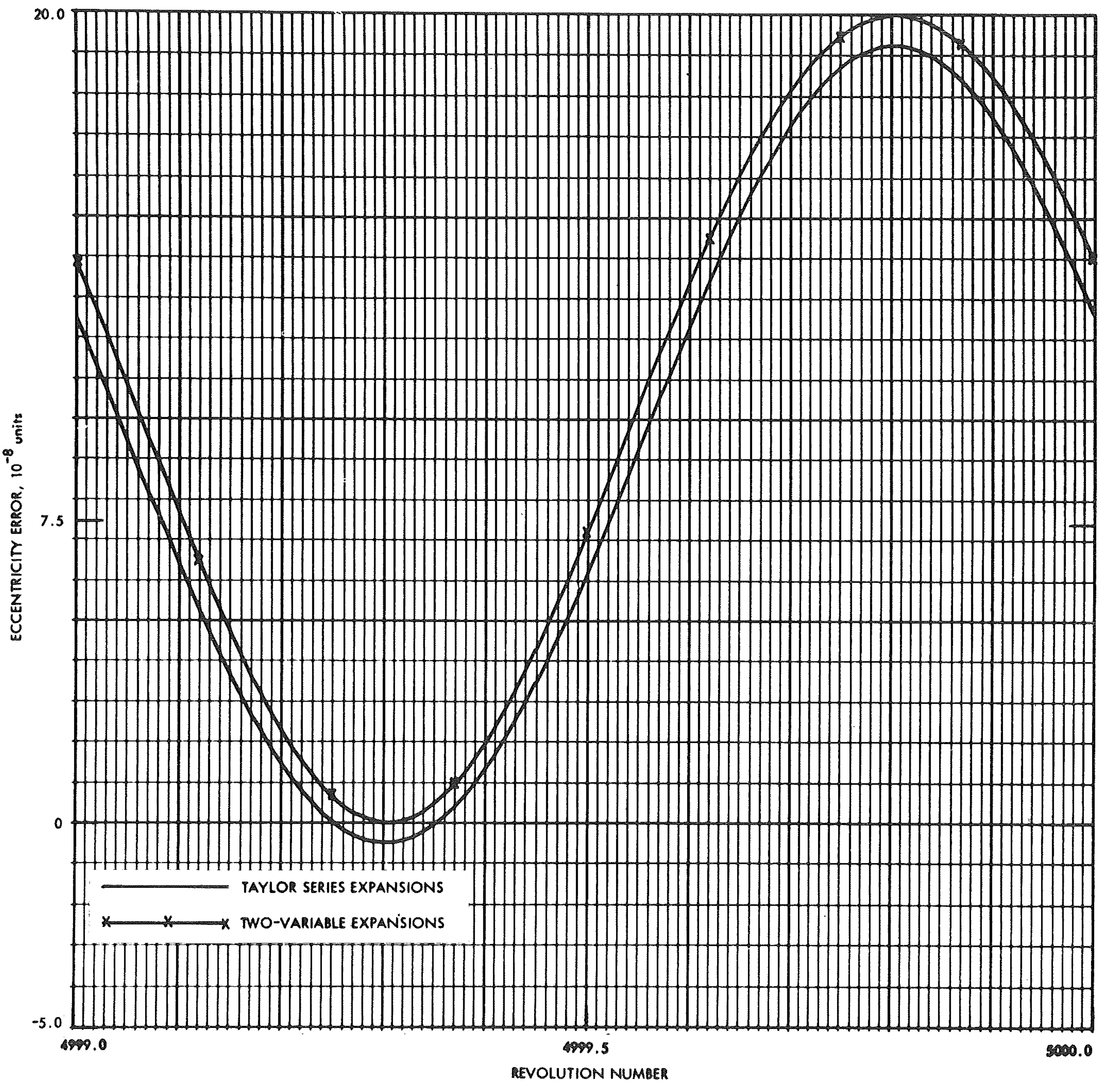


Fig. G-16. Eccentricity error in revolution 5000, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

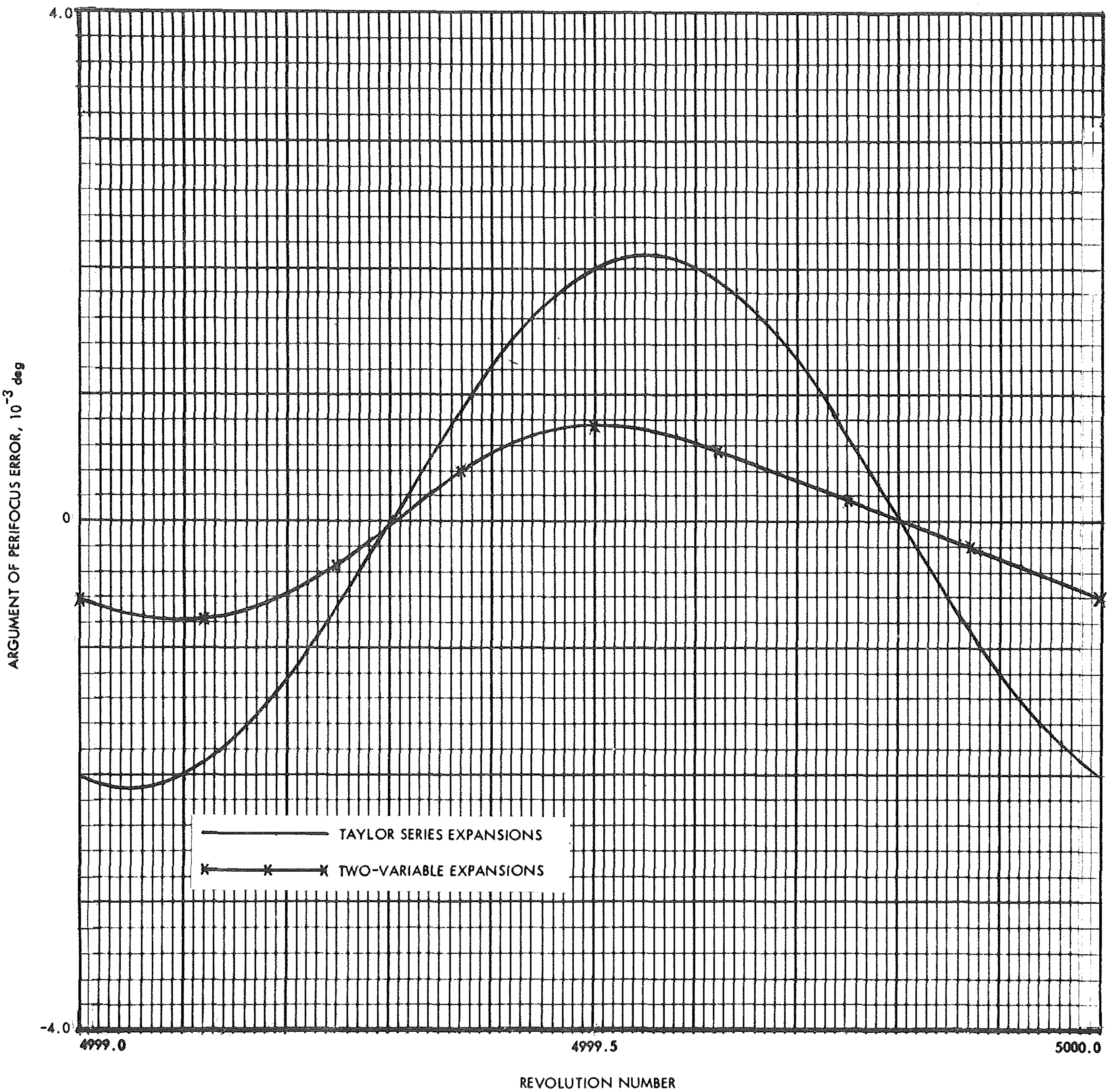


Fig. G-17. Argument of perifocus error in revolution 5000, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

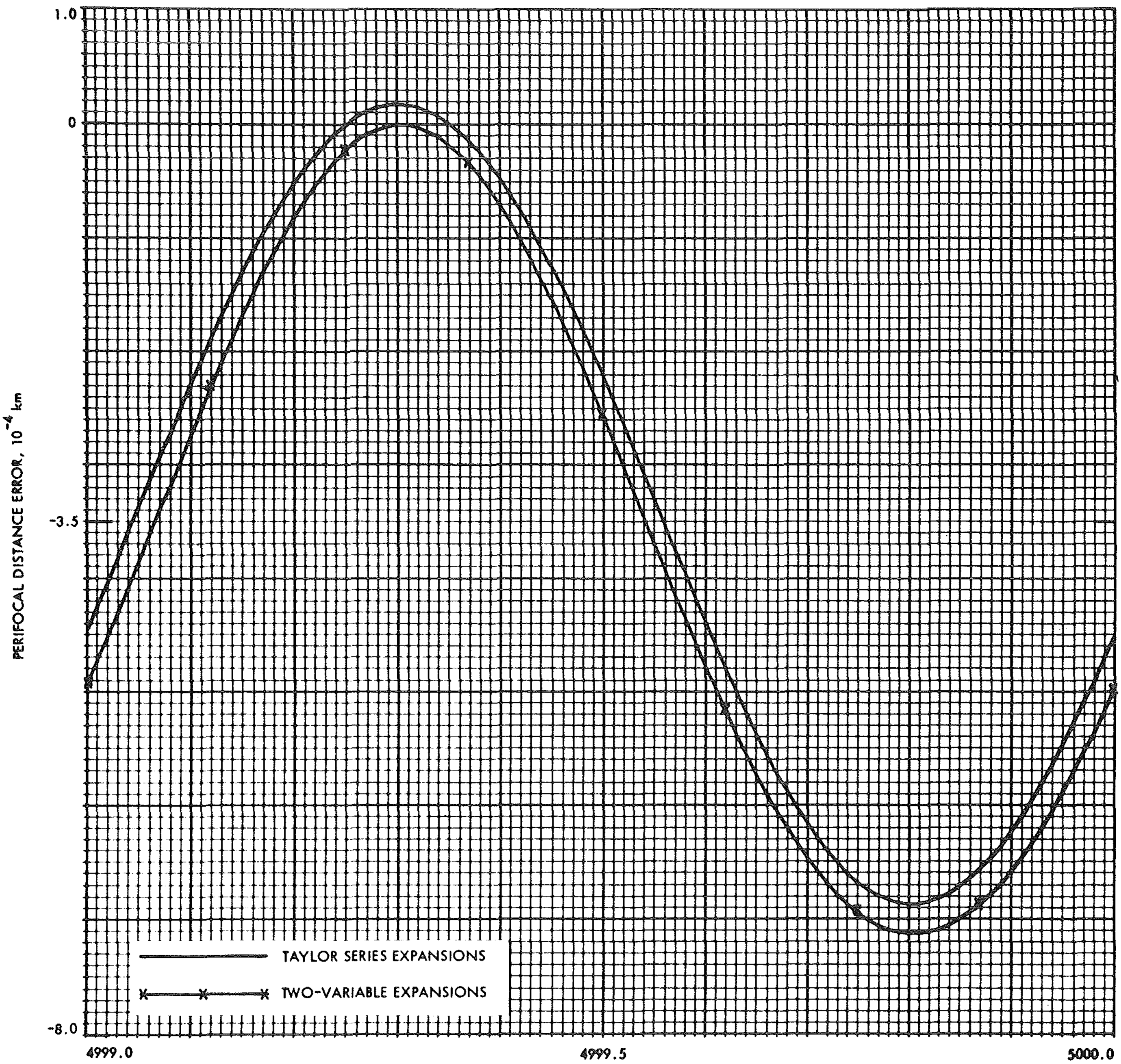


Fig. G-18. Perifocal distance error in revolution 5000, general perturbations solutions (e_0 = 0.002665, q_0 = 3578 km, v_0 = 0 deg)

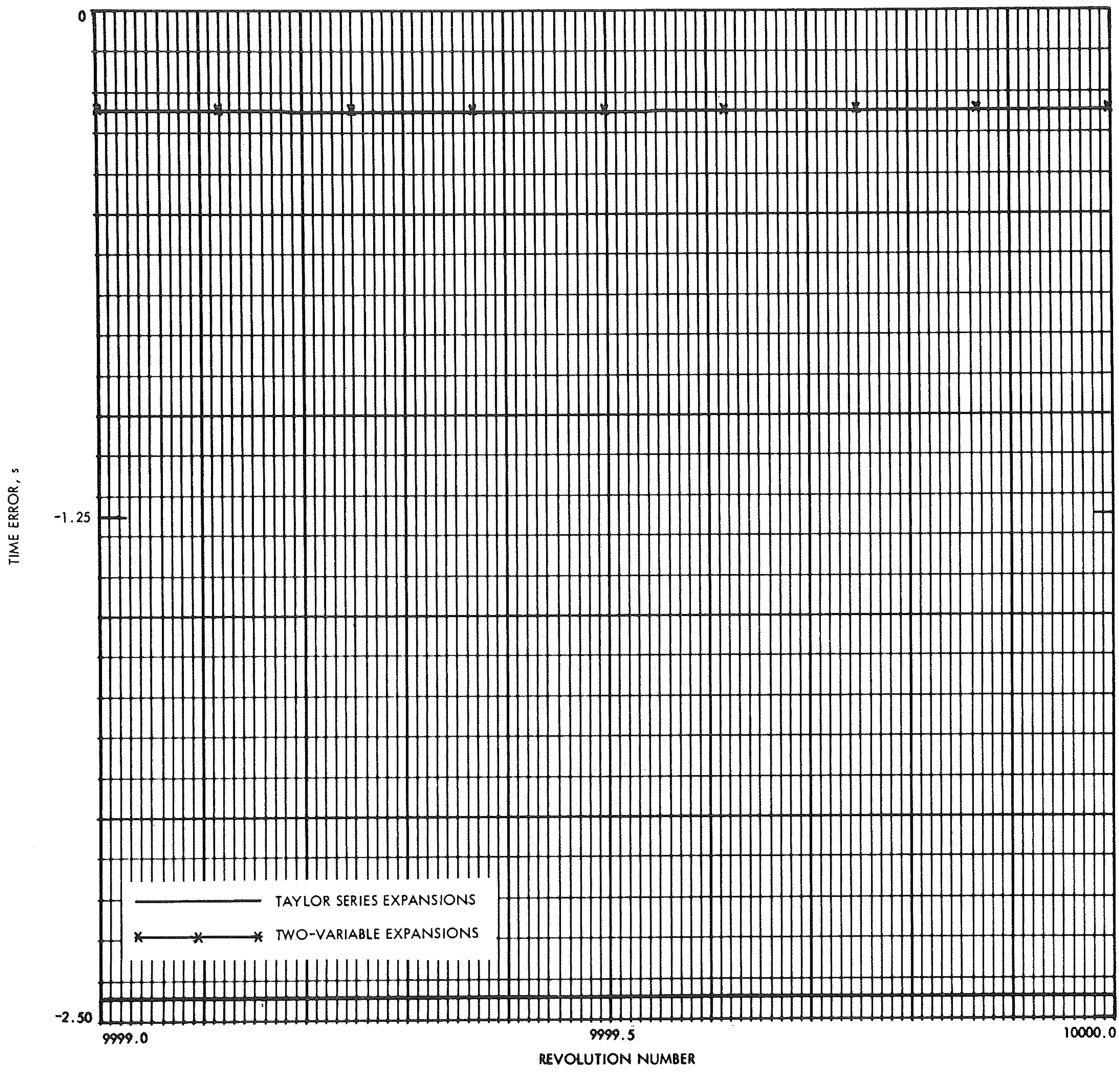


Fig. G-19. Time error in revolution 10,000, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

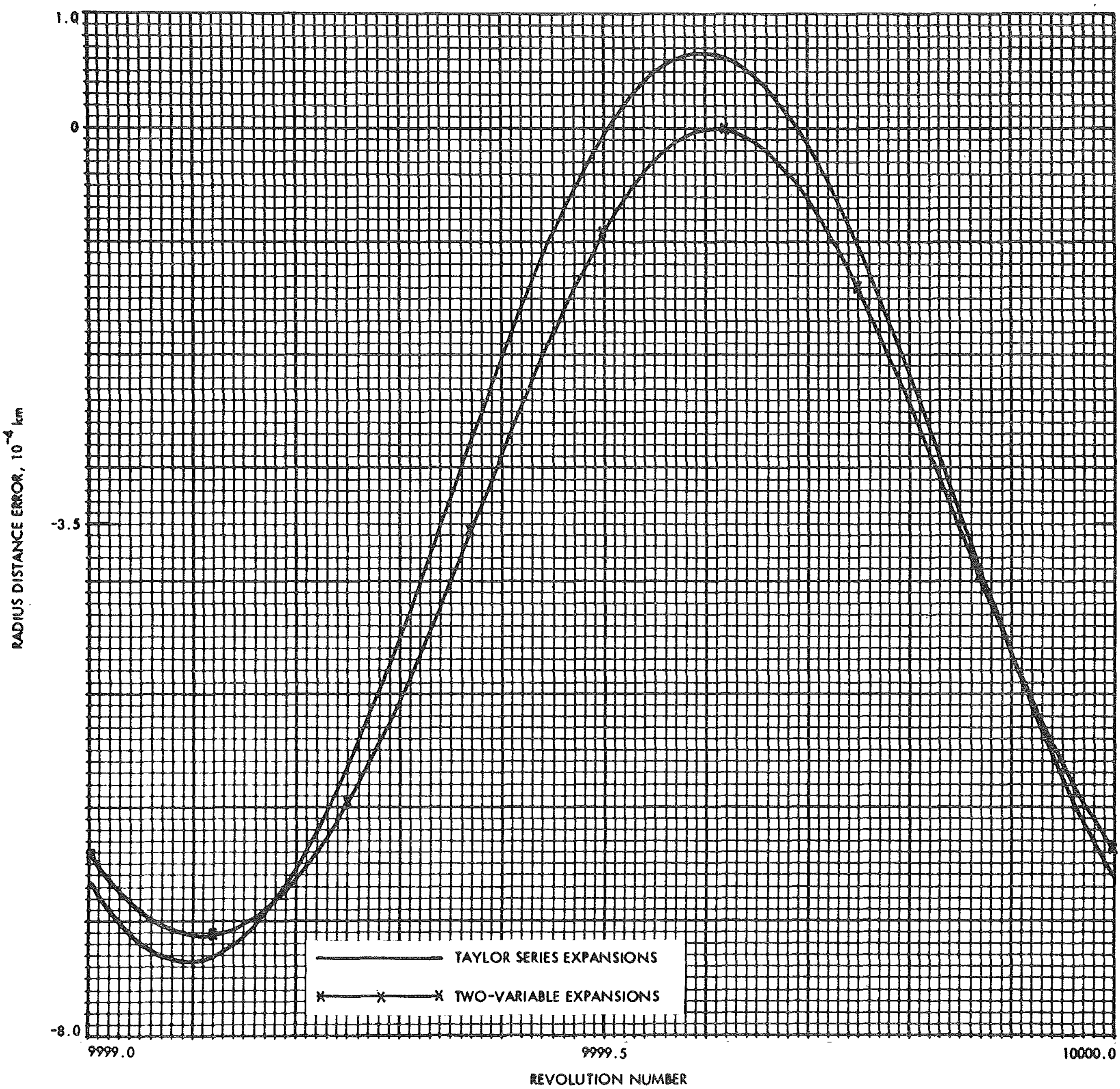


Fig. G-20. Radius distance error in revolution 10,000, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

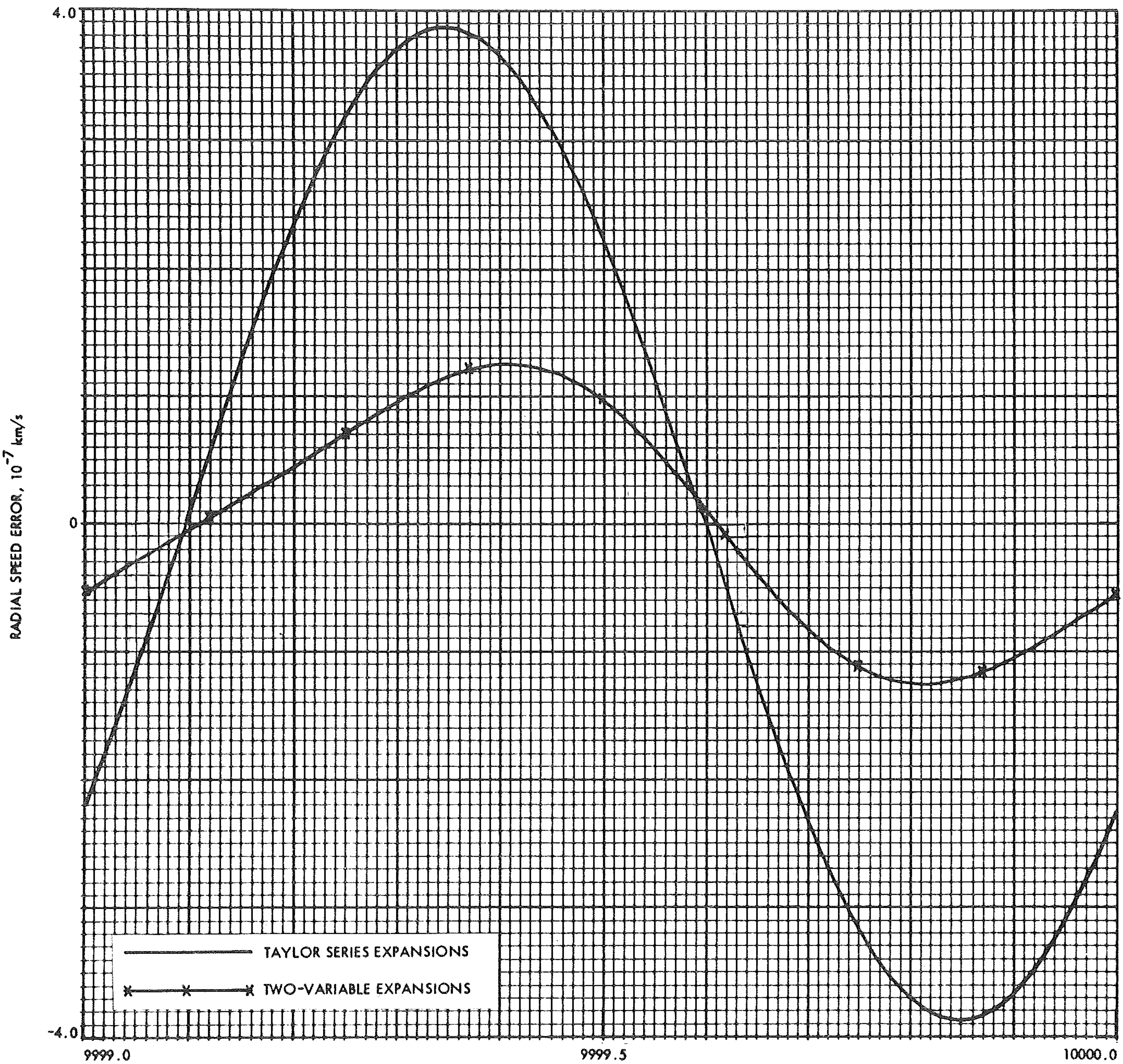


Fig. G-21. Radial speed error in revolution 10,000, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

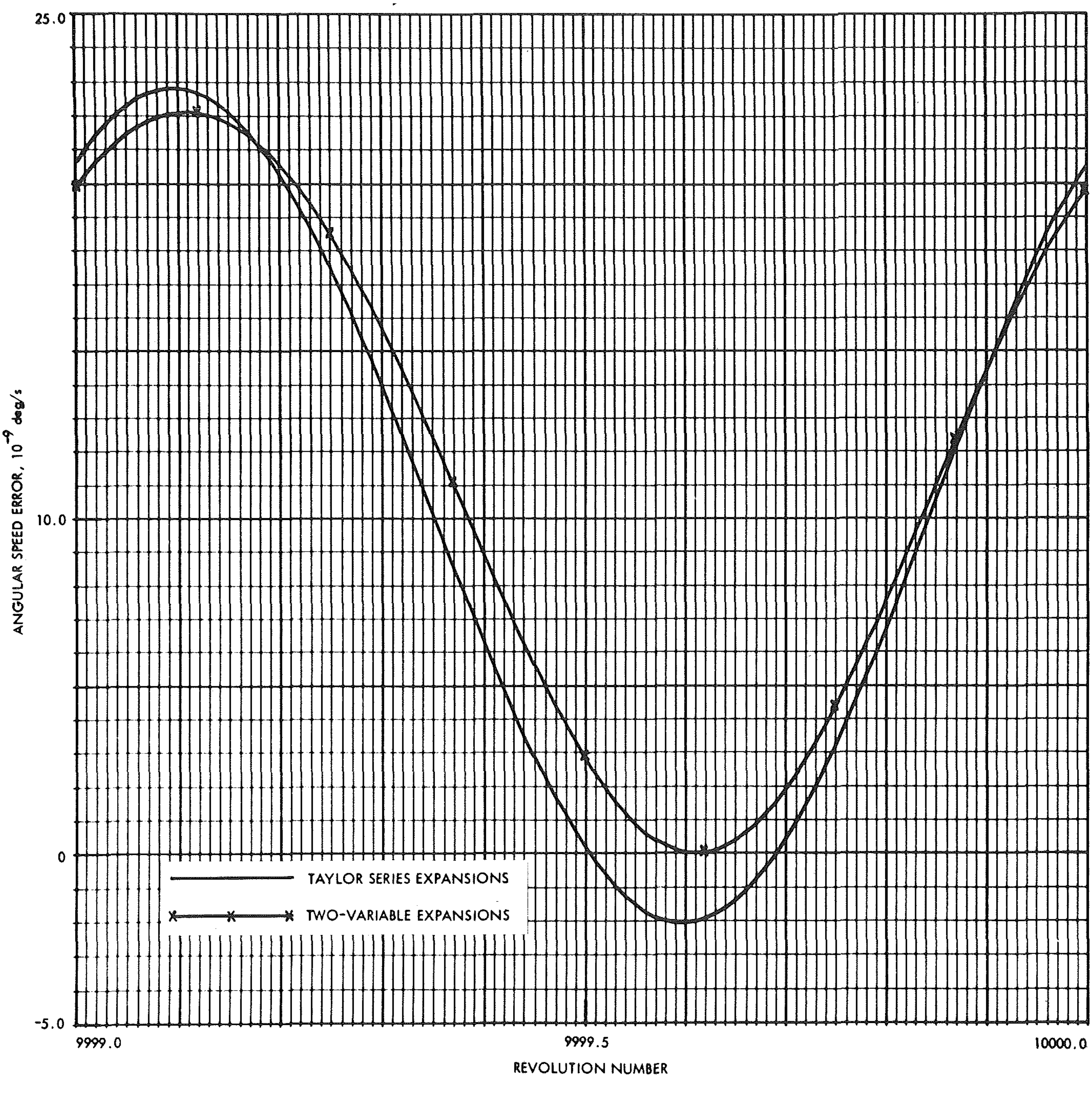


Fig. G-22. Angular speed error in revolution 10,000, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

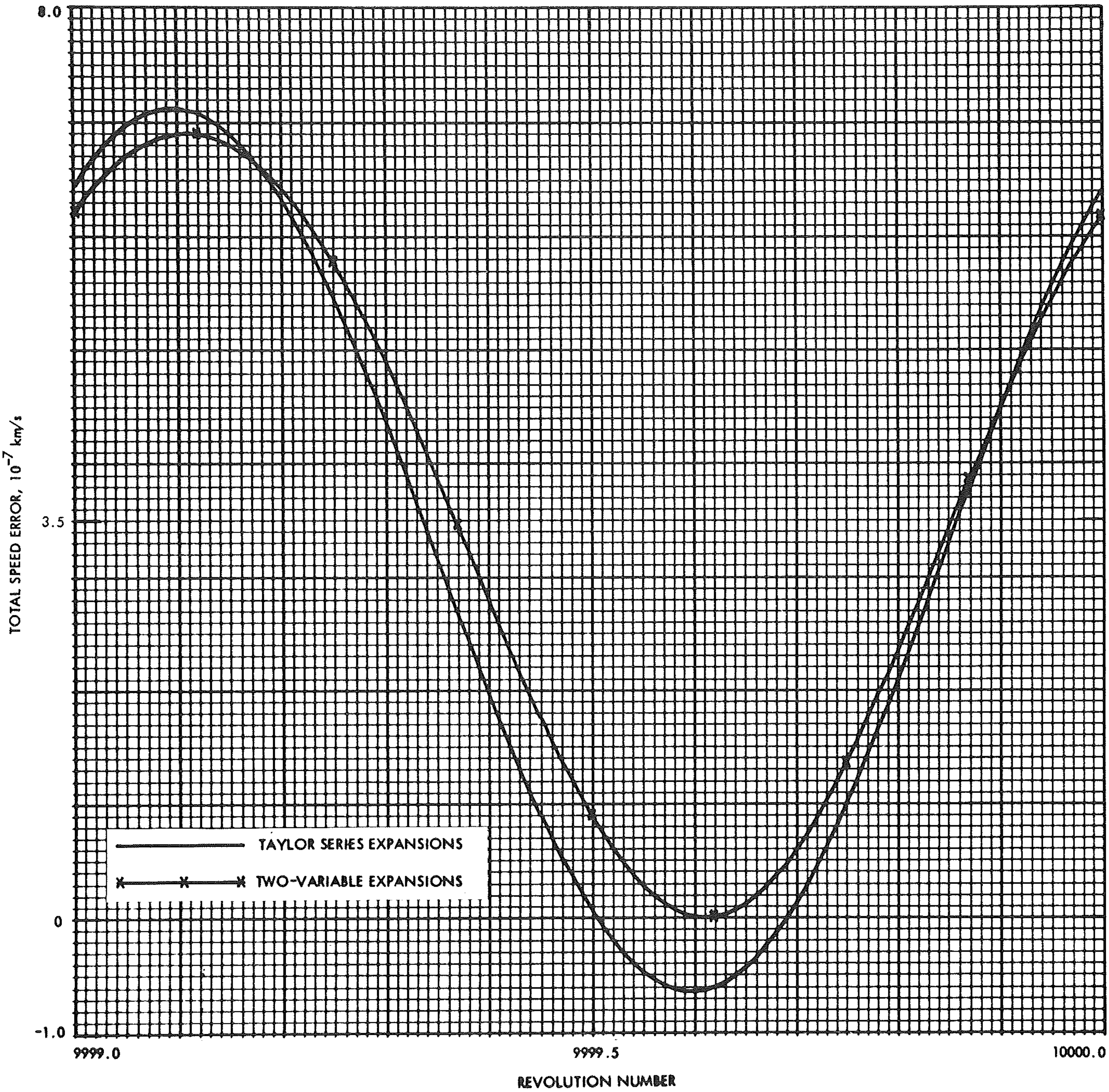


Fig. G-23. Total speed error in revolution 10,000, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

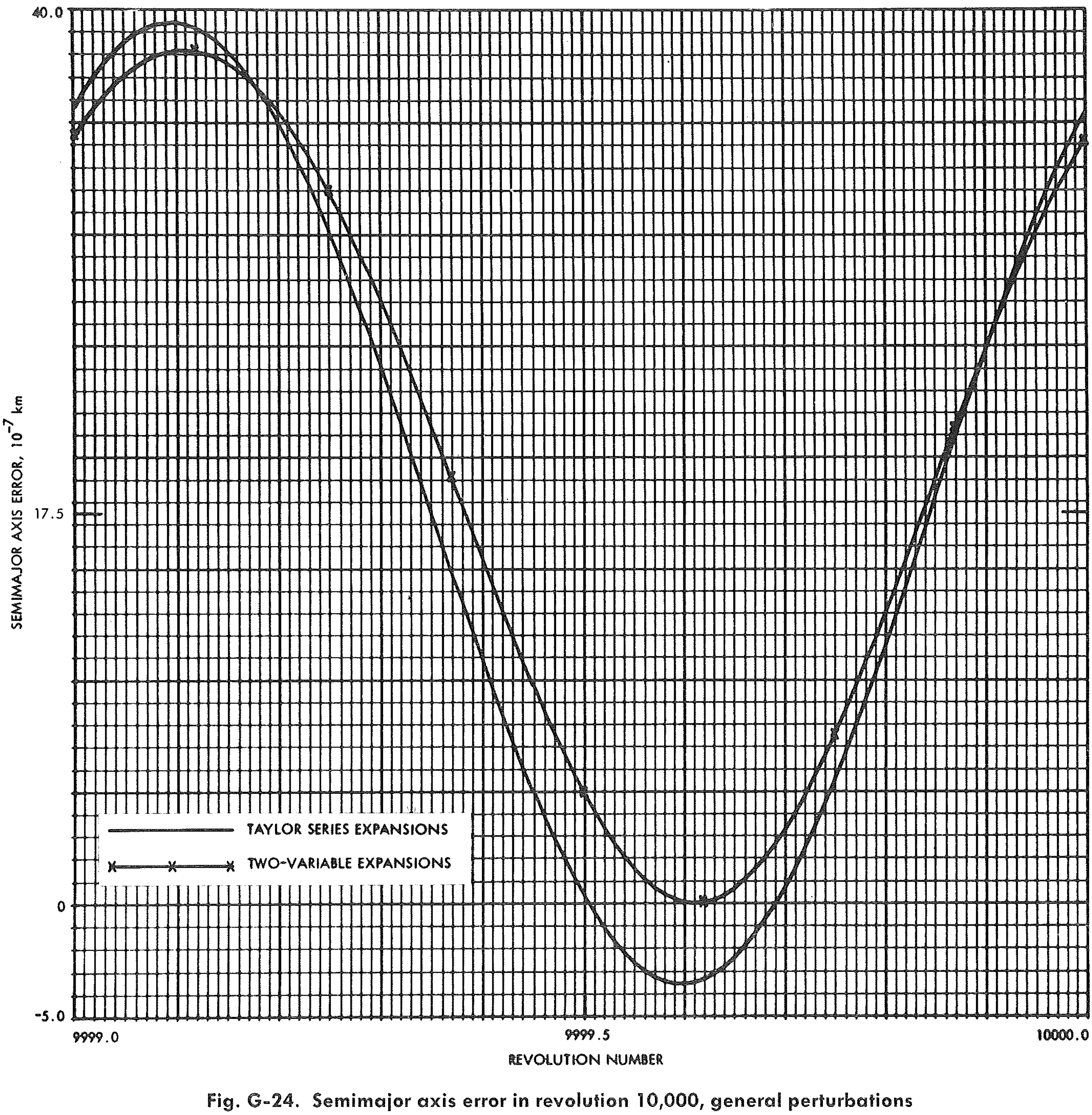


Fig. G-24. Semimajor axis error in revolution 10,000, general perturbations solutions (e_0 = 0.002665, q_0 = 3578 km, v_0 = 0 deg)

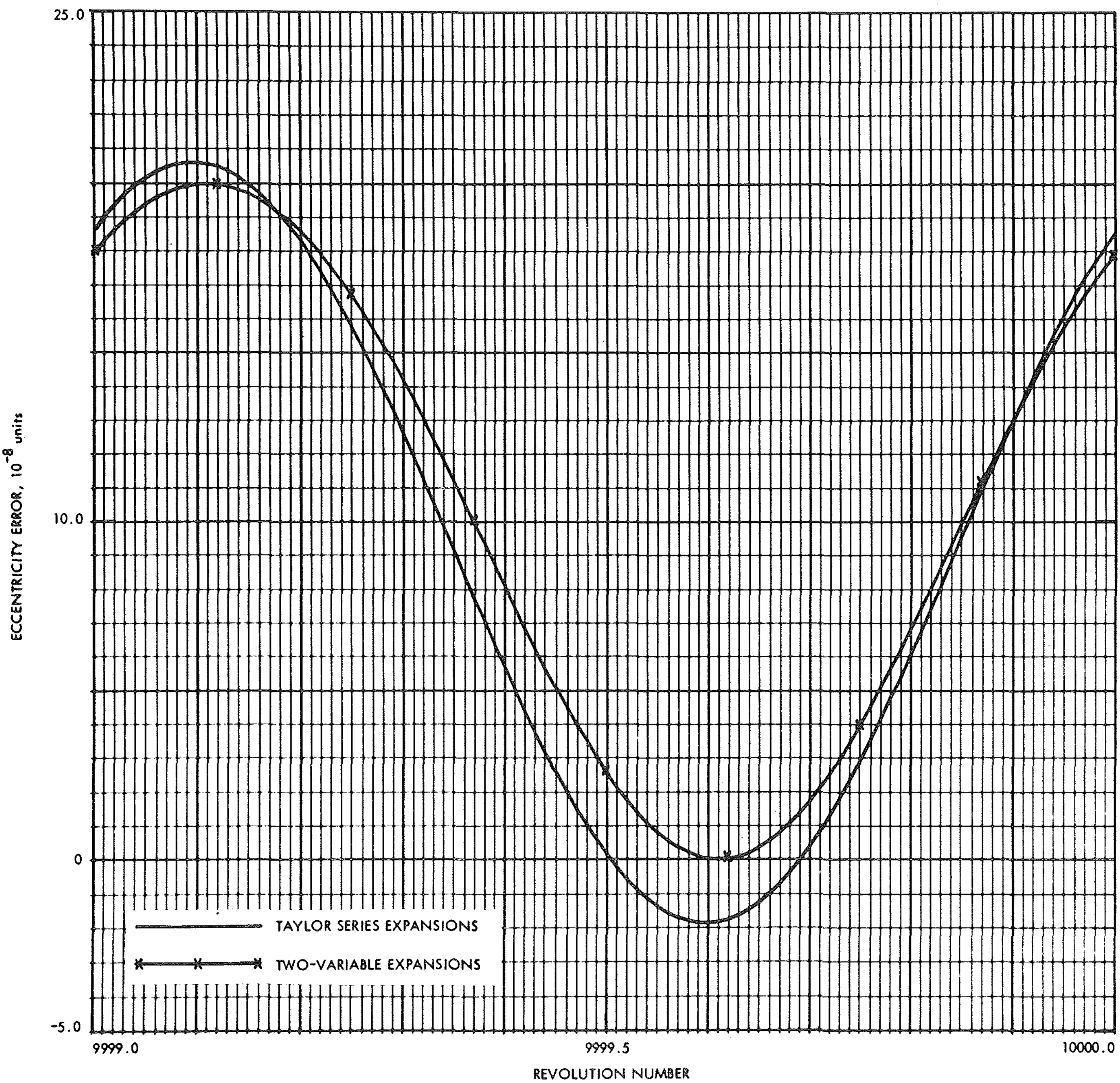


Fig. G-25. Eccentricity error in revolution 10,000, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

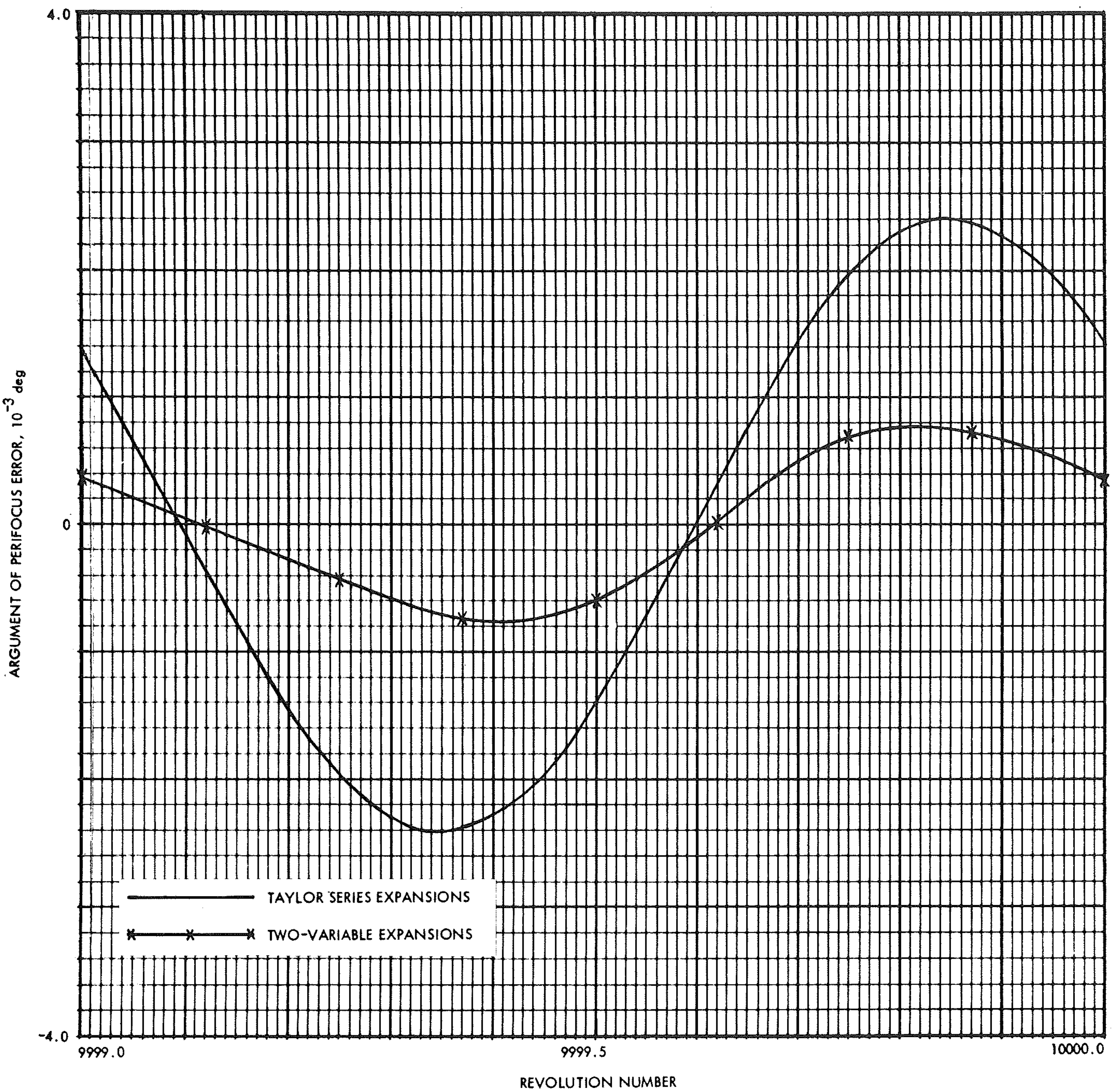


Fig. G-26. Argument of perifocus error in revolution 10,000, general perturbations solutions (e_0 = 0.002665, q_0 = 3578 km, v_0 = 0 deg)

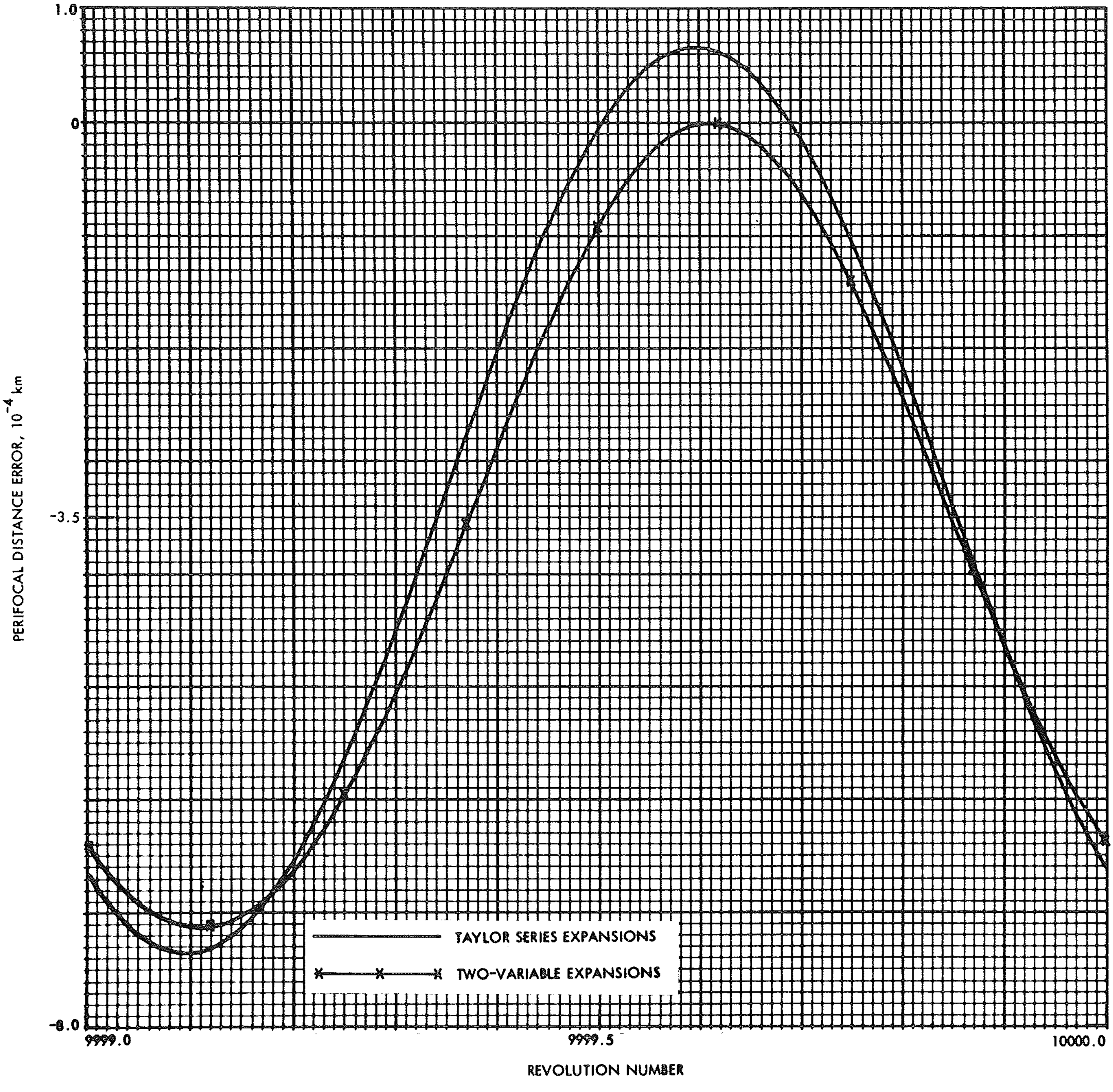


Fig. G-27. Perifocal distance error in revolution 10,000, general perturbations solutions ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

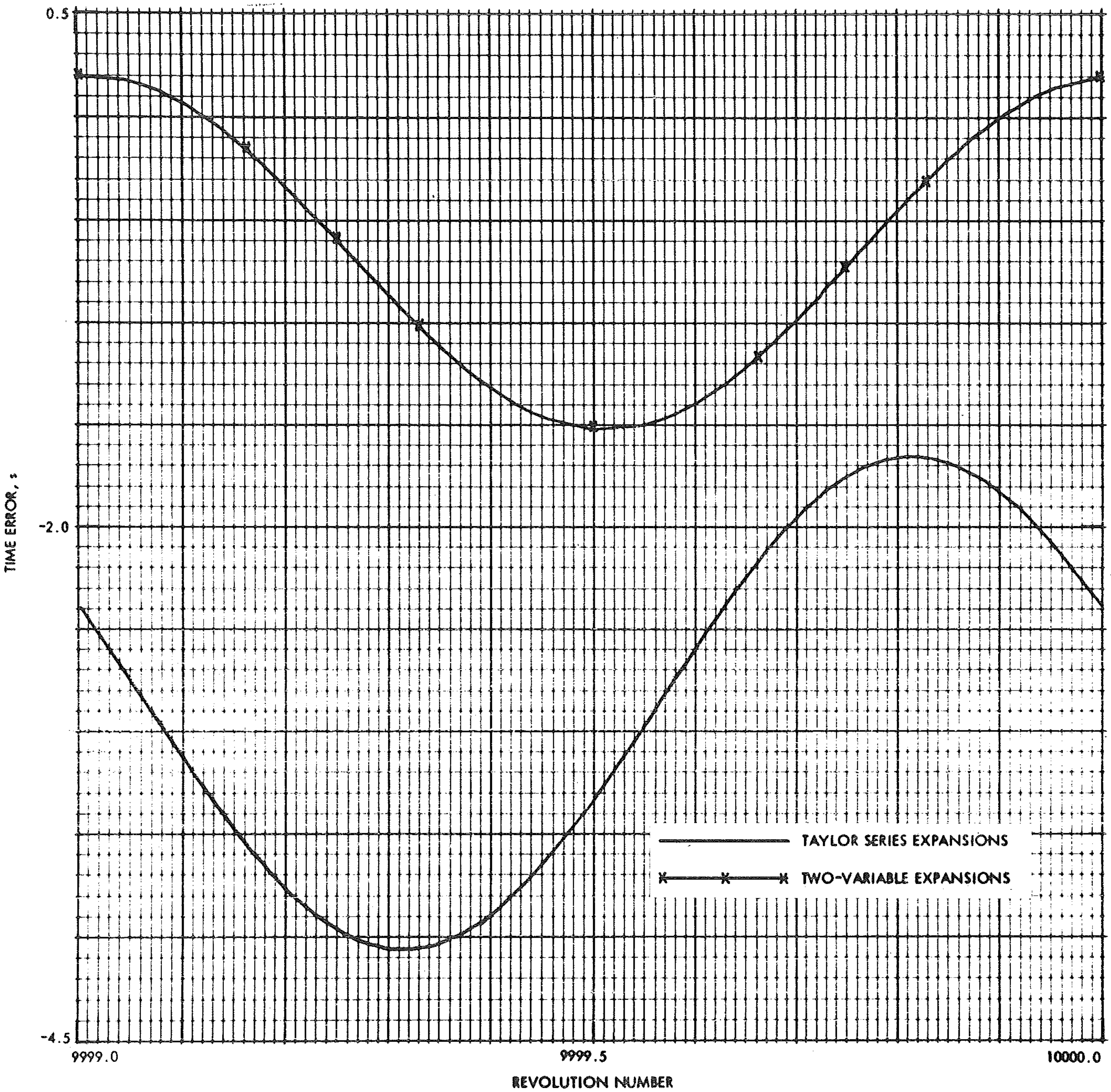


Fig. G-28. Time error in revolution 10,000, general perturbations solutions ($e_0 = 0$, $q_0 = 3578$ km, $v_0 = 180$ deg)

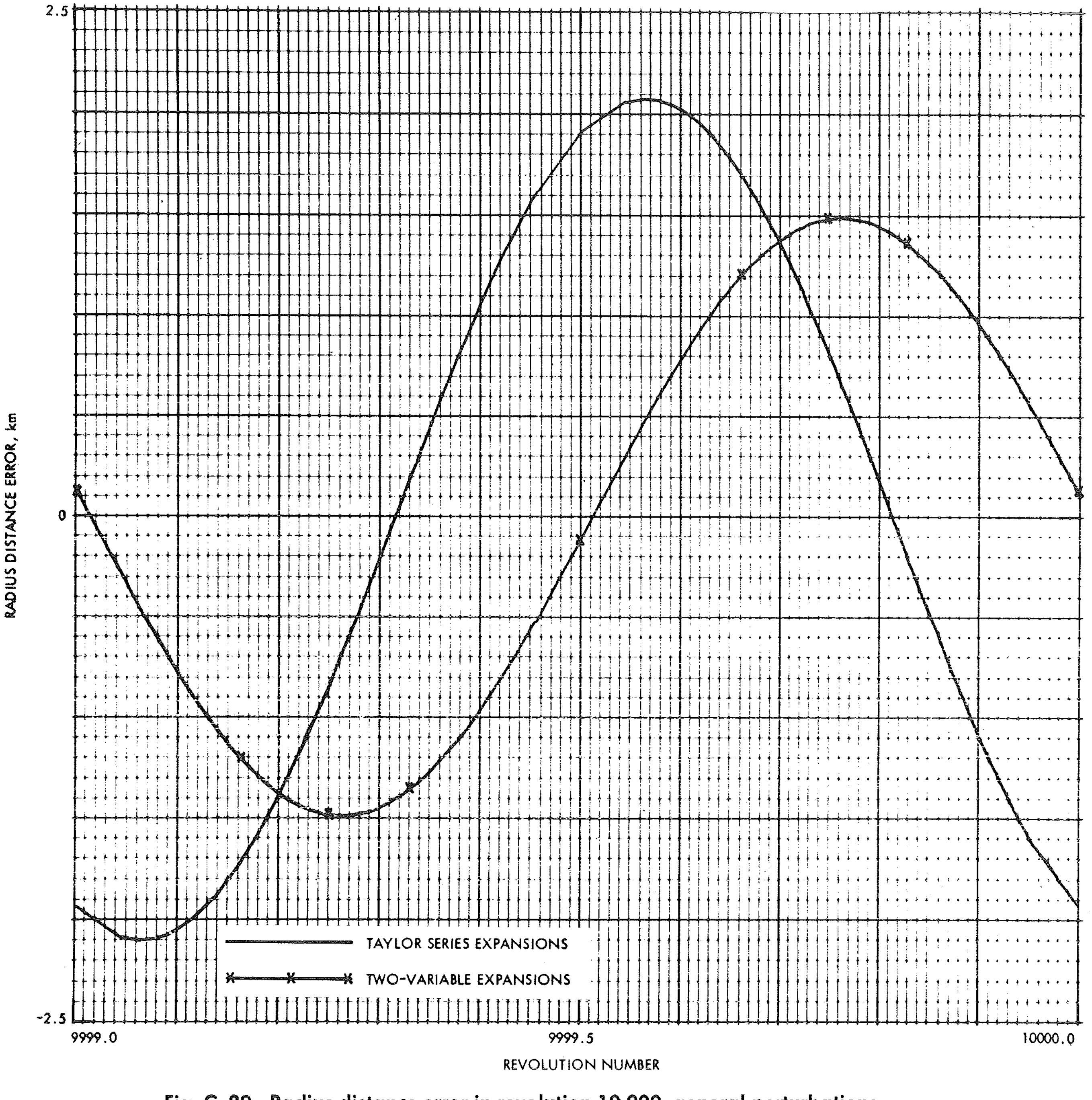


Fig. G-29. Radius distance error in revolution 10,000, general perturbations solutions ($e_0 = 0$, $q_0 = 3578$ km, $v_0 = 180$ deg)

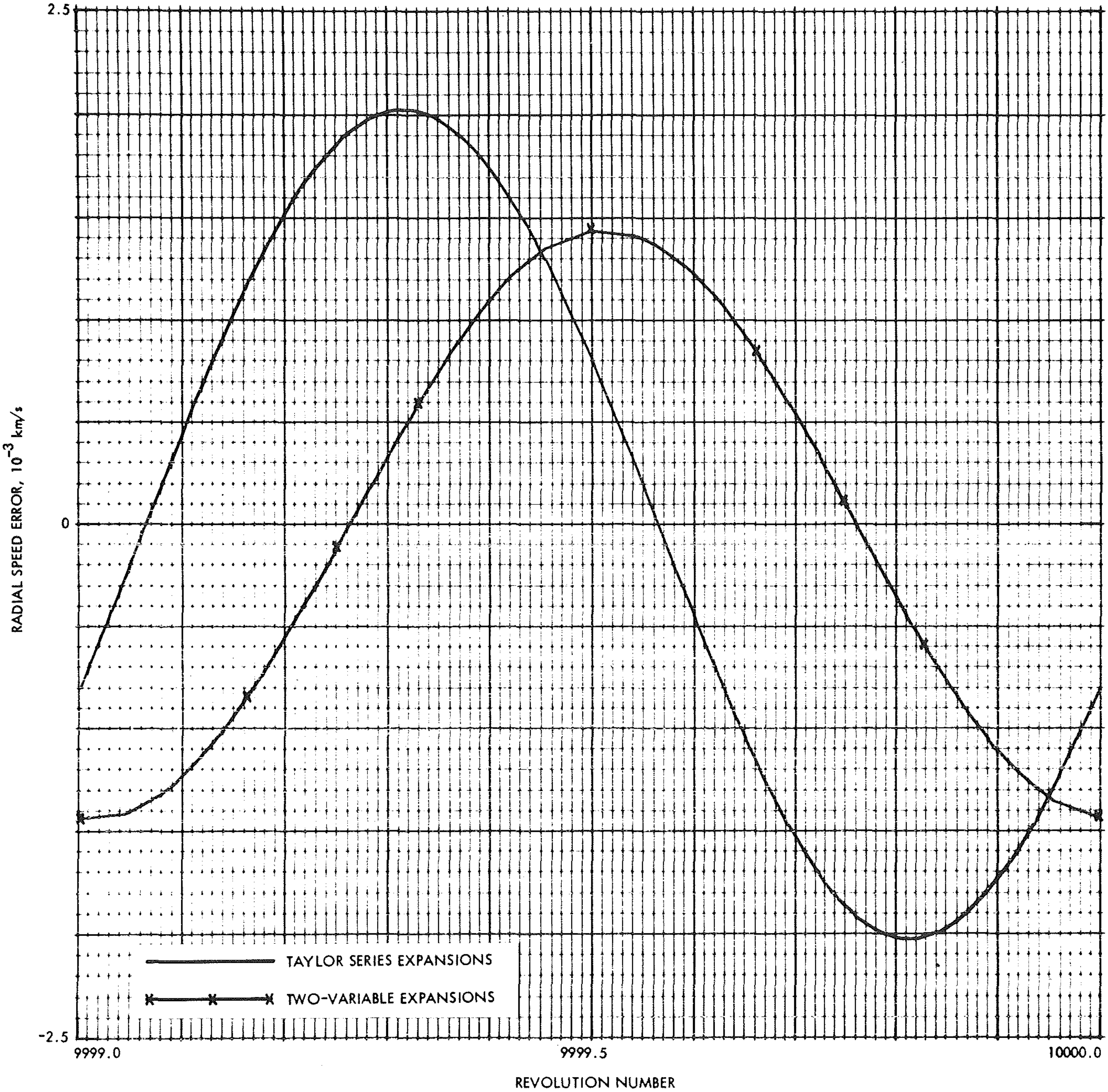


Fig. G-30. Radial speed error in revolution 10,000, general perturbations solutions ($e_0 = 0$, $q_0 = 3578$ km, $v_0 = 180$ deg)

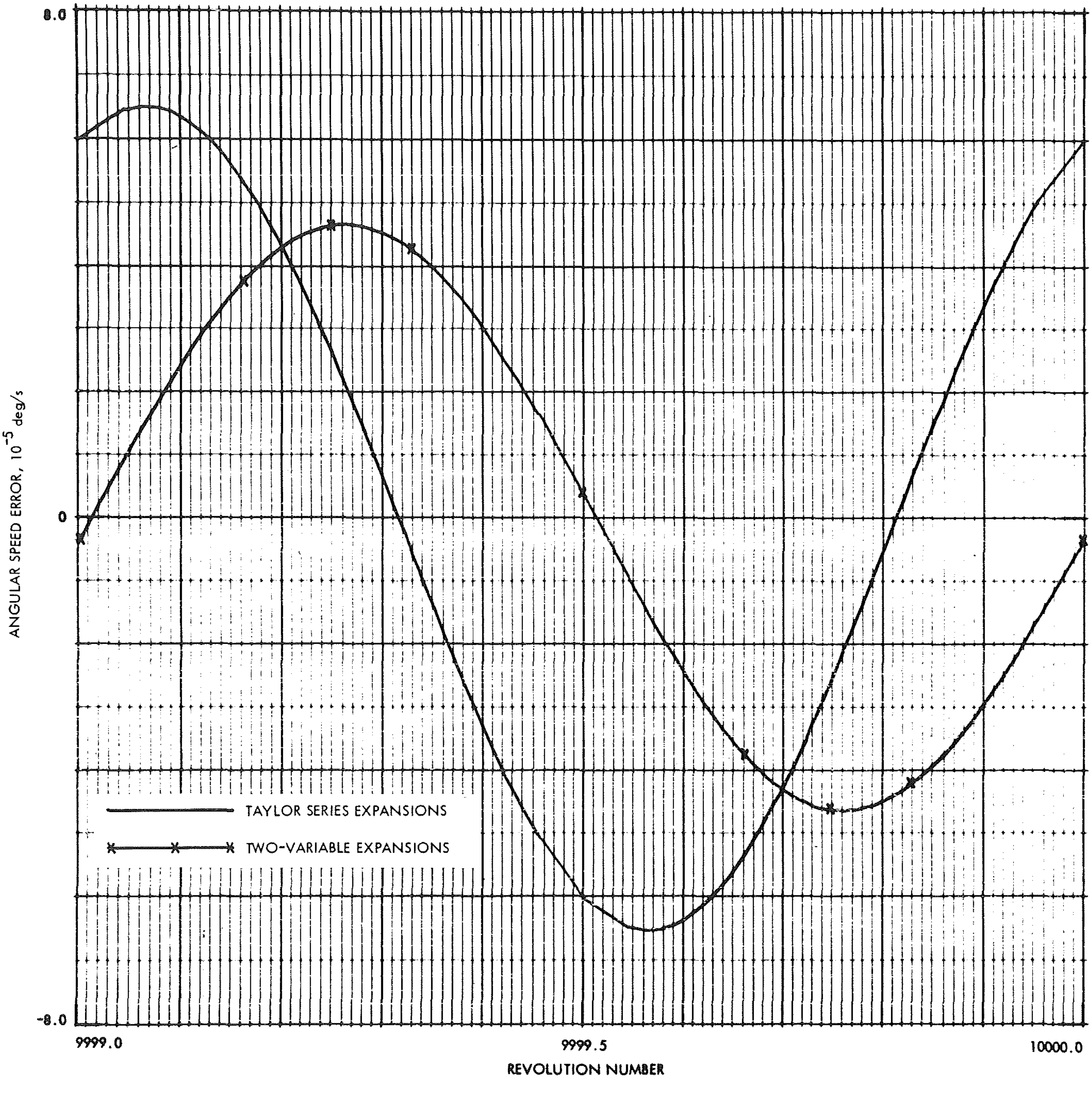


Fig. G-31. Angular speed error in revolution 10,000, general perturbations solutions ($e_0 = 0$, $q_0 = 3578$ km, $v_0 = 180$ deg)

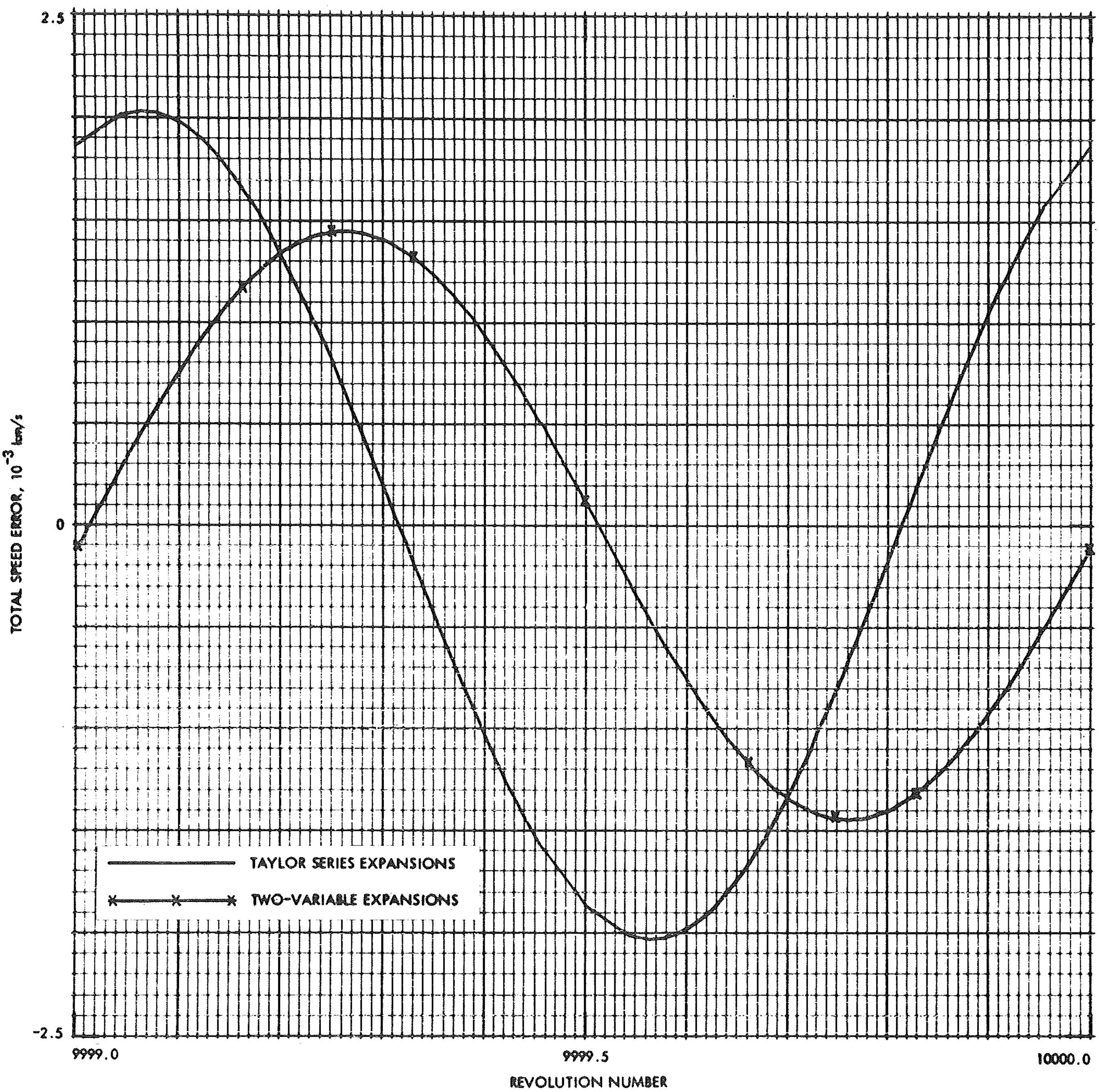


Fig. G-32. Total speed error in revolution 10,000, general perturbations solutions ($e_0 = 0$, $q_0 = 3578$ km, $v_0 = 180$ deg)

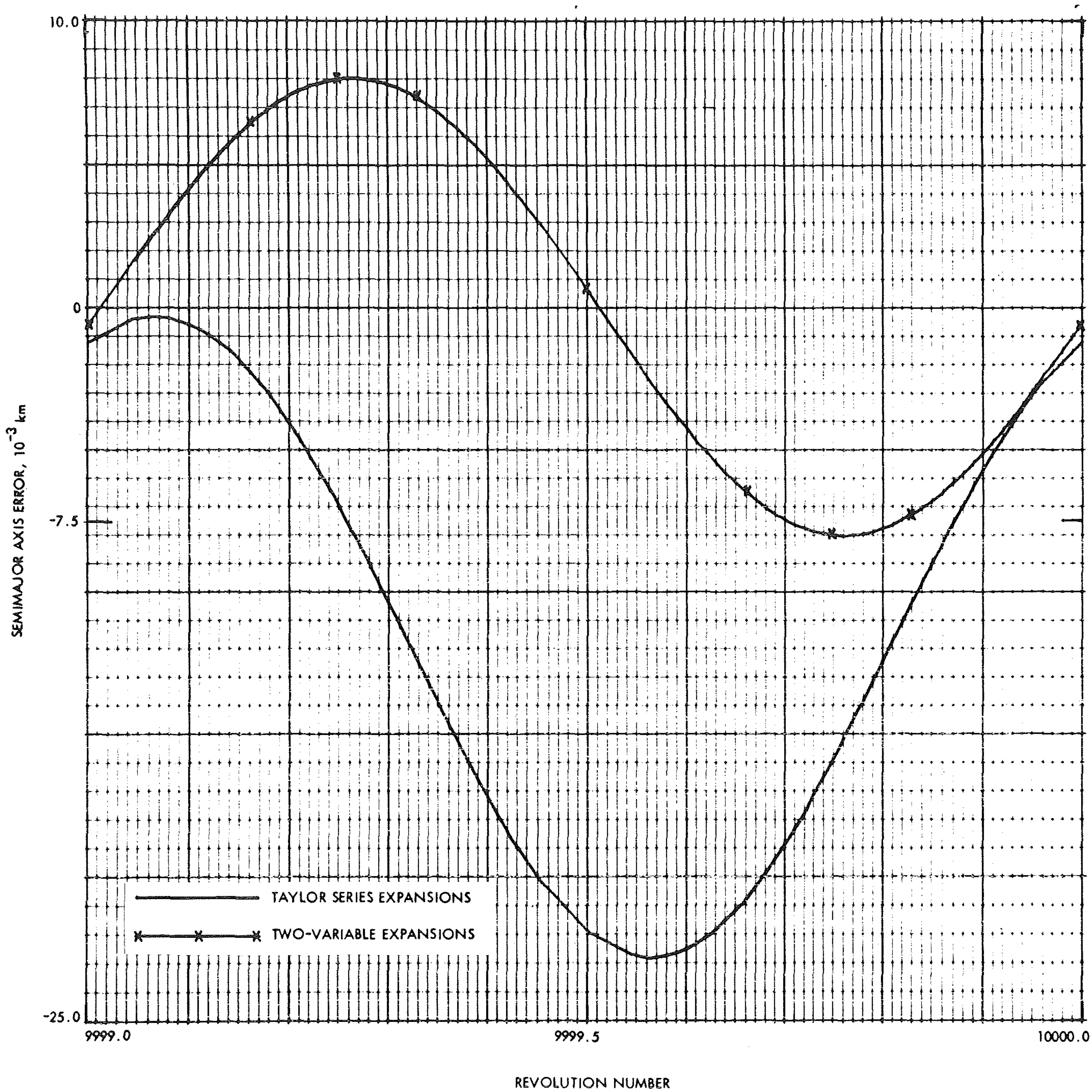


Fig. G-33. Semimajor axis error in revolution 10,000, general perturbations solutions ($e_0 = 0$, $q_0 = 3578$ km, $v_0 = 180$ deg)

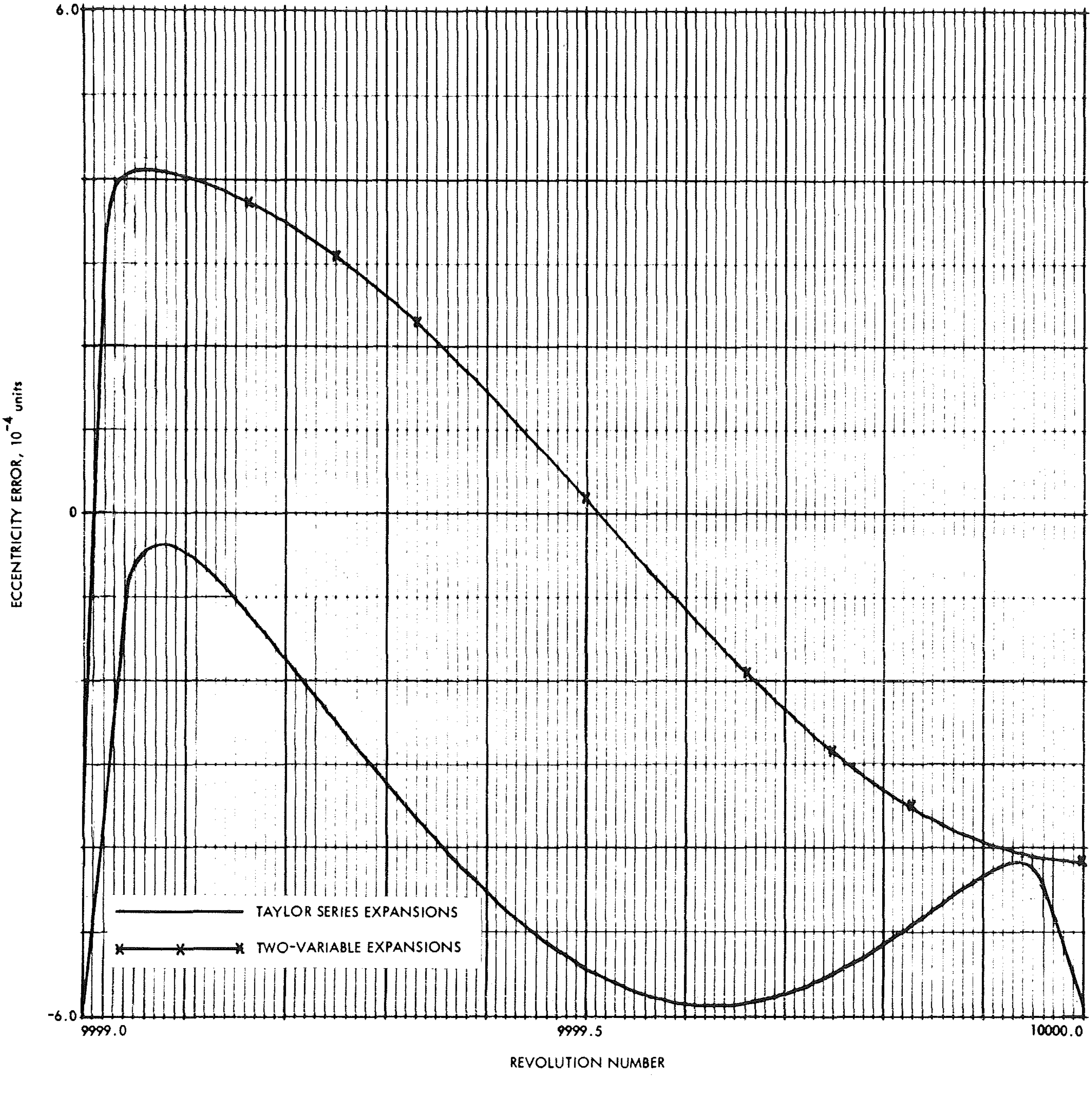


Fig. G-34. Eccentricity error in revolution 10,000, general perturbations solutions ($e_0 = 0$, $q_0 = 3578$ km, $v_0 = 180$ deg)

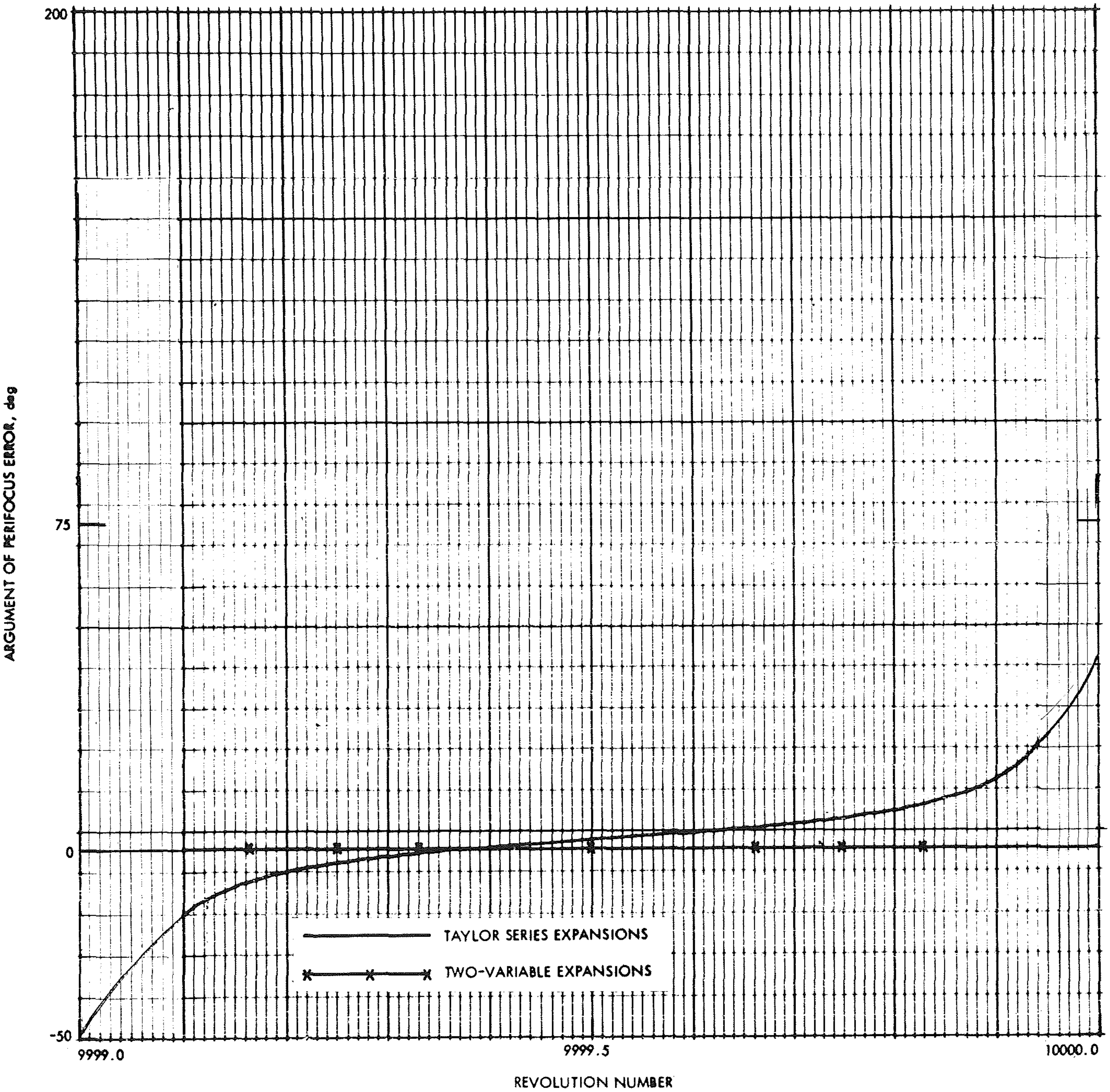


Fig. G-35. Argument of perifocus error in revolution 10,000, general perturbations solutions ($e_0 = 0$, $q_0 = 3578$ km, $v_0 = 180$ deg)

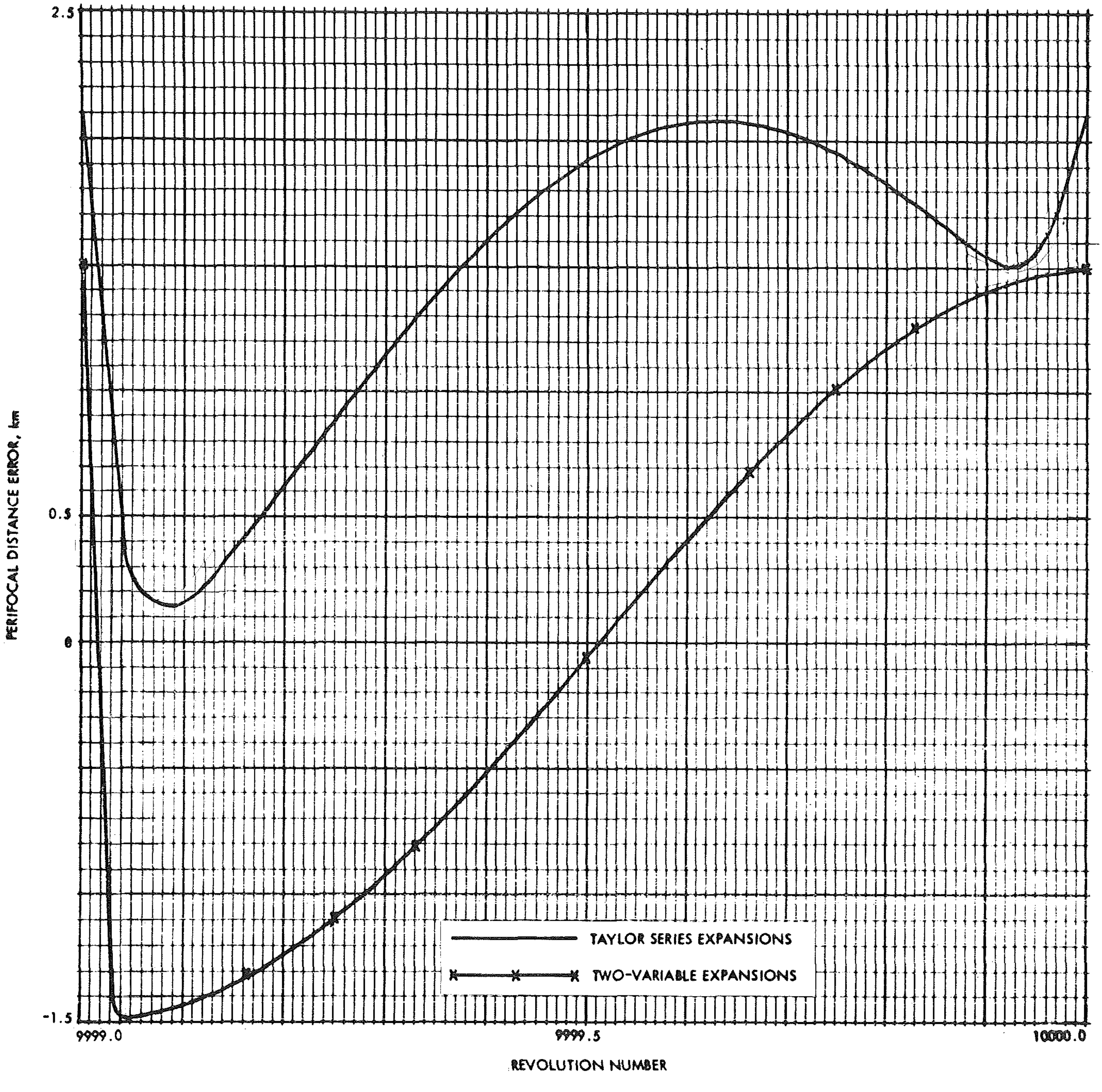


Fig. G-36. Perifocal distance error in revolution 10,000, general perturbations solutions ($e_0 = 0$, $q_0 = 3578$ km, $v_0 = 180$ deg)

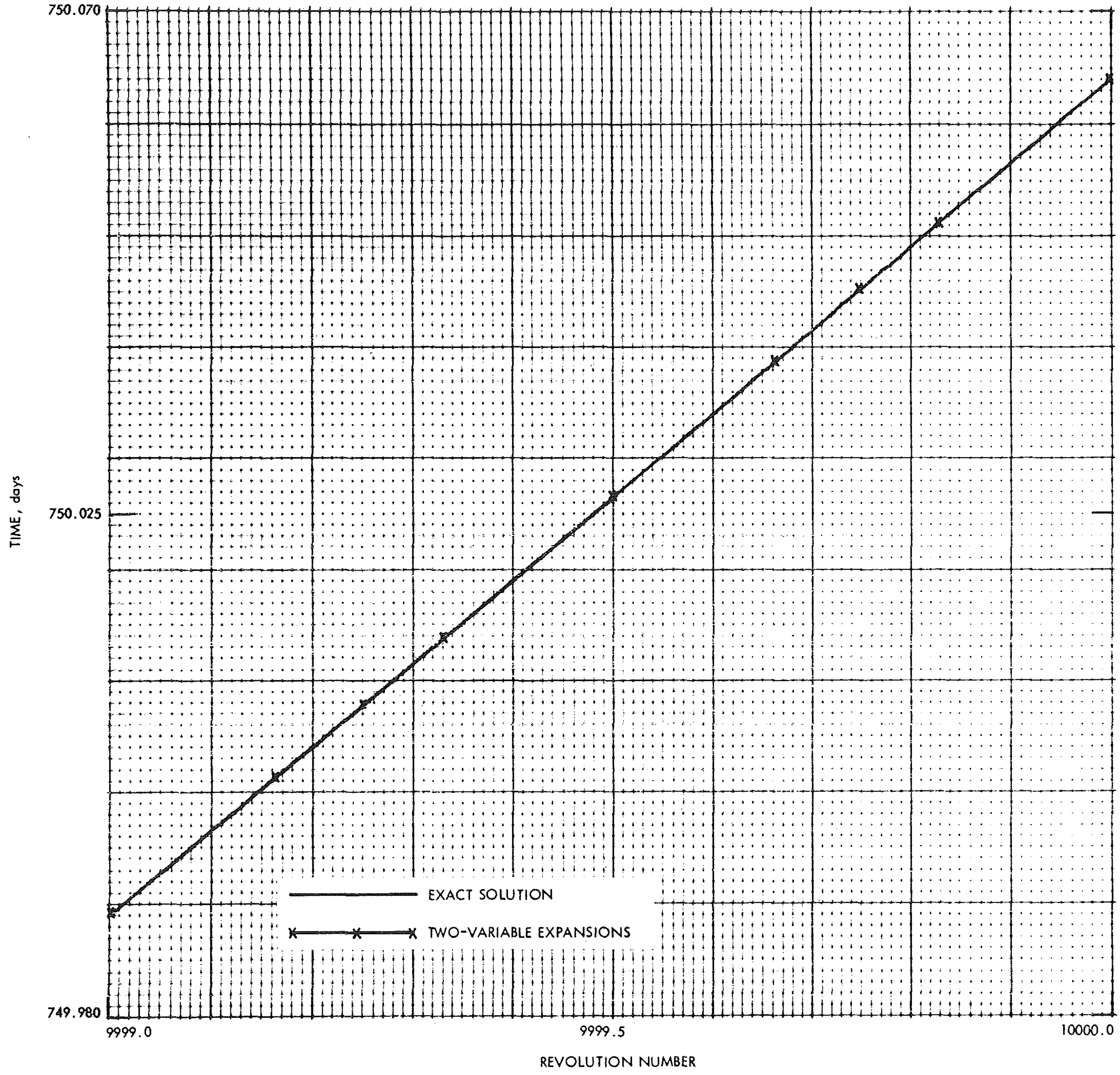


Fig. G-37. Variation of time in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0$, $q_0 = 3578$ km, $v_0 = 180$ deg)

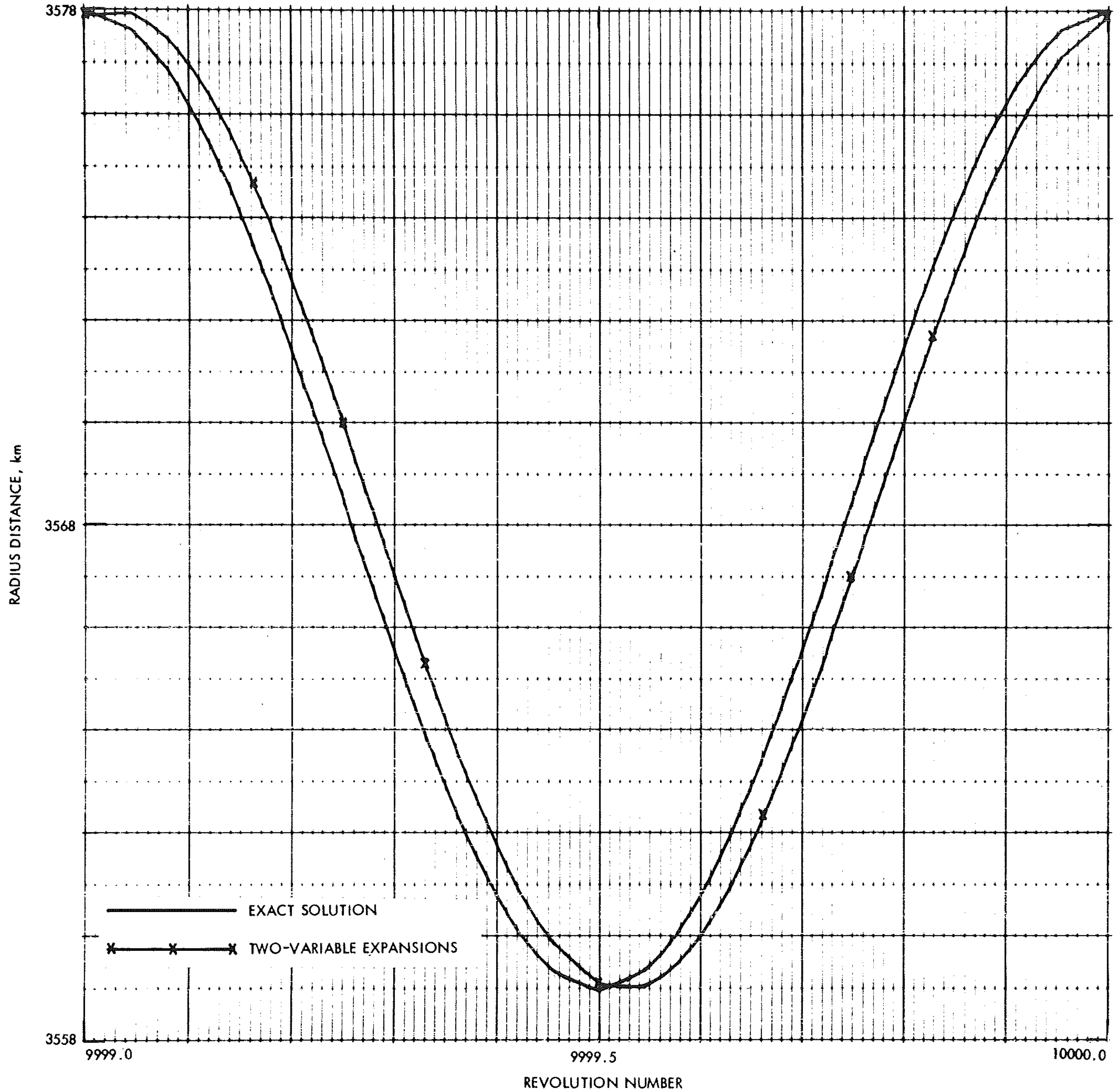


Fig. G-38. Variation of radius distance in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0$, $q_0 = 3578$ km, $v_0 = 180$ deg)

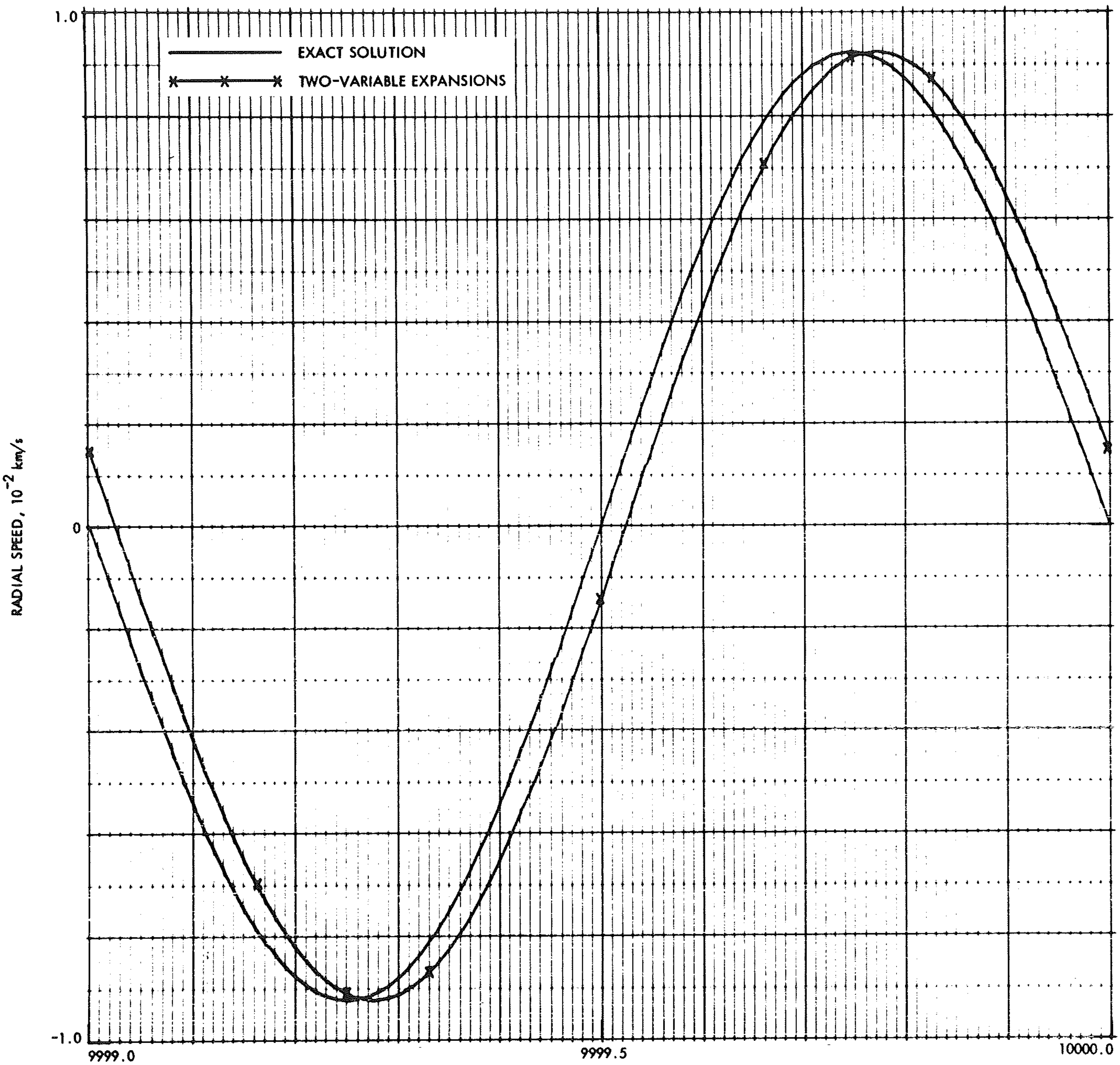


Fig. G-39. Variation of radial speed in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0$, $q_0 = 3578$ km, $v_0 = 180$ deg)

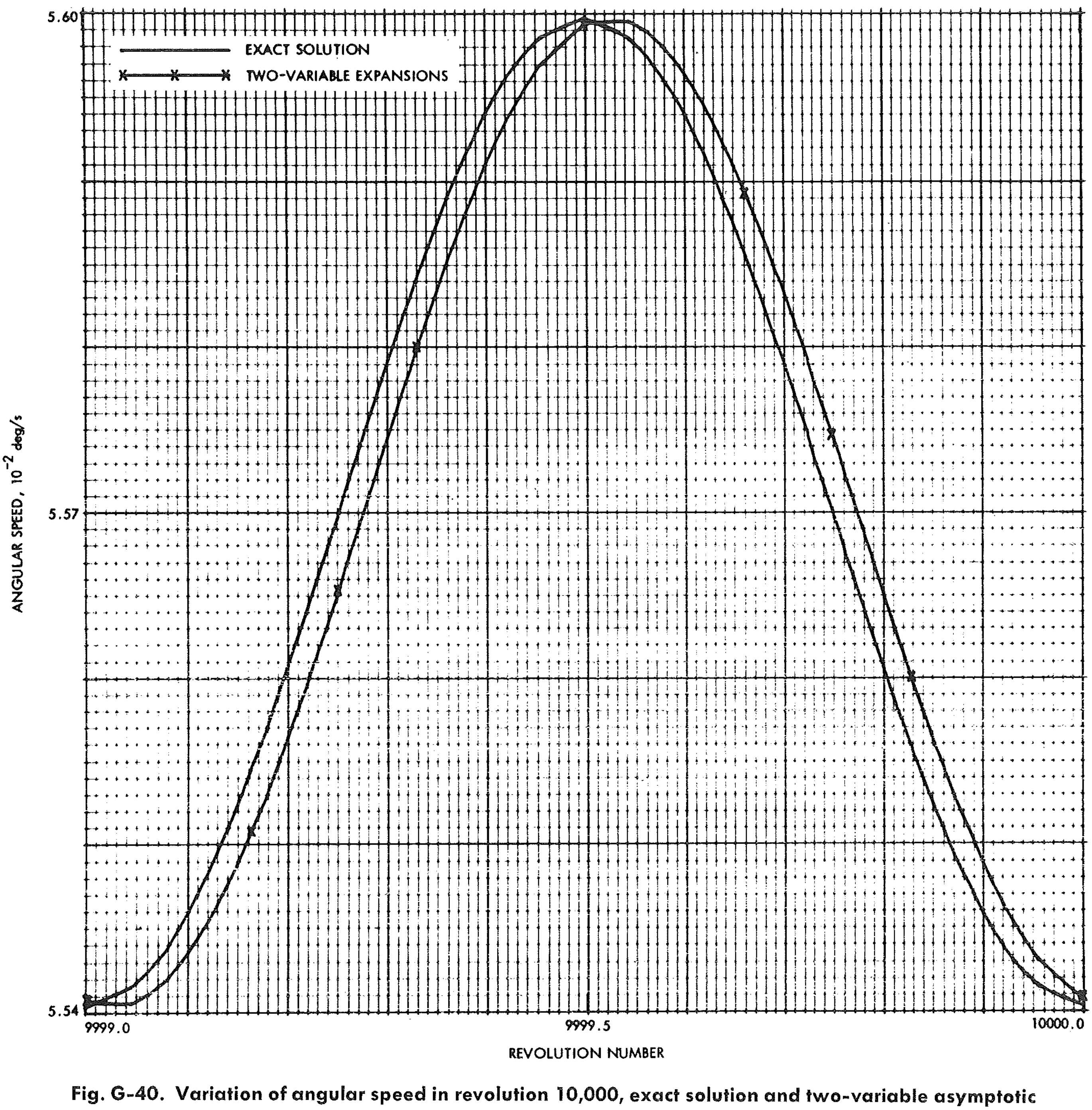


Fig. G-40. Variation of angular speed in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0$, $q_0 = 3578$ km, $v_0 = 180$ deg)

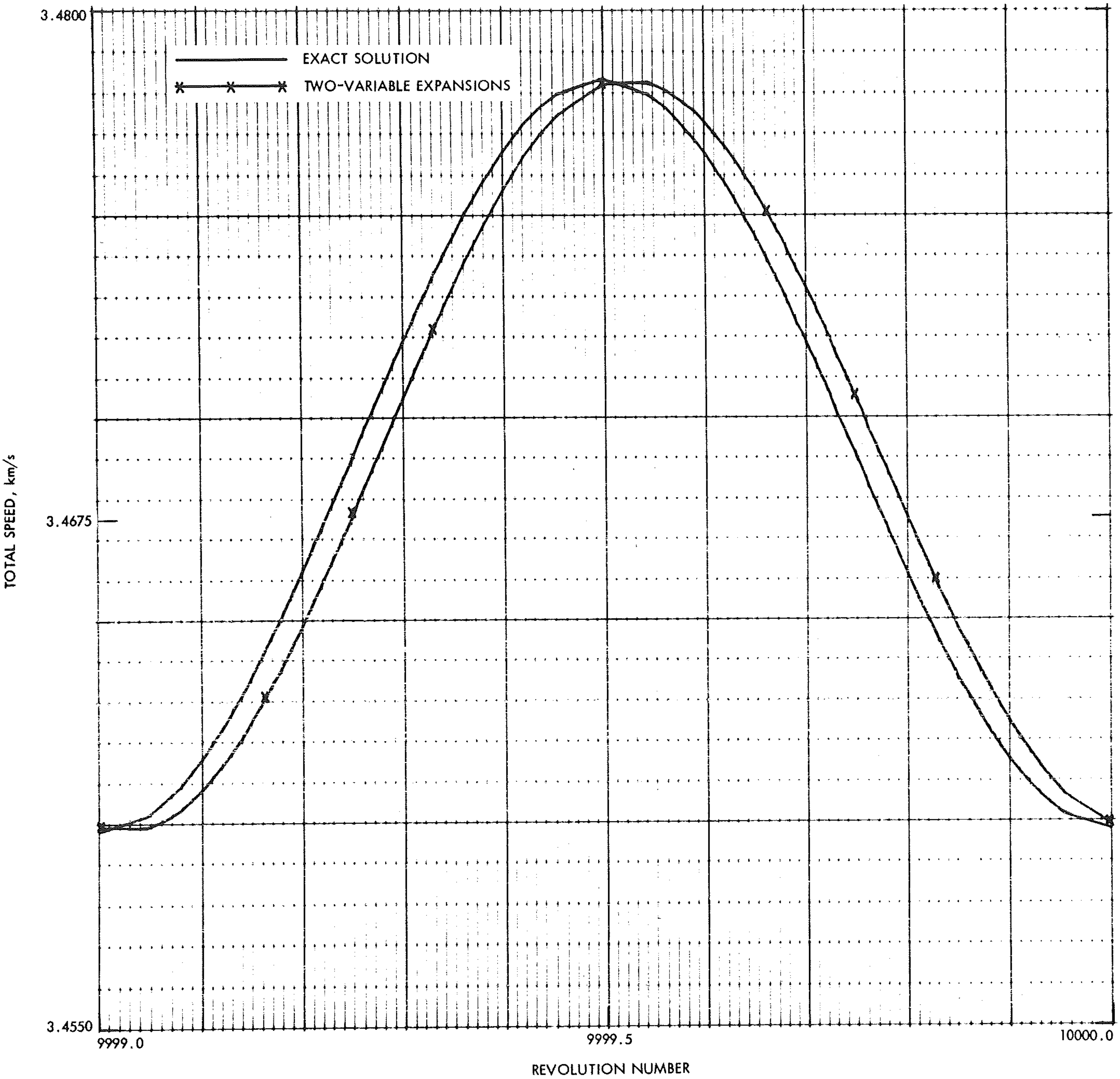


Fig. G-41. Variation of total speed in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0$, $q_0 = 3578$ km, $v_0 = 180$ deg)

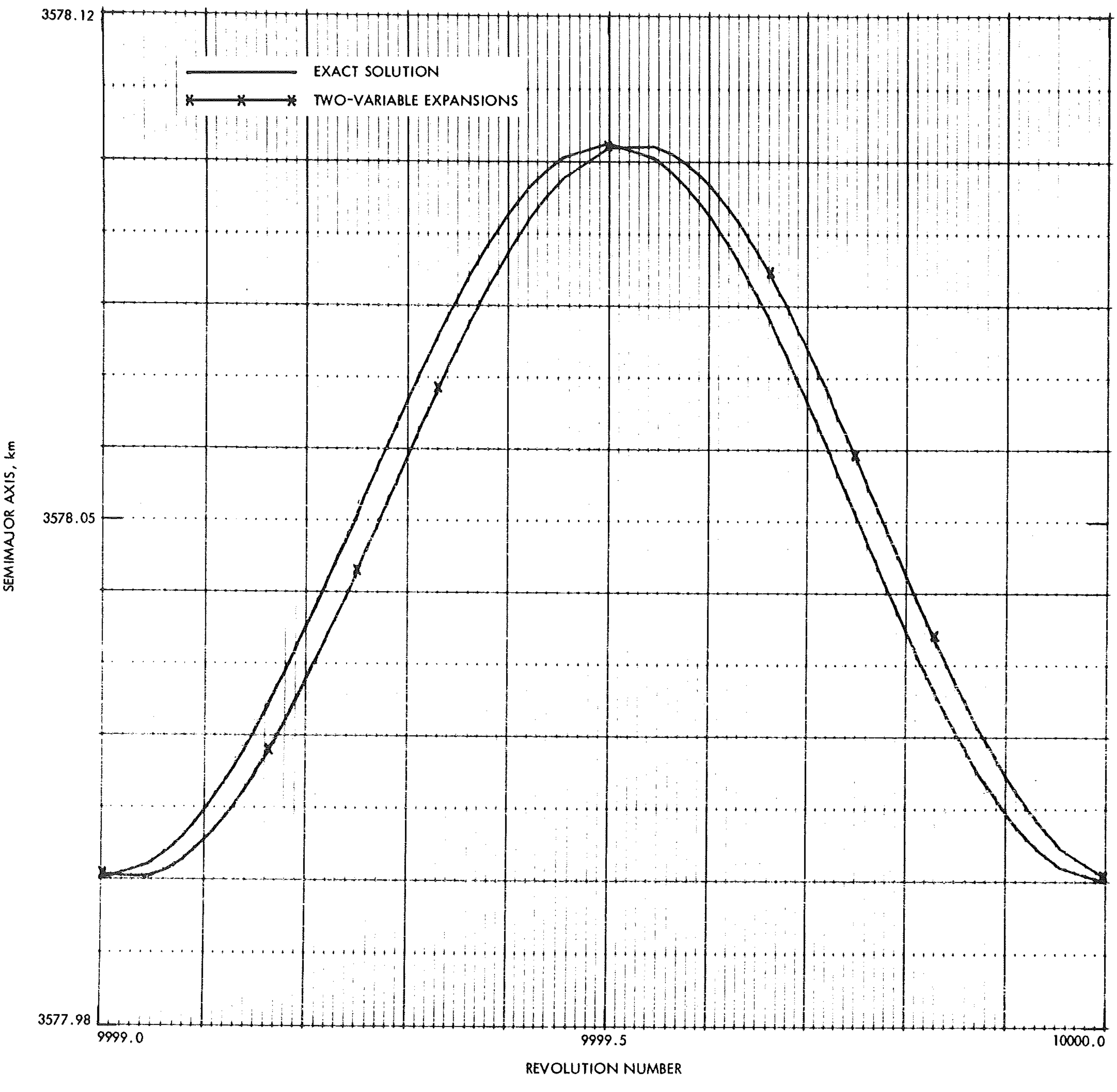


Fig. G-42. Variation of semimajor axis in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0$, $q_0 = 3578$ km, $v_0 = 180$ deg)

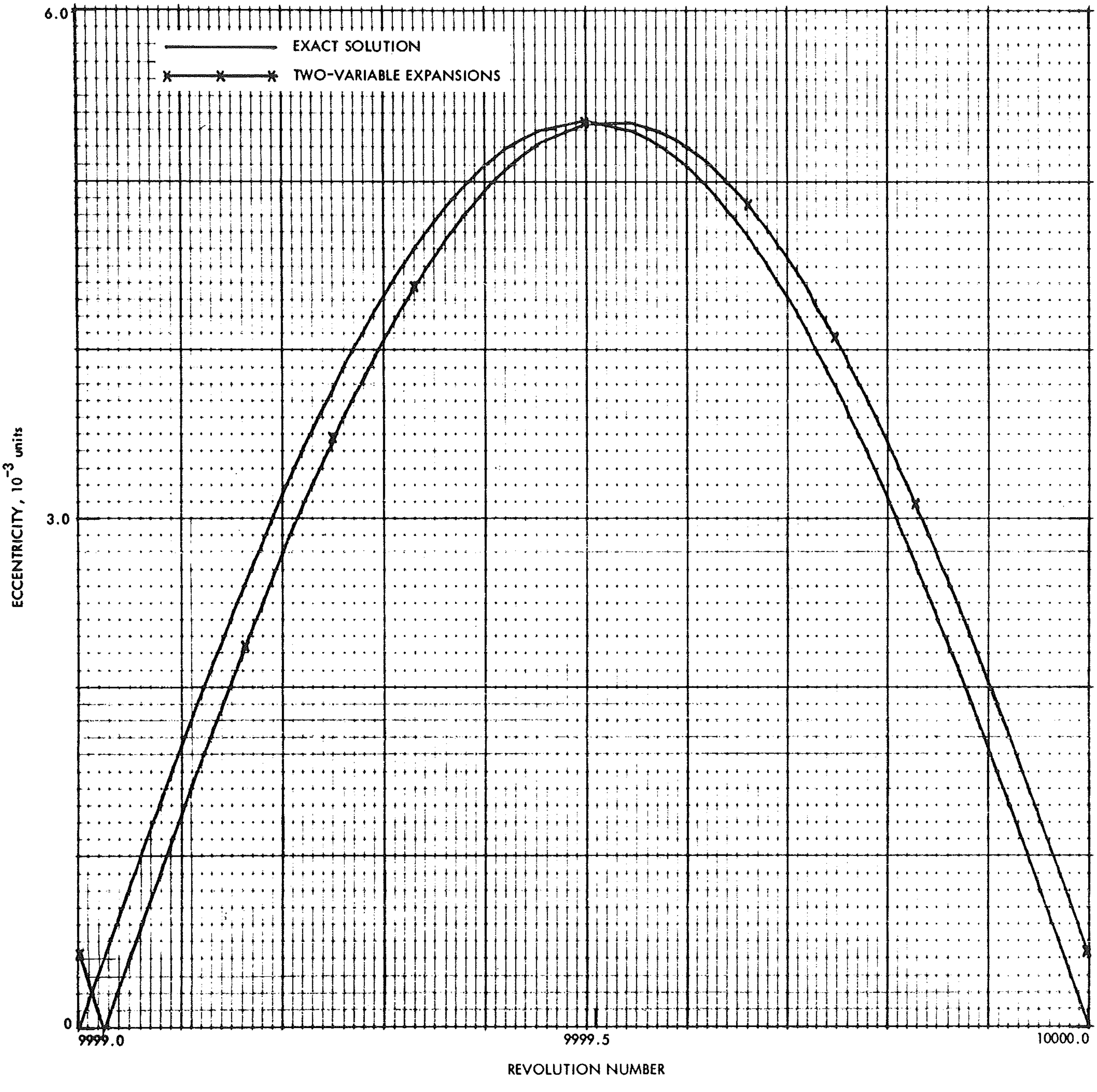


Fig. G-43. Variation of eccentricity in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0$, $q_0 = 3578$ km, $v_0 = 180$ deg)

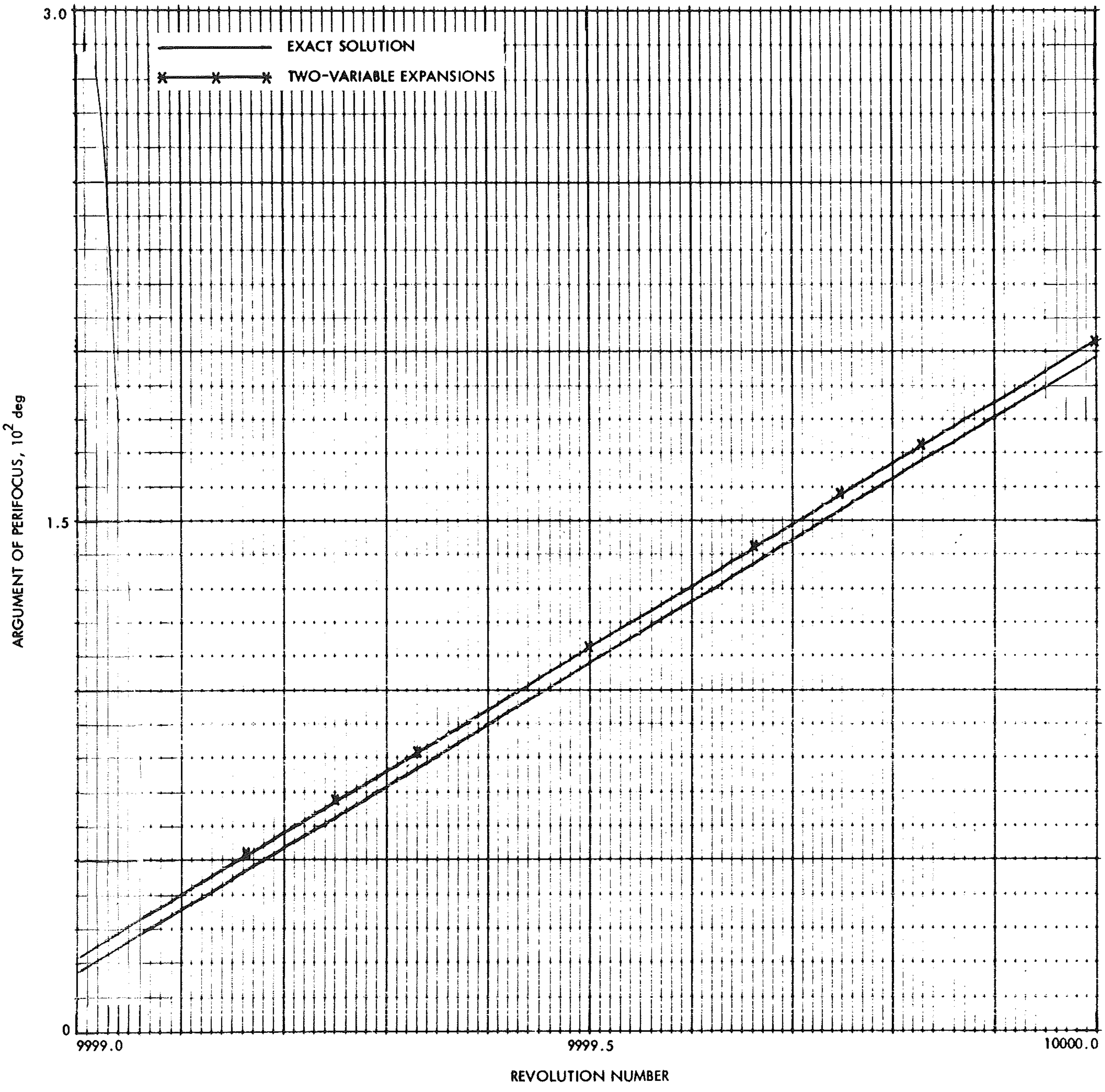


Fig. G-44. Variation of argument of perifocus in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0$, $q_0 = 3578$ km, $v_0 = 180$ deg)

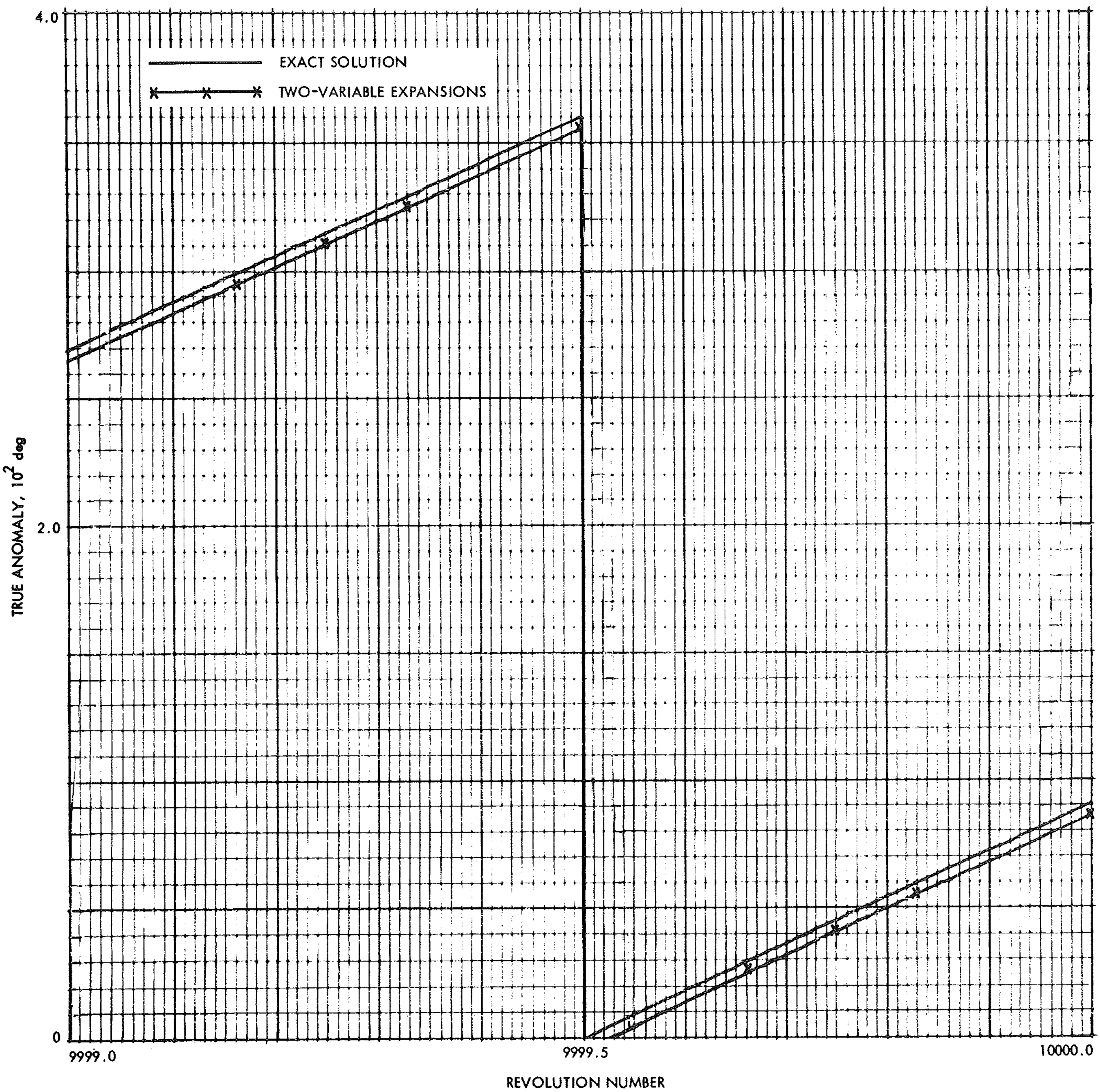


Fig. G-45. Variation of true anomaly in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0$, $q_0 = 3578$ km, $v_0 = 180$ deg)

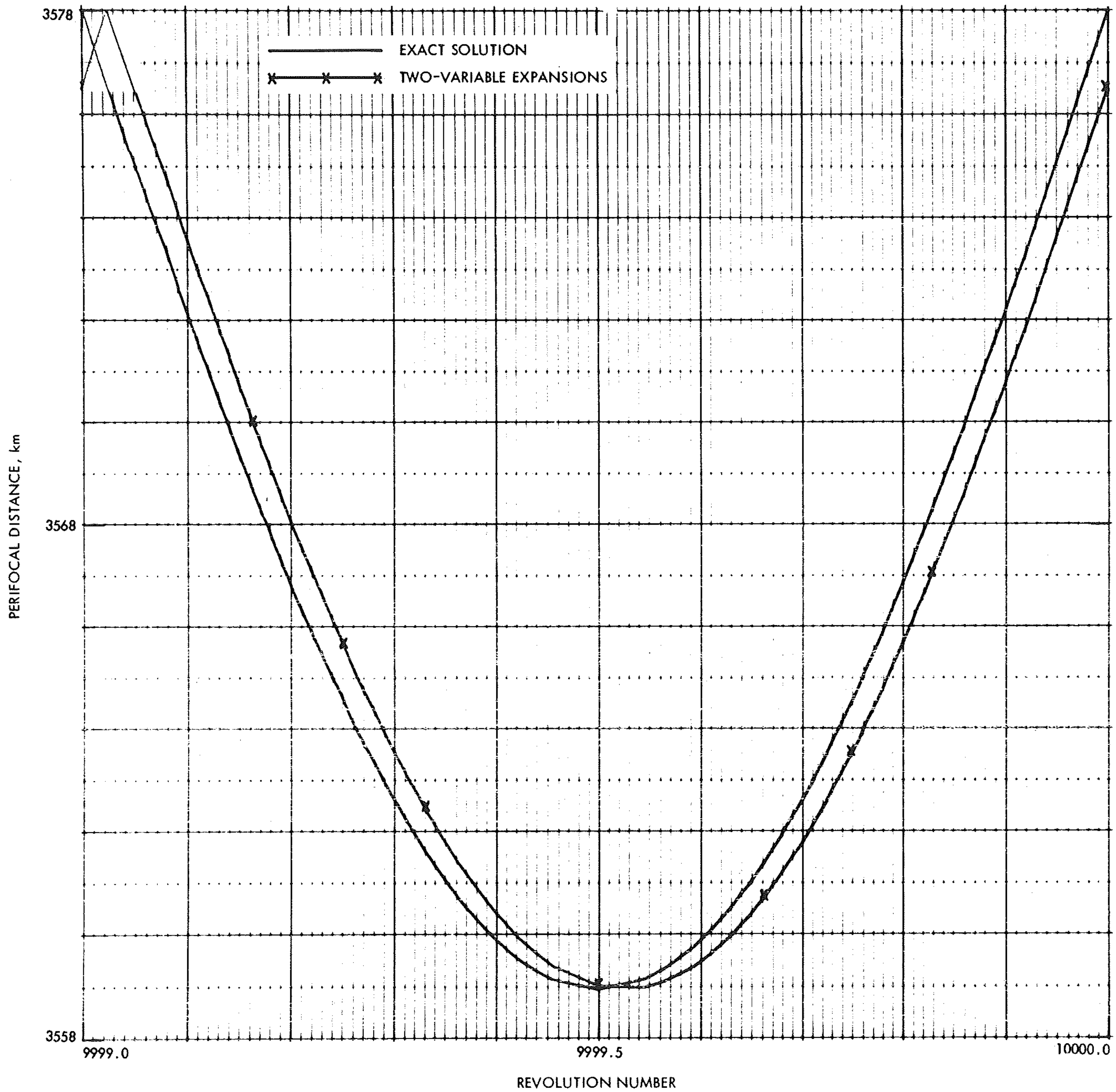


Fig. G-46. Variation of perifocal distance in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0$, $q_0 = 3578$ km, $v_0 = 180$ deg)

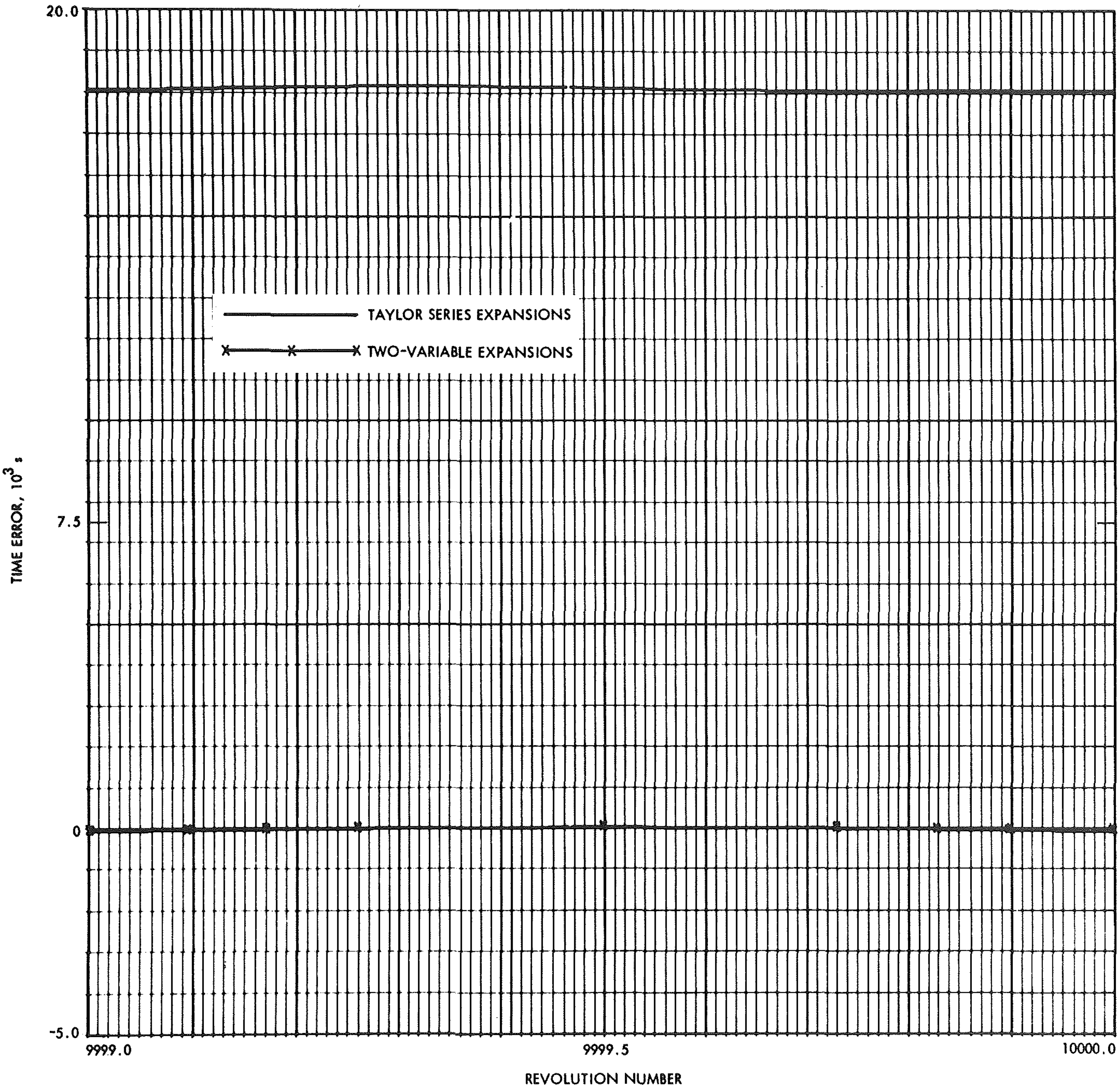


Fig. G-47. Time error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 3578$ km, $v_0 = 0$ deg)

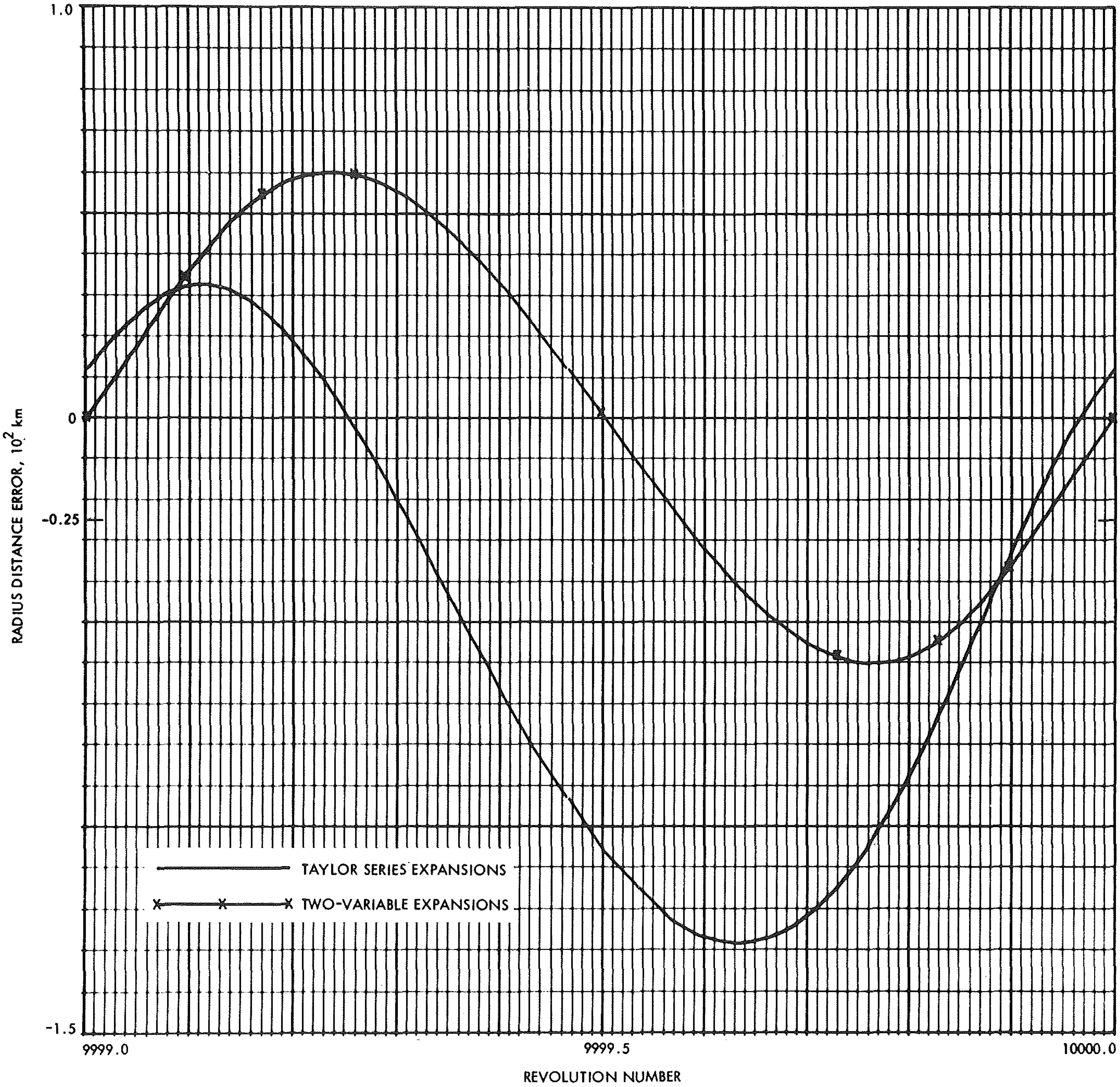


Fig. G-48. Radius distance error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 3578$ km, $v_0 = 0$ deg)

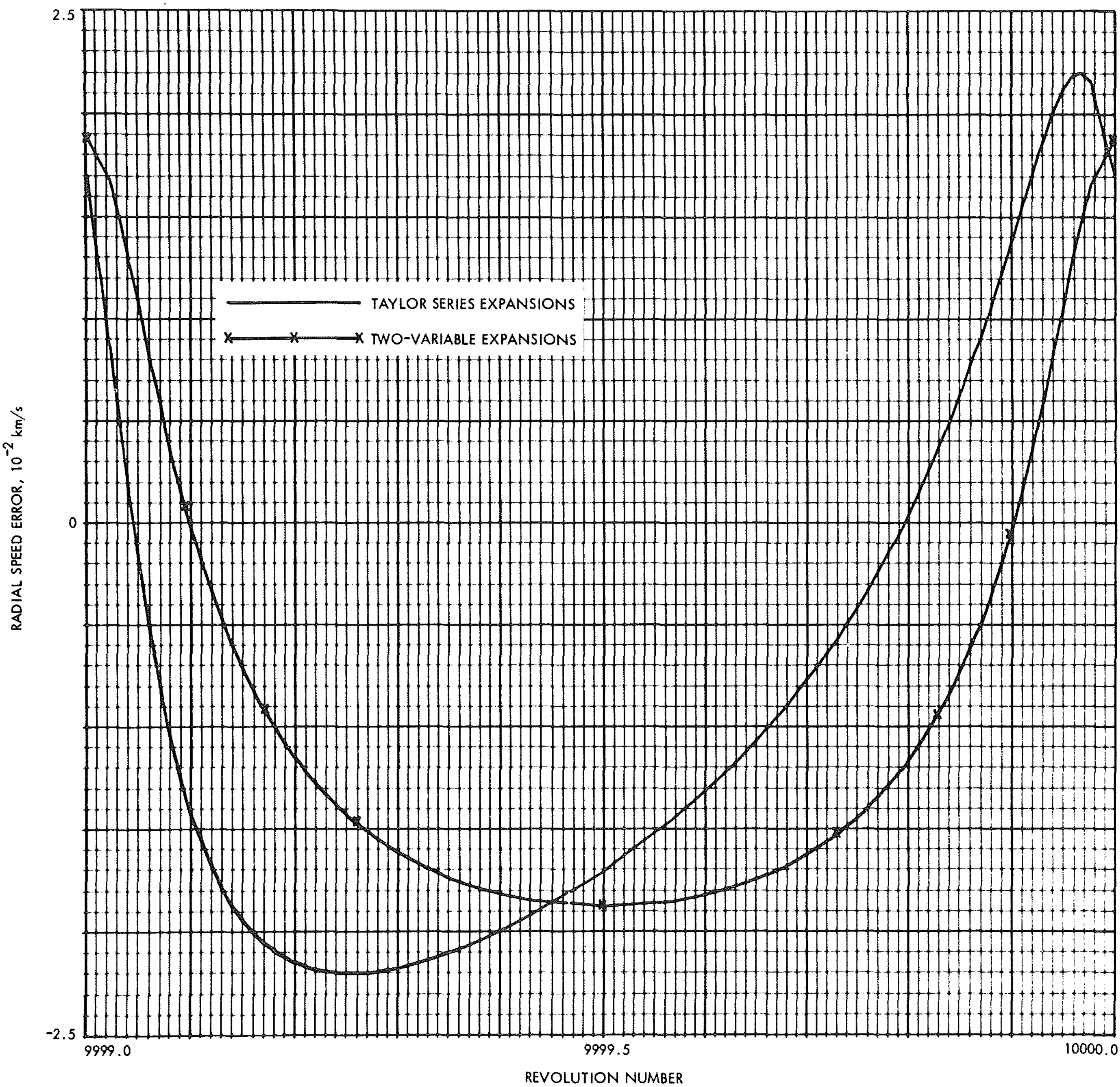


Fig. G-49. Radial speed error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 3578$ km, $v_0 = 0$ deg)

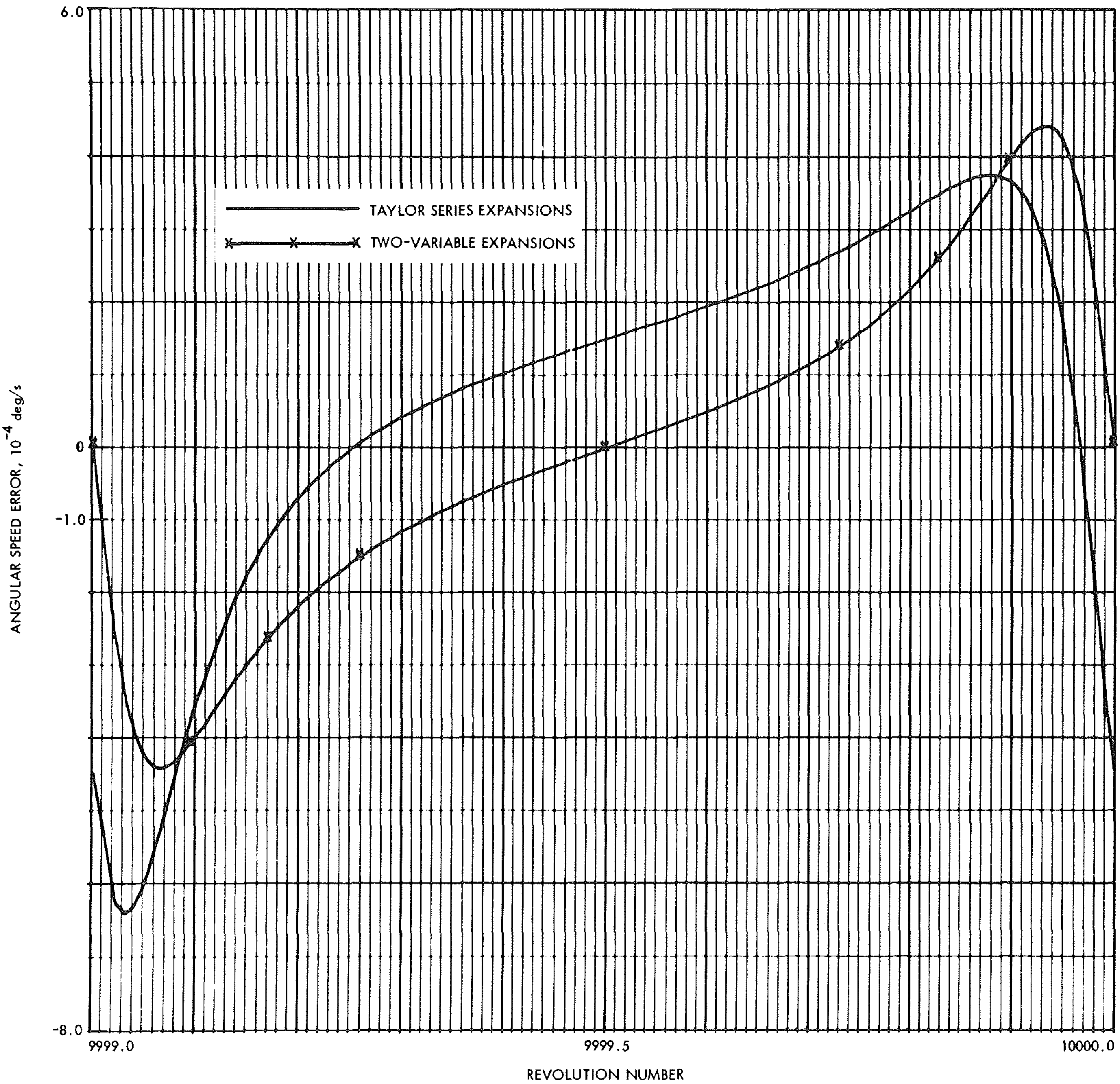


Fig. G-50. Angular speed error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 3578$ km, $v_0 = 0$ deg)

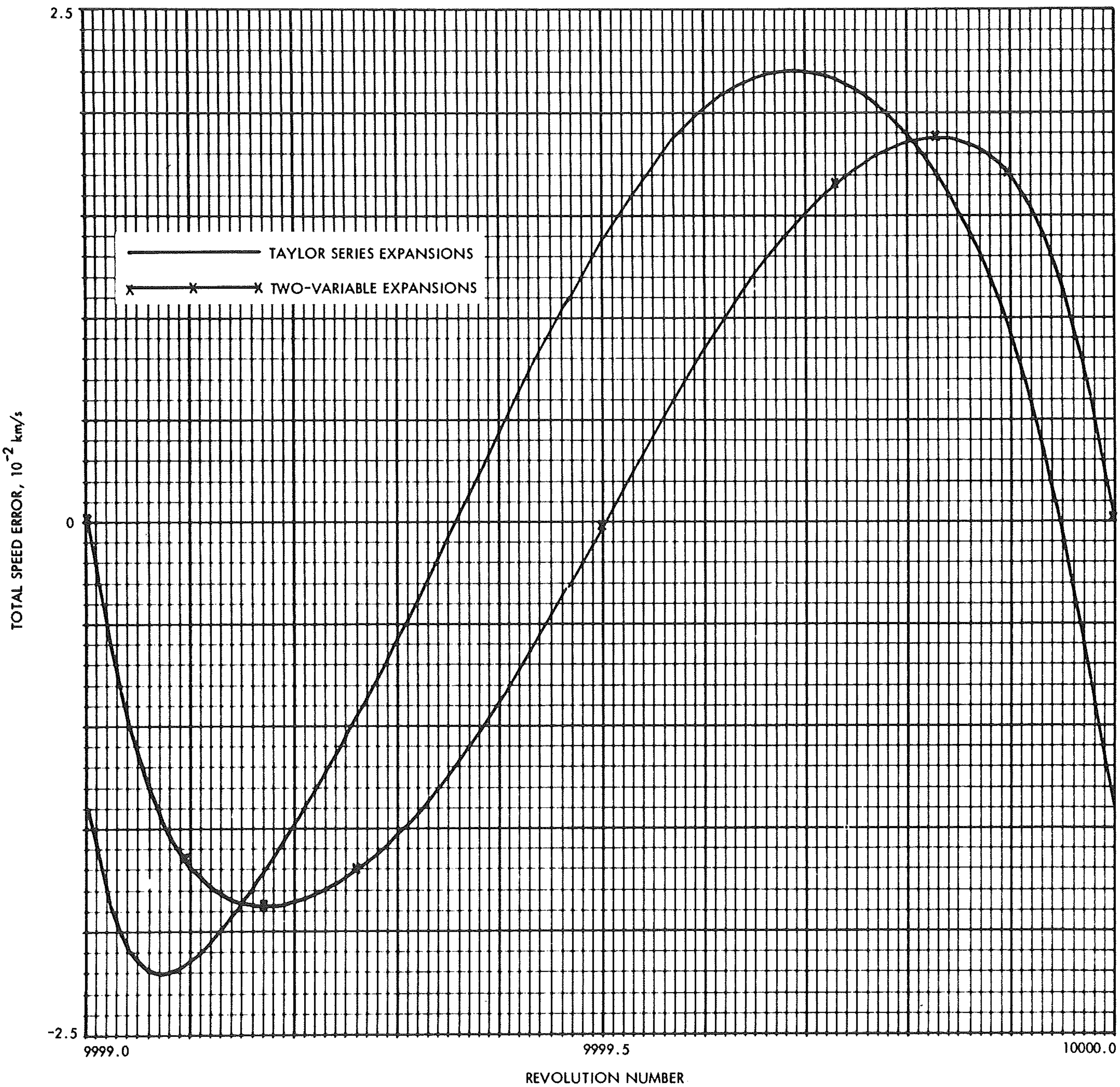


Fig. G-51. Total speed error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 3578$ km, $v_0 = 0$ deg)

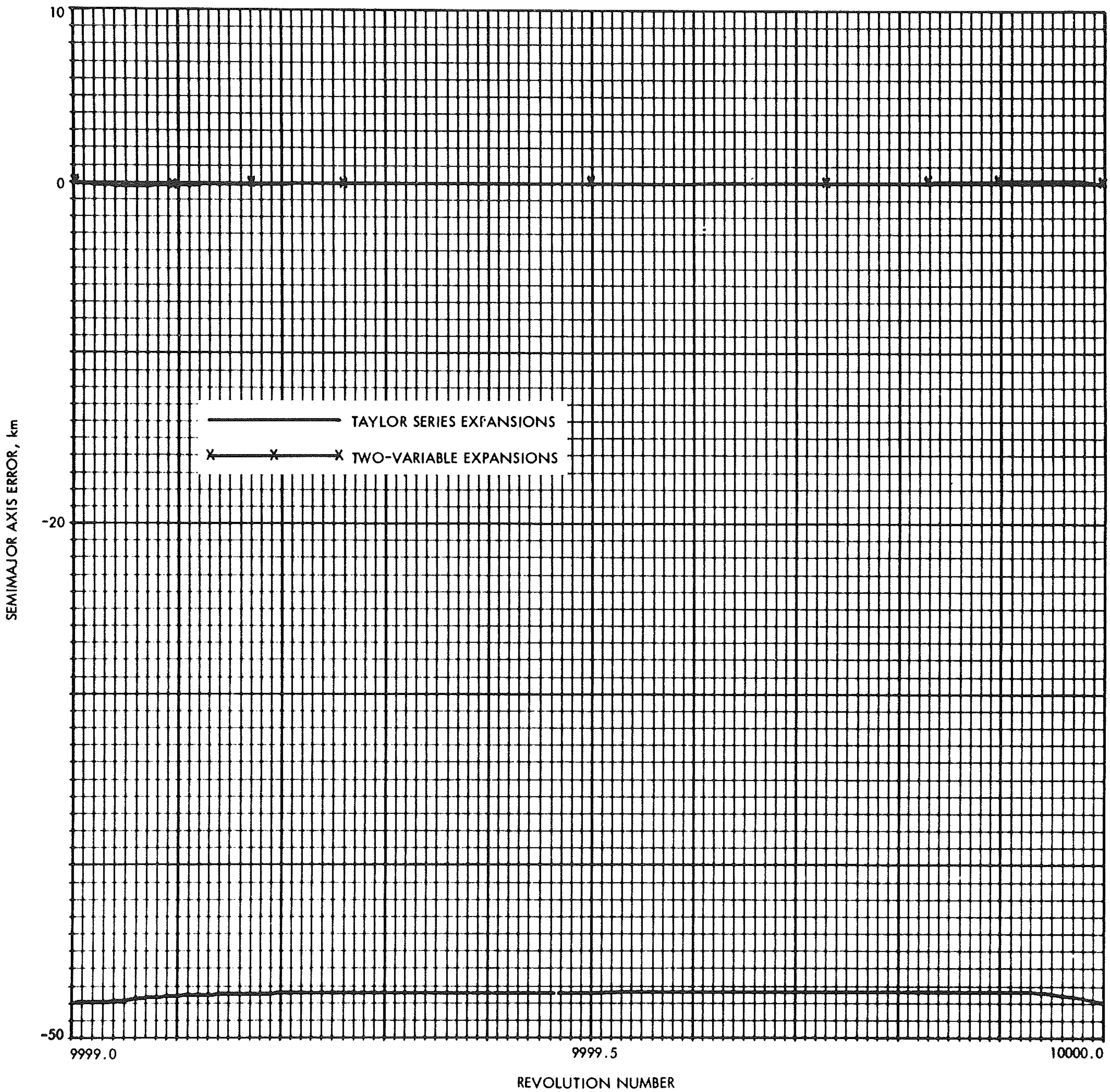


Fig. G-52. Semimajor axis error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 3578$ km, $v_0 = 0$ deg)

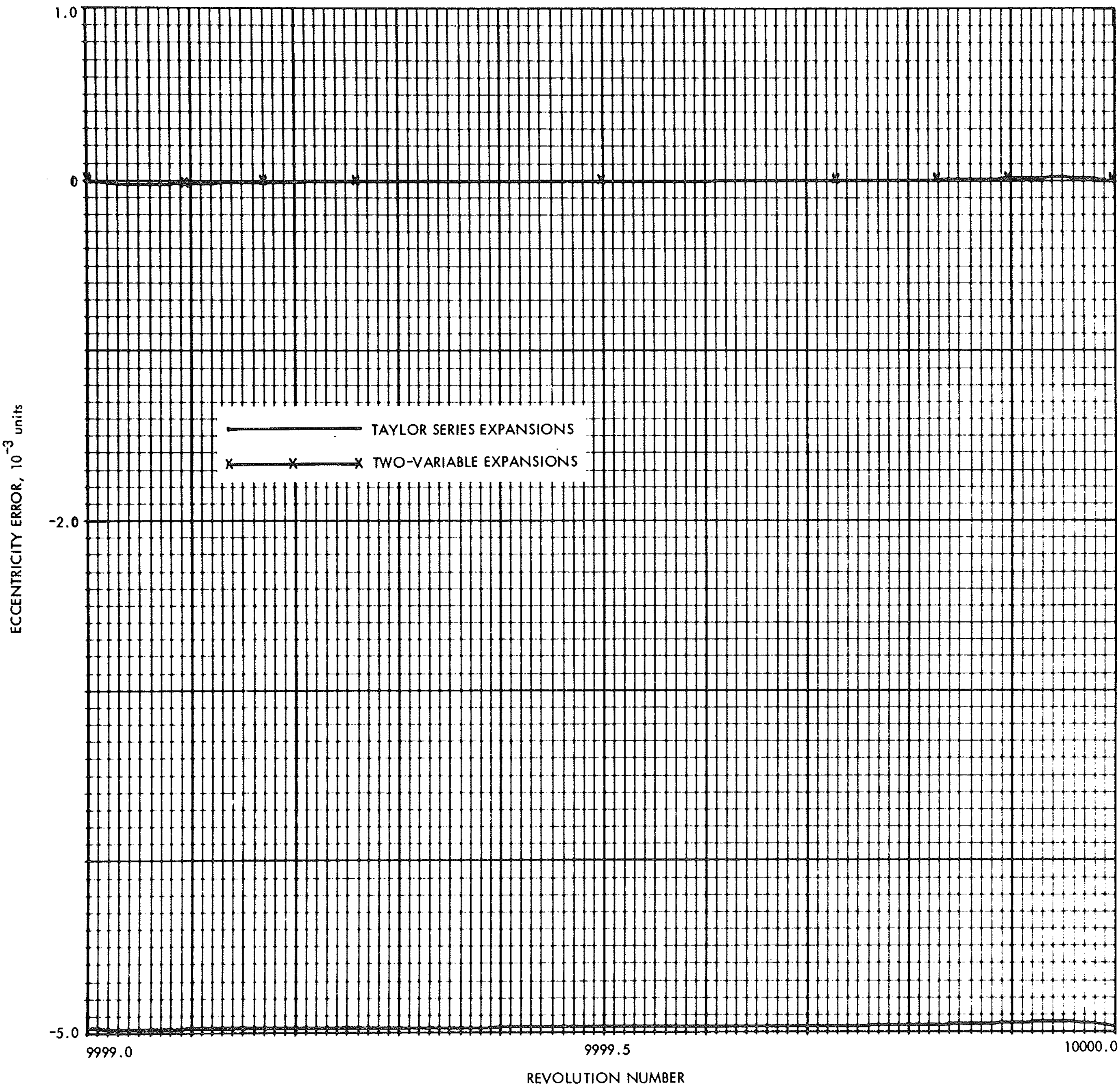


Fig. G-53. Eccentricity error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 3578$ km, $v_0 = 0$ deg)

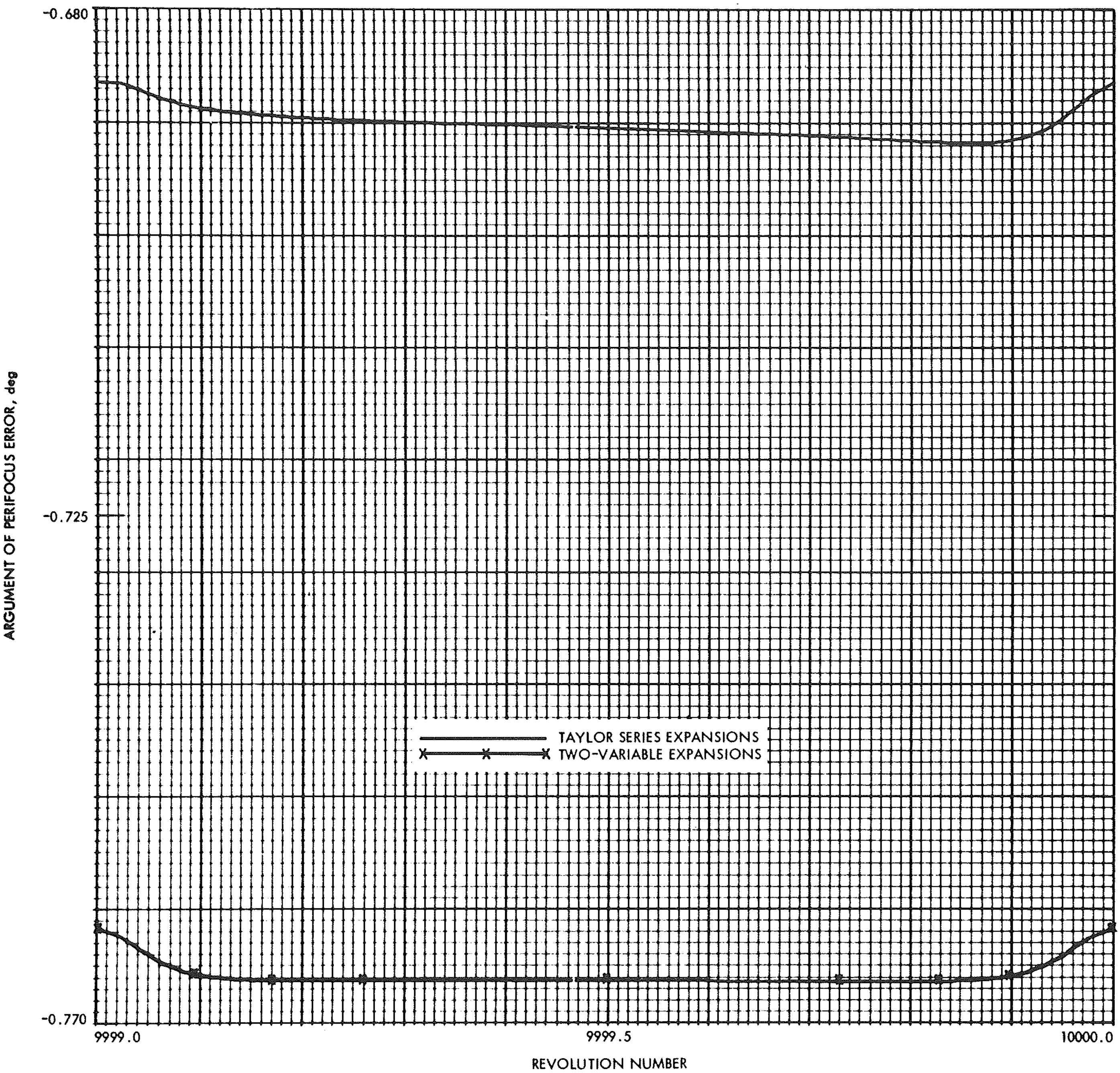


Fig. G-54. Argument of perifocus error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 3578$ km, $v_0 = 0$ deg)

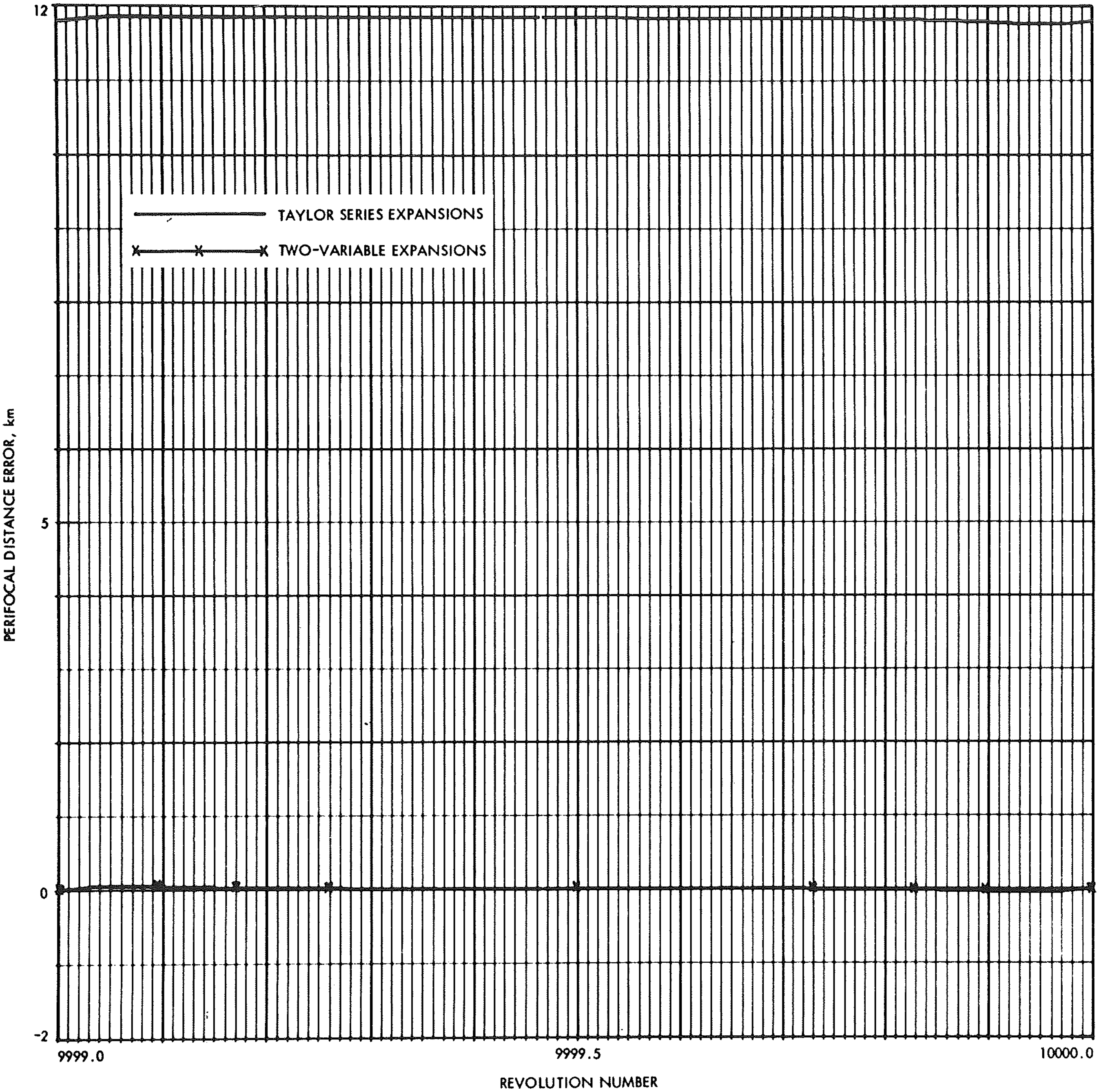


Fig. G-55. Perifocal distance error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 3578$ km, $v_0 = 0$ deg)

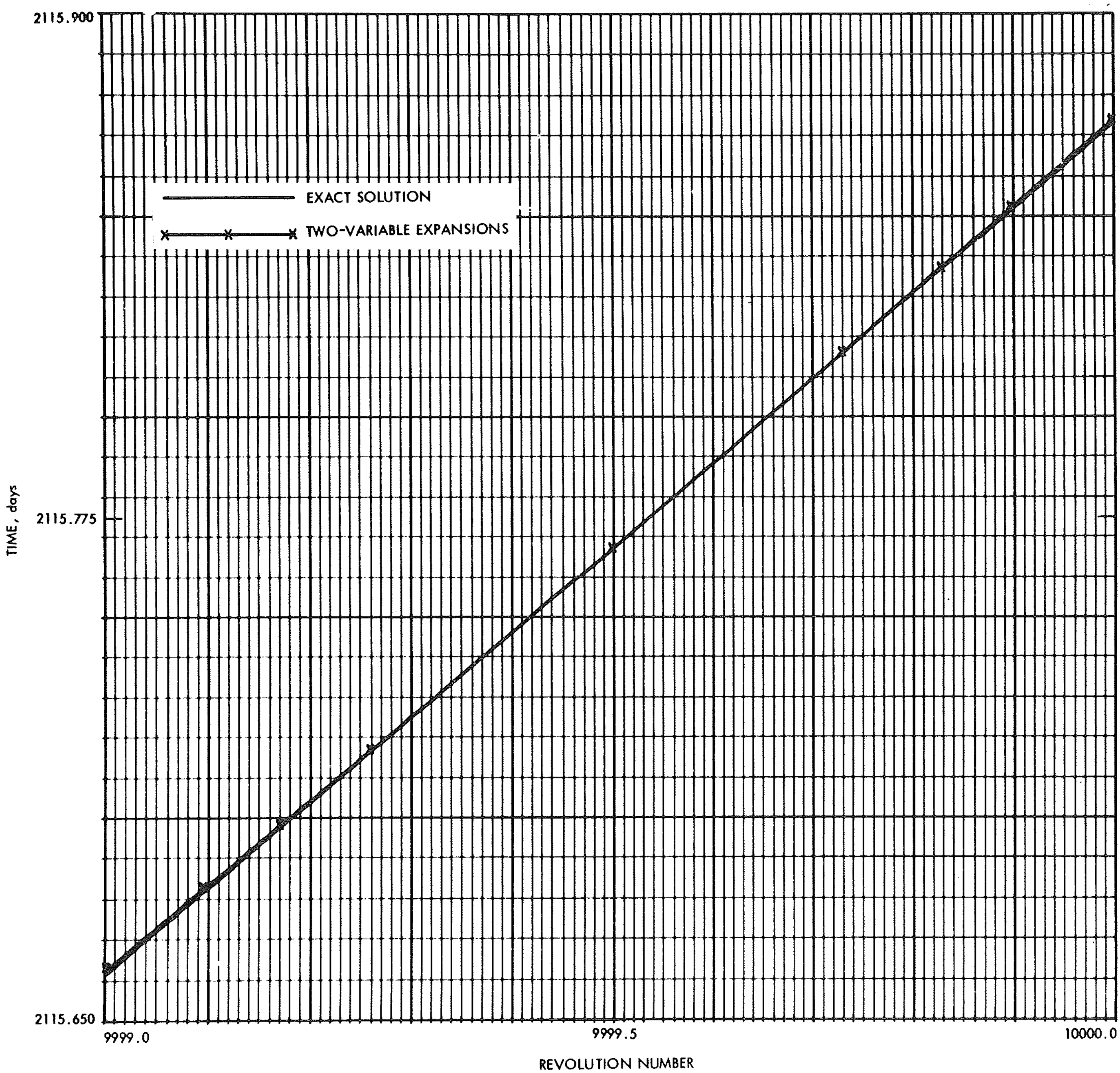


Fig. G-56. Variation of time in revolution 10,000, exact solution and two-variable asymptotic expansions solution (e_0 = 0.5, q_0 = 3578 km, v_0 = 0 deg)

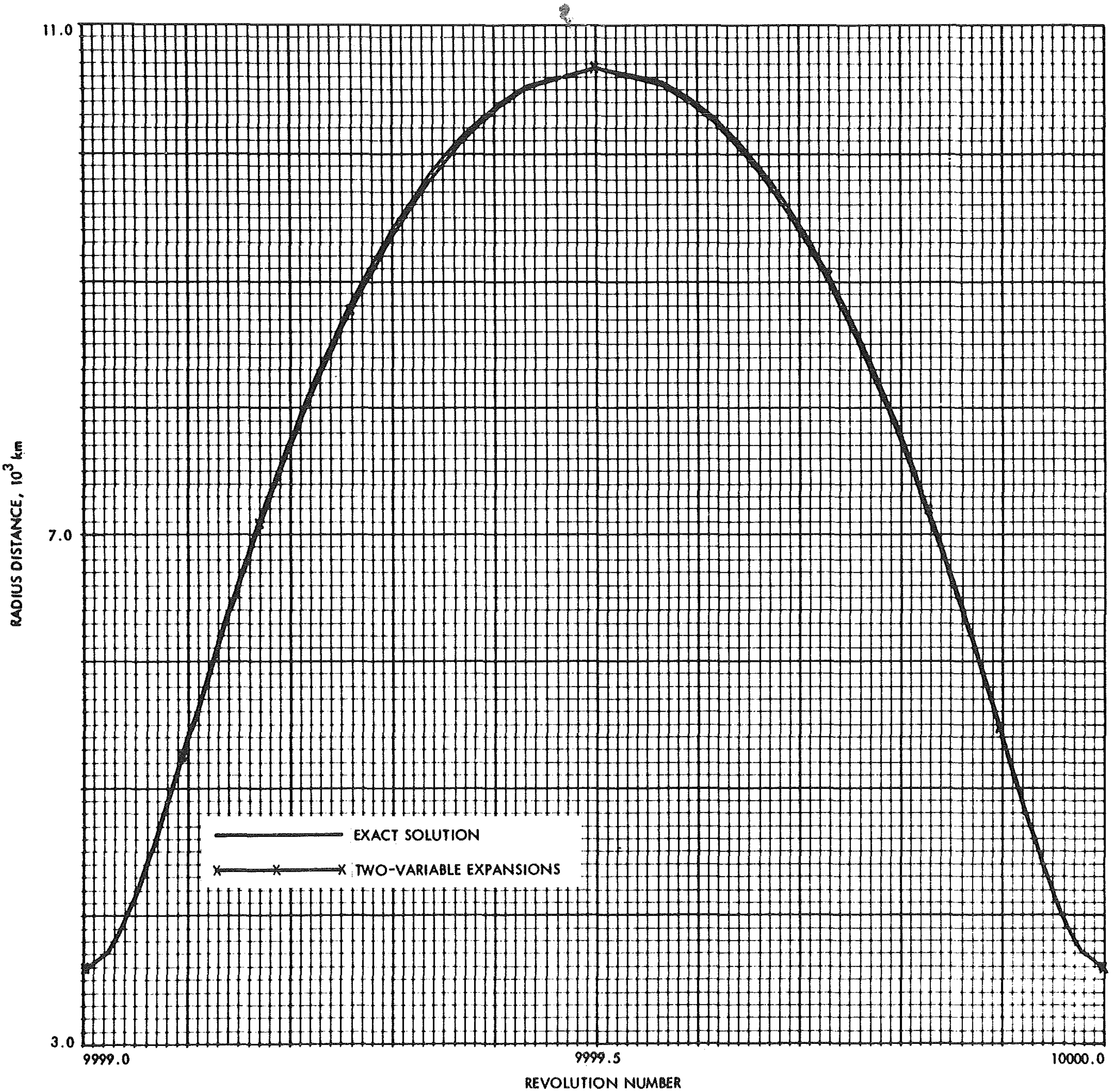


Fig. G-57. Variation of radius distance in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 3578$ km, $v_0 = 0$ deg)

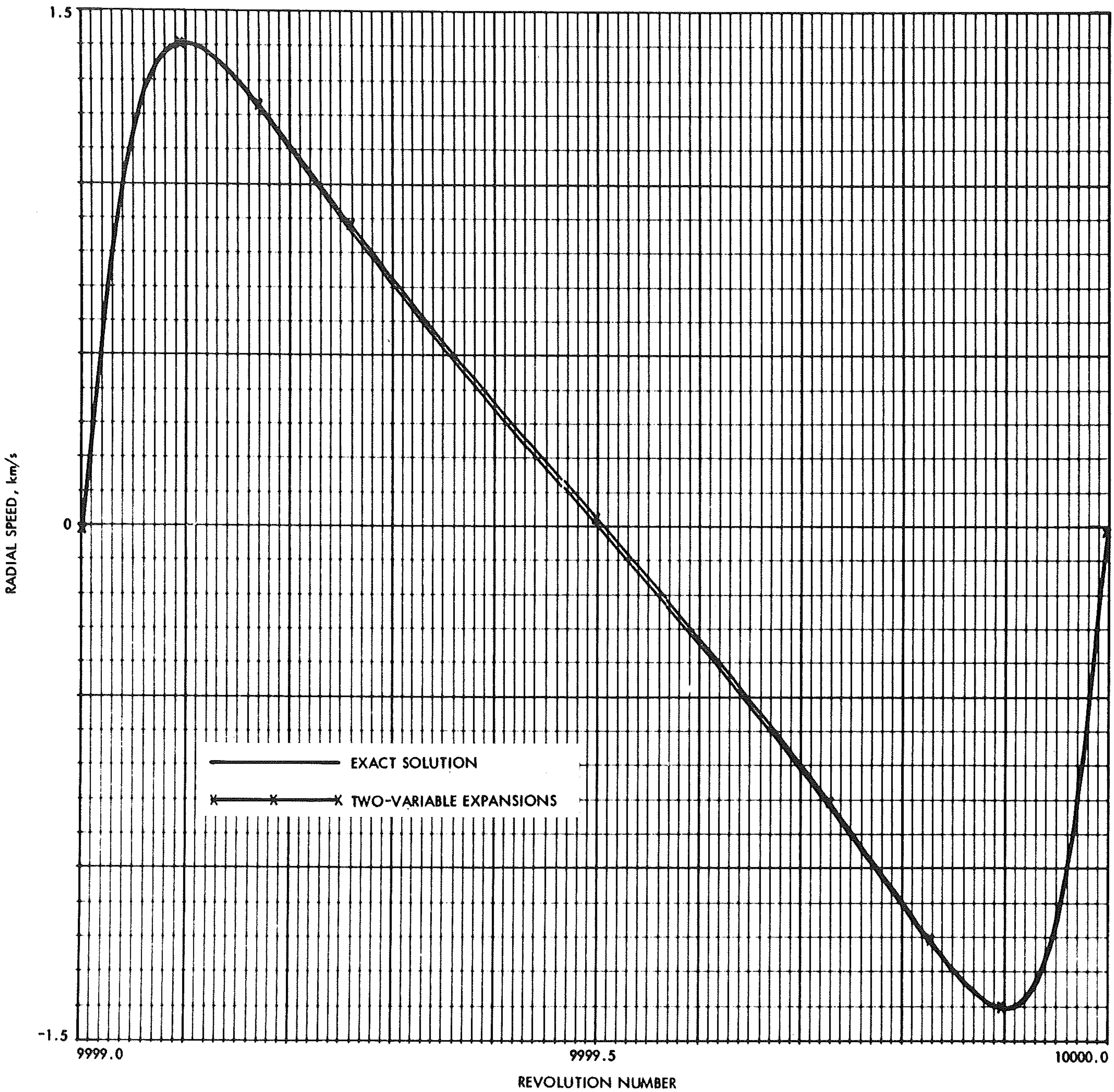


Fig. G-58. Variation of radial speed in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 3578$ km, $v_0 = 0$ deg)

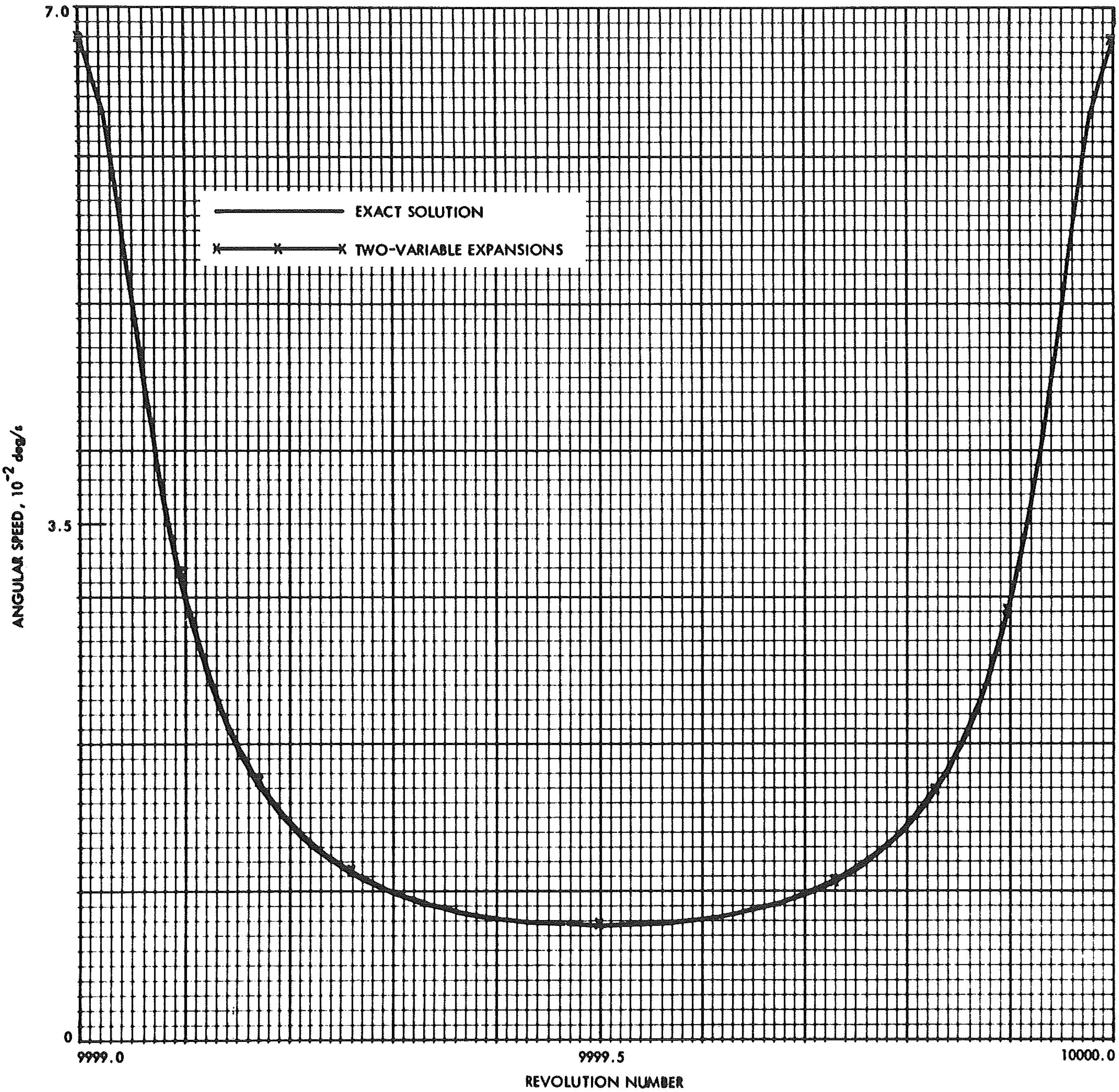


Fig. G-59. Variation of angular speed in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 3578$ km, $v_0 = 0$ deg)

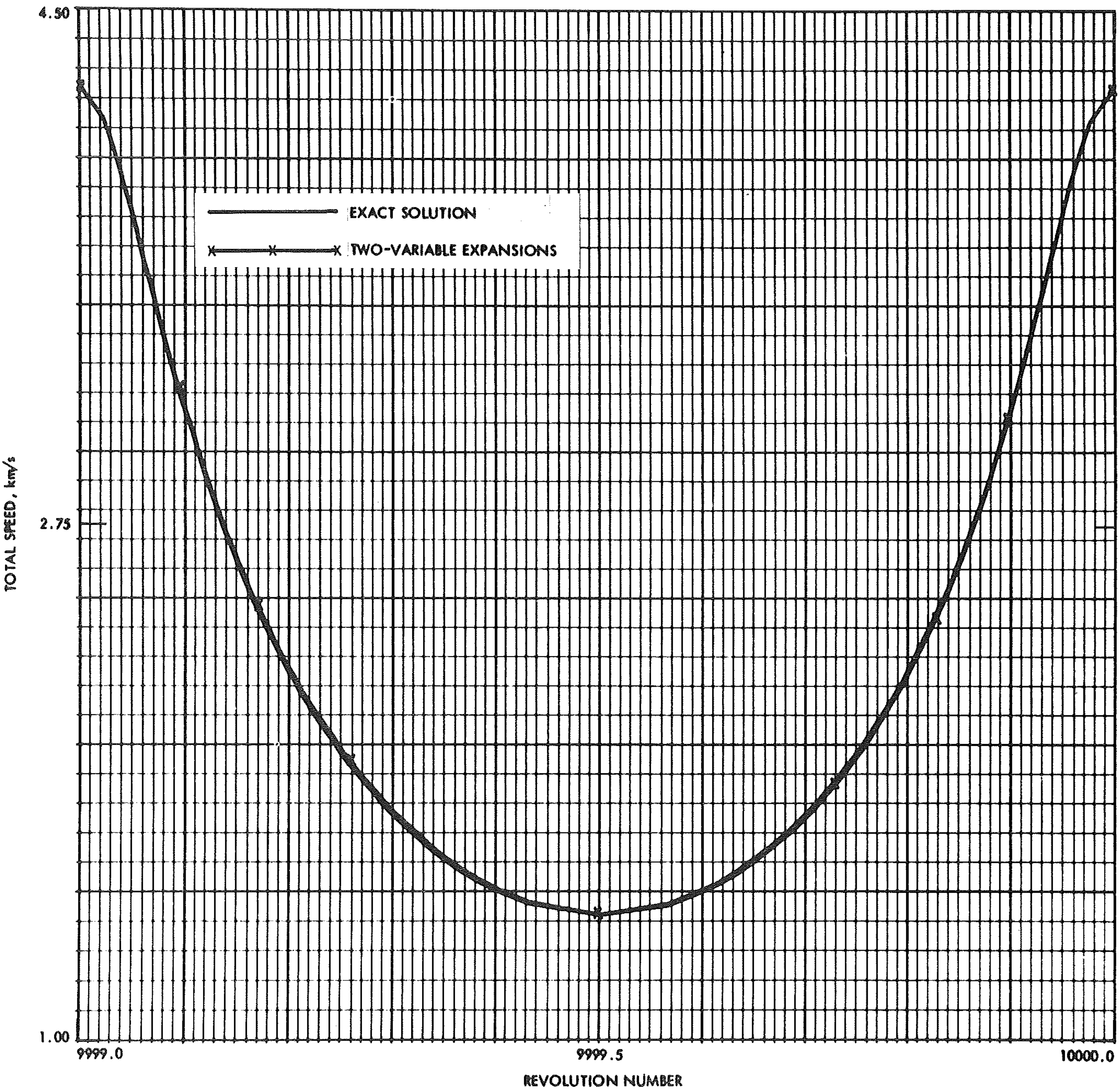


Fig. G-60. Variation of total speed in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 3578$ km, $v_0 = 0$ deg)

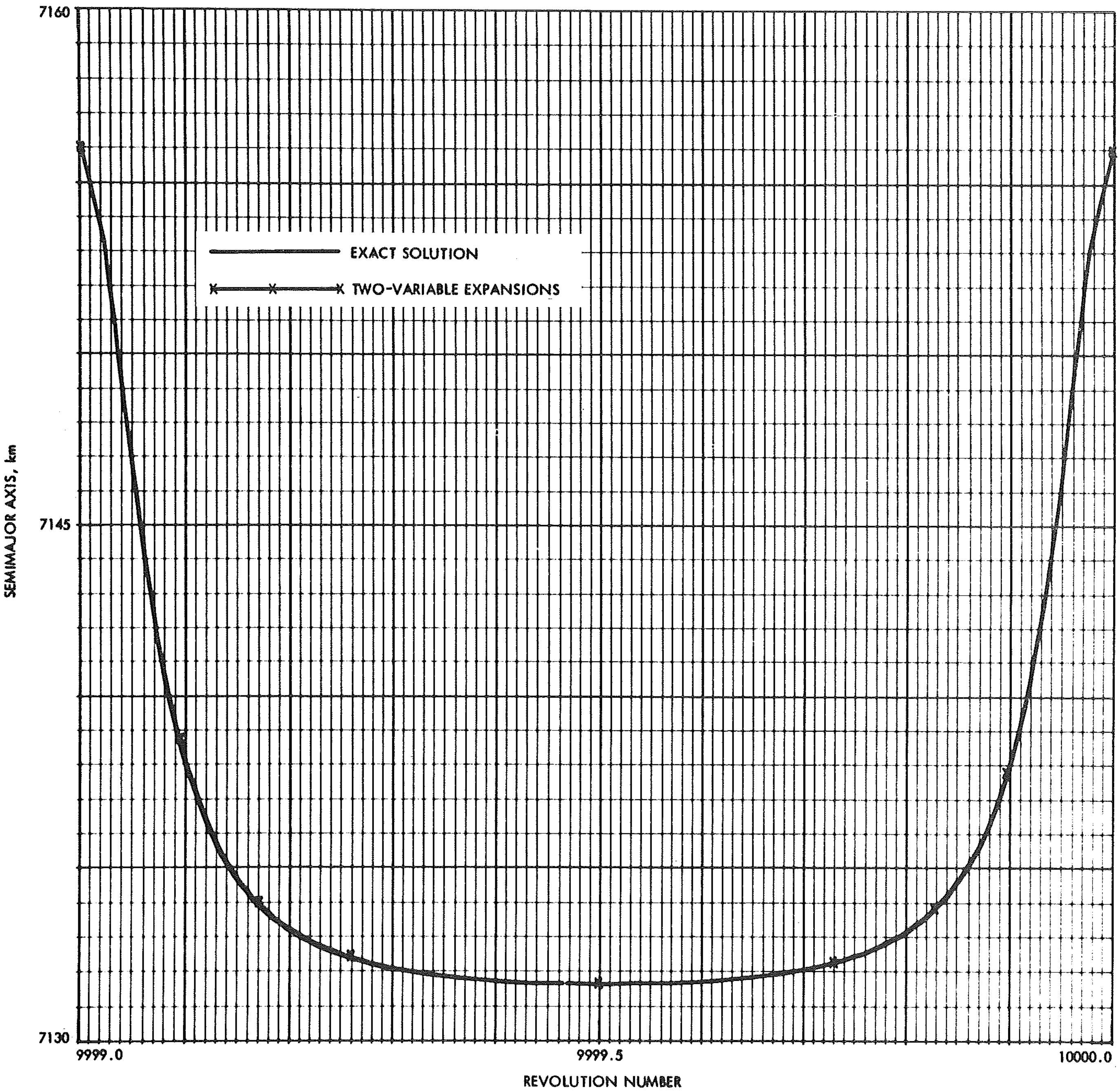


Fig. G-61. Variation of semimajor axis in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 3578$ km, $v_0 = 0$ deg)

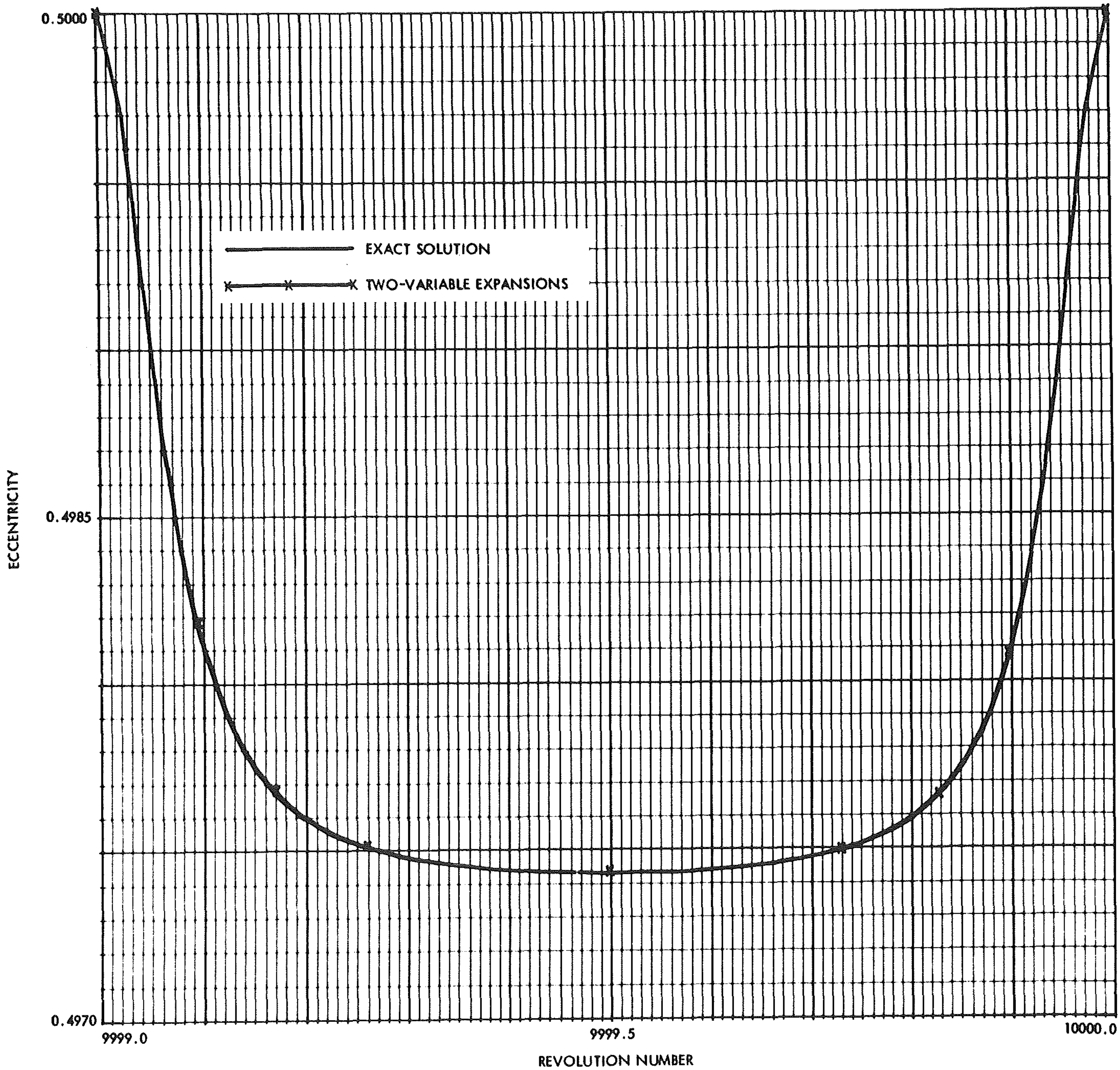


Fig. G-62. Variation of eccentricity in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 3578$ km, $v_0 = 0$ deg)

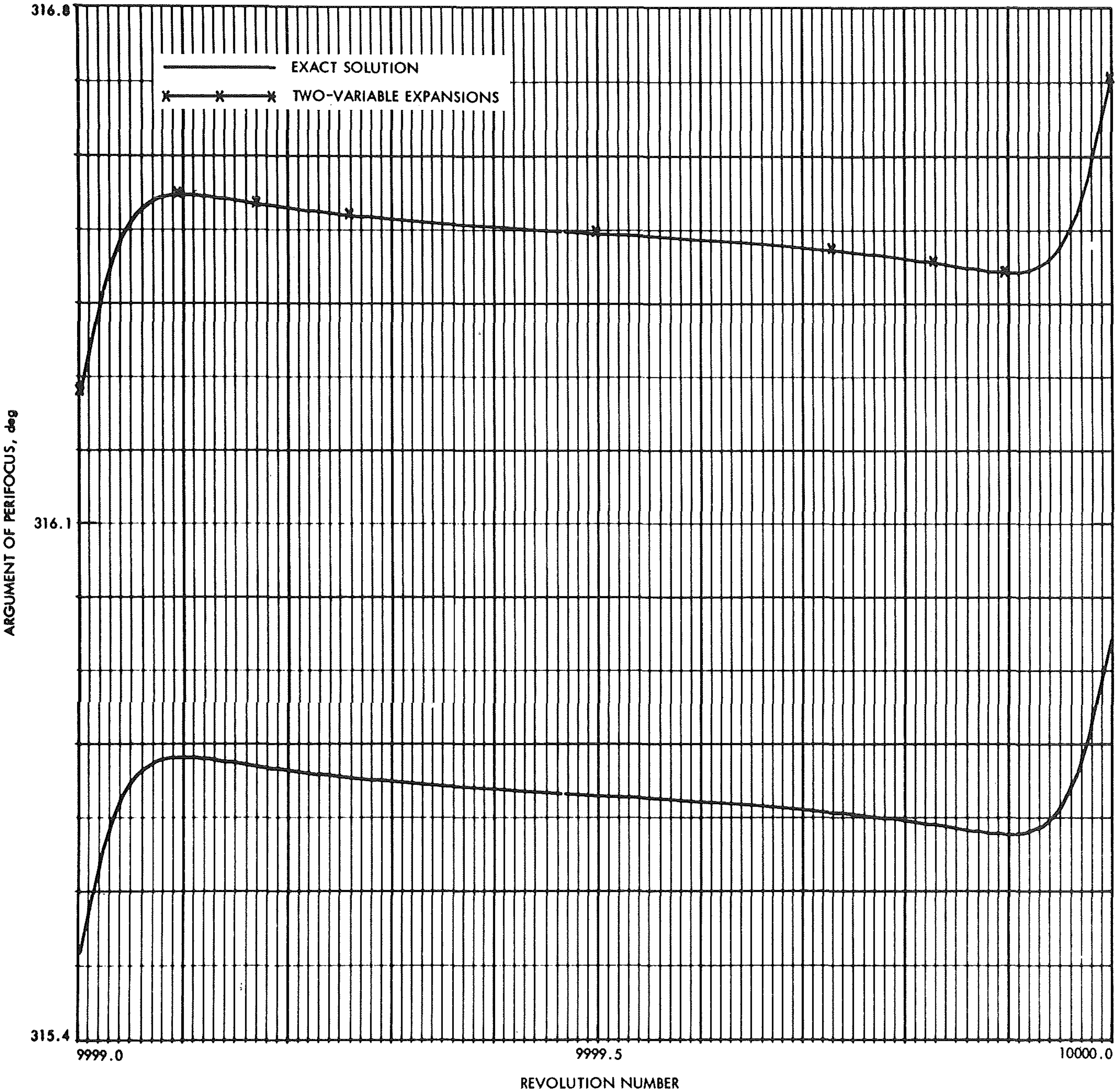


Fig. G-63. Variation of argument of perifocus in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 3578$ km, $v_0 = 0$ deg)

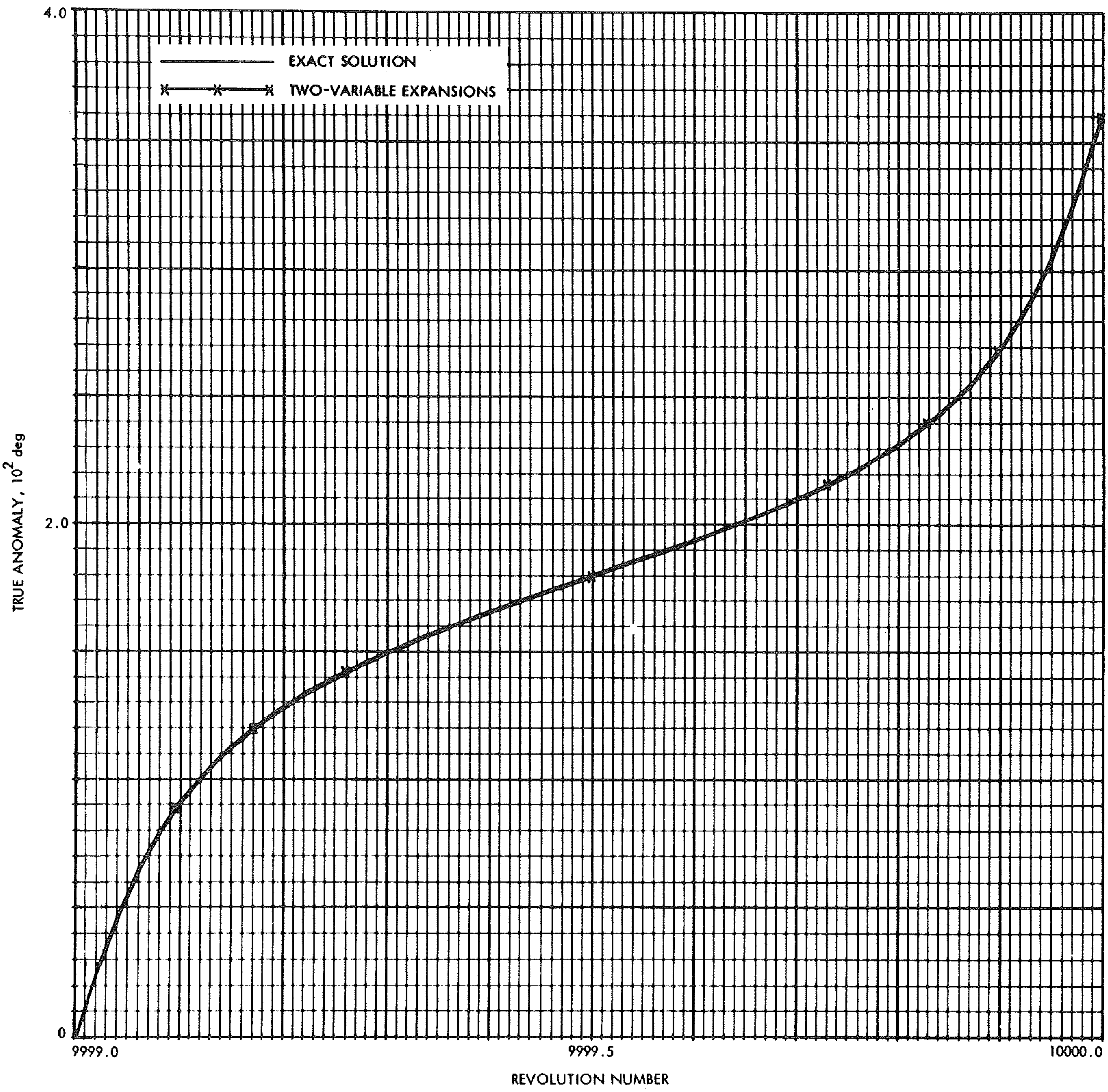


Fig. G-64. Variation of true anomaly in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 3578$ km, $v_0 = 0$ deg)

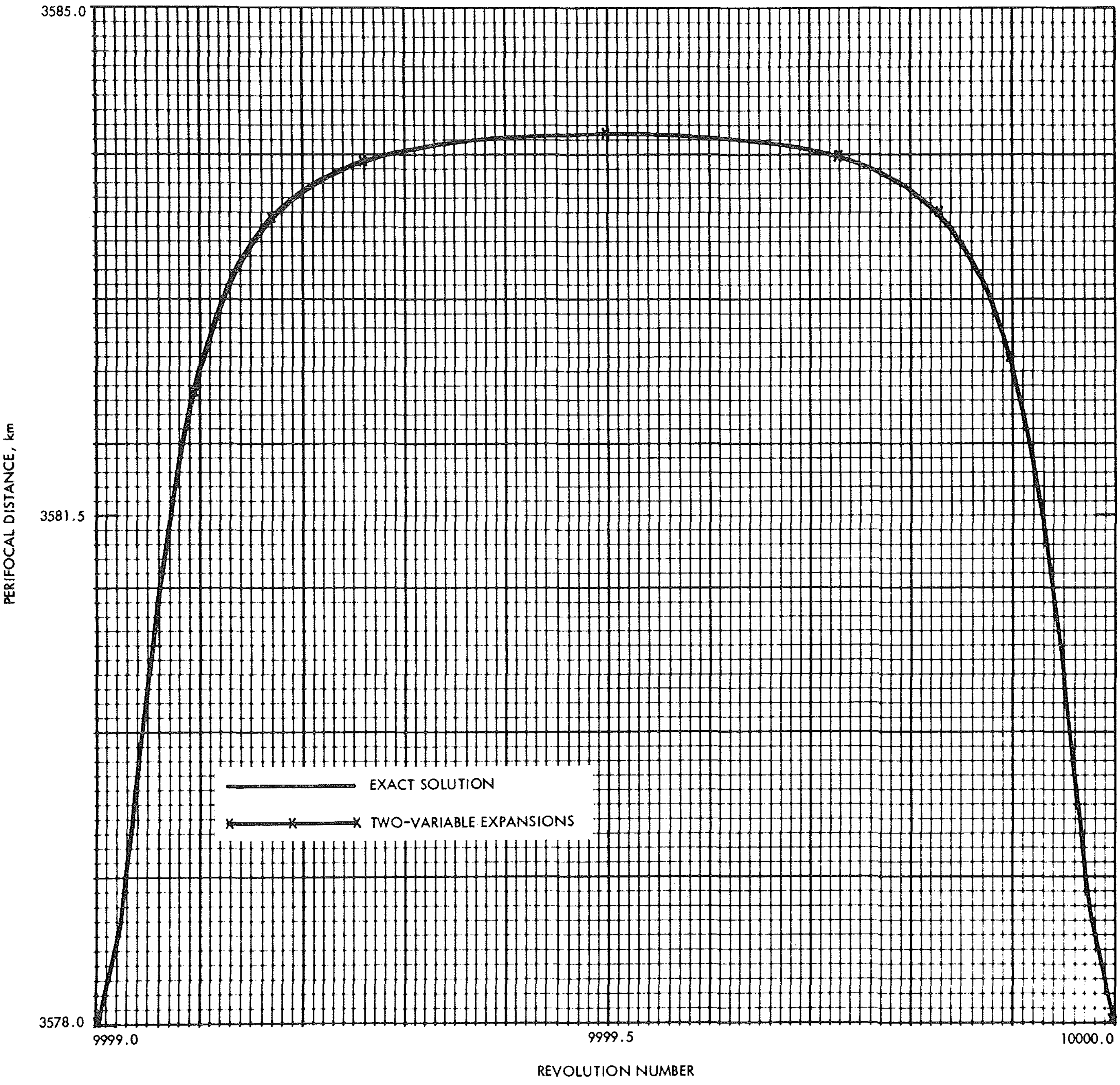


Fig. G-65. Variation of perifocal distance in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 3578$ km, $v_0 = 0$ deg)

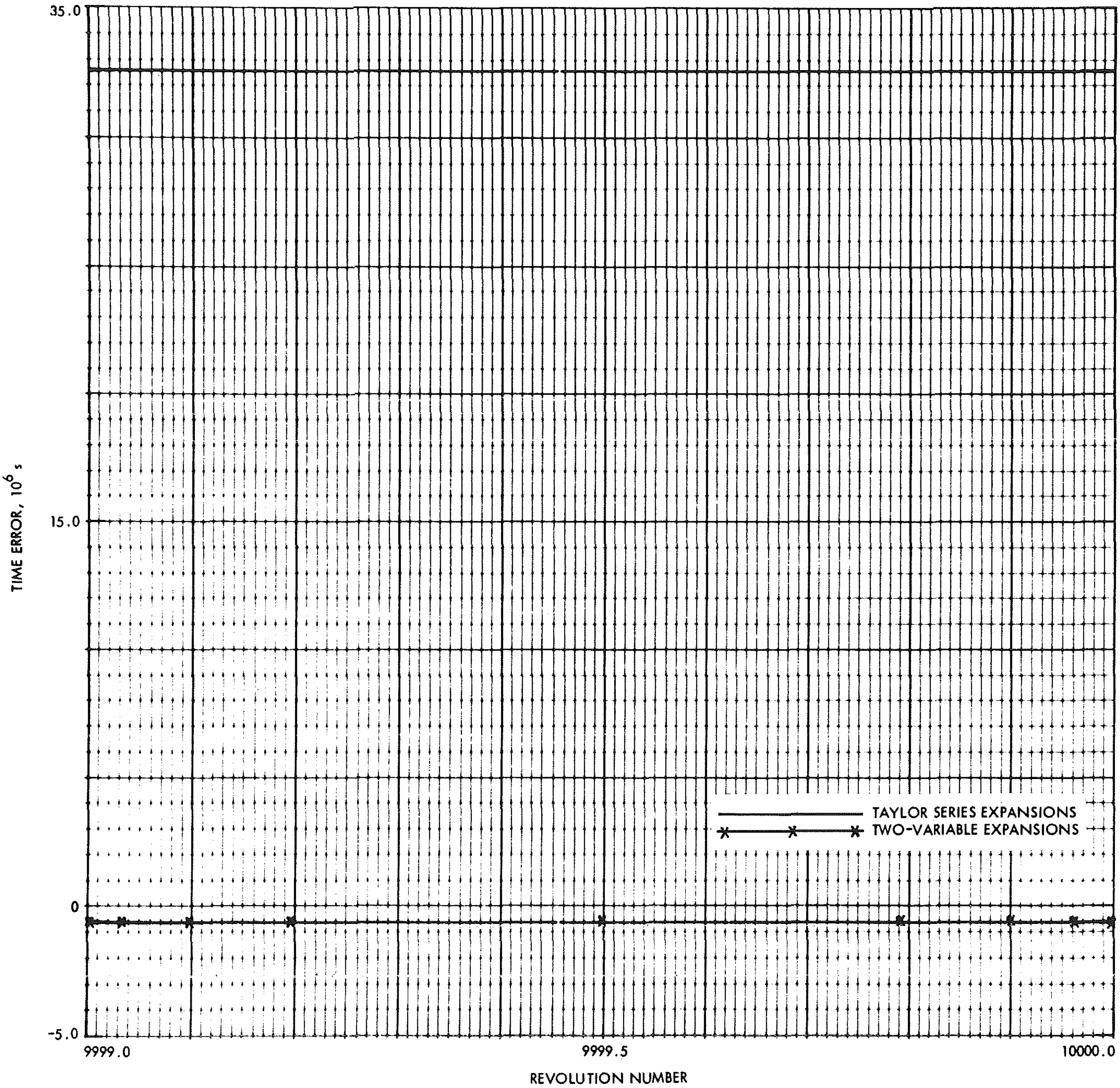


Fig. G-66. Time error in revolution 10,000, general perturbations solutions ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

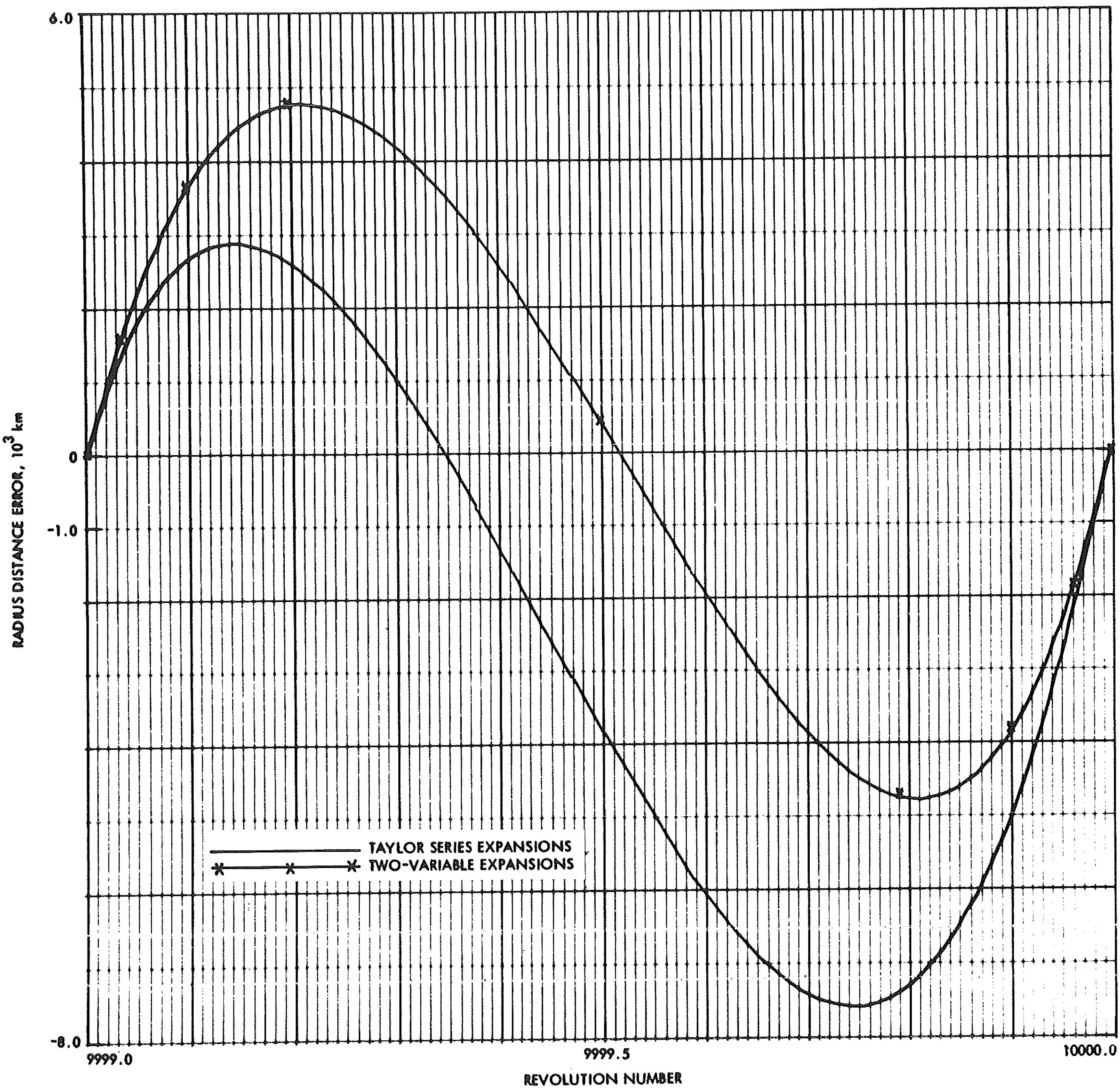


Fig. G-67. Radius distance error in revolution 10,000, general perturbations solutions ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

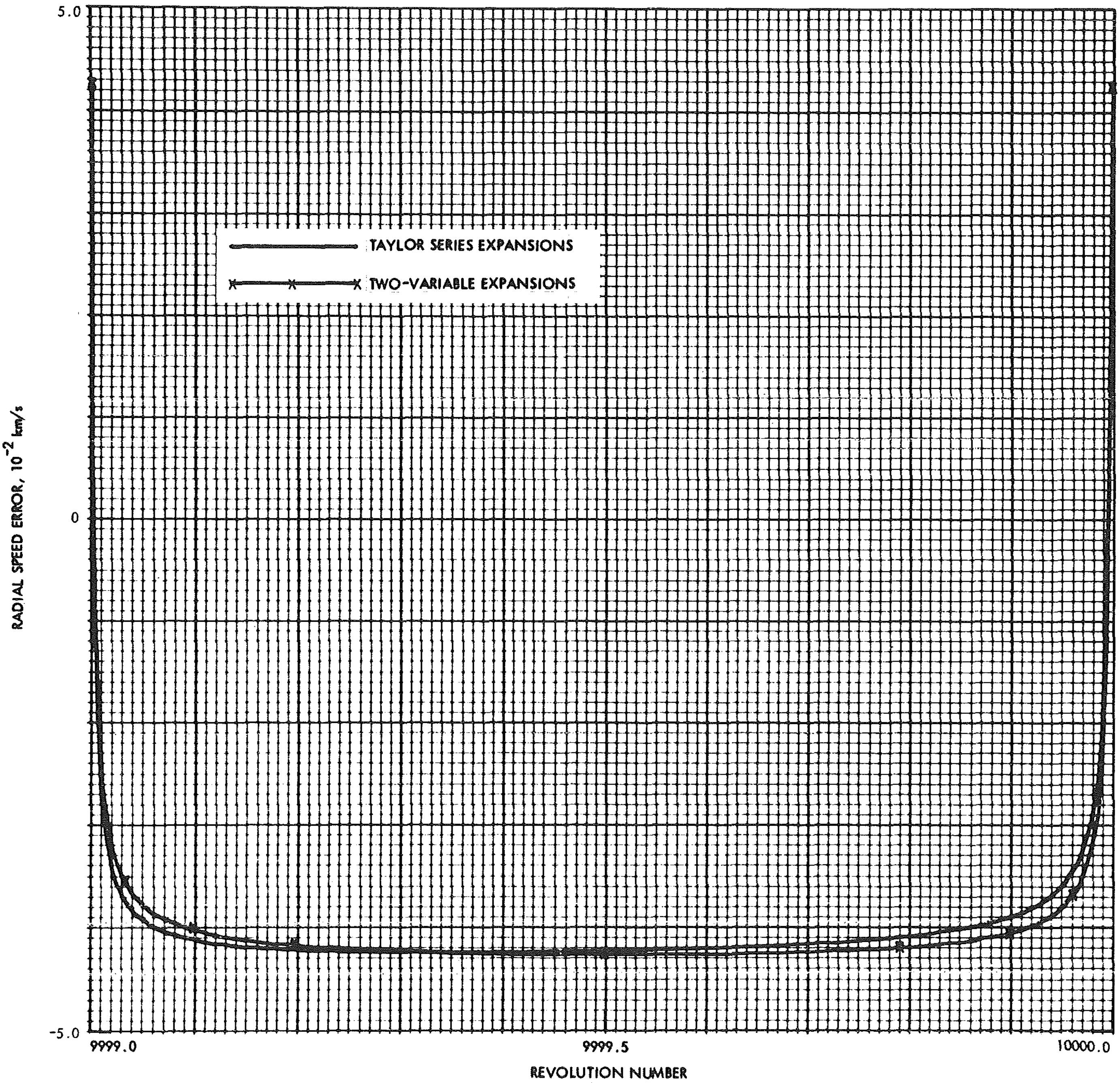


Fig. G-68. Radial speed error in revolution 10,000, general perturbations solutions ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

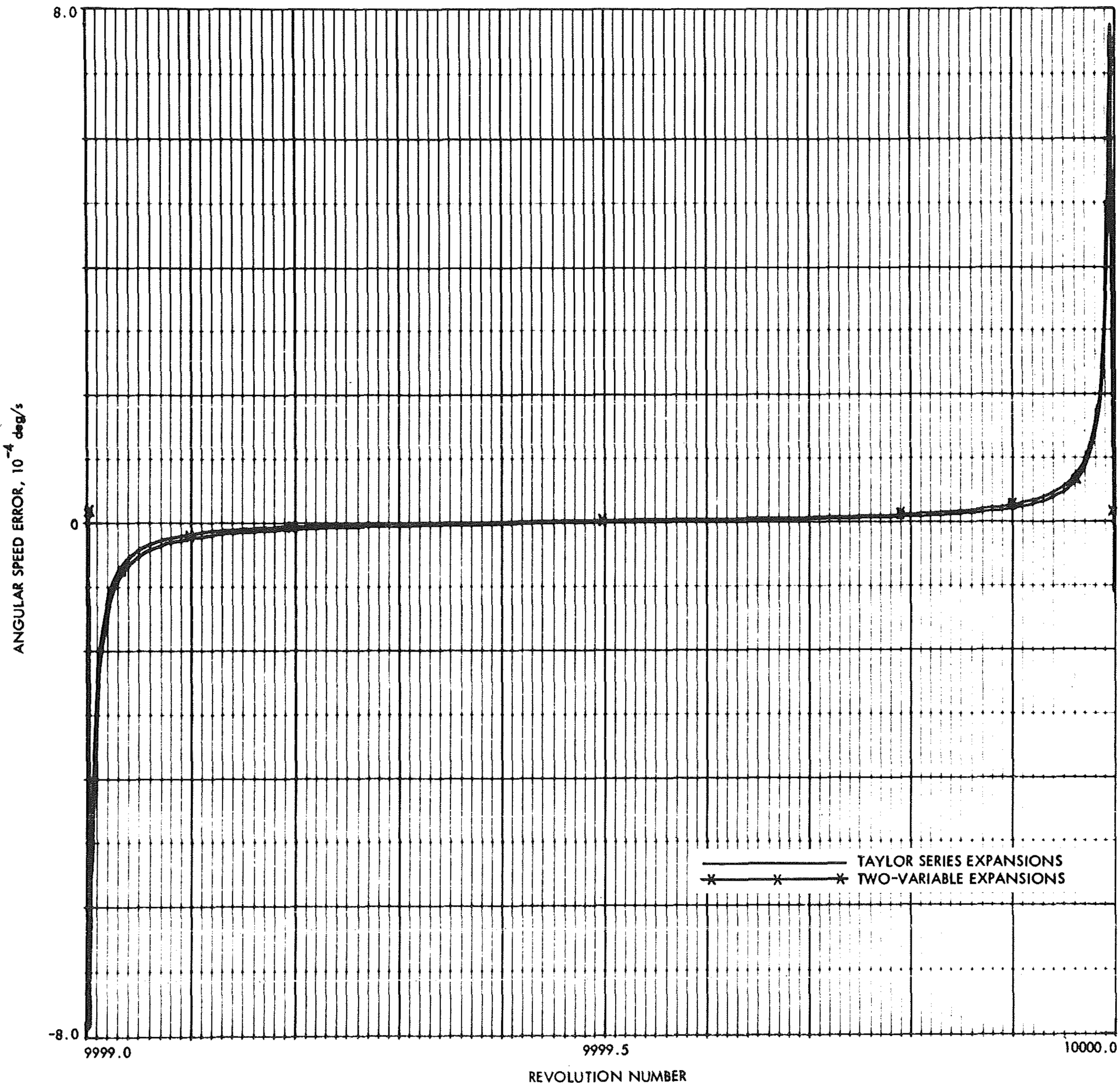


Fig. G-69. Angular speed error in revolution 10,000, general perturbations solutions ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

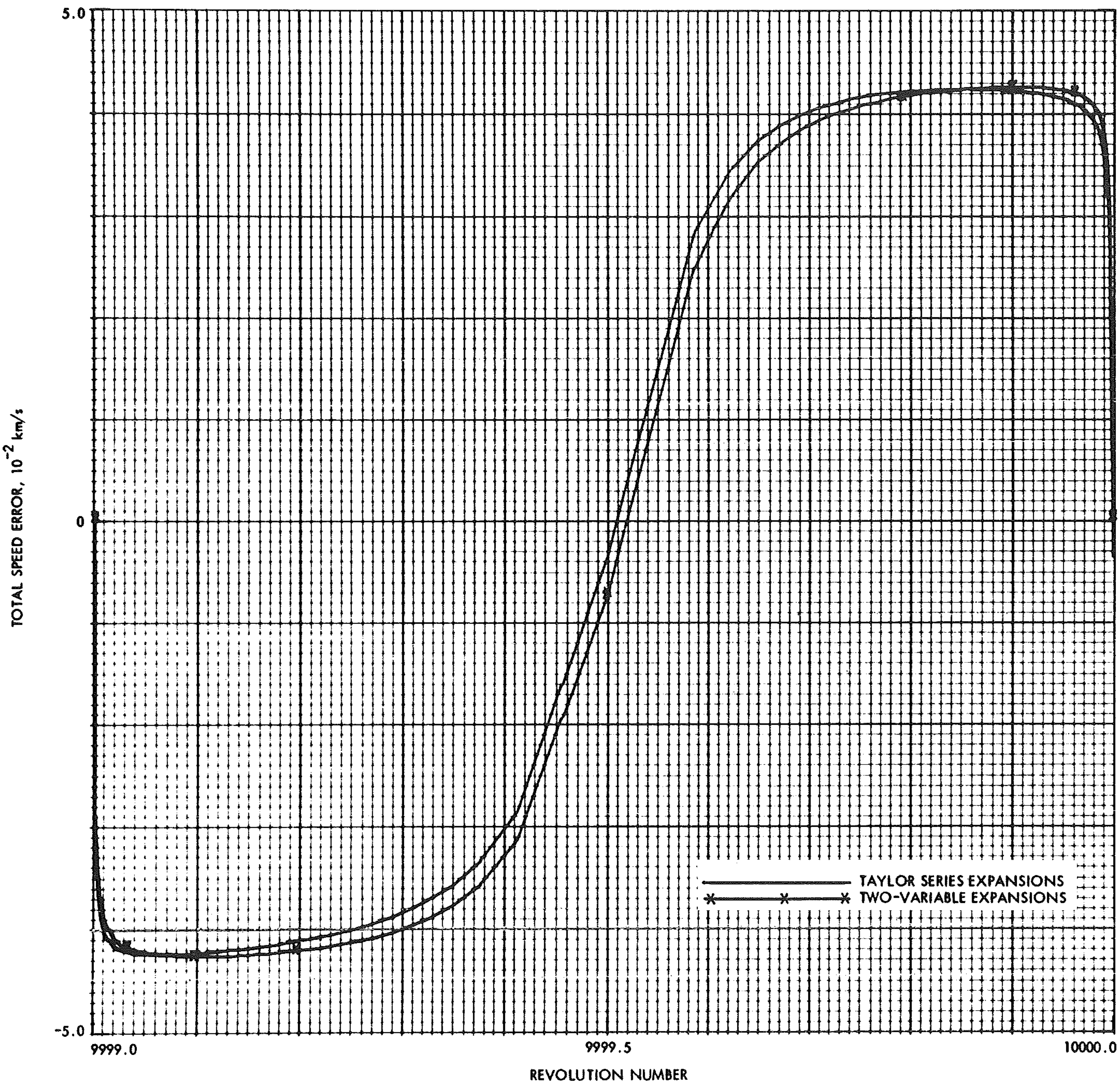


Fig. G-70. Total speed error in revolution 10,000, general perturbations solutions (e_0 = 0.95, q_0 = 3578 km, v_0 = 0 deg)

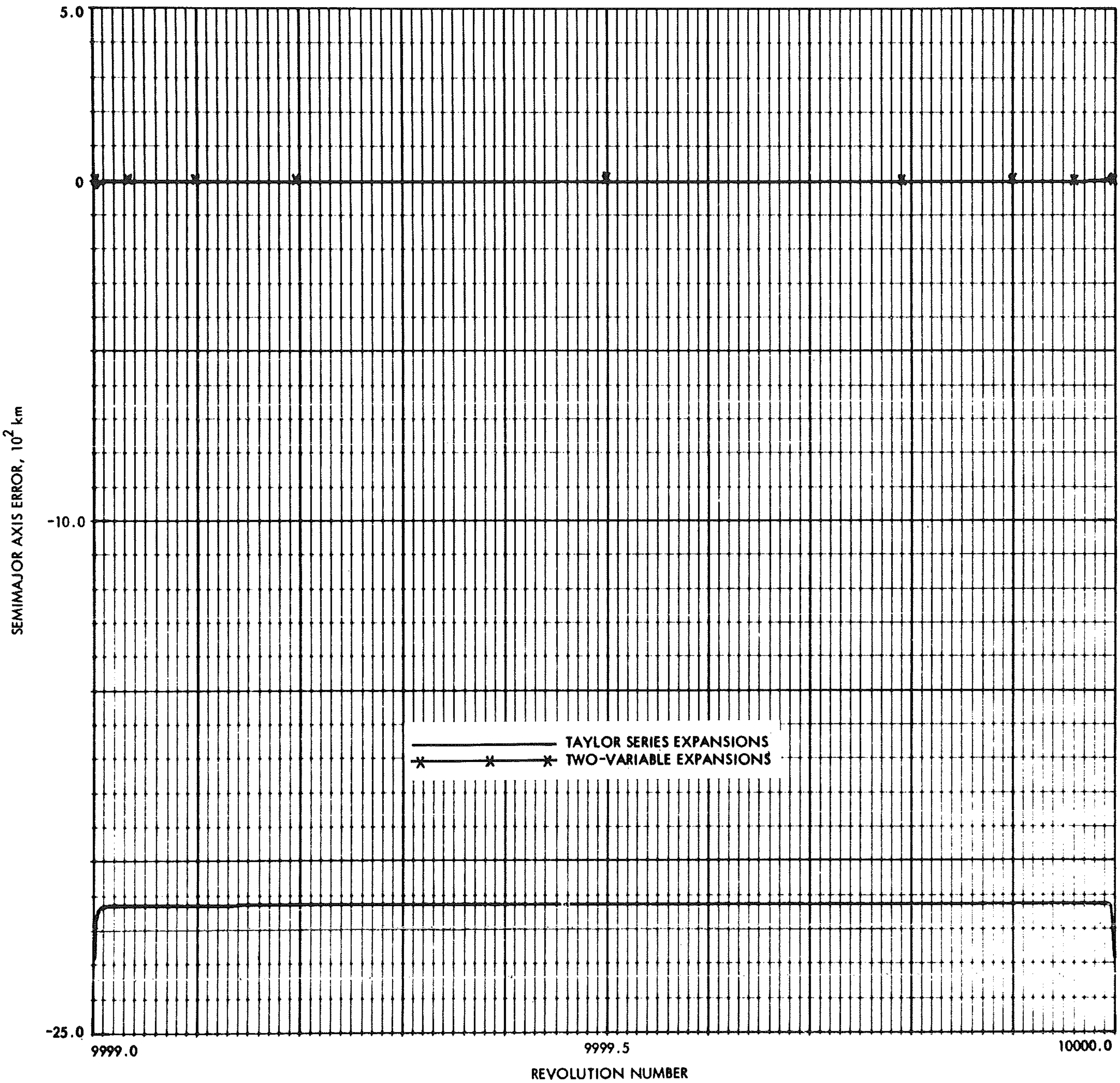


Fig. G-71. Semimajor axis error in revolution 10,000, general perturbations solutions ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

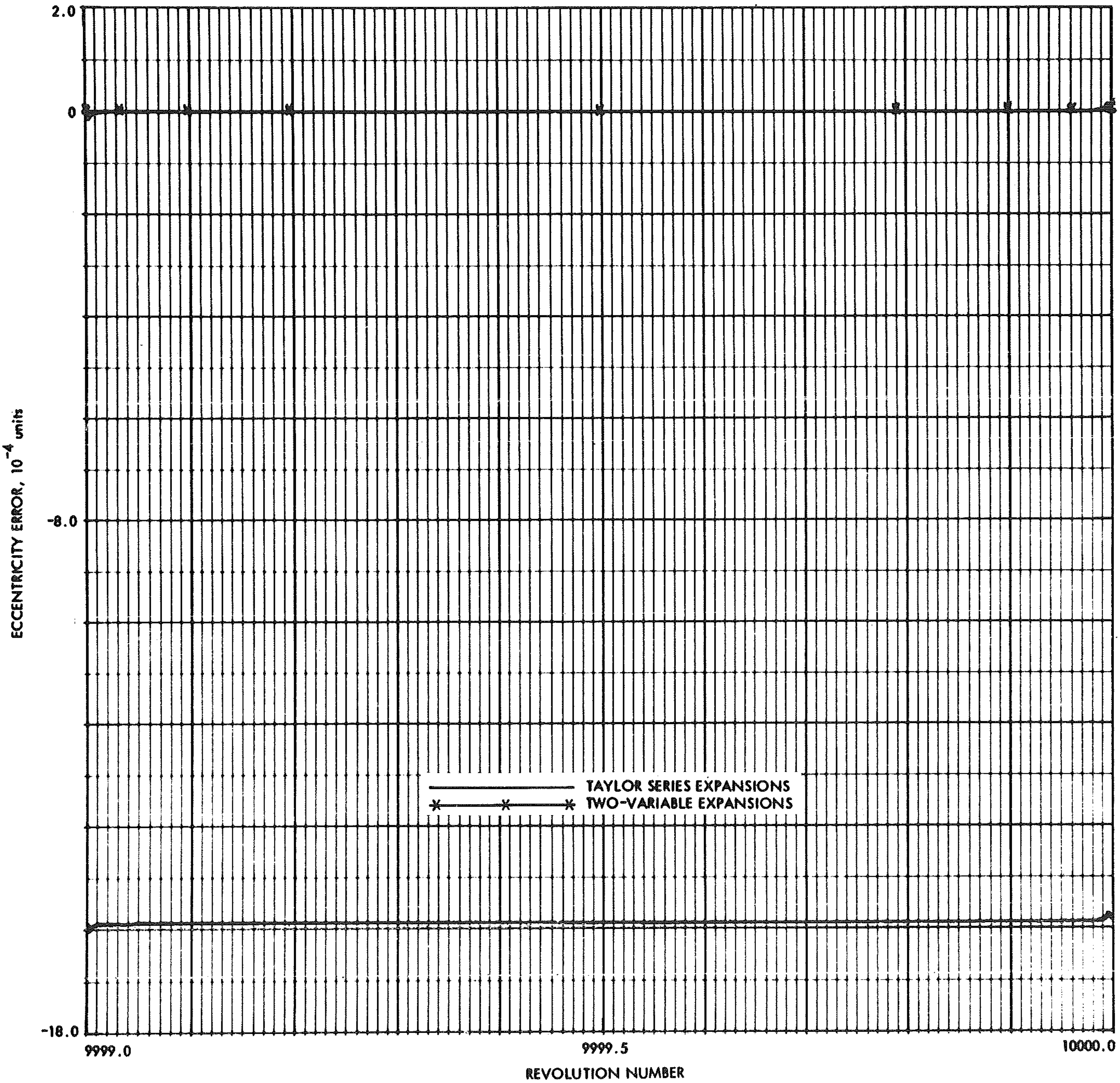


Fig. G-72. Eccentricity error in revolution 10,000, general perturbations solutions ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

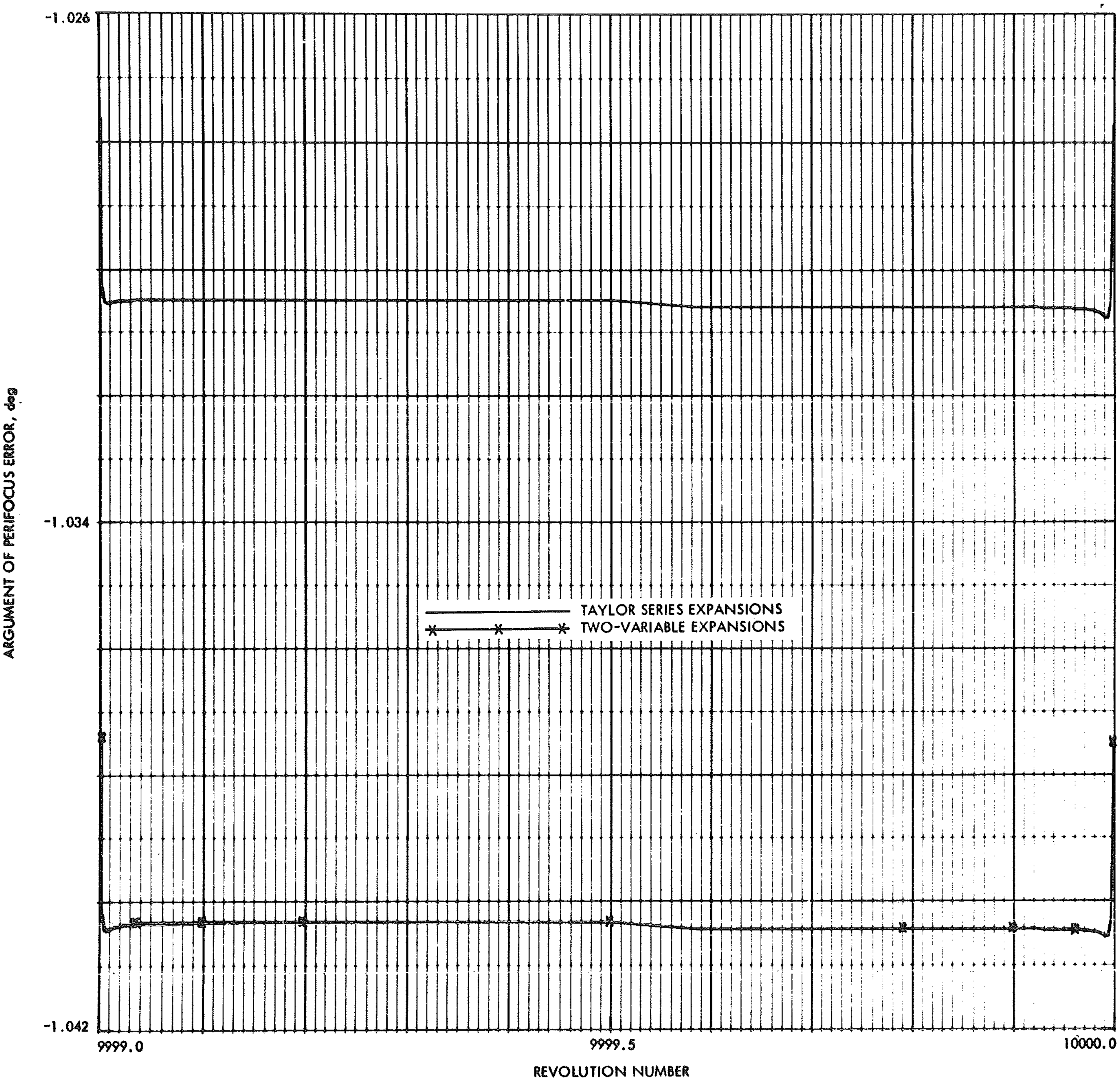


Fig. G-73. Argument of perifocus error in revolution 10,000, general perturbations solutions ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

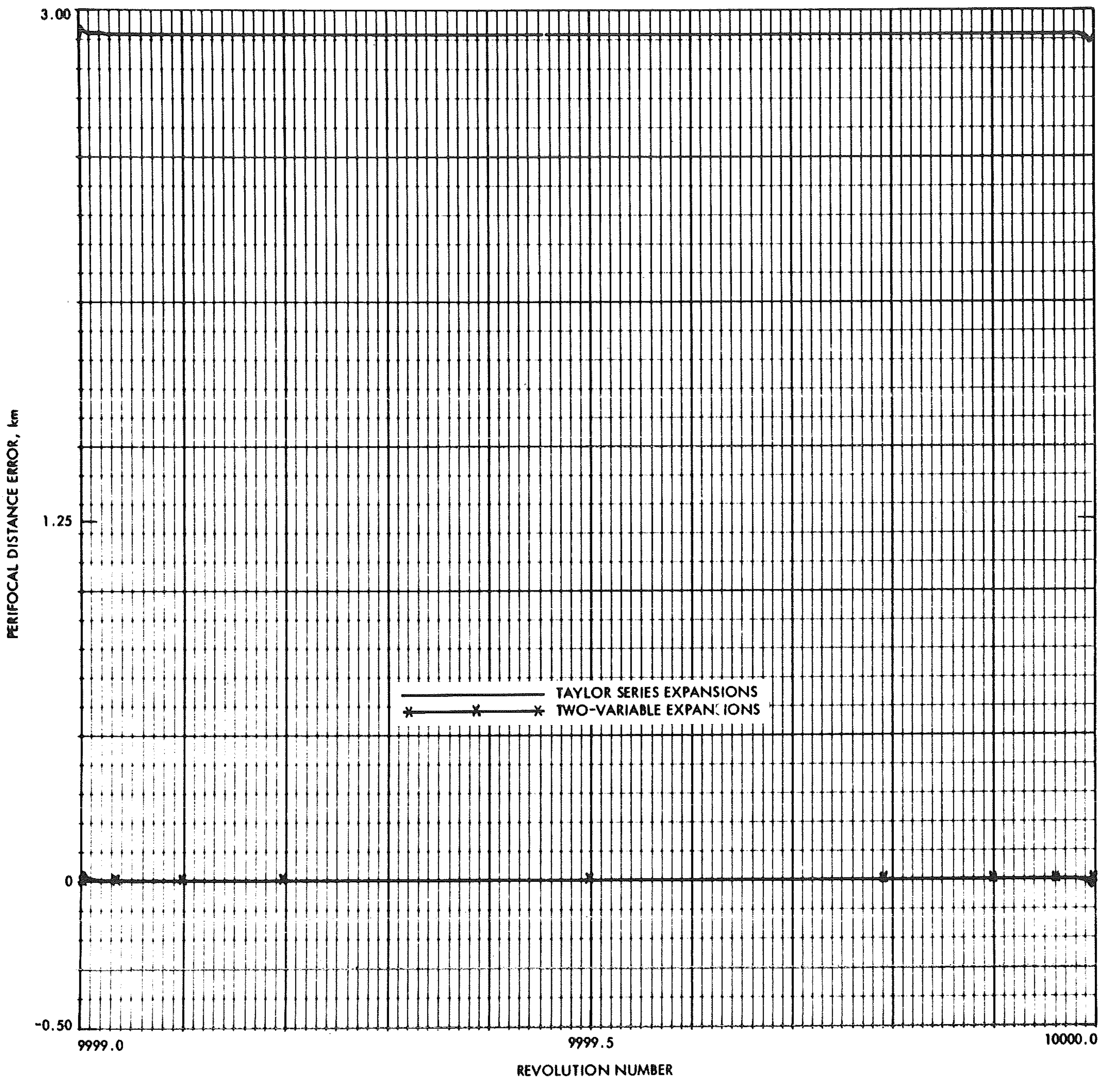


Fig. G-74. Perifocal distance error in revolution 10,000, general perturbations solutions ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

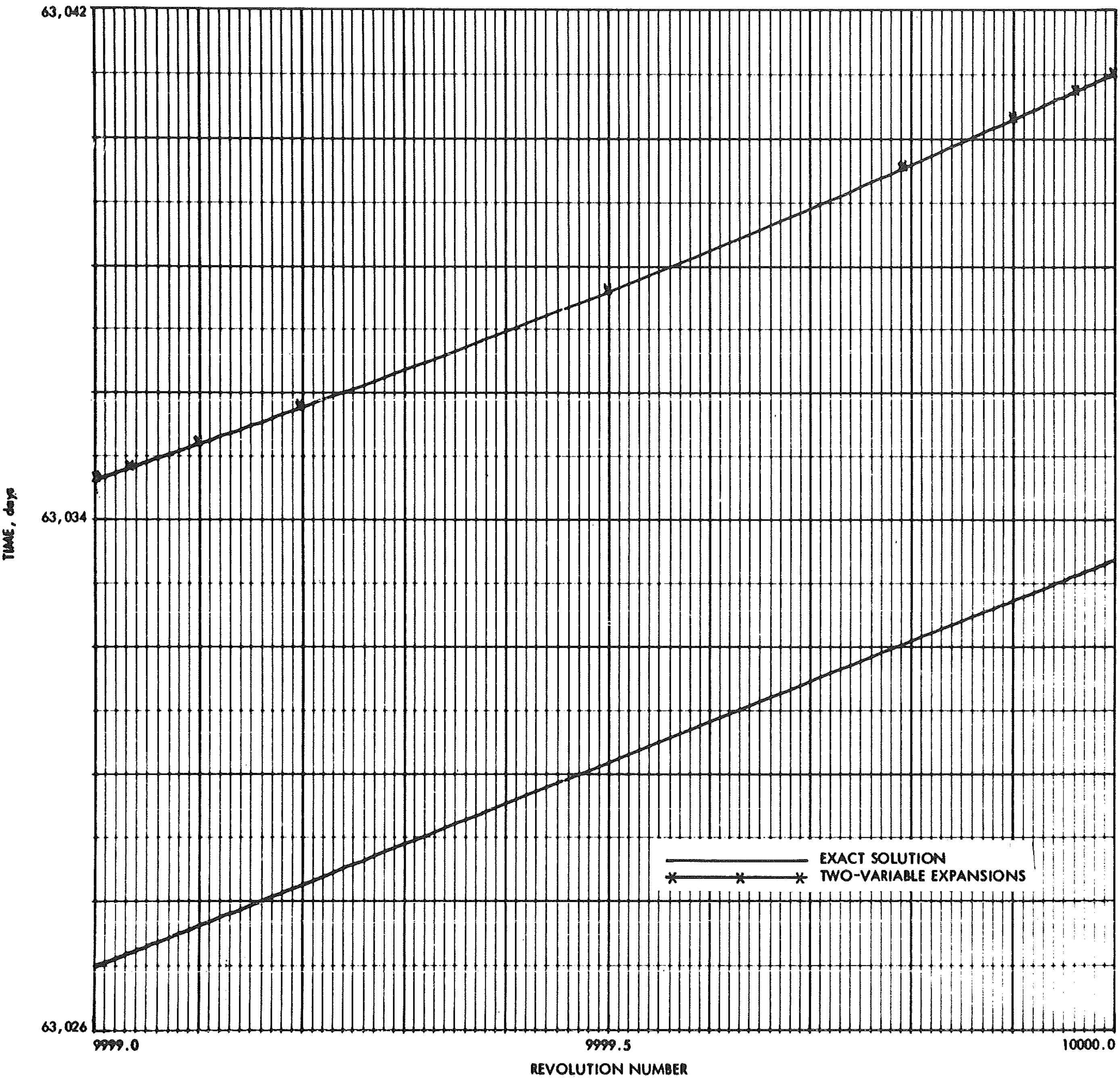


Fig. G-75. Variation of time in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

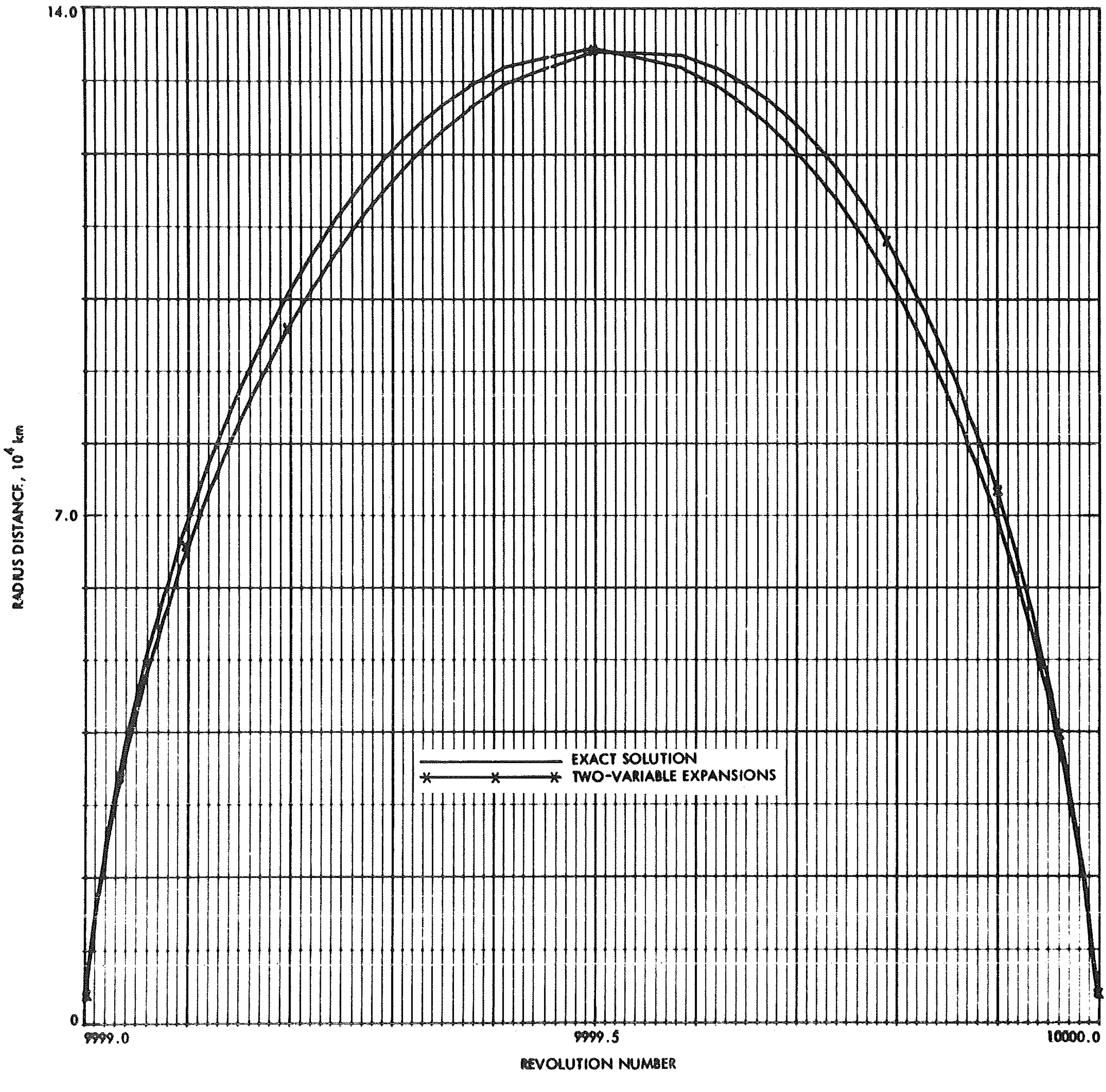


Fig. G-76. Variation of radius distance in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

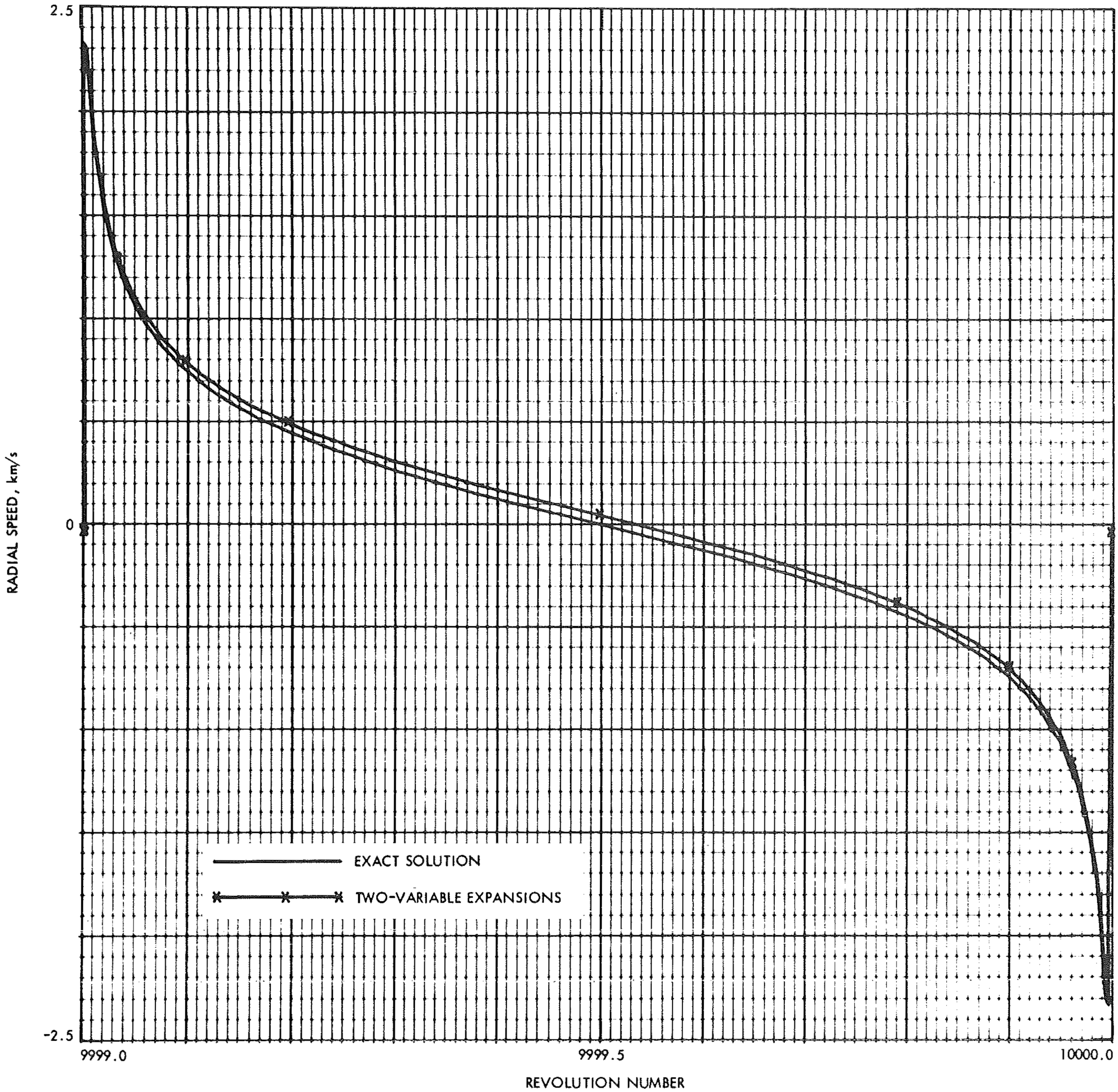


Fig. G-77. Variation of radial speed in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

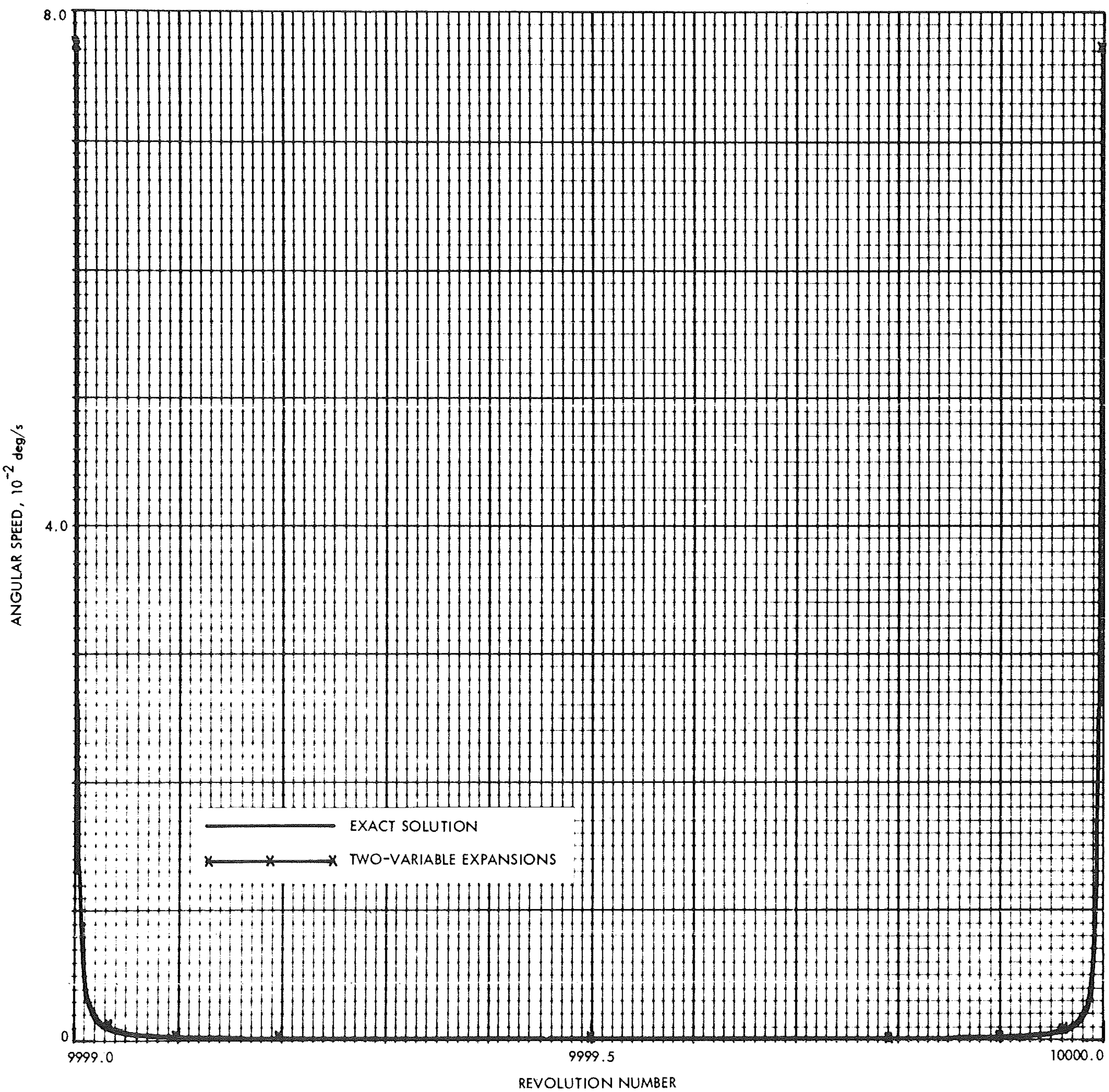


Fig. G-78. Variation of angular speed in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

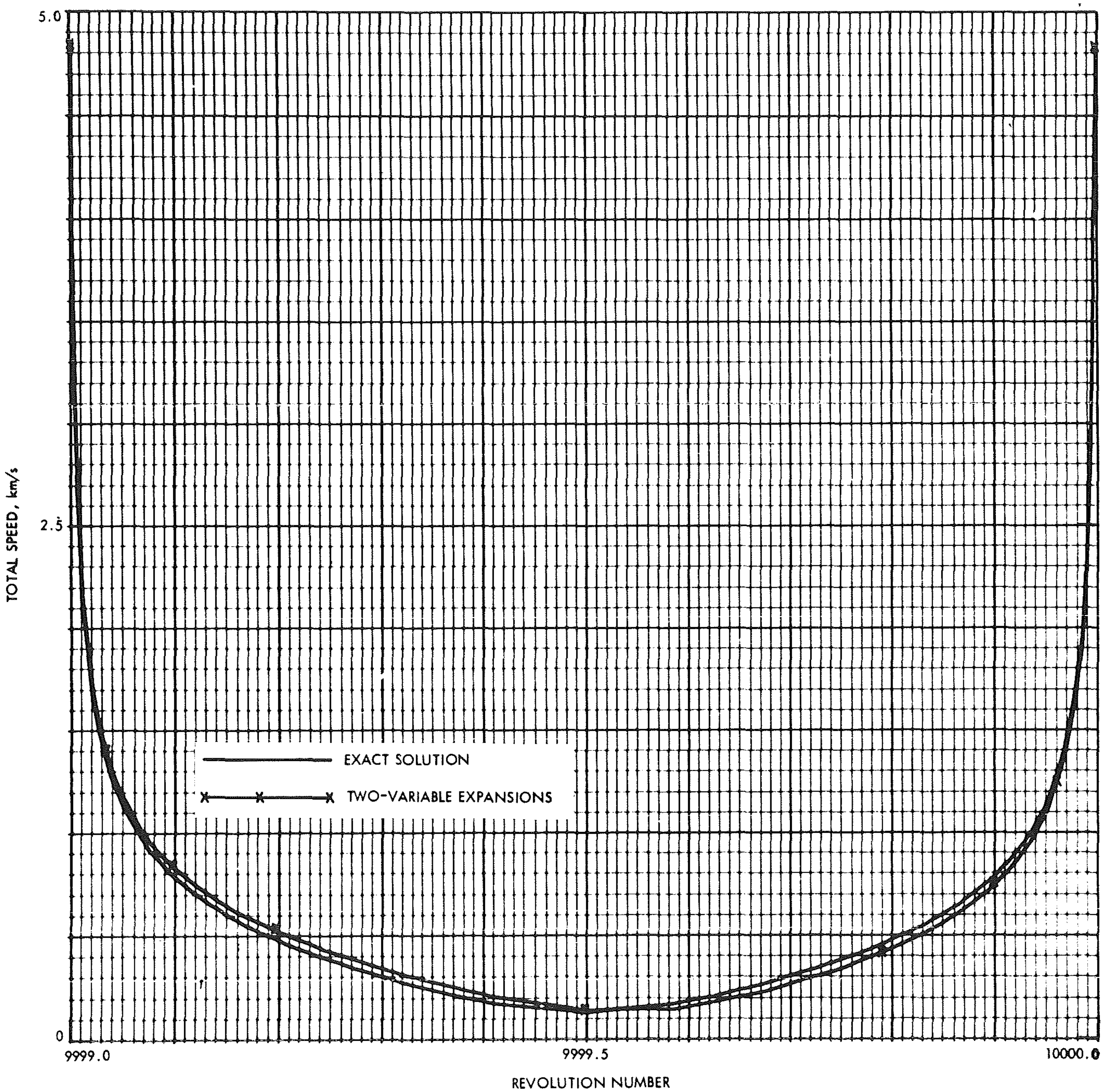


Fig. G-79. Variation of total speed in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

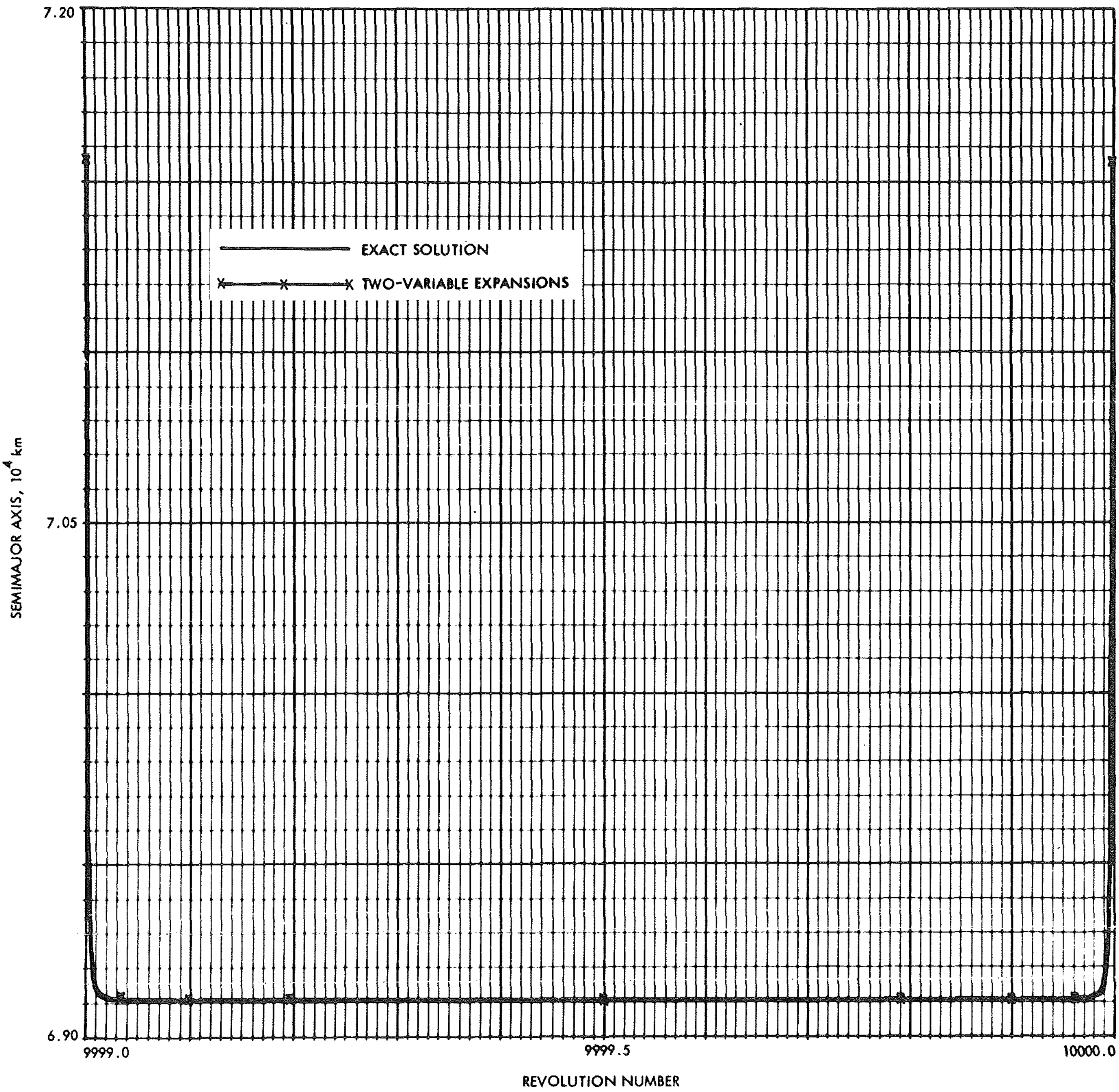


Fig. G-80. Variation of semimajor axis in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

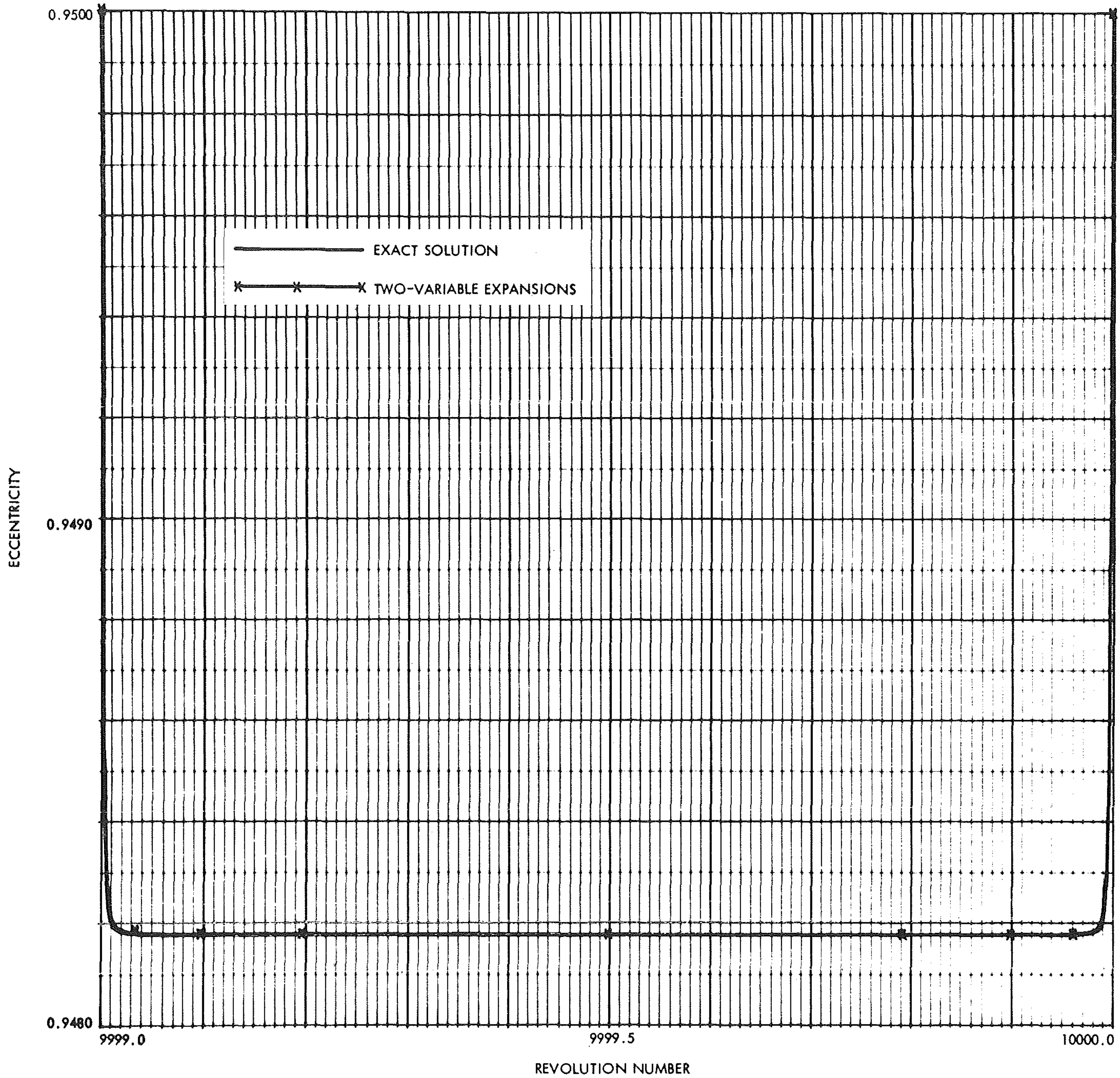


Fig. G-81. Variation of eccentricity in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

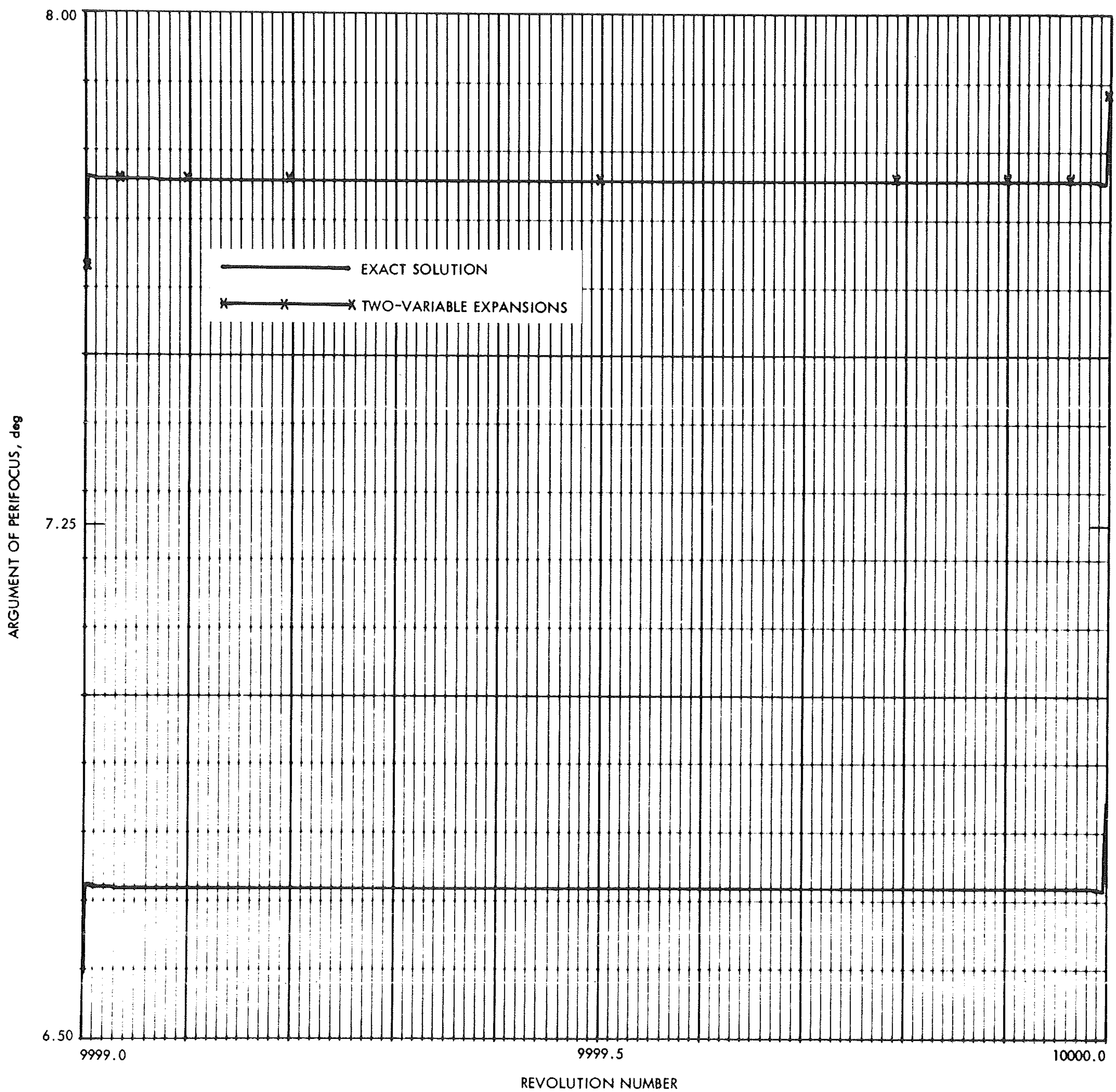


Fig. G-82. Variation of argument of perifocus in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

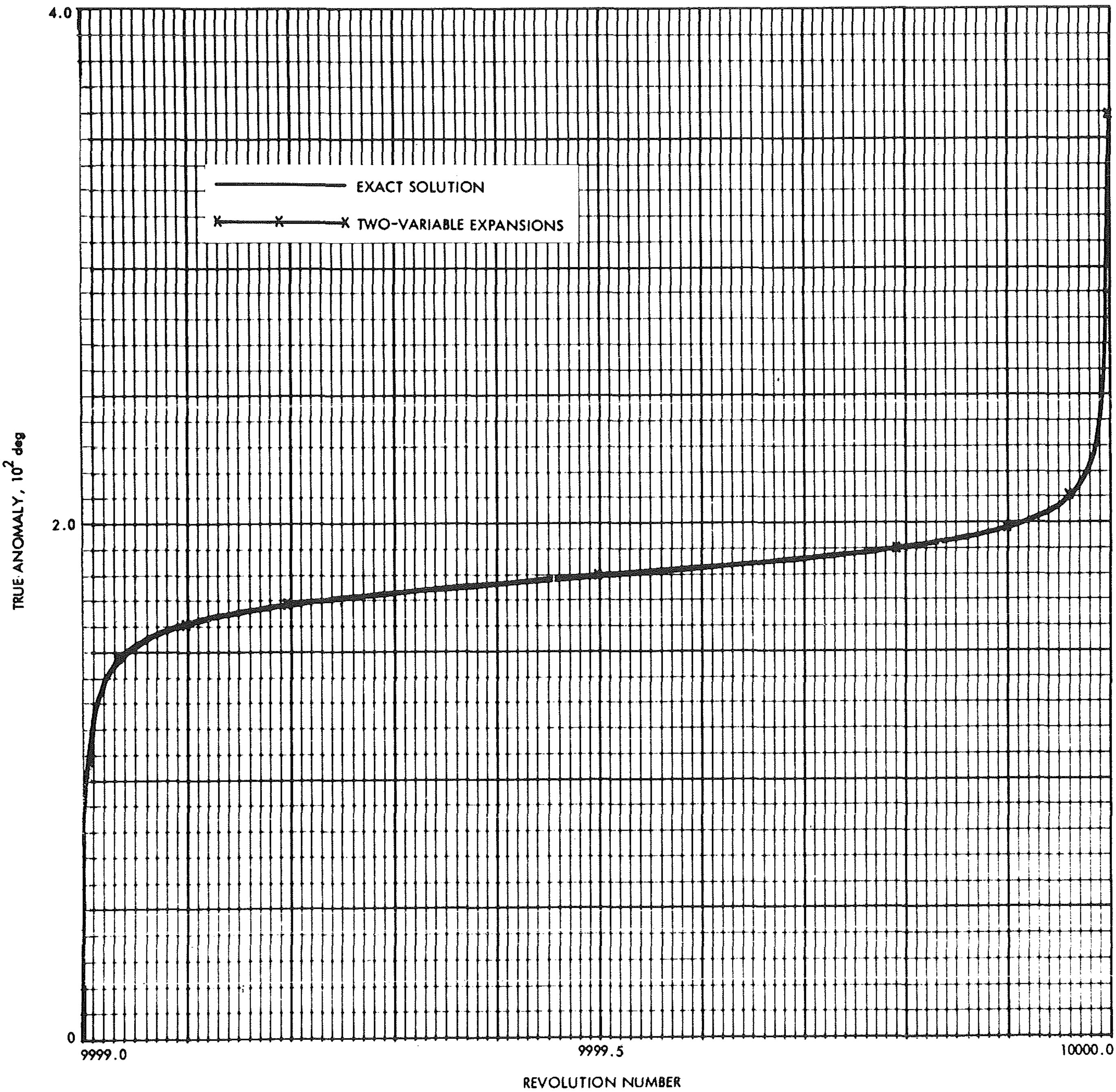


Fig. G-83. Variation of true anomaly in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

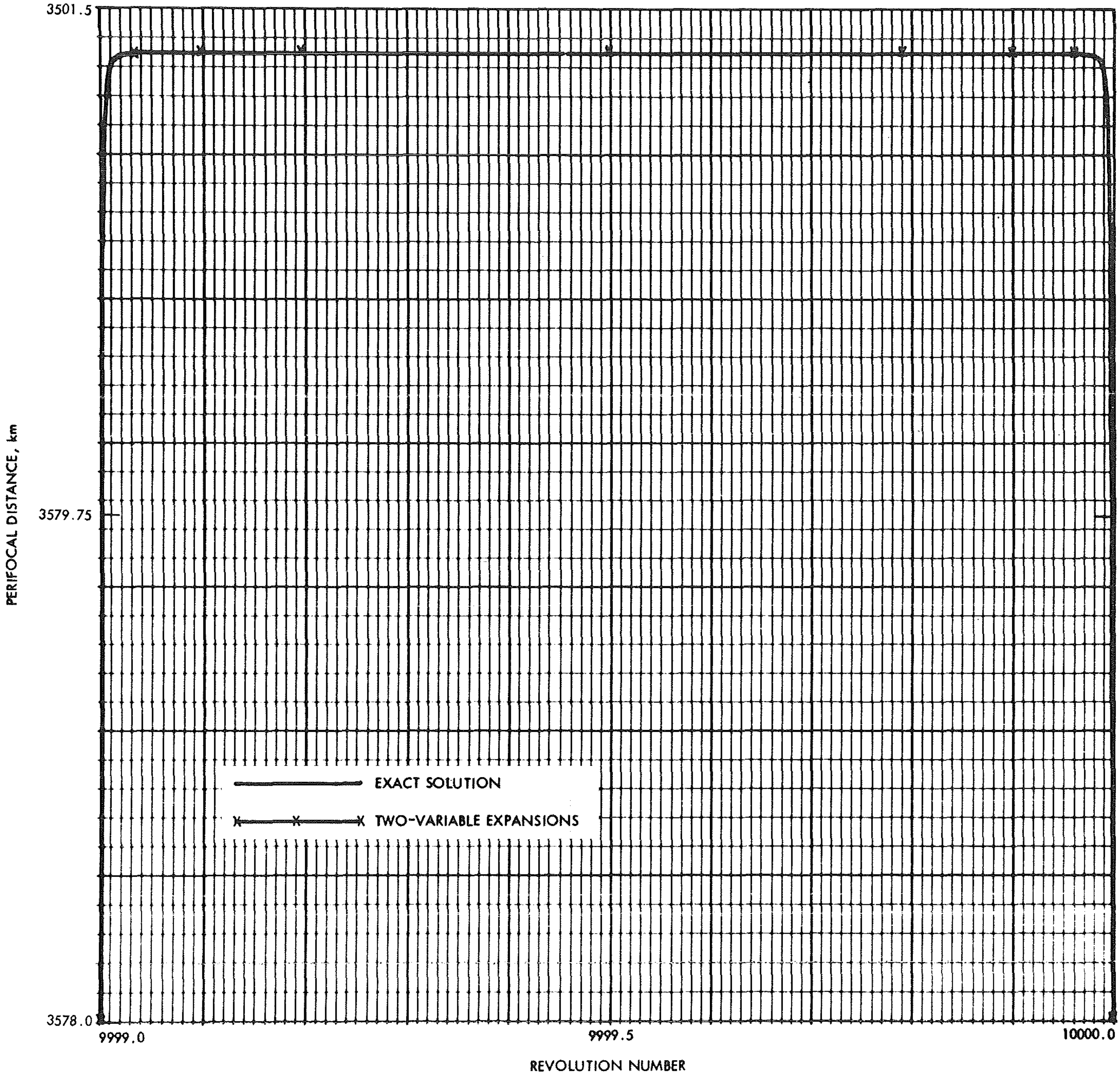


Fig. G-84. Variation of perifocal distance in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

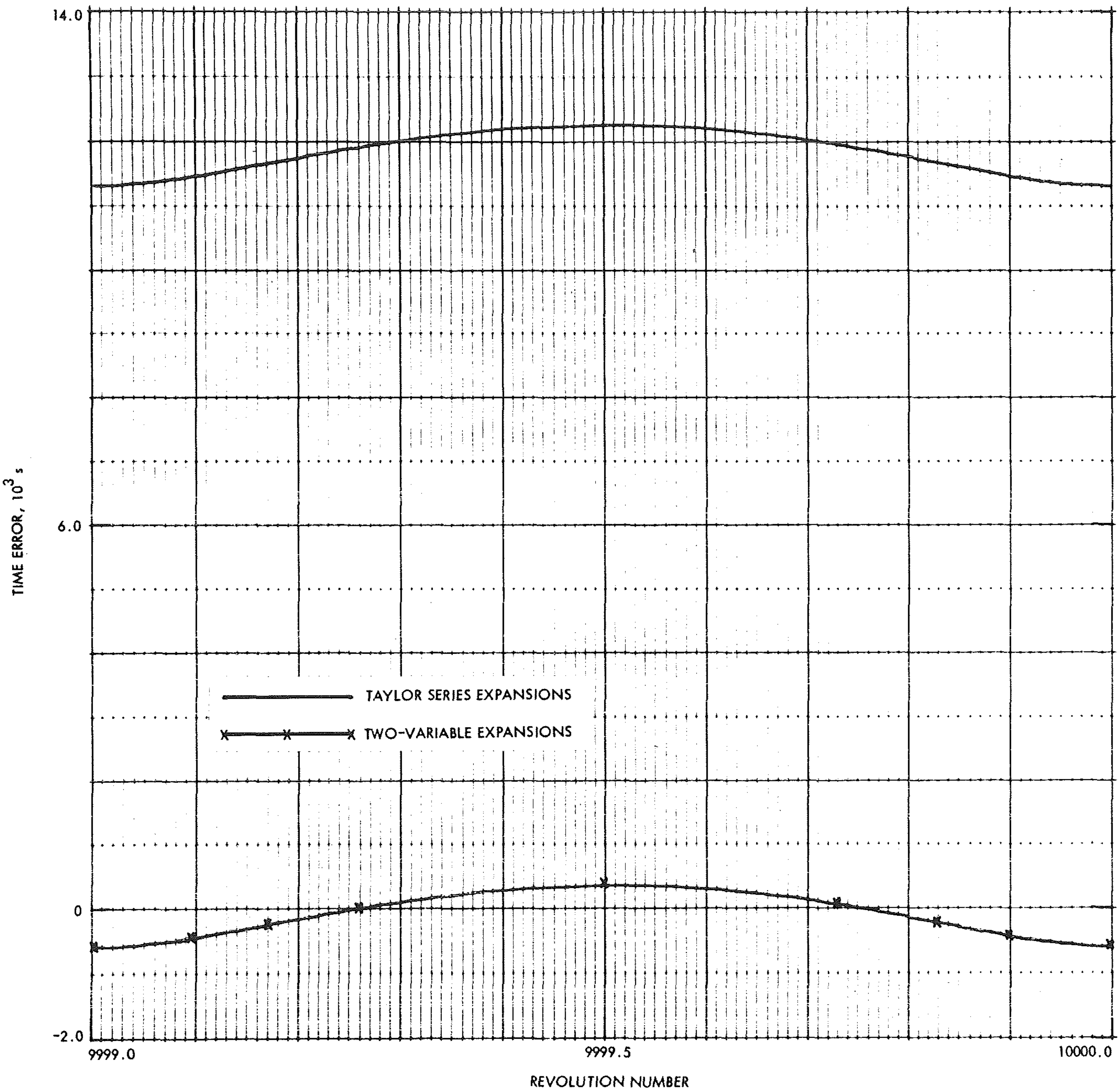


Fig. G-85. Time error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 8300$ km, $v_0 = 0$ deg)

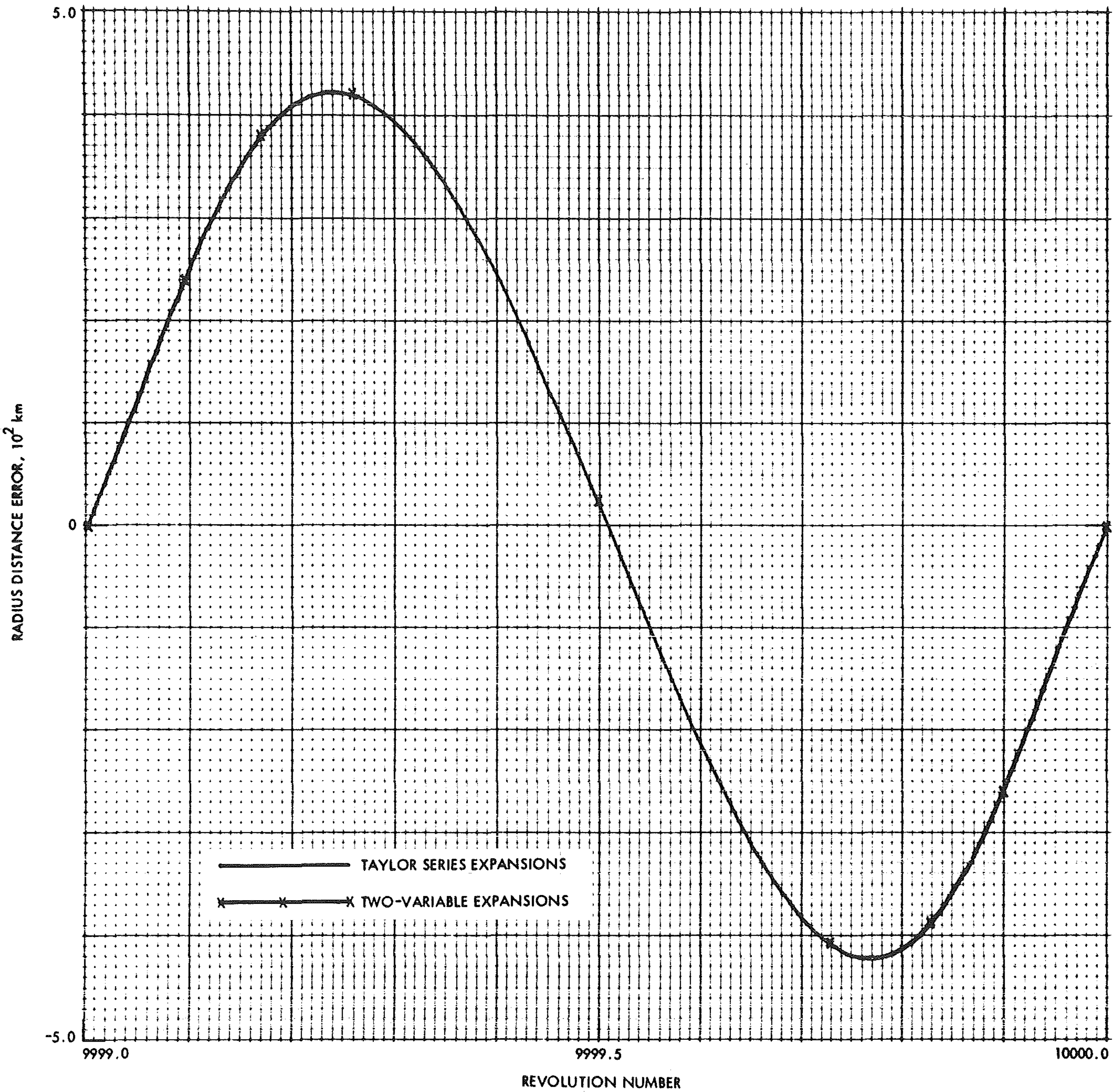


Fig. G-86. Radius distance error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 8300$ km, $v_0 = 0$ deg)

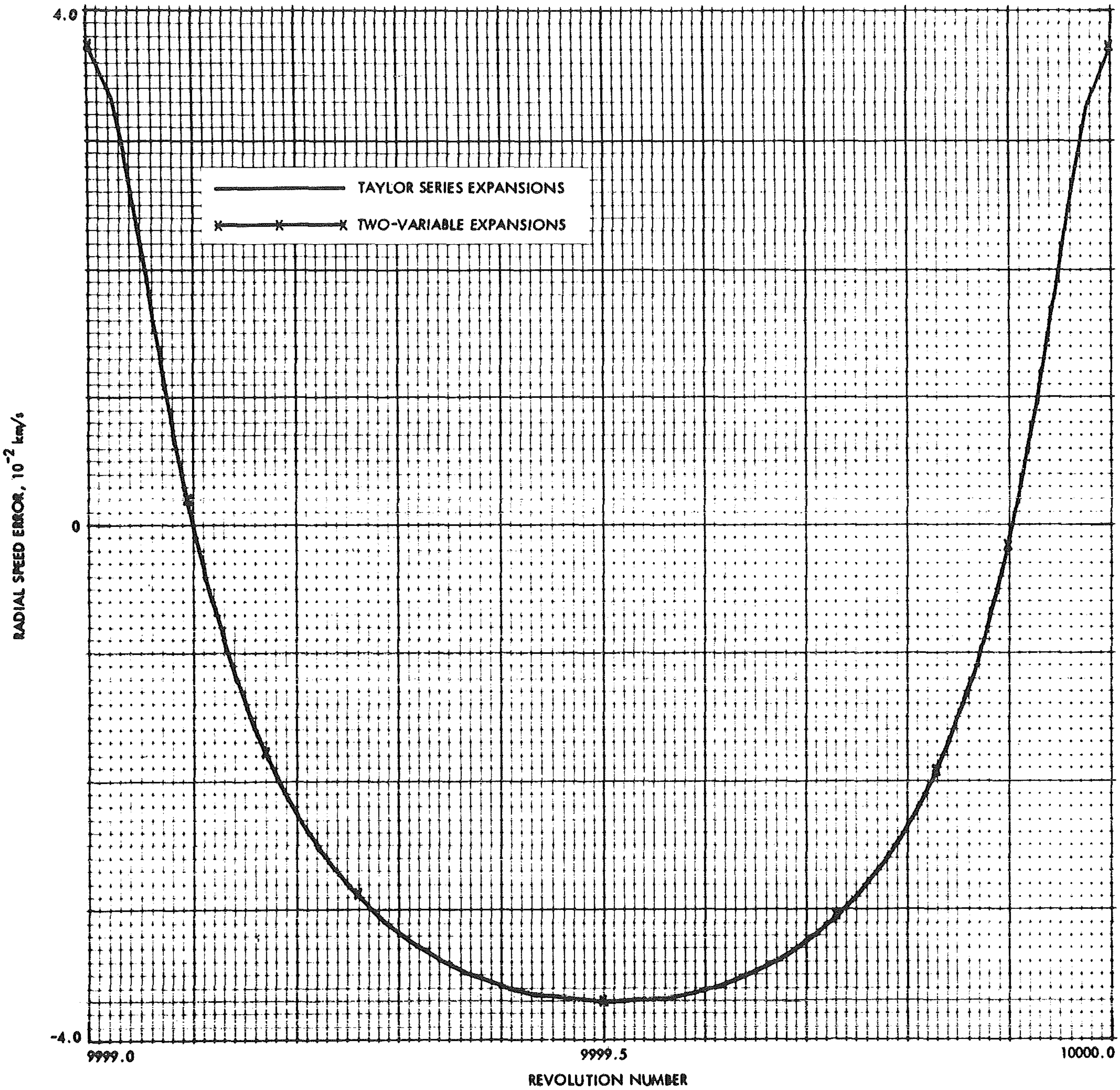


Fig. G-87. Radial speed error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 8300$ km, $v_0 = 0$ deg)

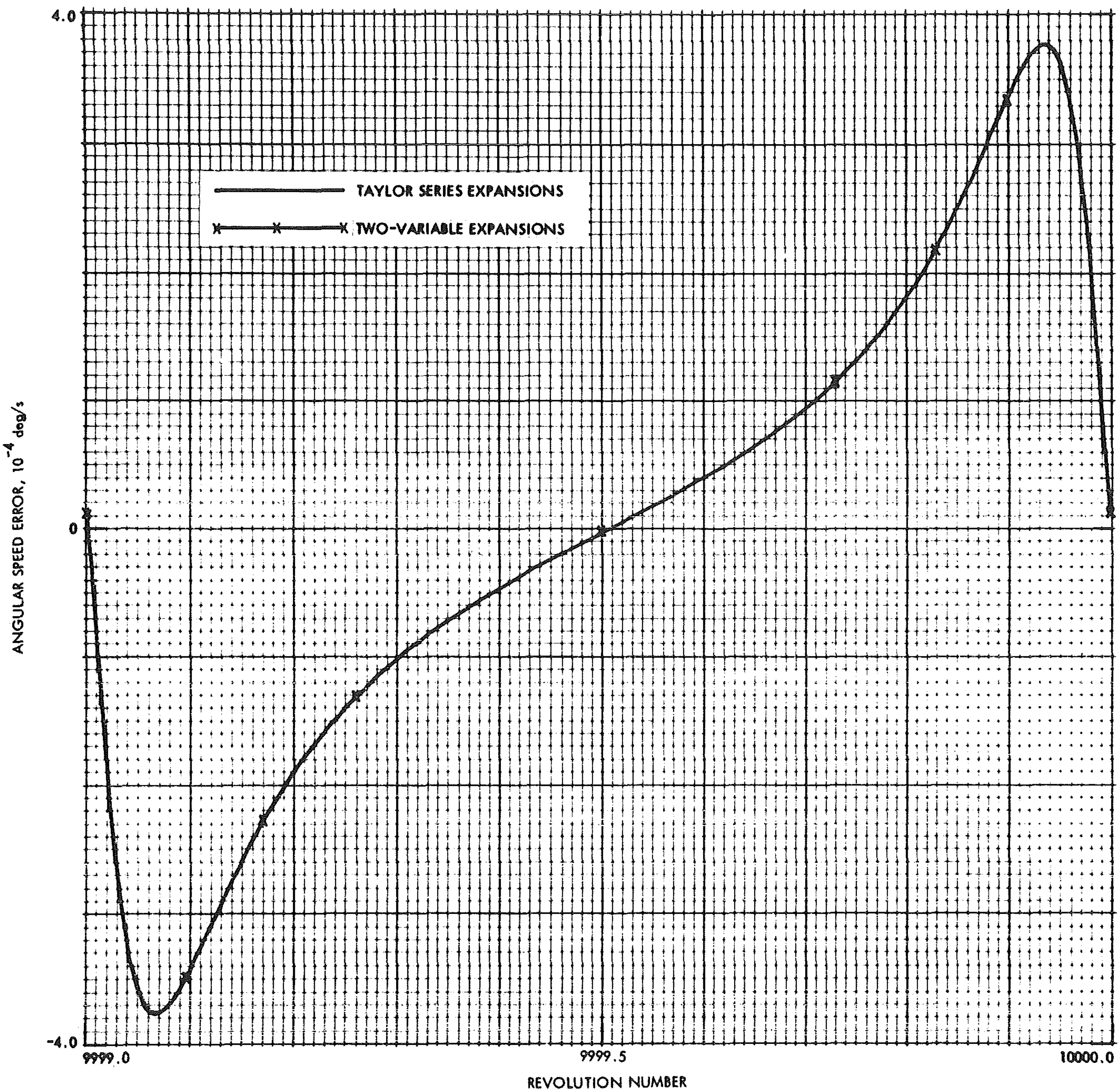


Fig. G-88. Angular speed error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 8300$ km, $v_0 = 0$ deg)

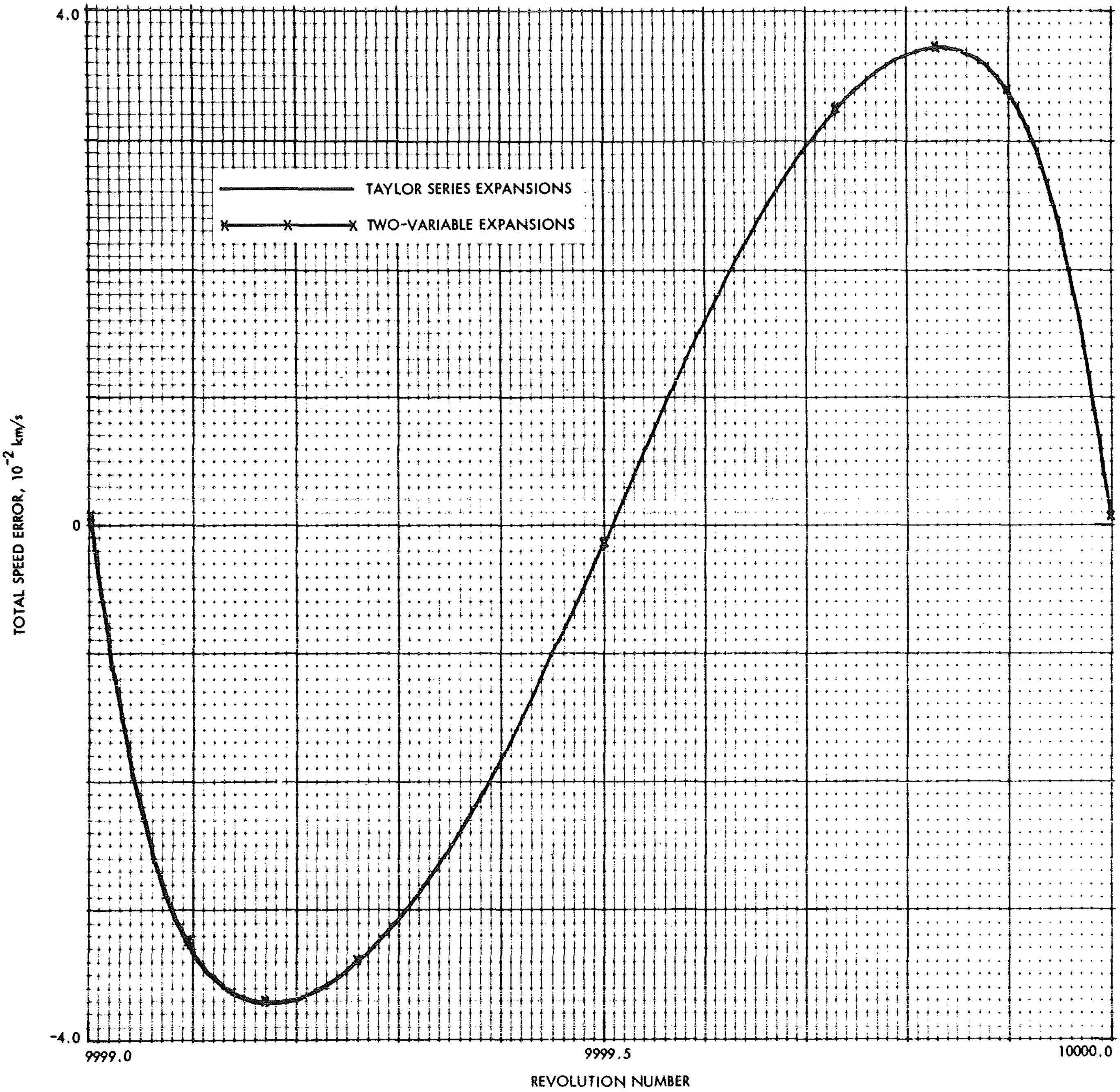


Fig. G-89. Total speed error in revolution 10,000, general perturbations solutions (e_0 = 0.5, q_0 = 8300 km, v_0 = 0 deg)

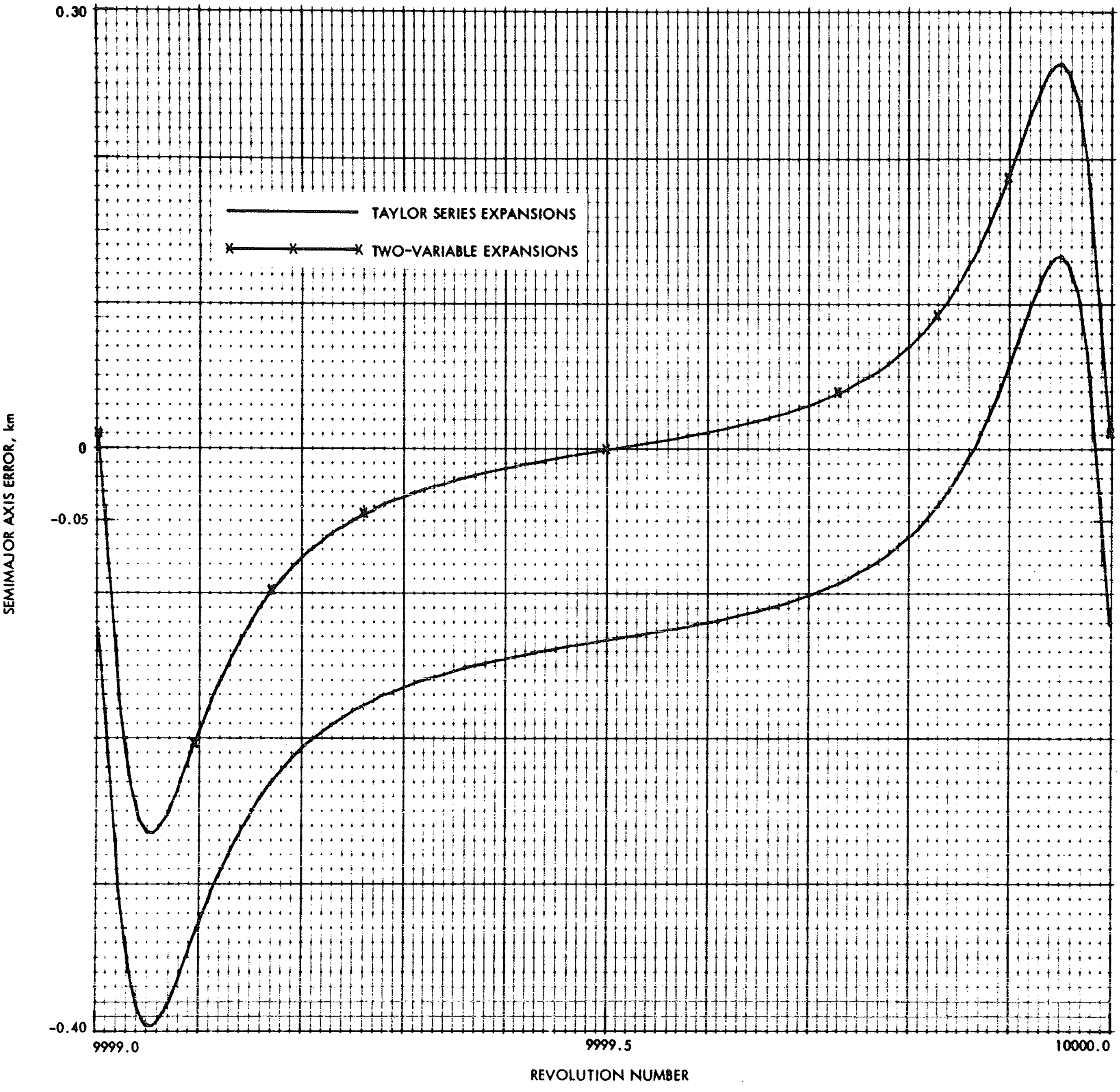


Fig. G-90. Semimajor axis error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 8300$ km, $v_0 = 0$ deg)

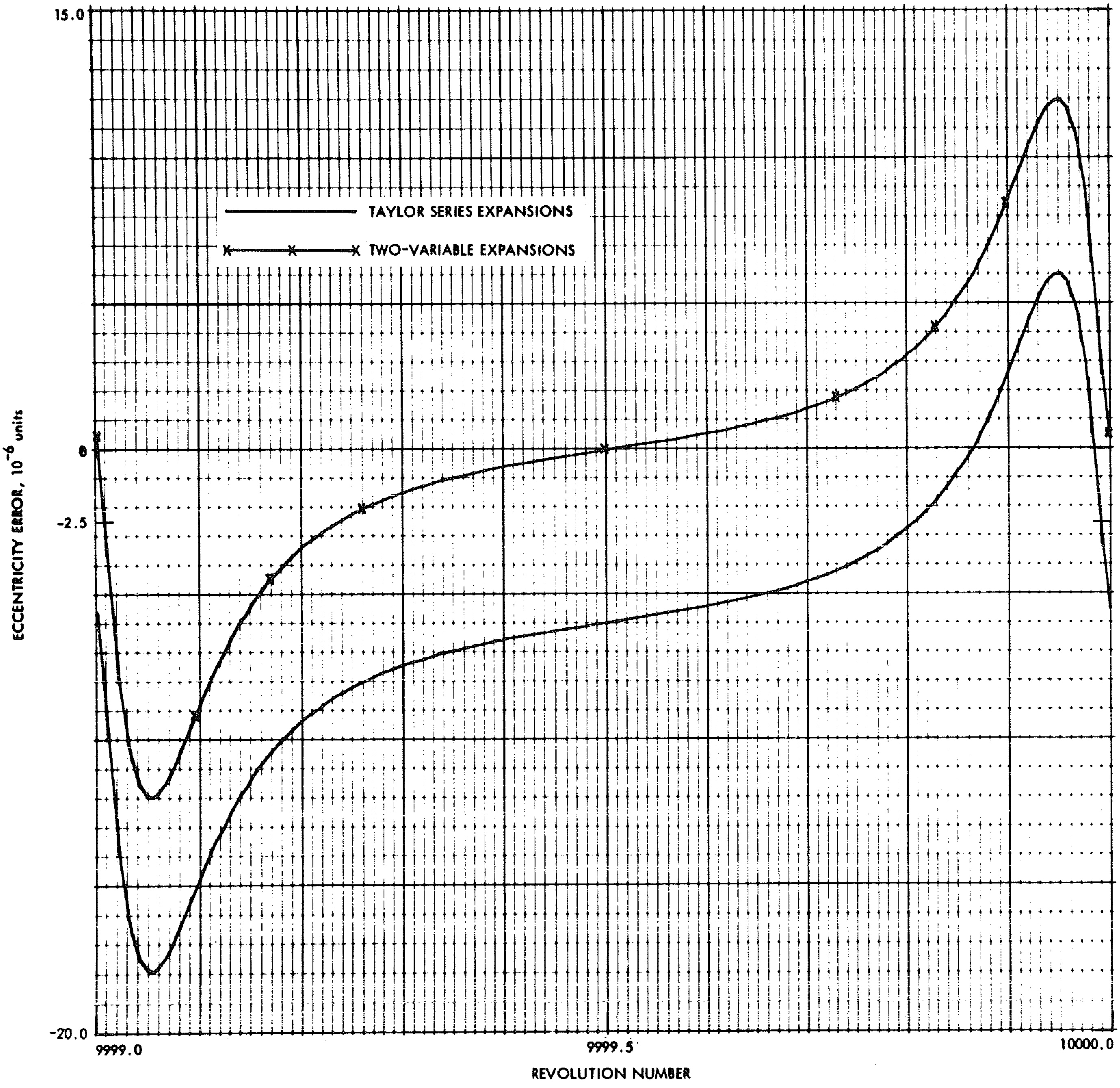


Fig. G-91. Eccentricity error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 8300$ km, $v_0 = 0$ deg)

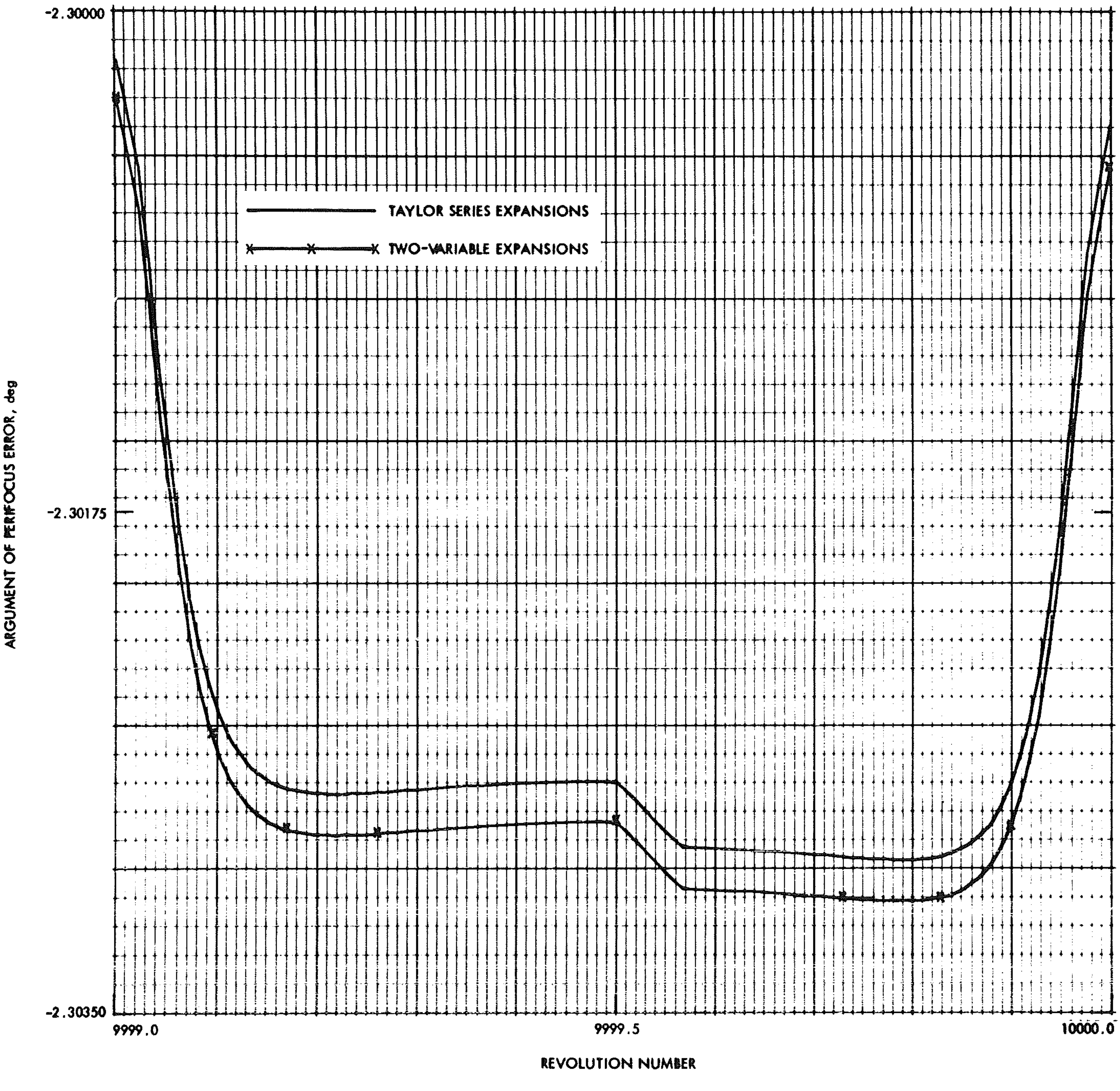


Fig. G-92. Argument of perifocus error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 8300$ km, $v_0 = 0$ deg)

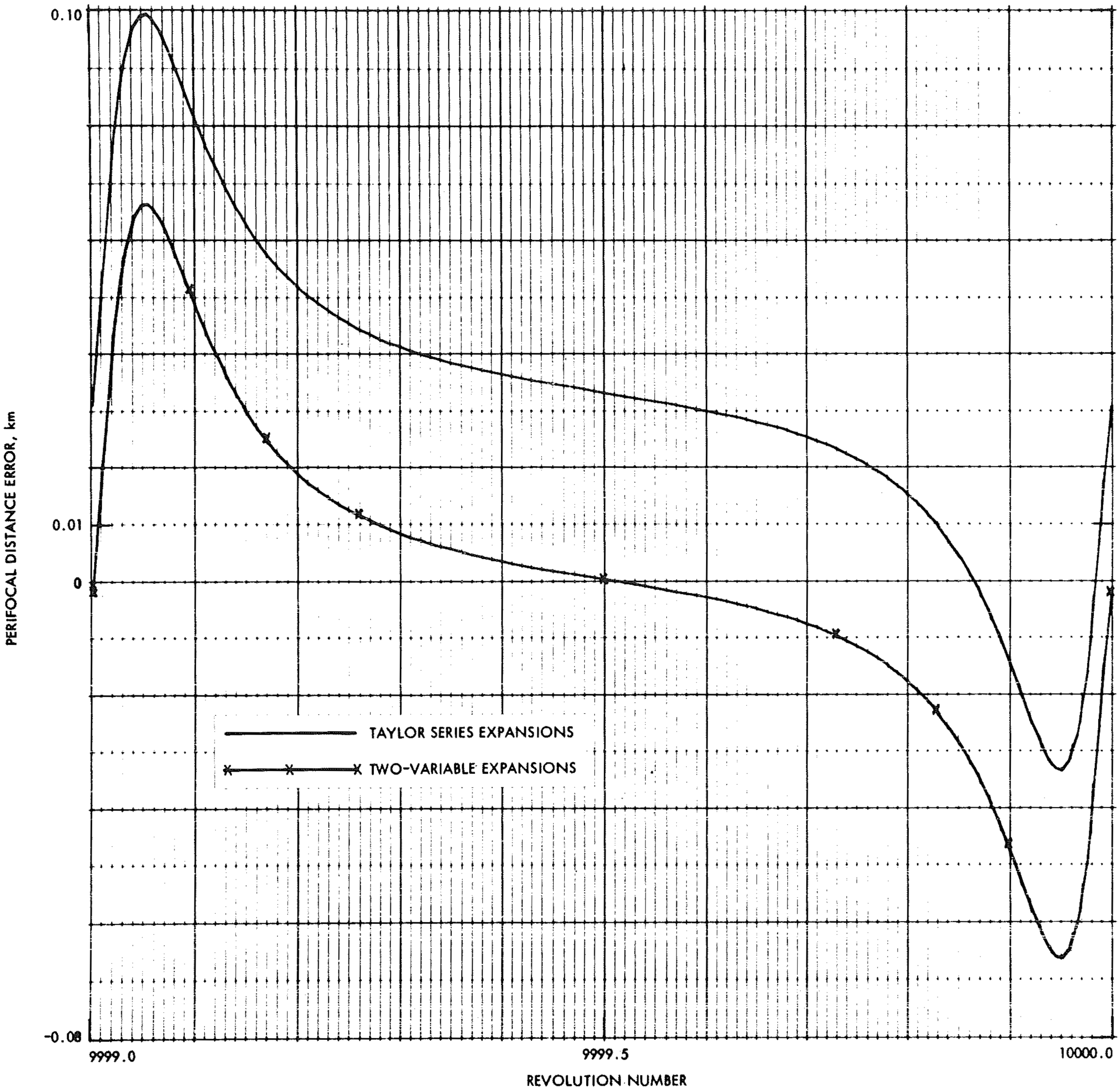


Fig. G-93. Perifocal distance error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 8300$ km, $v_0 = 0$ deg)

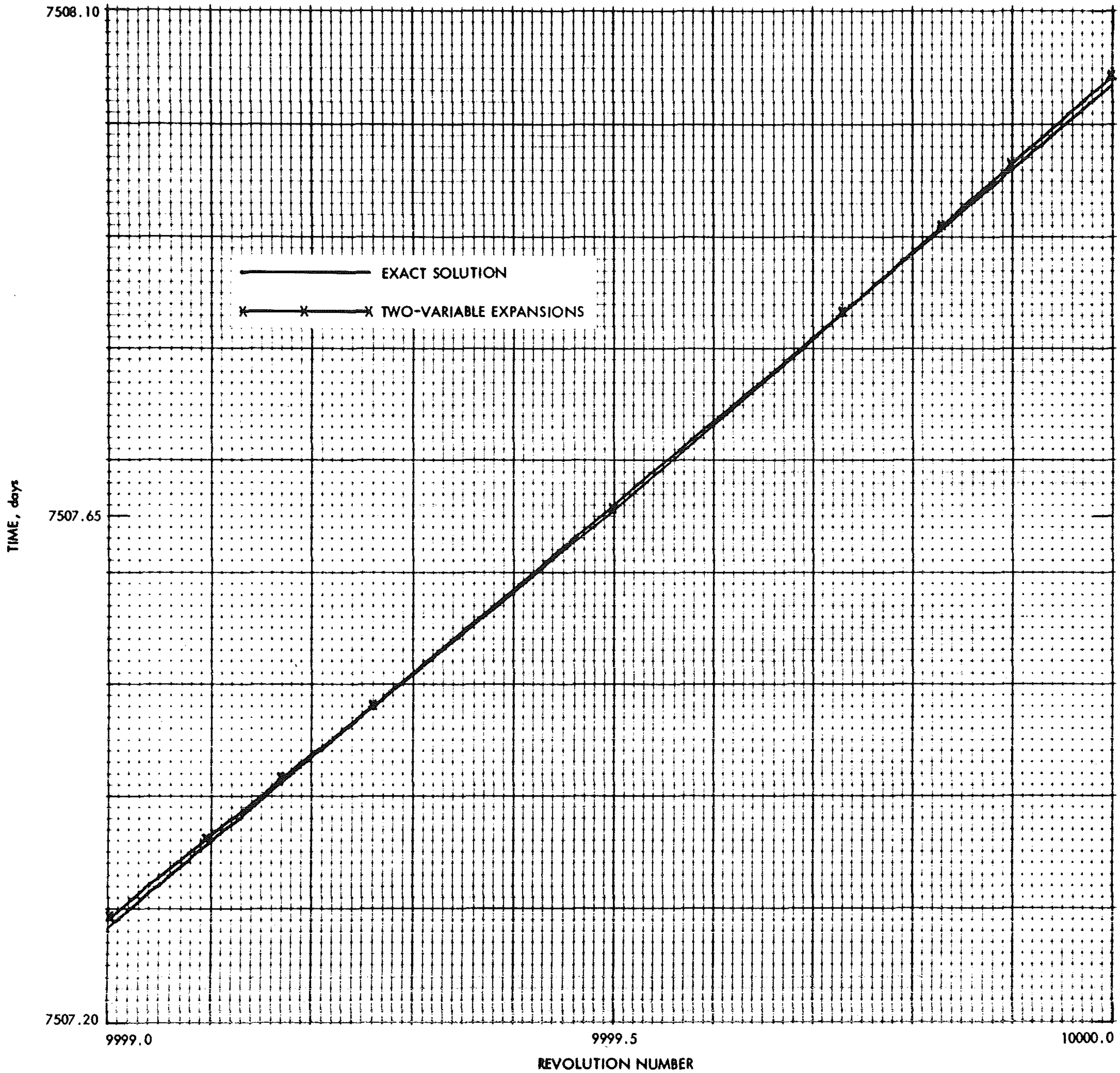


Fig. G-94. Variation of time in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 8300$ km, $v_0 = 0$ deg)

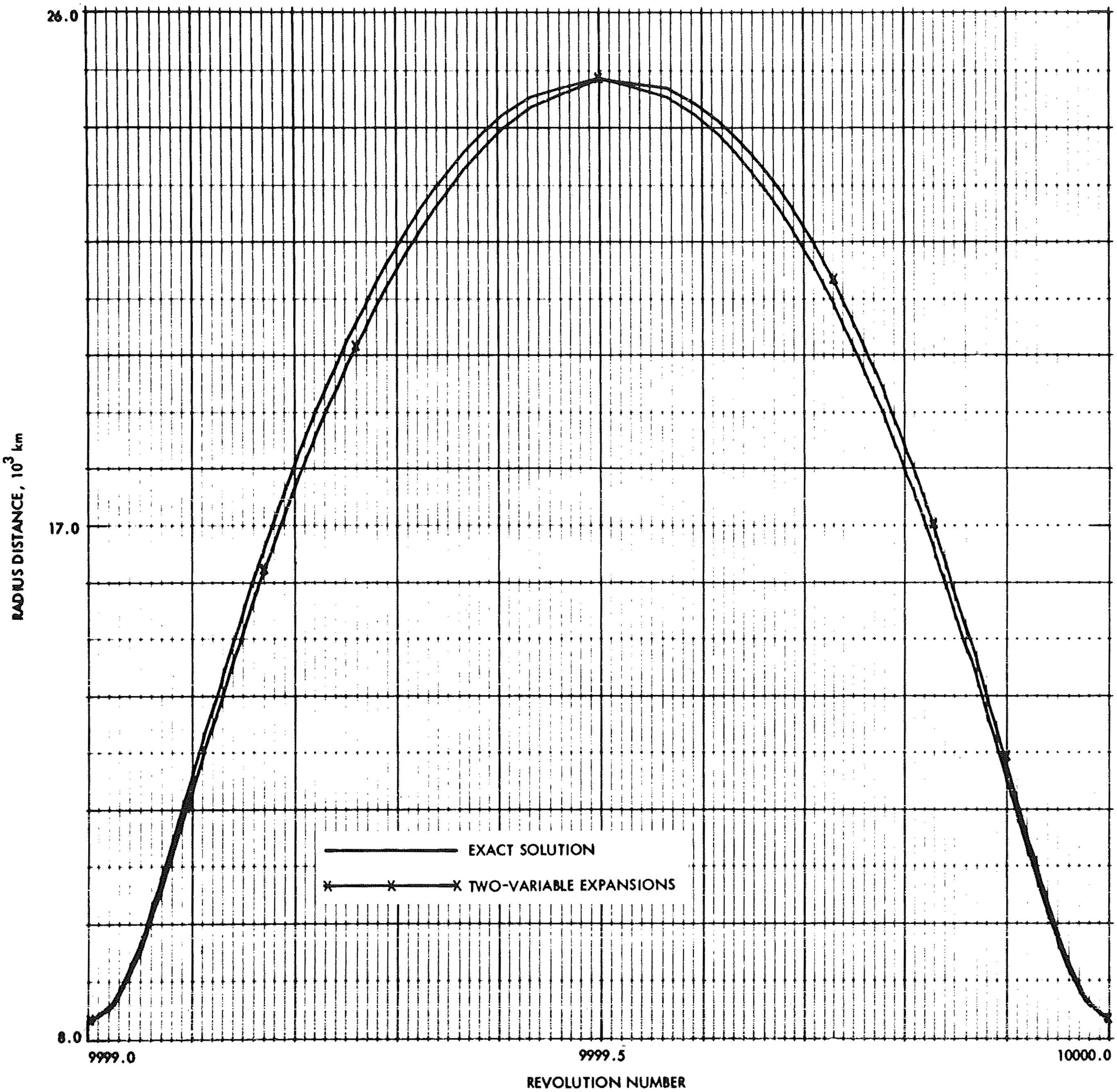


Fig. G-95. Variation of radius distance in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 8300$ km, $v_0 = 0$ deg)

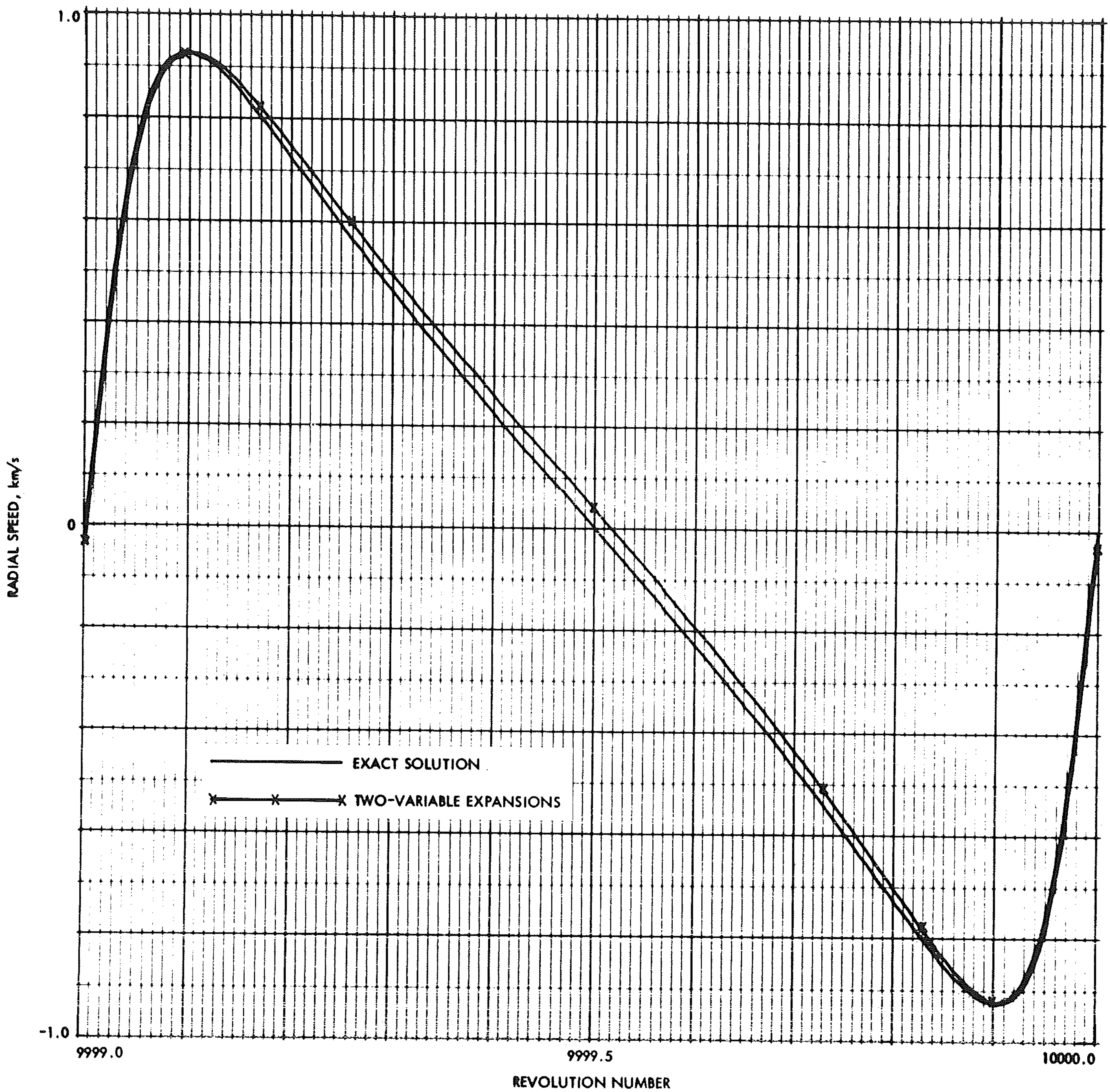


Fig. G-96. Variation of radial speed in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 8300$ km, $v_0 = 0$ deg)

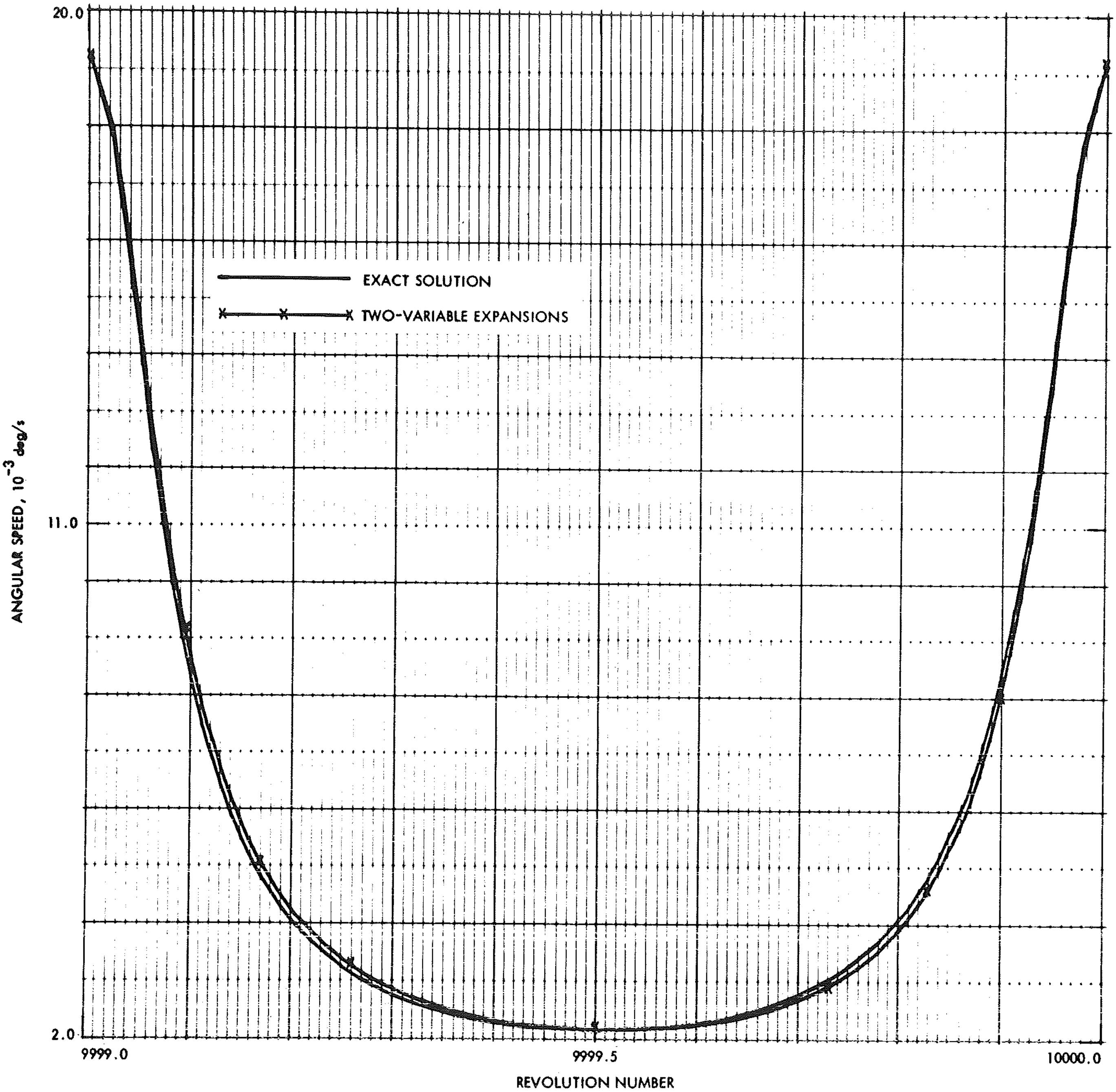


Fig. G-97. Variation of angular speed in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 8300$ km, $v_0 = 0$ deg)

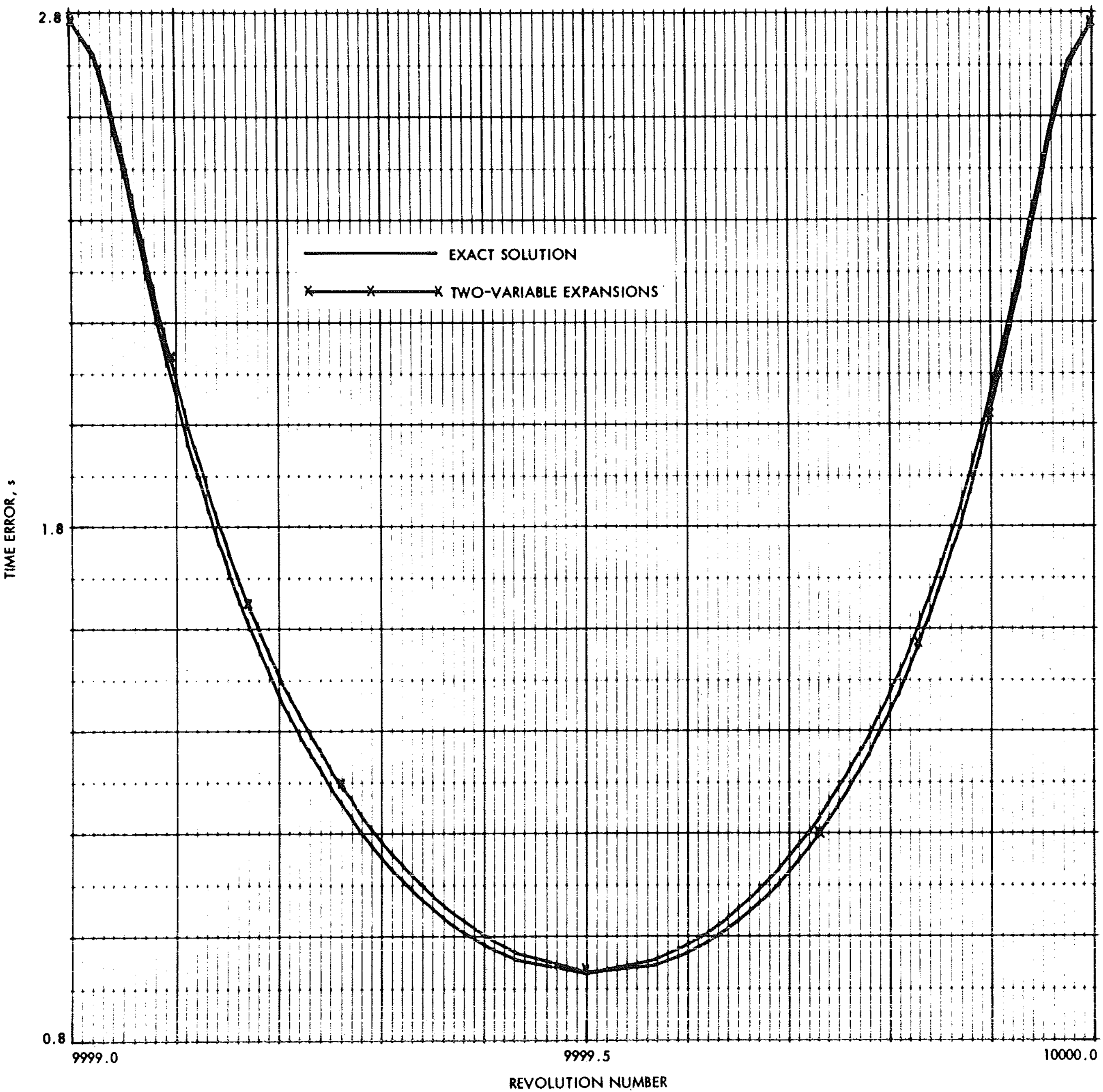


Fig. G-98. Variation of total speed in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 8300$ km, $v_0 = 0$ deg)

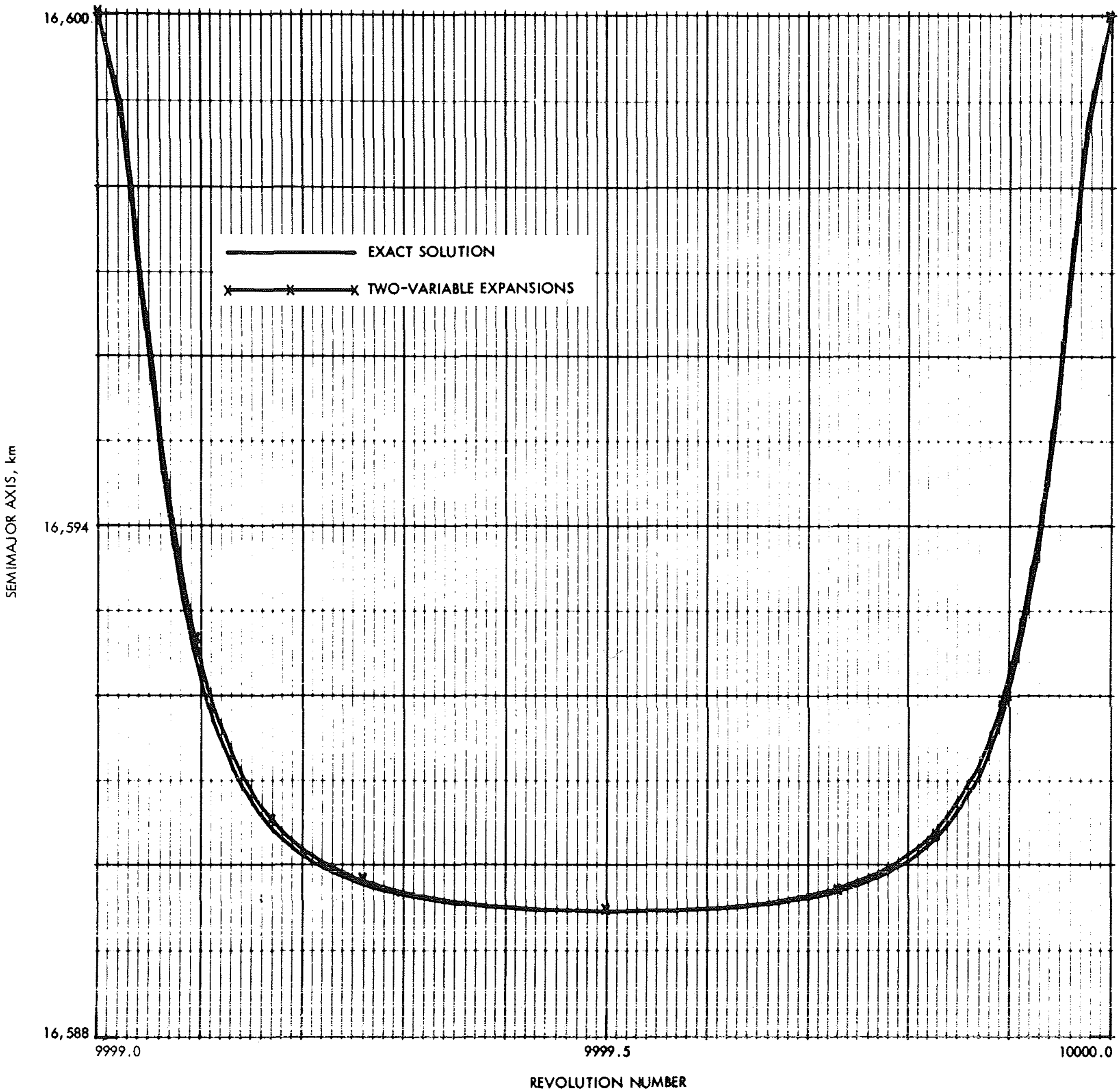


Fig. G-99. Variation of semimajor axis in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 8300$ km, $v_0 = 0$ deg)

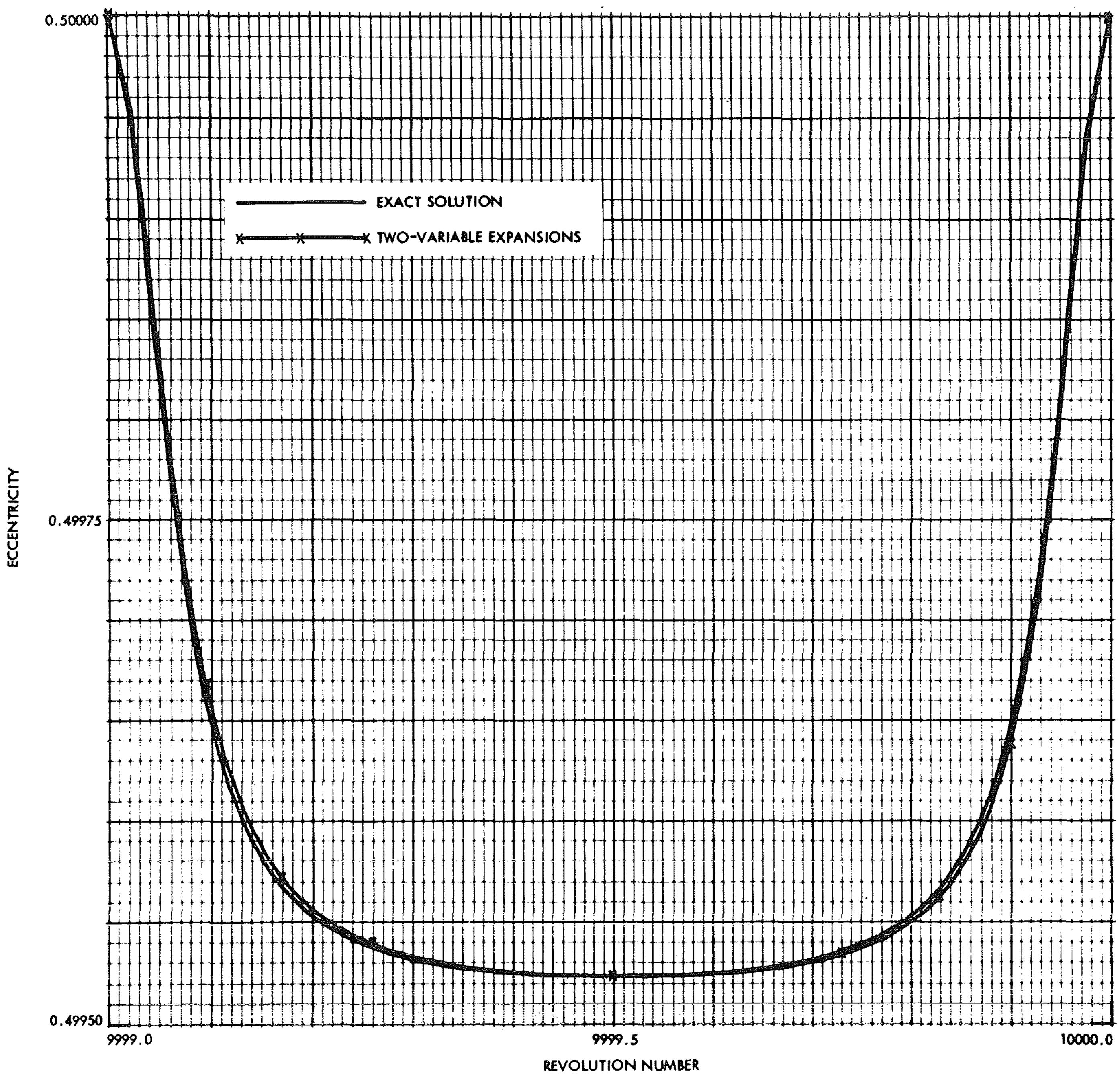


Fig. G-100. Variation of eccentricity in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 8300$ km, $v_0 = 0$ deg)

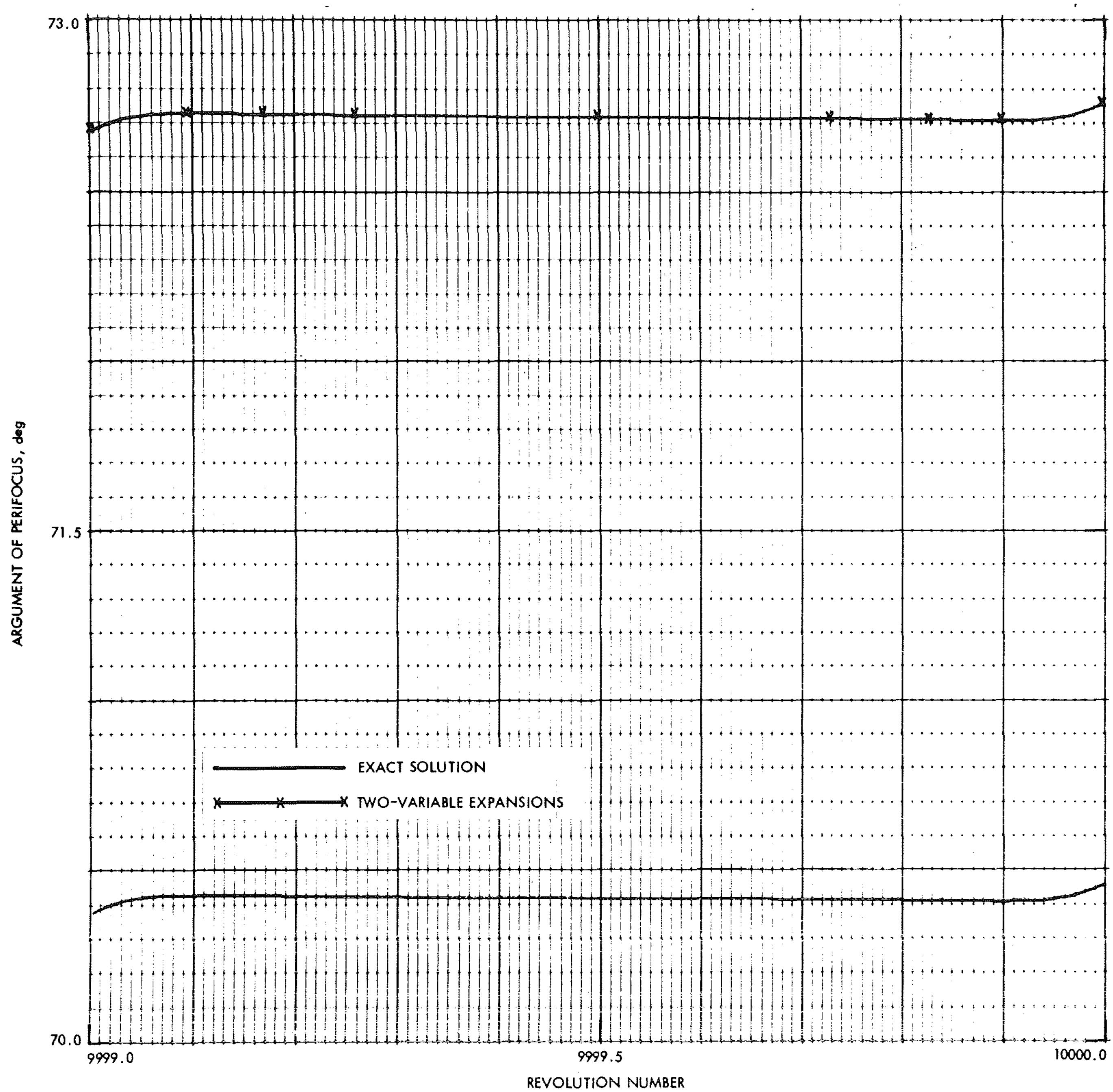


Fig. G-101. Variation of argument of perifocus in revolution 10,000, exact solution and two-variable asymptotic expansions solution (e_0 = 0.5, q_0 = 8300 km, v_0 = 0 deg)

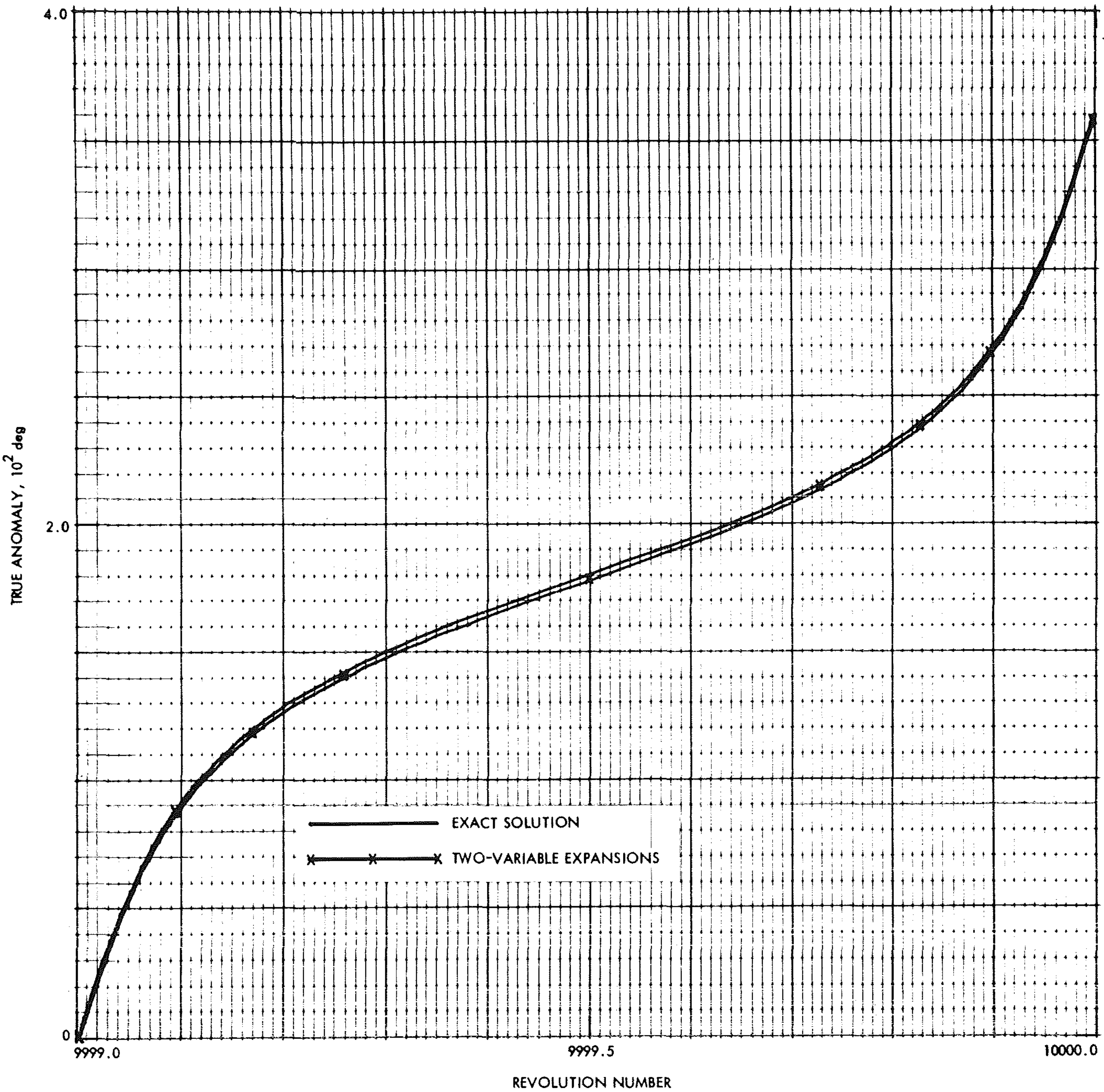


Fig. G-102. Variation of true anomaly in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 8300$ km, $v_0 = 0$ deg)

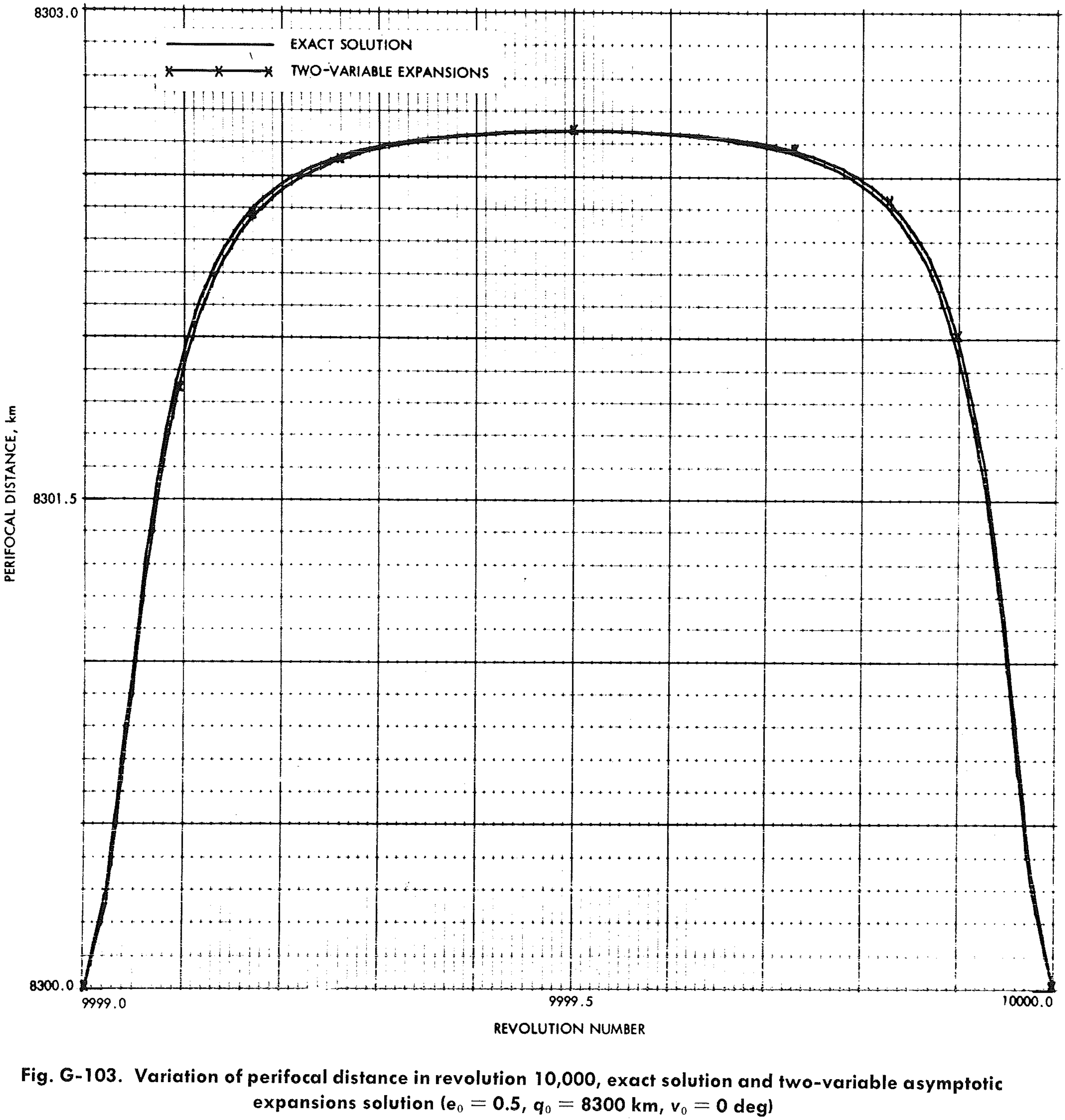


Fig. G-103. Variation of perifocal distance in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 8300$ km, $v_0 = 0$ deg)

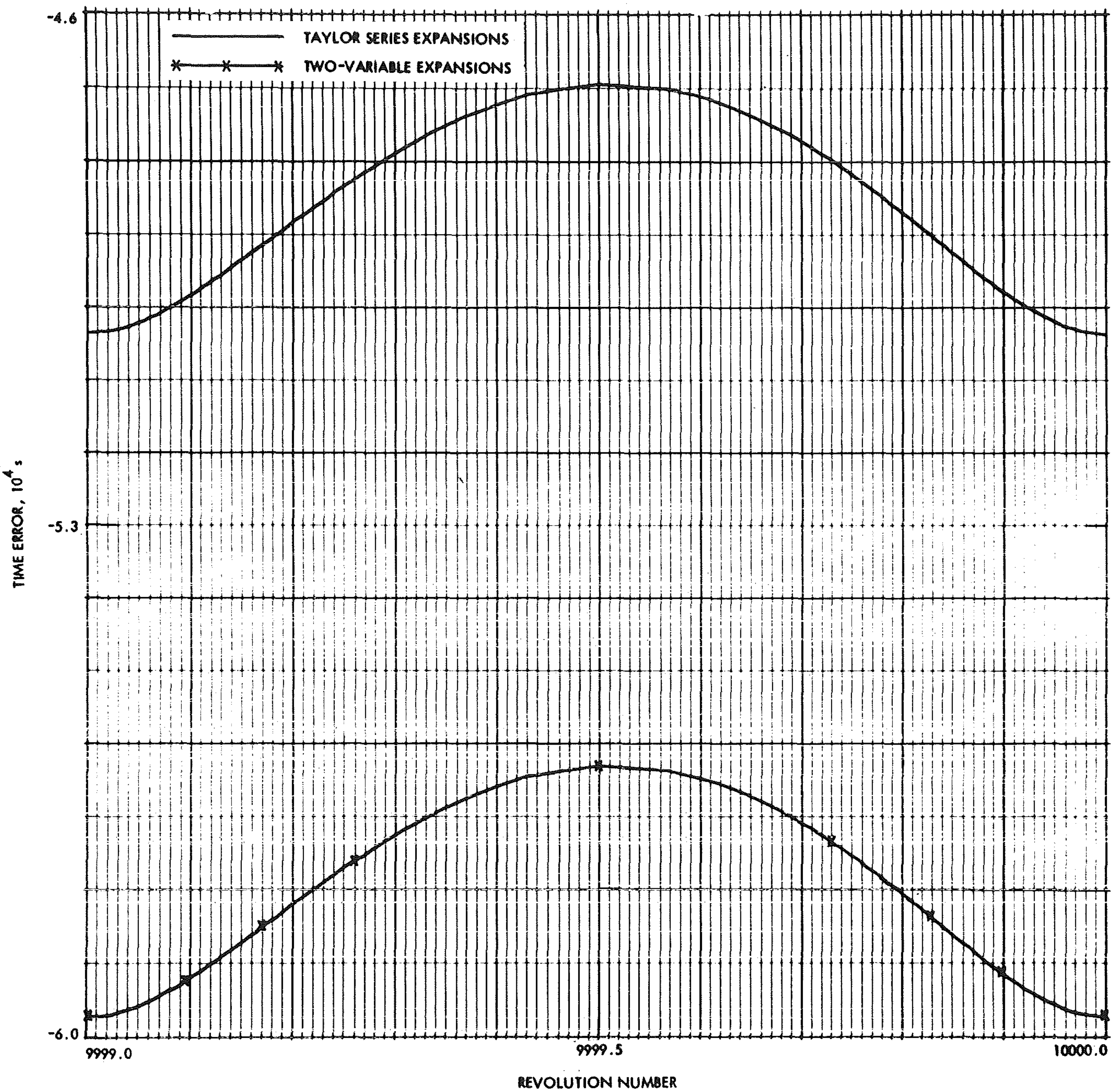


Fig. G-104. Time error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 13{,}393$ km, $v_0 = 0$ deg)

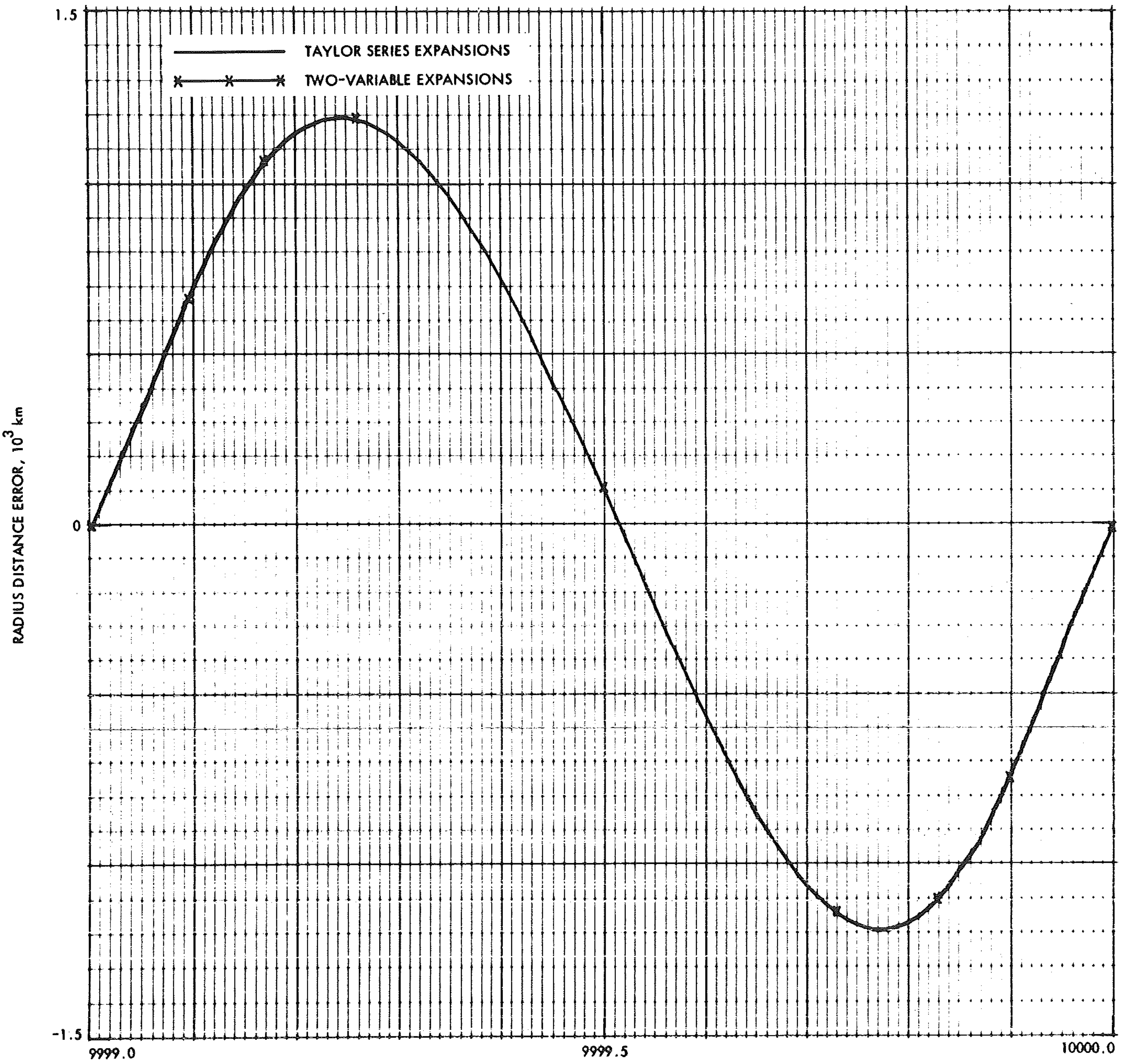


Fig. G-105. Radius distance error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 13{,}393$ km, $v_0 = 0$ deg)

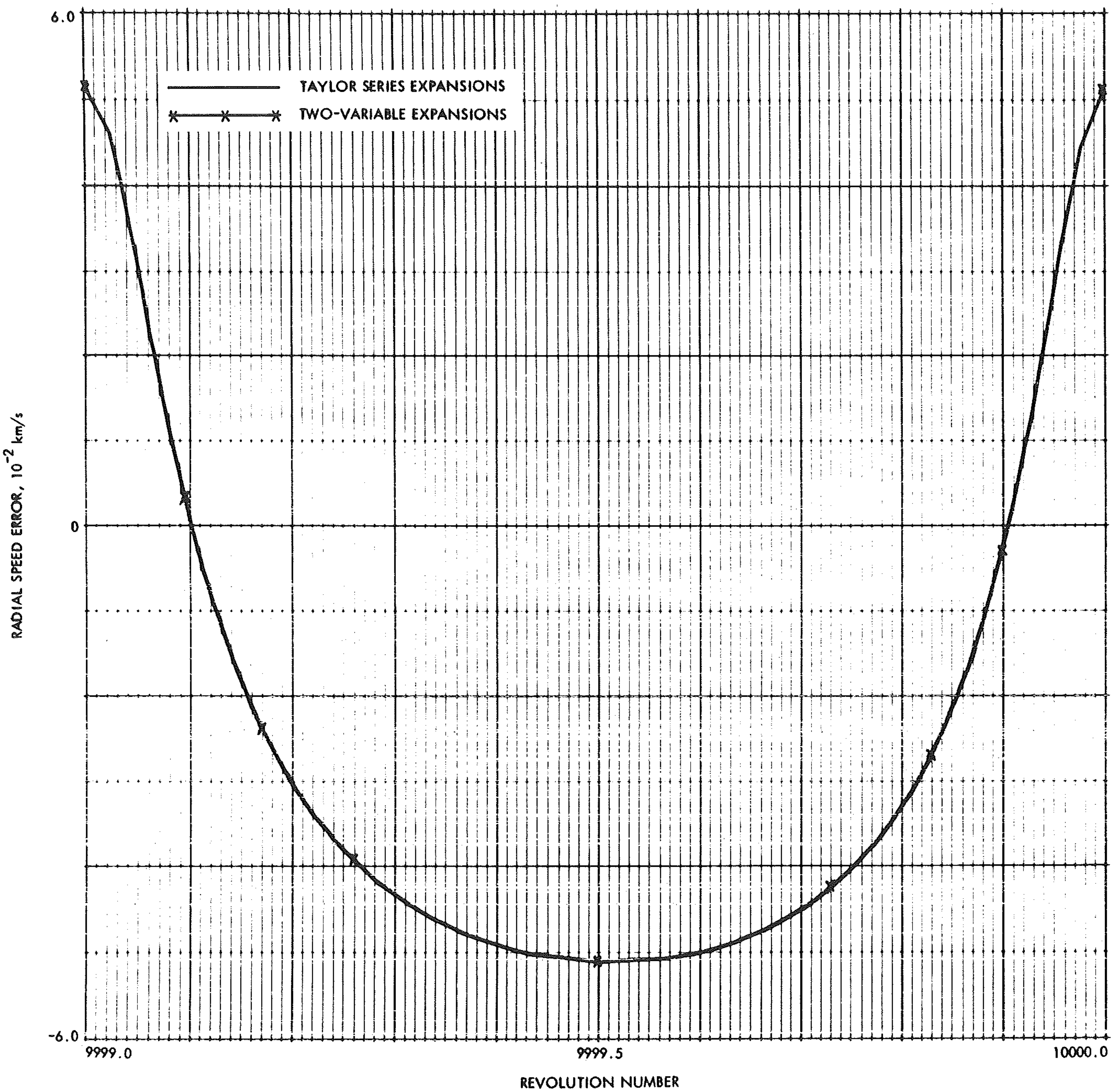


Fig. G-106. Radial speed error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 13{,}393$ km, $v_0 = 0$ deg)

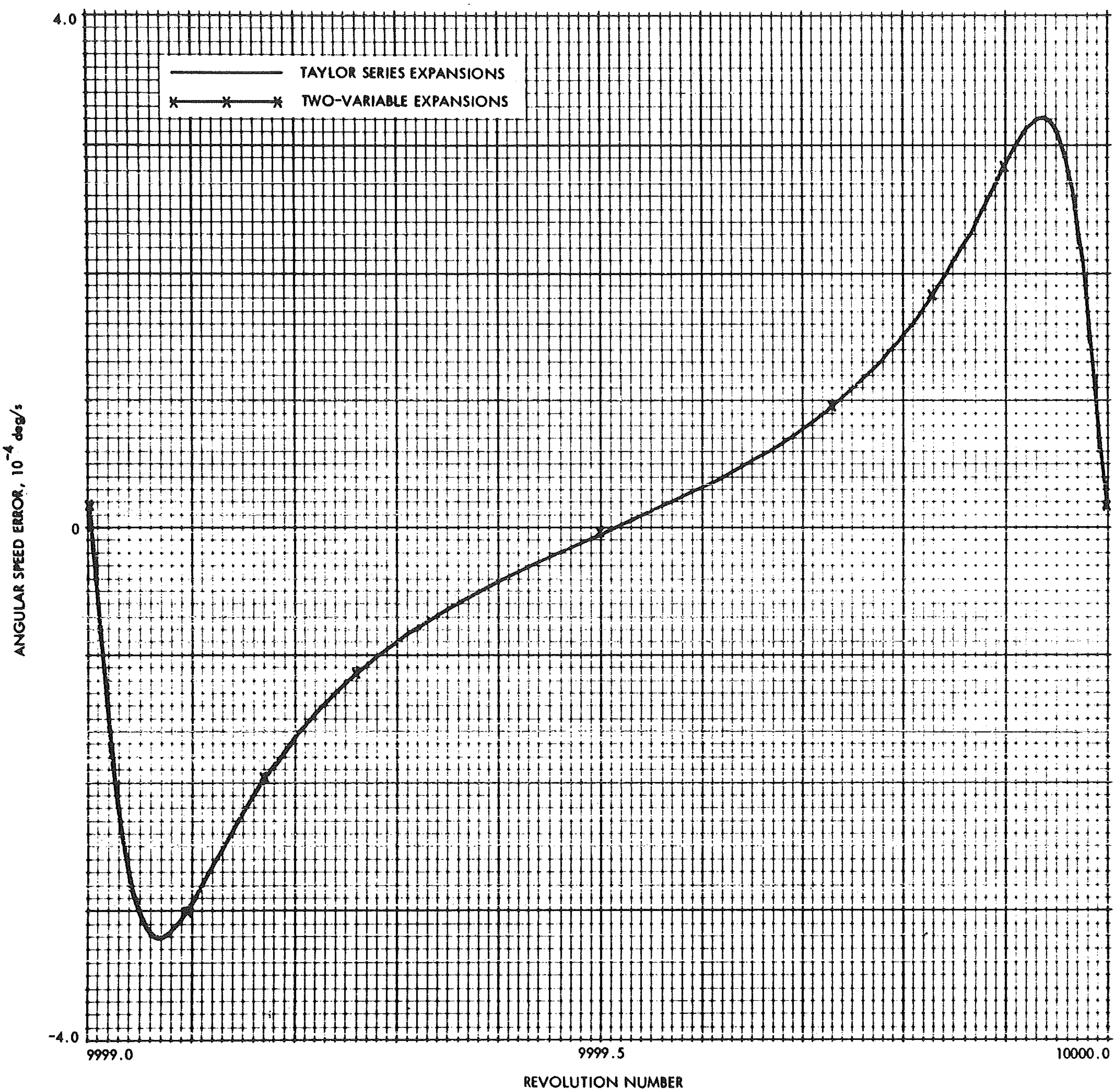


Fig. G-107. Angular speed error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 13{,}393$ km, $v_0 = 0$ deg)

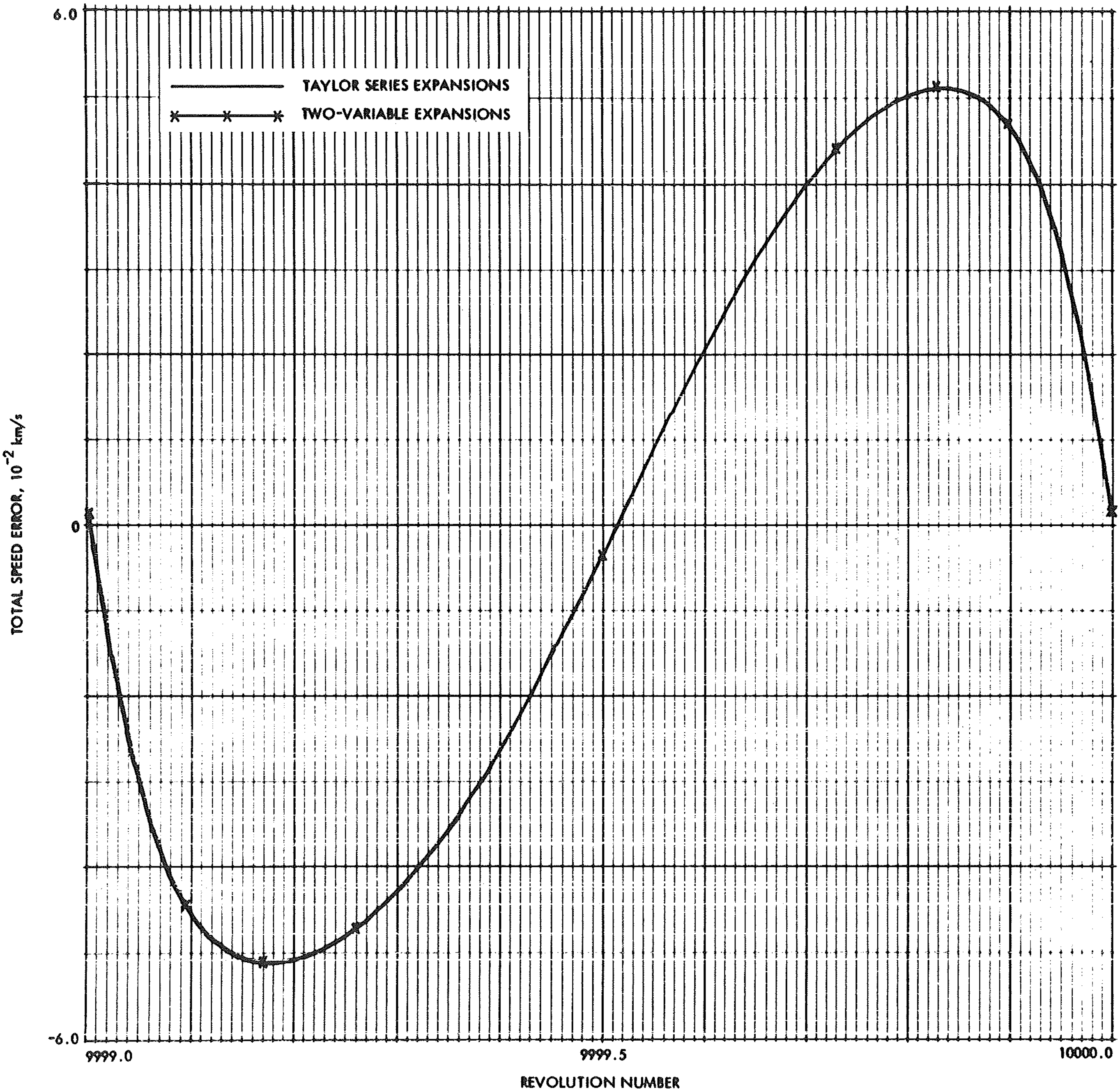


Fig. G-108. Total speed error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 13{,}393$ km, $v_0 = 0$ deg)

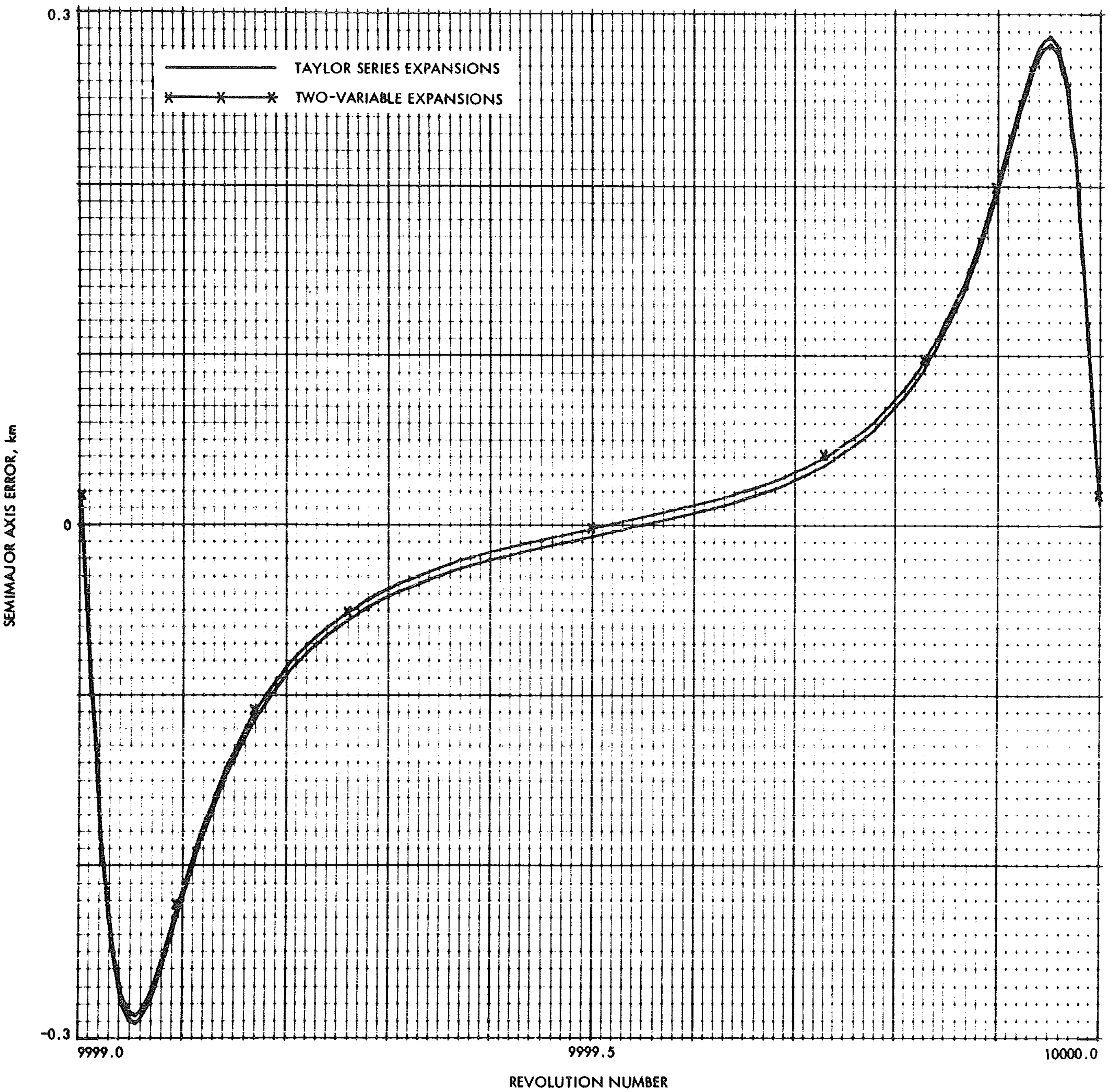


Fig. G-109. Semimajor axis error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 13{,}393$ km, $v_0 = 0$ deg)

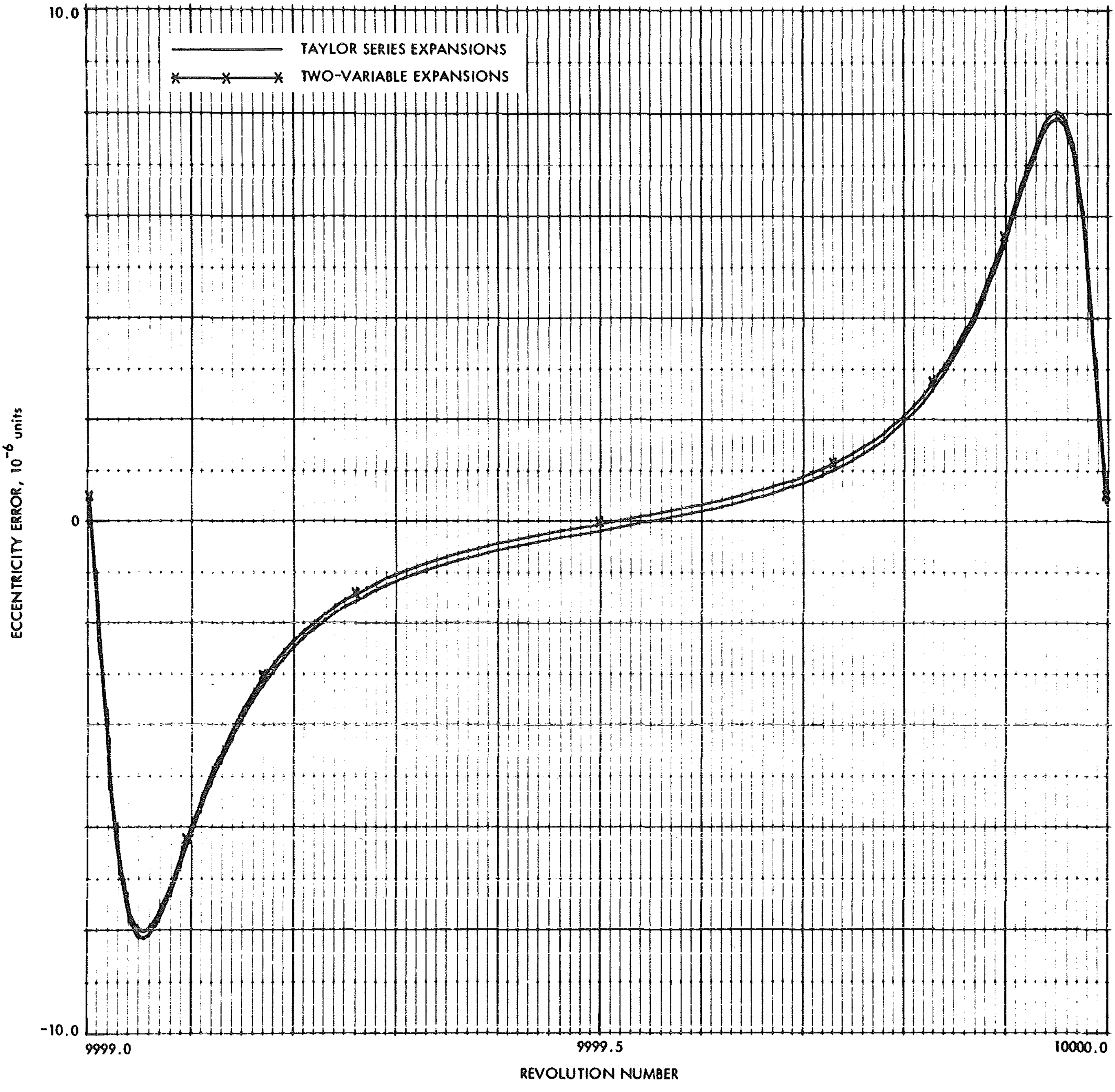


Fig. G-110. Eccentricity error in revolution 10,000, general perturbations solutions (e_0 = 0.5, q_0 = 13,393 km, v_0 = 0 deg)

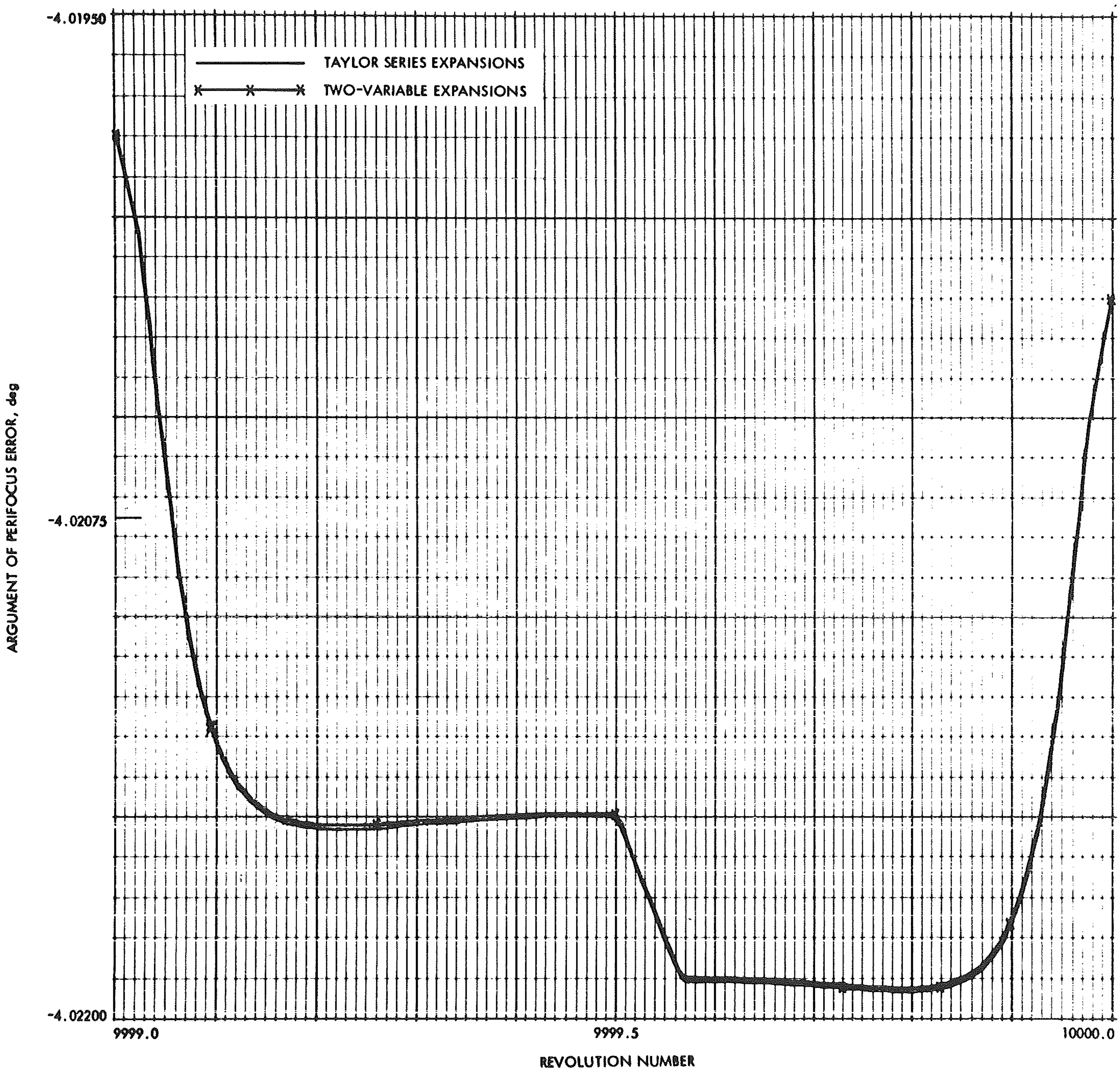


Fig. G-111. Argument of perifocus error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 13{,}393$ km, $v_0 = 0$ deg)

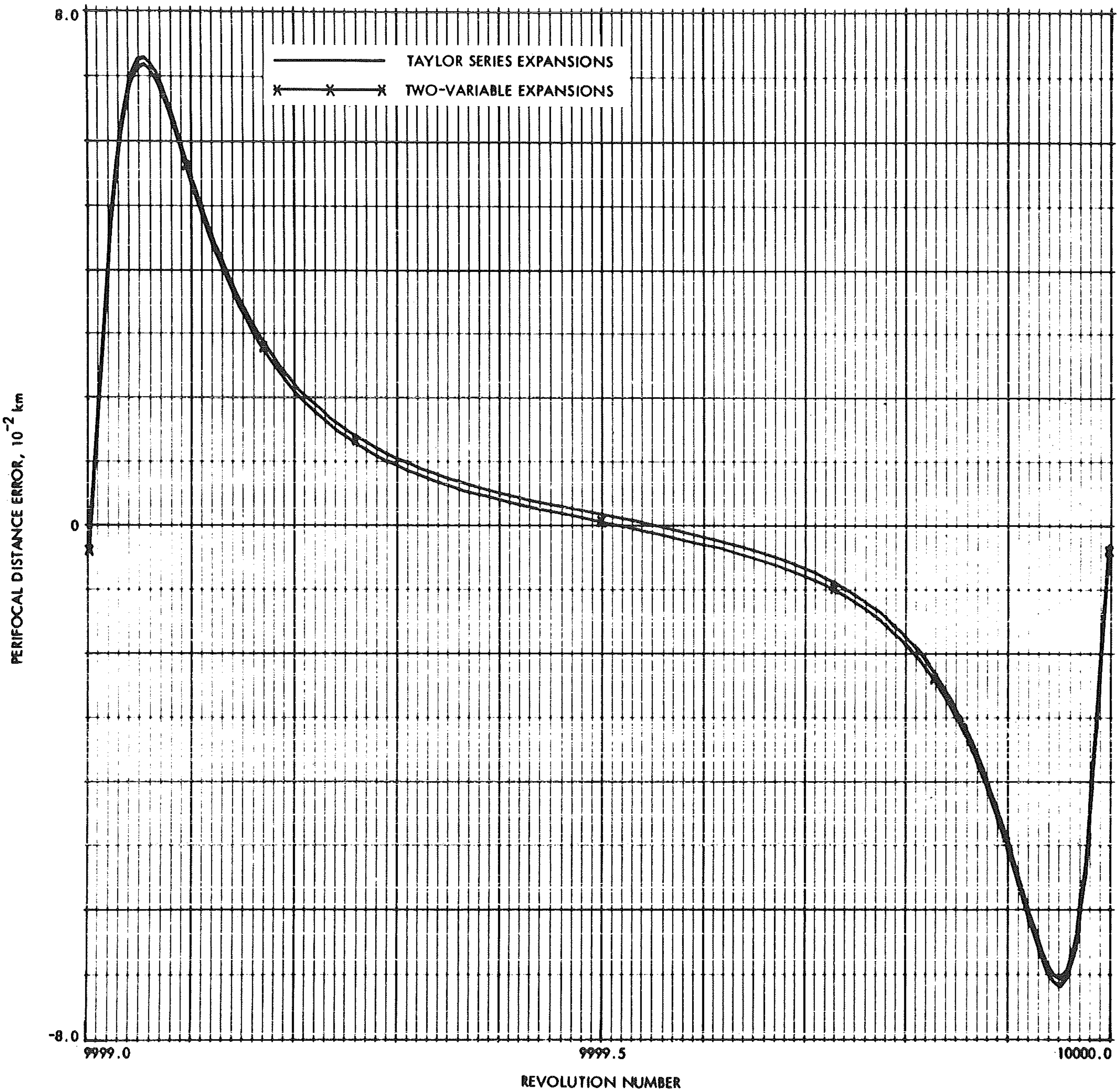


Fig. G-112. Perifocal distance error in revolution 10,000, general perturbations solutions ($e_0 = 0.5$, $q_0 = 13{,}393$ km, $v_0 = 0$ deg)

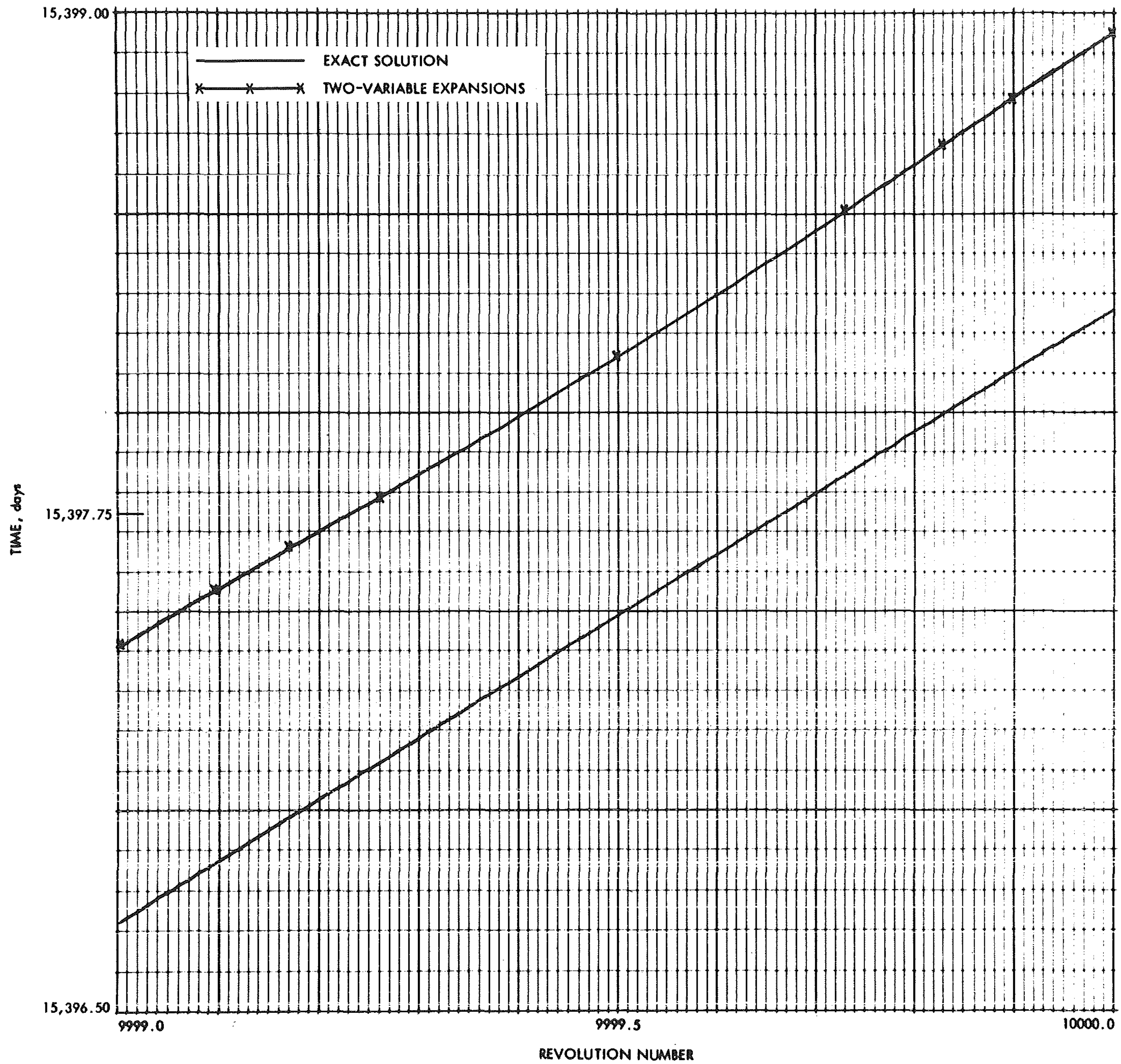


Fig. G-113. Variation of time in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 13{,}393$ km, $v_0 = 0$ deg)

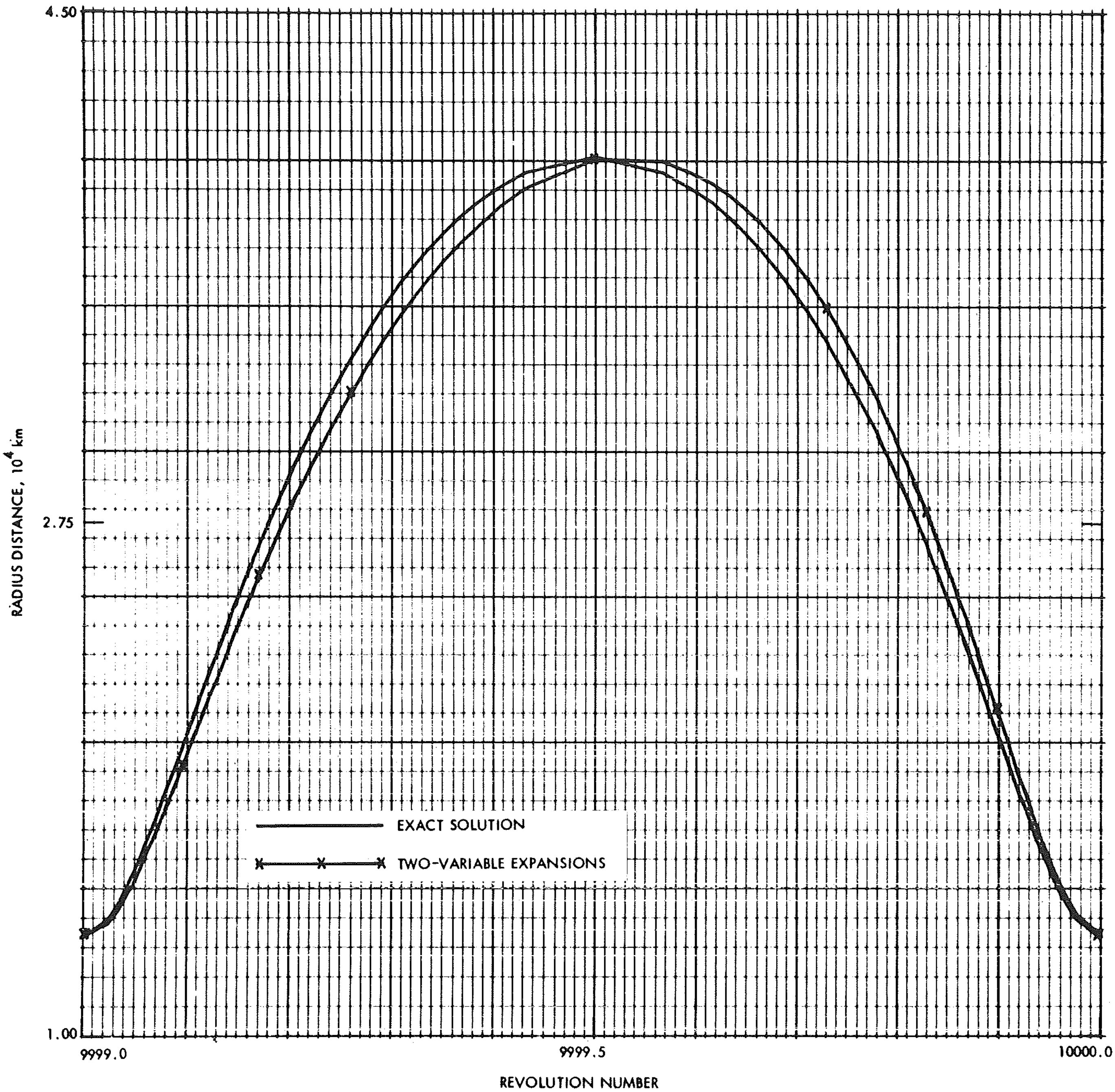


Fig. G-114. Variation of radius distance in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 13,393$ km, $v_0 = 0$ deg)

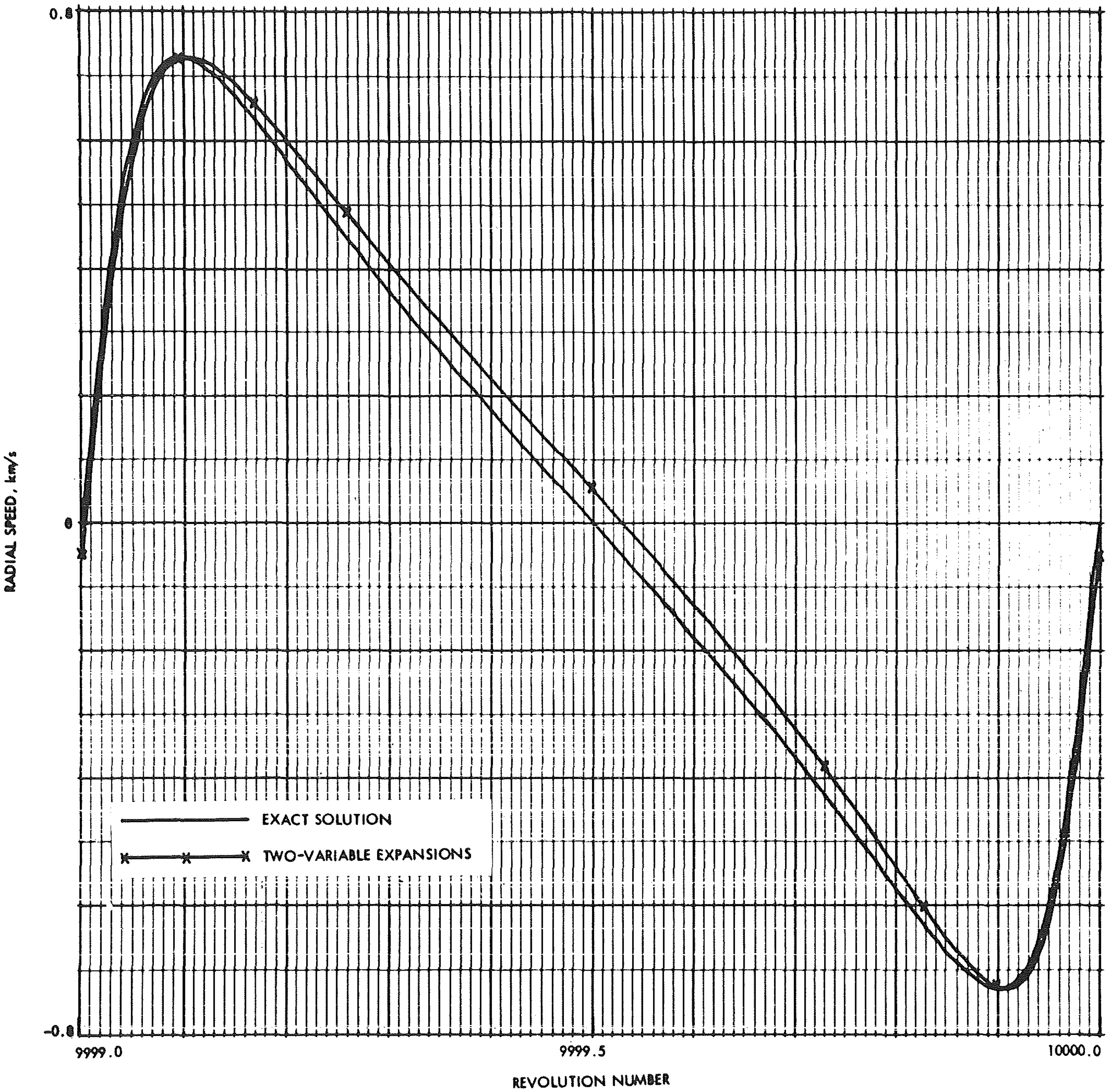


Fig. G-115. Variation of radial speed in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 13{,}393$ km, $v_0 = 0$ deg)

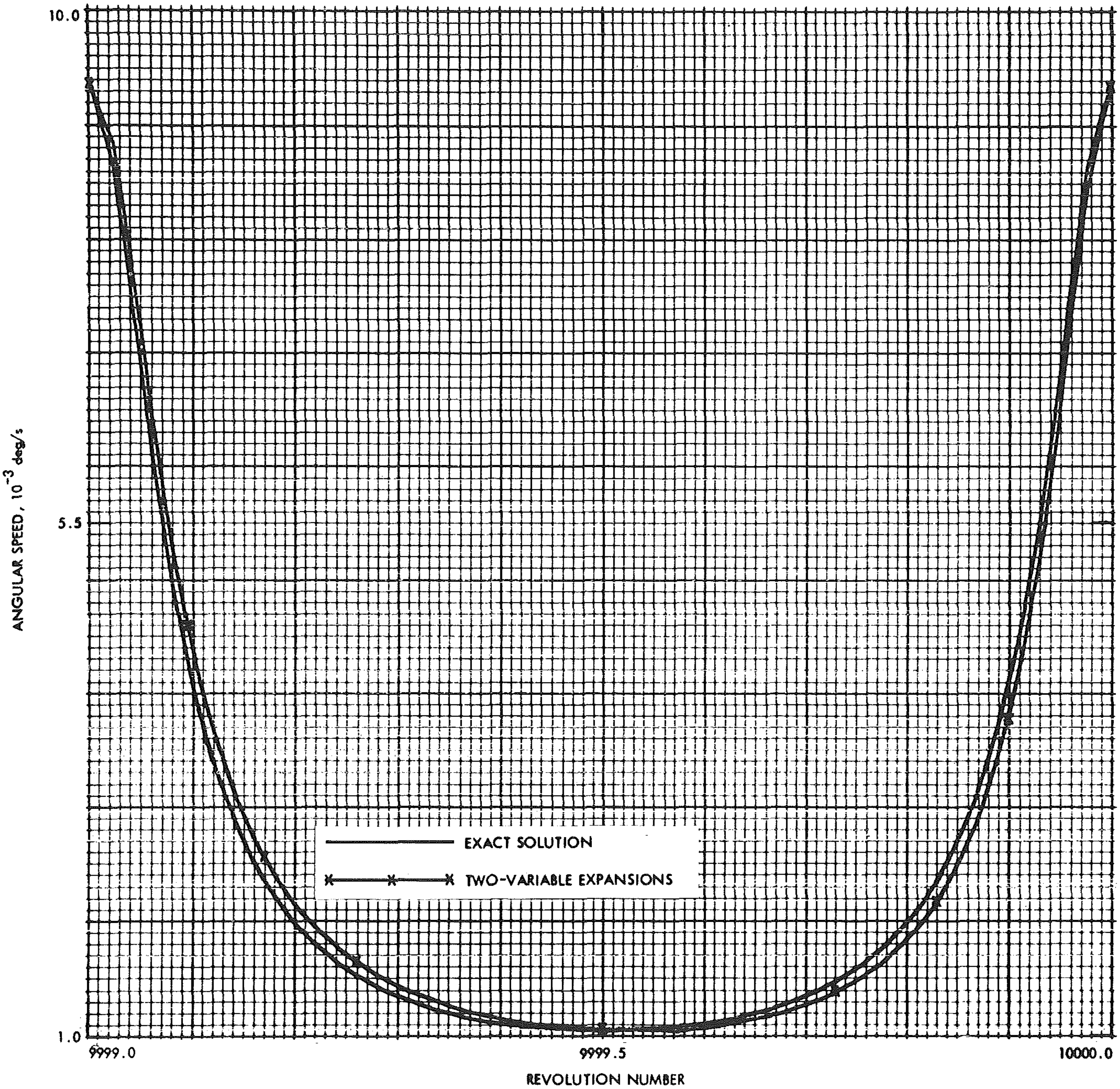


Fig. G-116. Variation of angular speed in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 13{,}393$ km, $v_0 = 0$ deg)

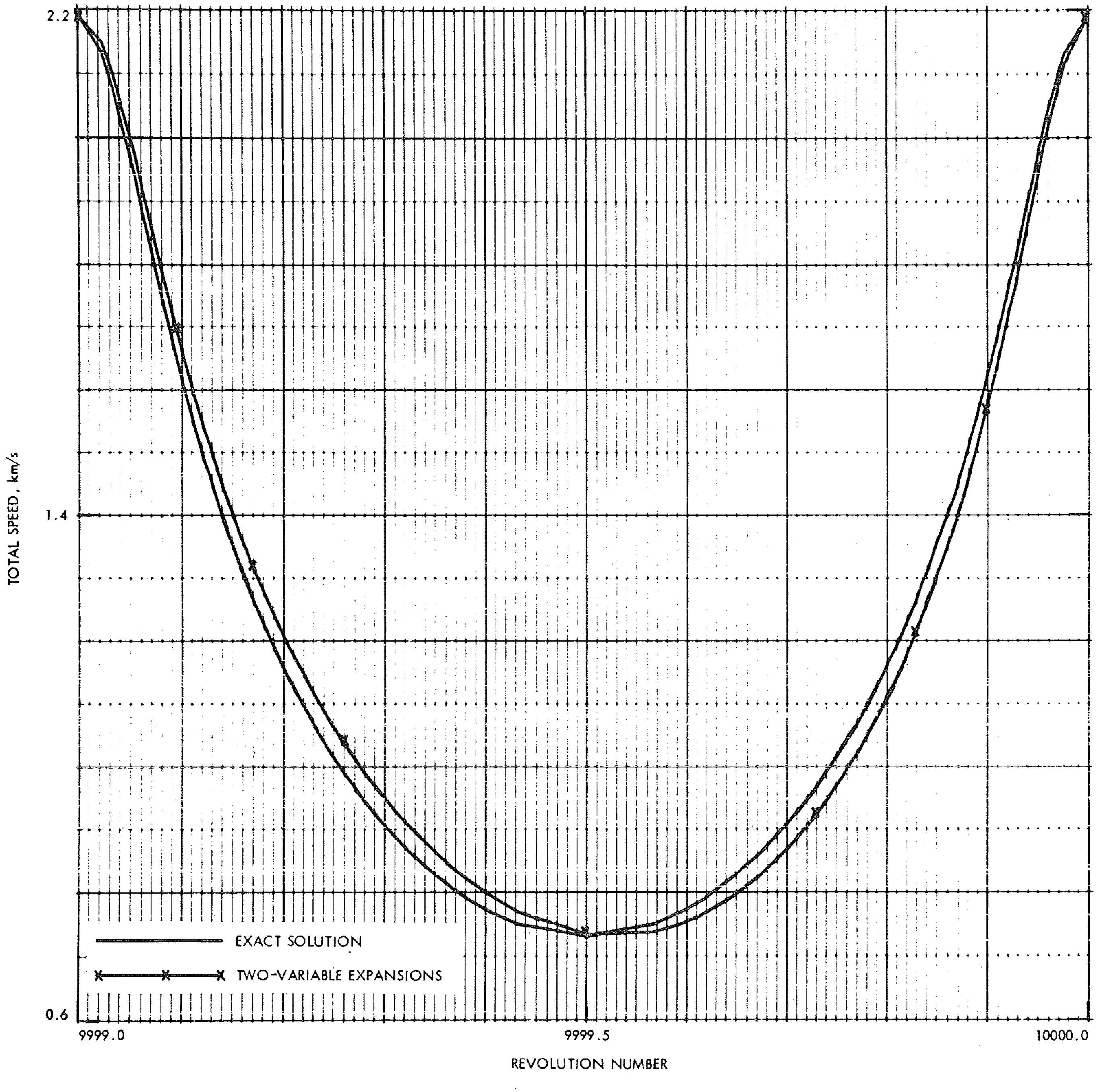


Fig. G-117. Variation of total speed in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 13{,}393$ km, $v_0 = 0$ deg)

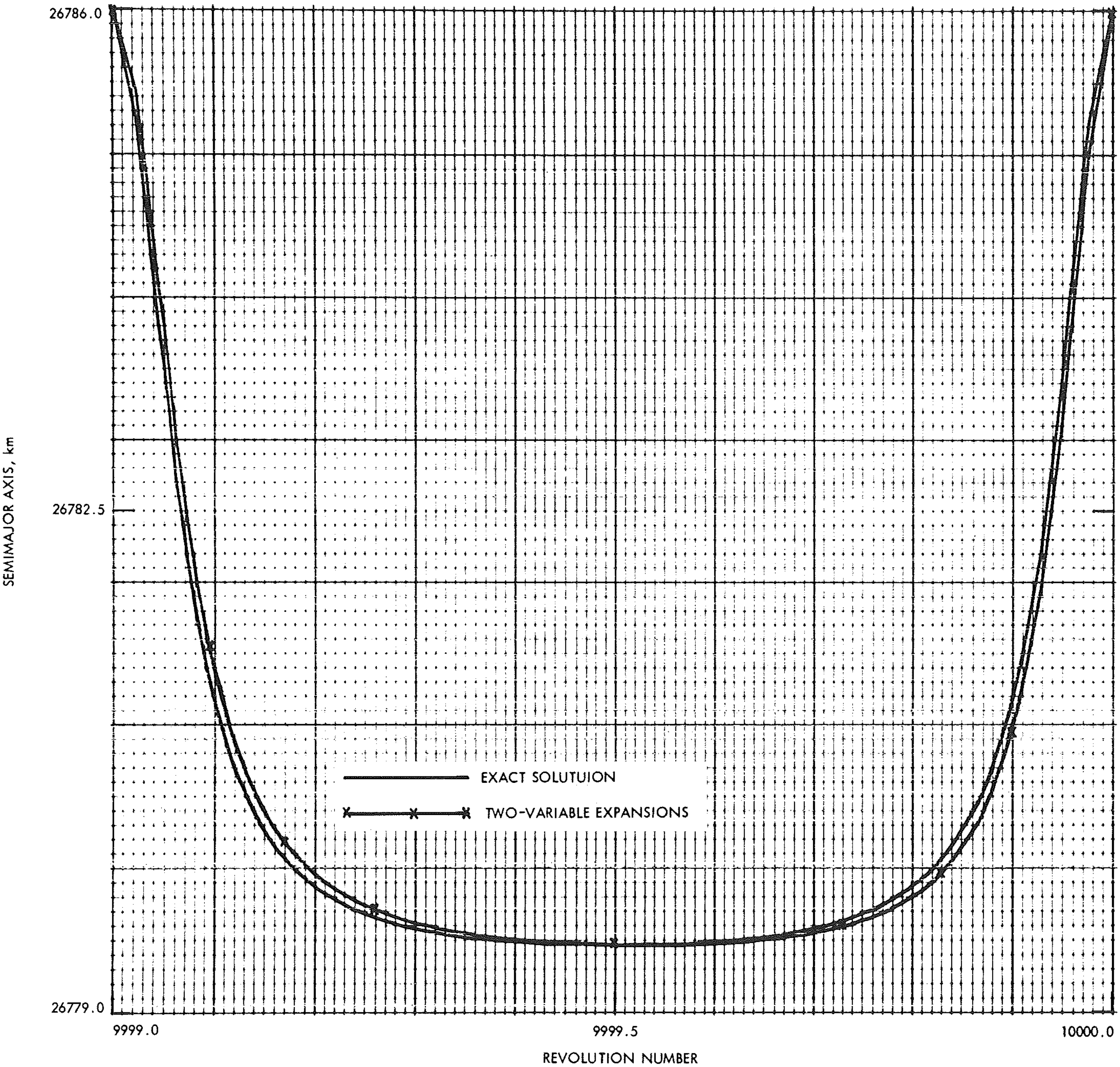


Fig. G-118. Variation of semimajor axis in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 13{,}393$ km, $v_0 = 0$ deg)

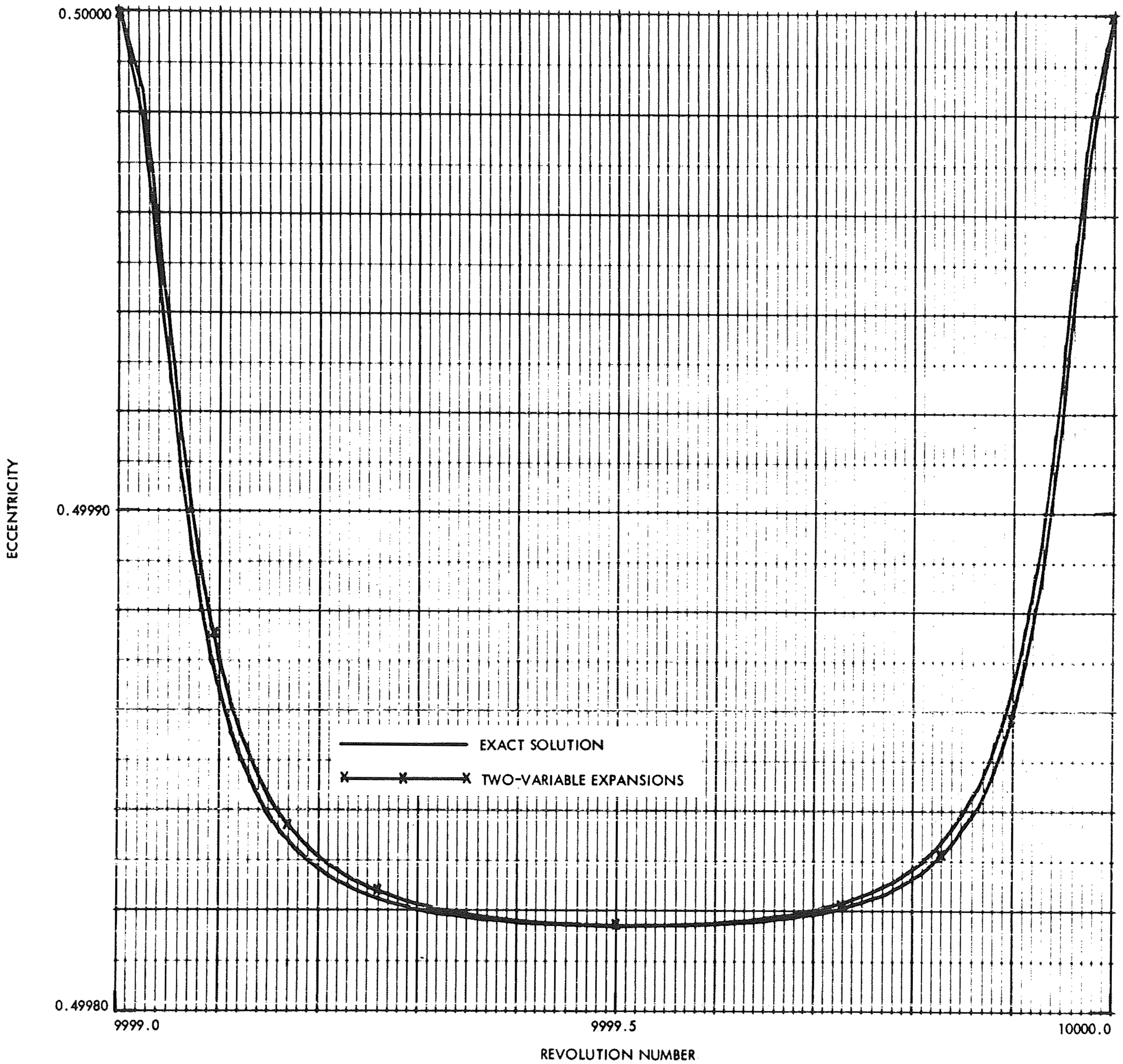


Fig. G-119. Variation of eccentricity in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 13{,}393$ km, $v_0 = 0$ deg)

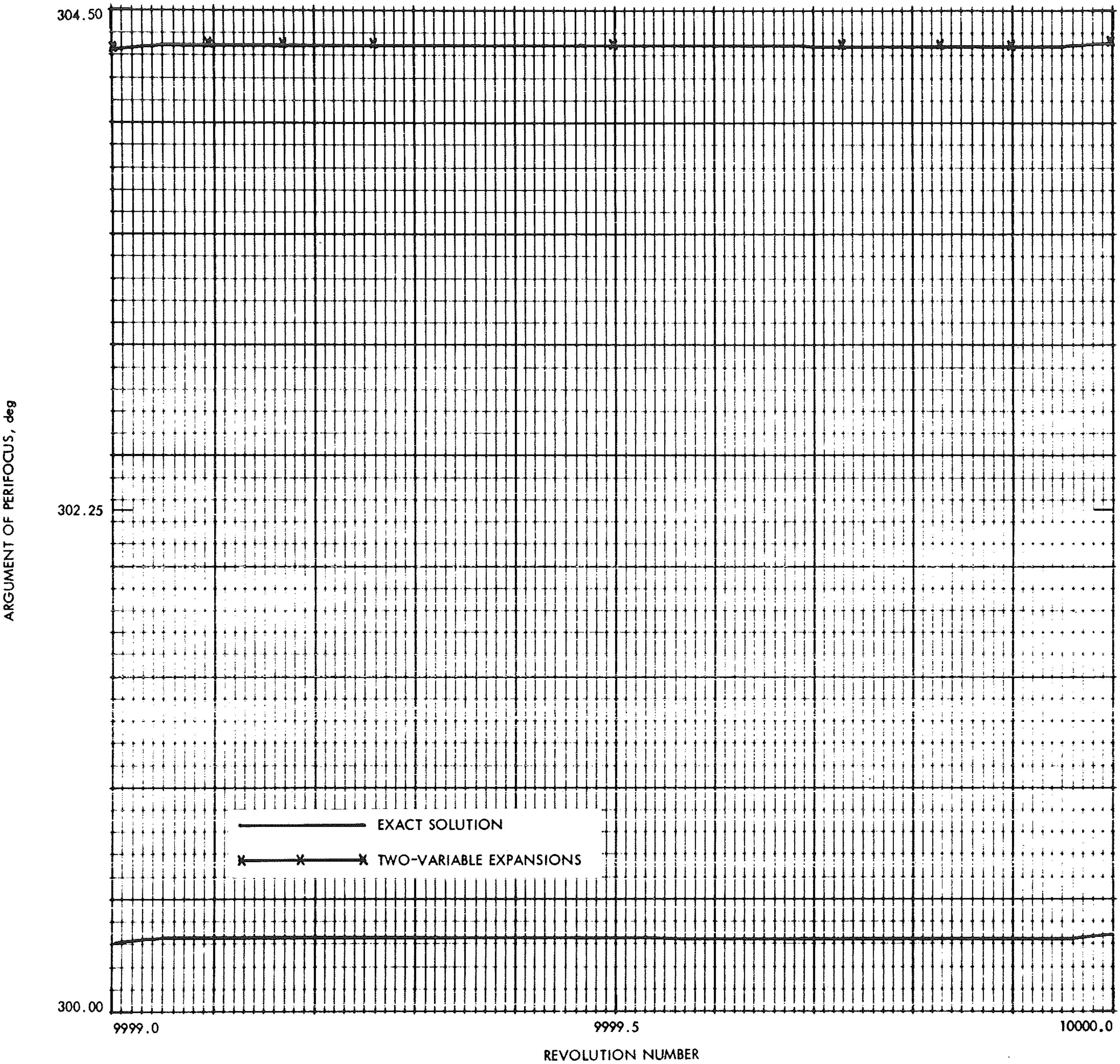


Fig. G-120. Variation of argument of perifocus in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 13{,}393$ km, $v_0 = 0$ deg)

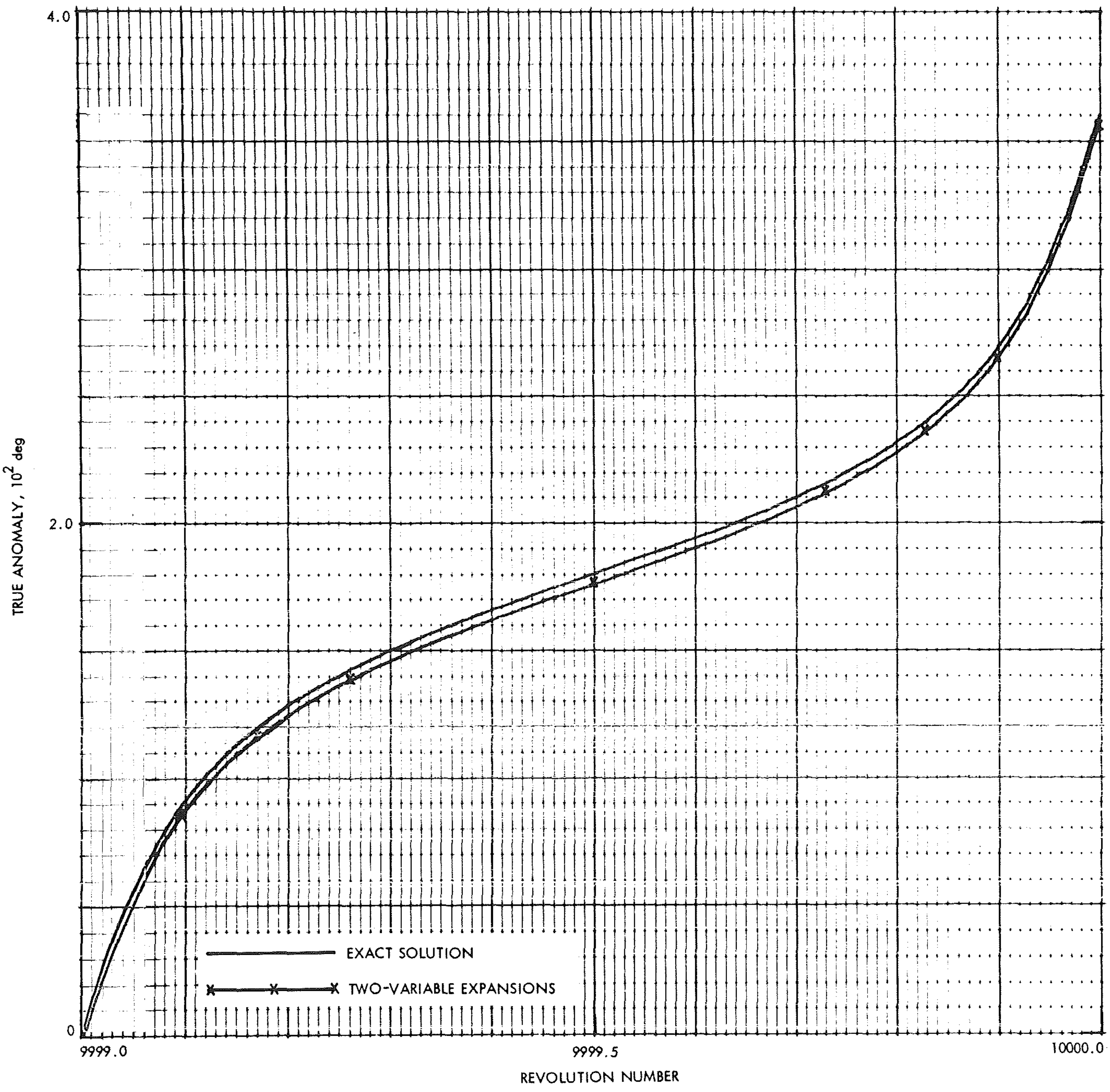


Fig. G-121. Variation of true anomaly in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 13{,}393$ km, $v_0 = 0$ deg)

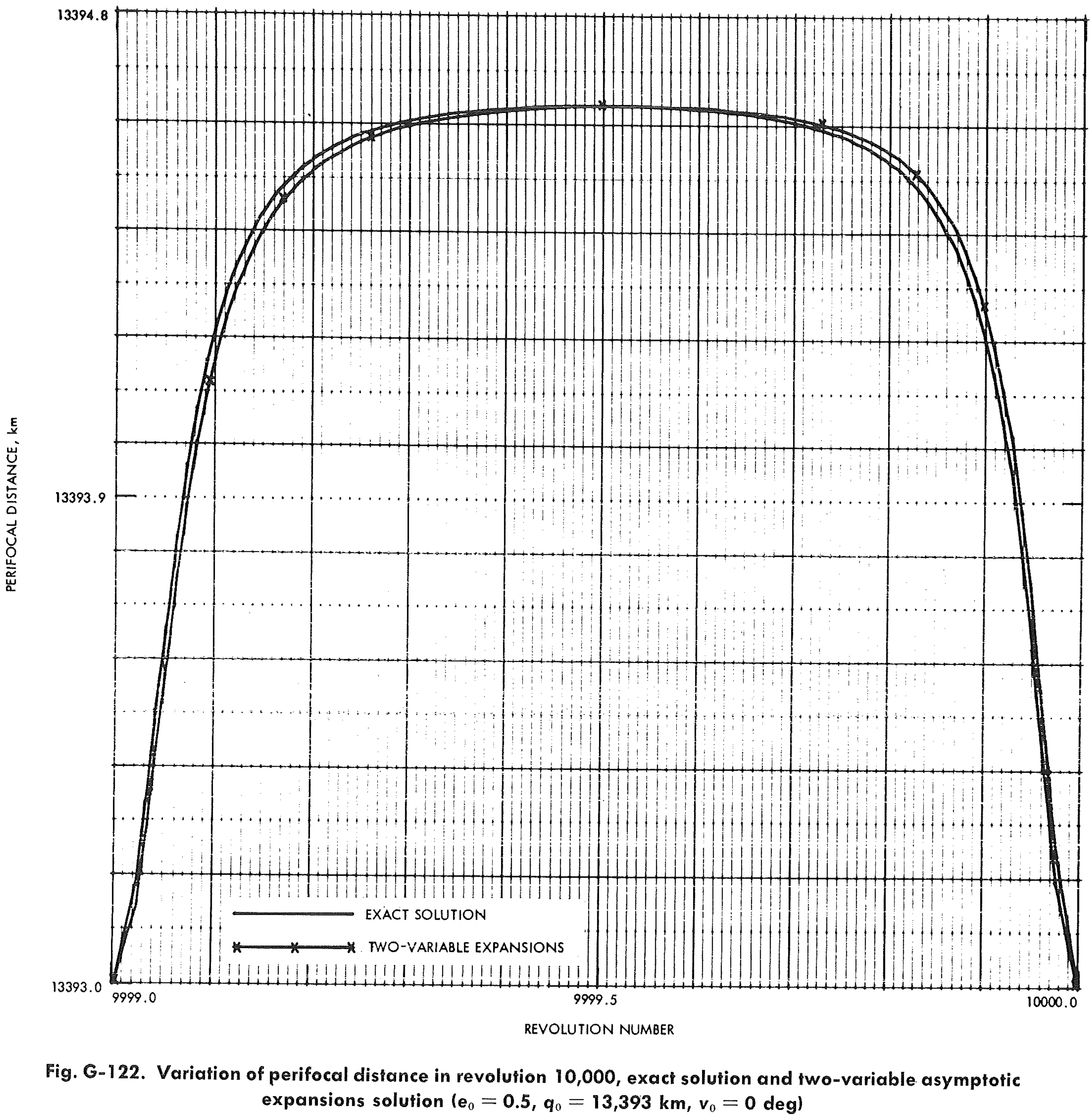


Fig. G-122. Variation of perifocal distance in revolution 10,000, exact solution and two-variable asymptotic expansions solution ($e_0 = 0.5$, $q_0 = 13{,}393$ km, $v_0 = 0$ deg)

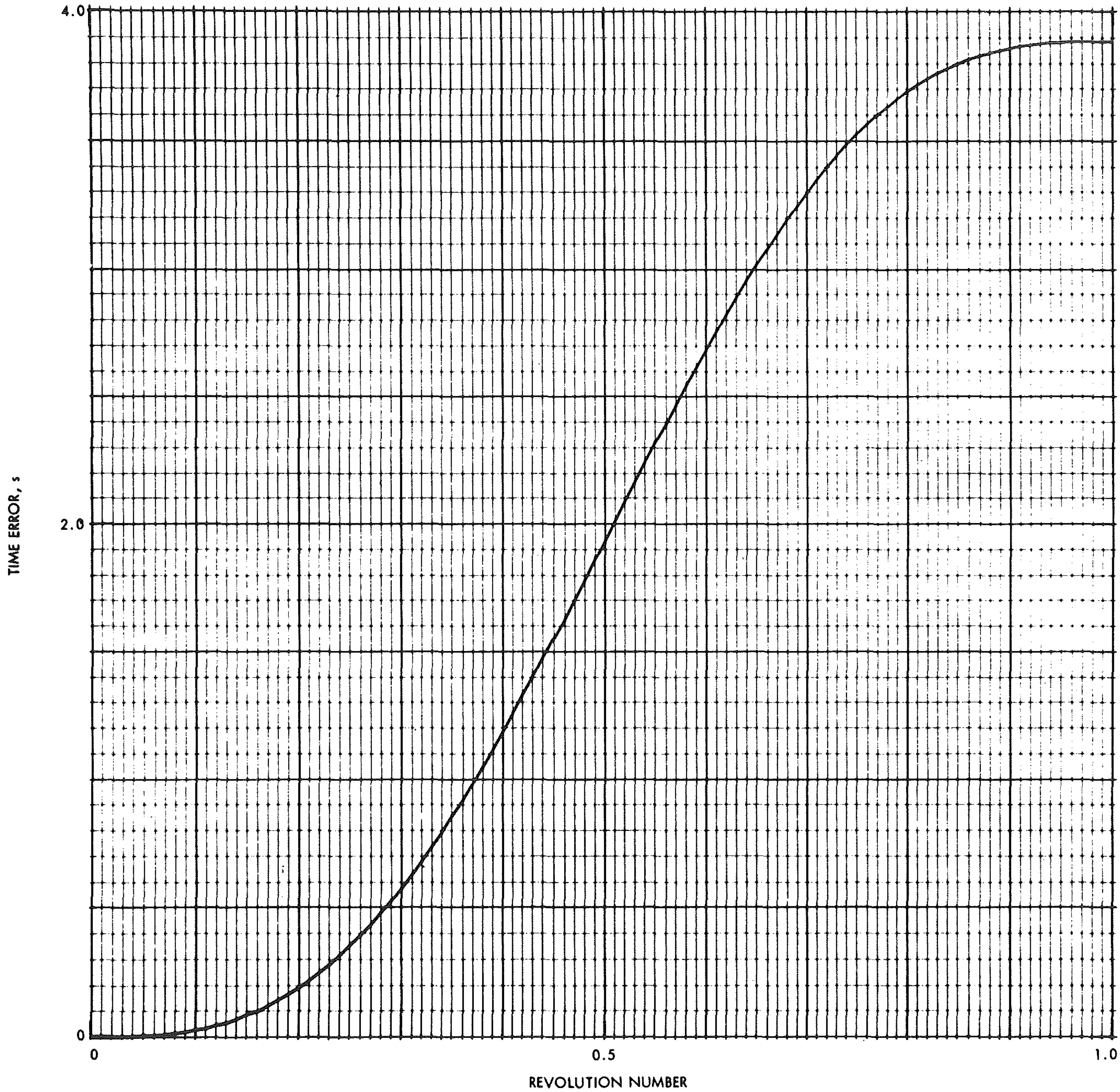


Fig. G-123. Time error in revolution 1, special perturbations solution ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

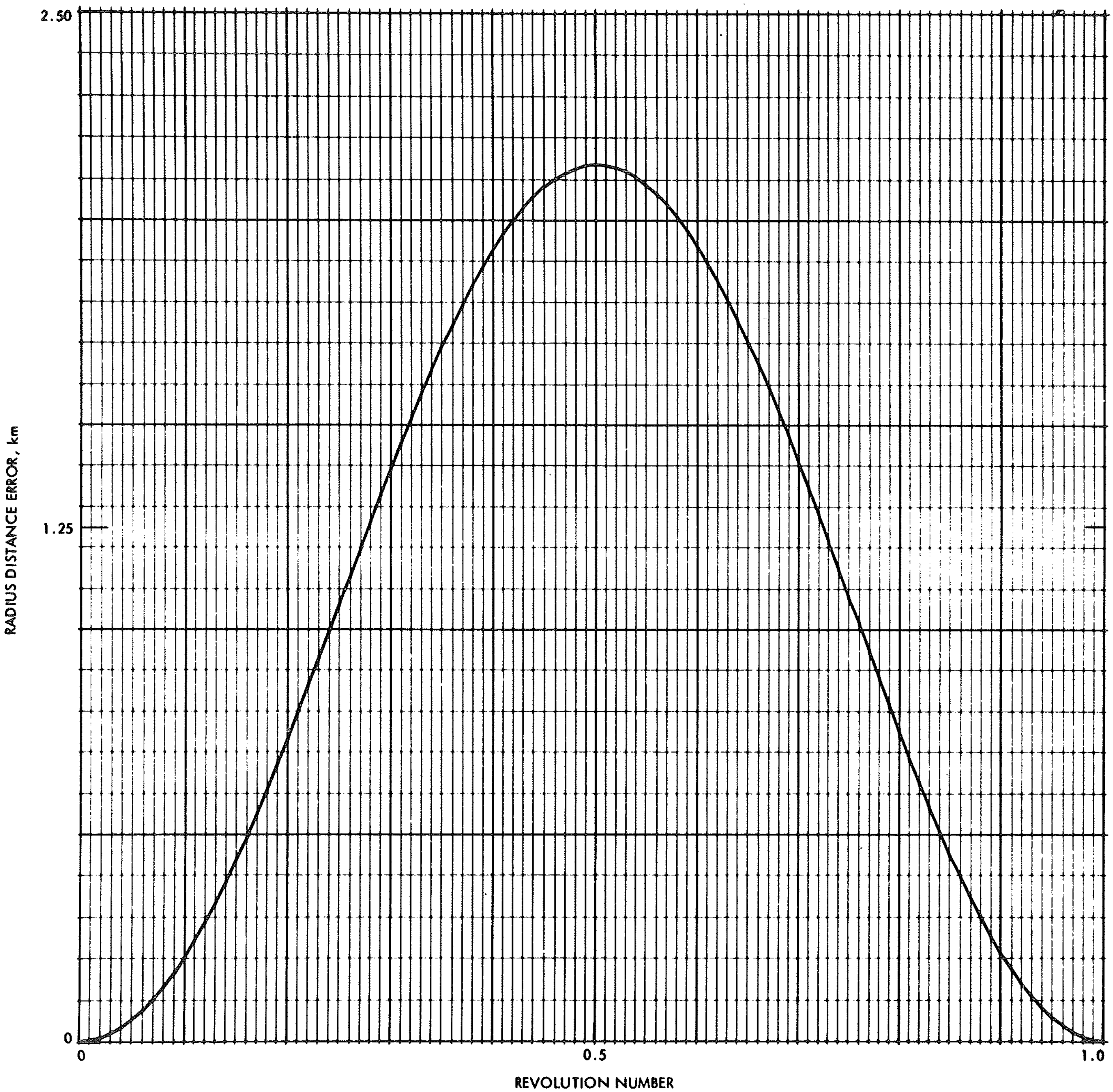


Fig. G-124. Radius distance error in revolution 1, special perturbations solution ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

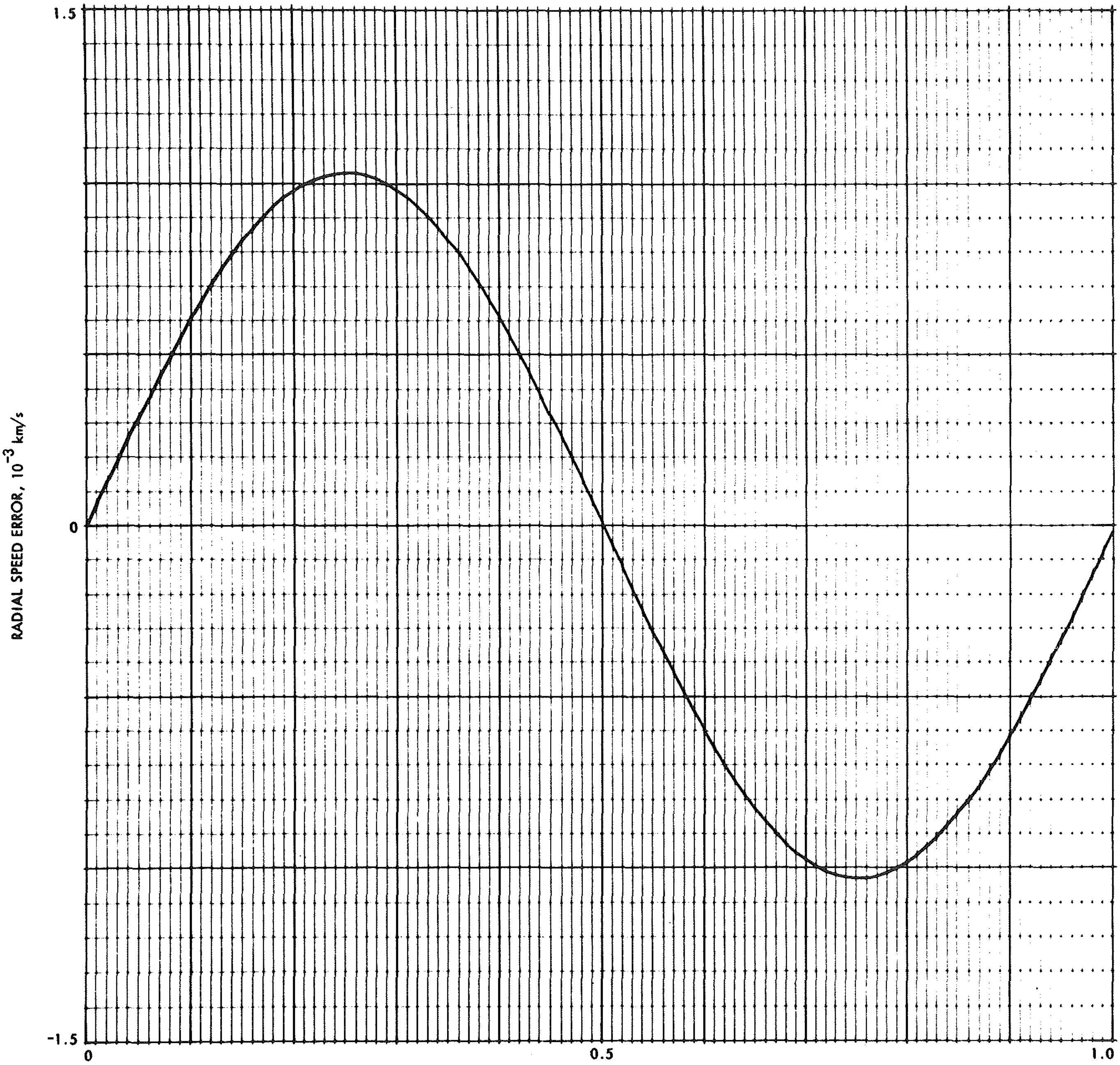


Fig. G-125. Radial speed error in revolution 1, special perturbations solution ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

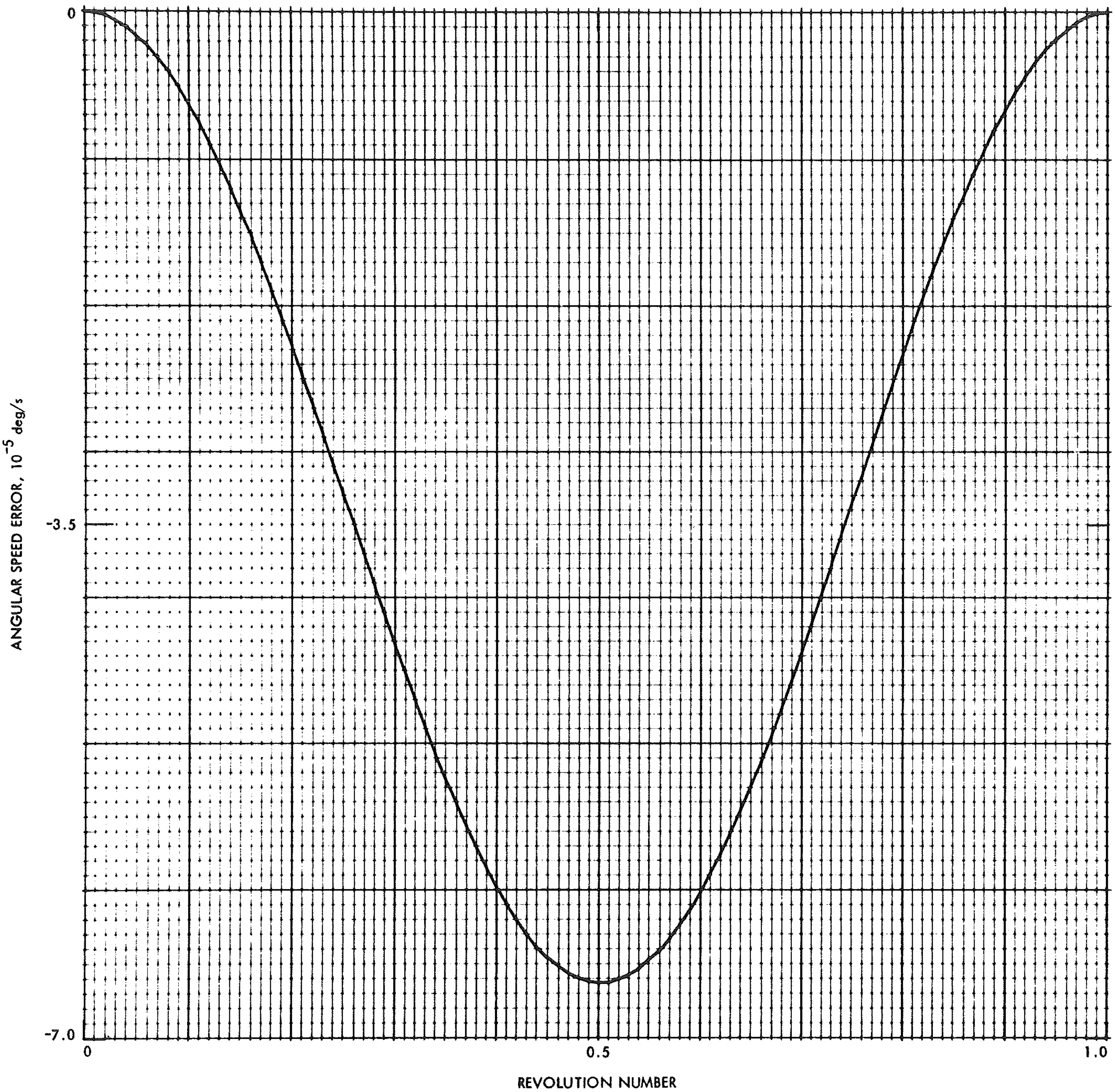


Fig. G-126. Angular speed error in revolution 1, special perturbations solution ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

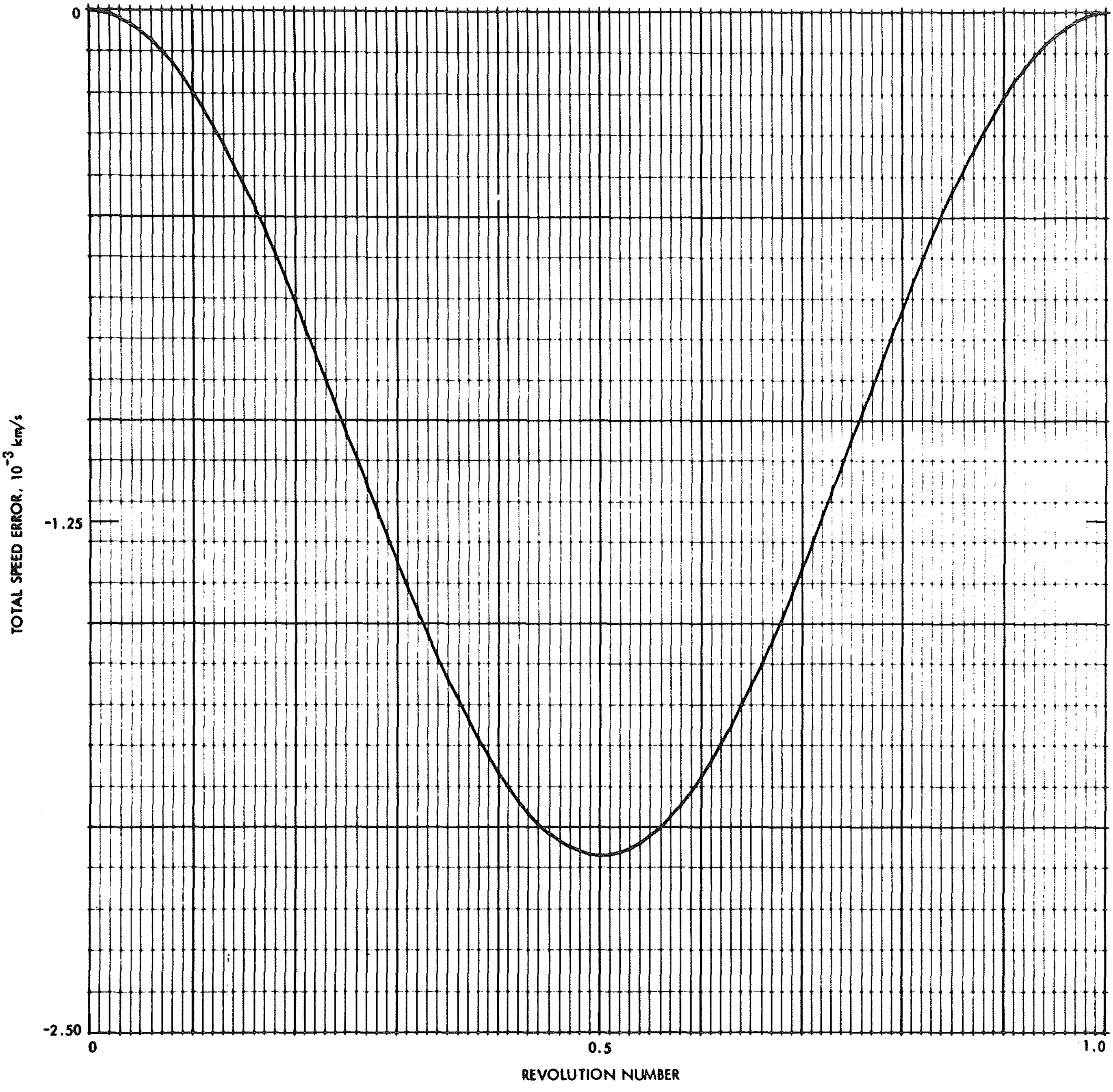


Fig. G-127. Total speed error in revolution 1, special perturbations solution ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

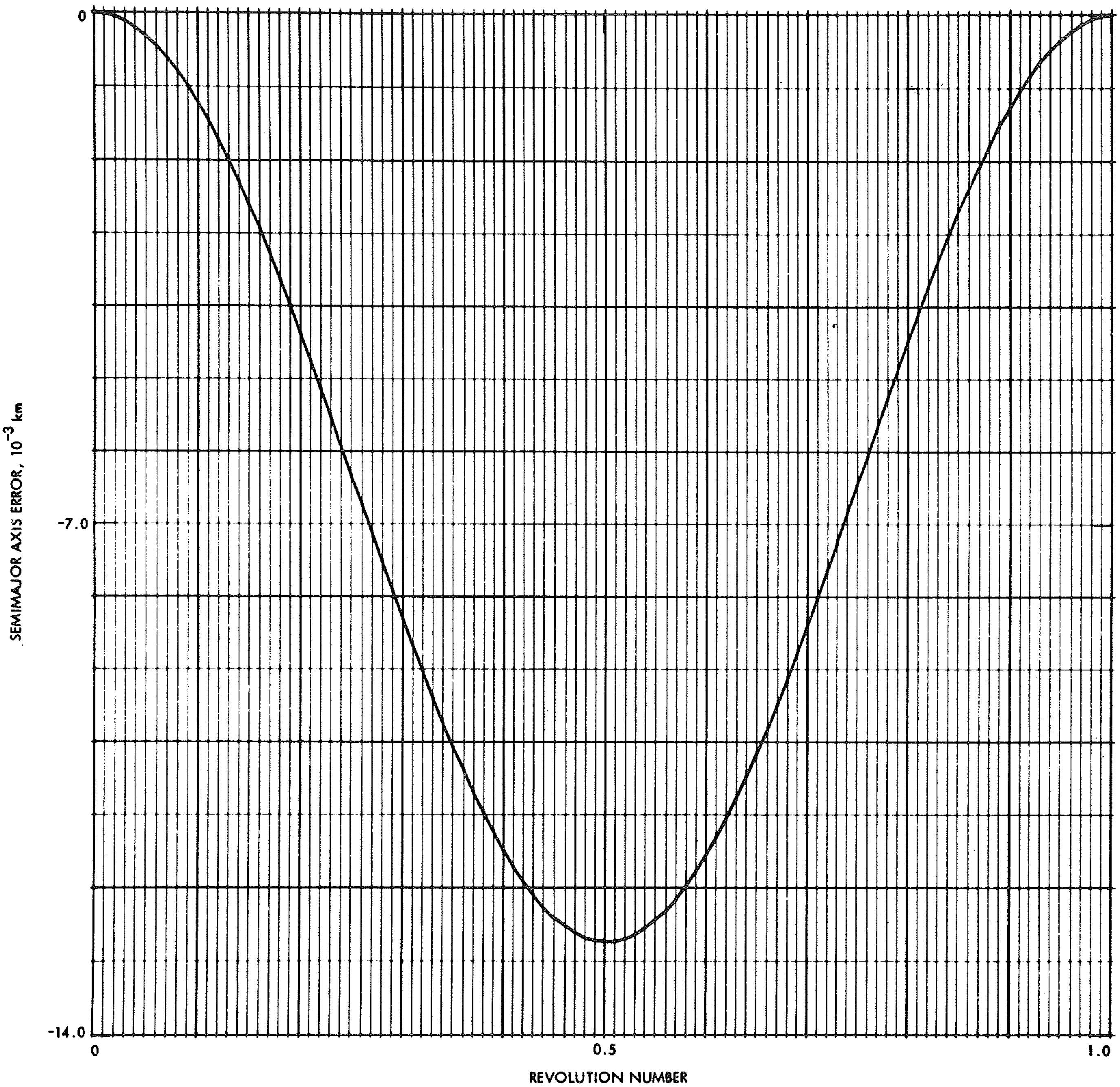


Fig. G-128. Semimajor axis error in revolution 1, special perturbations solution ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

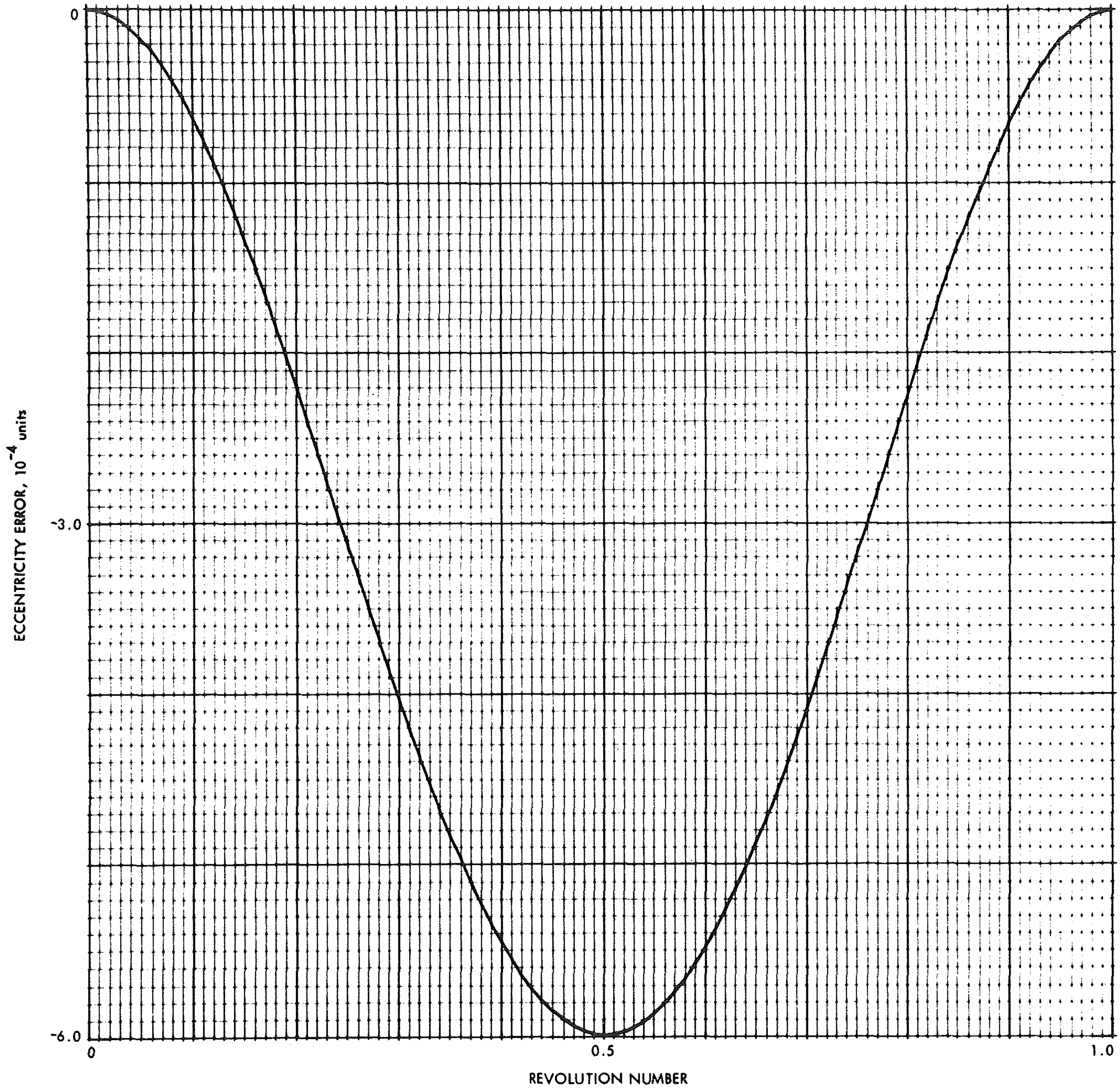


Fig. G-129. Eccentricity error in revolution 1, special perturbations solution ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

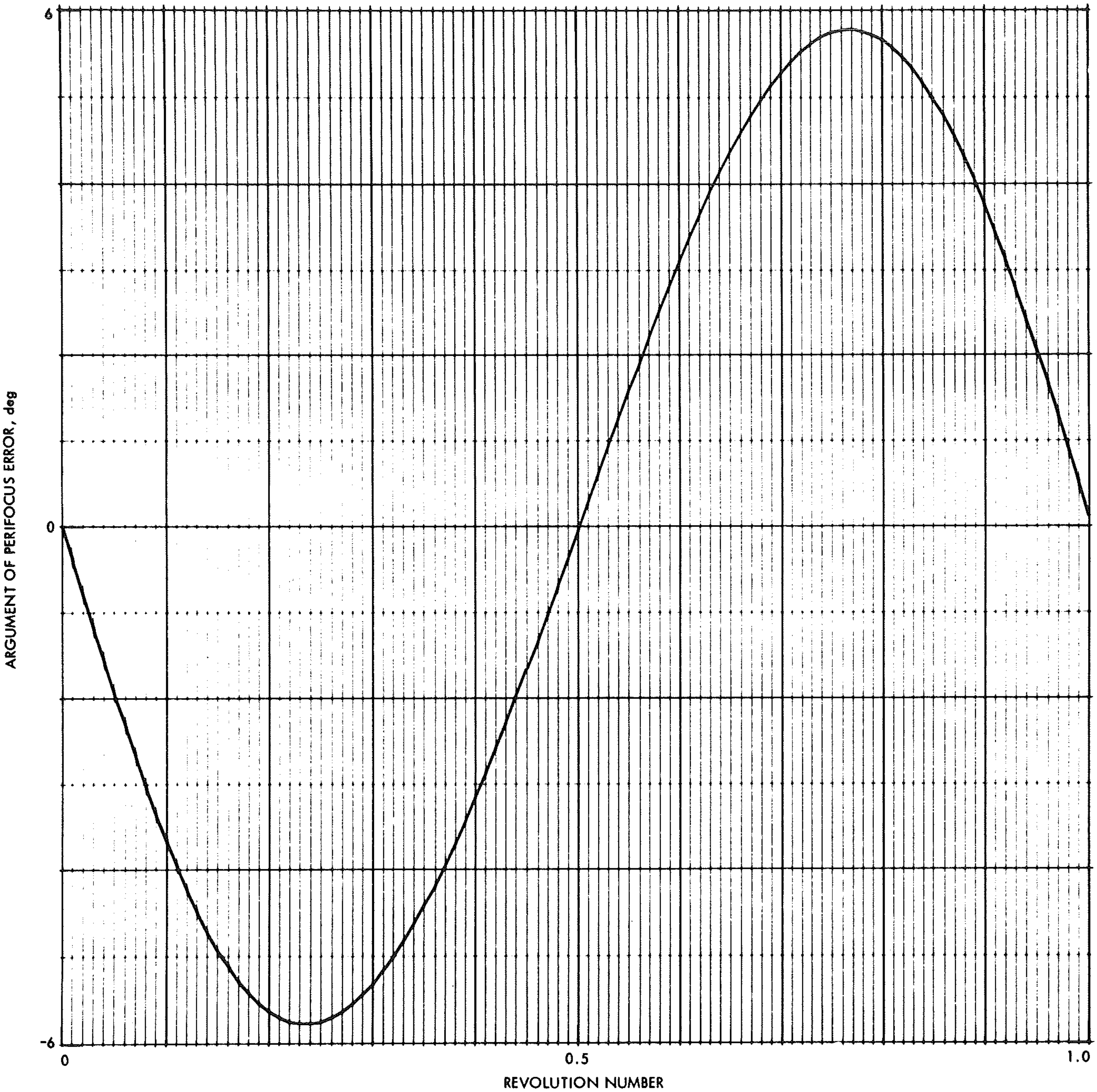


Fig. G-130. Argument of perifocus error in revolution 1, special perturbations solution ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

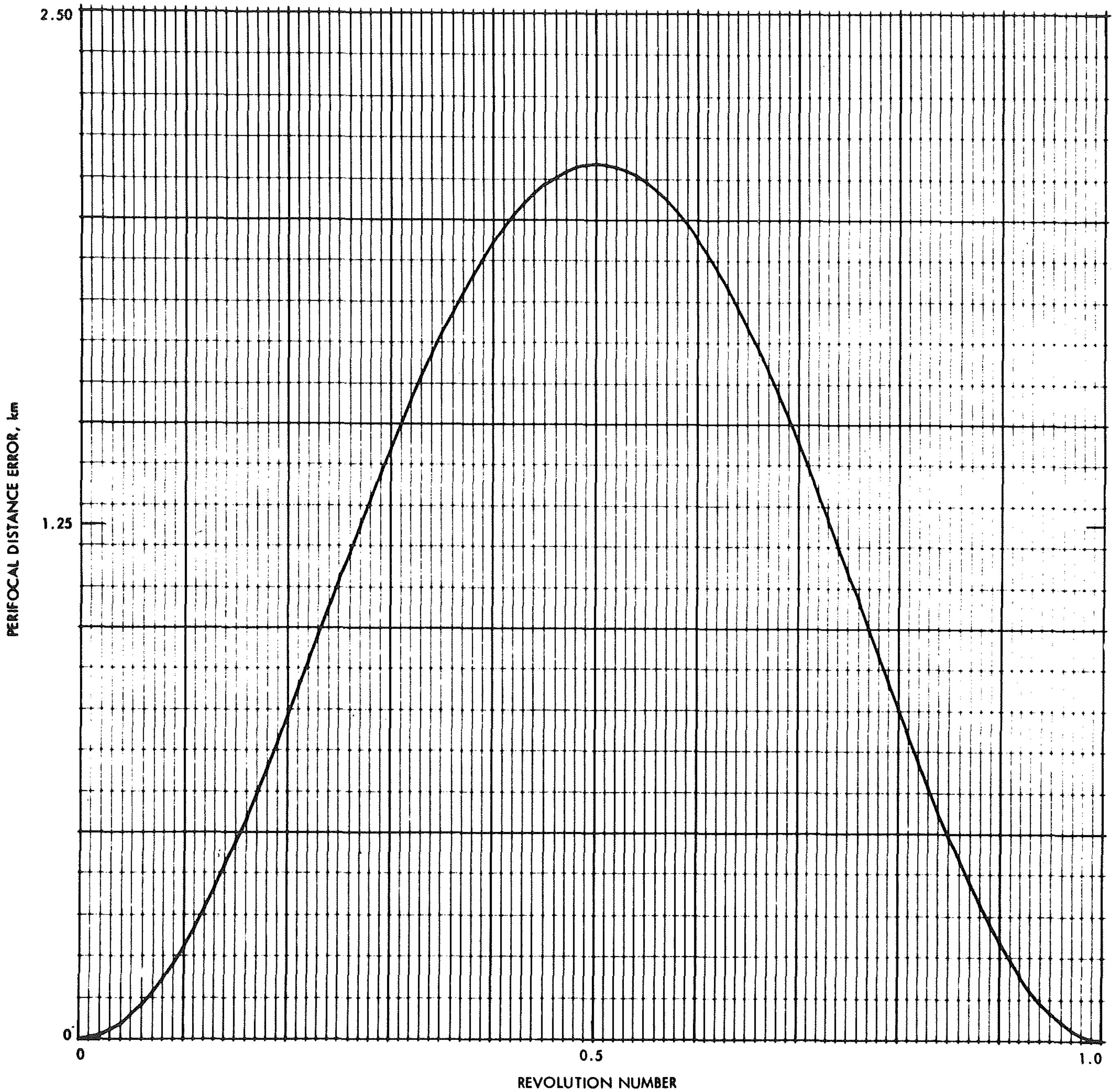


Fig. G-131. Perifocal distance error in revolution 1, special perturbations solution ($e_0 = 0.002665$, $q_0 = 3578$ km, $v_0 = 0$ deg)

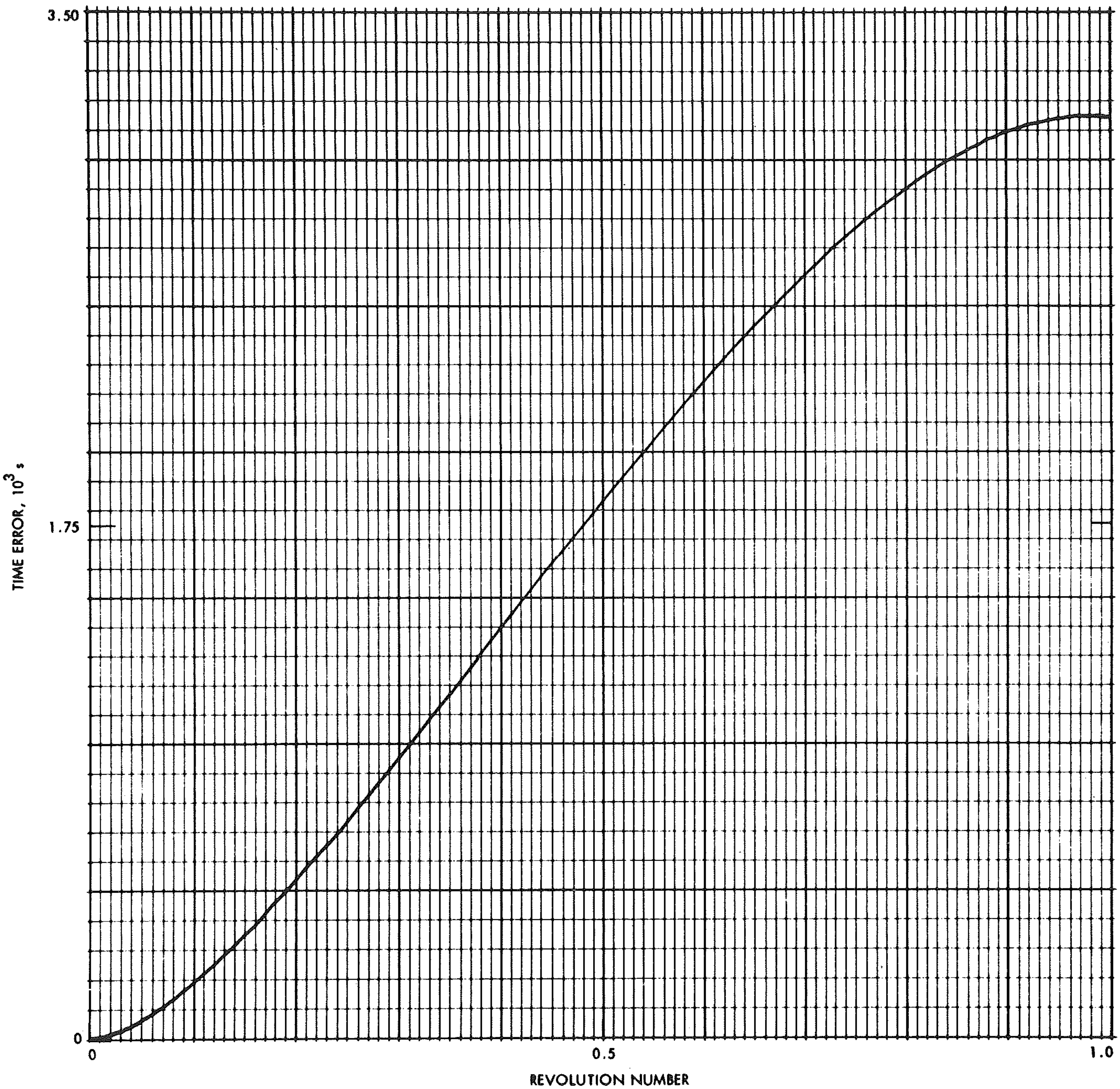


Fig. G-132. Time error in revolution 1, special perturbations solution ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

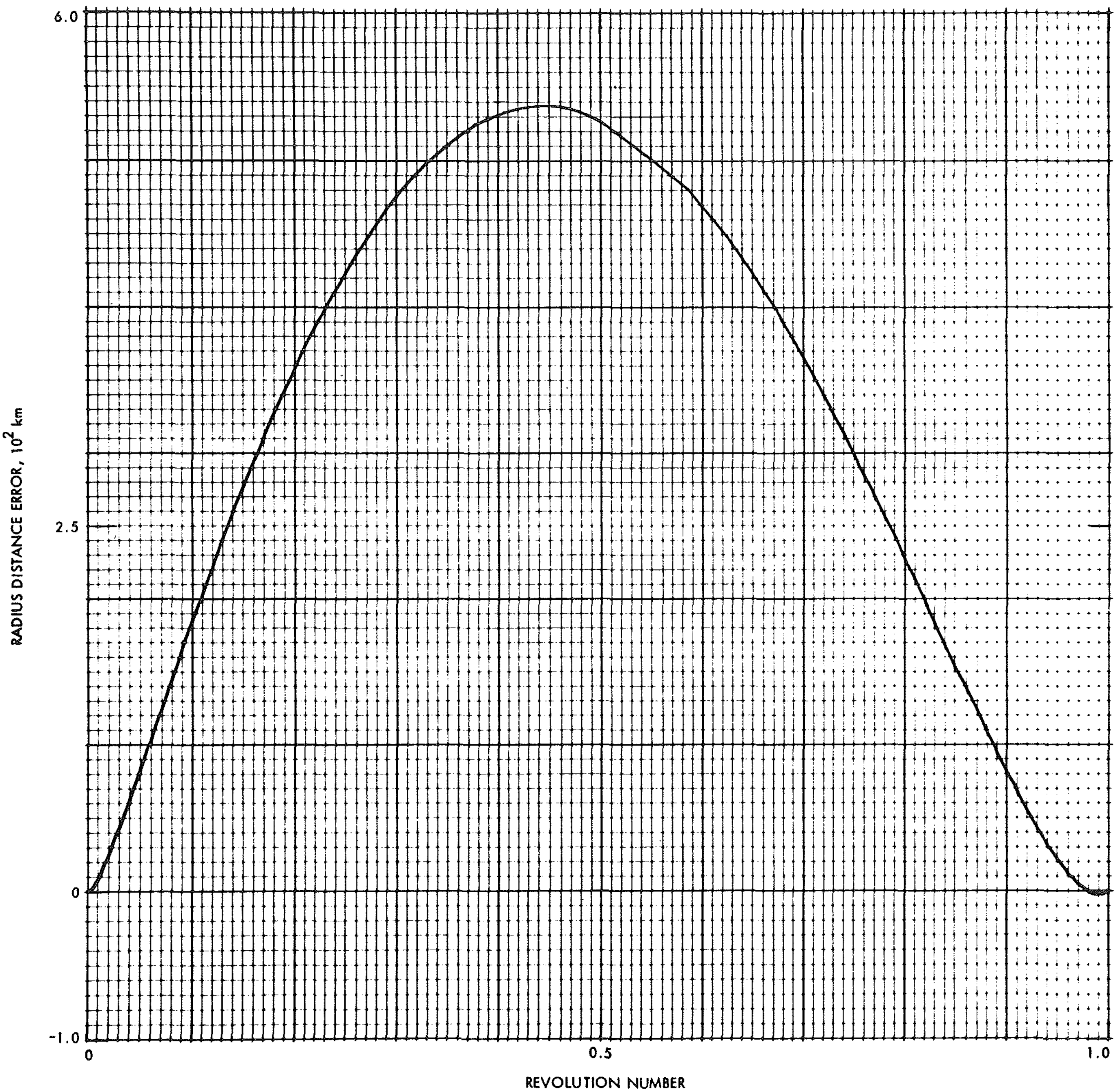


Fig. G-133. Radius distance error in revolution 1, special perturbations solution ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

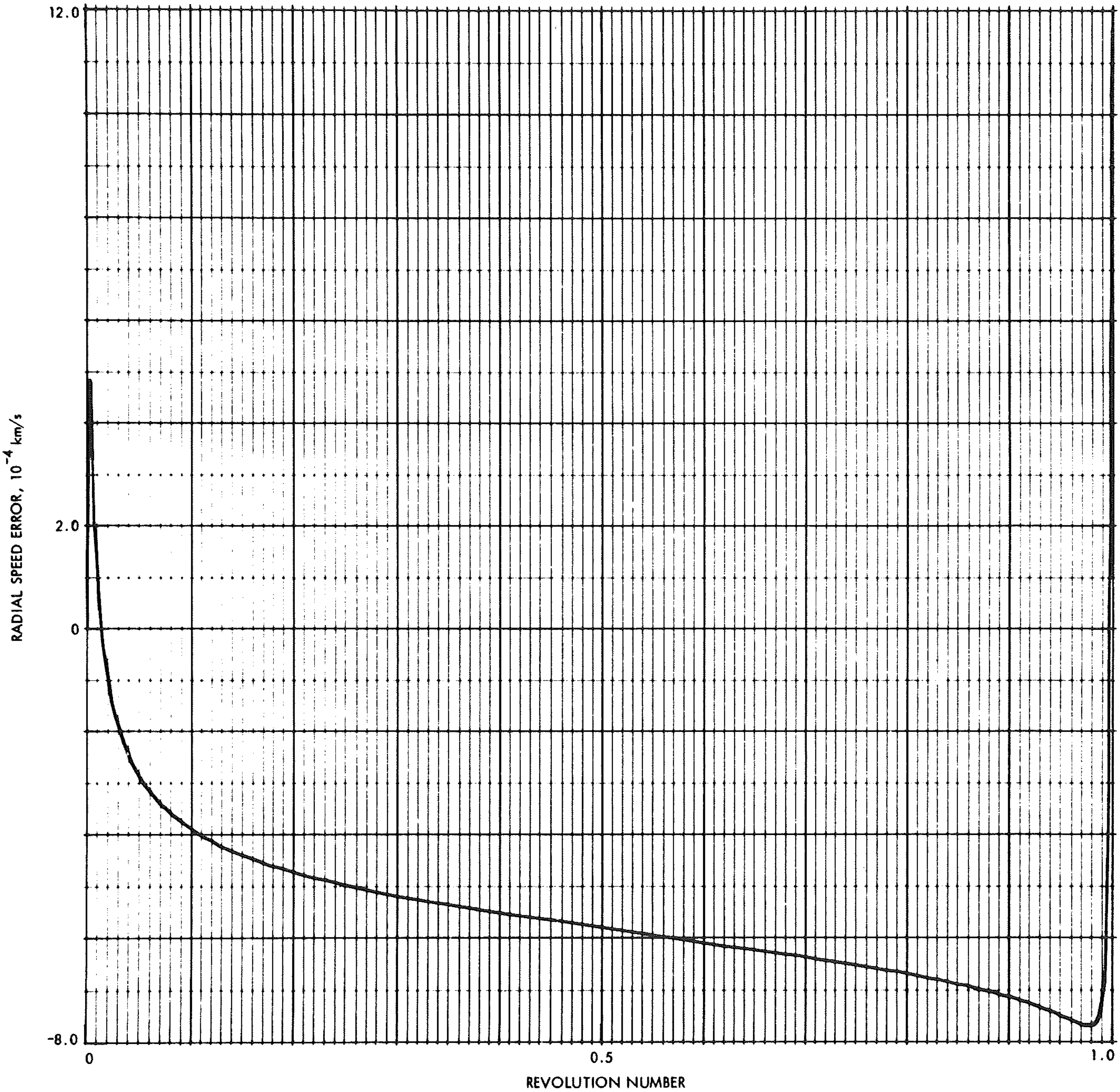


Fig. G-134. Radial speed error in revolution 1, special perturbations solution ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

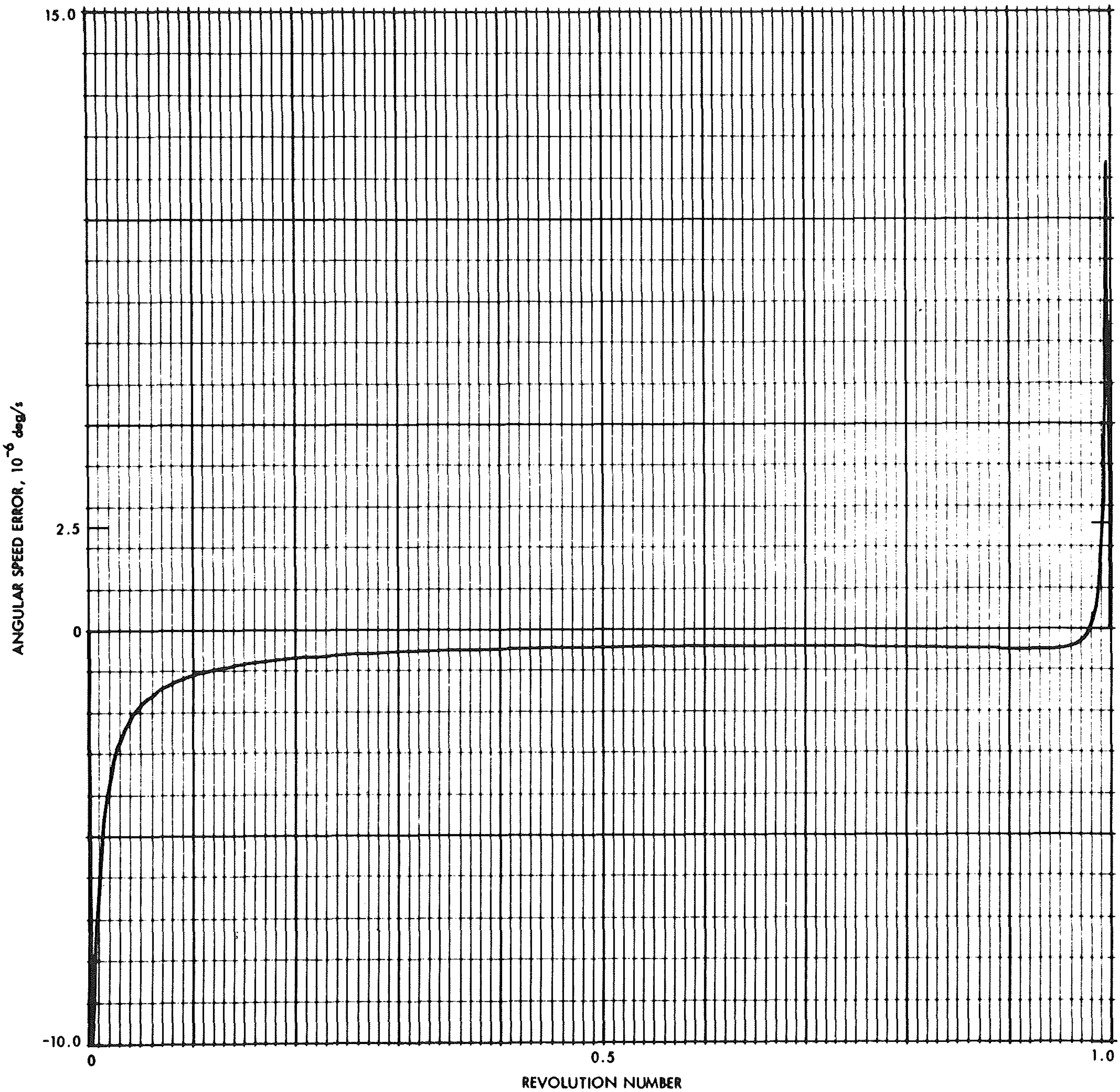


Fig. G-135. Angular speed error in revolution 1, special perturbations solution ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

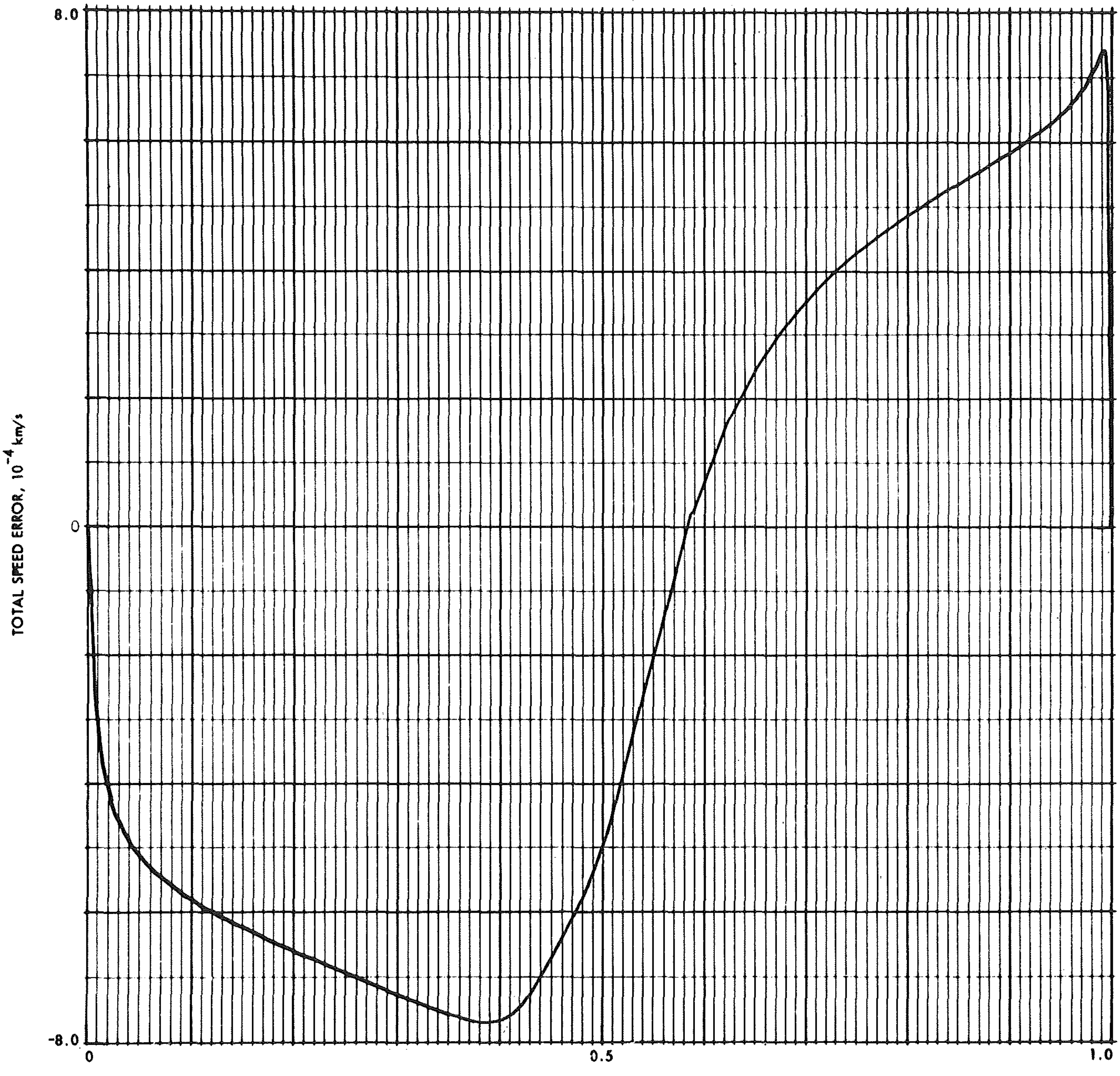


Fig. G-136. Total speed error in revolution 1, special perturbations solution ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

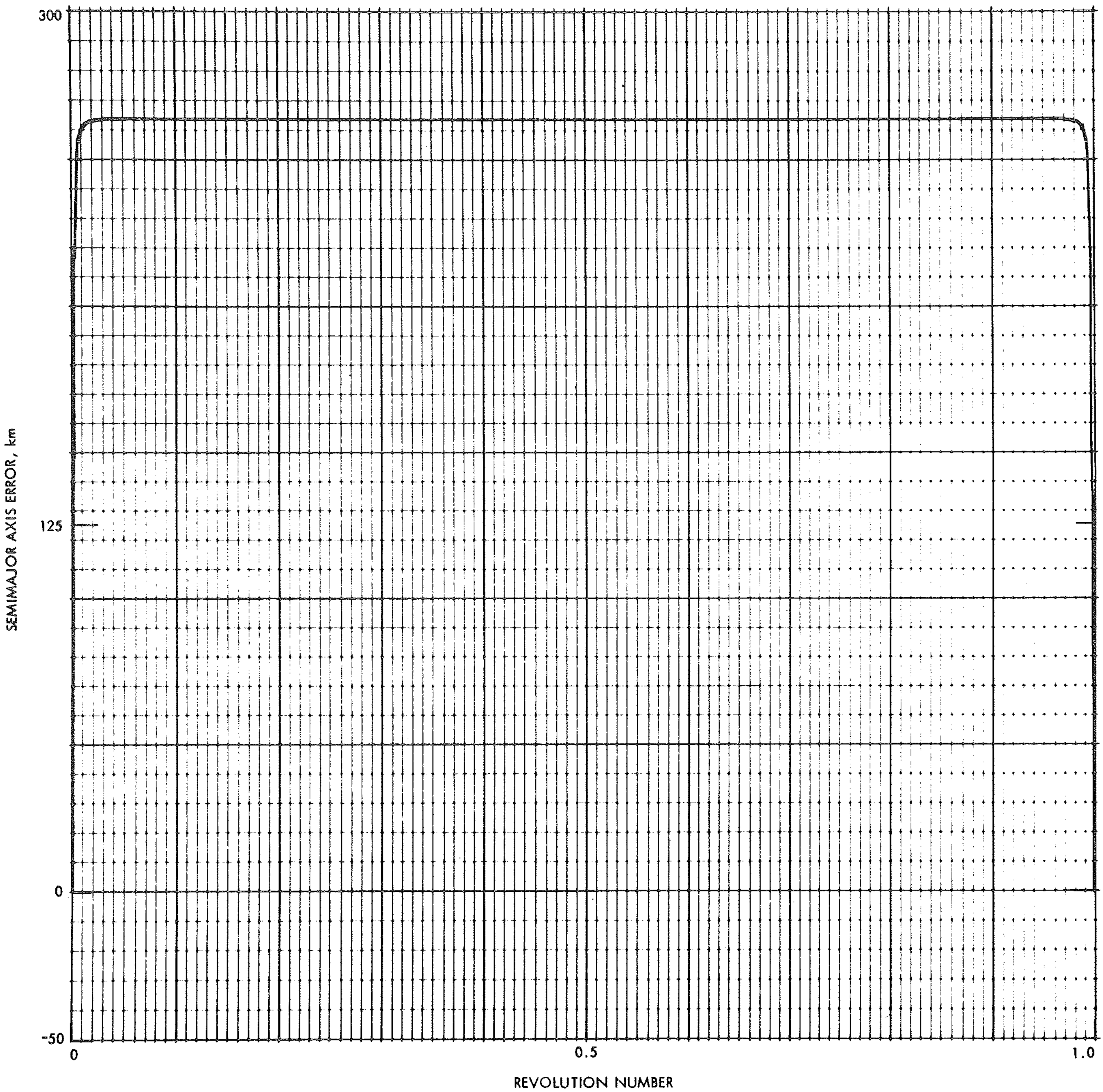


Fig. G-137. Semimajor axis error in revolution 1, special perturbations solution ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

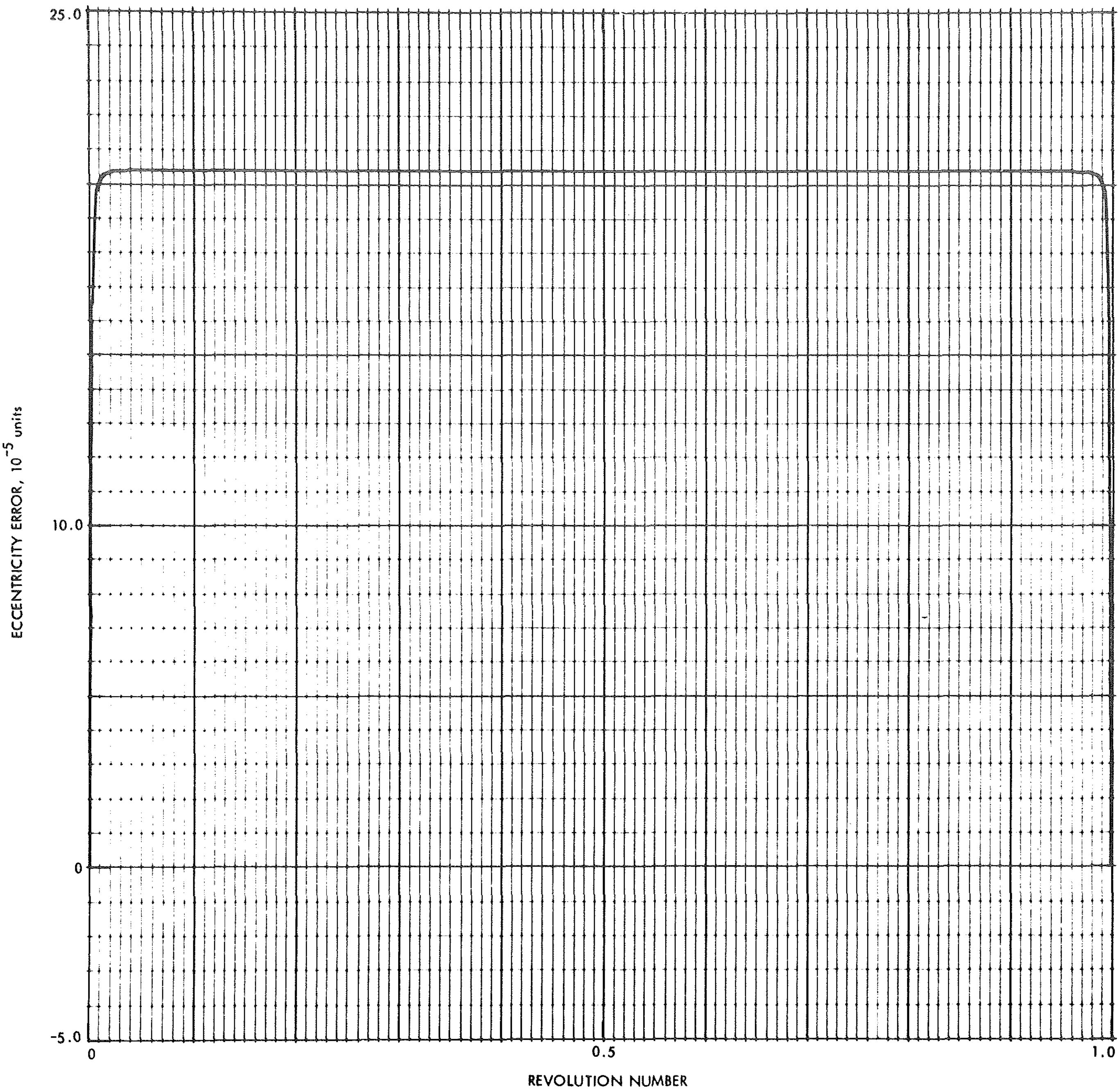


Fig. G-138. Eccentricity error in revolution 1, special perturbations solution ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

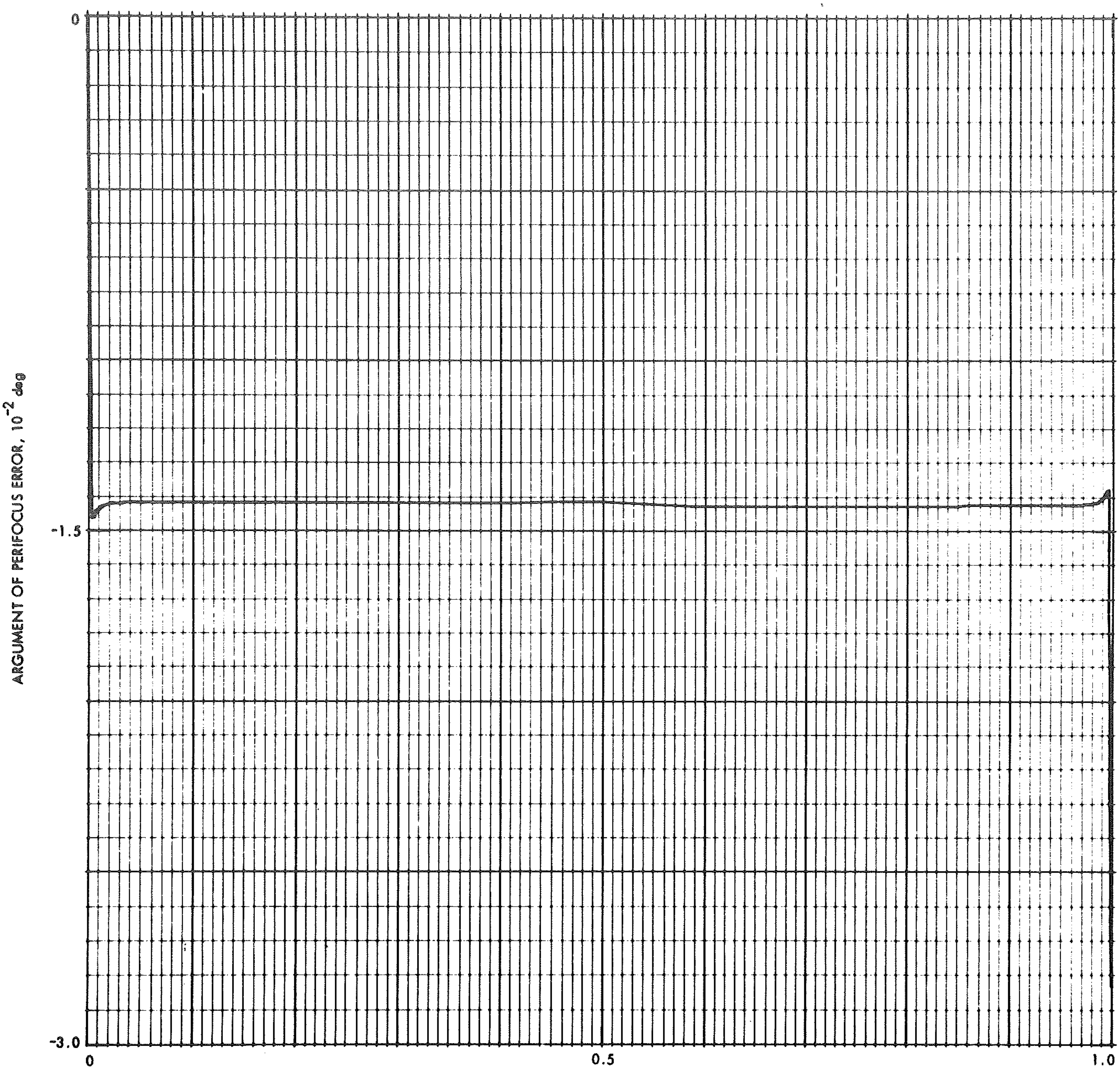


Fig. G-139. Argument of perifocus error in revolution 1, special perturbations solution ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

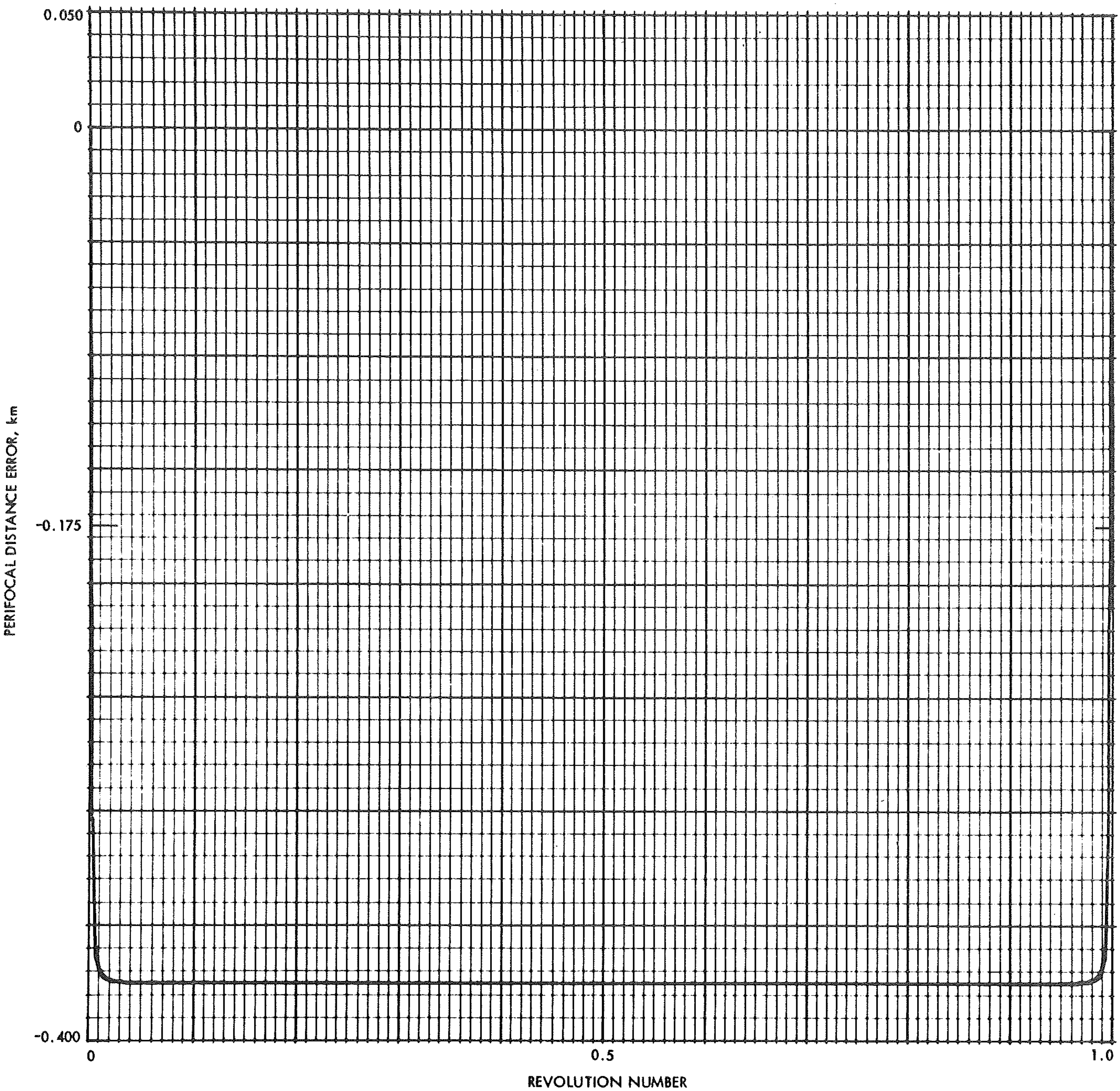


Fig. G-140. Perifocal distance error in revolution 1, special perturbations solution ($e_0 = 0.95$, $q_0 = 3578$ km, $v_0 = 0$ deg)

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