N70-26998

BELLCOMM, INC.
955 L'ENFANT PLAZA NORTH, S.W., WASHINGTON, D.C. 20024

COVER SHEET FOR TECHNICAL MEMORANDUM

TITLE- Characterizing the Behavior of Phase Lock Loops

TM- 70-2034- 2

DATE- February 20, 1970

FILING CASE NO(S)- 900

AUTHOR(S)- L. Schuchman

FILING SUBJECT(S) (ASSIGNED BY AUTHOR(S))- Communication Theory
Phase Lock Loops
Nonlinear Tracking Devices
Diffusion Processes

ABSTRACT

The order of a phase lock loop is determined by the linear filter in the loop. Thus we say a phase lock loop is of n+1 order when there are n finite poles in the Laplace transfer function of its linear filter under the restriction that the number of zeros of this Laplace transform is no greater than n.

In a previous memorandum (TM-69-2034-8) the behavior of a first and a particular second order phase lock loop was described in terms of the statistical behavior of two parameters, namely the steady state phase error ϕ_e and the time to cycle slip $\hat{T}$. This memorandum generalizes the work just described in that the statistical behavior of ϕ_e and $\hat{T}$ is characterized for a very large class of $n^{\underline{th}}$ order phase lock loops.

It is to be noted that the procedure developed in this memo can be used to characterize the behavior of several other nonlinear devices as, for example, the delay lock loop.

SEE REVERSE SIDE FOR DISTRIBUTION LIST

DISTRIBUTION

COMPLETE MEMORANDUM TO

CORRESPONDENCE FILES:

OFFICIAL FILE COPY
plus one white copy for each additional case referenced

TECHNICAL LIBRARY (4)

NASA Headquarters
S. W. Fordyce/MLA
H. S. Fosque/TA
T. A. Keegan/MA
J. T. McClanahan/MOR
W. E. Miller, Jr./MOG
L. M. Robinson/TS

GSFC
T. Roberts/810
J. P. Shaughnessy/834
W. P. Varson/830

MSC
H. C. Kyle/EE
R. W. Moorehead/EB-2
L. Packham/EE
P. Vavra/EB
J. M. West/AE

MSFC
T. A. Barr/R-ASTR-IR
L. B. Malone, Jr./R-ASTR-IRC
J. T. Powell, Jr./R-ASTR-I

ARC
W. C. Coombs

BTL-HO
F. T. Andrews
D. Gillette
I. Jacobs

BTL-MH
E. W. Gilbert
S. O. Rice
D. Slepian

JPL
D. W. Boyd
M. Easterling
E. Posner
R. C. Tausworthe

Philco-Ford
J. J. Spilker

UCLA
A. J. Viterbi

USC
W. C. Lindsey

Univ. of Texas
D. Lainiotis

Bellcomm, Inc.
G. M. Anderson
D. O. Baechler*
W. J. Benden
A. P. Boysen, Jr.
R. K. Chen
L. A. Ferrara, Jr.
D. R. Hagner
H. A. Helm
J. J. Hibbert
B. T. Howard
J. E. Johnson*
H. Kraus*
S. Y. Lee
J. P. Maloy*
J. Z. Menard
L. D. Nelson
B. F. O'Brien*
J. T. Raleigh
I. I. Rosenblum
I. M. Ross
P. S. Schaenman*
N. W. Schroeder
R. L. Selden
J. L. Strand
J. W. Timko
B. P. Tunstall
R. L. Wagner
A. G. Weygand
M. P. Wilson
W. D. Wynn
Dept. 1024 File

*Abstract Only

BELLCOMM, INC.
955 L'ENFANT PLAZA NORTH, S.W. WASHINGTON, D.C. 20024

SUBJECT: Characterizing the Behavior of Phase Lock Loops - Case 900

DATE: February 20, 1970

FROM: L. Schuchman

TM-70-2034-2

TECHNICAL MEMORANDUM

The phase lock loop to be discussed is shown in Figure 1

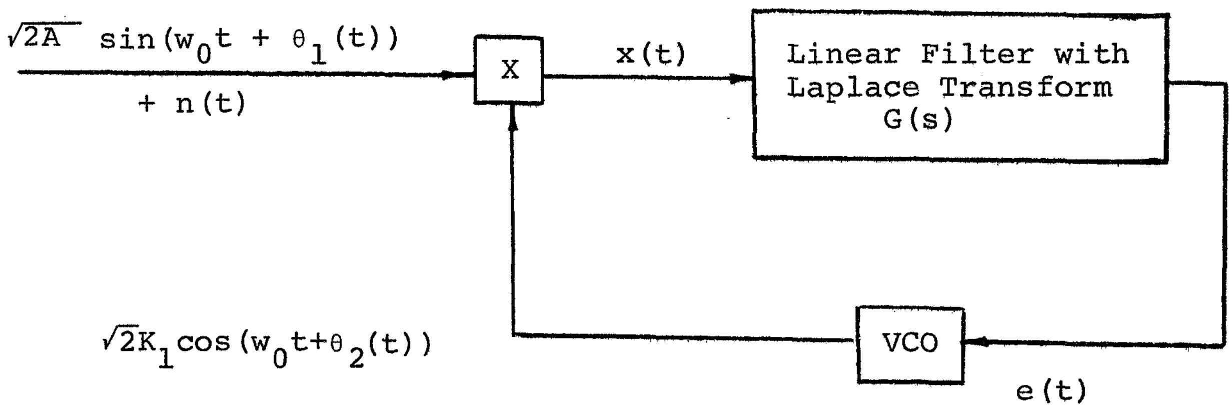

Figure 1 Phase Lock Loop

The loop consists of three components, a multiplier, a time-invariant linear filter g(t) whose Laplace transform is G(s), and a voltage controlled oscillator (VCO). If the loop is operating on a noiseless sine wave, then the steady state output of the VCO is a quadrature replica of the input signal which may differ only in amplitude. The multiplier output x(t), under these conditions, will have only a double frequency term which the linear filter and VCO configuration will not pass. The linear filter has a second and equally important function which is to reduce the effects of noise that normally gets into the loop.

In my previous memorandum[1], the behavior of a first and a particular second order loop was described in terms of the statistics of two parameters; namely, the loop stationary phase error $\phi(t) = \theta_1(t) - \theta_2(t)$, and the time to cycle slip $\hat{T}$. Since both these parameters are random, their behavior was characterized statistically. That is $\phi(t)$ was described in terms of its probability density function while $\hat{T}$ was given in terms of its first four moments which were shown to fit those of a Pearson Type III density function.

At this point, the work of two other researchers is briefly described. Lindsey[2] recently published a paper in which he obtained solutions for the time to cycle slip and the steady state density of the phase error for a N^{th} order generalized tracker. His approach is an extension of Viterbi's[3] work in which an n^{th} order phase lock loop is characterized by N first order Markov differential equations. This leads to the statistical characterization of the tracker by a N dimensional Fokker-Planck equation (forward Kolmogorov equation). In addition, Lindsey elegantly demonstrates the analogy between the theory of continuous Markov processes and Maxwell's wave equations.

Tausworthe[4] has obtained results for the mean time to cycle slip for the N^{th} order loop. In a footnote to his paper, he describes a differential equation that the characteristic function of the time to cycle slip parameter satisfies. Use of this result could also lead to my results for the n^{th} moment of the time to cycle slip parameter for an arbitrary N^{th} order loop.

In this technical memorandum, the behavior of the phase lock loop is characterized by a Markov like process which leads quickly and simply to the statistical characterization of the loop by both the forward and backward Kolmogorov differential equations. We then make use of the work of Darling and Siegert[5] to derive results for the n^{th} moment of the time to cycle slip parameter for an N^{th} order loop. This result is in the form of a differential equation. It has been found by the author that although a solution to this equation can be obtained, the evaluation of the solution in most cases is a significant task, while direct evaluation of the differential equation by simulation on a digital computer is relatively simple.

The class of linear filters for which our results will be valid is defined as

$$G(s) = C + H(s) \qquad (1)$$

where C is a constant and

$$H(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i} \qquad \text{with } m<n \qquad (2)$$

the sets of $[a_i]$ and $[b_i]$ are constants chosen so that the phase lock loop is stable and physically realizable.

It is assumed that the loop input is a sine wave plus a stationary narrow band Gaussian process n(t) of zero mean. The noise n(t) may be represented as

$$n(t) = \sqrt{2}\ [n_1(t)\sin\omega_0 t + n_2(t)\cos\omega_0 t] \qquad (3)^*$$

where $n_1(t)$ and $n_2(t)$ are independent Gaussian processes of zero mean and identical low pass spectral densities. The parameter $\omega_o \frac{rad}{sec}$ is taken as the quiescent frequency output at the VCO. The spectral densities of n(t), $n_1(t)$ and $n_2(t)$ are assumed flat over a sufficiently wide range so that with respect to the closed loop bandwidth they can be approximated as white noise with the same magnitude of $N_o/2$ watts/Hz.**

---

*The reader is referred to section 2.7 of Reference 3 for a more detailed development of the noise model.

**$N_0 = KT^o$ watts/ Hz and is the one sided noise spectral density where K = Boltzmann's constant and T^o = system noise temperature, degrees Kelvin.

If we define the input to the phase lock loop as $\sqrt{2}\, A \sin[\omega_o t + \theta_2(t)]$ then we may write the multiplier output x(t) as

$$x(t) = AK_1 \sin\phi(t) - K_1 n_1(t) \sin\theta_2(t) + K_1 n_2(t) \cos\theta_2(t)$$

$$+ AK_1 \sin[2\omega_o t + \theta_1(t) + \theta_2(t)] + K_1 n_1(t) \sin[2\omega_o t + \theta_2(t)]$$

$$+ K_1 n_1(t) \cos[2\omega_o t + \theta_2(t)] \qquad (4)$$

Since the VCO will not pass the double frequency terms, they can be ignored. When the control signal to the VCO is applied at t=0, the VCO frequency becomes $\omega_o + k_2 e(t)$ where K_2 is a proportionality constant at the VCO whose dimensions are radians per second per volt. Thus the time derivative of the loop phase error is

$$\frac{d\phi}{dt} = \frac{d}{dt}[\theta_1 - \theta_2] = \frac{d\theta_1(t)}{dt} - K_2 e(t) \qquad (5)$$

where

$$e(t) = K_1 \int_0^t g(t-u) \; [A\sin\phi(u) + n'(u)]du \qquad (6)$$

with

$$n'(t) = -n_1(t)\sin\theta_2(t) + n_2(t)\cos\theta_2(t)$$

Using equation (1), we can write equation (5) as

$$\frac{d\phi}{dt} = \frac{d\theta_1(t)}{dt} - KC[A\sin\phi(t) + n'(t)] - f_t \qquad (7)$$

where

$$f_t = K \int_0^t h(t-u)[A\sin\phi(u) + n'(u)]du \qquad (8)^*$$

$K = K_1 K_2$ defined as the loop again

$h(t)$ is the inverse Laplace transform of $H(s)$.

This work parallels that given in Reference 1, and much of the detail is omitted from the derivations that follow since such detail can be obtained from the referenced work.

We define four variables descriptively in Figure 2 and write their joint density distribution as

$$P(\phi,\phi_1,f_t,\phi_0) = P(\phi_1,\phi_0,f_t) \cdot P(\phi|\phi_1,\phi_0,f_t) \qquad (9)$$

As is shown in Appendix C of Reference 1, integrating both sides with respect to ϕ_1 and then factoring out $P(\phi_0,f_t)$ leads to

$$P(\phi|\phi_0,f_t) = \int_{-\infty}^{+\infty} P(\phi_1|\phi_0,f_t)\ P(\phi|\phi_1,f_t)d\phi_1 \qquad (10)$$

---

*f_t differs from its definition given in TM-69-2034-8 only by a constant multiplier for the second order loop.

Thus we see that the conditional transitional probabilities satisfy the Chapman Kolmogorov equations given by equation (11)

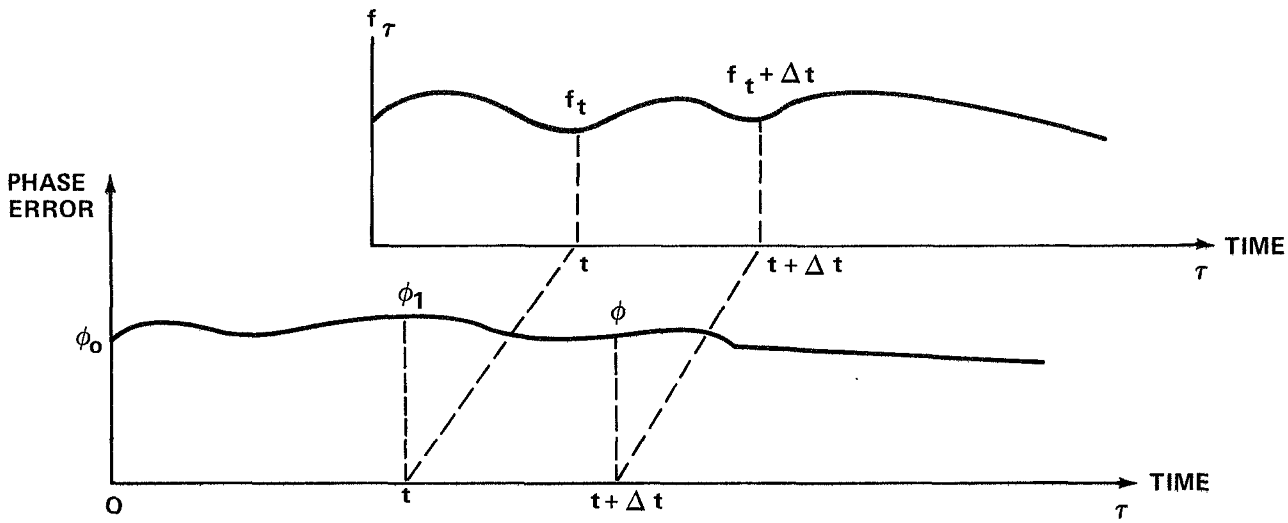

FIGURE 2 - PHASE ERROR AND f_τ AS A FUNCTION OF TIME (τ)

Proceeding as we did in Appendix C of Reference 1 we have that

$$\frac{\partial P(\phi_1|\phi_0,f_t)}{\partial t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial \phi^n}[\gamma_n(\phi_1,f_t)P(\phi_1|\phi_0,f_t)] +$$

$$\frac{df_t}{dt} \frac{\partial P}{\partial f_t} (\phi_1|\phi_0,f_t) \tag{11}$$

where

$$\gamma_n(\phi_1,f_t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{-\infty}^{+\infty} (\phi-\phi_1)^n P(\phi|\phi_1,f_t)d\phi \quad n \geq 1 \tag{12}$$

But the left side of equation (11) can be expanded with the result that

$$\frac{\partial P(\phi_1|\phi_0,f_t)}{\partial t} = \frac{\partial P(\phi_1|\phi_0,f_t)}{\partial \phi_1} \frac{d\phi_1}{dt} + \frac{\partial P(\phi_1|\phi_0,f_t)}{\partial f_t} \frac{df_t}{dt} \tag{13}$$

Equating the right side of equations (11) and (13) we have

$$\frac{\partial P(\phi_1|\phi_0,f_t)}{\partial \phi_1} \frac{d\phi_1}{dt} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial \phi^n}[\gamma_n(\phi_1,f_t)P(\phi_1|\phi_0,f_t)) \tag{14}$$

It is easy to show that

$$-\gamma_1(\phi_1,f_t) = \text{AKC} \sin \phi_1 t + f_t - \frac{d\theta_1}{dt}$$

$$\gamma_2(\phi_1,f_t) = \frac{C^2K^2N_0}{2}$$

$$\gamma_n(\phi_1,f_t) = 0 \qquad \text{for } n > 2$$

Equation (14) then reduces to

$$\frac{\partial P(\phi_1|\phi_0,f_t)}{\partial \phi_1} \frac{d\phi_1}{dt} = [\text{CAK}\sin\phi_1(t) + f_t - \frac{d\theta_1}{dt}]P(\phi_1|\phi_0,f_t) + \frac{C^2K^2N_0}{4} \frac{\partial^2 P(\phi_1|\phi_0,f_t)}{\partial \phi_1^2} \tag{15}$$

We next multiply both sides of equation (15) by $p(f_t)$ and then integrate over all values of f_t with the result that

$$\frac{dP(\phi_1|\phi_0,)}{dt} = \frac{\partial}{\partial\phi_1}\left[AK\sin\phi_1 + E[f_t|\phi_1] - \frac{d\theta_1}{dt}\right]P(\phi_1|\phi_0,)$$

$$+ \frac{K^2N_0}{4}\frac{\partial^2 P(\phi_1|\phi_0,)}{\partial\phi_1^2} \tag{16}$$

where $E[f_t|\phi_1]$ is the expectation of f_t given the phase of the received signal at time t. The conditional expectation of f_t cannot be evaluated exactly since it requires a knowledge of the probability density distribution of ϕ_1 which we obviously do not know.

It has been shown[1,4] that for a second order phase lock loop the approximation

$$E[f_t|\phi_1] \approx -e\phi_1(t) \tag{17}$$

gives results which agree excellently with experimental findings. Tauseworthe[4] has shown that e can be obtained from a linear approximation of the phase lock loop with the result that

$$e = AKC - \frac{\ell^2(0)}{4B_L} \tag{18}$$

where $\ell(0) = \lim_{s\to\infty} s\left[\frac{AKG(s)/s}{1+AKG(s)/s}\right]$.

The loop-noise bandwidth B_L is defined as

$$B_L = \frac{1}{4\pi i} \int_{-i\infty}^{i\infty} \left[\frac{AKG(s)/s}{1+AKG(s)/s}\right] \left[\frac{AKG(-s)/s}{AK\,\frac{G(-s)}{s} - 1}\right] ds \qquad (19)$$

Viterbi computed B_L and tabulated his results for several filters in Reference 3.

To illustrate let

$$G(s) = 1 + \frac{\alpha}{s}\text{ , then}$$

$$e = AK - \frac{(AK)^2}{4\left(\frac{AK+\alpha}{4}\right)} = \frac{\alpha}{AK+\alpha}$$

This is in agreement with the result presented in Reference 1.

If the linear approximation can be made or, more generally, when $E[f_t|\phi_1]$ can be shown to be independent of t then equation (16) defines a stationary Markov Process so that we can use a result by Darling and Siegert[5] to obtain the following recursive relationship for the moments of the time to cycle slip parameter $\hat{T}$.

$$\frac{C^2K^2N_0}{2}\,\frac{d^2E[\hat{T}^n]}{d^2\phi_0} - \left[AK\sin\phi_0 + E[f_t|\phi_0] - \frac{d\theta_1}{dt}\right]\frac{dE[\hat{T}^n]}{d\phi_0} = -nE[\hat{T}^{n-1}] \qquad (20)$$

where $E[\hat{T}^\circ] = 1$

$$E[\hat{T}^n(2\pi)] = E[\hat{T}^n(-2\pi)] = 0$$

2034-LS-jf

L. Schuchman

Attachment
References

REFERENCES

1. L. Schuchman, "Time to Cycle Slip in First and Second Order Phase Lock Loops", Bellcomm Technical Memorandum TM-69-2034-8, December 19, 1969.

2. N. C. Lindsey, "Nonlinear Analysis of Generalized Tracking Systems", Proceedings of the IEEE, Vol. 57, No. 10, pp. 1705-17022, Oct. 1969.

3. A. J. Viterbi,"Principles of Coherent Communications", Chapter 2, McGraw-Hill, 1966.

4. R. C. Tausworthe, "Cycle Slipping in Phase-Locked Loops", IEEE Transactions on Communication Technology, Vol. COM-15, No. 3, pp. 417-421, June, 1967.

5. D. A. Darling and A. J. E. Siegert,"The First Passage Time Problem for a Continuous Markov Process", Am. Math. Stat., Vol. 24, pp. 624-639, 1953.