N70-27026

BELLCOMM, INC.
955 L'ENFANT PLAZA NORTH, S.W., WASHINGTON, D.C. 20024

COVER SHEET FOR TECHNICAL MEMORANDUM

TITLE– Explicit Inverse Fourier Transformation of a Rational Function and a New Theorem

TM– 70-1033-2

DATE– February 2, 1970

FILING CASE NO(S)– 103-6

AUTHOR(S)– S. Y. Lee

FILING SUBJECT(S) (ASSIGNED BY AUTHOR(S))– Fourier and Inverse Fourier Transforms
Digital Inverse Fourier Transforms

ABSTRACT

Analyses in areas such as communication, network, and control theory are often performed in the frequency domain. Many solutions are not complete until the inverse Fourier transformation of a rational algebraic fraction has been successfully carried out. With the results derived in this memorandum, we can obtain the inverse Fourier transform h(t) of any general rational algebraic fraction directly from a table after partial fraction expansion of H(jω) has been performed without any further calculation. A new theorem for the evaluation of the inverse Fourier transforms of the terms consisting of poles on the jω axis has been developed. With the addition of this theorem, the inverse Fourier transform of any rational algebraic fraction can be evaluated directly from its inverse Laplace transform.

A computer program has been developed for obtaining the inverse Fourier transformation of a general rational algebraic fraction where the output of this method is a closed form solution in time. Numerical examples are provided to show the usage of this computer program.

SEE REVERSE SIDE FOR DISTRIBUTION LIST

DISTRIBUTION

COMPLETE MEMORANDUM TO

CORRESPONDENCE FILES:

OFFICIAL FILE COPY
plus one white copy for each additional case referenced

TECHNICAL LIBRARY (4)

A. T. Ackerman
G. M. Anderson
D. O. Baechler
I. Y. Bar-Itzhack
W. J. Benden
A. P. Boysen, Jr.
F. A. Brewer
J. O. Cappellari
K. R. Carpenter
G. K. Chang
R. K. Chen
C. L. Davis
D. A. DeGraaf
J. P. Downs
B. D. Elrod
J. J. Fearnsides
A. J. Ferrari
D. R. Hagner
W. G. Heffron
A. Heiber
H. A. Helm
J. J. Hibbert
N. W. Hinners
W. W. Hough
B. T. Howard
D. B. James
J. Kranton
P. F. Long
W. Levidow
H. S. London
K. E. Martersteck
D. L. Mather
R. K. McFarland
J. Z. Menard
L. D. Nelson
J. M. Nervik
B. G. Niedfeldt
E. A. Nussbaumer
J. J. O'Connor
M. P. Odle
G. T. Orrok
J. T. Raleigh
R. J. Ravera
I. M. Ross
J. W. Schindelin
N. W. Schroeder
L. Schuchman
R. L. Selden
P. F. Sennewald
P. G. Smith
R. V. Sperry
W. Strack
W. B. Thompson
J. W. Timko
B. P. Tunstall
R. L. Wagner
J. E. Waldo
R. A. Wenglarz
M. P. Wilson
D. P. Woodard
W. D. Wynn
All Members of Department 1033
Department 1024 File

BELLCOMM, INC.
955 L'ENFANT PLAZA NORTH, S.W. WASHINGTON, D.C. 20024

SUBJECT: Explicit Inverse Fourier Transformation of a Rational Function and a New Theorem - Case 103-6

DATE: February 2, 1970

FROM: S. Y. Lee

TM-70-1033-2

TECHNICAL MEMORANDUM

I. INTRODUCTION

Analyses in areas such as communication, network, and control theory are often performed in the frequency domain (e.g., Fourier or Laplace transforms). However, many solutions are not complete until the inverse Fourier transformation of a rational algebraic fraction is obtained. Although the literature on Fourier transformation and inverse Fourier transformation is quite extensive, there are some problems that have not been explicitly considered. The subject of this memorandum is, to the author's knowledge, one of them. We consider the following problem. Given a general rational algebraic fraction $H(j\omega)$ we would like to obtain the the inverse Fourier transformation $h(t)$ directly from a table after partial fraction expansion on $H(j\omega)$ has been performed without essentially anymore manipulations. In the process of deriving the results of this memorandum, a new theorem is developed for obtaining the inverse Fourier transform of a rational algebraic fraction when its poles are on the $j\omega$ axis directly from its inverse Laplace transform.

The motivation for considering this problem is to develop a computer program for obtaining the inverse Fourier transformation of a general rational algebraic fraction where the output of this method is a closed form solution (not discrete digital value) in time. This program can be developed only if the explicit inverse Fourier transforms of the general terms after partial fraction expansion on $H(j\omega)$ are known. Up to this writing, explicit inverse Fourier transforms of a rational algebraic fraction when its poles are on the $j\omega$ axis (with the exception at the origin) have not been published. This is due to the difficulties which arise with the convergence of the integral in the inversion formula.

II. PROPER AND IMPROPER FRACTIONS, DISTINCT AND MULTIPLE ROOTS

We shall now direct our attention of a general rational algebraic fraction of the following form:

$$H(p) = \frac{KN(p)}{D(p)} = \frac{a_m p^m + a_{m-1} p^{m-1} + \ldots + a_1 p + a_0}{b_n p^n + b_{n-1} p^{n-1} + \ldots + b_1 p + b_0} \qquad (1)$$

where $p=j\omega$, the coefficients a_i and b_i are all real constants, and m and n are positive integers. Let us distinguish two different cases of H(p), namely, when $m<n$ and $m\geq n$. For the case $m<n$, we say that H(p) is a proper fraction. While when $m\geq n$ we have an improper fraction. We see that an improper fraction can always be transformed into the sum of a polynomial in p and a proper fraction by simply dividing N(p) by D(p) until the degree of the numerator of the remainder is less than the degree of D(p). It is known that a proper fraction can be expanded by partial fraction expansion method into the summation of the following terms: $\frac{K}{p}$, $\frac{K}{p+\sigma}$, $\frac{Kp}{p^2+\omega_0^2}$, $\frac{K\omega_0}{p^2+\omega_0^2}$, $\frac{K(p+\sigma)}{(p+\sigma)^2+\omega_0^2}$ and $\frac{K\omega_0}{(p+\sigma)^2+\omega_0^2}$, if the roots of D(p) are all distinct. When $D(p)=0$ has multiple roots and H(p) is a proper fraction then the partial fraction expansion of H(p) will consist of all the above terms plus additional terms of the form $\frac{K}{(p+\sigma+j\omega_0)^n}$ and $\frac{K}{(p+j\omega_0)^n}$. As stated previously, the basic difference between an improper fraction and a proper fraction is that the expansion of the improper fractions contains a polynomial in p whereas the expansion of the proper fraction does not. Thus, when H(p) is an improper fraction the terms of the form of p^n must be included. Hence, to find the inverse Fourier transformation of a rational algebraic fraction we must derive the inverse Fourier transform of each of the above terms.

III. INVERSE FOURIER TRANSFORMATION OF $\frac{K}{p+\sigma}$, $\frac{K(p+\sigma)}{(p+\sigma)^2+\omega_0^2}$, AND $\frac{K\omega_0}{(p+\sigma)^2+\omega_0^2}$

Let $H^*(s)$ denote the Laplace transform of $h(t)$ where $s=\sigma+j\omega$. Then it is known that if the region of convergence of $H^*(s)$ contains the $j\omega$ axis in its interior, (i.e., if $\sigma<0$), then the Fourier transform $H(p)$ is a special case of the Laplace transform $H^*(s)$. Furthermore, if the $j\omega$ axis is outside the region of convergence of $H^*(s)$, (i.e., if $\sigma>0$), then the inverse Fourier transform is equal to the inverse Laplace transform of $H(-s)$ with t replaced by $-t$. This result can be readily seen by examining the time scaling theorem. Thus, from these statements we can conclude that

$$F^{-1}[H(p)] = L^{-1}[H^*(s)] \quad , \quad p=s \tag{2}$$

for general terms of $\frac{K}{p+\sigma}$, $\frac{K(p+\sigma)}{(p+\sigma)^2+\omega_0^2}$, and $\frac{K\omega_0}{(p+\sigma)^2+\omega_0^2}$ when $\sigma>0$. And when $\sigma<0$,

$$F^{-1}[H(p)] = L^{-1}[H(-s)]_{\text{replacing } t \text{ by } -t} \tag{3}$$

The inverse Laplace transforms of the above terms are well known. Hence

$$F^{-1}\left[\frac{K}{p+\sigma}\right] = Ke^{-\sigma t}u(t) \quad \text{where} \quad \sigma>0 \tag{4}$$

$$= -Ke^{-\sigma t}u(-t) \quad \text{where} \quad \sigma<0$$

$$F^{-1}\left[\frac{K(p+\sigma)}{(p+\sigma)^2+\omega_0^2}\right] = Ku(t)e^{-\sigma t}\cos\omega_0 t \text{ where } \sigma>0 \tag{5}$$

$$= -Ku(-t)^{-\sigma t}\cos\omega_0 t \text{ where } \sigma<0$$

$$F^{-1}\left[\frac{K\omega_0}{(p+\sigma)^2+\omega_0^2}\right] = Ku(t)e^{-\sigma t}\sin\omega_0 t \text{ where } \sigma>0 \tag{6}$$

$$= -Ku(-t)e^{-\sigma t}\sin\omega_0 t \text{ where } \sigma<0$$

and u(t) is a unit step function.

IV. <u>INVERSE FOURIER TRANSFORMATION OF</u> $\frac{K}{p}$, $\frac{Kp}{p^2+\omega_0^2}$ and $\frac{K\omega_0}{p^2+\omega_0^2}$

When $\sigma=\gamma=0$, at least one of the singular points of h(p) lies on the $j\omega$ axis, therefore no general conclusion about the convergence of H(p) can be drawn. To derive the inverse Fourier transforms of $\frac{K}{p}$ we substitute $G(\omega) = \frac{K}{j\omega}$ into the inversion formula,

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{K}{j\omega}e^{j\omega t}d\omega \tag{7}$$

Simplifying, (7) becomes

$$g(t) = \frac{K}{2\pi}\int_{-\infty}^{\infty}\frac{\sin t\omega}{\omega}d\omega \tag{8}$$

or

$$g(t) = \frac{K}{\pi}\int_{0}^{\infty}\frac{\sin t\omega}{\omega}d\omega \tag{9}$$

From an integral table or performing a contour integration and applying Jordan's lemma, (9) is equal to $\frac{K}{2}$ if $t>0$, minus $\frac{K}{2}$ if $t<0$, and zero if $t=0$. Thus,

$$F^{-1}[\frac{K}{p}] = \frac{K}{2} \text{ sgn } (t) \tag{10}$$

where sgn (t) is shown in Figure 1.

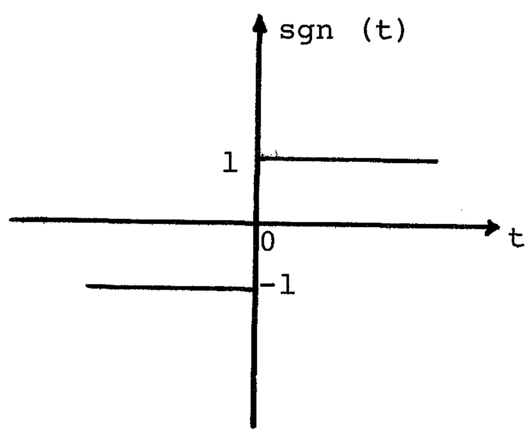

Figure 1

Using the result of (10), we obtain

$$F[\text{sgn } (t)] = \frac{2}{j\omega} \tag{11}$$

The function u(t) can be written in terms of sgn (t) as

$$u(t) = \frac{1}{2} + \frac{1}{2} \text{ sgn } (t) \tag{12}$$

where

$$u(t) = \begin{cases} 0 \ , & t<0 \\ 1 \ , & t>0 \\ \frac{1}{2} \ , & t=0 \quad \text{by preceding definition (12).} \end{cases}$$

By knowing $F[\frac{1}{2}] = \pi\delta(\omega)$, where $\delta(\omega)$ is the delta function introduced as a generalized function, and

$$\int_{-\infty}^{\infty} \delta(t)f(t)dt = f(0) \quad , \tag{13}$$

Papoulis derives the Fourier transform of u(t)

$$F[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}^{*} \tag{14}$$

Next we will derive the inverse Fourier transform of $\frac{Kp}{p^2+\omega^2}$. It has been shown in Reference [1]

$$F[u(t)\cos(\omega_0 t)] = \frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)] + \frac{j\omega}{\omega_0^2-\omega^2} \tag{15}$$

where

$$\int_{-\infty}^{\infty} \delta(\omega-\omega_0)f(\omega)d\omega = f(\omega_0) \tag{16}$$

---

*Note: many publications give $F[u(t)] = \frac{1}{j\omega}$ which is incorrect because $F^{-1}[\frac{1}{j\omega}] \neq u(t)$ but equals $\frac{1}{2}$ sgn (t). This can be readily seen by (7).

and

$$F[\cos\omega_0 t] = \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)] \tag{17}$$

Thus from (15) we have

$$\frac{j\omega}{\omega_0^2-\omega^2} = F[u(t)\cos\omega_0 t] - \frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)] \tag{18}$$

or from (17) the inverse Fourier transform of $\frac{j\omega}{\omega_0^2-\omega^2}$ is

$$F^{-1}[\frac{j\omega}{\omega_0^2-\omega^2}] = u(t)\cos\,\omega_0 t - \frac{1}{2}\cos\,\omega_0 t = \frac{1}{2}\,\text{sgn}\,(t)\,\cos\,\omega_0 t \tag{19}$$

or

$$F^{-1}[\frac{Kp}{\omega_0^2+p^2}] = \frac{K}{2}\,\text{sgn}\,(t)\,\cos\,\omega_0 t \tag{20}$$

The inverse Fourier transform of $\frac{K\omega_0}{p^2+\omega_0^2}$ can be derived similarly by the following relationships given in Reference [1].

$$F[u(t)\sin\omega_0 t] = \frac{\pi}{2j}\,[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)] + \frac{\omega_0}{\omega_0^2-\omega^2} \tag{21}$$

and

$$F[\sin\omega_0 t] = j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)] \tag{22}$$

Thus from (21) we have

$$F[u(t)\sin\omega_0 t] - \frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)] = \frac{\omega_0}{\omega_0^2-\omega^2} \tag{23}$$

or from (22) the inverse Fourier transform of $\frac{K\omega_0}{p^2+\omega_0^2}$ is

$$F^{-1}[\frac{\omega_0}{\omega_0^2-\omega^2}] = u(t)\sin\omega_0 t - \frac{1}{2}\sin\omega_0 t = \frac{1}{2}\,\mathrm{sgn}(t)\sin\omega_o t \tag{24}$$

or

$$F^{-1}[\frac{K\omega_0}{p^2+\omega_0^2}] = \frac{K}{2}\,\mathrm{sgn}(t)\sin\omega_0 t \tag{25}$$

In this section we derived the inverse Fourier transforms of $\frac{K}{p}$, $\frac{Kp}{p^2+\omega_0^2}$, and $\frac{K\omega_0}{p^2+\omega_0^2}$. The inverse Laplace transforms of these functions are well known, they are

$$L^{-1}[\frac{K}{p}] = Ku(t) \tag{26}$$

$$L^{-1}[\frac{Kp}{p^2+\omega_0^2}] = Ku(t)\cos\omega_0 t \tag{27}$$

and

$$\mathcal{L}^{-1}\left[\frac{K\omega_0}{p^2+\omega_0^2}\right] = Ku(t)\ \sin\omega_0 t \qquad (28)$$

By comparing (10) with (26), (20) with (27), and (25) with (28), a new theorem is developed for obtaining the inverse Fourier transforms of the terms consisting of simple poles on the $j\omega$ axis. This new theorem can be stated as follows:

Theorem:

The inverse Fourier transforms of the terms consisting of simple poles on the $j\omega$ axis can be evaluated as one half of their inverse Laplace transforms with u(t) replaced by sgn (t).

For example: Evaluate the inverse Fourier transform of $\frac{1}{p+j\omega_0}$. By inspection, the pole of this function is on the $j\omega$ axis therefore the theorem above can be applied. It is known that the inverse Laplace transform of $\frac{1}{s+j\omega_0}$ is $e^{-j\omega_0 t}u(t)$, thus

$$F^{-1}\left[\frac{1}{p+j\ \omega_0}\right] = \frac{1}{2}\ e^{-j\omega_0 t}\ \text{sgn}\ (t) \qquad (29)$$

This solution can be verified in a more complicated procedure; first multiply u(t) by $e^{-j\omega_0 t}$ which is equivalent to knowing the inverse Laplace transform of $\frac{1}{s+j\omega_0}$. Then by applying the frequency shifting theorem

$$F[e^{-j\omega_0 t}g(t)] = G(\omega+\omega_0) \qquad (30)$$

to (14) thereby obtaining

$$F[e^{-j\omega_0 t}u(t)] = \pi\delta(\omega+\omega_0) + \frac{1}{j\omega+j\omega_0} \qquad (31)$$

Next, solve for $\frac{1}{j\omega+j\omega_0}$ in (31), take the inverse Fourier transform of both sides which requires knowing $F^{-1}[\pi\delta(\omega+\omega_0)]$ and finally simplify to obtain (29).

IV. INVERSE FOURIER TRANSFORMATION OF $\frac{K}{(p+\sigma+j\omega_0)^n}$ AND $\frac{K}{(p+j\omega_0)^n}$

As stated in section III, if the region of convergence of H*(s) contains the $j\omega$ axis in its interior, (i.e., $\sigma>0$) then the Fourier transform and inverse Fourier transform are special cases of Laplace transform and inverse Laplace transform. Furthermore, when the $j\omega$ axis is outside the region of convergence of H*(s), (i.e., $\sigma<0$), then the Fourier transform is equal to the inverse Laplace transform with t replaced by minus t. Thus, the inverse Fourier transform of $\frac{K}{(p+\sigma+j\omega_0)^n}$ is

$$F^{-1}\left[\frac{K}{(p+\sigma+j\omega_0)^n}\right] = \frac{Kt^{n-1}}{(n-1)!} e^{-(\sigma+j\omega_0)t} u(t) \text{ for } \sigma>0 \qquad (32)$$

$$= \frac{-K(-t)^{n-1}}{(n-1)!} e^{-(\sigma+j\omega_0)t} u(-t) \text{ for } \sigma<0$$

To derive the inverse Fourier transform of $\frac{K}{(p+j\omega_0)^n}$, we first apply the frequency differentiation theorem

$$F[(-jt)^n g(t)] = \frac{d^n G(\omega)}{d\omega^n} \qquad (33)$$

to obtain

$$F\left[\frac{t^{n-1}}{(n-1)!} e^{-j\omega_0 t} u(t)\right] = \frac{\pi j^{n-1}}{(n-1)!} \delta^{n-1}(\omega+\omega_0) + \frac{1}{(j\omega+j\omega_0)^n} \qquad (34)$$

where $\delta^{n-1}(\omega+\omega_0)$ is the n-1 derivative of $\delta(\omega+\omega_0)$ and the derivatives exist only in the sense of distributions or

$$F^{-1}\left[\frac{1}{(j\omega+j\omega_0)^n}\right] = \frac{t^{n-1}}{(n-1)!} e^{-j\omega_0 t} u(t) - F^{-1}\left[\frac{\pi j^{n-1}}{(n-1)!} \delta^{n-1}(\omega+\omega_0)\right] \qquad (35)$$

The last term on the right hand side of (35) can be evaluated by (33) which is

$$F^{-1}\left[\frac{\pi j^{n-1}}{(n-1)!} \delta^{n-1}(\omega+\omega_0)\right] = \frac{t^{n-1}}{2(n-1)!} e^{-j\omega_0 t} \qquad (36)$$

Substituting (36) into (35) and simplifying we obtain

$$F^{-1}\left[\frac{K}{(j\omega+j\omega_0)^n}\right] = \frac{K}{2} \frac{t^{n-1}}{(n-1)!} e^{-j\omega_0 t} \operatorname{sgn}(t) \qquad (37)$$

It should be noted that inverse Laplace transform of $\frac{1}{(j\omega+j\omega_0)^n}$ is

$$L^{-1}\left[\frac{K}{(s+j\omega_0)^n}\right] = \frac{Kt^{n-1}}{(n-1)!} e^{-j\omega_0 t} u(t) \qquad (38)$$

By comparing (37) and (38), we see that the theorem developed in section III holds for multiple poles on the $j\omega$ axis as well as simple poles on the $j\omega$ axis. This theorem can now be restated as follows:

Theorem:

The inverse Fourier transforms of the terms consisting of poles on the $j\omega$ axis can be evaluated as one half of their inverse Laplace transforms with u(t) replaced by sgn (t).

V. INVERSE FOURIER TRANSFORMATION OF p^n

The results of sections II, III and IV enables us to evaluate the inverse Fourier transform of any general rational algebraic proper fraction. However, as previously stated, in order to evaluate the inverse Fourier transform of a general rational algebraic improper fraction we must include the terms of the form of p^n. The inverse Fourier transform of the term p^n can be easily evaluated by applying the time differentiation theorem

$$F[\frac{d^n g}{dt^n}] = (j\omega)^n G(\omega) \tag{39}$$

Thus

$$F^{-1}[(p)^n] = \frac{d^n \delta(t)}{dt^n} \tag{40}$$

where the derivatives exist only in the sense of distributions.

VI. COMPUTER PROGRAM AND NUMERICAL EXAMPLES

The results of this paper are given in Table 1. With these results, a digital program was developed for obtaining the inverse Fourier transformation of a general rational algebraic fraction. The inputs to this computer program are the gain K, the roots of N(p) and D(p) as given in equation (1). The output is the inverse Fourier transform of H(p). Here are some of the numerical examples which have been done using the computer program.

(1) $$F^{-1}\left[\frac{(p+2)}{p(p+1)^2(p+3)}\right] = \frac{1}{3}\,\mathrm{sgn}(t) - \left[\left(\frac{1}{2}t+\frac{3}{4}\right)e^{-t}+\frac{1}{12}e^{-3t}\right]u(t)$$

(2) $$F^{-1}\left[\frac{\frac{1}{3}}{p^2(p^2+4)}\right] = \frac{1}{24}\left[t-\frac{1}{2}\sin(2t)\right]\mathrm{sgn}(t)$$

(3) $$F^{-1}\left[\frac{1}{p(p+1)^3(p+2)}\right] = \frac{1}{4}\,\mathrm{sgn}(t) - \left[\left(\frac{1}{2}t^2+1\right)e^{-t}+\frac{1}{2}e^{-2t}\right]u(t)$$

(4) $$F^{-1}\left[\frac{1}{p(p+1)(p^2+4)}\right] = \frac{1}{4}\,\mathrm{sgn}(t) - \frac{1}{5}e^{-t}-\frac{1}{20}\left[\frac{1}{2}\cos(2t)+\sin(2t)\right]\mathrm{sgn}(t)$$

(5) $$F^{-1}\left[\frac{2}{-p^2+1}\right] = e^{t}u(-t)+e^{-t}u(t) = e^{-|t|}$$

(6) $$F^{-1}\left[\frac{1}{p(p+1)(p^2+4)^3[(p+3)^2+1]}\right] = \left[\frac{1}{1280}+\left(\frac{1}{9600}\sin 2t-\frac{1}{12800}\cos 2t\right)t^2\right.$$

$$+\left(\frac{1}{12454}\sin 2t+\frac{1}{3972}\cos 2t\right)t - \left.\left(\frac{1}{11850}\sin 2t-\frac{1}{27686}\cos 2t\right)\right]\mathrm{sgn}(t)$$

$$-\left[\frac{1}{625}e^{-t}+\left(\frac{1}{30000}\sin t+\frac{1}{20769}\cos t\right)e^{-3t}\right]u(t)$$

Table 1 - Complete Table for the Evaluation of Inverse Fourier Transform of Rational Functions

G(p)	g(t)
p^n	$\frac{d^n}{dt^n}\,\delta(t)$
$\frac{K}{p}$	$\frac{K}{2}$ sgn (t)
$\frac{K}{p+\sigma}$	$K\,e^{-\sigma t}\,u(t)$ where $\sigma>0$ $-K\,e^{-\sigma t}\,u(-t)$ where $\sigma<0$
$\frac{Kp}{p^2+\omega_0^2}$	$\frac{K}{2}$ sgn (t) cos $\omega_0 t$
$\frac{K\omega_0}{p^2+\omega_0^2}$	$\frac{K}{2}$ sgn (t) sin $\omega_0 t$
$\frac{K(p+\sigma)}{(p+\sigma)^2+\omega_0^2}$	$Ku(t)e^{-\sigma t}\cos\omega_0 t$ where $\sigma>0$ $-Ku(-t)e^{-\sigma t}\cos\omega_0 t$ where $\sigma<0$
$\frac{K\omega_0}{(p+\sigma)^2+\omega_0^2}$	$Ku(t)e^{-\sigma t}\sin\omega_0 t$ where $\sigma>0$ $-Ku(-t)e^{-\sigma t}\sin\omega_0 t$ where $\sigma<0$

Table 1 (Cont.)

$\frac{K}{(p+\sigma+j\omega_0)^n}$	$\frac{Kt^{n-1}}{(n-1)!}e^{-(\sigma+j\omega_0)t}\,u(t)$ where $\sigma>0$ $-\frac{K(-t)^{n-1}}{(n-1)!}e^{-(\sigma+j\omega_0)t}\,u(-t)$ where $\sigma<0$
$\frac{K}{(p+j\omega_0)^n}$	$\frac{Kt^{n-1}}{2(n-1)!}e^{-j\omega_0 t}\,\mathrm{sgn}\,(t)$

Theorem: The inverse Fourier transforms of the terms consisting of poles on the $j\omega$ axis is equal to one half of their inverse Laplace transforms with u(t) replaced by sgn (t).

VII. SUMMARY

With the results derived in this memorandum, the inverse Fourier transform h(t) of any general rational algebraic fraction can be obtained directly from a table after partial fraction expansion of H(jω) has been performed without any further calculation. A new theorem for the evaluation of the inverse Fourier transforms of the terms consisting of the poles on the jω axis has been developed. With the addition of this theorem, the inverse Fourier transform of any rational algebraic fraction can be evaluated directly from its inverse Laplace transform. That is, the inverse Fourier transform is equal to the inverse Laplace transform when the poles are in the left half of p-plane; the inverse Fourier transform is equal to the inverse Laplace transform with t replaced by minus t when the poles are in the right half of p-plane; and the inverse Fourier transform is equal to one half of its inverse Laplace transform with u(t) replaced by sgn (t) when the poles are on the jω axis.

A computer program has been developed for obtaining the inverse Fourier transformation of a general rational algebraic fraction where the output of this method is a closed form solution in time. Numerical examples are provided to show the usage of this computer program.

ACKNOWLEDGMENT

The author would like to thank Mr. G. C. Reis for his many helpful suggestions and criticism. The able programming assistance of Mr. R. R. Singers is gratefully acknowledged. Thanks are due to Mr. L. D. Nelson for his careful review of this memorandum.

1033-SYL-jf S. Y. Lee

Attachment
References

REFERENCES

1. Papoulis, A., The Fourier Integral and Its Applications, McGraw-Hill Book Company, New York, 1962.

2. Bracewell, R. M., The Fourier Transform and Its Applications, McGraw-Hill Book Company, New York, 1965.

3. Ku, Y. H., Transient Circuit Analysis, D. Van Nostrand Company, Princeton, New Jersey, 1961.

4. Arsac, J., Fourier Transforms and the Theory of Distributions, Prentice-Hall, Englewood Cliffs, New Jersey, 1966.

5. Campbell, G. A. and Foster, R. M., Fourier Integral for Practical Applications, Bell Telephone System Technical Publications, 1942.

6. Lighthill, M. J., Introduction to Fourier Analysis and Generalized Functions, Cambridge University Press, 1958.

7. Heffron, W. G., Program for Analysis of Linear Control Systems, Bellcomm Memorandum for File, August 25, 1964.