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# FOUNDATIONS OF AEROELASTIC OPTIMIZATION AND SOME APPLICATIONS TO CONTINUOUS SYSTEMS



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# ABSTRACT

Various optimization problems will be presented here, applications of the general methods developed in optimum control theory [Refs. 3, 10, 11, 12] and based on the Hamiltonian formulation of variational calculus for structural optimization with aeroelastic constraints.

The problem, common for all the applications, may be stated in a general form: given a reference structure (cantilever beam or two-dimensional plate) with uniform structural properties and specified aeroelastic requirements (such as a given divergence speed or flutter speed), find the structure with minimal weight satisfying the same requirements.

This report will be divided into two parts, the first one dealing with static, the second one with dynamic aeroelastic problems (and more precisely flutter problems). This division is not arbitrary, since two out of the three problems of Part A will be found to have a simple analytical solution confirmed by numerical methods, whereas we have to rely on numerical integration mainly in Part B, the torsional-flutter case being excepted. A very powerful numerical procedure, the transition-matrix algorithm, will be described in detail and applied wherever possible. Its limitations in the more complicated case of panel flutter are emphasized.

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## NOMENCLATURE

| | |
|---|---|
| $a$ | width of two-dimensional panel (flutter case) |
| $a_o$ | lift-curve slope |
| $A$ | constant of integration |
| $A_1$ | proportionality constant |
| $B$ | constant of integration |
| $B_1$ | width of two-dimensional plate (chordwise divergence case) |
| $C$ | dimensional chord |
| $C_1$ | constant of integration |
| $D$ | constant of integration |
| $E$ | dimensional distance between elastic axis and line of aerodynamic centers, positive for a. c. line forward of e. a. |
| $E_1$ | reduced Young's modulus for the state of plane strain |
| $EI$ | flexural rigidity |
| $GJ$ | torsional rigidity |
| $H$ | Hamiltonian |
| $I$ | moment of inertia per unit span |
| $I_\alpha$ | section-mass moment of inertia about elastic axis |
| $J$ | torsion constant |
| $k$ | constant |
| $k_1$ | constant |
| $K$ | constant |
| $L$ | dimensional span |
| $m$ | integer |
| $M$ | mass ratio |
| $M$ | free-stream Mach number |
| $n$ | integer |

| | |
|---|---|
| $N$ | number of iterations |
| $p$ | auxiliary variable |
| $p_a$ | pressure |
| $q$ | dynamic pressure |
| $r$ | auxiliary variable |
| $s$ | auxiliary variable |
| $t$ | dimensionless thickness |
| $t_1$ | dimensionless thickness constraint |
| $T$ | dimensional thickness |
| $\underset{\sim}{T}$ | transition matrix |
| $w$ | non-dimensional deflection |
| $W$ | dimensional deflection |
| $V$ | speed of the air stream |
| $x$ | dimensionless space coordinate |
| $X$ | dimensional space coordinate —— spanwise for wing, chordwise for panel |
| $\underset{\sim}{X}$ | state-variables column matrix |
| $Z_t(X)$ | half thickness of panel at station X |
| $Z_{t_r}$ | half thickness of panel at the root |
| $\alpha$ | proportionality constant |
| $\alpha_o$ | mean incidence at midchord |
| $\alpha_e$ | bending slope |
| $\gamma$ | ratio of specific heat at constant pressure to that at constant volume for free stream (1.4 for air) |
| $\delta_1$ | fraction of the mass that is structural |
| $\delta_2$ | fraction of the mass that is non-structural |
| $\Delta(\ \ )$ | increment in a variable |

$\epsilon$ multiplicative constant referring to the variation on the initial guess in the transition-matrix procedure, chosen $(0 < \epsilon \lesssim 1)$

$\kappa$ Bredt's formula proportionality coefficient

$\underset{\sim}{\lambda}$ Lagrange multipliers column matrix

$\lambda_p$ Lagrange multiplier adjoint to p

$\lambda_q$ Lagrange multiplier adjoint to q

$\lambda_r$ Lagrange multiplier adjoint to r

$\lambda_s$ Lagrange multiplier adjoint to s

$\lambda_w$ Lagrange multiplier adjoint to w

$\lambda_{\alpha_e}$ Lagrange multiplier adjoint to $\alpha_e$

$\mu$ Lagrange multiplier, adjoint to the minimum-thickness constraint

$\nu$ Poisson's ratio

$\rho$ density of the free stream

$\theta$ amplitude of section rotation

$\omega$ frequency

$\omega_\alpha$ torsional frequency

Subscripts and Superscripts

$(\ )_D$ value of the quantity at which divergence occurs

$(\ )_O$ quantity for reference system, system with uniform thickness and same aeroelastic eigenvalues as optimized system

$(\ )^T$ transposed matrix

$(\ )^*$ quantity relative to the structural part of the mass

$(\bar{\ })$ complex quantity

$(\ )'$ differentiation with respect to x

$(\ )^{\cdot}$ differentiation with respect to $(1 - x)$

$\delta(\ )$ first variation of a quantity (in the calculus of variations sense)

## A. STATIC AEROELASTIC PROBLEMS

For this kind of problems, time does not appear as an independent variable and therefore vibratory inertial forces are eliminated from the equilibrium equations. Aerodynamic forces can be based then upon well-known steady-flow results rather than the more complex unsteady flow theory.

The static aeroelastic instabilities known as torsional divergence and supersonic chordwise divergence normally would occur at such high flight speeds as not to have a direct influence on structural design[1]. Nevertheless, the speed $V_D$ at which divergence occurs is a good enough measure of the general stiffness level of a lifting surface that a mass minimization based on holding it constant could have practical interest.

## 1. OPTIMIZATION OF A RECTANGULAR WING FOR A GIVEN TORSIONAL DIVERGENCE SPEED

This example, although dealing with a very simplified problem, will serve our purpose to state the problem of optimization in its general analytical form and to outline a method of solution. The problems we will encounter appear also to arise in more complex optimization problems, and the simplicity of the solution here will illuminate them rather than hiding them amidst cumbersome computations. The analytical solution found will be checked against numerical methods used in modern control theory.

### 1.1 Statement of the Problem

Consider (Fig. 1.1)+a cantilever straight wing with elastic axis perpendicular to the free stream. The wing cross-sectional profile, constant along the span, is characterized by a lift-coefficient slope $a_o$. The other parameters and variables are defined in Fig. 1.1.

With the use of aerodynamic strip theory, the problem reduces to the eigenvalue problem

$$\frac{d}{dX}\left(GJ\frac{d\theta}{dX}\right) + qCEa_o\theta = 0$$

where

$$q = \frac{1}{2}\rho V^2$$

is the dynamic pressure ($\rho$ is the density of the free stream), and with the boundary conditions:

$$\theta(0) = 0,$$

stating that the wing is built-in at the end $X = 0$,

$$GJ\frac{d\theta}{dX}\bigg|_{X = L} = 0,$$

no torque applied at $X = L$. If we assume that the torsional stiffness of the wing is dominated by the contribution from the skin, then from Bredt's formula (Ref. 4, p. 44) the torsion constant $J$ is directly proportional to the thickness $T$ of the

+Figures will be found at the end of each chapter.

skin

$$J = \kappa T$$

If we introduce the dimensionless quantities

$$x = \frac{X}{L}$$

$$t = \frac{T}{T_o},$$

where $T_o$ is the (constant) skin thickness of the reference wing with identical cross-section, the problem is rewritten as

$$(t\theta')' + \omega^2\theta = 0$$

$$\theta(0) = 0$$

$$t\theta' \Big|_{x = 1} = 0$$

where $(\ \ )'$ denotes the differentiation with respect to $x$ and

$$\omega^2 = \frac{qCEa_o}{G\kappa T_o}L^2 = \frac{qCEa_o}{GJ_o}L^2$$

$J_o$ being the torsion constant of the reference wing.

For the uniform chord cantilever wing with uniform thickness $T_o$,

$$t = 1$$

and the amplitude of the section rotation is given by

$$\theta(x) = A \sin \omega x + B \cos \omega x.$$

With the boundary conditions

$$\theta(0) = 0 \qquad \theta'(1) = 0,$$

then:

$$B = 0$$

$$\cos \omega = 0$$

A solution thus exists only when

$$\omega = (2n + 1)\frac{\pi}{2} \qquad n = 0, 1, 2, \ldots..$$

The smallest of these values, $\omega = \frac{\pi}{2}$, corresponds to the significant torsional-divergence dynamic pressure. The corresponding dynamic pressure is

$$q_D = \frac{GJ_o\pi^2}{4CEa_oL^2},$$

and the torsional divergence speed is

$$V_D = \frac{\pi}{2L}\sqrt{\frac{2GJ_o}{CEa_o\rho}}.$$

The associated deflection mode at divergence is

$$\theta(x) = A \sin \frac{\pi x}{2}$$

We now are able to state the optimization problem. We want to minimize the integral

$$M = \int_o^1 t(x)dx$$

representing the dimensionless skin mass, with $t(x)$ being subjected to the additional condition $(t\theta')' + \omega^2\theta = 0$, on $[0, 1]$, with the boundary conditions $\theta(0) = 0$ and $(t\theta')\Big|_{x=1} = 0$.

To reduce this problem to a more conventional type, let us introduce the new variable

$$s = t\theta'$$

Now we want to minimize

$$M = \int_o^1 tdx$$

subjected to the constraints

$$\theta' = \frac{s}{t}$$

$$s' = -\omega^2\theta$$

with the boundary conditions

$$\theta(0) = 0 \qquad s(1) = 0$$

This is a problem of the calculus of variations, of a kind often encountered in optimal control theory (Refs. 3, 9, 10). Of course, the role of time for continuous systems will be played by the spanwise dimensionless coordinate x.

The general problem may be stated as follows:

Minimize the scalar performance index

$$M = \int_0^1 t(x)dx$$

for the system described by the following differential equations, written in matrix form

$$\underset{\sim}{X}' = \underset{\sim}{f}[\underset{\sim}{X}(x), t(x), x], \qquad 0 \le x \le 1$$

$\underset{\sim}{X}$ being an (m x 1) column matrix; some $X_i$ given equal to zero at $x = 0$, some others at $x = 1$

$$X_i(0) = 0 \qquad i = 1, \ldots\ldots, n$$

$$X_j(1) = 0 \qquad j = n+1, \ldots\ldots, m$$

The $X_i$ are the state variables and t is the control variable, following the optimal control theory appellations. We define a scalar function, the Hamiltonian by

$$H[\underset{\sim}{X}(x), t(x), x] = t(x) + \underset{\sim}{\lambda}^T(x) \cdot \underset{\sim}{f}[\underset{\sim}{X}(x), t(x), x]$$

To find a control function t(x) that produces a stationary value of M, we must solve the following differential system constituting the necessary conditions for an extremal:

$$X' = f[X, t, x]$$

$$\lambda' = -\left(\frac{\partial H}{\partial X}\right)^T$$

where $t(x)$ is determined by the so-called control equation

$$\frac{\partial H}{\partial t} = 0,$$

the boundary conditions being the previous ones on $X$

$$X_i(0) = 0 \qquad i = 1, \ldots\ldots, n$$

$$X_j(1) = 0 \qquad j = n+1, \ldots\ldots, m,$$

to which are added the conditions on the $\lambda$' s

$$\lambda_i(1) = 0 \qquad i = 1, \ldots\ldots, n$$

corresponding to the given values of the state variables at $x = 0$

$$\lambda_j(0) = 0 \qquad j = n+1, \ldots\ldots, m$$

corresponding to the given values of the state variables at $x = 1$. (These conditions are often referred to as transversality conditions.) For our particular problem $\theta$ and $s$ are assimilated to the state variables.

For our particular problem on hand, the Hamiltonian is:

$$H = t + \lambda_\theta \frac{s}{t} - \lambda_s \omega^2 \theta$$

We have the following system of differential equations:

$$\theta' = \frac{s}{t}$$

$$s' = -\omega^2 \theta$$

$$\lambda'_\theta = \omega^2 \lambda_s$$

$$\lambda'_s = -\frac{\lambda_\theta}{t}$$

$$1 - \frac{\lambda_\theta s}{t^2} = 0$$

with the boundary conditions

$$\theta(0) = 0$$

$$s(1) = 0$$

$$\lambda_\theta(1) = 0$$

$$\lambda_s(0) = 0$$

1.1.1 Analytical Approach

The solution proceeds very easily by eliminating $\lambda_s$ and $\theta$, which step yields the equations:

$$s'' + \frac{\omega^2}{t} s = 0$$

$$\lambda_\theta'' + \frac{\omega^2}{t} \lambda_\theta = 0$$

with the boundary conditions

$$s(1) = 0$$

$$\lambda_\theta(1) = 0$$

$s$ and $\lambda_\theta$ satisfy the same homogeneous differential equation; thus a solution of the problem is such that $\lambda_\theta$ and $s$ are proportional:

$$\lambda_\theta = \frac{s}{A^2}$$

Therefore

$$t^2 = \lambda_\theta s = \frac{s^2}{A^2}$$

$$t = \frac{s}{A}$$

Now from the first equation, $\theta' = A$ so that, using the boundary condition $\theta(0) = 0$, we find $\theta = Ax$, $s' = -\omega^2 Ax$ and, using $s(1) = 0$, $s = \frac{\omega^2 A}{2}(1 - x^2)$. It follows that

$$t = \frac{s}{A} = \frac{\omega^2}{2}(1 - x^2)$$

We now have to go back to the main object of the problem: for a given divergence speed, we want to minimize the mass of a wing with given geometrical characteristics. If our reference thickness is $T_o$, the thickness of the wing with the same geometrical characteristics and constant thickness which has the same divergence speed, then

$$\omega = \frac{\pi}{2}$$

and

$$t = \frac{T}{T_o} = \frac{\pi^2}{8}(1 - x^2) = 1.2337(1 - x^2)$$

The dimensionless mass is then

$$M = \int_o^1 t dx = \frac{\pi^2}{8}\int_o^1 (1 - x^2)dx = \frac{\pi^2}{12} = 0.8225,$$

which corresponds to a mass saving of 17.75%. We recall that $T_o$ is given, as a function of $V_D$ and the geometrical characteristics of the wing, by

$$T_o = \frac{4q_D CEa_o}{\pi^2 G\kappa} L^2$$

where the quantities appearing on the right-hand side have been previously defined.

1.1.2 Numerical Approach: The Transition-Matrix Procedure

The transition-matrix approach (as named by Bryson and Ho (Ref. 13)) to solve the system of optimizing equations has proven very powerful in this particular case. The method, developed by Bryson, is described in Section 7.3 of Ref. 3.

Let us show how the method applies in this particular case. We have the following system of differential equations

$$\theta' = \frac{s}{t}$$

$$s' = -\omega^2\theta$$

$$\lambda'_\theta = \omega^2\lambda_s$$

$$\lambda'_s = -\frac{\lambda_\theta}{t}$$

$$t = \sqrt{\lambda_\theta s}$$

with the two-point boundary conditions:

$$\theta(0) = 0 \qquad s(1) = 0$$

$$\lambda_s(0) = 0 \qquad \lambda_\theta(1) = 0$$

The idea is to integrate the equations on $[0,1]$ starting from the left, by guessing some initial values for $s$ and $\lambda_\theta$. The algorithm proceeds as follows:

a.) We guess the 2 unspecified conditions, i.e. the values of $s$ and $\lambda_\theta$ at $x = 0$.

b.) We now integrate the system of equations above from 0 to 1, the last equation giving the value of $t$, auxiliary variable. This gives us values for $s(1)$ and $\lambda_\theta(1)$.

c.) We determine the 2 x 2 transition matrix T.

$$T = \begin{bmatrix} \frac{\partial s(1)}{\partial s(0)} & \frac{\partial s(1)}{\partial \lambda_\theta(0)} \\ \frac{\partial \lambda_\theta(1)}{\partial s(0)} & \frac{\partial \lambda_\theta(1)}{\partial \lambda_\theta(0)} \end{bmatrix}$$

such that

$$\begin{bmatrix} \delta s(1) \\ \delta\lambda_\theta(1) \end{bmatrix} = T \begin{bmatrix} \delta s(0) \\ \delta\lambda_\theta(0) \end{bmatrix}$$

which is a measure of the variations of the final values of $s$ and $\lambda_\theta$ when the initial boundary conditions are perturbed.

To determine T, we begin by computing the first variations $\delta\theta, \delta s, \delta\lambda_\theta, \delta\lambda_s$ of the 4 quantities $\theta, s, \lambda_\theta, \lambda_s$ using the system of differential equations defining them; we then obtain

$$\delta\theta' = \frac{1}{2}\sqrt{\frac{s}{\lambda_\theta}}\left(\frac{\delta s}{s} - \frac{\delta\lambda_\theta}{\lambda_\theta}\right)$$

$$\delta s' = -\omega^2 \delta\theta$$

$$\delta\lambda'_\theta = \omega^2 \delta\lambda_s$$

$$\delta\lambda'_s = \frac{1}{2}\sqrt{\frac{\lambda_\theta}{s}}\left(\frac{\delta s}{s} - \frac{\delta\lambda_\theta}{\lambda_\theta}\right)$$

If $\delta s(0)$ is set equal to unity, all the other components being zero, integration of this system together with the initial system on $[0,1]$ will give us values for $\delta s(1)$ and $\delta\lambda_\theta(1)$ which will be the first column of our matrix T. Similarly, the second column of T is obtained by integrating this system and the initial one with, as boundary conditions, $\delta\lambda_\theta(0)$ set equal to 1 and all the others equal to zero. The boundary conditions for the initial system will be of course those guessed in step a.), and step b.) can be skipped as we will integrate two times the original system in this step.

d.) We now choose $\delta s(1)$ and $\delta\lambda_\theta(1)$ so as to bring the next solution closer to the desired values 0 of $s(1)$ and $\lambda_\theta(1)$. We choose

$$\delta s(1) = -\epsilon s(1)$$

$$\delta\lambda_\theta(1) = -\epsilon\lambda_\theta(1) \qquad 0 \le \epsilon \le 1$$

where $s(1)$ and $\lambda_\theta(1)$ have been computed in step b.).

e.) We now have to invert T with chosen values of $\delta s(1)$ and $\delta\lambda_\theta(1)$ from step d.) to find $\delta s(0)$ and $\delta\lambda_\theta(0)$ by

$$\begin{bmatrix} \delta s(0) \\ \delta\lambda(0) \end{bmatrix} = T^{-1} \begin{bmatrix} \delta s(1) \\ \delta\lambda(1) \end{bmatrix}$$

f.) Using $s(0)_{new} = s(0)_{old} + \delta s(0)$,

$$\lambda_\theta(0)_{new} = \lambda_\theta(0)_{old} + \delta\lambda_\theta(0)$$

we repeat steps a.) through e.) until $s(1)$ and $\lambda_\theta(1)$ have the specified values 0 to the desired accuracy.

The main problem in this specific case is that $t$ appears on the denominator of the differential system and therefore cannot go to zero on the interval $[0,1]$, except at $x = 1$, so that we must start the algorithm with values of $s(0)$ and $\lambda_\theta(0)$ giving a $t(0)$ quite high so that the convergence is from the upper side of the actual curve. We started with an $\epsilon$ equal to 1, and values of $s(0)$ and $\lambda_\theta(0)$ equal to 0.3 and 6 respectively. An algorithm was designed to reduce $\epsilon$ by one half everytime the thickness distribution would reach the value zero somewhere on $[0,1]$ and start again from the preceding step with this smaller $\epsilon$(*). With the use of this process, convergence was very smoothly obtained from the upper half of the $(x,t)$ plane to the actual thickness distribution in 6 iterations, with a relative error of 0.04% and a final $\epsilon$ of 0.125. This precision may be rendered even better by decreasing $\epsilon$ and using a few more integrations. The differential equations were integrated using a subroutine of the Stanford Computation Center library called DFEQS1, making use of a Runge-Kutta method.

The successive results of the iteration and the actual thickness distribution are plotted in Fig. 1.2.

## 1.2 Optimization With a Minimum-Thickness Constraint

### 1.2.1 Analytical Solution

In the unconstrained case, $t$ was found to be equal to zero at the tip, and thus the problem of optimization with a constraint on the value of the thickness is of greater practical interest.

(*)It would seem that the more we approach the actual thickness distribution, the closer $\epsilon$ should be to unity as then the transition matrix, found by using a linear approximation, would be very accurate. This cannot be true here for the reason explained above, that $t$ cannot approach 0 on $[0,1]$ and so that the convergence has to be non-oscillatory.

We want to minimize the integral

$$M = \int_0^1 t dx$$

subject to the constraints

$$\theta' = \frac{s}{t}$$

$$s' = -\omega^2 \theta$$

with the boundary conditions

$$\theta(0) = 0 \qquad s(1) = 0$$

and the inequality constraint

$$t \geq t_1$$

where $t_1$ is a given real quantity between 0 and 1

Following Bryson and Ho (Ref. 3), Section 3.8, we define the Hamiltonian in this case as

$$H = t + \lambda_\theta \frac{s}{t} - \lambda_s \omega^2 \theta + \mu(t_1 - t)$$

where

$$\mu(x) \geq 0 \quad \text{if} \quad t = t_1$$

$$\mu(x) = 0 \quad \text{if} \quad t > t_1$$

The necessary conditions for an extremal are expressed by the system of equations

$$\theta' = \frac{s}{t}$$

$$s' = -\omega^2 \theta$$

$$\lambda'_\theta = \omega^2 \lambda_s$$

$$1 - \frac{\lambda_\theta s}{t^2} - \mu = 0$$

together with the boundary conditions

$$\theta(0) = 0 \qquad \lambda_s(0) = 0$$

$$s(1) = 0 \qquad \lambda_\theta(1) = 0$$

A solution of the system is obviously such that

$$\lambda_\theta = \frac{s}{\alpha} \qquad \lambda_s = -\frac{\theta}{\alpha},$$

where $\alpha$ is a constant, so that we only have to solve the simpler system

$$\theta' = \frac{s}{t}$$

$$s' = -\omega^2\theta$$

$$t^2 = \frac{s^2}{\alpha(1-\mu)}$$

with the boundary conditions

$$\theta(0) = 0 \qquad s(1) = 0$$

For $t > t_1$, i.e. $\mu = 0$, then the last equation becomes

$$t^2 = \frac{s^2}{\alpha}$$

which implies that the constant $\alpha$ has to be positive, say, $\alpha = A^2$:

$$t = \frac{s}{A}$$

$$\theta' = A$$

$$\theta = A(x + B)$$

and

$$s' = -\omega^2 A(x + B)$$

$$s = -\omega^2 A\left(\frac{x^2}{2} + Bx - 2C\right)$$

and

$$t = -\omega^2\left(\frac{x^2}{2} + Bx - 2C\right),$$

where B and C are constants of integration.

If $t = t_1$, then the first two equations are

$$\theta' = \frac{s}{t_1}$$

$$s' = -\omega^2\theta$$

so that, with 2 constants of integration D and E

$$\theta = D\left(\cos\frac{\omega}{\sqrt{t_1}}x + E\sin\frac{\omega}{\sqrt{t_1}}x\right)$$

$$s = D\omega\sqrt{t_1}\left(-\sin\frac{\omega}{\sqrt{t_1}}x + E\cos\frac{\omega}{\sqrt{t_1}}x\right)$$

Now the boundary condition $\theta(0) = 0$ can be satisfied only in the case $\mu = 0$, so that at the end $x = 0$ the extremum thickness is larger than $t_1$; it then increases as $x$ increases according to the distribution

$$t = +\frac{\omega^2}{2}(C - x^2)$$

with

$$\theta = Ax$$

$$s = +\frac{\omega^2}{2}A(C - x^2)$$

The optimum thickness $t$ will reach the value $t_1$ at $x = X$ such that

$$X^2 = C - \frac{2}{\omega^2}t_1$$

and will then remain equal to $t_1$. The constant E is determined by the boundary condition $s(1) = 0$, so that on $[X, 1]$

$$t = t_1$$

$$\theta = \frac{D}{\cos\frac{\omega}{\sqrt{t_1}}} \cos\frac{\omega}{\sqrt{t_1}}(1 - x)$$

$$s = \frac{D\omega\sqrt{t_1}}{\cos\frac{\omega}{\sqrt{t_1}}} \sin\frac{\omega}{\sqrt{t_1}}(1 - x)$$

Continuity of $\theta$ and $\theta'$ at $x = X$ requires that $X$ is a solution of the transcendental equation.

$$\frac{\omega}{\sqrt{t_1}} X = \operatorname{cotan}\frac{\omega}{\sqrt{t_1}}(1 - X)$$

and the optimal thickness distribution is

$$0 \le x \le X \qquad t = t_1 - \frac{\omega^2}{2}(x^2 - X^2)$$

$$X \le x \le 1 \qquad t = t_1$$

The non-dimensional mass is found to be

$$M = \int_0^1 t dx = t_1 + \frac{\omega^2 X^3}{3}$$

The transcendental equation for $X$ may be written as

$$X = \frac{\sqrt{t_1}}{\omega} \operatorname{cotan}\frac{\omega}{\sqrt{t_1}}(1 - X)$$

and the solution on $[0, 1]$ will be graphically found at the intersection of the curve with equation

$$y = \frac{\sqrt{t_1}}{\omega} \operatorname{cotan}\frac{\omega}{\sqrt{t_1}}(1 - X)$$

and the bissectrix of the first quadrant of the $(X, y)$ plane. Recall that $\omega$ is constant and equal to $\frac{\pi}{2}$, and $t_1$ is a parameter varying from 1 to 0.

As $t_1$ is decreased, there is only one solution to the equation between 0 and 1, until $t_1$ reaches the value 0.11111; there are then two solutions for $t_1$ less than this value, three for $t_1$ less than 0.0400, four for $t_1$ less than 0.020408, five for $t_1$ less than 0.01234508, and so on, and at the limit for $t_1 = 0$ an infinity of solutions (Fig. 1.3).

It seems then that our hypothesis of the uniqueness of the solution is not valid, and that we are going to obtain considerable mass savings reaching 100%. However, a careful analysis shows that the numbers where the demultiplication of the roots occur are nothing but the fractions

$$\frac{1}{3^2}, \frac{1}{5^2}, \frac{1}{7^2}, \frac{1}{9^2}, \ldots\ldots, \frac{1}{(2k+1)^2}, \ldots\ldots$$

For $t_1 > 0.11111$, the non-constant part of the thickness is very close to the solution of the unconstrained case, which is the parabola we found before. As $t_1$ is decreased, $\chi$ increases and the nonconstant part gets more important. For $t_1 = 0.1111$ a second solution consisting of a constant thickness with this value arises. For this value of $t_1$, the constrained equation becomes:

$$\frac{\theta''}{9} + \omega^2\theta = 0$$

and a solution exists only when

$$3\omega = (2n+1)\frac{\pi}{2} \qquad n = 0, 1, 2, \ldots\ldots$$

or

$$\omega = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \ldots\ldots$$

so that in that case the lowest eigenvalue is no longer $\frac{\pi}{2}$ but $\frac{\pi}{6}$, corresponding to a divergence speed of one-third of the given one. As $t_1$ is decreased, the solutions belonging to this family will all correspond to this eigenvalue $\frac{\pi}{6}$. For $t_1 = 0.04 = \frac{1}{5^2}$ a new constant-thickness solution appears, and the new eigenvalues corresponding to this case are

$$\omega = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \ldots\ldots$$

The lowest one here is now $\frac{\pi}{10}$, corresponding to a divergence speed of one-fifth of the desired value.

The only solution of interest to us is thus the solution corresponding to the eigenvalue $\frac{\pi}{2}$, which is very similar to the unconstrained case. Some typical solutions are plotted (Fig. 1.4). In dotted lines in the figure are two of the spurious solutions that appeared when solving the equation for $X$.

The optimal mass is also plotted versus $t_1$ (Fig. 1.5). We still have to check that $\mu$ is positive on $[X, 1]$ in order to satisfy the necessary conditions.

It is found to be

$$\mu = 1 - \frac{\sin^2 \frac{\pi}{2\sqrt{t_1}}(1 - x)}{\sin^2 \frac{\pi}{2\sqrt{t_1}}(1 - X)}$$

As $(1 - X) < \sqrt{t_1}$ for every value of $t_1$ on $[0, 1]$,

$$0 < \frac{\pi}{2\sqrt{t_1}}(1 - X) \leq \frac{\pi}{2}$$

Moreover, on $[X, 1]$, $x > X$ so that

$$0 < \frac{\pi}{2\sqrt{t_1}}(1 - x) \leq \frac{\pi}{2\sqrt{t_1}}(1 - X) \leq \frac{\pi}{2}$$

and

$$\sin \frac{\pi}{2\sqrt{t_1}}(1 - x) \leq \sin \frac{\pi}{2\sqrt{t_1}}(1 - X)$$

Therefore,

$$\mu \geq 0$$

and the condition is satisfied.

As we can see, the best saving is obtained in the unconstrained case. However, the kind of thickness distribution encountered in the constrained case has much more practical interest; between a $t_1$ of 0.3 and the unconstrained case ($t_1 = 0$) the difference in mass saving is only 1.28%.

### 1.2.2 A Transition-Matrix Procedure

For a given $t_1$, the problem is exactly the same as before, except that t is computed from

$$t = \sqrt{\lambda_\theta s}$$

and set equal to $t_1$ whenever the above computed value is less than $t_1$. Another difficulty is that s(1) may now take negative values, breaking up the convergence process. This may be easily overcome by diminishing the $\epsilon$ of the process (say, to one half) and starting the algorithm again. Initial values for s and $\lambda_\theta$ were chosen equal to 0.3 and 6.0 respectively, and $\epsilon$ set equal to 0.5. With $t_1 = 0.5$, the actual distribution was obtained after 7 integrations with a relative error of 0.01% and a final $\epsilon$ of 0.03125 (Fig. 1.6). Actually, the third iteration is so close to the exact result that the two are indistinguishable on the plot.

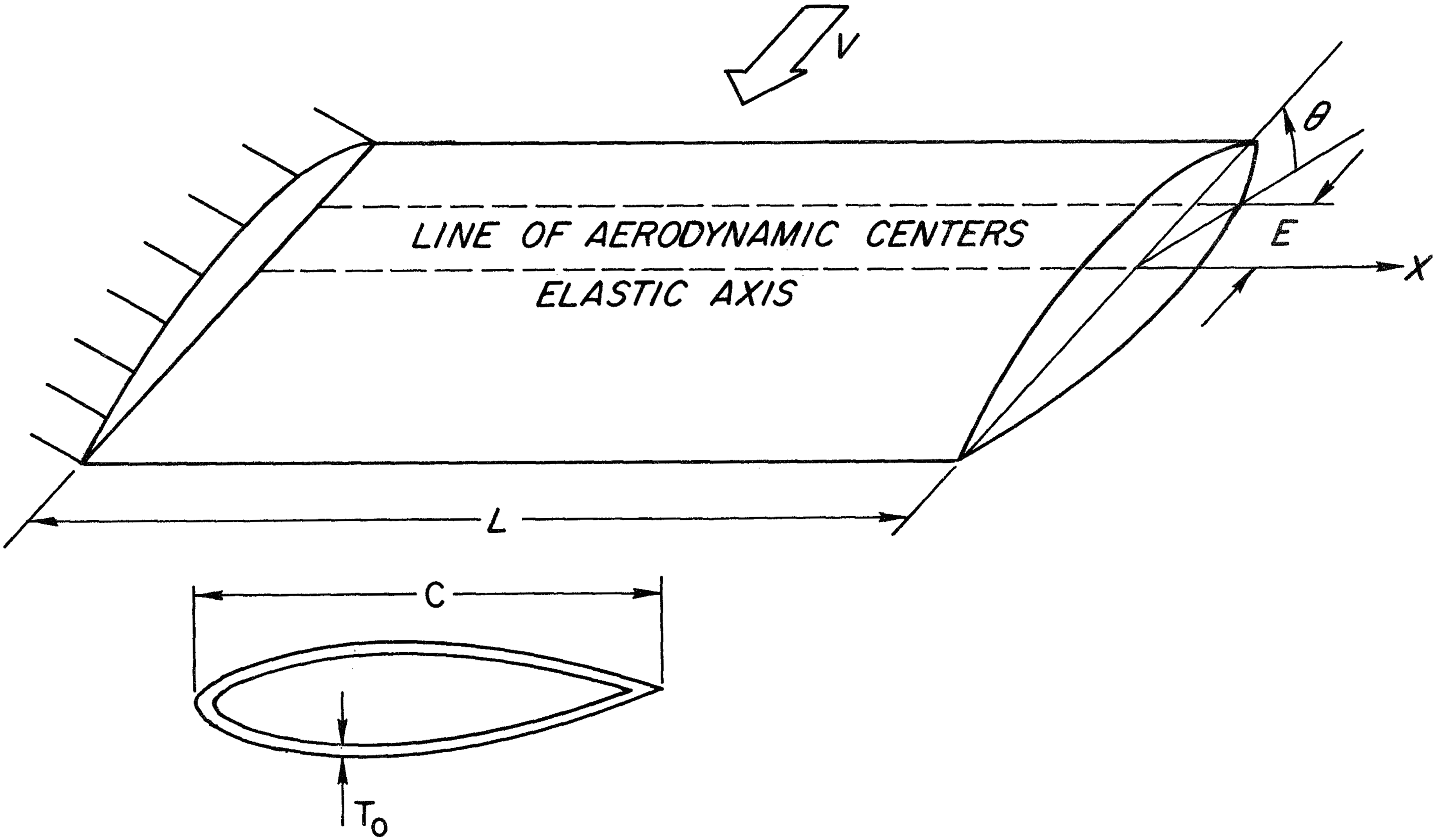


Fig. 1.1 Unswept cantilever wing with constant chord.

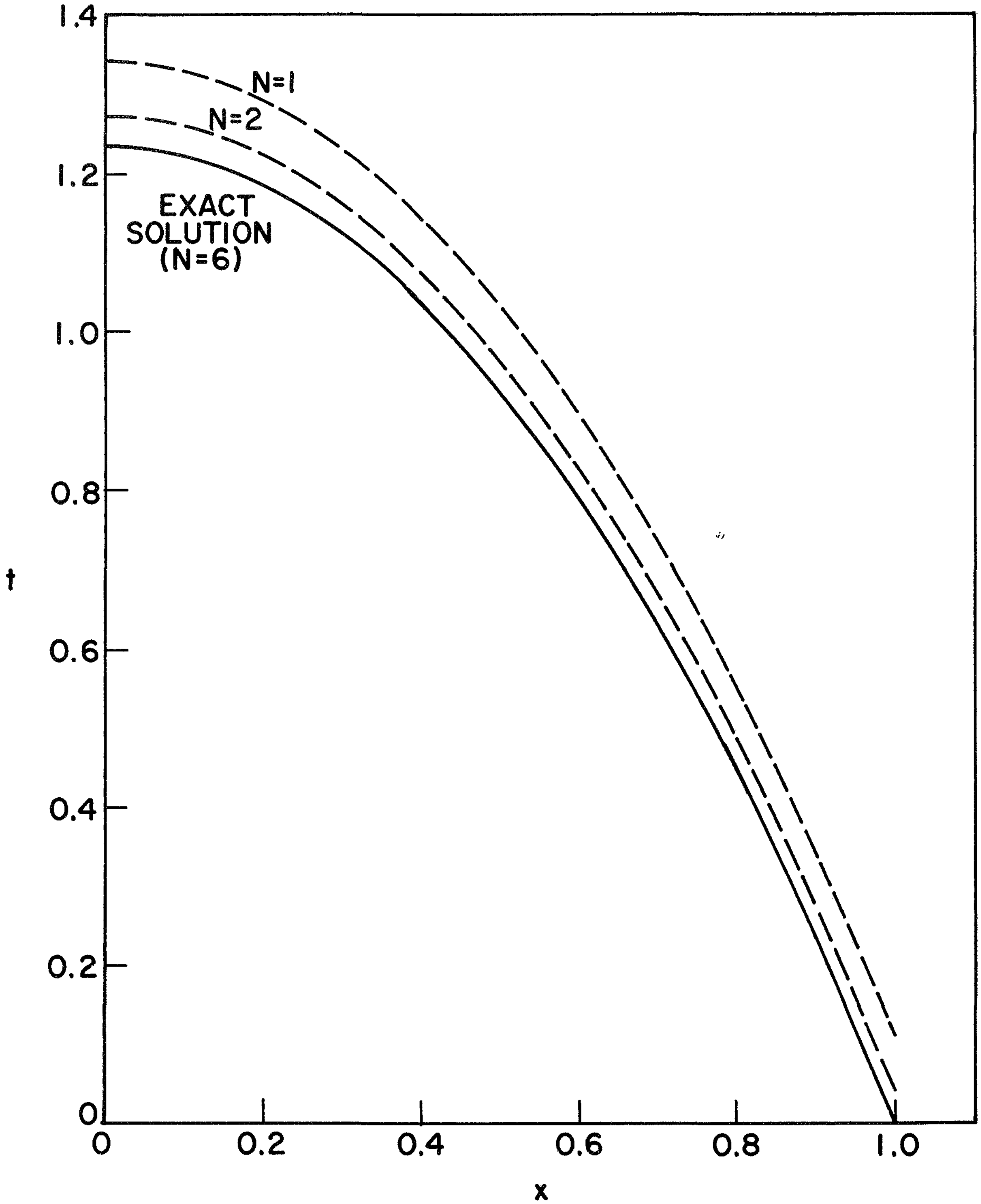


Fig. 1.2 A transition-matrix procedure —— torsional divergence with no minimal thickness constraint (N ≡ number of iterations).

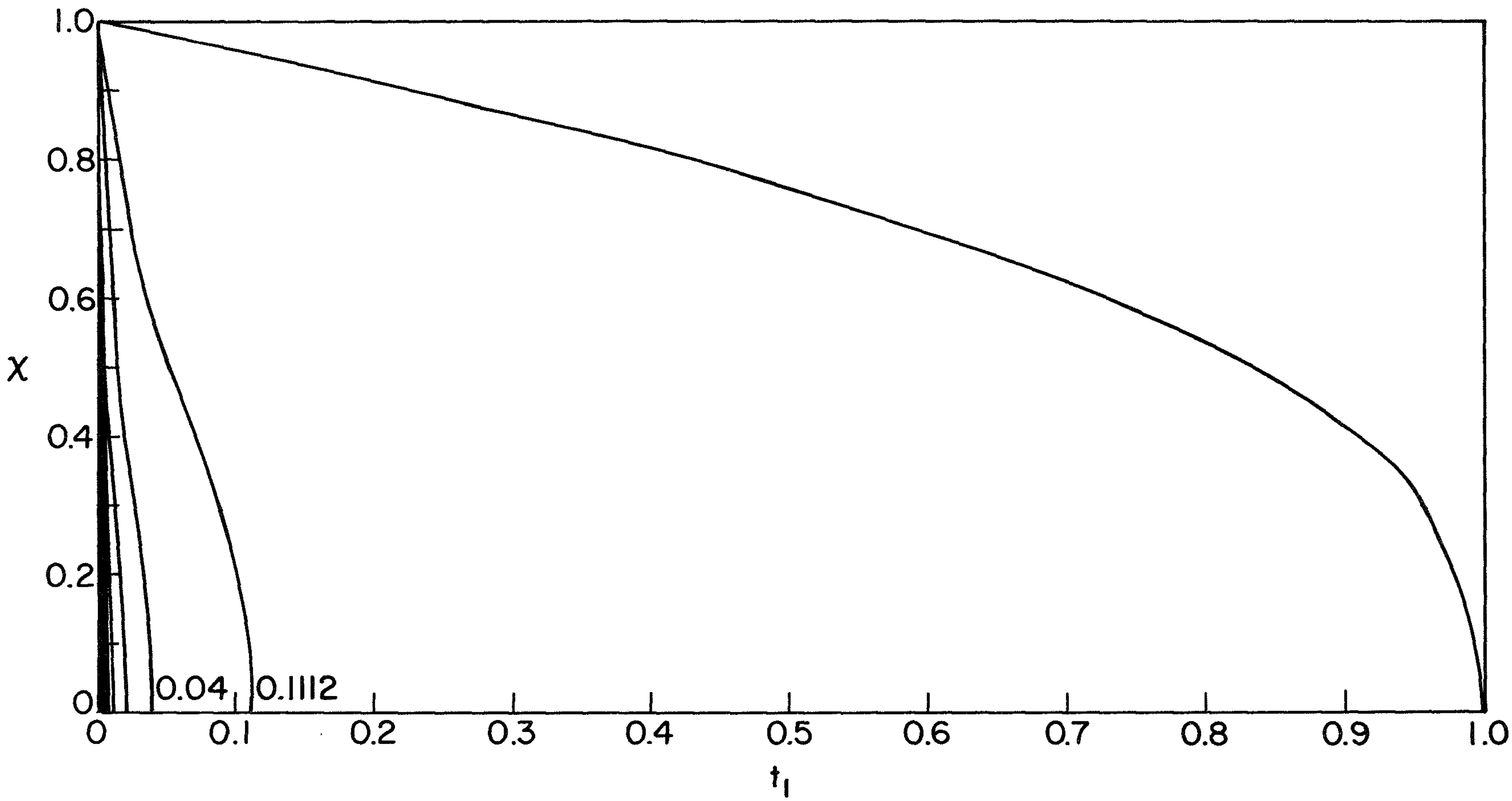


Fig. 1.3 Variation of $\chi$ with respect to the minimal thickness constraint $t_1$. $\chi$ is the root of $\chi = \frac{\sqrt{t_1}}{\omega} \cot \frac{\omega}{\sqrt{t_1}}(1 - \chi)$ lying between 0 and $1 (\omega = \frac{\pi}{2})$.

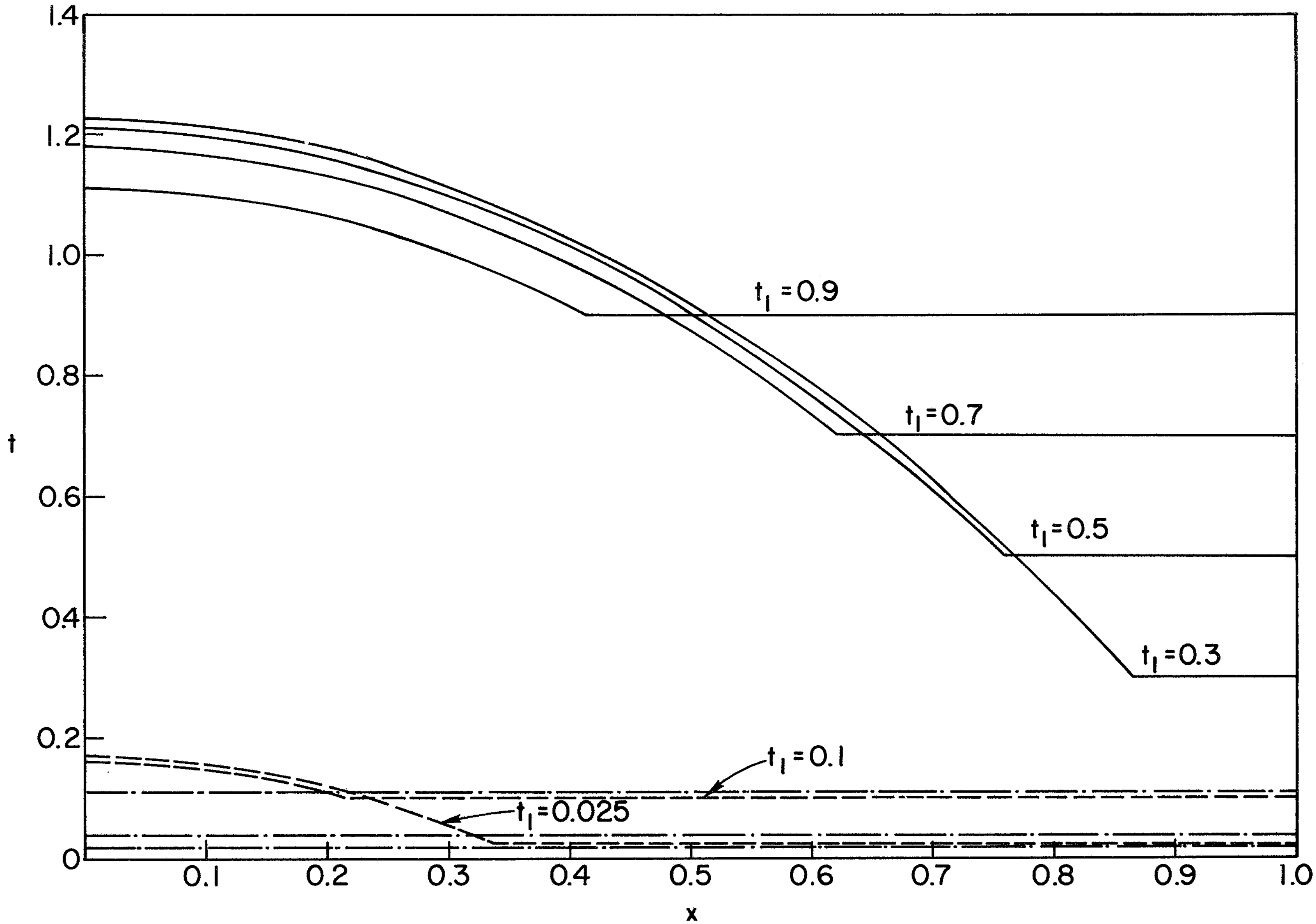


Fig. 1.4 Optimal thickness distributions. Torsional-divergence case with minimum-thickness constraint for different values of $t_1$ (the dotted lines are the distributions found for $t_1 < \frac{1}{3^2}$ and $t_1 < \frac{1}{5^2}$ respectively as described in Section 1.2.1).

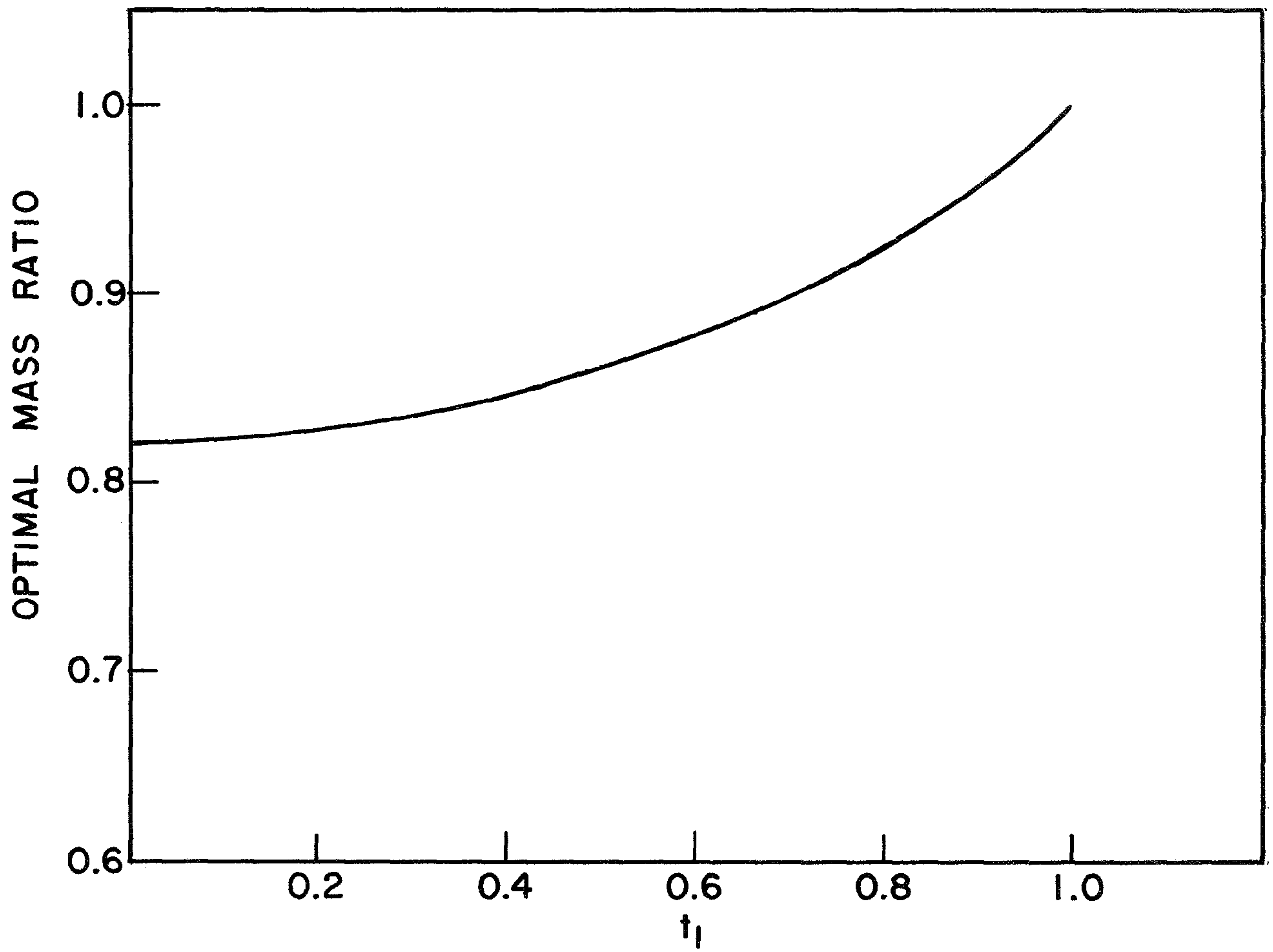


Fig. 1.5 Variation of the optimal mass ratio with the minimum thickness $t_1$ (torsional divergence case).

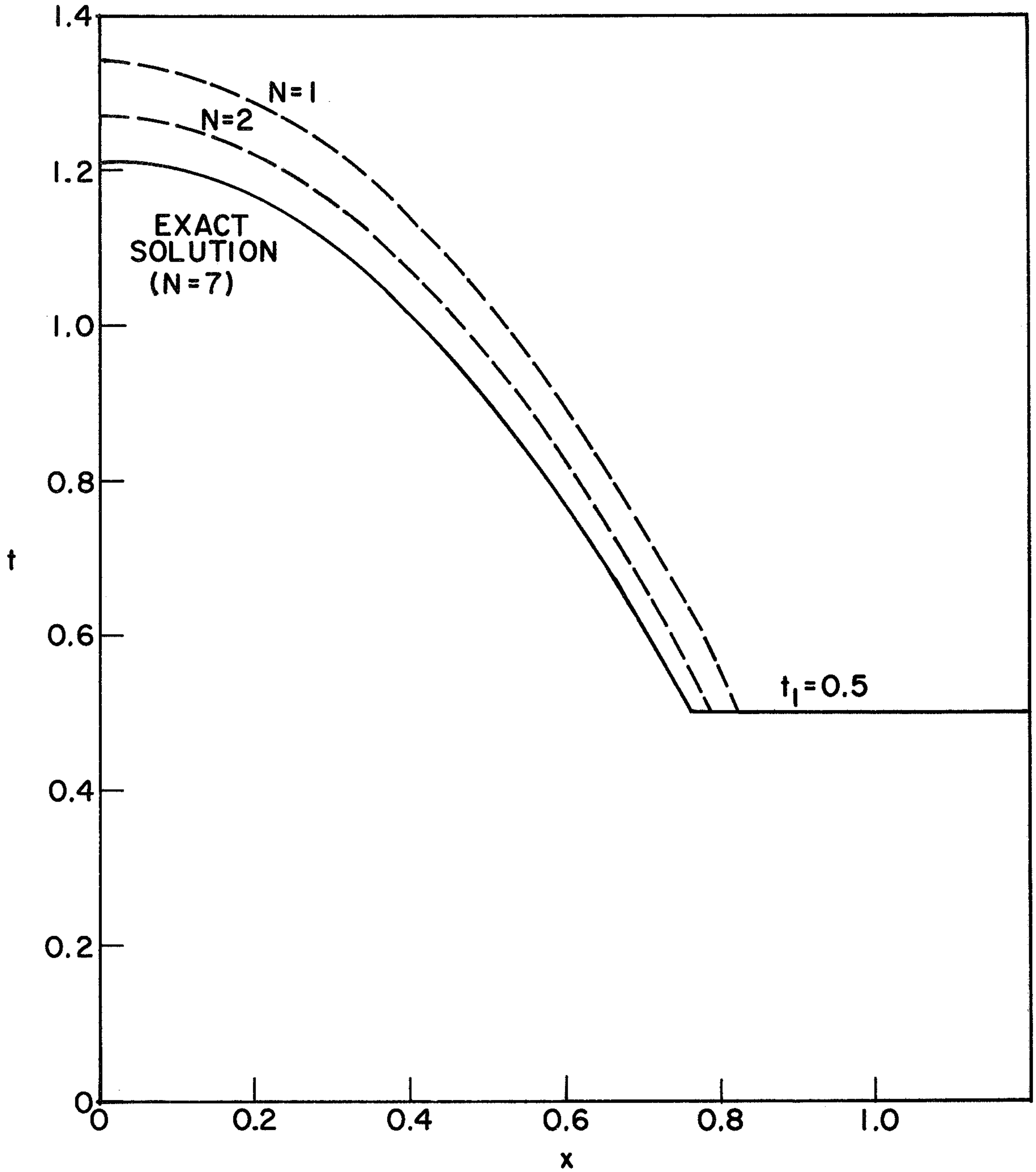


Fig. 1.6 A transition-matrix procedure for the case of torsional divergence with minimum thickness.

## 2. MINIMUM-WEIGHT CANTILEVER WING WITH A SPECIFIED TORSIONAL FREQUENCY

### 2.1 Statement of the Problem

The torsional vibrations are governed by the equation

$$\frac{d}{dX}\left(GJ\frac{d\theta}{dX}\right) + I_\alpha \omega_\alpha^2 \theta = 0$$

with the boundary conditions

$$\theta(0) = 0 \qquad GJ\frac{d\theta}{dX}\bigg|_{X = L} = 0$$

We assume that $I_\alpha(X)$ and $GJ(X)$ are determined primarily by the sectional skin thickness $T(X)$ and are in fact proportional to it. With the zero subscript denoting the properties of the reference wing, and with a dimensionless spanwise length $x = \frac{X}{L}$, the proportionality assumptions can be written as

$$I_\alpha(x) = I_{\alpha o} t(x)$$

$$GJ(x) = GJ_o t(x)$$

where

$$t(x) = \frac{T(x)}{T_o}$$

With primes denoting as usual differentiation with respect to $x$, the equation of the problem reduces to

$$(t\theta')' + \omega^2 t\theta = 0$$

$$\theta(0) = 0 \qquad t\theta'\bigg|_{x = 1} = 0$$

where

$$\omega^2 = \frac{\omega_\alpha^2 I_{\alpha_o}}{GJ_o} L^2$$

We introduce as usual the state variable

$$s = t\theta',$$

and the optimization problem reduces to the following:

Minimize the integral

$$M = \int_0^1 t(x)dx,$$

subject to the contraints

$$\theta' = \frac{s}{t}$$

$$s' = -\omega^2 t\theta$$

$$\theta(0) = 0, \qquad s(1) = 0$$

The Hamiltonian is

$$H = t + \lambda_\theta \frac{s}{t} - \omega^2 \lambda_s t\theta$$

The equations of the problem are

$$\theta' = \frac{s}{t}$$

$$s' = -\omega^2 t\theta$$

$$\lambda'_\theta = \omega^2 \lambda_s t$$

$$\lambda'_s = -\frac{\lambda_\theta}{t}$$

$$1 - \frac{\lambda_\theta s}{t^2} - \omega^2 \lambda_s \theta = 0$$

with the boundary conditions

$$\theta(0) = 0 \qquad \lambda_s(0) = 0$$

$$s(1) = 0 \qquad \lambda_\theta(1) = 0$$

Obviously, a solution is such that, $\alpha$ being an arbitrary constant,

$$\lambda_\theta = \frac{s}{\alpha}$$

$$\lambda_s = -\frac{\theta}{\alpha}$$

Also $t, \theta$ and $s$ are determined by the system

$$\theta' = \frac{s}{t}$$

$$s' = -\omega^2 t\theta$$

$$t = \frac{\epsilon s}{\sqrt{\alpha + \omega^2\theta^2}} \qquad (\epsilon = \pm 1)$$

$\theta$ is thus a solution of the differential equation

$$\theta' = \epsilon\sqrt{\omega^2\theta^2 + \alpha}$$

Integrating and introducing an integration constant $x_0$ gives

$$x - x_0 = \frac{\epsilon}{\omega}\log(\omega\theta + \sqrt{\omega^2\theta^2 + \alpha})$$

The boundary condition $\theta(0) = 0$ shows that $\alpha$ has to be positive and has the value

$$\alpha = e^{-2\epsilon\omega x_0}$$

$\theta$ is thus found to be

$$\theta = \frac{\epsilon}{\omega}e^{-\epsilon\omega x_0}\sinh(\omega x)$$

and $s$ and $t$ are then

$$s = \frac{K}{\cosh \omega x}$$

$$t = \epsilon K\frac{e^{\epsilon\omega x_0}}{\cosh^2 \omega x}$$

$K$ being another integration constant.

The boundary condition $s(1) = 0$ cannot be satisfied except for the trivial case $K = 0$, corresponding to a zero thickness distribution. This case appears to be similar to the case of the longitudinal vibrations of a bar investigated by Turner (Ref. 7), where the optimal thickness was found to vanish for a bar without a tip mass.

## 2.2 Minimum-Thickness Constraint

To overcome this difficulty, we introduce a minimum-thickness constraint by requiring $t$ to be always greater than or equal to a constant thickness $t_1$, $0 < t_1 < 1$.

Our augmented Hamiltonian is then

$$H = t + \lambda_\theta \frac{s}{t} - \omega^2 \lambda_s t\theta + \mu(t_1 - t)$$

where

$$\mu = 0 \quad \text{if} \quad t > t_1$$

$$\mu \geq 0 \quad \text{if} \quad t = t_1$$

The equations of the problem are

$$\theta' = \frac{s}{t}$$

$$s' = -\omega^2 t\theta$$

$$\lambda'_\theta = \omega^2 \lambda_s t$$

$$\lambda'_s = -\frac{\lambda_\theta}{t}$$

$$1 - \frac{\lambda_\theta s}{t^2} - \omega^2 \lambda_s \theta - \mu = 0$$

As noted before, there are very simple relations between $\theta, s$ and their adjoints, namely

$$\lambda_\theta = \frac{s}{\alpha}$$

$$\lambda_s = -\frac{\theta}{\alpha}$$

and $t, \theta, s$ are solutions of the system

$$\theta' = \frac{s}{t}$$

$$s' = -\omega^2 t\theta$$

$$1 - \frac{s^2}{\alpha t^2} + \frac{\omega^2\theta^2}{\alpha} - \mu = 0$$

with boundary conditions $\theta(0) = 0$, $s(1) = 0$. Along the portions of the $x$ interval for which $t$ will be larger than $t_1, \mu \equiv 0$ and the solution will be exactly the one we found in the unconstrained case. This solution cannot satisfy the $s(1) = 0$ requirement, so that it will be valid over a portion $[0, X]$ of the $x$ axis:

$$\theta = \frac{\epsilon}{\omega} e^{-\epsilon\omega x_o} \sinh(\omega x)$$

$$s = \frac{K}{\cosh(\omega x)}$$

$$t = \epsilon K \frac{e^{\epsilon\omega x_o}}{\cosh^2(\omega x)}$$

For $X \le x \le 1$, $t$ will be equal to $t_1$ and

$$\theta' = \frac{s}{t_1}$$

$$s' = -\omega^2 t_1 \theta$$

so that

$$\theta'' + \omega^2\theta = 0$$

and

$$\theta = A \cos \omega x + B \sin \omega x$$

$$s = \omega t_1 (B \cos \omega x - A \sin \omega x)$$

from the boundary condition $s(1) = 0$

$$B = A \tan \omega$$

and

$$\theta = \frac{A}{\cos \omega} \cos \omega(1 - x)$$

so that

$$s = \frac{\omega A t_1}{\cos \omega} \sin \omega(1 - x)$$

Continuity of $\theta$ and $\theta'$ at $x = X$ results in the two conditions

$$\frac{\epsilon}{\omega} e^{-\epsilon \omega x_o} \sinh(\omega X) = \frac{A}{\cos \omega} \cos \omega(1 - X)$$

$$\epsilon e^{-\epsilon \omega x_o} \cosh(\omega X) = \frac{A\omega}{\cos \omega} \sin \omega(1 - X)$$

so that $X$ is found to satisfy the transcendental equation

$$\tanh \omega X = \cot \omega(1 - X)$$

For the reference bar of uniform thickness, $\omega = \frac{\pi}{2}$ and the above equation reads

$$\tanh \frac{\pi}{2} X = \tan \frac{\pi}{2} X$$

This equation has only one solution on $[0, 1]$,

$$X = 0$$

which is of no interest to us as this corresponds to the bar of uniform thickness distribution.

In this case, the minimum-thickness constraint does not lead to a solution, and in order to solve the problem we have to introduce further hypotheses on the structure.

## 2.3 Structural Mass Hypothesis - No Minimum Thickness Constraint

One way to overcome the foregoing difficulty is, following Turner (Ref. 7), to make the assumption that the mass is made up of two parts: a constant fraction $\delta_2$ being nonstructural, the remaining part being allowed to vary. The thickness may be expressed as

$$t(x) = \delta_1 t^*(x) + \delta_2 \tag{2.3.1}$$

Under this hypothesis, the running GJ and $I_\alpha$ are expressed in function of the corresponding referred quantities as

$$GJ(x) = GJ_o t^*(x)$$

$$I_\alpha(x) = I_{\alpha_o}(\delta_1 t^*(x) + \delta_2)$$

$\delta_1$ and $\delta_2$ being two positive constants satisfying

$$\delta_1 + \delta_2 = 1$$

The inertial radii of gyration of the two portions of the mass distribution are seen to be assumed equal. The equation of the problem is now

$$(t^*\theta') + \omega^2(\delta_1 t^* + \delta_2)\theta = 0 \tag{2.3.2}$$

and we may set up the variational problem for our intermediate variable $t^*$.

As before, after we introduce the variable $s = t^*\theta'$, the problem reduces to the system of equations

$$\theta' = \frac{s}{t^*}$$

$$s' = -\omega^2\theta(\delta_1 t^* + \delta_2)$$

$$\lambda'_\theta = \omega^2\lambda_s(\delta_1 t^* + \delta_2)$$

$$\lambda'_s = -\frac{\lambda_\theta}{t^*}$$

$$1 - \frac{\lambda_\theta s}{t^{*2}} - \omega^2\delta_1\theta\lambda_s = 0$$

with the boundary conditions

$\theta(0) = 0 \qquad \lambda_s(0) = 0$

$s(1) = 0 \qquad \lambda_\theta(1) = 0$

2.3.1 Analytical Solution

A solution is again such that

$$\lambda_\theta = \frac{s}{\alpha}$$

$$\lambda_s = -\frac{\theta}{\alpha}$$

$\alpha$ being an arbitrary constant, and the problem is simplified into

$$\theta' = \frac{s}{t^*}$$

$$s' = -\omega^2 \theta(\delta_1 t^* + \delta_2)$$

$$t^* = \frac{\epsilon s}{\sqrt{\alpha + \omega^2 \delta_1 \theta^2}} \qquad (\epsilon = \pm 1).$$

$\theta(0) = 0 \qquad s(1) = 0$

The twist amplitude $\theta$ satisfies the differential equation

$$\theta' = \epsilon \sqrt{\alpha + \omega^2 \delta_1 \theta^2}$$

Therefore, upon introducing an integration constant $x_o$,

$$e^{\epsilon\omega\sqrt{\delta_1}(x - x_o)} = \omega\sqrt{\delta_1}\theta + \sqrt{\alpha + \omega^2 \delta_1 \theta^2}$$

The condition $\theta(0) = 0$ is expressed as

$$\alpha = e^{-2\epsilon\omega\sqrt{\delta_1}x_o}$$

so that

$$\theta = \frac{\epsilon e^{-\epsilon\omega\sqrt{\delta_1}x_o}}{\omega\sqrt{\delta_1}} \sinh(\omega\sqrt{\delta_1}x)$$

Now $s$ satisfies the linear first-order differential equation

$$s' + \omega\sqrt{\delta_1}\frac{\sinh(\omega\sqrt{\delta_1})}{\cosh(\omega\sqrt{\delta_1}x)}s + \epsilon\frac{\omega\delta_2}{\sqrt{\delta_1}}e^{-\epsilon\omega\sqrt{\delta_1}x_o}\sinh(\omega\sqrt{\delta_1}x) = 0$$

The general solution is

$$s = \epsilon\frac{\delta_2}{2\delta_1}e^{-\epsilon\omega\sqrt{\delta_1}x_o}\left(\frac{C}{\cosh(\omega\sqrt{\delta_1}x)} - \cosh(\omega\sqrt{\delta_1}x)\right)$$

$C$ being determined by the condition $s(1) = 0$. Therefore,

$$s = \epsilon\frac{\delta_2}{2\delta_1}e^{-\epsilon\omega\sqrt{\delta_1}x_o}\left(\frac{\cosh^2(\omega\sqrt{\delta_1})}{\cosh(\omega\sqrt{\delta_1}x)} - \cosh(\omega\sqrt{\delta_1}x)\right)$$

and the optimal thickness distribution is given by

$$t^* = \frac{\delta_2}{2\delta_1}\left[\left(\frac{\cosh(\omega\sqrt{\delta_1})}{\cosh(\omega\sqrt{\delta_1}x)}\right)^2 - 1\right]$$

The actual thickness distribution turns out to be

$$t = \frac{\delta_2}{2}\left[\left(\frac{\cosh(\omega\sqrt{\delta_1})}{\cosh(\omega\sqrt{\delta_1}x)}\right)^2 + 1\right]$$

The corresponding non-dimensional mass is found as

$$M = \frac{\delta_2}{2}\left(\frac{\sinh(2\omega\sqrt{\delta_1})}{2\omega\sqrt{\delta_1}} + 1\right)$$

For a uniform bar of thickness $t = 1$, we have $\omega = \frac{\pi}{2}$.

When $\delta_1$ increases from 0 to 1, this mass decreases from 1 to 0. The relation between $M$ and $\delta_1$ is plotted in Fig. 2.1.

At the limit $\delta_1 = 1$, where all the mass is structural and allowed to vary, we have the case already investigated in Section 2.1.

For the case $\delta_1 = 0.25$, $\delta_2 = 0.75$ where 75% of the mass is labeled "non-structural" and remains fixed, the optimal thickness distribution is as represented on Fig. 2.2 by a solid line. The other curve (dashed line) is the $t^*$ distribution corresponding to this case. The mass ratio is then equal to 0.9244 corresponding to an optimal saving of 7.56%. This saving, not considerable here, increases with $\delta_1$. An identical case with different values of $\delta_1$ and $\delta_2$ will be treated in Section 3, as we will see later.

2.3.2 A Transition-Matrix Procedure

Numerically, we will look for $t^*$ solution of the system (2.3.3). This procedure is very similar to the one already used. We begin by numerically integrating the system of differential equations

$$\theta' = \frac{s}{t^*}$$

$$s' = -\omega^2\theta(\delta_1 t^* + \delta_2)$$

$$\lambda'_\theta = \omega^2\lambda_s(\delta_1 t^* + \delta_2)$$

$$\lambda'_s = -\frac{\lambda_\theta}{t^*}$$

where

$$t^* = \sqrt{\frac{\lambda_\theta s}{1 - \omega^2\delta_1\theta\lambda_s}}$$

on $[0,1]$, with the initial boundary conditions

Two given $\quad \theta(0) = \lambda_s(0) = 0$

Two guessed $\quad s(0) = 0.5, \ \lambda_\theta(0) = 5$

$\omega$ is equal to $\frac{\pi}{2}$, and $\delta_1$ and $\delta_2$ were taken equal to 0.25 and 0.75, respectively.

The first column of the 2 x 2 transition matrix is given by the values at $x = 1$ of $\delta s$ and $\delta\lambda_\theta$. These are in turn solutions to the system of differential equations

$$(\delta\theta)' = \frac{\delta s}{t^*} - s\frac{\delta t^*}{t^{*2}}$$

$$(\delta s)' = -\omega^2\delta\theta(\delta_1 t^* + \delta_2) - \omega^2\delta_1\theta\delta t^*$$

$$(\delta\lambda_\theta)' = \omega^2\delta\lambda_s(\delta_1 t^* + \delta_2) + \omega^2\delta_1\lambda_s\delta t^*$$

$$(\delta\lambda_s)' = -\frac{\delta\lambda_\theta}{t^*} + \lambda_\theta \frac{\delta t}{t^{*2}}$$

where

$$\delta t^* = \frac{t^*(\omega^2 \delta_1 t^{*2} + 1)}{2\lambda_\theta s}(\lambda_\theta \delta s + s\delta\lambda_\theta)$$

satisfying the initial conditions

$$\delta s(0) = 1 \qquad \delta\lambda_\theta(0) = 0 \qquad \delta_\theta(0) = 0 \qquad \delta\lambda_s(0) = 0$$

Similarly, the second column is made of the values at 1 of $\delta s$ and $\delta\lambda_\theta$ satisfying the same system but with the initial conditions

$$\delta s(0) = 0 \qquad \delta\lambda_\theta(0) = 1 \qquad \delta\theta(0) = 0 \qquad \delta\lambda_s(0) = 0$$

As previously, the integration stops whenever $t^*$ is found to be equal to zero, so that we have to force convergence from above. The initial $\epsilon$, taken equal to 1, was decreased by half every time the integration was stopped, and the analytical solution was found with a precision of 0.4% after eight iterations. A precision of 0.004% was obtained after 13 iterations, with a final $\epsilon$ of 0.0625 (Fig. 3.3).

## 2.4 Structural Mass With a Minimum-Thickness Constraint

We require $t^*$ to be always greater than or equal to a constant thickness $t_1^*$, such that $0 < t_1^* < 1$. This case has some practical interest as the GJ at the extremity $x = 1$ was found equal to zero in the unconstrained case, due to the vanishing of the structural mass there. The augmented Hamiltonian reads

$$H = t^* + \lambda_\theta \frac{s}{t^*} - \omega^2 \lambda_s(\delta_1 t^* + \delta_2)\theta + \mu(t_1^* - t^*)$$

where

$$\mu = 0 \ \text{ if } \ t^* > t_1^*$$

$$\mu \geq 0 \ \text{ if } \ t^* = t_1^*$$

The equations of the problem are

$$\theta' = \frac{s}{t^*}$$

$$s' = -\omega^2(\delta_1 t^* + \delta_2)\theta$$

$$\lambda'_\theta = \omega^2(\delta_1 t^* + \delta_2)\lambda_s$$

$$\lambda'_s = -\frac{\lambda_\theta}{t^*}$$

$$1 - \frac{\lambda_\theta s}{t^{*2}} - \omega^2 \delta_1 \lambda_s \theta - \mu = 0$$

with the boundary conditions

$$\theta(0) = 0 \qquad \lambda_s(0) = 0$$

$$s(1) = 0 \qquad \lambda_\theta(1) = 0$$

2.4.1 Analytical Solution

A solution is seen, as previously, to be such that,

$$\lambda_\theta = \frac{s}{\alpha}, \qquad \lambda_s = -\frac{\theta}{\alpha}$$

$\alpha$ being a constant. The problem is rewritten as

$$\theta' = \frac{s}{t^*}$$

$$s' = -\omega^2(\delta_1 t^* + \delta_2)\theta$$

$$t^* = \frac{\epsilon s}{\sqrt{\alpha(1-\mu) + \omega^2 \delta_1 \theta^2}}$$

$$\theta(0) = 0$$

$$s(1) = 0$$

As in the previous cases encountered, we will start at $x = 0$ with a thickness greater than $t_1$, so that until $x$ reaches the value $\chi$

$$\mu = 0$$

$$\theta = \frac{\epsilon e^{-\epsilon\omega\sqrt{\delta_1} x_o}}{\omega\sqrt{\delta_1}} \sinh(\omega\sqrt{\delta_1} x)$$

$$s = \epsilon \frac{\delta_2}{2\delta_1} e^{-\epsilon\omega\sqrt{\delta_1} x_o} \left( \frac{C}{\cosh(\omega\sqrt{\delta_1} x)} - \cosh(\omega\sqrt{\delta_1} x) \right)$$

$$t^* = \frac{\delta_2}{2\delta_1} \left( \frac{C}{\cosh^2(\omega\sqrt{\delta_1} x)} - 1 \right)$$

The thickness $t^*$ decreases and reaches the value $t_1^*$ for $x = X$. Then it remains constant and equal to $t^*_1$ on $[X, 1]$, so that

$$\theta = \frac{A}{\omega t_1^* \sqrt{\delta_1 + \frac{\delta_2}{t_1^*}} \sin(\omega \sqrt{\delta_1 + \frac{\delta_2}{t_1^*}})} \cos \left[ \omega \sqrt{\delta_1 + \frac{\delta_2}{t_1^*}} (1 - x) \right]$$

$$s = \frac{A}{\sin \omega \sqrt{\delta_1 + \frac{\delta_2}{t_1^*}}} \sin \left[ \omega \sqrt{\delta_1 + \frac{\delta_2}{t_1^*}} (1 - x) \right]$$

$$t^* = t_1^*$$

Continuity of $\theta$ and $\theta'$ at $x = X$ requires that $X$ be a solution of the transcendental equation

$$\sqrt{1 + \frac{\delta_2}{\delta_1 t_1^*}} \tanh(\omega\sqrt{\delta_1} X) = \cot \left[ \omega \sqrt{\delta_1 + \frac{\delta_2}{t_1^*}} (1 - X) \right]$$

(Note that in the limit $\delta_1 = 0$ we get the same equation as in Section 1.2, and for $\delta_2 = 0$ the same as in Section 2.2.)

When $X$ is found from this equation, the $t^*$ distribution is given by

$0 \leq x \leq X$:

$$t^* = \frac{\delta_2}{2\delta_1} \left[ \frac{\cosh^2 \omega\sqrt{\delta_1} X}{\cosh^2 \omega\sqrt{\delta_1} x} (1 + 2\frac{\delta_1}{\delta_2} t_1^*) - 1 \right]$$

$X \leq x \leq 1$:

$$t^* = t_1^*$$

The optimal thickness distribution is as follows:

$0 \le x \le \chi$

$$t = \frac{\delta_2}{2}\left[\frac{\cosh^2 \omega\sqrt{\delta_1}\,\chi}{\cosh^2 \omega\sqrt{\delta_1}\,x}\left(1 + 2\frac{\delta_1}{\delta_2}t_1^*\right) + 1\right]$$

$\chi \le x \le 1$:

$$t = \delta_1 t_1^* + \delta_2$$

The optimal mass ratio is found to be

$$M = \frac{\delta_2}{2}\left(\frac{\sinh 2\omega\sqrt{\delta_1}\,\chi}{2\omega\sqrt{\delta_1}} + 2 - \chi\right) + \delta_1 t_1^*\left(\frac{\sinh 2\omega\sqrt{\delta_1}\,\chi}{2\omega\sqrt{\delta_1}} + 1 - \chi\right)$$

For fixed values of $\delta_1$ and $\delta_2$ chosen as 0.25 and 0.75, respectively, the thickness distribution corresponding to some typical values of $t_1$ is plotted in Fig. 2.4.

The variation of the optimal mass ratio with minimal thickness $t$, is represented in Fig. 2.5; as in the case of Chapter 1, the mass saving is maximum for $t_1 = 0$, corresponding to the unconstrained case.

2.4.2 <u>A Transition-Matrix Procedure</u>

This is exactly the same procedure as described in Section 1.2. With initial values of $s$ and $\lambda_\theta$ of 0.3 and 6 respectively, for the case $\delta_1 = 0.25$, $\delta_2 = 0.75$ and a chosen minimal thickness of 0.5, the $t^*$ distribution was obtained after 9 iterations with a relative error (in excess) of 0.01%. The starting $\epsilon$ was taken equal to 0.5, and the final $\epsilon$ was 0.03125.

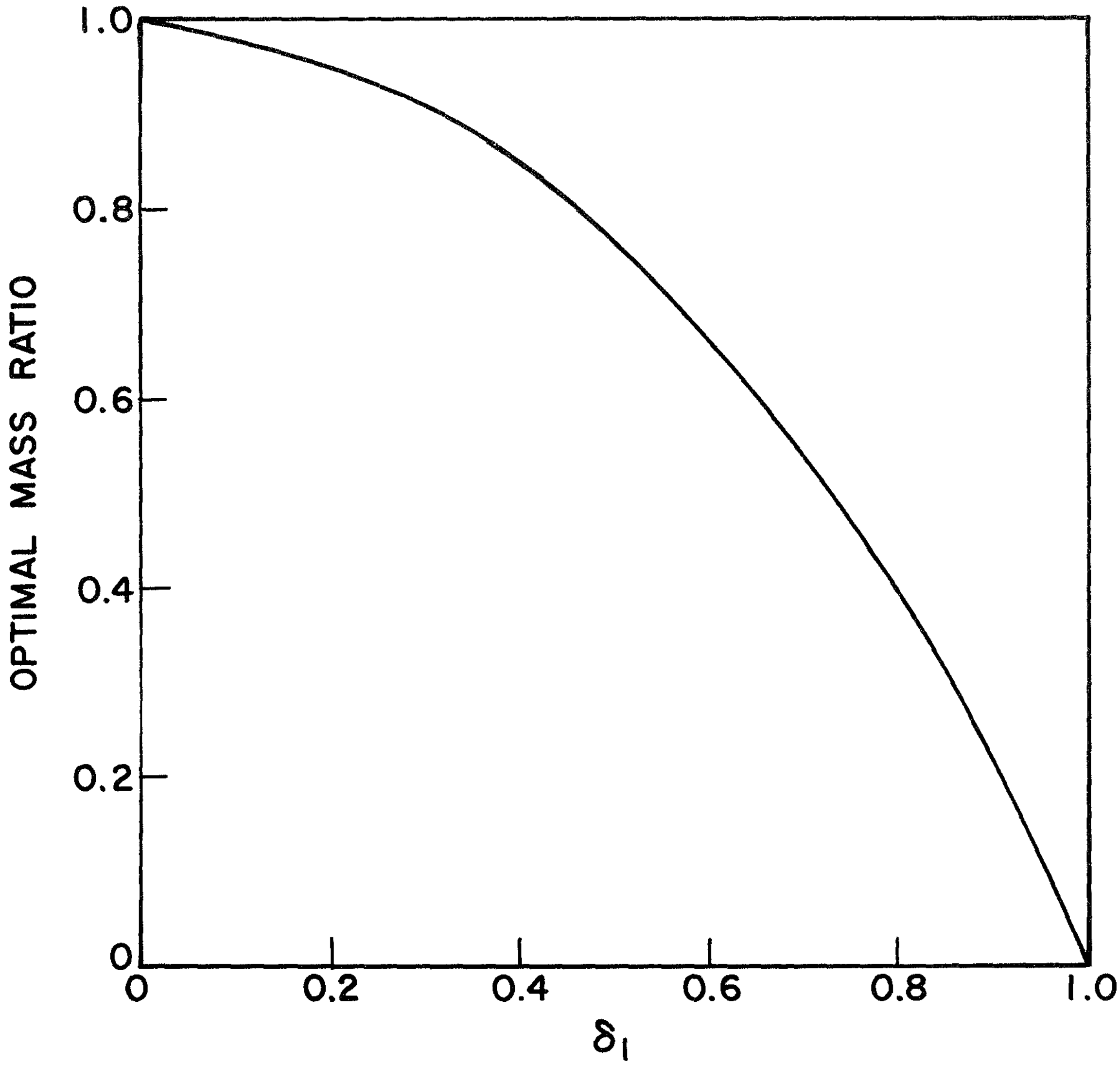


Fig. 2.1 Variation of the mass ratio with $\delta_1$, fraction of total mass allowed to vary —— fixed torsional frequency case.

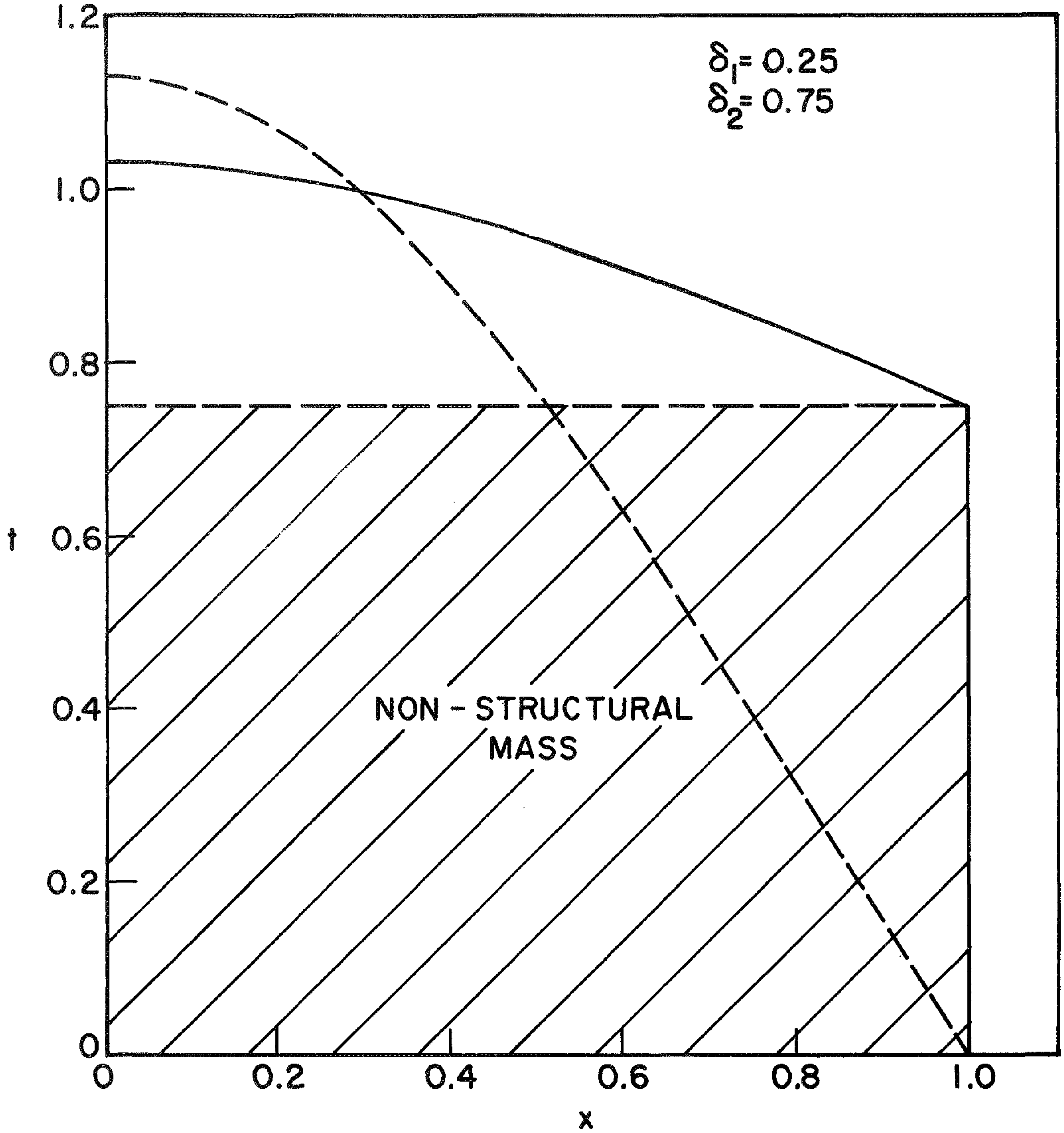


Fig. 2.2 Optimal thickness distribution — fixed torsional frequency.

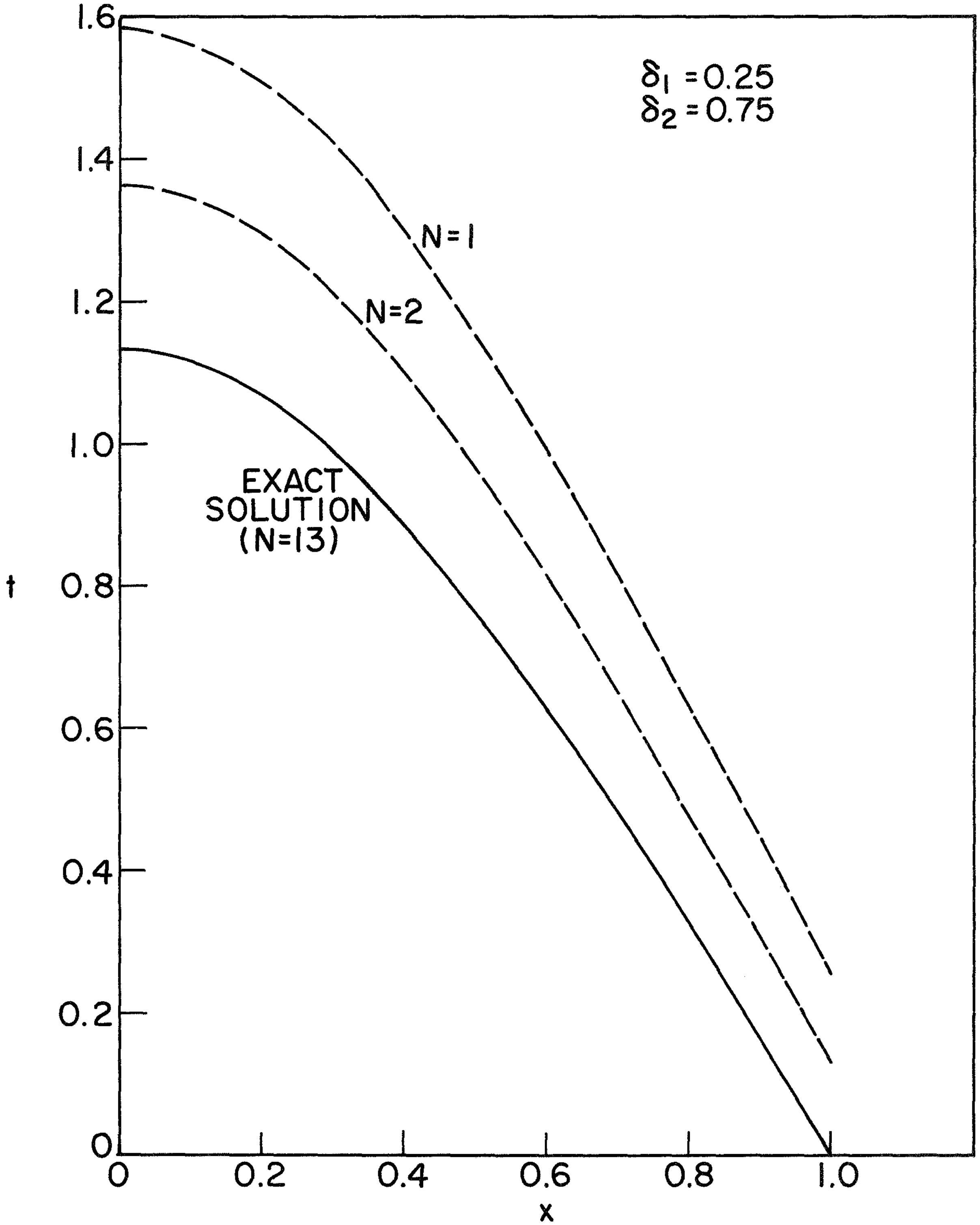


Fig. 2.3 A transition-matrix procedure —— fixed torsional frequency.

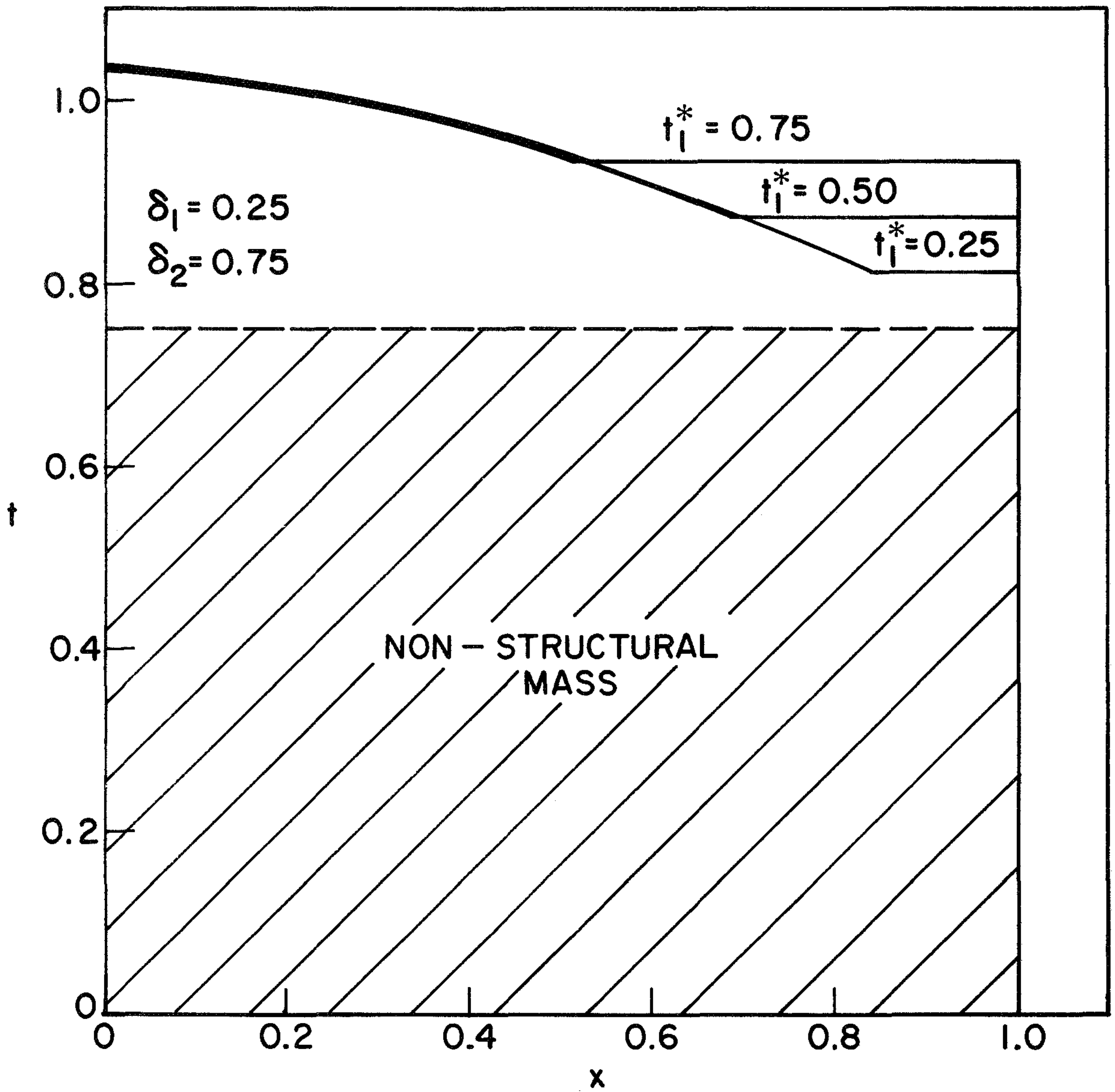


Fig. 2.4 Optimization for a fixed torsional frequency with minimum-thickness constraint on $t^*$.

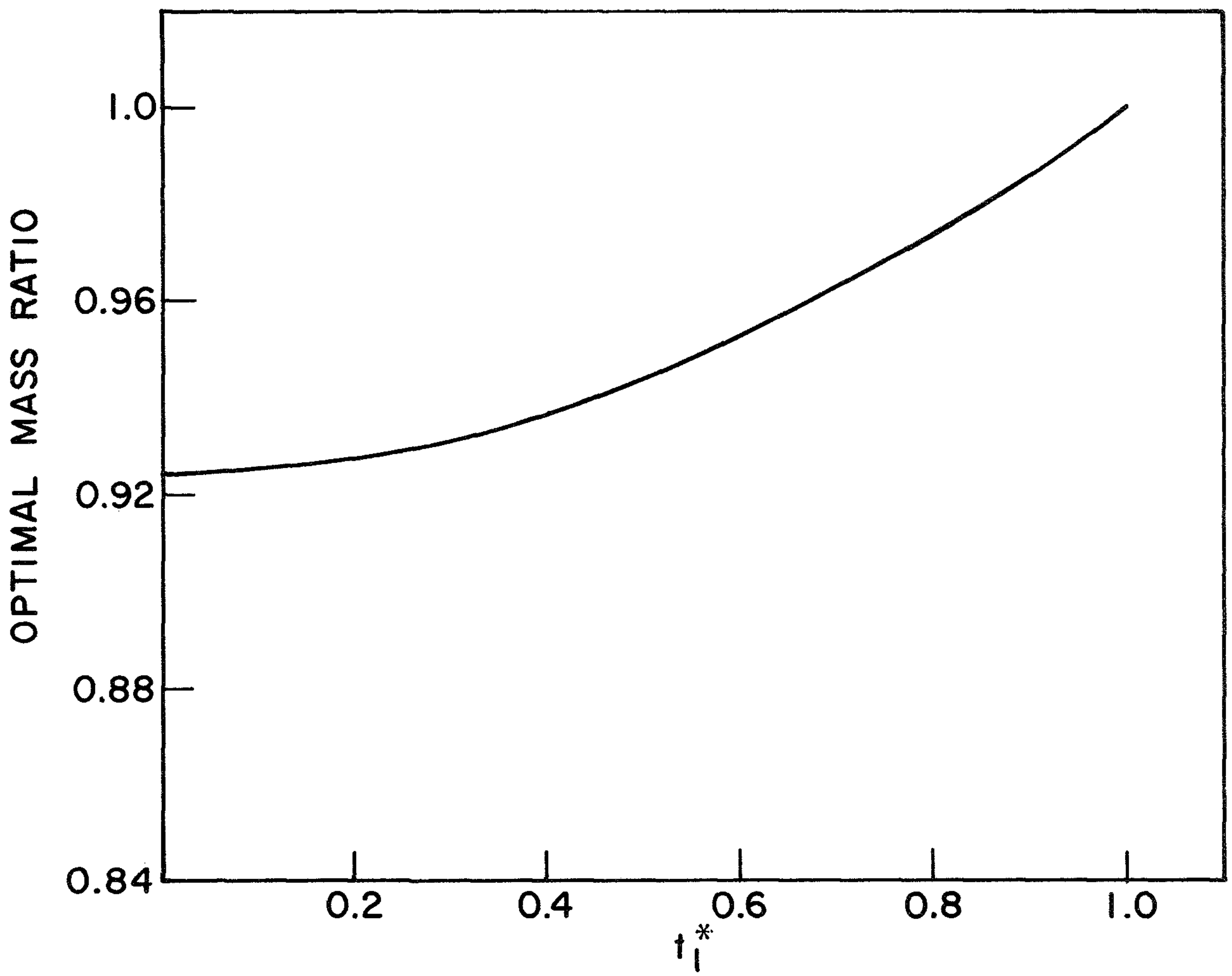


Fig. 2.5 Mass distribution vs. minimum thickness $t_1^*$.

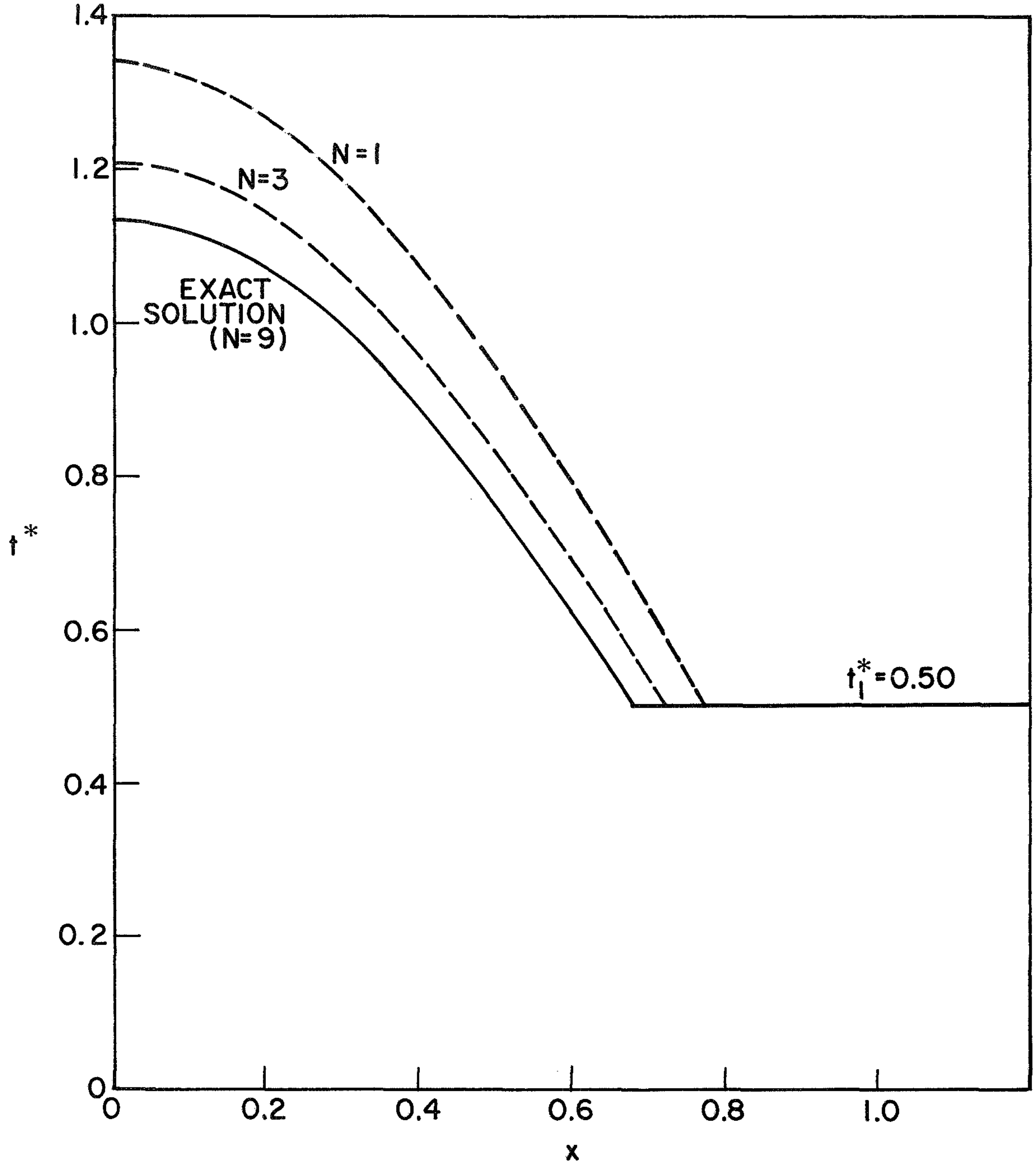


Fig. 2.6 A transition-matrix procedure for fixed torsional frequency with minimum-thickness constraint.

## 3. OPTIMIZATION OF A PLATE FOR FIXED CONDITION OF CHORDWISE DIVERGENCE

### 3.1 Statement of the Optimization Problem

The different quantities are defined in Fig. 3.1, approximating the cross section of the forward half of an airfoil undergoing chordwise bending in a two-dimensional supersonic airstream.

As shown in Ref. 6, Section 7-4(a), the area moment of inertia per unit span for solid sections is

$$I = \frac{2}{3} Z_t^3$$

For a thin-face-sheet sandwich with no contribution of the core to bending stiffness, the corresponding formula would be

$$I = 2 Z_t^2 T,$$

where T is the thickness of one face sheet. The governing differential equation for the bending slope $\alpha_e \equiv \frac{dW}{\partial X}$ reads

$$\frac{d^2}{dX^2}[E_1 I \frac{d\alpha_e}{dX}] = \Delta p_a(X) = \frac{4q}{\sqrt{M^2 - 1}} \left[ 1 + \frac{\gamma + 1}{2} M \frac{dZ_t}{dX} \right] (-\alpha_e + \alpha_o),$$

$\alpha_o$ being the mean incidence at midchord, and the aerodynamic loading $\Delta p_a(X)$ being approximated by piston theory.

In the first case above, the complete dimensionless problem reads, introducing the dimensionless length $x = \frac{X}{B_1}$,

$$\frac{d^2}{dx^2} \left[ \frac{Z_t^3}{Z_{tR}^3} \frac{d\alpha_e}{dx} \right] + k_1 \alpha_e = - k_1 \alpha_o$$

where

$$k_1 = \frac{6q}{E_1 \sqrt{M^2 - 1}} \left( \frac{B_1}{Z_{t_R}} \right)^3 \left[ 1 + \frac{\gamma + 1}{2} M \frac{dZ_t}{dx} \right]$$

Here $E_1 = \frac{E}{1 - \nu^2}$ is the effective Young's modulus, q the dynamic pressure, and M the Mach number. The quantity $k_1$ is actually a constant when the plate is a uniformly tapered wedge so that

$$\frac{Z_t}{Z_{t_R}} = x$$

In this case, the differential equation reads

$$\frac{d^2}{dx^2}\left[x^3 \frac{d\alpha_e}{dx}\right] + k_1\alpha_e = - k_1\alpha_o$$

Boundary conditions for the free edge at $x = a_1$ and clamped midchord line at $x = 1$ read

$$\alpha_e(1) = x^3 \frac{d\alpha_e}{dx}\bigg|_{x = a_1} = \frac{d}{dx}\left[x^3 \frac{d\alpha_e}{dx}\right]\bigg|_{x = a_1} = 0$$

The last differential equation above is equidimensional, and a complete solution (due to Biot) is given in the Ref. 6. These results are very complicated and cannot form the basis of a simple optimization problem (results could be obtained by numerical integration). Since the objective is to find situations with a probability of exact analytic solutions, we shall here seek out the simplest possible case which might be regarded as meaningful.

The case we will investigate here is that of a uniform-depth honeycomb with face-sheet thickness varied to seek minimum mass at fixed $q_D$, $M_D$. Since then $Z_t = Z_{t_R}$, we can set $a_1 = 0$, neglect any aerodynamic effects of the finite leading-edge bluntness, and write the dimensionless problem as

$$\frac{d^2}{dx^2}\left[t_1 \frac{d\alpha_e}{dx}\right] + k_2\alpha_e = - k_2\alpha_o$$

where

$$k_2 = \frac{2q}{D\sqrt{M^2 - 1}}\left(\frac{B_1}{Z_{t_R}}\right)^2$$

$$t_1 = \frac{T}{B_1} \equiv \frac{T}{B}$$

The reference case, with uniform thickness $t_o$, is a characteristic-value problem expressed by the equation

$$\frac{d^3\alpha_e}{dx^3} + k\alpha_e = 0$$

together with the boundary conditions

$$\alpha_e(1) = \alpha_e'(0) = \alpha_e''(0) = 0$$

with

$$k = \frac{2q}{E_1\sqrt{M^2 - 1}}\left(\frac{B_1^3}{T_o Z_{t_R}^2}\right)$$

This is solved, by analogy with the swept-forward-wing bending divergence problem, by

$$\alpha_e = A_1 e^{r_1 x} + A_2 e^{r_2 x} + A_3 e^{r_3 x}$$

where $r_1, r_2, r_3$ are the roots of the algebraic equation

$$r^3 + k = 0$$

Hence

$$r_1 = \sqrt[3]{-k}, \; r_2, r_3 = \tfrac{1}{2}[-1 \pm i\sqrt{3}]\sqrt[3]{-k}$$

The transcendental equation resulting from the three boundary conditions reads

$$1 + \frac{r_1}{r_2}\left[\frac{r_1 - r_3}{r_3 - r_2}\right] e^{r_2 - r_1} + \frac{r_1}{r_3}\left[\frac{r_2 - r_1}{r_3 - r_2}\right] e^{r_3 - r_1} = 0$$

It reduces on substitution to

$$e^{\frac{3}{2}\sqrt[3]{-k}} + 2\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{-k}\right) = 0$$

This has a fundamental root

$$k = k_D = 6.33$$

from which $T_o$ can be related to $q_D$ and other dimensions.

Now it appears that the optimum thickness distribution might be directly compared by keeping $k_D$ the same. That is, with $k_D = 6.33$, we seek to find $t$ that minimizes total face-sheet weight in

$$(t\alpha_e')'' + k\alpha_e = 0$$

with the boundary conditions

$$\alpha_e(1) = t\alpha_e'\Big|_{x=0} = (t\alpha_e')'\Big|_{x=0} = 0$$

Here $t = \frac{T}{T_o}$, so that $k$ retains the value 6.33. After we introduce the auxiliary variables

$$r = t\alpha_e'$$

$$s = (t\alpha_e')',$$

the optimization problem reads as follows.

Minimize the definite integral

$$M = \int_o^1 t(x)dx,$$

subject to the constraints

$$\alpha_e' = \frac{r}{t}$$

$$r' = s$$

$$s' = -k\alpha_e$$

with the boundary conditions

$$\alpha_e(1) = r(0) = s(0) = 0$$

The Hamiltonian is

$$H = t + \lambda_{\alpha_e} \frac{r}{t} + \lambda_r s - k\lambda_s \alpha_s,$$

and the Lagrange multipliers satisfy the necessary conditions for an optimum:

$$\lambda'_{\alpha_e} = -\frac{\partial H}{\partial \alpha_e} = k\lambda_s$$

$$\lambda'_r = -\frac{\partial H}{\partial r} = -\frac{\lambda_{\alpha_e}}{t}$$

$$\lambda'_s = -\frac{\partial H}{\partial s} = -\lambda_r$$

The control equation turns out to be

$$\frac{\partial H}{\partial t} \equiv 1 - \frac{\lambda_{\alpha_e} r}{t^2} = 0$$

The transversality conditions give

$$\lambda_{\alpha_e}(0) = \lambda_r(1) = \lambda_s(1) = 0$$

We have therefore to solve the system of six equations with six unknowns

$$\begin{aligned}
\alpha'_e &= \frac{r}{t} \\
r' &= s \\
s' &= -k\alpha_e \\
\lambda'_{\alpha_e} &= k\lambda_s \\
\lambda'_r &= -\frac{\lambda_{\alpha_e}}{t} \\
\lambda'_s &= -\lambda_r,
\end{aligned} \tag{3.1}$$

where

$$t = \lambda_{\alpha_e} r$$

The boundary conditions read

$$r(0) = s(0) = \lambda_{\alpha_e}(0) = 0$$

$$\alpha_e(1) = \lambda_r(1) = \lambda_s(1) = 0$$

## 3.2 Numerical Integration

No analytical solution can be easily found in this case; the proportionality between the variables and their adjoints does not hold any more (it worked for a system of 4 equations, where a variable was proportional to the adjoint of the other). In order to find a solution we have to use numerical integration.

Once again, the transition-matrix algorithm was proved very powerful. It has to be changed slightly here, as $t$ is zero at the end $x = 0$. In order to integrate the system on $[0, 1]$, we will assume values at the end $x = 1$ for the variables $r, s$ and $\lambda_{\alpha_e}$ and integrate backwards from 1 to 0.

The transition matrix $T$ is a 3 x 3 square matrix such that

$$\begin{bmatrix} \delta r(0) \\ \delta s(0) \\ \delta\lambda_{\alpha_e}(0) \end{bmatrix} = T \begin{bmatrix} \delta r(1) \\ \delta s(1) \\ \delta\lambda_{\alpha_e}(1) \end{bmatrix}$$

$T$ is formed as follows: we adjoin to the given system the system of equations satisfied by the first variations of the variables:

$$\begin{aligned}
\delta\alpha'_e &= \frac{r}{2t}\left(\frac{\delta r}{r} - \frac{\delta\lambda_\alpha}{\lambda_\alpha}\right) \\
\delta r' &= \delta s \\
\delta s' &= -k\delta\alpha_e \\
\delta\lambda'_{\alpha_e} &= k\delta\lambda_s \\
\delta\lambda'_r &= \frac{\lambda_\alpha}{2t}\left(\frac{\delta r}{r} - \frac{\delta\lambda_\alpha}{\lambda_\alpha}\right) \\
\delta\lambda'_s &= -\delta\lambda_r
\end{aligned} \qquad (3.2)$$

The first column of $T$ will be composed of the values at $x = 0$ of $\delta r, \delta s$ and

$\delta\lambda_{\alpha_e}$. These are solutions of the above system with all the functions but $\delta r$ being set equal to zero at $x = 1$, $\delta r(1)$ being set equal to unity. In a similar fashion, we obtain the second and third columns, respectively, with $\delta s(1) = 1$, then $\delta\lambda_{\alpha_e}(1) = 1$ being the only non-zero conditions at $x = 1$. Of course, the systems (3.1) and (3.2) have to be solved simultaneously, as the first variations are functions of the variables $\alpha_e, r, s$ and their adjoints.

We now invert T and compute the increments $\Delta r(1), \Delta s(1), \Delta\lambda_{\alpha_e}(1)$ by

$$\begin{bmatrix} \Delta r(1) \\ \Delta s(1) \\ \Delta\lambda_{\alpha_e}(1) \end{bmatrix} = -\epsilon T^{-1} \begin{bmatrix} r(0) \\ s(0) \\ \lambda_{\alpha_e}(0) \end{bmatrix}$$

Here the column on the right-hand side is formed of the values at $x = 0$ of $r, s, \lambda_{\alpha_e}$ using our initial assumptions, and $\epsilon$ is a positive number smaller than or equal to unity, taken equal to unity to start with.

We now start the integration again, with the new values for $r(1), s(1), \lambda_{\alpha_e}(1)$

$$r(1)_{NEW} = r(1)_{OLD} + \Delta r(1)$$

$$s(1)_{NEW} = s(1)_{OLD} + \Delta s(1)$$

$$\lambda_{\alpha_e}(1)_{NEW} = \lambda_{\alpha_e}(1)_{OLD} + \Delta\lambda_{\alpha_e}(1)$$

As the integration will stop whenever t is found equal to zero (cf. previous examples), it will be necessary in the course of the process to reduce $\epsilon$ to have smoother changes in the initial values at the end $x = 1$.

We chose the values 1, 1, 6 for $r, s$ and $\lambda_{\alpha_e}$ respectively at $x = 1$. The thickness distribution corresponding to this guess is labeled under $N = 1$ on Fig. 3.2 ($N \equiv$ number of iterations). The second iteration brings us closer to the solution, and the optimal thickness distribution is attained for $N = 6$ with a relative error of less than 1% (judged on how close $t(0)$ is from its actual value zero) and an $\epsilon$ of 0.125. For $N = 11$ the relative error is less than 0.2%, and $\epsilon$ is equal to 0.015625.

The area under the curve is equal to 0.690: this optimal solution represents a mass saving of 31.0%.

The case of the sandwich section with a minimum-thickness constraint $t_1$ has even more practical interest. The transition-matrix algorithm applies to this example with slight modifications; for $t_1 = 0.2$ and with initial values 1, 1, 6 for $r, s$ and $\lambda_{\alpha_e}$, respectively, and a starting $\epsilon$ of 0.5, the sequence of curves represented in Fig. 3.3 was found. The optimal thickness distribution obtained for $N = 7$ with a relative error smaller than 0.2% is plotted in solid lines. The optimal mass ratio is then equal to 0.704, so that this solution represents a mass saving of 29.6%.

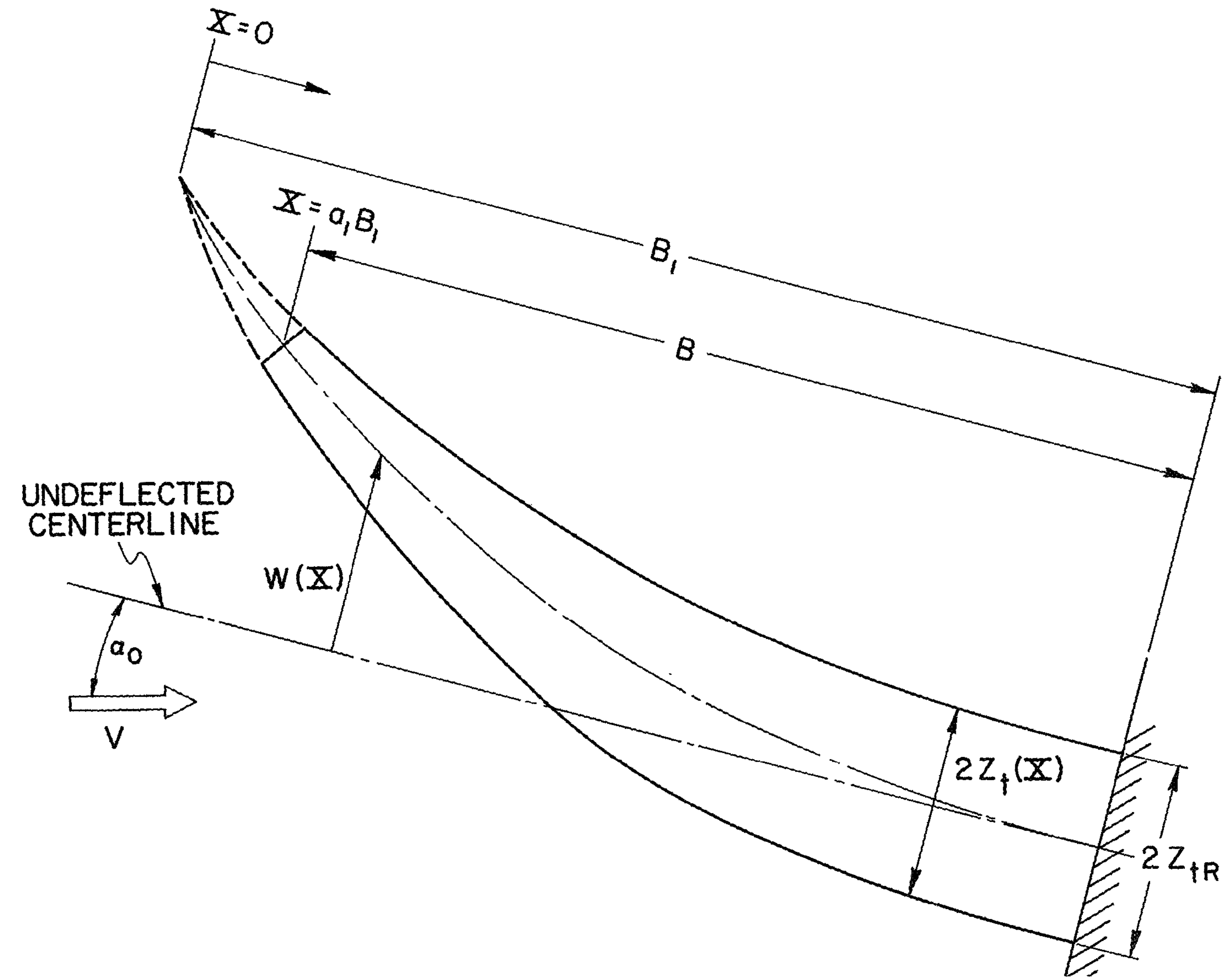


Fig. 3.1 Cross-section of forward half of a plate undergoing chordwise bending in a two-dimensional supersonic airstream.

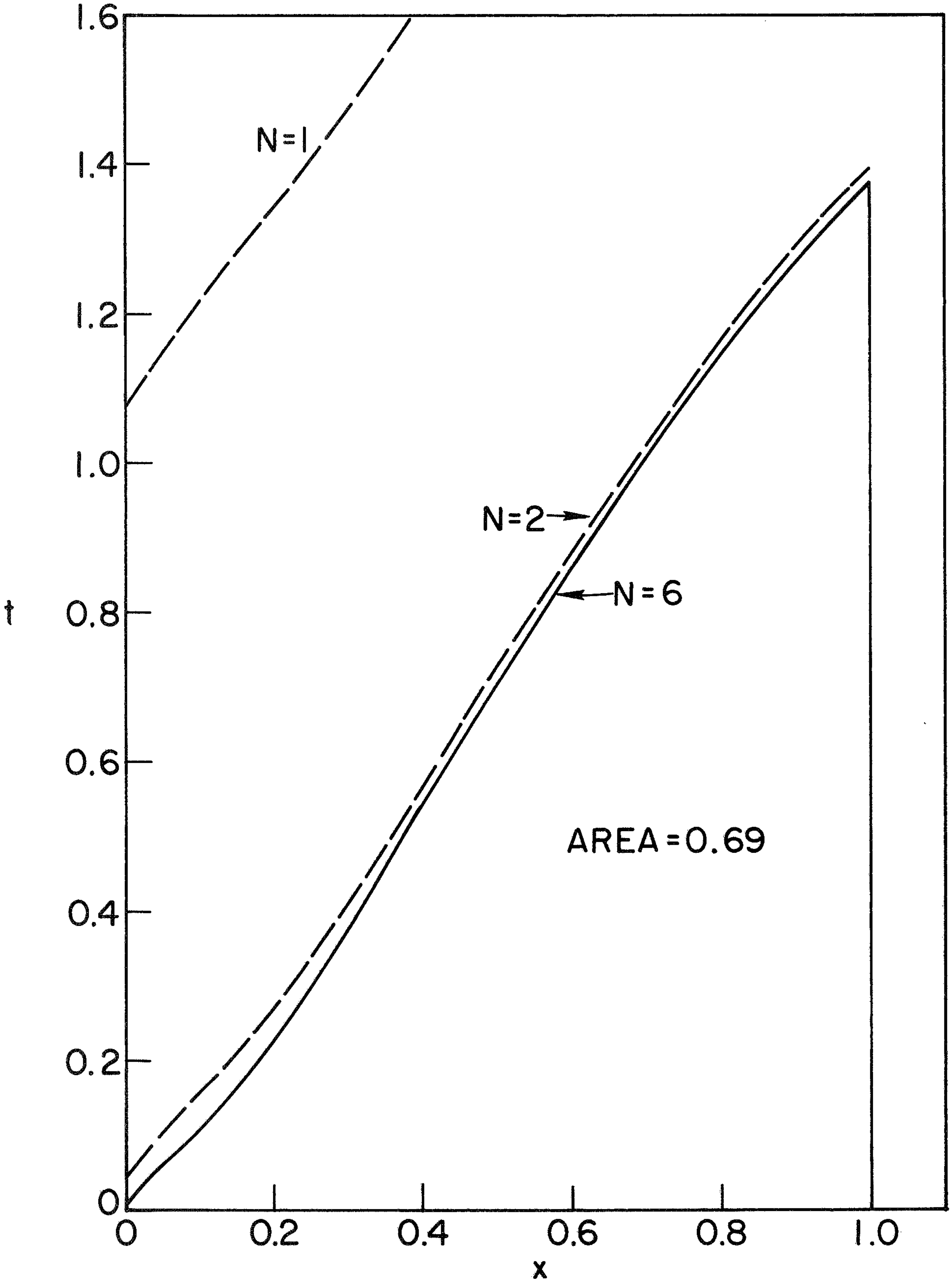


Fig. 3.2 Optimum thickness distribution —— plate for fixed condition of chord-wise divergence.

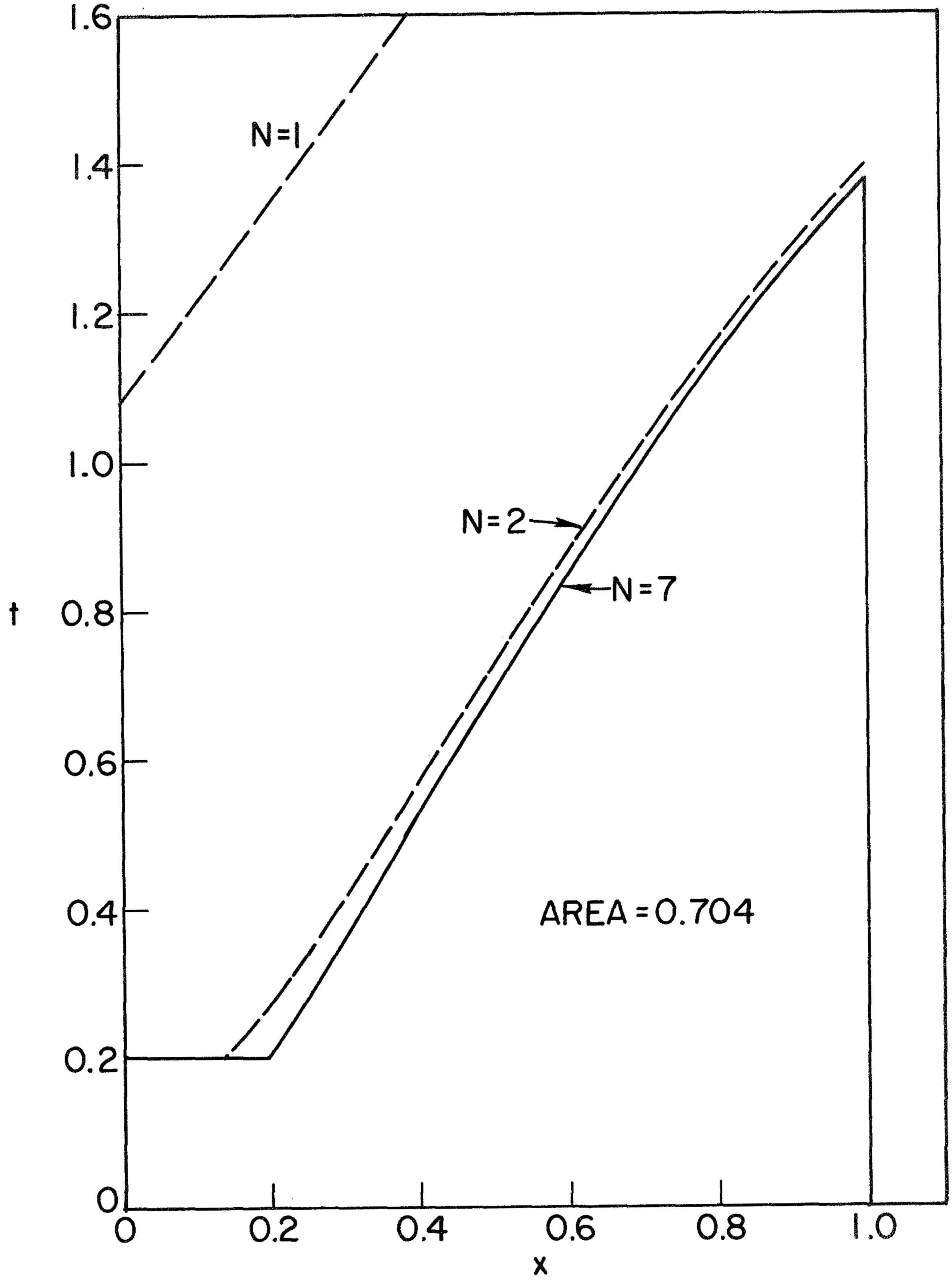


Fig. 3.3 Optimal thickness distribution for fixed condition of chordwise divergence with minimum thickness $t_1 = 0.2$.

## B. DYNAMIC AEROELASTIC PROBLEMS

The assumption of simple harmonic motion will reduce the partial differential equations of these dynamic cases to ordinary differential equations in complex dependent variables. However the following problems will be converted to a real form, and the optimization process will be similar to what has been already encountered.

## 4. PURE TORSIONAL FLUTTER OF A STRAIGHT WING

We consider, as before, a rectangular wing planform. When we make use of aerodynamic strip theory, the equation of motion reads

$$I_\alpha \frac{\partial^2 \theta}{\partial \tau^2} - \frac{\partial}{\partial X}\left[GJ \frac{\partial \theta}{\partial X}\right] = M_\alpha(X,\tau) \tag{4.1}$$

with boundary conditions

$$\theta(0,\tau) = 0$$

$$GJ \frac{\partial \theta}{\partial X}(b,\tau) = 0$$

and $\tau$ denoting the time.

We will adopt the standard procedure used in flutter problems, by assuming separation of variables for $\theta$; introducing the dimensionless length $x = \frac{X}{b}$, we look for $\theta$ in the form

$$\theta(X,\tau) = \text{Re}\left\{\overline{\theta}(x)e^{i\omega\tau}\right\}$$

where $\overline{\theta}(x)$ is a complex function of the real variable $x$.

A reference wing is characterized by the zero subscript, with the same proportionality assumptions as in Chapter 2,

$$EI_\alpha(x) = EI_{\alpha_o} t(x)$$

$$GJ(x) = GJ_o t(x)$$

where

$$t(x) = \frac{T(x)}{T_o}$$

If aerodynamic strip theory is adopted, the complex running moment $\overline{M}_\alpha$ is known to depend linearly on the complex amplitude $\overline{\theta}$, the proportionality factor being a function of the elastic-axis location, the reduced frequency $k = \frac{\omega b}{V}$, and the flight Mach number $M$. For incompressible flow, and with the notations used in AFTR 4798,

$$\overline{M}_\alpha(x) = \pi\rho b^4\omega^2 \left[ M_\alpha - (\frac{1}{2} + a)(\overline{L}_\alpha + \overline{M}_h) + (\frac{1}{2} + a)^2 \overline{L}_h \right] \overline{\theta}(x)$$

($\overline{L}_\alpha, \overline{L}_h, M_\alpha, \overline{M}_h$ are dimensionless complex functions of the reduced frequency $k$, tabulated in the report).

We introduce the dimensionless quantities

$$\omega_\theta = \sqrt{\frac{GJ_o}{I_{\alpha_o} \ell^2}} \qquad \mu = \frac{m}{\pi\rho b^2}$$

$$r_\alpha = \sqrt{\frac{I_{\alpha_o}}{mb^2}}$$

and the two parameters

$$\alpha = (\frac{\omega}{\omega_\theta})^2$$

$$\overline{\beta} = \frac{1}{\mu r_\alpha^2} \left(\frac{\omega}{\omega_\theta}\right)^2 \left\{ M_\alpha - (\frac{1}{2} + a)(\overline{L}_\alpha + \overline{M}_h) + (\frac{1}{2} + a)^2 \overline{L}_h \right\}$$

($\alpha$ real and $\overline{\beta}$ complex). Thus we reduce the problem, in complex form, to the equation

$$(t\overline{\theta}')' + (\alpha t + \overline{\beta})\overline{\theta} = 0 \tag{4.1'}$$

with the two boundary conditions

$$\overline{\theta}(0) = 0$$

$$t\overline{\theta}' \Big|_{x=1} = 0$$

where the primes denote differentiation with respect to $x$.

4.1 Solution for $t = 1$

A uniform-wing reference flutter case must be constructed to provide numerical coefficients for starting the computation. A key reference is Smilg's paper (Ref. 5), in which it is indicated that $\frac{I_{\alpha_o}}{\pi\rho b^4}$ must be very large and that the elastic axis must be ahead of the quarter-chordline ($a < -\frac{1}{2}$) before single-degree-of-freedom instability can occur. For $t = 1$, the equation reduces to

$$\bar{\theta}'' + (\alpha + \bar{\beta})\bar{\theta} = 0$$

with the boundary conditions

$$\bar{\theta}(0) = 0$$

$$\bar{\theta}'(1) = 0$$

The corresponding characteristic equation is

$$s^2 + (\alpha + \bar{\beta}) = 0$$

let $a + ib$ and $-(a + ib)$ be the two square roots of the complex number $-(\alpha + \bar{\beta})$; the general solution of (4.1') is then

$$\bar{\theta}(x) = Ae^{(a + ib)x} + Be^{-(a + ib)x},$$

$A$ and $B$ being two complex constants.

The boundary condition at zero leads to the condition

$$A + B = 0$$

so that

$$\bar{\theta}(x) = A \left[ e^{(a + ib)x} - e^{-(a + ib)x} \right]$$

The eigenvalues for flutter are determined from the remaining boundary condition

$$\bar{\theta}'(1) = 0$$

which gives

$$(a + ib) \left[ e^{a + ib} + e^{-(a + ib)} \right] = 0$$

The quantity $\alpha + \bar{\beta}$ is different from zero, so that $a + ib$ is also different from zero, and this reduces to

$$e^{a + ib} + e^{-(a + ib)} = 0$$

or

$$e^{2(a + ib)} + 1 = 0$$

or, equating real and imaginary parts,

$$e^{2a} \cos 2b = -1$$

$$e^{2a} \sin 2b = 0$$

from the second equation,

$$2b = k_o \pi \qquad k_o = 0, \pm 1, \pm 2, \ldots .$$

so that

$$\cos 2b = (-1)^{k_o}$$

and

$$e^{2a} = (-1)^{k_o + 1}$$

This implies

$$k_o = 2k + 1 \qquad (k_o \text{ odd})$$

$$a = 0$$

so that the two roots of $-(\alpha + \bar{\beta})$ are $\pm(2k+1)\frac{\pi}{2}i$, $k = 0, 1, 2, \ldots .$ This, in turn, implies that $\alpha + \bar{\beta}$ is real and equal to $(2k+1)^2 \frac{\pi^2}{4}(k = 0, 1, 2, \ldots)$. This result means that $\bar{\beta}$ has to be real or that

$$\mathrm{Im}\left\{\bar{\beta}\right\} = 0 \tag{4.2}$$

The problem therefore reduces to a real one. Equation (4.2) is identical with Smilg's equation (3) for vanishing of the imaginary part of the moments, so that the necessary conditions for neutral stability are obtainable from the solid curves in Smilg's Figs. 1 and 2.

We choose the smallest eigenvalue,

$$\alpha + \beta = \frac{\pi^2}{4} = 2.47$$

Therefore, given Smilg's charts, a reference eigenvalue can be estimated. Careful study of the problem shows that it is more convenient to choose the elastic axis at the leading edge; k is taken equal to 0.027. From Fig. 2A this corresponds to

$$\frac{I_{\alpha_o}}{\pi \rho b^4}\left[1 - 2.47\left(\frac{\omega_\theta}{\omega}\right)^2\right] = 595,$$

whence flutter for a uniform wing with $\frac{I_{\alpha_o}}{\pi \rho b^4} \cong 1500$ would occur at

$$2.47\left(\frac{\omega_\theta}{\omega}\right)^2 \simeq 0.603$$

or

$$\omega \simeq 2.81\omega_\theta$$

It follows that

$$\alpha = 4.08$$

$$\beta = -1.61$$

4.2 Optimization Process

The problem is to minimize the definite integral

$$M = \int_0^1 t(x)dx$$

subject to the constraints

$$\theta' = \frac{s}{t}$$

$$s' = -\theta(\alpha t + \beta)$$

with the boundary conditions

$$\theta(0) = 0$$

$$s(1) = 0$$

This is identical with the problem statement encountered in Section (2.3) for which an analytical solution was found. We introduce the new parameters

$$\delta_1 = \frac{4}{\pi^2}\alpha \qquad \delta_2 = \frac{4}{\pi^2}\beta,$$

such that $\delta_1 + \delta_2 = 1$.

The sum of the two parameters is still equal to 1, but the main difference is that now $\delta_2$ is negative; from the integration of the above equations (referenced in Section (2.3)), the optimal thickness appears to be negative, since $\delta_2$ appears in it as a multiplicative factor. As in the stability domain $\alpha$ is always positive and $\beta$ always negative, $\delta_2$ will always be negative and the problem stated under this form will have no realistic solution.

## 4.3 Optimization With a Nonstructual Part

One way to overcome the foregoing difficulty is as we already did in Chapter 2, to label a portion of the mass as "nonstructural" and to allow the rest to vary, the latter part only contributing to the structural properties (stiffness). Under this assumption, we may force the part of the $\theta$ coefficient which is independent of $t$ to be positive.

Introducing a parameter $\lambda$, $0 < \lambda < 1$, we set

$$t = \lambda + (1 - \lambda)t^*$$

and the equation of motion is rewritten as

$$(t^*\theta')' + \frac{\pi^2}{4}(\delta_1^* t^* + \delta_2^*)\theta = 0$$

where

$$\delta_1^* = \delta_1(1 - \lambda)$$

$$\delta_2^* = \delta_1\lambda + \delta_2$$

In order that $\delta_2^*$ be positive, $\lambda$ has to be chosen such that

$$\lambda > -\frac{\delta_2}{\delta_1}$$

With our values of $\delta_1$ and $\delta_2$,

$$\lambda > 0.393$$

If we choose $\lambda = 0.5$, i.e. if we allow 50% of the mass to vary, thus

$$\delta_1^* = 0.826$$

$$\delta_2^* = 0.174$$

The optimal thickness distribution is therefore given by

$$t^* = \frac{\delta_2^*}{2\delta_1^*}\left[\left(\frac{\cosh(\frac{\pi}{2}\sqrt{\delta_1^*})}{\cosh(\frac{\pi}{2}\sqrt{\delta_1^*}x)}\right)^2 - 1\right]$$

and the true thickness distribution by

$$t = \lambda + (1 - \lambda)\frac{\delta_2^*}{2\delta_1^*}\left[\left(\frac{\cosh(\frac{\pi}{2}\sqrt{\delta_1^*})}{\cosh(\frac{\pi}{2}\sqrt{\delta_1^* x})}\right)^2 - 1\right]$$

The optimal mass ratio is

$$M = \lambda + (1 - \lambda)\frac{\delta_2^*}{2\delta_1^*}\left[\frac{\sinh(\pi\sqrt{\delta_1^*})}{\pi\sqrt{\delta_1^*}} - 1\right]$$

The thickness distribution corresponding to the above data is represented in Fig. 4.1; the corresponding mass saving is found equal to 39.3%.

As the stiffness at the end $x = 1$ is zero, it may be interesting to apply a minimum-thickness constraint on $t^*$, looking for the optimum satisfying the inequality.

$$t^* \geq t_1^*$$

The problem is then the same as in Section (2.4). Some typical thickness profiles are represented in Fig. (4.2), and the variation of the optimal mass percentage with this minimal constraint $t_1^*$ is plotted in Fig. (4.3). As usual, the maximum saving is obtained for the unconstrained case.

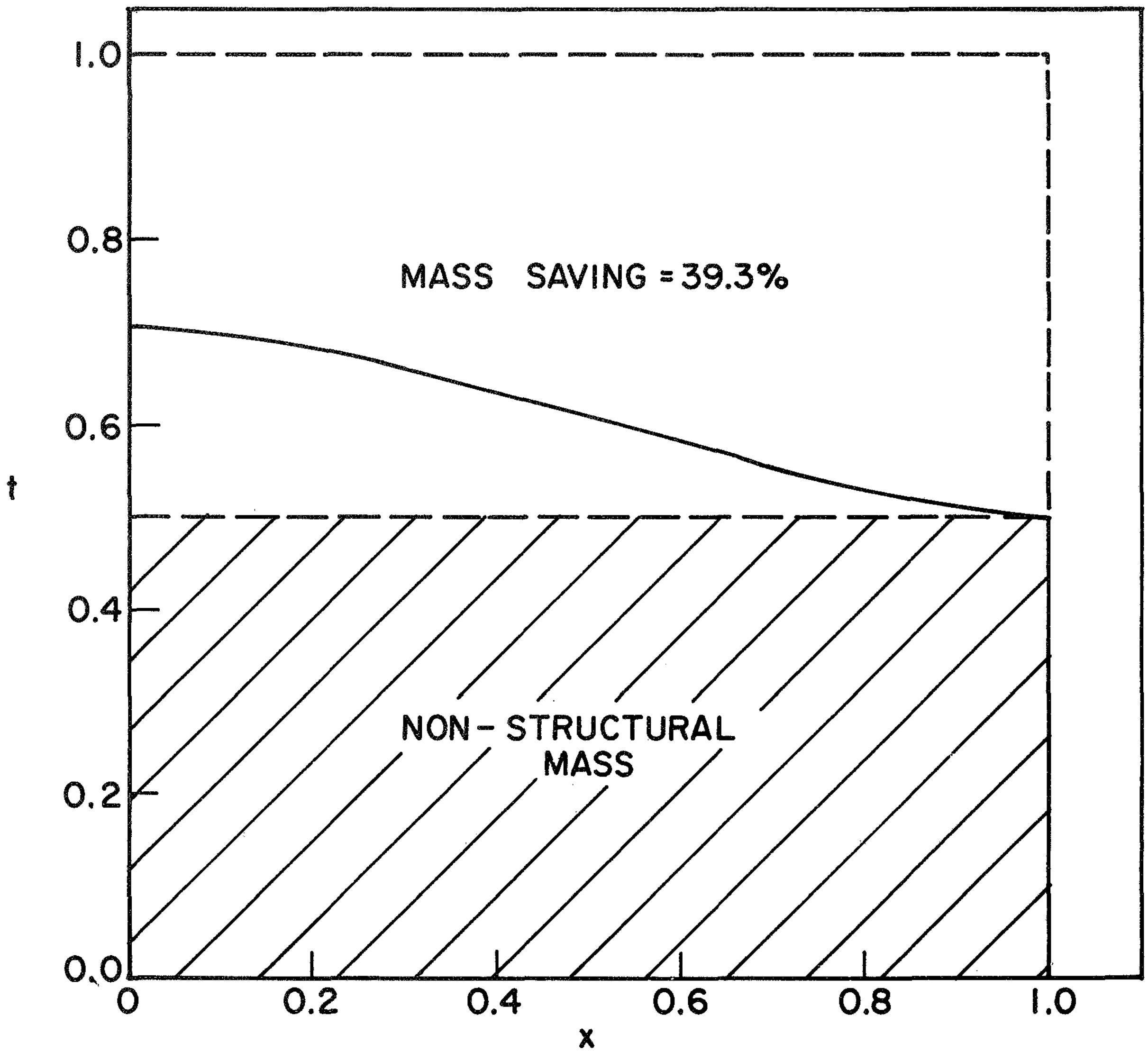


Fig. 4.1 Optimal thickness distribution for torsional flutter case, 50% of mass allowed to vary, $\alpha = 4.08$, $\beta = -1.61$.

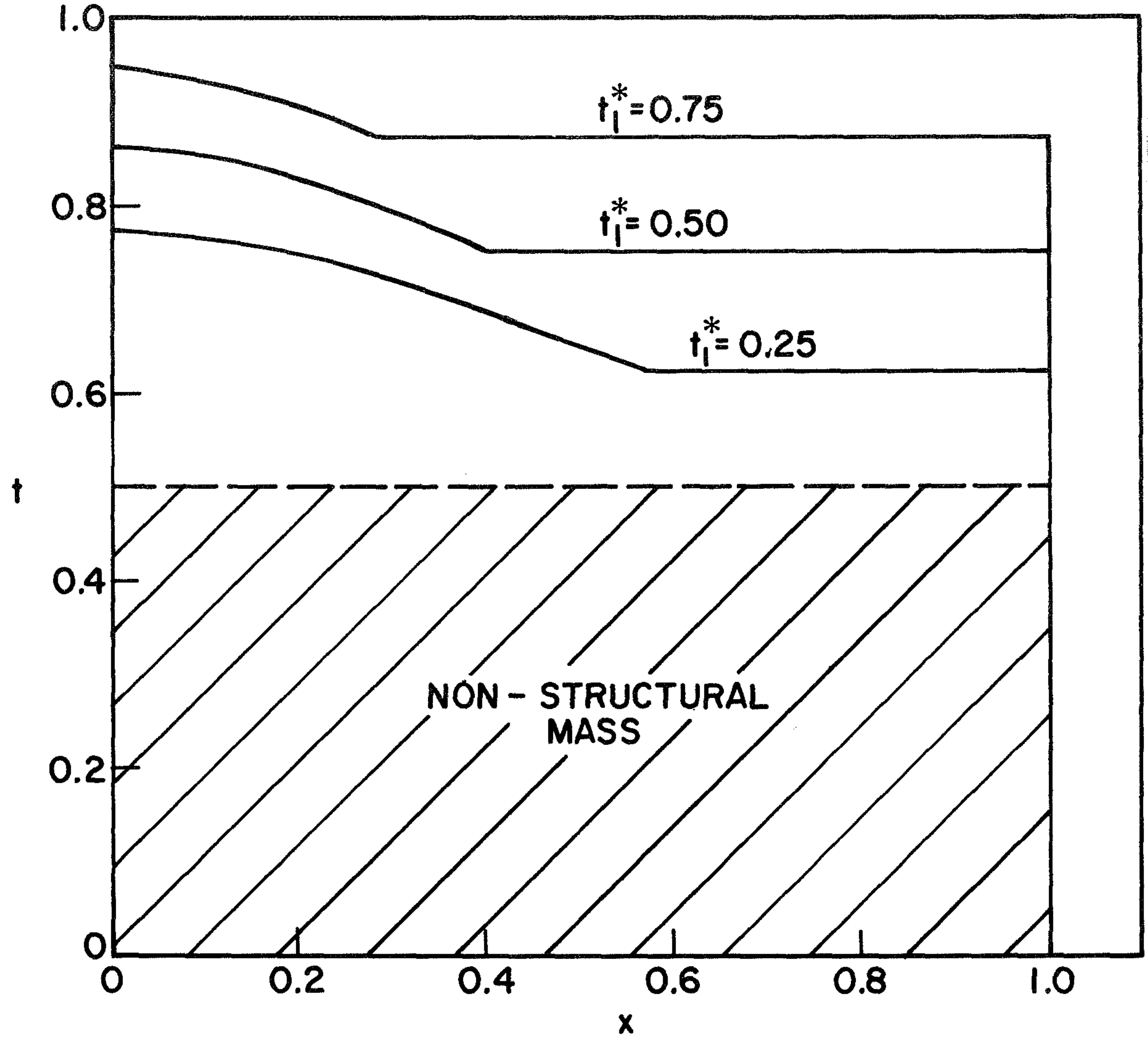


Fig. 4.2 Optimal thickness configuration for torsional flutter case with minimum-thickness constraint, $\alpha = 4.08$, $\beta = -1.61$, 50% of mass nonstructural.

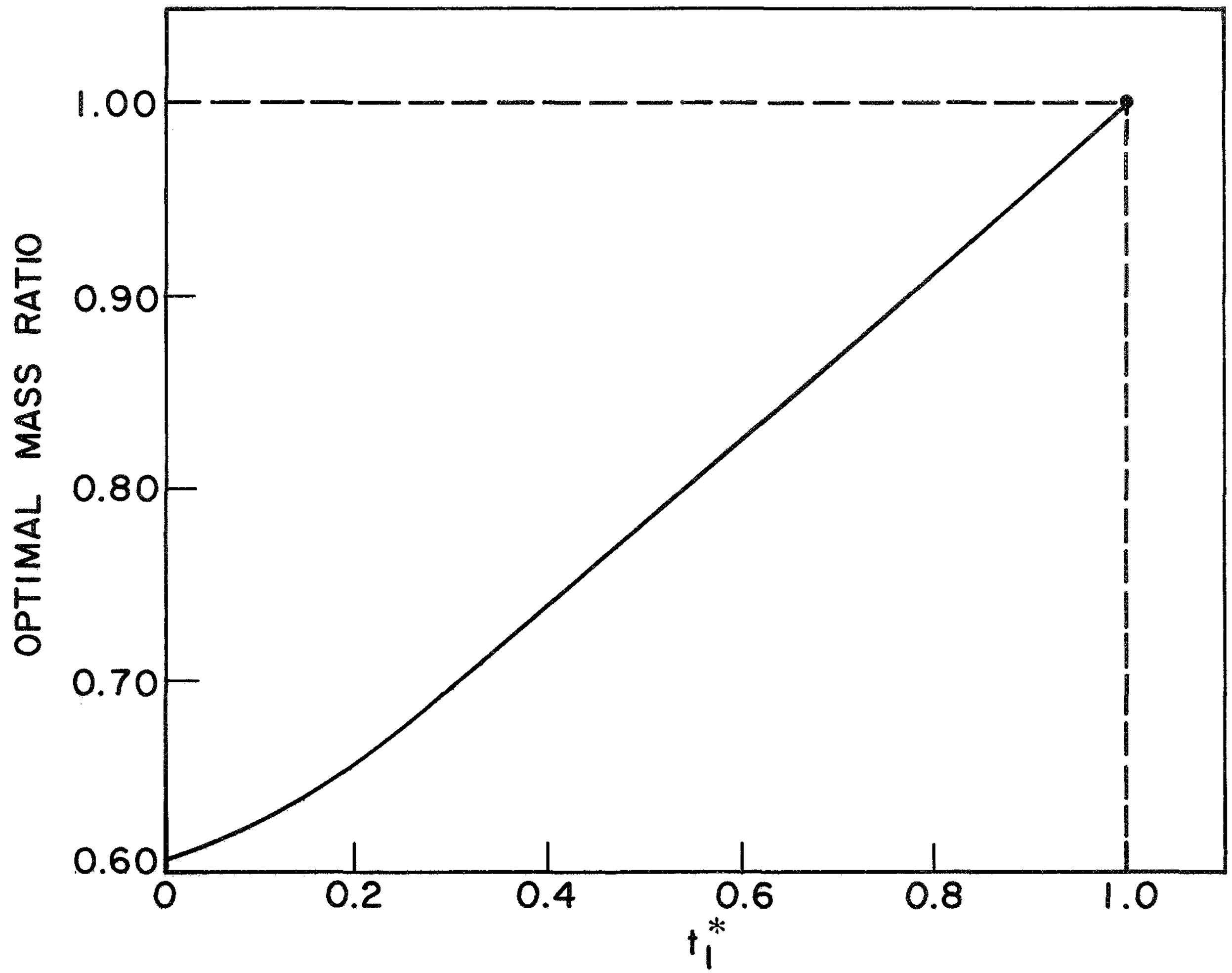


Fig. 4.3 Optimal mass ratio vs. minimum thickness $t_1^*$ (case of Fig. 4.2).

## 5. PANEL FLUTTER OPTIMIZATION

### 5.1 Statement of the Problem

For a two-dimensional panel of length $a$ with deflection $w(x,t) = \frac{W(X,t)}{a}$ and extending from $x = \frac{X}{a} = 0$ to $x = 1$ the dimensionless partial differential equation reads: (cf. Ref. 6, Section 8-5(a), pp. 419-423)

$$\frac{\partial^4 w}{\partial x^4} + R_{xx}\frac{\partial^2 w}{\partial x^2} + \lambda\frac{\partial w}{\partial x} + \lambda\frac{M^2 - 2}{M^2 - 1}\frac{a}{U}\frac{\partial w}{\partial t} + \frac{m_o a^4}{D}\frac{\partial^2 w}{\partial t^2} = 0 \tag{5.1}$$

Here we have defined the dimensionless parameters

$$R_{xx} = -\frac{N_{xx}^{(0)} a^2}{D} \qquad (N_{xx}^{(0)} \equiv \text{in-plane tensile force})$$

$$\lambda = \frac{2qa^3}{D\sqrt{M^2 - 1}}$$

where

$$D = \frac{Eh^3}{12(1 - \nu^2)}$$

For a simply-supported panel the boundary conditions read

$w(0) = w''(0) = 0$

$w(1) = w''(1) = 0$

If we assume simple harmonic motion

$w(x,t) = w^*(x)e^{i\omega t}$

(5.1) reduces to

$$w^{*(IV)} + R_{xx}w^{*''} + \lambda w^{*'} - kw^* = 0 \tag{5.2}$$

with boundary conditions: $w^*(0) = w^{*''}(0) = 0$

$w^*(1) = w^{*''}(1) = 0$

where $k$ is an eigenvalue of the form

$$k = \frac{m_o a^4}{D}\omega^2 - i\lambda \frac{M^2 - 2}{M^2 - 1}\frac{a}{U}\omega$$

$$= \pi^4 \left(\frac{\omega}{\omega_o}\right)^2 - i\pi^4 g_{\omega_o} \frac{\omega}{\omega_o}$$

Here

$$\omega_o = \pi^2 \sqrt{\frac{D}{m_o a^4}}$$

is the first natural frequency for a semi-infinite simply supported flat panel and

$$g_{\omega_o} = \frac{M^2 - 2}{(M^2 - 1)^{3/2}} \frac{\rho U}{m_o \omega_o}$$

is the damping coefficient based on $\omega_o$.

Solutions due to Houbolt, Hedgepeth and others are discussed in the cited reference. It appears that the most productive case to work on first would be the one where in-plane stress and aerodynamic damping are neglected (essentially

$$R_{xx} = 0 \qquad \text{and} \qquad k = \frac{m_o a^4}{D}\omega^2)$$

and simple support is assumed. Uniform-panel flutter then occurs in the fundamental mode at

$$\lambda = 3.52\pi^4 \cong 343$$

$$k = (1.9\pi)^4$$

For purposes of optimization, consider a panel that flutters at the same $q_\infty$. It will be interesting to see if $\omega$ can be allowed to vary. Let $R_{xx} = 0$, neglect the $\frac{\partial w}{\partial t}$ term, and allow $m(x)$ and $D(x)$ to vary. The equation of the problem is

$$\frac{d^2}{dx^2}\left[\frac{D(x)}{D_o}\frac{\partial^2 w^*}{\partial x^2}\right] + \frac{2q_\infty a^3}{D_o\sqrt{M^2 - 1}}\frac{dw^*}{dx} + \frac{m(x)}{m_o}\frac{m_o a^4}{D_o}(i\omega)^2 w^* = 0$$

with boundary conditions

$$w^*(0) = Dw^{*''}\Big|_{x=0} = 0$$

$$w^*(1) = Dw^{*''}\Big|_{x=1} = 0$$

Now suppose $t(x)$ and $\omega$ are allowed to vary in search of an optimum. Then, using subscript o to identify properties of the uniform panel of solid metal with the same $q_{\infty}$ , we have
D

$$\frac{D(x)}{D_o} = \left(\frac{T(x)}{T_o}\right)^3 = t^3(x)$$

$$\frac{m(x)}{m_o} = t(x)$$

Therefore the problem reduces to the equation (dropping the asterisk superscript for $w$ as no confusion will now be possible)

$$(t^3 w'')'' + \lambda_o w' - k_o t\left(\frac{\omega}{\omega_o}\right)^2 w = 0$$

subject to the boundary conditions

$$w(0) = t^3 w\Big|_{x=0} = 0 \qquad (5.3)$$

$$w(1) = t^3 w\Big|_{x=1} = 0$$

where

$$\lambda_o = \frac{2q_{\infty}a^3}{D_o\sqrt{M^2 - 1}} = \text{constant}$$

$$k_o = \frac{m_o a^4}{D_o}\omega_o^2 = \text{constant}$$

In the search for an optimum, the first step is to hold $\omega$ constant, to solve the simplified problem, and then to minimize algebraically with respect to this parameter afterward.

We define the auxiliary variables

$$p = w'$$

$$q = t^3 w''$$

$$r = (t^3 w'')'$$

and let

$$k = k_o \left(\frac{\omega}{\omega_o}\right)^2$$

## 5.2 Necessary Conditions for an Extremum

The problem is to minimize the definite integral

$$M = \int_o^1 t(x)dx$$

subject to the constraints

$$w' = p$$

$$p' = \frac{q}{t^3}$$

$$q' = r$$

$$r' = ktw - \lambda_o p$$

with boundary conditions

$$w(0) = q(0) = 0$$

$$w(1) = q(1) = 0$$

The Hamiltonian is constructed as follows:

$$H = t + \lambda_w p + \lambda_p \frac{q}{t^3} + \lambda_q r + \lambda_r (ktw - \lambda_o p)$$

The necessary conditions for an extremal read

$$\lambda'_w = -\frac{\partial H}{\partial w} = -kt\lambda_r$$

$$\lambda'_p = -\frac{\partial H}{\partial p} = -\lambda_w + \lambda_o\lambda_r$$

$$\lambda'_q = -\frac{\partial H}{\partial q} = -\frac{\lambda_p}{t^3}$$

$$\lambda'_r = -\frac{\partial H}{\partial r} = -\lambda_q$$

The control equation is

$$1 - \frac{3\lambda_p q}{t^4} + k\lambda_r w = 0$$

The transversality conditions give

$$\lambda_p(0) = \lambda_r(0) = 0$$

$$\lambda_p(1) = \lambda_r(1) = 0$$

Therefore, the whole problem reads

$$\begin{aligned}
& w' = p && \text{(i)} \\
& p' = \frac{q}{t^3} && \text{(ii)} \\
& q' = r && \text{(iii)} \\
& r' = ktw - \lambda_o p && \text{(iv)} \\
& \lambda'_w = -kt\lambda_r && \text{(v)} \\
& \lambda'_p = -\lambda_w + \lambda_o\lambda_r && \text{(vi)} \\
& \lambda'_q = -\frac{\lambda_p}{t^3} && \text{(vii)} \\
& \lambda'_r = -\lambda_q && \text{(viii)} \\
& 1 - \frac{3q\lambda_p}{t^4} + kw\lambda_r = 0 && \text{(ix)}
\end{aligned} \qquad (5.4)$$

$$w(0) = q(0) = \lambda_p(0) = \lambda_r(0) = 0$$

$$w(1) = q(1) = \lambda_p(1) = \lambda_r(1) = 0$$

From (ii) and (vii)

$$\frac{q\lambda_p}{t^4} = -t^2 p' \lambda_q'$$

and from (i) and (viii) this is also equal to $t^2 w'' \lambda_r''$, so that the control equation (ix) may be rewritten as

$$1 - 3t^2 w'' \lambda_r'' + kw\lambda_r = 0 \qquad \text{(x)}$$

We recall that $t$ and $w$ satisfy

$$(t^3 w'')'' + \lambda_o w' - ktw = 0$$

But from (v)

$$kt\lambda_r = -\lambda_w'$$

$$= \lambda_p'' - \lambda_o \lambda_r' \qquad \text{from (vi)}$$

$$= +(t^3 \lambda_r'')'' - \lambda_o \lambda_r' \qquad \text{from (vii) and (viii)}$$

Thus $t$ and $\lambda_r$ satisfy

$$(t^3 \lambda_r'')'' - \lambda_o \lambda_r' - kt\lambda_r = 0 \qquad \text{(xi)}$$

Therefore, $\lambda_r$, $w$, and $t$ satisfy the system

$$\begin{aligned} &1 - 3t^2 w'' \lambda_r'' + kw\lambda_r = 0 \\ &(t^3 w'')'' + \lambda_o w' - ktw = 0 \\ &(t^3 \lambda_r'')'' - \lambda_o \lambda_r' - kt\lambda_r = 0 \end{aligned} \qquad (5.5)$$

with boundary conditions

$$w(0) = w(1) = \lambda_r(0) = \lambda_r(1) = 0$$

Now, following a suggestion of Turner (Ref. 13), let us change $x$ into $1 - x$; if ( $\dot{}$ ) denotes the differentiation with respect to $1 - x$, we get the system

$$1 - 3t^2(1-x)\ddot{w}(1-x)\ddot{\lambda}_r(1-x) + kw(1-x)\lambda_r(1-x) = 0$$

$$(t^3(1-x)\ddot{w}(1-x))^{\cdot\cdot} - \lambda_o\dot{w}(1-x) - kt(1-x)w(1-x) = 0$$

$$(t^3(1-x)\ddot{\lambda}_r(1-x))^{\cdot\cdot} - \lambda_o\dot{\lambda}_r(1-x) - kt(1-x)w(1-x) = 0$$

If we adopt the notation

$$t(1-x) = \bar{t}(x)$$

$$w(1-x) = \overline{w}(x)$$

$$\lambda_r(1-x) = \bar{\lambda}_r(x)$$

this system can be written as

$$(\bar{t}^3\overline{w}'')'' - \lambda_o\overline{w}' - k\bar{t}\overline{w} = 0$$

$$(\bar{t}^3\bar{\lambda}_r'')'' + \lambda_o\bar{\lambda}_r' - k\bar{t}\bar{\lambda}_r = 0 \tag{5.6}$$

$$1 - 3\bar{t}^2\overline{w}''\bar{\lambda}_r'' + k\overline{w}\bar{\lambda}_r = 0$$

with boundary conditions

$$\overline{w}(0) = \overline{w}(1) = \bar{\lambda}_r(0) = \bar{\lambda}_r(1) = 0$$

A comparison of (5.5) and (5.6) shows that a possible solution of the problem is such that

$$t(x) = \bar{t}(x)$$

$$\lambda_r(x) = \alpha\overline{w}(x)$$

$$(\text{or } \; w(x) = \frac{1}{\alpha}\bar{\lambda}_r(x))$$

(here $\alpha$ is a constant), or that

$$t(x) = t(1-x)$$

$$\lambda_r(x) = \alpha w(1-x)$$

$$(\text{or } \; w(x) = \frac{1}{\alpha}\lambda_r(1-x))$$

This shows us that $t$, the optimal thickness distribution, has to be symmetrical with respect to the straight line $x = 1/2$. This result is true if the solution of the system is unique; if it is not, it still indicates that one of the optimal solutions has to be symmetrical (note that this symmetrical solution may correspond to a minimum or a maximum).

## 5.3 Tentative Solution: A Transition-Matrix Procedure

No analytical solution for system (5.4) has been found to date. A transition-matrix procedure seems indicated here. The algorithm has to be slightly modified, however, as the values of $p, r, \lambda_w, \lambda_q$ are unknown at both ends. Also, the fact that $q = t^3 w''$ is zero at both ends seems to indicate that $t$ also has to vanish at both ends (as found previously for one end). This means that the equations will be numerically integrable on $[0,1]$ in the case of a minimum-thickness constraint only,

$$t > t_1$$

The procedure is then as follows:

We guess four initial values for $p, r, \lambda_w, \lambda_q$ and integrate (5.4) from $x = 0$ to $x = 1$, recording the values at $x = 1$ of the variables $w, q, \lambda_p, \lambda_r$. The 4 x 4 transition matrix, connecting the first variations of $w, q, \lambda_p, \lambda_r$ at $x = 1$ with the first variations of $p, r, \lambda_w, \lambda_q$ at $x = 0$ through

$$\begin{bmatrix} \delta w(1) \\ \delta q(1) \\ \delta\lambda_p(1) \\ \delta\lambda_r(1) \end{bmatrix} = T \begin{bmatrix} \delta p(0) \\ \delta r(0) \\ \delta\lambda_w(0) \\ \delta\lambda_q(0) \end{bmatrix}$$

is formed as follows.

The first column will be composed of the values at 1 of $\delta w, \delta q, \delta\lambda_p, \delta\lambda_r$. These are solutions to the differential system (5.4) to which we adjoin the system satisfied by the first variations of the variables:

$$(\delta w)' = \delta p$$

$$(\delta p)' = \frac{\delta q}{t^3} - \frac{3q\delta t}{t^4}$$

$$(\delta q)' = \delta r$$

$$(\delta r)' = k(t\delta w + w\delta t) - \lambda_o \delta p$$

$$(\delta \lambda_w)' = - k(t\delta\lambda_r + \lambda_r \delta t)$$

$$(\delta\lambda_p)' = - \delta\lambda_w + \lambda_o \delta\lambda_r$$

$$(\delta\lambda_q)' = - \frac{\delta\lambda_p}{t^3} + \frac{3\lambda_p \delta t}{t^4}$$

$$(\delta\lambda_r)' = - (\delta\lambda_q)'$$

where

$$\delta t = \frac{3}{4t^3(1 + kw\lambda_r)^2}\left[(q\delta\lambda_p + \lambda_p \delta q)(1 + kw\lambda_r) - kq\lambda_p(w\delta\lambda_r + \lambda_r \delta w)\right]$$

For the first column of the transition matrix, the initial conditions for this system of 16 equations with 16 unknowns will be the previous ones for $w, q, \lambda_p, \lambda_r, p, r, \lambda_w, \lambda_q$ and $0, 0, 0, 0, 1, 0, 0, 0$ for their first variations, in that order.

The second column will be composed of the same values of the same variables solutions of the same system, $\delta p(0)$ being now zero and $\delta r(0)$ being set to unity; for the third column only $\delta\lambda_w(0)$ is unity, the other variations being all zero, and for the fourth $\delta\lambda_q(0)$ will be the only non-zero variation.

Choosing an $\epsilon (0 < \epsilon \leqslant 1)$, we now compute $\Delta p(0), \Delta r(0), \Delta\lambda_w(0), \Delta\lambda_q(0)$ by

$$\begin{bmatrix} \Delta p(0) \\ \Delta r(0) \\ \Delta\lambda_w(0) \\ \Delta\lambda_q(0) \end{bmatrix} = - \epsilon T^{-1} \begin{bmatrix} w(1) \\ q(1) \\ \lambda_p(1) \\ \lambda_r(1) \end{bmatrix},$$

where the column on the right-hand side is made up of the values at $x = 1$ of $w, q, \lambda_p, \lambda_r$, which are in turn solutions of (5.4) with the guessed boundary conditions. We start the procedure again with the new critical values

$$p(0)_{NEW} = p(0)_{OLD} + \Delta p(0)$$

$$r(0)_{NEW} = r(0)_{OLD} + \Delta r(0)$$

$$\lambda_w(0)_{NEW} = \lambda_w(0)_{OLD} + \Delta\lambda_w(0)$$

$$\lambda_q(0)_{NEW} = \lambda_q(0)_{OLD} + \Delta\lambda_q(0)$$

No solution has been calculated to date. The problem, in view of the complexity of the system and the large values of the coefficients $k$ and $\lambda_o$, is to guess the four initial boundary conditions such that the system may be numerically integrated on $[0, 1]$. It appears that some values become too large or too small during the process of integration, and that it stops before the end $x = 1$ is reached, even with a minimal thickness constraint.

To achieve the desired solution, it seems that we may have to make some more assumptions to simplify the system of differential equation (5.4). A clue can be derived from the fact that some coefficients are very big and some very small.

Another numerical method of the kind used in optimal control theory does not have to be used until the transition-matrix approach is proven to fail, and the main problem is as we said before to find a good start for the iterations.

Another problem of the same kind is that of a sandwich panel; the equations are then rather simpler (Ref. 2). However no solution had been found by numerical integration at the time of writing. The only known solution is that of Turner (Ref. 13) who uses a finite-element approach. His results do, in fact, confirm that the thickness distribution is symmetrical with respect to $x = 1/2$.

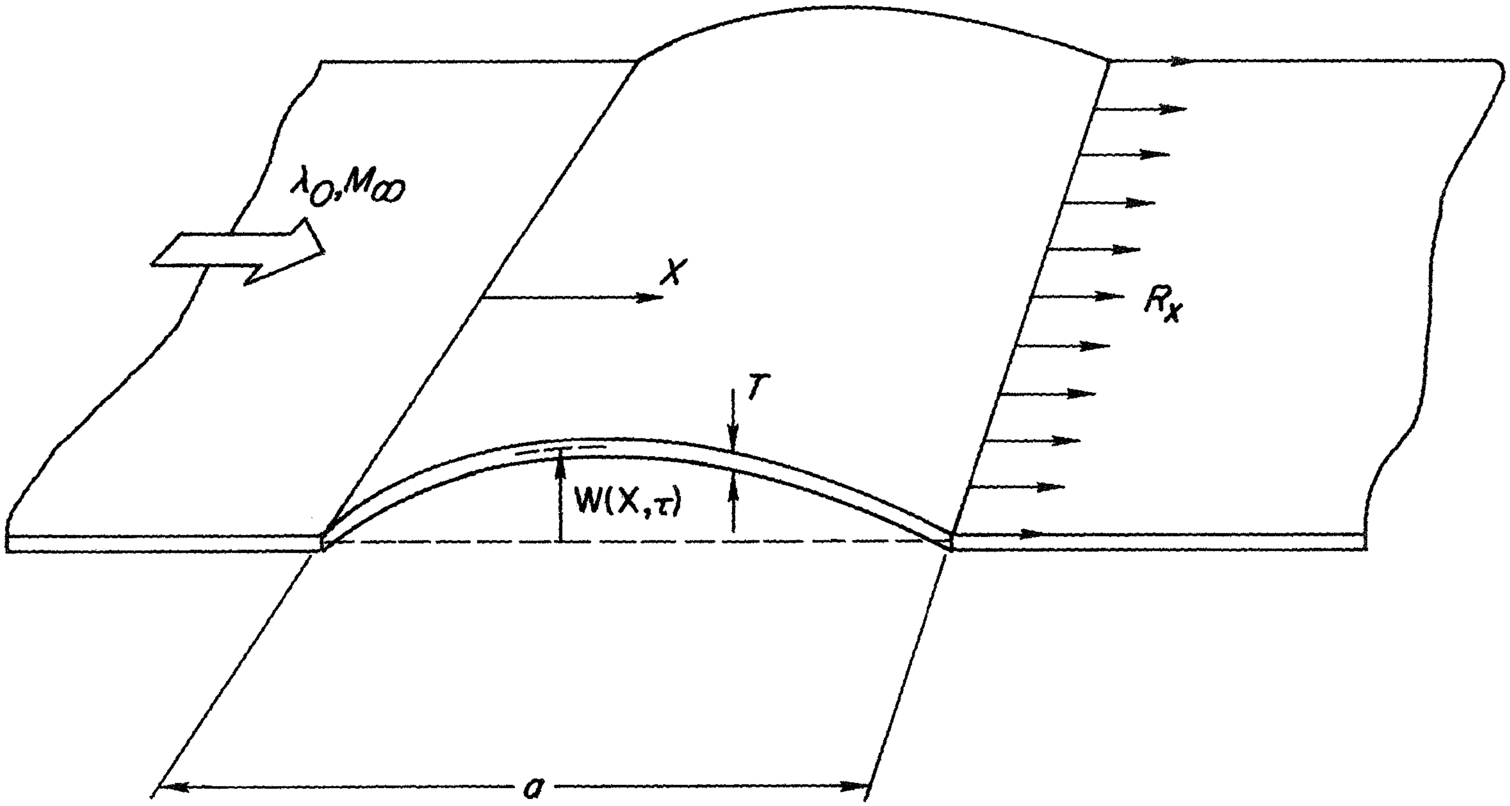


Fig. 5.1 Panel (plate-column) of infinite span, showing notation used for optimization with constant flutter eigenvalues.

## 6. CONCLUDING REMARKS

Despite the simplicity of the problems chosen here, the solutions require a great deal of numerical computation. In one case, no solution has been found to date (January 1970), the main difficulty being the complexity of the system of differential equations involved. The structures we considered were one-dimensional, so that we only had to deal with ordinary differential equations. However, for the more ambitious goal of optimizing plates or shells, the problem involves partial differential equations, and even for the simplest problems (square plate) a numerical solution of the optimizing equations seems very unlikely to be found in a way not involving excessive computations (an analytical solution seems sort of a Utopia in two-dimensional cases). For complex structures, such as actual airplane configurations, of course, it seems that then one has to turn to a more direct approach to the optimization problem by means of a finite-element discretization and more classical parameter optimization techniques (such as the gradient method). However, the techniques described in this report have not yet been widely used, and still present a lot of new openings. More work has to be completed in this area before a universal optimizing technique — if one exists! — can be devised, and until we exhaust the possibilities of the present methods.

## REFERENCES


1. H. Ashley and S. C. McIntosh, Jr., "Application of Aeroelastic Constraints in Structural Optimization," Proceedings of the 12th International Congress of Applied Mechanics, Springer, Berlin, 1969.
2. S. C. McIntosh, Jr., T. A. Weisshaar and H. Ashley, Progress in Aeroelastic Optimization — Analytical Versus Numerical Approaches, SUDAAR No. 383, Stanford University, Stanford, California, 1969.
3. A. E. Bryson, Jr. and Yu-Chi Ho, Applied Optimal Control, Blaisdell Publishing Company, Waltham, Massachusetts, 1968.
4. R. L. Bisplinghoff, H. Ashley and R. L. Halfman, Aeroelasticity, Addison-Wesley Publishing Company, Reading, Massachusetts, 1955.
5. R. L. Bisplinghoff and H. Ashley, Principles of Aeroelasticity, John Wiley and Sons, New York, 1962.
6. S. C. McIntosh, Jr., and F. E. Eastep, "Design of Minimum-Mass Structures with Specified Stiffness Properties," AIAA Journal, Vol. 6, No. 5, May 1968, pp. 962-964.
7. M. J. Turner, "Design of Minimum Mass Structures with Specified Natural Frequencies," AIAA Journal, Vol. 5, No. 3, March 1967, pp. 406-412.
8. B. Smilg, "The Instability of Pitching Oscillations of an Airfoil in Subsonic Incompressible Potential Flow," Journal of the Aeronautical Sciences, Vol. 16, No. 11, November 1949, pp. 691-696.
9. H. Ashley and S. C. McIntosh, Jr., "Expanding the Consciousness of the Aeroelastician," Fluid-Solid Interaction, published by the ASME, New York, 1967, pp. 120-143.
10. A. P. Sage, Optimum System Control, Prentice Hall, Englewood Cliffs, New Jersey, 1968.
11. Iu. P. Petrov, Variational Methods in Optimum Control Theory, Academic Press, New York, 1968.

12. I. M. Gelfand and S. V. Fomin, Calculus of Variations, Prentice Hall, Englewood Cliffs, New Jersey, 1963.

13. M. J. Turner, "Optimization of Structures to Satisfy Flutter Requirements," Proc. AIAA Structural Dynamics and Aeroelasticity Specialist Conference, New Orleans, Louisiana, April 16-17, 1969.

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13. ABSTRACT

Various optimization problems will be presented here, applications of the general methods developed in optimum control theory [Refs. 3,10,11,12] and based on the Hamiltonian formulation of variational calculus for structural optimization with aeroelastic constraints. The problem, common for all the applications, may be stated in a general form: given a reference structure (cantilever beam or two-dimensional plate) with uniform structural properties and specified aeroelastic requirements (such as a given divergence speed or flutter speed), find the structure with minimal weight satisfying the same requirements. This report will be divided into two parts, the first one dealing with static, the second one with dynamic aeroelastic problems (and more precisely flutter problems). This division is not arbitrary, since two out of the three problems of Part A will be found to have a simple analytical solution confirmed by numerical methods, whereas we have to rely on numerical integration mainly in Part B, the torsional-flutter case being excepted. A very powerful numerical procedure, the transition-matrix algorithm, will be described in detail and applied wherever possible. Its limitations in the more complicated case of panel flutter are emphasized.

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