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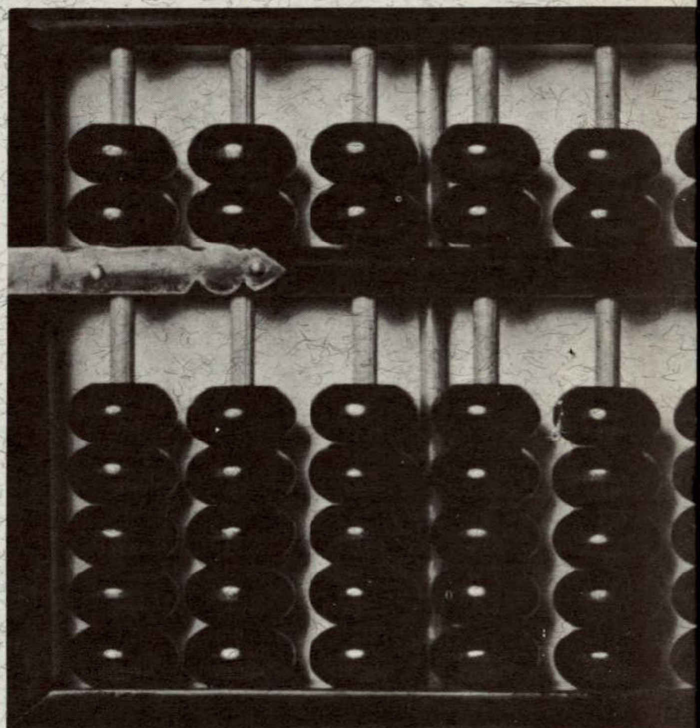

VOLUME 1: USER'S MANUAL

A COMPUTER PROGRAM FOR POWER SPECTRAL ANALYSIS OF UNEQUALLY SPACED POINTS

By

Matthew Lybanon

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ABSTRACT

A power spectral density computer program with some unique capabilities is described. The special features include

1. The spectrum of unequally spaced data may be calculated.
2. The spectrum can be evaluated at any frequencies.
3. Estimates of the parameters of dominant spectral terms can be improved.
4. Prewhitening can be done with no prior knowledge of the characteristics of the signal.

First, the mathematical basis of the conventional computer Fourier analysis technique is discussed, with special attention to the approximations involved. Next, the problem of spectral analysis is considered as a least-squares regression problem. Formulas are developed for continuous and discrete equally spaced data. Next, the interpretation of the power spectral estimates is explored, and the possibility of evaluating the spectrum at any set of frequencies is discussed. Then, the general formulas for unequally spaced points are derived.

After the mathematical formulation comes a description of the computer program implementing the theory. This includes a discussion of the program and its use, a description of the input cards and of the form of the output, and instructions for execution. Appendices include a flowchart and a sample run.

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INTRODUCTION

This report describes a power spectral density computer program which has several features not commonly found in such programs. Perhaps the most unusual feature is its ability to analyze unequally spaced data. In addition, the spectrum can be evaluated at frequencies other than harmonics of the span of data, the maximum frequency at which the spectrum is evaluated can be chosen to be other than one-half the average data sampling rate, and the initial estimates of the frequencies and coefficients of dominant spectral terms can be improved. This last feature in effect permits a reconstruction of the original data as a sum of sinusoidal terms (not a classical Fourier series), which may accurately represent the signal without noise. In addition, it is used for "prewhitening" the spectrum, a procedure recommended by several authors.

CONVENTIONAL SPECTRAL ANALYSIS — THE DISCRETE FOURIER TRANSFORM

The scientist or engineer has an intuitive feeling about what he means by a "spectrum." However, there are certain differences between the Fourier transform and a computer-calculated spectrum. At present, the latter very often is obtained by the fast Fourier transform algorithm [References 1-5], which is an efficient way of calculating the finite or discrete Fourier transform. If $X(j)$, $j = 0, 1, \ldots, N-1$, is a sequence of N complex numbers, the discrete Fourier transform of $X(j)$ is defined

$$A(n) = \frac{1}{N} \sum_{j=0}^{N-1} X(j) \exp(-2\pi i n j/N) \tag{1}$$

$X(j)$ can be expressed as the inverse finite Fourier transform of $A(n)$

$$X(j) = \sum_{n=0}^{N-1} A(n) \exp(2\pi i n j/N) \tag{2}$$

$A(n)$ and $X(j)$ are a transform pair, as can be seen by substituting (1) into (2) and using the orthogonality relationship

$$\sum_{j=0}^{N-1} \exp(2\pi i n j/N) \exp(-2\pi i m j/N) = \begin{cases} N & \text{if } n = m \bmod N \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Let $A(n)$ and $X(j)$ be considered to be defined by (1) and (2) for all integers n and j. Then it follows from the definitions that

$$\left.\begin{aligned} X(j) &= X(kN+j) \\ A(n) &= A(kN+n) \end{aligned}\right\} \quad k = 0, \pm 1, \pm 2, \ldots \tag{4}$$

That is, $X(j)$ and $A(n)$ are both periodic, of period N.

Equations (1) and (2) are simply operations performed on sets of numbers, but form (1) is reminiscent of the Fourier integral transform, or of Fourier series coefficients. The discrete Fourier transform possesses an inverse, (2), it involves functions which satisfy an orthogonality relation, (3). The nature of the relationship between the Fourier transform and Fourier series on the one hand, and the discrete Fourier transform on the other, will now be examined further.

Two pertinent theorems involve the concept of a periodic aliased function. The periodic aliased version $a_p(f)$ of a function $a(f)$ is defined to be

$$a_p(f) = \sum_{k=-\infty}^{\infty} a(f+kF) \tag{5}$$

where F is some interval. It is proved in References 2 and 3 that

Theorem 1 If $X(t)$, $-\infty<t<\infty$, and $a(f)$, $-\infty<f<\infty$, are a Fourier integral transform pair, then $TX_p(j\Delta t) = T\sum_{l=-\infty}^{\infty} X(j\Delta t+lT)$, $j=0,1,\ldots,N-1$, and $a_p(n\Delta f) = \sum_{k=-\infty}^{\infty} a(n\Delta f+kF)$, $n=0,1,\ldots,N-1$, are a finite Fourier transform pair. In these expressions $F = 1/\Delta t = N\Delta f$, $T = N\Delta t = N/N\Delta f = 1/\Delta t$, $FT=N$.

It is also shown in these references that

Theorem 2 If the periodic function $X(t)$ with period T has the Fourier series expansion $c(n)$ (i.e. $c(n)$ are the coefficients in the complex form of the Fourier series, $X(t) = \sum_{n=-\infty}^{\infty} c(n)e^{2\pi i(n+/T)}$), then the periodic sequence $X(j\Delta t)$ of period N, where $\Delta t = T/N$, has the finite Fourier transform

$$c_p(n) = \sum_{l=-\infty}^{\infty} c(n+lN).$$

These two theorems appear to say essentially the same thing. The resemblance is more than apparent, as the following heuristic argument shows. The usual "derivation" of the Fourier transform, by going to the limit in which the number of terms in the Fourier series increases while the frequency spacing decreases, and replacing the sum by an integral, shows that the Fourier series and integral are indeed very similar. The basic difference is that the integral transform maps one nondenumerably infinite number of points into another nondenumerably infinite set of numbers, whereas the Fourier series specifies the function by a denumerably infinite set of numbers. (Not all functions have Fourier series representations, only periodic ones.) In a numerical problem, one has a finite number of data points (containing a finite amount of information, hence an incomplete specification of the physical process), which can be completely specified by another equal-sized set of numbers. Whether these numbers are regarded as estimates of Fourier series coefficients, or of points on the Fourier transform curve, is largely a matter of taste. (This situation is discussed further in the next section.) At various places in the subsequent discussion, whichever supposition is more useful will be used.

The first problem is a consideration of how to apply Theorems 1 and 2 to the calculation of a spectrum when all that one has is a finite-length sample of experimental data, a situation which often arises in practice. If $X(t)$ actually vanishes outside the interval $0 \le t < T$, then $X_p(j\Delta t) = X(j\Delta t)$, $j=0, 1, \ldots, N-1$. At this point it should be noted that it follows directly from (4) that

$$A(-n) = A(N-n) \tag{6}$$

Using this result it can be seen that if the Fourier transform $a(f)$ is nonzero only for $|f| < F/2$ (recall that $a(f)$ is defined for negative as well as positive f, and $|a(-f)| = |a(f)|$ only if $X(t)$ is real), then $a_p(n\Delta f) = a(n\Delta f)$, $n=0, 1, \ldots, N-1$. So in this case, Theorem 1 can be restated

If $X(t)$, non-zero only in the range $0 \le t < T$, and $a(f)$, non-zero over $-F/2 < f < F/2$, are a Fourier integral transform pair, then $TX(j\Delta t)$, $j=0, 1, \ldots, N-1$, and $a(n\Delta f)$, $n=0, 1, \ldots, N-1$, are a finite Fourier transform pair.

For the simple case considered here, the relationship between the Fourier transform and the discrete Fourier transform can easily be found. The Fourier integral transform is

$$a(f) = \int_{-\infty}^{\infty} X(t) \exp(-2\pi i f t)\, dt \tag{7}$$

For $X(t)$ non-zero only in the range $0 \le t < T$,

$$a(f) = \int_{0}^{T} X(t) \exp(-2\pi i f t)\, dt \tag{8}$$

The integral can be approximated by a sum

$$\begin{aligned} a(f) &\approx \sum_{j=0}^{N-1} X(j\Delta t) \exp(-2\pi i f j \Delta t)\, \Delta t \\ &= \frac{1}{N} \sum_{j=0}^{N-1} \left(N\Delta t\right) X(j\Delta t) \exp(-2\pi i f j \Delta t) \end{aligned} \tag{9}$$

For $f=n\Delta f$, $n=0, 1, \ldots, N-1$,

$$a(n\Delta f) \approx \frac{1}{N} \sum_{j=0}^{N-1} (N\Delta t)\, X(j\Delta t) \exp(-2\pi i n\Delta f j\Delta t)$$

Using the relations $T=N\Delta t$, $\Delta f = F/N = 1/N\Delta t$,

$$a(n\Delta f) \approx \frac{1}{N} \sum_{j=0}^{N-1} T\, X(j\Delta t) \exp(-2\pi i n j/N) \tag{10}$$

But by the modified Theorem 1 stated above, the equality in (10) is exact, that is, at the points $f=n\Delta f$, $n=0, 1, \ldots, N-1$, no error is involved in replacing the integral of equation (8) by the sum of equation (9).

If $X(t)$ does not vanish outside the interval $0 \le t < T$, equation (8) is the Fourier transform of the product of $X(t)$ and a square pulse of value unity on $0 \le t < T$, and value zero elsewhere. This transform is the convolution of the transform of $X(t)$ with the transform of the square pulse. The effect of this is to "smear out" each line in the spectrum of $X(t)$.

This case can be viewed in another way. The equality in (10) is not necessarily exact, as was the case for the previous example, because $X(j\Delta t) \neq X_p(j\Delta t)$ (and $a(n\Delta f)$ will not equal $a_p(n\Delta f)$ in an arbitrary case). So there will be some error in representing the Fourier transform by the discrete Fourier transform, in general.

In fact there is always an error if $a(f)$ does not vanish for all $|f| > F/2$. It is convenient for this demonstration to discuss Fourier series (as was stated before, as far as representing them by the discrete Fourier transform is concerned, Fourier series and the Fourier integral are essentially the same thing). The complex form of the Fourier series representation of a function $X(t)$ is

$$X(t) = \frac{1}{2} \sum_{n=-\infty}^{\infty} C_n \exp(2\pi int/T)$$

$$\therefore X(j\Delta t) = \frac{1}{2} \sum_{n=-\infty}^{\infty} C_n \exp(2\pi inj/N), \quad j=0, 1, \ldots, N-1 \tag{11}$$

Comparison with equation (2), using relation (6), shows that the finite Fourier transform gives a finite, i. e. truncated version of the Fourier series representation of a function, limited to frequencies under F/2. Reference 8 thoroughly discusses the error that this causes. Furthermore, the second of equations (4), A(n)=A(kN+n), k=0, ±1, ±2, ...,(and Theorems 1 and 2) implies that if there are frequencies above F/2, the discrete Fourier transform will show these frequencies as occurring below F/2. This is the well known phenomenon of aliasing, the possibility of which always exists when one is dealing with equally spaced data. This result can also be proved in another manner. [7]

At the conclusion of a discussion of aliasing in their important book [9], Blackman and Tukey make the comment "...It is not infrequently suggested that there should be a workable scheme of taking discrete data in some definite, but not uniformly spaced pattern, and estimating the power spectrum without aliasing. No such scheme seems so far to have been developed..." The subject of this report is a method of estimating a power spectrum from unequally spaced samples, in which case indeed there is no aliasing. However, before getting to a discussion of this technique, a few more preliminary remarks will be made.

Blackman and Tukey also discuss the obvious possibility that the length of the sampling interval might not be ideally chosen. For instance, a seemingly aperiodic function might be seen to repeat exactly if it were sampled long enough, or a seemingly periodic function might stop repeating eventually. Of course, it is impossible to know from examining a data sample what might happen if the function were sampled for a longer time, but another example can be given of a possible difficulty related to the size of the data interval. In Reference 7 the function

$$y(t) = B \cos\left[\frac{2\pi(k_o + 1/2)}{T} t\right], \quad 0 \le t < T \tag{12}$$

where k_o is an integer, is considered. The key fact is that the interval does not contain a whole number of cycles of the cosine. The Fourier series coefficients of this function are

$$\begin{aligned} a_k &= 0 \\ b_k &= \frac{B}{\pi}\left[\frac{1}{k+(k_o + 1/2)} + \frac{1}{k-(k_o + 1/2)}\right], \quad k = 1, 2, 3, \ldots \end{aligned} \tag{13}$$

Although the function in closed form is a cosine, the only non-zero Fourier series coefficients are the sine coefficients. But there is no mistake. The (infinite) Fourier series converges to y(t) at every interior point of the interval [10], and if y(t) is sampled at N equally spaced points spanning the interval, the (finite) discrete Fourier transform will specify a trigonometric series passing exactly through the N points.

The paradox is only apparent. Because the frequency of the cosine in (12) is not a harmonic of the basic period T, the Fourier series does not contain any terms of that frequency. The function y(t) may continue outside $0 \le t < T$. However, the Fourier series is periodic with period T, so it does not represent y(t) in the interval $T \le t < 2T$, for instance.

The (conventional, or restricted) sampling theorem summarizes some of the properties and limitations of conventional spectral analysis.[9] References 11 and 12 discuss the generalization of the sampling theorem to the case of unequally spaced samples, and applications to the reconstruction of signals from non-uniform samples. It is pointed out that a generalized sampling theorem was known to Cauchy over a hundred years ago.

CONTINUOUS AND EQUALLY SPACED DATA

Derivation of Formulas

The Fourier integral and Fourier series converge at every point to the functions they represent in many cases, and in every case they minimize the mean square error. Furthermore, the sum of the first M terms of a Fourier series, where M is any positive integer, is the best least-squares fit of a trigonometric sum of that order (involving those frequencies). [10]

The following discussion is presented in greater detail in References 6 and 7. Suppose the problem is approached from the opposite direction, as a regression problem in which it is desired to approximate a function y(t) in an interval $0 \le t < T$ by a trigonometric sum,

$$y_c(t) = \frac{a_0}{2} + \sum_{k=1}^{L} \left(a_k \cos 2\pi kft + b_k \sin 2\pi kft\right), \quad f=1/T \tag{14}$$

Applying the requirement that the mean square error on the interval be a minimum gives the following formulas for the coefficients

$$a_o = \frac{2}{T} \int_0^T y(t)\,dt$$

$$\left.\begin{aligned} a_k &= \frac{2}{T} \int_0^T y(t) \cos 2\pi kft\,dt \\ b_k &= \frac{2}{T} \int_0^T y(t) \sin 2\pi kft\,dt \end{aligned}\right\} \quad k=1, 2, \ldots, L \tag{15}$$

which are the usual expressions for Fourier series coefficients. (The well-known orthogonality relationships make it unnecessary to invert a large matrix to obtain these formulas.)

Suppose we consider only one term of the series in (14),

$$y_k(t) = a_k \cos 2\pi kft + b_k \sin 2\pi kft \tag{16}$$

Let $\epsilon(t) = y(t) - y_k(t)$. Then a straightforward calculation gives

$$\frac{1}{T}\left[\int_0^T y^2(t)dt - \int_0^T \epsilon^2 dt\right] = \frac{1}{2}\left(a_k^2 + b_k^2\right) \tag{17}$$

If y(t) represents voltage, for instance, then $y^2(t)$ is the power dissipated in a unit resistance, and $\epsilon^2(t)$ is the power not contained in the term of frequency kf. Therefore, the left-hand side of (17) is the average power contained in the term (16). This analogy is the justification for plotting $a_k^2 + b_k^2$ against k or f and calling it the power spectrum of y(t).

The case of discrete equally spaced data is very similar. We want to fit a model

$$\hat{y}(n) = \frac{a_o}{2} + \sum_{k=1}^{L}\left(a_k \cos 2\pi kfn + b_k \sin 2\pi kfn\right), \quad f=1/N \tag{18}$$

to a function $y_n = y(n\Delta t)$ which takes on values at n=1, 2, . . . , N (the y_n are though of as samples of a function y(t) of a continuous time variable). To do so we make use of identities (orthogonality relations) completely analogous to those for the continuous case,

$$\begin{aligned}
&\sum_{n=1}^{N} \cos(2\pi kn/N) = \sum_{n=1}^{N} \sin(2\pi kn/N) = 0 \\
&\sum_{n=1}^{N} \cos^2(2\pi kn/N) = \sum_{n=1}^{N} \sin^2(2\pi kn/N) = N/2 \\
&\sum_{n=1}^{N} \cos(2\pi kn/N)\cos(2\pi pn/N) = 0 \quad \text{if } p \neq k \\
&\sum_{n=1}^{N} \sin(2\pi kn/N)\sin(2\pi pn/N) = 0 \quad \text{if } p \neq k \\
&\sum_{n=1}^{N} \sin 2\pi kn/N \cos 2\pi pn/N = 0
\end{aligned} \tag{19}$$

where k and p are integers. Applying these to the problem gives

$$a_0 = \frac{2}{N} \sum_{n=1}^{N} y_n$$

$$\left.\begin{aligned} a_k &= \frac{2}{N} \sum_{n=1}^{N} y_n \cos(2\pi k n/N) \\ b_k &= \frac{2}{N} \sum_{n=1}^{N} y_n \sin(2\pi k n/N) \end{aligned}\right\} k=1, \ldots, L \tag{20}$$

The similarity to (15) should be noted.

At this point it is appropriate to again mention <u>aliasing,</u> which is caused by the fact that $\cos\left[2\pi\left(\frac{N}{2} + \epsilon\right) fn + \phi\right]$ and $\cos\left[2\pi\left(\frac{N}{2} - \epsilon\right) fn - \phi\right]$ are equal for $f = 1/N$ and $n=1, 2, \ldots, N$. So there is a frequency $f_{max} = 1/2\Delta t$ such that frequencies above f_{max} appear to be below f_{max}, possibly with a phase shift. This is often discussed in connection with the sampling theorem, for equally spaced data, it is necessary to sample at a rate of at least twice the highest frequency in the signal to extract all the information. The frequency $f_{max} = 1/2\Delta t$ is known as the <u>Nyquist</u> or <u>folding</u> frequency.

From the point of view of fitting a function to N data points, aliasing is related to the fact that only N independent fit parameters can be found. If N is odd, $(N-1)/2$ is the greatest integer not exceeding $N/2$. If this value is chosen for L in (18), there is one a_0 coefficient, $(N-1)/2$ a_k coefficients, and $(N-1)/2$ b_k coefficients, making N in all If N is even, $N/2$ is an integer. It can be seen from (20) that $b_k = 0$ if $k = N/2$, so with $L = N/2$

there is one a_0, $N/2$ a_k, and $\left(\frac{N}{2} - 1\right)$ b_k, making N all together.

Proceeding exactly as was done in obtaining (17), if $y_k(n) = a_k \cos 2\pi kfn + b_k \sin 2\pi kfn$ and $\epsilon_n = y_n - y_k(n)$, then it is easy to show that

$$\frac{1}{N}\left[\sum_{n=1}^{N} y_n^2 - \sum_{n=1}^{N} \epsilon_n^2\right] = \frac{1}{2}\left(a_k^2 + b_k^2\right) \tag{21}$$

There are seen to be great similarities between the continuous case (Fourier series) and the discrete equally spaced case. Hamming [13], in discussing the use of orthogonal functions, explains that it is often difficult to decide between the two approaches of assuming that one has continuous functions and approximating the integrals giving the coefficients by numerical methods, or using the discrete orthogonal set of functions to fit the discrete samples. But he goes on to point out that in the Fourier series case these two approaches produce exactly the same computations.

Effectiveness of Spectrum Estimates

Before proceeding to the case of unequally spaced discrete data, some additional discussion of the interpretation of the power spectrum will be presented. Equations (17) and (21) are special ways of writing Parseval's theorem. This theorem, in the continuous frequency case (Fourier transform), can be written

$$\int_{-\infty}^{\infty} \left[X(t)\right]^2 dt = \int_{-\infty}^{\infty} |a(f)|^2 \, df \tag{22}$$

where the notation is that of Theorem 1 (X(t) is considered real). In the discrete frequency case (Fourier series),

$$\frac{1}{T}\int_0^T \left[y(t)\right]^2 dt = \frac{1}{2}\left[\frac{a_0^2}{2} + \sum_{k=1}^{\infty}\left(a_k^2 + b_k^2\right)\right] \tag{23}$$

In case one is dealing with discrete equally spaced samples of data,

$$\frac{1}{N}\sum_{n=1}^{N} y_n^2 = \frac{1}{2}\left[\frac{a_0^2}{2} + \sum_{k=1}^{N}\left(a_k^2 + b_k^2\right)\right] \tag{24}$$

The discussion concerning equations (14) - (17) shows that obtaining the Fourier transform of a time series is equivalent to fitting a series of independent functions of the form

$$y_k(t) = a_k \cos 2\pi kft + b_k \sin 2\pi kft \tag{25}$$

where $f = 1/T$, T the length of the data interval. It was pointed out that the actual power in the signal might occur at any frequency. The digitized data to be analyzed in a practical case consists of discrete samples of some continuous process sampled over a time T. This time limit T may be imposed by practical considerations having nothing to do with true periodicities of the signal, the frequencies in the data may not be convenient integral multiples of $1/T$. It would be wise, therefore, to explore the sensitivity of discrete spectrum estimates to the continuum of frequencies which can occur in the signal itself.

For simplicity (25) will be rewritten in the equivalent form

$$y_k(t) = A_k \sin(2\pi kft + \phi) \tag{26}$$

where $A_k = \left(a_k^2 + b_k^2\right)^{\frac{1}{2}}$ and $\phi = \tan^{-1}\left(a_k/b_k\right)$. The absolute phase ϕ will not affect the results to be obtained and will be dropped.

From this point on, in anticipation of certain results, it will not be assumed that $f = 1/T$ necessarily. The integral to be minimized in performing the least squares fit is

$$I = \int_0^T \epsilon^2 \, dt = \int_0^T [y(t) - A_k \sin 2\pi kft]^2 dt \tag{27}$$

The value of A_k which minimizes I is

$$A_k = \frac{\int_0^T y(t) \sin 2\pi kft \, dt}{\int_0^T \sin^2 2\pi kft \, dt} \tag{28}$$

It can easily be shown that, if $F = (k+\delta)/T$, where k is an integer and $|\delta| < 1$,

$$J = \int_0^T \sin^2 2\pi Ft \, dt = \frac{T}{2} - \frac{T \sin 4\pi\delta}{8\pi(k+\delta} = \frac{T}{2}\left[1 - \frac{\sin 4\pi\delta}{4\pi(k+\delta)}\right] \tag{29}$$

If $\delta = 0$ (or $\pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}$), $J = T/2$, a result which was used in obtaining equations (15). For arbitrary values of δ, even when k is a fairly small integer, the second term in (29) is much smaller than the first. So J is seldom very different from $T/2$. Assuming $J = T/2$, equation (28) becomes

$$A_k = \frac{2}{T} \int_0^T y(t) \sin 2\pi kft \, dt \tag{30}$$

Substituting the expression (28) for A_k into (27), the result is

$$\int_0^T \epsilon^2 dt = \int_0^T [y(t) - A_k \sin 2\pi kft]^2 dt$$

$$= \int_0^T y^2(t) \, dt - \left[\int_0^T y(t) \sin 2\pi kft \, dt\right]^2 \Big/ J \tag{31}$$

Then as the discussion following equation (17) shows, the power associated with $y_k(t)$ is

$$P_k = \frac{1}{T}\left[\int_0^T y^2(t)\,dt - \int_0^T \epsilon^2\,dt\right] = \frac{1}{T}\left[\int_0^T y(t)\sin 2\pi kft\,dt\right]^2 \Big/ J \quad (32)$$

Using the value J=T/2,

$$P_k = \frac{2}{T^2}\left[\int_0^T y(t)\sin 2\pi kft\,dt\right]^2 \quad (33)$$

Suppose y(t) contains a component of frequency θ, with relative phase α, so that the contribution of this term to the total signal is

$$y(\theta,\alpha) = C \sin(2\pi\theta t+\alpha) \quad (34)$$

The contribution that this term makes to P_k is

$$P_k(\theta,\alpha) = \frac{C^2}{TJ}\left[\int_0^T \sin 2\pi kft \sin(2\pi\theta t+\alpha)\,dt\right]^2 \quad (35)$$

The expression in brackets gives the functional dependence on θ and α

$$\left[\int_0^T \sin 2\pi kft \sin(2\pi\theta t+\alpha)\,dt\right]^2$$
$$= \tfrac{1}{4}\left\{\int_0^T \cos[2\pi(kf-\theta)t-\alpha]\,dt - \int_0^T \cos[2\pi(kf+\theta)t+\alpha]\,dt\right\}^2$$
$$= \tfrac{1}{4}\left\{\frac{\sin[2\pi(kf-\theta)T-\alpha]+\sin\alpha}{2\pi(kf-\theta)} - \frac{\sin[2\pi(kf+\theta)T+\alpha]-\sin\alpha}{2\pi(kf+\theta)}\right\}^2 \quad (36)$$

$P_k(\theta,\alpha)$ is a weighting function in a sum or integral over θ, to give the power estimate at frequency kf due to contributions from all possible frequencies θ. (A sum over α is also implied.) Unless kf is small, this weighting function will be relatively insignificant except for values of θ near kf. For such values of θ the second term is much smaller than the first. If the second term is dropped,

$$P_k(\theta,\alpha) = \frac{C^2}{4TJ} \left\{\frac{\sin[2\pi(kf-\theta)T-\alpha]+\sin\alpha}{2\pi(kf-\theta)}\right\}^2$$

$$\left\{\sin[2\pi(k-\theta)T-\alpha]+\sin\alpha\right\}^2$$

$$= \left\{\sin[2\pi(kf-\theta)T]\cos\alpha - \cos[2\pi(kf-\theta)T]\sin\alpha+\sin\alpha\right\}^2$$

$$= \left\{2\sin[2\pi(kf-\theta)T/2]\cos[2\pi(kf-\theta)T/2]\cos\alpha + 2\sin^2[2\pi(kf-\theta)T/2]\sin\alpha\right\}^2$$

$$= 4\sin^2[2\pi(kf-\theta)T/2]\left\{\cos[2\pi(kf-\theta)T/2]\cos\alpha + \sin[2\pi(kf-\theta)T/2]\sin\alpha\right\}^2$$

$$= 4\sin^2[2\pi(kf-\theta)T/2]\cos^2[2\pi(kf-\theta)T/2-\alpha]$$

$$P_k(\theta,\alpha) = \frac{C^2}{TJ}\sin^2[2\pi(kf-\theta)T/2]\cos^2[2\pi(kf-\theta)T/2-\alpha]/[2\pi(kf-\theta)]^2 \tag{37}$$

Examination of the exact expression (36) shows that for kf a multiple of $1/T$, $P_k(\theta,\alpha)$ vanishes when θ is also a multiple of $1/T$. This is a manifestation of the orthogonality of the terms in a Fourier series. Equation (37) shows that this property is more general, $P_k(\theta,\alpha)$ vanishes whenever $kf-\theta$ is any integral multiple of $1/T$, regardless of the values of kf and θ (unless kf is small and (37) doesn't hold).

The detailed behavior of $P_k(\theta,\alpha)$ depends on α. For instance, if $\alpha = 0$ equation (37) has zeros when $kf-\theta$ is any multiple of $1/2T$, twice as often as in the general case. For certain values of α (depending on θ and T) $P_k(\theta,\alpha)$ is identically zero. But for an arbitrary signal the contribution at frequency θ may have any phase, so an expression independent of α would be useful. A reasonable estimate for the effect of the contribution at θ on the power estimate at kf is the expected value $P_k(\theta) = E_\alpha\left[P_k(\theta,\alpha)\right]$, where the subscript α denotes that the average is over α. Assuming all values of α are equally likely, this involves

$$\frac{1}{2\pi}\int_0^{2\pi}\left\{\cos\left[2\pi(kf-\theta)T/2\right]\cos\alpha+\sin\left[2\pi(kf-\theta)T/2\right]\sin\alpha\right\}^2 d\alpha = \frac{1}{2} \tag{38}$$

Therefore, the expected value of the contribution which a component of amplitude c, frequency θ, and arbitrary phase makes to the power estimate at frequency kf is

$$P_k(\theta) = \frac{Tc^2}{8J}\left[\frac{\sin 2\pi(kf-\theta)T/2}{2\pi(kf-\theta)T/2}\right]^2 \tag{39}$$

The gain factor is actually independent of T, since J is proportional to T/2.

Equation (39) can be written in a useful form by defining δ,

$$kf - \theta = \delta/T \tag{40}$$

and using the (very good) approximation that J=T/2. Then

$$P_k(\theta) = \frac{c^2}{4}\left[\frac{\sin\pi\delta}{\pi\delta}\right]^2 \tag{41}$$

The foregoing shows that the mathematical method of power spectrum analysis gives essentially the same result as filtering the signal through a bank of filters, each with frequency response given by (41), centered at the frequencies kf, k=1, 2,... The power estimate at kf is the power at the output of the filter centered at kf. The shape (in particular, the width) of the filter is independent of the center frequency. This process of filtering is precisely the analog technique of spectral analysis. [9, 14]

An examination of the function $\left(\frac{\sin\pi\delta}{\pi\delta}\right)^2$ shows that the estimate of power at the frequency kf includes most of the power at frequencies close to kf. The power at frequencies $\pm 1/T$ away is completely filtered out, and frequencies farther away than this make very little contribution. (An extremely large component occurring near the peak of one of the closest side lobes may cause an erroneously high power estimate at kf.) A "worst-case" example is f=1/T, and a component of the signal at (k+1/2)/T. For this case, about 41% of the component's power would be seen at k/T and the same amount at (k+1)/T.

The analysis from the paragraph following equation (25) to this point has shown that one need not be restricted to making power estimates only at the frequencies k/T, k=1,2,... If estimates are made at frequencies closer together than this the neighboring estimates will be correlated, but finer resolution of the spectrum will be obtained. If the spectrum varies slowly with frequency, computer time can be saved by making estimates further apart than 1/T, the correlation between estimates will be slight, if not zero. If there is some advance knowledge of the spectrum (perhaps based on the properties of the physical system producing the signal), f can be chosen equal to a fundamental frequency, possibly far from 1/T. In the case considered in equations (12) and (13) f (or kf, k=2,3...) could be chosen equal to $(k_0 + 1/2)/T$, with results more appealing to intuition.

The approximations used in obtaining equations (37), (39), and (41) are good, except for values of k and f such that kf is of the order of or smaller than 1/T, particularly for $\theta < f$. Physically, this is connected with the fact that a very low frequency must be observed for a very long time to be accurately estimated. Therefore, a constant (bias, d.c. component) may "contaminate" low frequency power estimates and should be subtracted off before spectral analysis.

UNEQUALLY SPACED DATA

The mathematical problem for the unequally spaced case is almost the same as for points equally spaced in time. Formulas will be obtained proceeding by analogy with the equally spaced case, keeping in mind the results of the previous section.

A set of points is given at times t_i with values $y_i = y(t_i)$, $i=1, 2, \ldots, N$, sampled during an interval T. The model to be fit is

$$\hat{y}(t_i) = \frac{a_0}{2} + \sum_{k=1}^{L} \left(a_k \cos 2\pi k f t_i + b_k \sin 2\pi k f t_i \right) \tag{42}$$

where, keeping in mind the discussion of the previous section, no restrictions have been placed on the value of f.

The least squares equations can be formed as was done in obtaining (20). However, in the general unequally spaced case there are no orthogonality relationships like (19), that is,

$$\sum_{i=1}^{N} \cos 2\pi k f t_i \neq 0 \qquad \sum_{i=1}^{N} \sin 2\pi k f t_i \neq 0$$

$$\sum_{i=1}^{N} \cos^2 2\pi k f t_i \neq N/2 \qquad \sum_{i=1}^{N} \sin^2 2\pi k f t_i \neq N/2$$

$$\sum_{i=1}^{N} \cos 2\pi k f t_i \cos 2\pi p f t_i \neq 0 \quad \text{if } p \neq k \tag{43}$$

$$\sum_{i=1}^{N} \sin 2\pi k f t_i \sin 2\pi p f t_i \neq 0 \quad \text{if } p \neq k$$

$$\sum_{i=1}^{N} \sin 2\pi k f t_i \cos 2\pi p f t_i \neq 0$$

even for f=1/T. In other words, the estimates of a_k, b_k are not independent of a_j, b_j for $j \neq k$, and a_k is not independent of b_k.

At this point, it appears that it is necessary to invert a large matrix in order to fit (42) to the data. However, making use of the filter concept of the previous section, it will be assumed that frequencies 1/T apart do not not interfere with one another. (The same assumptions as before concerning changing the frequency spacing will also be made.) Therefore, the problem considered is that of fitting a series of (assumed) independent functions of the form

$$y_k(t_i) = a_k \cos 2\pi kft_i + b_k \sin 2\pi kft_i \tag{44}$$

The least squares normal equations are (dropping subscripts on a and b)

$$\begin{bmatrix} \sum_i \cos^2 2\pi kft_i & \sum_i \sin 2\pi kft_i \cos 2\pi kft_i \\ \sum_i \sin 2\pi kft_i \cos 2\pi kft_i & \sum_i \sin^2 2\pi kft_i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_i y_i \cos 2\pi kft_i \\ \sum_i y_i \sin 2\pi kft_i \end{bmatrix} \tag{45}$$

with solutions

$$a = \frac{\sum_i \sin^2 2\pi kft_i \sum_i y_i \cos 2\pi kft_i - \sum_i \sin 2\pi kft_i \cos 2\pi kft_i \sum_i y_i \sin 2\pi kft_i}{\sum_i \cos^2 2\pi kft_i \sum_i \sin^2 2\pi kft_i - \left(\sum_i \sin 2\pi kft_i \cos 2\pi kft_i\right)^2} \tag{46}$$

$$b = \frac{\sum_i \cos^2 2\pi kft_i \sum_i y_i \sin 2\pi kft_i - \sum_i \sin 2\pi kft_i \cos 2\pi kft_i \sum_i y_i \cos 2\pi kft_i}{\sum_i \cos^2 2\pi kft_i \sum_i \sin^2 2\pi kft_i - \left(\sum_i \sin 2\pi kft_i \cos 2\pi kft_i\right)^2} \tag{47}$$

These are the estimates of the coefficients at a given frequency. To go with them we need an expression for the power at that frequency (in the sense discussed in the previous section). Since spectral analysis is being considered as a least-squares regression problem, the natural measure of the power at frequency kf is the variance $[E(y^2)-E(\epsilon^2)]$, where $\epsilon_i = y(t_i)-y_k(t_i)$, and $y_k(t_i)$ is defined by equation (44). It is shown in Reference 6 that this definition makes possible meaningful estimates of the significance of contributions from specific frequencies. Furthermore, it will be shown that this definition leads to results consistent with equations (17) and (21), which express power as might be expected from an engineering point of view.

Before working out the expression for power, it is convenient to express a and b in slightly different forms

$$a = \frac{\sum_i y_i \cos 2\pi kft_i - \delta \sum_i y_i \sin 2\pi kft_i}{\sum_i \cos^2 2\pi kft_i - \delta \sum_i \sin 2\pi kft_i \cos 2\pi kft_i} \quad (48a)$$

$$\delta = \frac{\sum_i \sin 2\pi kft_i \cos 2\pi kft_i}{\sum_i \sin^2 2\pi kft_i} \quad (48b)$$

$$b = \frac{\sum_i y_i \sin 2\pi kft_i - \Delta \sum_i y_i \cos 2\pi kft_i}{\sum_i \sin^2 2\pi kft_i - \Delta \sum_i \sin 2\pi kft_i \cos 2\pi kft_i} \quad (49a)$$

$$\Delta = \frac{\sum_i \sin 2\pi kft_i \cos 2\pi kft_i}{\sum_i \cos^2 2\pi kft_i} \quad (49b)$$

It should be noted that, for equally spaced data and $f=1/T$, $\delta=\Delta=0$ and equations (48a) and (49a) reduce exactly to equations (20).

Let t_i be considered a random variable satisfying a uniform distribution. Using some trigonometric identities

$$\delta = \frac{\sum_i \sin 2\pi kft_i \cos 2\pi kft_i}{\sum_i \sin^2 2\pi kft_i} = \frac{\frac{1}{2}\sum_i \sin 2\pi(2kf)t_i}{\frac{1}{2}\sum_i \left[1-\cos 2\pi(2kf)t_i\right]} = \frac{\sum_i \sin\theta_i}{\sum_i \left(1-\cos\theta_i\right)} \qquad (50)$$

where now $\theta_i = 2\pi(2kf)t_i$ is regarded as a random variable. The sum $\sum_i \sin\theta_i$ is N times an estimate of the expected value of $\sin\theta$. Likewise, $\sum_i \cos\theta_i =$ $NE(\cos\theta)$. Therefore,

$$\delta = E(\sin\theta)/[1-E(\cos\theta)] \qquad (51)$$

If the interval contains any whole number of cycles, $E(\sin\theta) = E(\cos\theta) = 0$. If the interval is not an integral number of periods in length, $E(\sin\theta)$ and $E(\cos\theta)$ differ from zero by an amount of the order of $1/N$ or less. Therefore, (unless N is a small number) δ is a quantity of order $1/N$. Likewise Δ is of the same magnitude.

Returning now to the expression for power, letting $\omega = 2\pi kf$ for conciseness.

$$\sum_i y_i^2 - \sum_i \epsilon_i^2 = \sum_i y_i^2 - \sum_i \left(y_i - a\cos\omega t_i - b\sin\omega t_i\right)^2$$

$$= 2a\sum_i y_i \cos\omega t_i + 2b\sum_i y_i \sin\omega t_i - a^2\sum_i \cos^2\omega t_i$$

$$- 2ab\sum_i \sin\omega t_i \cos\omega t_i - b^2\sum_i \sin^2\omega t_i \qquad (52)$$

Substituting (48a) - (50),

$$\sum_i y_i^2 - \sum_i \epsilon_i^2 = 2a \sum_i y_i \cos\omega t_i + 2b \sum_i y_i \sin\omega t_i - 2ab \sum_i \sin\omega t_i \cos\omega t_i$$

$$- a \left[\frac{\sum_i y_i \cos\omega t_i - \delta \sum_i y_i \sin\omega t_i}{\sum_i \cos^2 \omega t_i - \delta \sum_i \sin\omega t_i \cos\omega t_i} \right] \sum_i \cos^2 \omega t_i$$

$$- b \left[\frac{\sum_i y_i \sin\omega t_i - \Delta \sum_i y_i \cos\omega t_i}{\sum_i \sin^2 \omega t_i - \Delta \sum_i \sin\omega t_i \cos\omega t_i} \right] \sum_i \sin^2 \omega t_i$$

$$= 2a \sum_i y_i \cos\omega t_i + 2b \sum_i y_i \sin\omega t_i - 2ab \sum_i \sin\omega t_i \cos\omega t_i$$

$$- a \left[\frac{\sum_i y_i \cos\omega t_i - \delta \sum_i y_i \sin\omega t_i}{1 - \delta\Delta} \right]$$

$$- b \left[\frac{\sum_i y_i \sin\omega t_i - \Delta \sum_i y_i \cos\omega t_i}{1 - \delta\Delta} \right]$$

Since both δ and Δ were found to be small quantities, terms of order $\delta\Delta$ will be neglected.

$$\therefore \sum_i y_i^2 - \sum_i y_i^2 \ \epsilon_i^2 = a \sum_i \cos\omega t_i + b \sum_i y_i \sin\omega t_i + a\,\delta \sum_i y_i \sin\omega t_i$$

$$+ b\Delta \sum_i y_i \cos\omega t_i - 2ab \sum_i \sin\omega t_i \cos\omega t_i$$

$$= a \sum_i y_i \cos\omega t_i + b \sum_i y_i \sin\omega t_i - 2ab \sum_i \sin\omega t_i \cos\omega t_i$$

$$+ a \left[\frac{\left(\sum_i \sin\omega t_i \cos\omega t_i\right)\left(\sum_i y_i \sin\omega t_i\right)}{\sum_i \sin^2\omega t_i} \right]$$

$$+ b \left[\frac{\left(\sum_i \sin\omega t_i \cos\omega t_i\right)\left(\sum_i y_i \cos\omega t_i\right)}{\sum_i \cos^2\omega t_i} \right]$$

$$= a \sum_i y_i \cos\omega t_i + b \sum_i y_i \sin\omega t_i$$

$$+ \sum_i \sin\omega t_i \cos\omega t_i \left[\frac{\sum_i y_i \sin\omega t_i}{\sum_i \sin^2\omega t_i} + b \frac{\sum_i y_i \cos\omega t_i}{\sum_i \cos^2\omega t_i} - 2ab \right] \tag{53}$$

After further manipulation, neglecting terms of order $\delta\,\Delta$, the entire last term vanishes, leaving

$$\sum_i y_i^2 - \sum_i \epsilon_i^2 = a \sum_i y_i \cos\omega t_i + b \sum_i y_i \sin\omega t_i \tag{54}$$

For equally spaced data, with f=1/T, (54) becomes identical with (21).

COMPUTER PROGRAM

General Information

The logical structure of the computer program implementing the mathematical method described in the previous sections has three main parts

1. The power spectrum of the data is calculated.
2. Improved estimates of the coefficients and frequencies of dominant spectral terms are obtained.
3. The dominant terms are subtracted from the data (this is a form of prewhitening) and the spectrum of the residuals is calculated.

The user may control the calculations, and specify the form of the output, by means of parameters on input cards. Appendix 1 is a diagram showing the overall logic of the program.

The second step in the above list is done by searching the spectrum ("coarse spectrum") for relative maxima whose powers exceed a certain level and using the coefficients and frequencies of the maxima as initial estimates in a simultaneous least squares solution for the dominant terms. Specifically, the function is represented by

$$\hat{y}(t) = \sum_{i=1}^{M} \left(a_k \sin 2\pi f_k t + b_k \cos 2\pi f_k t\right) \tag{55}$$

in a least squares fit to the data, in which a_k, b_k, and f_k, k=1, . . . , M are allowed to vary. Here $M \leq 10$ is the number of relative maxima found in searching the coarse spectrum. Peaks are chosen in order of size, the largest first. It should be noted that in equation (55) a_k is the coefficient of the sine term and b_k that of the cosine term. This convention, which is opposite to that followed up to now in this report, is the one generally used in the program. Up to ten terms (the number is an input parameter) may be solved for simultaneously. The threshold which must be exceeded for a peak to be selected ("removal tolerance") may either be specified as input or computed by the program.

Parts 2 and 3 in the above list are done in a cyclic fashion until no more peaks can be found which exceed the removal tolerance. Then the removal tolerance is cut in half (once) and the process is continued until no more relative maxima can be found. The process is also terminated in the event that the number of terms removed exceeds 20% of the number of terms in the coarse spectrum.

This search-and-removal process is actually a method of prewhitening, a technique which is recommended for increasing the accuracy of power spectral density analysis. [9, 14] Prewhitening means filtering the data prior to analysis to remove spectral peaks and obtain as smooth a power spectrum as possible. One way to prewhiten data is to filter the signal as it is measured. To do this, of course, one must have some foreknowledge of the frequency content. In this program, the "preprocessing" is done in the course of the computations, and no advance knowledge of the properties of the signal is required. The terms removed, plus the spectrum of the prewhitened signal ("final spectrum"), give the most accurate representation of the frequency content of the signal

The subroutine which improves the estimates of dominant spectral terms uses an iterative linearized least squares formulation, employing Newton's method, to find the coefficients and frequencies of up to ten terms simultaneously There are several conditions besides convergence which will cause a return to the main program·

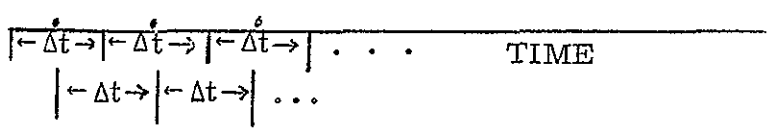
1. If the frequency corrections to one or more terms are more than one-half of the frequency spacing of the coarse spectrum, those terms are tagged as false peaks. Upon return to the main program, the spectrum is searched for the ten largest relative maxima - not including those which are tagged - whose powers exceed the removal tolerance.

2. After the first two iterations the least squares process is checked for convergence (by observing the change in variance). If there is divergence on the nth iteration, the values for the (n-1) st iteration are chosen for the coefficients and frequencies.

3. If the convergence test is not satisfied after 50 iterations, an error message is printed and execution is terminated. The convergence criterion is that the relative change in Δy must be less than 10^{-8} (and Δy must decrease), where Δy is the sum of squares of residuals. The error message includes a list of the old and new values of the fit parameters and Δy.

Reference 6 shows how significance testing can be applied to the spectral powers computed by formula (54). On the program's graphical output, a dotted line appears approximately at a 95% significance level.

Let T(1), ... , T(N) be the times (for simplicity the independent variable will henceforth be called time) associated with the N values of the observed quantity. The program calculates T(N) - T(1) as the duration of the signal. However, in order that the statement following equation (49b) be true, that the general formulas for the coefficients reduce to equations (20) in the case of equally spaced data and f=1/T, [N/(N-1)]x duration must be used for T. This is equivalent to regarding the observations as forming a histogram.



For equally spaced times the time axis is divided into N intervals, each of width Δt (the sampling increment), with the sampling points at the midpoint of each interval. Thus, T must include portions before T(1) and after T(N). The basic frequency f is given in the program by (regardless of the time spacing)

$$f = \frac{1}{S\left(\frac{N}{N-1}\right)[T(N) - T(1)]} \qquad (56)$$

where S is a factor which allows f to be different from 1/T. The spectrum is evaluated at kf, k=1, ... , k_{max}, where

$$k_{max} = RSN/2 \qquad (57)$$

Here R is a factor which determines the maximum frequency at which the spectrum is evaluated, the range factor. When R=1 this maximum is the Nyquist frequency. Because of its role in determining frequency spacing, S is called the spacing factor. When S=1 the frequencies are harmonics of the fundamental period T.

A few comments on the use of unequally spaced data are appropriate. When the times are randomly spaced there is no aliasing. Therefore, components above the Nyquist frequency can be detected. When the times are in some regular pattern (but not equally spaced) there may be some misleading effects. For example, if the times are equally spaced except for regularly occurring gaps of a fixed size, there is an effect something like modulation by a square wave.

Often in using the program, changing one or more of the quantities S, R, the limit on the number of terms in the simultaneous solution (MTERMS), and the removal tolerance and rerunning will yield better results. Using a very large value of S may reveal several peaks so close together that they prevent proper operation of the removal feature with a spacing factor near unity. Setting MTERMS = 1 may help in this instance. A smaller removal tolerance may lead to the discovery of further underlying structure in the signal. It is recommended that the user make several runs with known data to gain familiarity with the operation of the program.

An option is available to smooth the spectrum estimates with low-pass convolution filters of up to 13 points. This feature is not currently activated, but the FORTRAN statements are in the program as comments. To activate this feature the "C" in column 1 should be removed.

The following sections contain a description of the system of programs and its use. Appendix 1 contains a flowchart showing the overall logic of the program. Appendix 2 is a sample run. A listing of the program which generated the (unequally spaced) data is given, along with the output. All of the output options are exercised. Values of S and R different from 1.0 are used. Removal is done, with the program calculating the removal tolerance.

For more information on the program the user is referred to Volume 2 of this report, the Maintenance Manual.

Functions of Programs in System

CUCPSA (main program)
- Reads program control cards
- Reads data cards (or calls tape read subroutine)
- Calculates mean and a.c. power of total signal
- Calculates coefficients, power, and percent power of spectral terms.
- Prints power spectrum
- Calls two subroutines (GRAPH and RPLOT) to plot power spectrum
- Searches "coarse" spectrum for peaks.
- Calls subroutine to solve for coefficients and frequencies of dominant terms.
- Removes dominant spectral terms.

GRAPH
- Lists power and plots (on printer) percent power vs. frequency, 53-58 points/page. A dotted line is drawn approximately at a 95% significance level.

RPLOT
- Plots the spectrum on one page, with a significance indicator.

READ
- Reads data from tape.

CAROL3

Performs simultaneous least squares fit of up to ten sinusoidal functions, solving for coefficients and frequencies.

BIORTH

Matrix inversion routine called by CAROL3.

Input Cards

There are two types of input cards program control cards and data cards. The latter may be replaced by a tape.

A description of the input cards follows. Everything pertaining to the spectrum smoothing feature is given in brackets. It should be noted that if the smoothing feature is not activated, card 4 is not used and there are four program control cards per case.

Card 1. Format Card

1 – 80
(13A6, A2)

Observation card format. This is a FORTRAN FORMAT statement without the statement label and the word "FORMAT." It includes the beginning and ending parentheses. It must specify 3 double precision fields. This card should be left blank if the data are to be read from tape.

Card 2. Observation Definition Card

1 – 5	6 – 10	11 – 15	16 – 25	26 – 35
(I5)	(I5)	(I5)	(F10.5)	(F10.5)
Field number for observation (IY)	Field number for time (IT)	Field number for day (ID)	Time units per computational unit (TPC)	Computational units per day (CPD)

NOTE "Field" means fields on observation cards as given on the Format Card.

$$t = T/TPC + (D - D_o)\ CPD$$

where t - time used in program

T - time on observation card

TPC - time scale factor

D - day on observation card

D_o - day on 1st observation card

CPD - day scale factor

For example, if the program is to compute in minutes and the times on the observation cards are in seconds and whole days, then TPC=60 and CPD=1440

If the data are on tape rather than cards there is no "day" variable. ID and CPD are ignored by the program. Subroutine READ assumes that the tape contains a file consisting of N (the number of points) records, each containing 7 (or more) single-precision words. The tape should be binary (unformatted). IT tells which of the 7 words is time, IY identifies the observed quantity. Time is calculated by $t = T/TPC$.

Card 3. Program Option Card

Columns	Format	Field
1–18	(3A6)	Data set name (NAME)
19–23	(I5)	Number of observations (N)
24–33	(F10.5)	Spacing factor (S)
34–43	(F10.5)	Range factor (R)
44	(I1)	Plot flag (JPO)
45	(I1)	Removal flag (JF)
46–55	(F10.5)	Removal tolerance (TOL)
56	(I1)	Print flag (JS)

NAME identifies data

N is number of observations (limited to 2000)

S is spacing factor in equation (56)

R is range factor in equation (57)

JPO = 0 - program plots rough [and smooth] power spectrum

= 1 - plots rough power spectrum only

= 2 - [plots smooth power spectrum only]

= 3 - no plots

JF = 0 - program removes terms whose percent powers exceed TOL

= 1 - no removal

TOL is calculated by program if not inputted

JS = 0 - program prints power spectrum

= 1 - program does not print spectrum

NOTES The default value of TOL is the same as the 95% significance level (shown on the graphs by a dotted line). However, in this context it should not be regarded as a 95% significance level.

Combinations of N, S, and R which would produce more than 1000 spectral terms are not allowed. In such a case, the program will compute the first 1000 terms.

TOL is a fraction which is used to compute a power level, that fraction of the total power of the signal. After removal of terms that level is not re-evaluated as a fraction of the remaining power, except for being halved once as described previously.

[Card 4. Spectrum Smoothing Filter Card

1–5	6 →
(I5)	(7 F 10. 5)
Number of filter factors on card (LEND)	Filter factors beginning with the center factor and proceeding to the last factor. The filter used by the program will be symmetrical, with 2xLEND-1 points. (FILTER)

This card cannot be used unless the smoothing statements are activated and $S \neq 1$. If S=1 the values 1/4, 1/2, 1/4 ("Hanning") will be used for the filter.]

Card 5. Simultaneous Removal Card

1 — 5
(I 5)

Maximum number of terms solved for simultaneously in fine fit (MTERMS)

NOTE MTERMS may not exceed 10.

These cards are followed by N observation cards, in the format specified on the Format Card, unless the data are on tape. Any number of sets of data may be analyzed sequentially by stacking the input cards to form one large data deck. After the last case, two more cards are needed to terminate execution The first may be blank. The second should have -999 punched in columns 2-5.

Output

The first page of output for each case gives a list of input quantities. The first group on this page lists, in order, NAME, N, S, R, JPO, JF, TOL, JS, and MTERMS, one per line. [If the spectrum smoothing feature is activated, the next group is a list of the filter factors.] The final group consists of FMT, IY, IT, ID, TPC, CPD, and DUR=T(N)-T(1), also one per line.

On the next page the title

INITIAL DATA SPECTRUM FOR SET {NAME} (TOTAL POWER = {YYBAR2})

is printed. Here {X} means "the value of the quantity X." YYBAR2 is the total power, the sum of squares of the observations (after subtracting the mean) divided by N/2. This normalization for power is somewhat arbitrary, it amounts to a choice of the unit of power. The same normalization is used for the terms in the power spectrum, equation (54) is divided by N/2.

If the print option is used (JS=0), this is followed by a tabulation of the spectrum, 50-52 lines per page. The quantity C given at the top of each page is the mean of the observations. The first of the seven columns lists the frequency index k, which goes from 1 to k_{max} (equation (57)). The second column gives the period, the reciprocal of the frequency of the kth term. The third column is ω, 2π x frequency. The fourth and fifth columns list a_k , the coefficient of the sine term (called b in equation (47)), and b_k , the coefficient of the cosine term (equation (46)). Once again attention is called to the reversal of the usual naming convention for the coefficients. The next column lists the regression power of each term, equation (52) divided by N/2. The last column gives percent (really fractional) power, the preceding quantity divided by the total power YYBAR2.

Depending on whether the spectrum smoothing feature is activated, and on what plot option is specified, plots (on the printer) may be made of the raw spectrum, the smoothed spectrum, both, or neither. If plots of both the raw and smoothed spectrum are requested, the former appear first. Plots are made in pairs The first is on a single page, power vs. k. The second may occupy several pages, with the first 53 points on the first page and 58 on each succeeding page. The expanded plot shows percent power vs. k, with the k axis running down the page. To the left of the plot are two columns of figures giving the regression power and frequency corresponding to each value of k.

Automatic scale selection is done for each plot, and on both plots a dotted line shows an approximate 95% significance level.

After plotting the spectrum, the program may be finished with that case. However, if the search-and-removal option is specified the title

REMOVAL TABLE FOR SET {NAME},

column headings, and the first line of the removal table are printed This first line consists only of YYBAR2, the total power, under the heading "REMAINING TOTAL POWER."

The remainder of the removal table lists the results, if any, of the simultaneous least squares solution for dominant terms. The first five columns, in order, list the period, frequency, angular frequency (2 x frequency), and the sine and cosine coefficients of the terms in the solution. The column headed "SPECTRUM POWER" gives the regression power of each term. On the last line of a group of terms solved for simultaneously, the column headed "REMAINING TOTAL POWER" gives the power of the residuals after subtracting the solution terms from the original data. On every line of this group, the column headed "C" lists the mean of these residuals. There may be several such groups of terms. After each subtraction the residuals are treated as data, the spectrum is calculated and searched for peaks as before. When no more valid peaks exceeding the removal tolerance can be found, TOL is cut in half and the message "SEARCH WILL NOW BE MADE WITH SMALLER TOL" is printed Then the search-and-removal procedure continues until no more peaks are found, and the values found are printed in the removal table.

If at least one term is found and removed from the data, the spectrum of the residuals is printed and/or plotted according to the output options specified The form of this output is just the same as for the spectrum of the original data, except that the heading of the tabulation specifies that it is the "FINAL DATA SPECTRUM."

Executing the Program

The program is on the FASTRAND file LSQFIT*MLTSAN. The name of the absolute element is LSQFIT*MLTSAN.APSD. If input is from tape rather than cards, the tape is considered to be on logical unit 3. If it is necessary to reMAP the program for some reason, the MAP source element LSQFIT*MLTSAN.PSAMAP may be used.

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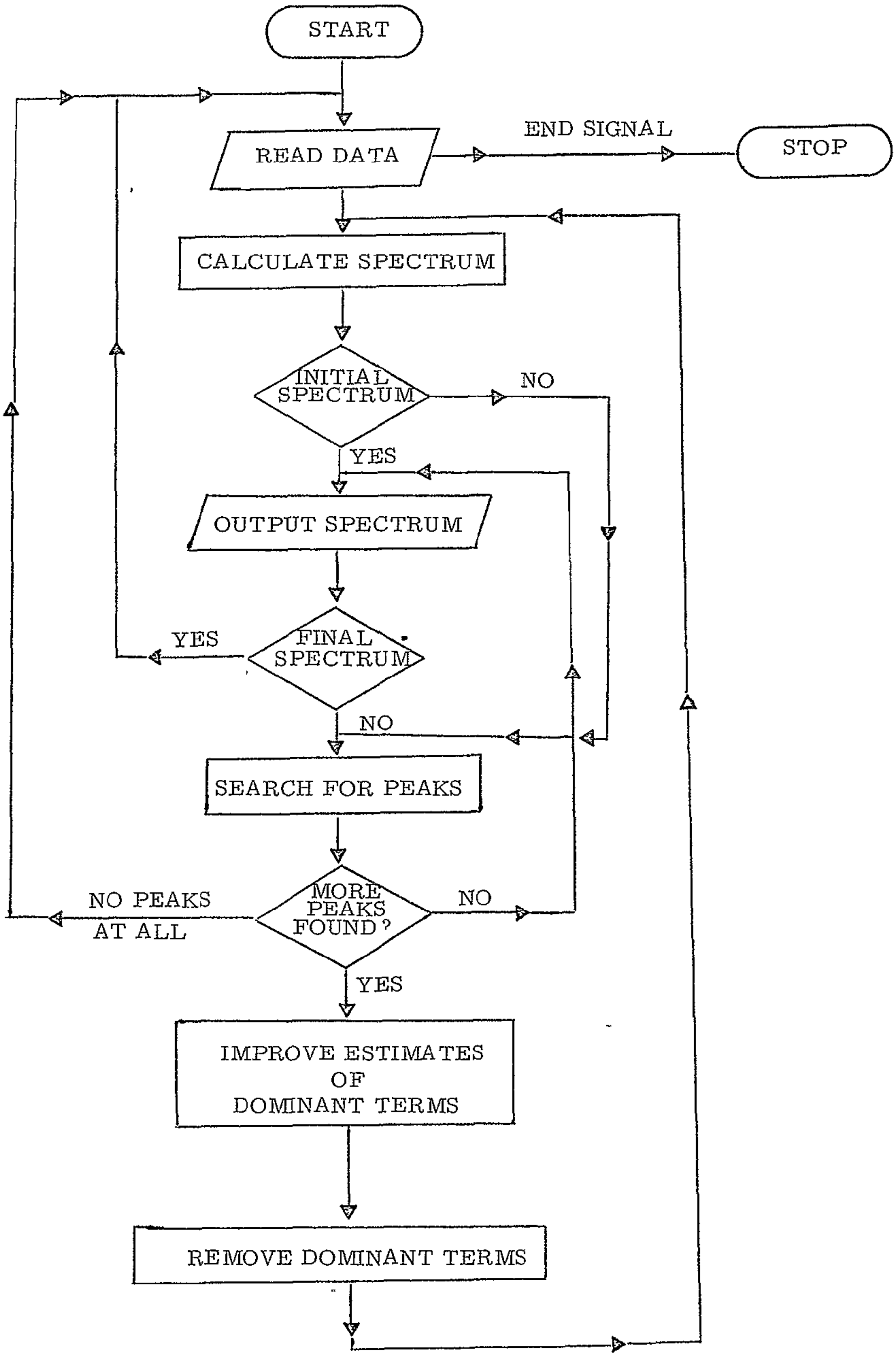
APPENDIX 1

Overall Logic Flowchart


START
READ DATA
END SIGNAL
STOP
CALCULATE SPECTRUM
INITIAL SPECTRUM
NO
YES
OUTPUT SPECTRUM
FINAL SPECTRUM
YES
NO
SEARCH FOR PEAKS
MORE PEAKS FOUND ?
NO PEAKS AT ALL
NO
YES
IMPROVE ESTIMATES OF DOMINANT TERMS
REMOVE DOMINANT TERMS

APPENDIX 2

Sample Run

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@       RUN      MXL,D01,SPANAL,10,50
@       HDG      GENERATE RANDOMLY SPACED DATA FOR SPECTRUM PROGRAM
@       FOR,IS   RANDSD,RANDSD
C **  PROGRAM TO GENERATE KNOWN DATA FOR SPECTRAL ANALYSIS PROGRAM
C **
C **                             M
C **  Y = YBAR + SLOPE*T + SUM(A(I)*SIN(2*PI*F(I)*T)
C **                            I=1
C **
C **                                 + B(I)*COS(2*PI*F(I)*T))
C **
      DIMENSION A(10),B(10),F(10),X(10),RANDNO(1000)
C **
C **  TIMES ARE RANDOMLY SPACED OVER (T0,T0 + 1.0).
C **  THE PSEUDO - NYQUIST FREQUENCY IS (N - 1) / 2 CPS.
C **
 1000 FORMAT (1H1,21X,I2/72H Y = YBAR + SLOPE*T + SUM(A(I)*SIN(2*PI*F(I)
     .*T) + B(I)*COS(2*PI*F(I)*T))/22X3HI=1//10X7HYBAR = ,F6.3,10X8HSLOP
     .E = ,F6.3//8X1HF,10X1HA,10X1HB/ (3(5XF6.3)/))
 1001 FORMAT (5XF5.2,5XE13.8)
 1002 FORMAT (3E20.8)
 1004 FORMAT (F10.0,I5)
 1005 FORMAT (I5)
 1006 FORMAT (8X1HT,13X1HY)
      READ (5,1002) YBAR,SLOPE
      READ (5,1005) M                    @    NUMBER OF FREQUENCIES
      READ (5,1002) (F(I),A(I),B(I),I=1,M)
      WRITE (6,1000) M,YBAR,SLOPE,(F(I),A(I),B(I),I=1,M)
      WRITE (6,1006)
C **
C **  T0 IS THE INITIAL TIME.
C **  N IS THE NUMBER OF POINTS.
      READ (5,1004) T0,N
      RANDNO(1) = 6.2831853076
      NM2 = N - 2
      CALL RANDU (RANDNO,NM2)            @    GENERATE ARRAY OF RANDOM NUMBERS.
      T = T0
      NM = 0
      GO TO 30
   10 CONTINUE
      NM = NM + 1
      T = T0 + RANDNO(NM)
      GO TO 30
   20 CONTINUE
      NM = NM + 1
      T = T0 + 1.0
   30 CONTINUE
```

```
      BIAS=YBAR+SLOPE*T
      Y=BIAS
      DO 50 J=1,M
      X(J)=6.2831853076*F(J)*T
   50 Y=Y+A(J)*SIN(X(J))+B(J)*COS(X(J))
      PRINT 1001,T,Y
      PUNCH 1002,T,Y
      IF (NM - NM2) 10,20,40
   40 CONTINUE
      STOP
      END
@       XQT
                     5.                    0.
    5
                    36.                    3.                    5.
                   158.                    9.                    6.
                   220.                    3.                    0.
                   250.                    0.                    5.
                   281.                    7.                    2.
        0.   500
@ FIN
@@
@@
```

SAMPLE SPECTRUM PROGRAM RUN - UNEQUALLY SPACED DATA

```
SET NAME    = RANDOM TIME 310568
NO. POINTS  =    500
SPACING     = .500000000
RANGE       = 1.30000000
PLOT FLAG   =      0
REMOVAL     =      0
TOLERANCE   =      .02000
PRINT FLAG  =      0
MTERMS      =     10

FORMAT      = (3D20.0)
Y FIELD     =      2
T FIELD     =      1
DAY FIELD   =      3
TIME SCALE  =  1.00000000    TIME UNITS PER COMPUTATIONAL UNIT
DAY SCALE   =  .000000000    COMPUTATIONAL UNITS PER DAY

DURATION    =      1.0000 COMPUTATIONAL UNITS
```

SAMPLE SPECTRUM PROGRAM RUN - UNEQUALLY SPACED DATA

INITIAL DATA SPECTRUM FOR SET RANDOM TIME 310568 (TOTAL POWER = .24051+03)

K	PERIOD	OMEGA	A	B	REGRESSION POWER	PERCENT POWER
1	.5010	12.541238	.38427+00	.19315-01	.149625	.000622
2	.2505	25.082476	.13933+00	.78377+00	.635199	.002641
3	.1670	37.623714	.38234+00	-.19485+00	.181005	.000753
4	.1253	50.164951	.22804+00	-.10337+01	1.15615	.004807
5	.1002	62.706189	-.14735+00	.54906+00	.323728	.001346
6	.0835	75.247427	.71437+00	-.49764-01	.531742	.002211
7	.0716	87.788665	.86443+00	.65185+00	1.16105	.004827
8	.0626	100.329903	.51587+00	.67216+00	.702510	.002921
9	.0557	112.871141	.26782+00	-.34871+00	.185700	.000772
10	.0501	125.412379	.54781+00	-.26621+00	.364523	.001516
11	.0455	137.953617	.60625+00	.28622+00	.457869	.001904
12	.0418	150.494854	.24629+00	.57047+00	.380864	.001584
13	.0385	163.036092	.12736+00	.69258+00	.504103	.002096
14	.0358	175.577330	.29731+00	.14428+00	.109238	.000454
15	.0334	188.118568	-.49089+00	.13375+00	.248718	.001034
16	.0313	200.659806	.35153+00	.25307+00	.185071	.000769
17	.0295	213.201044	.80349+00	.57537+00	.984454	.004093
18	.0278	225.742282	.12964+01	.56823+01	33.9198	.141031
19	.0264	238.283520	-.49421+00	-.86820+00	.960870	.003995
20	.0251	250.824757	-.14717+00	-.39713+00	.175213	.000728
21	.0239	263.365995	.79963+00	-.10048+01	1.69353	.007041
22	.0228	275.907233	-.10662+01	.17452+00	1.20290	.005001
23	.0218	288.448471	.31171+00	-.31962+00	.196311	.000816
24	.0209	300.989709	.21072+00	-.26428+00	.116106	.000483
25	.0200	313.530947	-.14969+01	-.17975+00	2.31140	.009610
26	.0193	326.072185	-.54436+00	-.10201+01	1.34717	.005601
27	.0186	338.613423	-.41553+00	-.78994-03	.171249	.000712
28	.0179	351.154660	.68447+00	.15912+01	2.95500	.012286
29	.0173	363.695898	.37035+00	-.25410+00	.202355	.000841
30	.0167	376.237136	.38030+00	.61740+00	.529262	.002201
31	.0162	388.778374	-.11352+01	.15919+00	1.23824	.005148
32	.0157	401.319612	.25589+00	-.95412-01	.758469-01	.000315
33	.0152	413.860850	-.91126+00	.75881+00	1.38307	.005751
34	.0147	426.402088	.73870+00	.77695+00	1.16997	.004864
35	.0143	438.943326	-.47313+00	-.10268+01	1.27766	.005312
36	.0139	451.484563	.10143+01	-.82577-01	1.01926	.004238
37	.0135	464.025801	.16536+01	-.64477+00	3.04091	.012643
38	.0132	476.567039	.52508-01	.50513+00	.257990	.001073
39	.0128	489.108277	-.29884+00	-.47652+00	.308960	.001285
40	.0125	501.649515	-.83427+00	-.10728+01	1.85894	.007729
41	.0122	514.190753	.76261+00	.20013-01	.600436	.002496
42	.0119	526.731991	-.12241+00	.65674+00	.432352	.001798
43	.0117	539.273229	.35417+00	-.28731+00	.205242	.000853
44	.0114	551.814466	-.12447+01	.22179+00	1.72498	.007172
45	.0111	564.355704	-.40730+00	.38661+00	.299123	.001244
46	.0109	576.896942	-.38045+00	-.65127+00	.759659	.003158
47	.0107	589.438180	.87716+00	-.56095+00	1.00642	.004184
48	.0104	601.979418	-.85103+00	.87704+00	1.49809	.006229
49	.0102	614.520656	-.66880+00	.36267+00	.571532	.002376
50	.0100	627.061894	.22249+00	-.78249+00	.645434	.002684

C = .44449+01

SAMPLE SPECTRUM PROGRAM RUN - UNEQUALLY SPACED DATA

C = .44449+01

K	PERIOD	OMEGA	A	B	REGRESSION POWER	PERCENT POWER
51	.0098	639.603132	-.94956-01	.10066+01	1.02168	.004248
52	.0096	652.144369	.26399+00	.93334+00	.941328	.003914
53	.0095	664.685607	-.56910+00	.66005+00	.796572	.003312
54	.0093	677.226845	-.11276+01	.11309+01	2.53085	.010523
55	.0091	689.768083	-.10431+01	.54313+00	1.47515	.006133
56	.0089	702.309321	-.95657+00	.27778+00	.977725	.004065
57	.0088	714.850559	.42312+00	-.62821+00	.575362	.002392
58	.0086	727.391797	.45594+00	.54623+00	.504476	.002097
59	.0085	739.933035	-.85177+00	.25604+00	.784240	.003261
60	.0084	752.474272	-.98026+00	.45258+00	1.11401	.004632
61	.0082	765.015510	.47677+00	.18912+00	.267772	.001113
62	.0081	777.556748	.26826+00	-.42422+00	.251261	.001045
63	.0080	790.097986	.10300+01	.66934+00	1.48658	.006181
64	.0078	802.639224	.51498+00	.89129-01	.266735	.001109
65	.0077	815.180462	-.22314+00	-.21640+00	.952897-01	.000396
66	.0076	827.721700	-.57385+00	.92601+00	1.21919	.005069
67	.0075	840.262938	.56949+00	.45505+00	.542610	.002256
68	.0074	852.804175	.43582+00	-.74372+00	.744540	.003096
69	.0073	865.345413	.15415+00	.40925-01	.252330-01	.000105
70	.0072	877.886651	-.83827+00	.21704+00	.767279	.003190
71	.0071	890.427889	-.33299+00	.19359+00	.148060	.000616
72	.0070	902.969127	.11274+01	-.28067+00	1.41627	.005889
73	.0069	915.510365	.48744+00	.38295+00	.387168	.001610
74	.0068	928.051603	-.15094+01	.55817-01	2.31363	.009620
75	.0067	940.592841	-.46897+00	.31220+00	.309903	.001289
76	.0066	953.134078	.97677+00	.14951+01	3.08130	.012811
77	.0065	965.675316	-.66324+00	.66178+00	.883476	.003673
78	.0064	978.216554	.43452-01	.10478+01	1.11218	.004624
79	.0063	990.757792	-.44950+00	.92348+01	80.8693	.336237
80	.0063	1003.299030	-.52987+00	-.16084+01	3.00653	.012500
81	.0062	1015.840268	-.14327+00	-.62794+00	.422924	.001758
82	.0061	1028.381506	.85875+00	-.10720+01	1.95762	.008139
83	.0060	1040.922744	-.77599+00	.79756+00	1.20470	.005009
84	.0060	1053.463981	-.37793+00	.11232+00	.150910	.000627
85	.0059	1066.005219	-.70092+00	-.13981+00	.518351	.002155
86	.0058	1078.546457	.50951+00	-.43658+00	.439145	.001826
87	.0058	1091.087695	.57415-02	.72225+00	.555543	.002310
88	.0057	1103.628933	-.39169+00	-.11261+00	.173927	.000723
89	.0056	1116.170171	.89217+00	-.14781+00	.840841	.003496
90	.0056	1128.711409	-.63873+00	-.53326-01	.401949	.001671
91	.0055	1141.252647	-.53876+00	-.35865+00	.402116	.001672
92	.0054	1153.793884	.43258+00	-.19874+01	4.23148	.017594
93	.0054	1166.335122	-.35796+00	-.97009+00	1.09472	.004552
94	.0053	1178.876360	.71831-01	-.21185-02	.524018-02	.000022
95	.0053	1191.417598	-.14378-01	.35002+00	.119684	.000498
96	.0052	1203.958836	-.38674+00	-.98856+00	1.15680	.004810
97	.0052	1216.500074	.82687+00	-.45741+00	.865888	.003600
98	.0051	1229.041312	-.86982+00	-.17566+00	.809704	.003367
99	.0051	1241.582550	-.34946-01	-.27344+00	.776204-01	.000323
100	.0050	1254.123787	-.48395+00	-.10364+01	1.37539	.005719
101	.0050	1266.665025	-.88731+00	-.26013+00	.910934	.003787
102	.0049	1279.206263	-.34879+00	.74295+00	.669674	.002784

SAMPLE SPECTRUM PROGRAM RUN -- UNEQUALLY SPACED DATA

K	PERIOD	OMEGA	A	B	REGRESSION POWER	PERCENT POWER
103	.0049	1291.747501	-.24400+00	.84841+00	.767720	.003192
104	.0048	1304.288739	-.24012+00	.13456+01	1.88536	.007839
105	.0048	1316.829977	-.29756+00	.92211+00	.946866	.003937
106	.0047	1329.371215	.59488+00	-.35728-01	.345136	.001435
107	.0047	1341.912453	.73795-01	.75390+00	.563381	.002342
108	.0046	1354.453690	.33153+00	-.74528+00	.638448	.002655
109	.0046	1366.994928	-.14302+00	.13249+01	1.77845	.007394
110	.0046	1379.536166	.51714+00	.31820+01	9.88808	.041112
111	.0045	1392.077404	-.40759+00	-.19174+01	3.97349	.016521
112	.0045	1404.618642	-.41319+00	.28124-01	.167113	.000695
113	.0044	1417.159880	.59903+00	.73706+00	.924658	.003845
114	.0044	1429.701118	.27838-01	.12339+01	1.57934	.006567
115	.0044	1442.242356	.31064+00	.40138+00	.261149	.001086
116	.0043	1454.783593	-.83642+00	-.55767+00	.982655	.004086
117	.0043	1467.324831	-.17758+00	-.61605+00	.426968	.001775
118	.0042	1479.866069	.85080+00	-.81191+00	1.37112	.005701
119	.0042	1492.407307	-.71873+00	.10616+01	1.75754	.007307
120	.0042	1504.948545	-.46421+00	.18446+00	.240911	.001002
121	.0041	1517.489783	.13792+00	.12885+01	1.72020	.007152
122	.0041	1530.031021	-.79100+00	-.27901+00	.686745	.002855
123	.0041	1542.572258	.50355+00	-.47417+00	.469223	.001951
124	.0040	1555.113496	-.78053+00	.30043+00	.662731	.002755
125	.0040	1567.654734	-.27506+01	.72037+00	8.01840	.033339
126	.0040	1580.195972	.15665+01	-.65317+00	3.01800	.012548
127	.0039	1592.737210	.22458+00	.13651+01	1.92079	.007986
128	.0039	1605.278448	.31494+00	.65564+00	.542729	.002257
129	.0039	1617.819686	-.65080+00	-.53412-01	.415851	.001729
130	.0039	1630.360924	.38081+00	-.29434+00	.240958	.001002
131	.0038	1642.902161	-.67699+00	-.80907+00	1.09074	.004535
132	.0038	1655.443399	.51012+00	.27073+00	.353060	.001468
133	.0038	1667.984637	.15230+01	.69721+00	3.06986	.012764
134	.0037	1680.525875	-.15742+00	.30219+00	.114383	.000476
135	.0037	1693.067113	-.22318+00	.89068+00	.832057	.003460
136	.0037	1705.608351	.14442-01	-.29624+00	.875075-01	.000364
137	.0037	1718.149589	.67508+00	-.16723+00	.492132	.002046
138	.0036	1730.690827	.10075+01	.19629+01	5.05962	.021037
139	.0036	1743.232064	-.76118+00	.69200+00	1.07396	.004465
140	.0036	1755.773302	.38359-01	.15876+01	2.52726	.010508
141	.0036	1768.314540	.19750+01	-.48866+01	26.8603	.111679
142	.0035	1780.855778	.81933+00	-.91042+00	1.52042	.006322
143	.0035	1793.397016	-.74828+00	.85166-01	.593278	.002467
144	.0035	1805.938254	-.26801+00	-.45227+00	.279174	.001161
145	.0035	1818.479492	.83221+00	.17848+00	.719649	.002992
146	.0034	1831.020730	.66772+00	-.28876+00	.526707	.002190
147	.0034	1843.561967	-.95728+00	-.56962+00	1.18280	.004918
148	.0034	1856.103205	.53216+00	.23699+00	.326881	.001359
149	.0034	1868.644443	.13840+01	.74113-01	1.86596	.007758
150	.0033	1881.185681	-.85560+00	.51725+00	.971919	.004041
151	.0033	1893.726919	-.43287+00	.89613+00	.981018	.004079
152	.0033	1906.268157	.49291+00	-.11639+01	1.60112	.006657
153	.0033	1918.809395	-.11164+00	-.11084+00	.243027-01	.000101
154	.0033	1931.350633	-.30456+00	-.23951+00	.147113	.000612

C = .44449+01

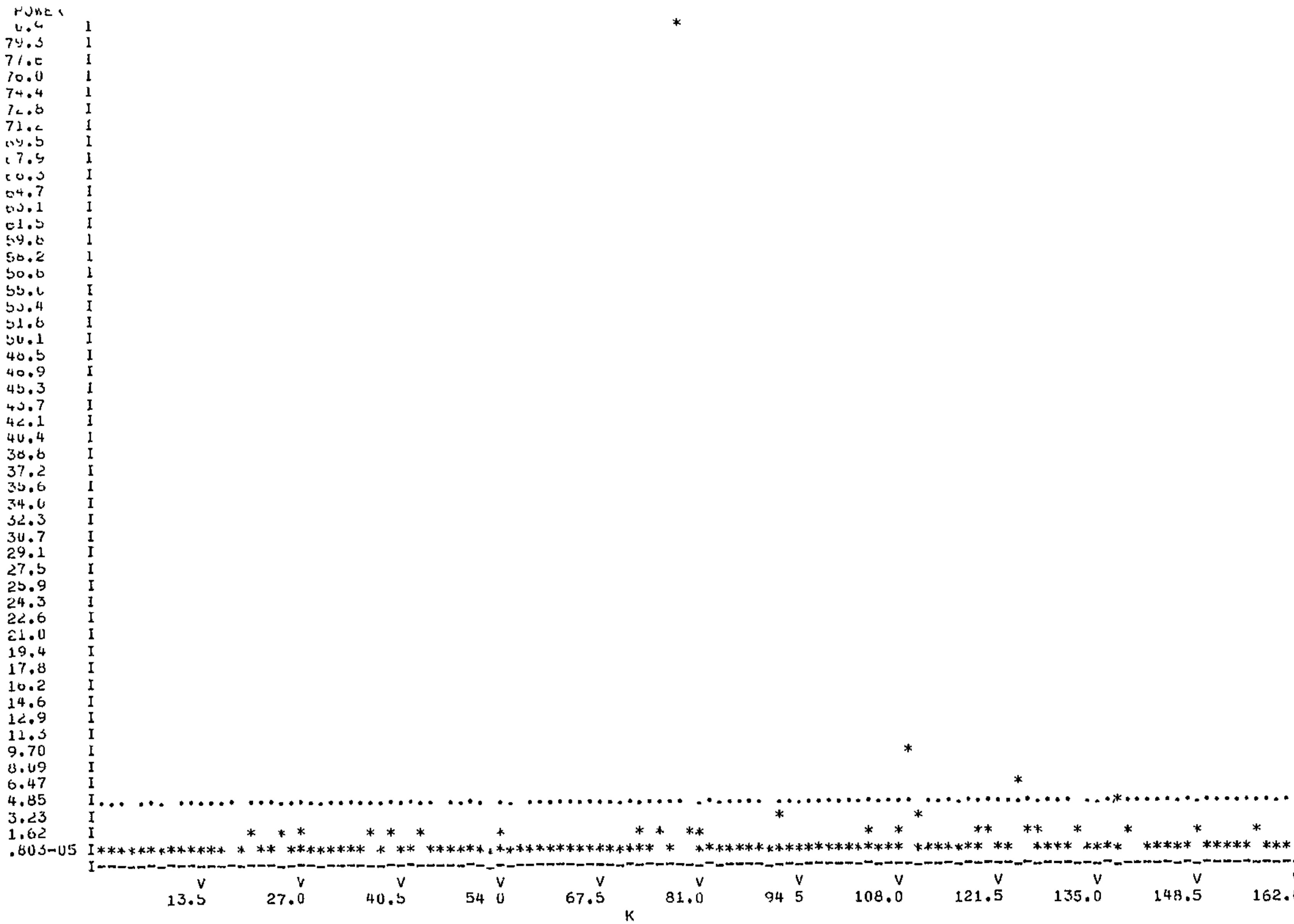
SAMPLE SPECTRUM PROGRAM RUN - UNEQUALLY SPACED DATA

K	PERIOD	OMEGA	A	B	REGRESSION POWER	PERCENT POWER
155	.0032	1943.891870	.86103-01	-.43136+00	.188547	.000784
156	.0032	1956.433108	-.87079-01	.39746+00	.166493	.000692
157	.0032	1968.974346	.69862+00	-.12843+01	2.24692	.009342
158	.0032	1981.515584	.58080+00	-.21431+00	.382052	.001588
159	.0032	1994.056822	-.17789+00	.17584+00	.634460-01	.000264
160	.0031	2006.598060	-.39421+00	-.14294+00	.175356	.000729
161	.0031	2019.139298	.10091+00	-.22574+00	.583952-01	.000243
162	.0031	2031.680536	.45036-01	-.46554+00	.214520	.000892

C = .44449+01

SIMPLE SPECTRUM PROGRAM RUN - UNEQUALLY SPACED DATA

RANDOM TIME 310500 P VS K



SAMPLE SPECTRUM PROGRAM RUN - UNEQUALLY SPACED DATA

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RANDOM TIME 310568 P VS K                                   PERCENT POWER

                                  0    2    4    6    8    10   12   14   16   18   20   22   24   26   28   30   32   34   36   38   40
    POWER        FREQUENCY     K  V    V    V    V    V    V    V    V    V    V    V    V    V    V    V    V    V    V    V    V    V
                                 I------------------------------------------------------------------------------------------------------
 .1496245         1.99600      1 I *    .
 .6351986         3.99200      2 I *    .
 .1810053         5.98800      3 I *    .
 1.156155         7.98400      4 I  *   .
 .3237283         9.98000      5 I *    .
 .5317422        11.9760       6 I *    .
 1.161054        13.9720       7 I  *   .
 .7025102        15.9680       8 I *    .
 .1857000        17.9640       9 I *    .
 .3645231        19.9600      10 I *    .
 .4578688        21.9560      11 I *    .
 .3808638        23.9520      12 I *    .
 .3041026        25.9480      13 I *    .
 .1092384        27.9440      14 I *    .
 .2487183        29.9400      15 I *    .
 .1850710        31.9360      16 I *    .
 .9844543        33.9320      17 I  *   .
 33.91976        35.9280      18 I      .
 .9608699        37.9240      19 I  *   .
 .1752131        39.9200      20 I *    .
 1.693526        41.9160      21 I   *  .
 1.202902        43.9120      22 I  *   .
 .1963107        45.9080      23 I *    .
 .1161055        47.9040      24 I *    .
 2.311395        49.9000      25 I   *  .
 1.347175        51.8960      26 I  *   .
 .1712490        53.8920      27 I *    .
 2.954997        55.8880      28 I    * .
 .2023546        57.8840      29 I *    .
 .5292624        59.8800      30 I *    .
 1.238237        61.8760      31 I  *   .
 .7564689-01     63.8720      32 I *    .
 1.383071        65.8680      33 I  *   .
 1.169970        67.8640      34 I  *   .
 1.277655        69.8600      35 I  *   .
 1.019257        71.8560      36 I  *   .
 3.040913        73.8520      37 I    * .
 .2579902        75.8480      38 I *    .
 .3089603        77.8440      39 I *    .
 1.858935        79.8400      40 I   *  .
 .6004357        81.8360      41 I *    .
 .4323525        83.8320      42 I *    .
 .2052419        85.8280      43 I *    .
 1.724981        87.8240      44 I   *  .
 .2991233        89.8200      45 I *    .
 .7596589        91.8160      46 I  *   .
 1.006424        93.8120      47 I  *   .
 1.498088        95.8080      48 I  *   .
 .5715316        97.8040      49 I *    .
 .6454344        99.8000      50 I *    .
 1.021681       101.796       51 I  *   .
 .9413276       103.792       52 I  *   .
 .7965723       105.788       53 I  *   .
```
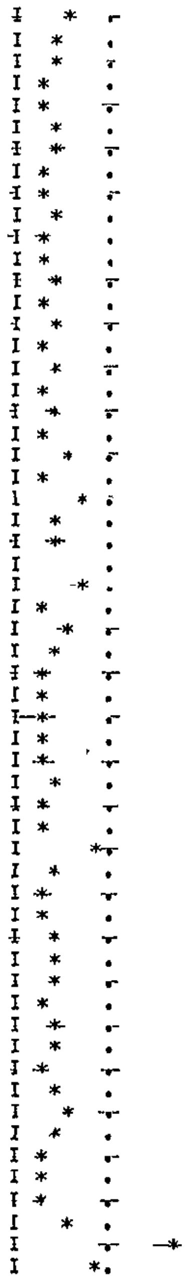

SAMPLE SPECTRUM PROGRAM RUN — UNEQUALLY SPACED DATA

2.530853	107.784	54	
1.475148	109.780	55	
.9777251	111.776	56	
.5753624	113.772	57	
.5044759	115.768	58	
.7842400	117.764	59	
1.114008	119.760	60	
.2677717	121.756	61	
.2512610	123.752	62	
1.486583	125.748	63	
.2667546	127.744	64	
.9528975-01	129.740	65	
1.219190	131.736	66	
.5426096	133.732	67	
.7445398	135.728	68	
.2523301-01	137.724	69	
.7672794	139.720	70	
.1480595	141.716	71	
1.416267	143.712	72	
.3871678	145.708	73	
2.313627	147.704	74	
.3099029	149.700	75	
3.081305	151.696	76	
.8834761	153.692	77	
1.112179	155.688	78	
80.86932	157.684	79	
3.006326	159.680	80	
.4229238	161.676	81	
1.957619	163.672	82	
1.204703	165.668	83	
.1509102	167.664	84	
.5183510	169.660	85	
.4391449	171.656	86	
.5555428	173.652	87	
.1739272	175.648	88	
.8408406	177.644	89	
.4019491	179.640	90	
.4021158	181.636	91	
4.231475	183.632	92	
1.094715	185.628	93	
.5240179-02	187.624	94	
.1196840	189.620	95	
1.156798	191.616	96	
.8658885	193.612	97	
.8097044	195.608	98	
.7762041-01	197.604	99	
1.375393	199.600	100	
.9109336	201.596	101	
.6696741	203.592	102	
.7677196	205.588	103	
1.885356	207.584	104	
.9468658	209.580	105	
.3451364	211.576	106	
.5633812	213.572	107	
.6384485	215.568	108	
1.778452	217.564	109	
9.888076	219.560	110	
3.973488	221.556	111	



SAMPLE SPECTRUM PROGRAM RUN - UNEQUALLY SPACED DATA

```
.1671130       223.552    112 I *    .
.9246581       225.548    113 I  *   .
1.579335       227.544    114 I  *   .
.2611486       229.540    115 I *    .
.9826546       231.536    116 I  *   .
.4269678       233.532    117 I *    .
1.371117       235.528    118 I  *   .
1.757539       237.524    119 I   *  .
.2409107       239.520    120 I *    .
1.720197       241.516    121 I   *  .
.6867451       243.512    122 I *    .
.4692231       245.508    123 I *    .
.6627310       247.504    124 I *    .
8.018402       249.500    125 I      .  *
3.017999       251.496    126 I    * .
1.920786       253.492    127 I   *  .
.5427287       255.488    128 I *    .
.4158510       257.484    129 I *    .
.2409580       259.480    130 I *    .
1.090745       261.476    131 I  *   .
.3530597       263.472    132 I *    .
3.069857       265.468    133 I    * .
.1143833       267.464    134 I *    .
.8320566       269.460    135 I  *   .
.8750748-01    271.456    136 I *    .
.4921323       273.452    137 I *    .
5.059622       275.448    138 I      *
1.073956       277.444    139 I  *   .
2.527255       279.440    140 I   *  .
26.86028       281.436    141 I      .
1.520425       283.432    142 I  *   .
.5932779       285.428    143 I *    .
.2791740       287.424    144 I *    .
.7196486       289.420    145 I *    .
.5267068       291.416    146 I *    .
1.182803       293.412    147 I  *   .
.3268812       295.408    148 I *    .
1.865957       297.404    149 I   *  .
.9719191       299.400    150 I  *   .
.9810183       301.396    151 I  *   .
1.601124       303.392    152 I  *   .
.2430267-01    305.388    153 I *    .
.1471134       307.384    154 I *    .
.1885470       309.380    155 I *    .
.1664933       311.376    156 I *    .
2.246916       313.372    157 I   *  .
.3820518       315.368    158 I *    .
.6344597-01    317.364    159 I *    .
.1753564       319.360    160 I *    .
.5839524-01    321.356    161 I *    .
.2145198       323.352    162 I *    .
```

SAMPLE SPECTRUM PROGRAM RUN - UNEQUALLY SPACE DATA

REMOVAL TABLE FOR SET RANDOM TIME 310568

PERIOD	FREQUENCY	OMEGA	A	B	C	SPECTRUM POWER	REMAINING TOTAL POWER
							.240513+03
.0063	158.000090	992.743847	.89995+01	.59415+01	.53828+00	.114821+03	
.0278	35.999333	226.190484	.30407+01	.50085+01	.53828+00	.340945+02	
.0036	280.997330	**********	.69338+01	.20961+01	.53828+00	.539825+02	
.0045	220.012693	**********	.29707+01	-.71298-01	.53828+00	.119490+02	
.0040	250.001495	**********	.57892-01	.50101+01	.53828+00	.250682+02	.180425-01

SEARCH WILL NOW BE MADE WITH SMALLER TOL.

SAMPLE SPECTRUM PROGRAM RUN - UNEQUALLY SPACED DATA

FINAL DATA SPECTRUM FOR SET RANDOM TIME 310468 (TOTAL POWER = .18043-01)

C = .53828+00

K	PERIOD	OMEGA	A	B	REGRESSION POWER	PERCENT POWER
1	.5010	12.541238	.36604-02	.55947-05	.135935-04	.000753
2	.2505	25.082476	-.31807-02	.33627-02	.220645-04	.001223
3	.1670	37.623714	.43095-03	.69240-02	.482660-04	.002675
4	.1253	50.164951	.11154-01	-.39911-03	.124620-03	.006907
5	.1002	62.706189	.12462-02	.90598-03	.234729-05	.000130
6	.0835	75.247427	.44771-03	.61719-02	.368800-04	.002044
7	.0716	87.788665	-.44042-03	.16946-02	.293500-05	.000163
8	.0626	100.329903	.14194-02	-.17993-02	.540332-05	.000299
9	.0557	112.871141	.43593-03	-.15018-01	.226702-03	.012565
10	.0501	125.412379	-.35427-02	-.49189-02	.368973-04	.002045
11	.0455	137.953617	.16881-02	.81117-03	.357162-05	.000198
12	.0418	150.494854	.17947-02	.19678-02	.706216-05	.000391
13	.0385	163.036092	-.23749-02	-.76750-02	.660643-04	.003662
14	.0358	175.577330	-.11793-01	-.18792-02	.140186-03	.007770
15	.0334	188.118568	-.38260-02	-.75570-02	.743457-04	.004121
16	.0313	200.659806	-.10638-02	-.55307-02	.307016-04	.001702
17	.0295	213.201044	.22748-02	-.80740-02	.720281-04	.003992
18	.0278	225.742282	-.44742-01	-.80156-02	.204654-02	.113429
19	.0264	238.283520	-.91116-03	.89261-03	.166606-05	.000092
20	.0251	250.824757	.67349-02	.68709-02	.911221-04	.005050
21	.0239	263.365995	.21716-02	-.30738-02	.145042-04	.000804
22	.0228	275.907233	-.58120-02	.26491-02	.414267-04	.002296
23	.0218	288.448471	.49607-02	.50234-02	.505900-04	.002804
24	.0209	300.989709	.34973-02	-.37464-02	.264944-04	.001468
25	.0200	313.530947	.14611-02	-.94732-03	.297267-05	.000165
26	.0193	326.072185	.58990-04	-.49394-02	.240527-04	.001333
27	.0186	338.613423	-.68323-02	.79889-02	.105308-03	.005837
28	.0179	351.154660	.98155-02	.43504-02	.110993-03	.006152
29	.0173	363.695898	-.11121-02	-.42481-02	.189510-04	.001050
30	.0167	376.237136	.22107-02	.55350-02	.354877-04	.001967
31	.0162	388.778374	-.48751-02	.10269-01	.134366-03	.007447
32	.0157	401.319612	.25084-02	-.41371-02	.229554-04	.001272
33	.0152	413.860850	.62061-02	-.11953-02	.382463-04	.002120
34	.0147	426.402088	.53587-02	-.96269-02	.122139-03	.006770
35	.0143	438.943326	-.14145-01	-.46597-02	.213411-03	.011828
36	.0139	451.484563	.11613-02	.31665-02	.112551-04	.000624
37	.0135	464.025801	.10166-01	-.77149-02	.158153-03	.008766
38	.0132	476.567039	.10008-01	-.21881-02	.105096-03	.005825
39	.0128	489.108277	.13542-02	-.46820-02	.224732-04	.001246
40	.0125	501.649515	-.38146-02	-.35364-02	.273796-04	.001518
41	.0122	514.190753	.83216-02	.10634-01	.181522-03	.010061
42	.0119	526.731991	-.50989-02	.79273-02	.852024-04	.004722
43	.0117	539.273229	.99858-03	-.46805-02	.213173-04	.001182
44	.0114	551.814466	-.31725-02	-.58790-02	.420738-04	.002332
45	.0111	564.355704	-.82422-03	.65512-02	.449862-04	.002493
46	.0109	576.896942	-.32568-02	-.29220-02	.191217-04	.001060
47	.0107	589.438180	.39541-03	-.55276-02	.313596-04	.001738
48	.0104	601.979418	.38372-02	.45657-02	.356193-04	.001974
49	.0102	614.520656	-.20494-02	.13936-02	.611097-05	.000339
50	.0100	627.061894	.18480-03	-.39966-02	.154631-04	.000857

SAMPLE SPECTRUM PROGRAM RUN - UNEQUALLY SPACED DATA

C = .53828+00

K	PERIOD	OMEGA	A	B	REGRESSION POWER	PERCENT POWER
51	.0098	639.603132	.48871-03	-.64025-02	.411603-04	.002281
52	.0096	652.144369	-.48842-02	-.19121-02	.281386-04	.001560
53	.0095	664.685607	.50246-02	.60103-02	.591330-04	.003277
54	.0093	677.226845	.87000-02	.22884-02	.841406-04	.004663
55	.0091	689.768083	-.88014-03	-.39134-02	.150495-04	.000834
56	.0089	702.309321	.55371-02	-.42472-02	.479847-04	.002660
57	.0088	714.850559	.10833-01	-.21317-02	.126403-03	.007006
58	.0086	727.391797	.32955-02	.74291-02	.648317-04	.003593
59	.0085	739.933035	-.84419-04	-.36654-02	.133760-04	.000741
-60-	.0084	752.474272	-.57578-02	-.82593-03	.331952-04	.001840
61	.0082	765.015510	.79837-03	-.19190-02	.424344-05	.000235
62	.0081	777.556748	.11519-01	.99014-02	.232015-03	.012859
63	.0080	790.097986	.32564-02	.68476-02	.550246-04	.003050
64	.0078	802.639224	-.41838-02	.11970-02	.178006-04	.000987
65	.0077	815.180462	.18188-02	.57891-02	.373420-04	.002070
66	.0076	827.721700	-.67738-02	-.57007-02	.776867-04	.004306
67	.0075	840.262938	.40491-02	-.73343-02	.688730-04	.003817
68	.0074	852.804175	-.66944-02	.15856-03	.451363-04	.002502
69	.0073	865.345413	-.51923-03	.36359-03	.399017-06	.000022
70	.0072	877.886651	-.11343-01	.73963-02	.185850-03	.010301
71	.0071	890.427889	.11900-02	-.54398-03	.169553-05	.000094
72	.0070	902.969127	.31620-02	-.40154-02	.261932-04	.001452
73	.0069	915.510365	.56108-02	-.70879-02	.811013-04	.004495
74	.0068	928.051603	-.10674-01	.10740-02	.116529-03	.006459
75	.0067	940.592841	-.10961-01	.14366-01	.321257-03	.017806
76	.0066	953.134078	.10699-02	-.29179-02	.998185-05	.000553
77	.0065	965.675316	.97568-03	.13988-01	.196965-03	.010917
78	.0064	978.216554	-.58344-02	-.48183-02	.570979-04	.003165
79	.0063	990.757792	-.37681-01	.30395-01	.231805-02	.128477
80	.0063	1003.299030	.96033-02	-.80374-02	.149520-03	.008287
81	.0062	1015.840268	.36703-02	-.10891-01	.138322-03	.007666
82	.0061	1028.381506	-.44279-02	-.74204-03	.197587-04	.001095
83	.0060	1040.922744	-.89354-02	-.34567-02	.866869-04	.004805
84	.0060	1053.463981	-.56546-02	.18568-02	.343264-04	.001903
85	.0059	1066.005219	-.37236-02	-.56623-02	.444317-04	.002463
86	.0058	1078.546457	.49218-02	-.70035-02	.733271-04	.004064
87	.0058	1091.087695	.37566-02	-.25572-02	.199806-04	.001108
88	.0057	1103.628933	.28263-02	-.70363-03	.873306-05	.000484
89	.0056	1116.170171	.50517-02	-.17119-01	.313193-03	.017359
90	.0056	1128.711409	.34614-02	.10629-02	.127837-04	.000709
91	.0055	1141.252647	.13235-01	-.27572-02	.172708-03	.009572
92	.0054	1153.793884	-.72534-03	.10669-02	.170063-05	.000094
93	.0054	1166.335122	-.25732-03	-.10827-01	.122059-03	.006765
94	.0053	1178.876360	.20585-04	-.10956-02	.118266-05	.000066
95	.0053	1191.417598	-.28823-02	.93771-02	.927000-04	.005138
96	.0052	1203.958836	.25479-02	-.57725-02	.410322-04	.002274
97	.0052	1216.500074	.60530-02	.92479-02	.126447-03	.007008
98	.0051	1229.041312	-.53317-02	-.46559-02	.507892-04	.002815
99	.0051	1241.582550	-.18470-02	-.31031-02	.130678-04	.000724
100	.0050	1254.123787	-.58454-02	.10648-02	.326388-04	.001809
101	.0050	1266.665025	-.68326-02	-.76763-04	.491394-04	.002724
102	.0049	1279.206263	.53029-02	.30732-02	.515329-04	.002856

SAMPLE SPECTRUM PROGRAM RUN - UNEQUALLY SPACED DATA

K	PERIOD	OMEGA	A	B	REGRESSION POWER	PERCENT POWER
103	.0049	1291.747501	.58503-03	.91631-02	.831803-04	.004610
104	.0048	1304.288739	-.71659-02	.10223-01	.162088-03	.008984
105	.0048	1316.829977	.10627-02	.60907-02	.662760-04	.003673
106	.0047	1329.371215	.16429-01	.86941-03	.265164-03	.014698
107	.0047	1341.912453	.85170-02	.16582-02	.762674-04	.004227
108	.0046	1354.453690	-.50107-02	-.28433-02	.346298-04	.001919
109	.0046	1366.994928	-.61829-02	-.22530-02	.433206-04	.002401
110	.0046	1379.536166	.44983-01	.50461-01	.447864-02	.248238
111	.0045	1392.077404	-.10325-02	.32456-02	.121941-04	.000676
112	.0045	1404.618642	-.21453-01	.83114-02	.507071-03	.028104
113	.0044	1417.159880	.77047-03	-.98852-03	.154437-05	.000086
114	.0044	1429.701118	.48615-02	.24293-02	.298742-04	.001656
115	.0044	1442.242356	.42651-02	-.53134-02	.469464-04	.002602
116	.0043	1454.783593	.26351-02	-.14178-01	.207840-03	.011519
117	.0043	1467.324831	.50405-02	-.45674-02	.441526-04	.002447
118	.0042	1479.866069	.14937-02	.10385-01	.110834-03	.006143
119	.0042	1492.407307	.40518-02	-.47298-02	.414913-04	.002300
120	.0042	1504.948545	-.29658-02	-.11802-02	.100096-04	.000555
121	.0041	1517.489783	-.88015-02	.81503-02	.148297-03	.008219
122	.0041	1530.031021	-.59101-02	-.90406-03	.355013-04	.001968
123	.0041	1542.572258	.27946-03	-.97102-02	.891132-04	.004939
124	.0040	1555.113496	-.35095-02	.25970-02	.184626-04	.001023
125	.0040	1567.654734	-.95547-02	-.11981-01	.241806-03	.013402
126	.0040	1580.195972	.45377-03	.10188-01	.102334-03	.005672
127	.0039	1592.737210	.41888-02	.14526-01	.229546-03	.012722
128	.0039	1605.278448	.24240-02	-.50797-02	.301005-04	.001668
129	.0039	1617.819686	.51747-03	.48330-02	.239555-04	.001328
130	.0039	1630.360924	-.15773-03	.70792-02	.489358-04	.002712
131	.0038	1642.902161	-.32202-02	.57156-02	.417246-04	.002313
132	.0038	1655.443399	.57049-02	-.90465-02	.107981-03	.005985
133	.0038	1667.984637	.48570-02	.53843-02	.558756-04	.003097
134	.0037	1680.525875	-.28200-03	.38799-02	.142918-04	.000792
135	.0037	1693.067113	.78113-03	.15002-04	.617366-06	.000034
136	.0037	1705.608351	.59086-02	-.59709-02	.665268-04	.003687
137	.0037	1718.149589	.14447-01	-.43797-02	.232524-03	.012888
138	.0036	1730.690827	-.13349-02	.79498-02	.663906-04	.003680
139	.0036	1743.232064	.21768-02	.67415-02	.512524-04	.002841
140	.0036	1755.773302	.75124-02	.14790-01	.279768-03	.015506
141	.0036	1768.314540	-.14647-01	-.59307-01	.354561-02	.196525
142	.0035	1780.855778	-.11935-01	-.12597-01	.299257-03	.016586
143	.0035	1793.397016	.89901-02	.61263-02	.117195-03	.006495
144	.0035	1805.938254	-.17016-02	-.85949-03	.360934-05	.000200
145	.0035	1818.479492	-.90808-03	.41100-02	.178370-04	.000989
146	.0034	1831.020730	.18815-02	-.35125-02	.154321-04	.000855
147	.0034	1843.561967	-.48906-02	.58768-02	.604962-04	.003353
148	.0034	1856.103205	-.97464-02	-.49469-02	.115755-03	.006416
149	.0034	1868.644443	.25549-02	-.68544-02	.568220-04	.003149
150	.0033	1881.185681	-.73747-02	-.88659-02	.136162-03	.007547
151	.0033	1893.726919	-.59506-02	-.17528-02	.390510-04	.002164
152	.0033	1906.268157	.25820-02	-.13871-01	.195199-03	.010819
153	.0033	1918.809395	.17045-04	.85616-03	.767083-06	.000043
154	.0033	1931.350633	.10149-01	-.26119-02	.108074-03	.005990

C = .53828+00

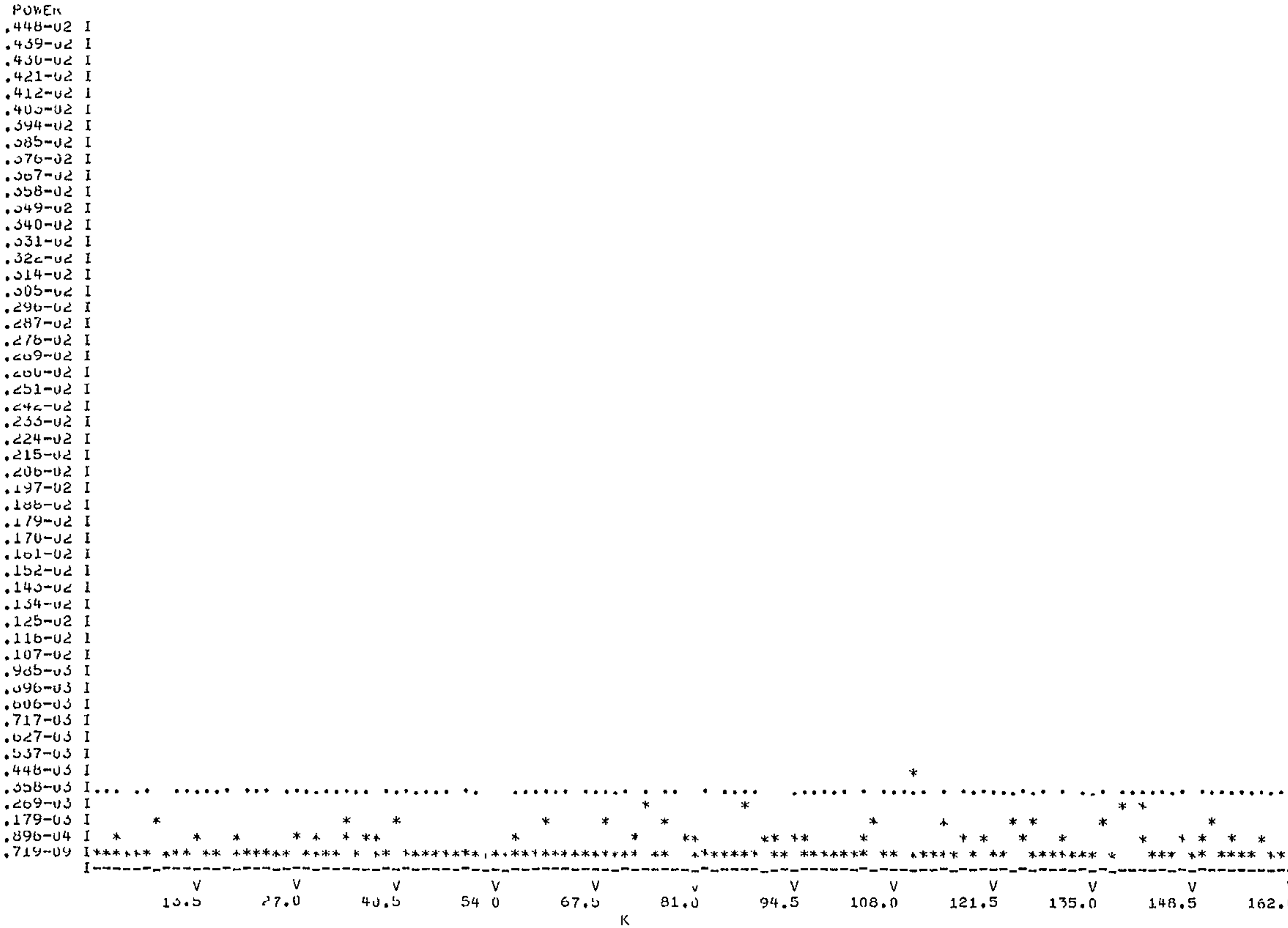
SAMPLE SPECTRUM PROGRAM RUN - UNEQUALLY SPACED DATA

C = .53828+00

K	PERIOD	OMEGA	A	B	REGRESSION POWER	PERCENT POWER
155	.0032	1943.891870	.39724-02	-.24817-02	.225792-04	.001251
156	.0032	1956.433108	.53312-02	-.32057-02	.379312-04	.002102
157	.0032	1968.974346	.86146-02	-.18286-02	.802928-04	.004450
158	.0032	1981.515584	.10552-01	-.53961-02	.140244-03	.007773
159	.0032	1994.056822	-.99427-02	.52826-02	.131549-03	.007291
160	.0031	2006.598060	-.14332-02	-.13484-02	.395745-05	.000219
161	.0031	2019.139298	.16569-02	.68240-02	.500789-04	.002776
162	.0031	2031.680536	.89621-02	-.35528-03	.813186-04	.004507

SAMPLE SPECTRUM PROGRAM RUN - UNEQUALLY SPACED DATA

RANDOM TIME 510568 P VS K



SAMPLE SPECTRUM PROGRAM RUN - UNEQUALLY SPACED DATA

```
RANDOM TIME 310568 P VS K                                 -PERCENT POWER

                                0    2    4    6    8    10   12   14   16   18   20   22   24   26   28   30   32   34   36   38   40
   POWER       FREQUENCY     K  V    V    V    V    V    V    V    V    V    V    V    V    V    V    V    V    V    V    V    V    V
                               I-----------------------------------------------------------------------------------------------------
.1359351-04  1.99600         1 I *    .
.2206446-04  3.99200         2 I *    .
.4826602-04  5.98800         3 I *    .
.1246200-03  7.98400         4 I  *   .
.2347286-05  9.98000         5 I *    .
.3688001-04  11.9760         6 I *    .
.2934997-05  13.9720         7 I *    .
.5403319-05  15.9680         8 I *    .
.2267025-03  17.9640         9 I    * .
.3689729-04  19.9600        10 I *    .
.3571624-05  21.9560        11 I *    .
.7062157-05  23.9520        12 I *    .
.6606427-04  25.9480        13 I  *   .
.1401860-03  27.9440        14 I   *  .
.7434568-04  29.9400        15 I  *   .
.3070155-04  31.9360        16 I *    .
.7202810-04  33.9320        17 I  *   .
.2046545-02  35.9280        18 I      .
.1666058-05  37.9240        19 I *    .
.9112208-04  39.9200        20 I  *   .
.1450422-04  41.9160        21 I *    .
.4142672-04  43.9120        22 I *    .
.5059003-04  45.9080        23 I *    .
.2649445-04  47.9040        24 I *    .
.2972669-05  49.9000        25 I *    .
.2405268-04  51.8960        26 I *    .
.1053083-03  53.8920        27 I  *   .
.1109926-03  55.8880        28 I  *   .
.1895104-04  57.8840        29 I *    .
.3548767-04  59.8600        30 I *    .
.1343657-03  61.8760        31 I   *  .
.2295538-04  63.8720        32 I *    .
.3824631-04  65.8680        33 I *    .
.1221393-03  67.8640        34 I  *   .
.2134106-03  69.8600        35 I    * .
.1125510-04  71.8560        36 I *    .
.1581525-03  73.8520        37 I   *  .
.1050956-03  75.8480        38 I  *   .
.2247323-04  77.8440        39 I *    .
.2737959-04  79.8400        40 I *    .
.1815218-03  81.8360        41 I   *  .
.8520237-04  83.8320        42 I  *   .
.2131729-04  85.8280        43 I *    .
.4267362-04  87.8240        44 I *    .
.4498619-04  89.8200        45 I *    .
.1912169-04  91.8160        46 I *    .
.3135956-04  93.8120        47 I *    .
.3561930-04  95.8080        48 I *    .
.6110971-05  97.8040        49 I *    .
.1546309-04  99.8000        50 I *    .
.4116030-04  101.796        51 I *    .
.2813635-04  103.792        52 I *    .
.5913296-04  105.788        53 I  *   .
```

SAMPLE SPECTRUM PROGRAM RUN - UNEQUALLY SPACED DATA

```
 .8414059-04  107.784      54 I  *    .
 .1504955-04  109.780      55 I *     .
 .4798471-04  111.776      56 I *     .
 .1264030-03  113.772      57 I   *   .
 .6483166-04  115.768      58 I  *    .
 .1337595-04  117.764      59 I *     .
 .3319524-04  119.760      60 I *     .
 .4243437-05  121.756      61 I *     .
 .2320152-03  123.752      62 I     * .
 .5502456-04  125.748      63 I  *    .
 .1780065-04  127.744      64 I *     .
 .3734200-04  129.740      65 I *     .
 .7768667-04  131.736      66 I  *    .
 .6887302-04  133.732      67 I  *    .
 .4513628-04  135.728      68 I *     .
 .3990174-06  137.724      69 I *     .
 .1858499-03  139.720      70 I   *   .
 .1695533-05  141.716      71 I *     .
 .2619319-04  143.712      72 I *     .
 .8110125-04  145.708      73 I  *    .
-.1165286-03  147.704      74 I  *    .
 .3212573-03  149.700      75 I      *.
 .9981852-05  151.696      76 I *     .
 .1969652-03  153.692      77 I   *   .
 .5709790-04  155.688      78 I  *    .
 .2318050-02  157.684      79 I       .
 .1495202-03  159.680      80 I   *   .
 .1383224-03  161.676      81 I   *   .
 .1975875-04  163.672      82 I *     .
 .8668885-04  165.668      83 I  *    .
 .3432642-04  167.664      84 I *     .
 .4443169-04  169.660      85 I *     .
 .7332715-04  171.656      86 I  *    .
 .1998664-04  173.652      87 I *     .
 .8733064-05  175.648      88 I *     .
 .3131926-03  177.644      89 I      *.
 .1278373-04  179.640      90 I *     .
 .1727077-03  181.636      91 I   *   .
 .1700632-05  183.632      92 I *     .
 .1220587-03  185.628      93 I  *    .
 .1182656-05  187.624      94 I *     .
 .9270005-04  189.620      95 I  *    .
 .4103220-04  191.616      96 I *     .
 .1264471-03  193.612      97 I   *   .
 .5078920-04  195.608      98 I *     .
 .1306782-04  197.604      99 I *     .
 .3263881-04  199.600     100 I *     .
 .4913939-04  201.596     101 I *     .
 .5153294-04  203.592     102 I *     .
 .8318033-04  205.588     103 I  *    .
 .1620879-03  207.584    -104 I   *   .
 .6627601-04  209.580     105 I  *    .
 .2651841-03  211.576     106 I    *  .
 .7626743-04  213.572     107 I  *    .
 .3462985-04  215.568     108 I *     .
 .4332060-04  217.564     109 I *     .
 .4478839-02  219.560     110 I       .
 .1219407-04  221.556     111 I *     .
```

SAMPLE SPECTRUM PROGRAM RUN - UNEQUALLY SPACED DATA

```
.5070707-03   223.552    112 I        .  *
.1544367-05   225.548    113 I *      .
.2987420-04   227.544    114 I *      .
.4694641-04   229.540    115 I *      .
.2078403-03   231.536    116 I     *  .
.4415257-04   233.532    117 I *      .
.1108337-03   235.528    118 I  *     .
.4149130-04   237.524    119 I *      .
.1000959-04   239.520    120 I *      .
.1482972-03   241.516    121 I   *    .
.3550133-04   243.512    122 I *      .
.8911321-04   245.508    123 I  *     .
.1846263-04   247.504    124 I *      .
.2418063-03   249.500    125 I     *  .
.1023336-03   251.496    126 I  *     .
.2295457-03   253.492    127 I     *  .
.3010049-04   255.488    128 I *      .
.2395549-04   257.484    129 I *      .
.4893581-04   259.480    130 I *      .
.4172458-04   261.476    131 I *      .
.1079808-03   263.472    132 I  *     .
.5587557-04   265.468    133 I  *     .
.1429183-04   267.464    134 I *      .
.6173660-06   269.460    135 I *      .
.6652684-04   271.456    136 I  *     .
.2325240-03   273.452    137 I     *  .
.6639060-04   275.448    138 I  *     .
.5125242-04   277.444    139 I *      .
.2797676-03   279.440    140 I       *.
.3545810-02   281.436    141 I        .
.2992572-03   283.432    142 I       *.
.1171946-03   285.428    143 I  *     .
.3609345-05   287.424    144 I *      .
.1783697-04   289.420    145 I *      .
.1543208-04   291.416    146 I *      .
.6049625-04   293.412    147 I  *     .
.1157547-03   295.408    148 I  *     .
.5682200-04   297.404    149 I  *     .
.1361618-03   299.400    150 I   *    .
.3905103-04   301.396    151 I *      .
.1951993-03   303.392    152 I   *    .
.7670826-06   305.388    153 I *      .
.1080738-03   307.384    154 I  *     .
.2257922-04   309.380    155 I *      .
.3793119-04   311.376    156 I *      .
.8029276-04   313.372    157 I  *     .
.1402444-03   315.368    158 I   *    .
.1315485-03   317.364    159 I   *    .
.3957450-05   319.360    160 I *      .
.5007891-04   321.356    161 I *      .
.8131857-04   323.352    162 I  *     .
```

@ FIN