N 70 27950

TTU-ES-70-1

NASA CR 109880

STRESS WAVES IN MULTIPLE LAMINATES

by

RAY KINSLOW

Prepared For

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

MANNED SPACECRAFT CENTER

HOUSTON, TEXAS

GRANT NUMBER NGR 43-003-007

MARCH,1970

TENNESSEE TECHNOLOGICAL UNIVERSITY

DEPARTMENT OF ENGINEERING SCIENCE

COOKEVILLE, TENNESSEE 38501

TTU-ES-70-1

STRESS WAVES IN MULTIPLE LAMINATES

by

Ray Kinslow

Prepared For

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

MANNED SPACECRAFT CENTER

HOUSTON, TEXAS

GRANT NUMBER NGR 43-003-007

MARCH, 1970

TENNESSEE TECHNOLOGICAL UNIVERSITY

DEPARTMENT OF ENGINEERING SCIENCE

COOKEVILLE, TENNESSEE 38501

ABSTRACT

Fractures caused by stress waves resulting from high velocity impact are sometimes reduced and at other times are increased by the use of laminated materials. This is a theoretical study of the propagation and reflections of stress waves in laminated targets and an analysis of the effects of material properties, target geometry, and wave characteristics. The multiple reflections and transmitted components at the interfaces bounding each layer of material may combine in phase so as to create stresses much greater than would have been produced in a homogeneous target; or a laminate may be designed so that the wave amplitude will be rapidly attenuated, resulting in a greatly reduced stress.

CONTENTS

TABLES

ILLUSTRATIONS

Figure

INTRODUCTION

When structures are subjected to high velocity impact or other impulsive loading, the stress waves generated may cause damage at points some distance from that of the load application due to reflections from free surfaces or boundaries (Ref. 1). It has been shown that such damage may be reduced or entirely eliminated by the use of layered or laminated materials (Ref. 2 and 3). This suggests the use of laminates as hulls and windows of spacecraft, as well as for EVA suits and visors which are to be exposed to the possible impact of meteorites. It is important to realize, however, that the resulting damage is not always reduced by the use of laminates, but in some instances the damage may actually be increased (Ref. 4 and 5).

This report deals with a theoretical analysis of the transmission and reflection of stress waves in multiple laminates. A report presenting experimental results and the correlation of theory and experiment is in preparation.

ASSUMPTIONS AND LIMITATIONS

In most theoretical studies, assumptions are made either to simplify mathematical derivations and relations or because of a lack of information or knowledge in certain areas. The following assumptions have been prompted by both of these reasons and are the same that have been given in previous reports and papers by the author

(Ref. 3 and 6).

1. This report is limited to the action of elastic waves in the targets. When a solid body is impacted by a hyper-velocity projectile, a crater is formed which expands very rapidly at first. As the velocity of crater formation decreases, a shock wave is detached from its surface but soon decays into a spherical elastic wave which continues to travel through the target (Ref. 6). The relative distance covered by the plastic shock wave is small, assuming a thick target, and fractures produced within this distance consist mainly of deformation of the target and ejection of target material by forces other than those attributed to the shock wave.

2. Closely related to the above is the assumption that the velocity of the elastic wave is a constant through any given material and is specified by the relation

$$c = \left[\frac{E(1 - \nu)}{\rho(1 + \nu)(1 - 2\nu)}\right]^{1/2} \tag{1}$$

where c is the wave velocity, E is Young's modulus, ν is Poisson's ratio, and ρ is the material density. This assumption is made for both incident and reflected waves. Since the reflected wave will traverse material through which the incident wave has recently passed, it is acknowledged that the density and elastic constants may change slightly, perhaps because of the heating of the material. (It has been shown that the velocity of the reflected wave in Lucite (acrylic resin) is about

80 percent of that of the incident wave (Ref. 3)). The reason for the assumption of constant velocity of the elastic wave is that, at present, the changes in material properties and temperatures are not known for the various materials.

3. Proceeding from the above, it is further assumed that the value of the product, ρc, called the "characteristic impedance" of the medium, remains constant.
4. The primary purpose of this study is to compare the stresses produced in laminates with the stresses produced in targets of a homogeneous material. This comparison can be made by neglecting the stress attenuation attributed to distance. Except as noted, this attenuation is neglected in this report.
5. Only the stresses along the axis perpendicular to the lamination interfaces will be considered. Unless the lateral dimensions of the target are very small, the maximum stress will be produced on this axis.
6. Energy losses are neglected.

IMPEDANCE MISMATCH EQUATIONS

If P_o is the amplitude of a pressure pulse or wave that encounters an abrupt change in the physical properties of a material, the amplitudes of the transmitted and reflected waves are specified by the relations

$$P_t = \left[\frac{2\rho_t c_t}{\rho_t c_t + \rho_o c_o}\right] P_o \tag{2}$$

$$P_r = \left[\frac{\rho_t c_t - \rho_o c_o}{\rho_t c_t + \rho_o c_o}\right] P_o \tag{3}$$

where the subscripts o and t refer to the original material and the material into which the wave is transmitted, respectively. These are usually referred to as the "impedance mismatch" equations.

Dividing Equation (2) by Equation (3)

gives

$$\frac{P_t}{P_r} = \frac{2\rho_t c_t}{\rho_t c_t - \rho_o c_o} \tag{4}$$

These equations may be simplified by letting

$$K = \frac{\rho_t c_t}{\rho_o c_o} \tag{5}$$

giving

$$P_t = \left[\frac{2K}{K + 1}\right] P_o \tag{6}$$

$$P_r = \left[\frac{K - 1}{K + 1}\right] P_o \tag{7}$$

and

$$\frac{P_t}{P_r} = \frac{2K}{K - 1} \tag{8}$$

MULTIPLE LAMINATIONS

A laminated structure is made up of N interfaces joining N + 1 layers of material. The structure may consist of any number of materials; but for the present, it will be assumed that only two different materials are employed as this is the most common type of laminate and is the only one for which it is practical to develop general relations. The first layer of material in which the pulse amplitude is P_o is designated as N = 0, followed by N = 1, 2, 3, - - - . The material of which the layers N = 0, 2, 4, 6, - - - is composed has a characteristic impedance of $\rho_o c_o$ and the alternate layers, N = 1, 3, 5, 7, - - -, are of a material having an impedance of $\rho_1 c_1$.

If $K = \rho_1 c_1/\rho_o c_o$, the amplitude of the pulse transmitted past the first interface is, according to Eq. (6),

$$P_t\Big]_{N = 1} = \left[\frac{2K}{K + 1}\right] P_o \tag{9}$$

and that reflected is

$$P_r\Big]_{N = 1} = \left[\frac{K - 1}{K + 1}\right] P_o \tag{10}$$

As the transmitted pulse reaches the second interface, the impedance mismatch becomes $\rho_o c_o/\rho_1 c_1$, or 1/K and the transmitted and reflected components are

$$P_t\Big]_{N = 2} = \left[\frac{4K}{(K + 1)^2}\right] P_o \tag{11}$$

$$P_r\Big]_{N = 2} = \left[\frac{2K(1 - K)}{(K + 1)^2}\right] P_o \tag{12}$$

It has been shown (Ref. 3) that the magnitude of the pulse transmitted past the Nth interface is given by the relation

$$P_t\Big]_N = \left[\frac{2}{1 + K}\right]^N K^{\frac{N}{2}} P_o \tag{13}$$

if N is an even number, and

$$P_t\Big]_N = \left[\frac{2}{1 + K}\right]^N K^{\frac{N+1}{2}} P_o \tag{14}$$

if the value of N is odd. This will be referred to as the primary pulse.

Similarly, the pulse reflected from the Nth interface has an amplitude of

$$P_r\Big]_N = \left[\frac{1 - K}{2}\right]\left[\frac{2}{1 + K}\right]^N K^{\frac{N}{2}} P_o \tag{15}$$

for even values of N, and

$$P_r\Big]_N = \left[\frac{K - 1}{2}\right]\left[\frac{2}{1 + K}\right]^N K^{\frac{N-1}{2}} P_o \tag{16}$$

for odd values of N.

SECONDARY PRESSURE WAVES

The portion of a pulse reflected at the Nth interface will, upon reaching the (N - 1)th interface, be again partially reflected and partially transmitted. As the pulse continues to be reflected back and forth between the Nth and the (N - 1)th interfaces, its amplitude attenuates very rapidly. This also applies to the multiple reflections from the two interfaces bounding any layer of material. Although it may be said, "Stress waves never die, they just fade away", they cannot be neglected in the design of laminated structures. If several of these "fading" waves meet at some later time, their total effect may be quite significant (Ref. 7).

Laminates are generally constructed with alternate layers of equal thickness. Figure 1-a is a distance-time plot of the front of a pressure wave or pulse (P) through a multi-layered structure in which the even-numbered laminations are all of the same material and have equal thickness. The odd-numbered layers are of another material and are also of equal thickness. In this illustration, the wave velocity is greater in the even numbered layers than in the odd ($C_1 < C_o$). The many reflected and transmitted waves are also shown. These will be referred to as secondary waves. As it is impossible to consider the effects of all of these waves, only the ones making the greater contribution to the total stress will be considered.

In Figure 1-b, the wave designated as S_1 is shown. This is the wave that undergoes two reflections in lamination number one. The wave that is reflected two times in lamination number two is called S_2 and is shown in Figure 1-c. The wave that is reflected four times in lamination number one is indicated by S_{11}; the one reflected four times in laminate two is called S_{22}; and the wave that undergoes two reflections in laminate number one and two reflections in number two is designated as S_{12}. These are shown in Figures 1-d, 1-e, and 1-f, respectively.

Figure 2 is a series of photographs made in a dynamic polariscope showing a stress wave in a laminated material of which the layers N = 0, 2, and 4 are of a transparent acrylic material and layers 1 and 3 are aluminum. Of course only the waves in the transparent material can be observed by this method. The waves transmitted through the aluminum and those reflected from the aluminum can both be seen in these photographs.

S_1 was defined as the wave that has been reflected two times within lamination number one, but this wave is joined at various points by waves that have been reflected twice within other odd-numbered laminations (Figure 1). Similarily, S_2 is joined by waves that have been reflected twice within other even-numbered laminations. S_{11} is made up of all the waves that have been reflected four times within odd-numbered laminations, regardless whether the four reflections were in the same or in different laminations. The wave S_{22} is the sum of all waves that have been reflected four times within the even-numbered layers. Likewise, S_{12} is made up of all waves having been reflected two times within an even-numbered and two times within an odd-numbered lamination. Due to these many waves meeting at various times, the amplitudes of the secondary waves become significant and must be taken into consideration in the design of laminated structures that may be subjected to impulsive loadings. There are fifteen waves, for example, that combine to make up the wave S_{12} in the sixth lamination. These are shown in Figure 3. The number of combinations making up these secondary waves in the first sixteen laminations are given in Table I.

Formulas have been derived for the amplitudes of the secondary waves as functions of the lamination number (N) and impedance mismatch (K). These are given in Table II for even values of N and in Table III for odd values of N.

The relations given in Tables II and III have been computed for values of K from 0.1 to 10.0, and for values of N up to twenty-four laminations. The results are shown in Figures 4 through 9. The amplitude of the initial pulse (P_o) is assumed to be one. The pulse or wave amplitudes for values of K less than one and for even numbered laminations are shown in Figure 4. It is seen that the primary pulse (P) decays slowly for high values of K but very rapidly for small values of K. Values of S_1, S_2, S_{11}, and S_{22} are always positive, indicating compression, but values of S_{12} are usually negative denoting tensile stresses. Values of both primary and secondary waves are always less than P_o. Values of pulse amplitude in the odd numbered layers are shown in Figure 5. Figures 6 and 7 give the pulse amplitudes when the values of K are greater than one. It will be noted that values of P and S_1 may be greater than P_o in the odd-numbered laminations. Figures 8 and 9 show the same information as the previous graphs but comparisons of the primary and secondary wave amplitudes may be made for certain values of K.

EFFECTS OF LAMINATION THICKNESS

If the thickness of the odd-numbered laminations is small compared with that of the even-numbered, the resulting stresses may be greatly increased. Figure 10-a shows that the fronts of waves P, S_1, and S_{11} come very close together for this condition. In fact, if the thickness of these odd-numbered laminations are very small, the waves will practically coincide, resulting in a wave having an amplitude of P + S_1 + S_{11} followed by another with amplitude of S_2 + S_{12} and then by S_{22}. Likewise, if the thickness of the even-numbered laminations are very small (Figure 10-b) the first wave would have an amplitude of P + S_2 + S_{22} followed by one of amplitude S_1 + S_{12} and then by S_{11}.

If the thickness ratio of the laminations are equal to the ratio of the wave velocities in the two materials, the secondary waves would combine as shown in Figure 10-c. The primary wave (P) would be followed by one of amplitude S_1 + S_2 and then by another of amplitude S_{11} + S_{12} + S_{22}.

By comparing the amplitudes of the secondary waves and their various combinations with that of the primary wave, it is seen that the stresses produced in many cases are much greater than that which would have been computed if the secondary waves had not been taken into consideration. In fact, these stresses are sometimes greater than would have occurred in a solid target of either material.

Unless the initial pressure pulse is very short, the primary and secondary waves will overlap, resulting in a wave amplitude greater than any of the combinations previously considered. This will also be true in the case where the wave velocity in one of the materials is very large compared with the velocity in the other material. For this reason, the sum of the primary wave and the five secondary wave amplitudes have been computed. This is designated as ΣS.

LOCATION OF MAXIMUM AND MINIMUM STRESS

Because structural damage is usually produced by the reflection of stress waves from a free surface, a laminated plate or shield subjected to impact should not have a thickness so that the maximum stress would occur at this surface. For the same reason, a good design would probably be one for which the minimum stress would occur at this point.

The relations for pulse amplitudes given in Tables II and III have been differentiated with respect to N and equated to zero, giving the locations of the maximum and minimum stress amplitude. These are given in Tables IV and V and shown graphically in Figure 11.

DISTANCE AND TIME RELATIONSHIPS

As stated in the assumptions, attenuation of the wave amplitudes due to distance is not considered in this study. If this is to be taken into account, it should be noted that the distances traveled by the secondary waves are not the same as the distance from point of impact. The actual distances traveled by each wave and the times required to reach any lamination are given in Table VI.

EFFECT OF REFLECTIONS FROM FRONT SURFACE

It probably has been noticed that the reflections from the front surface of the target at the point of impact were not considered. The reason for this is that upon impact of a hypervelocity projectile, a crater is usually formed and such reflections would be from the expanding crater surface instead of from a stress-free plane surface. Such reflections would not affect the magnitude of any of the secondary waves that have been considered. These reflections would, however, contribute to their sum and should probably be taken into account in the cases where the waves overlap. For this reason, the sum of the primary and secondary waves have been computed by including the effects of these reflections. This is denoted by $\Sigma S'$. The ratio of $\Sigma S'/\Sigma S$ is given and plotted in Figure 12 for various values of K. It is seen that the effects of these reflections do make a difference, increasing the value of stress when K is less than one and decreasing the computed stress when K is greater than one.

CONCLUSION

It is believed that this study will answer some of the questions as to how and why laminates act as they do when subjected to impact. It attempts to explain why some laminated targets have resisted fracture and others have suffered greater damage than solid targets when hit by hypervelocity projectiles. It is hoped that the designer of spacecraft and equipment will be alerted to the possibility of utilizing laminated material to reduce the probability of damage from meteorites but will also realize that an improperly designed laminate may pose an additional hazard.

As previously stated, further experimental work is being conducted to verify the validity of the assumptions made and the conclusions reached in this study.

REFERENCES

1. Kinslow, Ray. "Properties of Reflected Stress Waves". AEDC-TR-67-112 (AD818630), August 1967, p. 21.

2. Summers, James L. and Nysmith, C. Robert. "The Resistance of Composite Space Structures to Hypervelocity Impact." AIAA Fifth Annual Structures and Materials Conference Proceedings, April 1964, p. 386.

3. Kinslow, Ray. "Stress Waves in Composite Laminates." AEDC-TR-65-69, June 1965.

4. Kinslow, Ray. "Observations of Hypervelocity Impact of Transparent Plastic Targets." AEDC-TDR-64-49 (AD 438947), May 1964, p. 7, 16.

5. Gehring, J. W., Christman, D. R., and McMillan, A. R. "Hypervelocity Impact Studies Concerning the Meteroid Hazard to Aerospace Materials and Structures." AIAA Fifth Annual Structures and Materials Conference Proceedings, April 1964, p. 78.

6. Kinslow, Ray. "Properties of Spherical Stress Waves Produced by Hypervelocity Impact." AEDC-TDR-63-197 (AD 421578), October 1963.

7. Hutchinson, J. R. "Stress Waves in Layered Materials," AIAA Journal, Vol. 7, No. 4, April 1969, p. 786.

TABLE I

WAVE COMBINATIONS

N	S_1	S_2	S_{11}	S_{22}	S_{12}	ΣS
2	1	1	1	1	3	8
3	2	1	3	1	5	13
4	2	2	3	3	8	19
5	3	2	6	3	11	26
6	3	3	6	6	15	34
7	4	3	10	6	19	43
8	4	4	10	10	24	53
9	5	4	15	10	29	64
10	5	5	15	15	35	76
11	6	5	21	15	41	89
12	6	6	21	21	48	103
13	7	6	28	21	55	118
14	7	7	28	28	63	134
15	8	7	36	28	71	151
16	8	8	36	36	80	169

TABLE II

STRESS AMPLITUDES

(EVEN VALUES OF N)

P	$= \left(\frac{2\sqrt{K}}{1+K}\right)^{N} P_o$
$S_1 \;=\; S_2$	$= \frac{N}{2}\left(\frac{1-K}{1+K}\right)^{2}\left(\frac{2\sqrt{K}}{1+K}\right)^{N} P_o$
$S_{11} \;=\; S_{22}$	$= \frac{N(N+2)}{8}\left(\frac{2\sqrt{K}}{1+K}\right)^{N}\left(\frac{1-K}{1+K}\right)^{4} P_o$
S_{12}	$= -\left(\frac{2\sqrt{K}}{1+K}\right)^{N}\left(\frac{\sqrt{1-K}}{1+K}\right)^{4}\left[4NK - \frac{N^2}{4}(1-K)^2\right] P_o$
$S_1 + S_2$	$= N\left(\frac{1-K}{1+K}\right)^{2}\left(\frac{2\sqrt{K}}{1+K}\right)^{N} P_o$
$S_{11}+S_{12}+S_{22}$	$= \left(\frac{2\sqrt{K}}{1+K}\right)^{N}\left(\frac{\sqrt{1-K}}{1+K}\right)^{4}\left[\frac{N(N+2)(1-K)^2}{4} - 4NK + \frac{N^2(1-K)^2}{4}\right] P_o$
$P+S_1+S_{11}$	$= \left\{\left(\frac{2\sqrt{K}}{1+K}\right)^{N} + \left(\frac{2\sqrt{K}}{1+K}\right)^{N}\left(\frac{1-K}{1+K}\right)^{2}\left[\frac{N}{2} + \left(\frac{1-K}{1+K}\right)^{2}\frac{N(N+2)}{8}\right]\right\} P_o$
$P+S_2+S_{22}$	$= \left\{\left(\frac{2\sqrt{K}}{1+K}\right)^{N} + \left(\frac{2\sqrt{K}}{1+K}\right)^{N}\left(\frac{1-K}{1+K}\right)^{2}\left[\frac{N}{2} + \left(\frac{1-K}{1+K}\right)^{2}\frac{N(N+2)}{8}\right]\right\} P_o$
$S_1+S_{12} = S_2+S_{12}$	$= \left(\frac{2\sqrt{K}}{1+K}\right)^{N}\left(\frac{1-K}{1+K}\right)^{2}\left[\frac{N}{2} - \frac{4NK}{(1+K)^2} + \frac{N^2}{4}\left(\frac{1-K}{1+K}\right)^{2}\right] P_o$
ΣS	$= \left\{\left(\frac{2\sqrt{K}}{1+K}\right)^{N} + \left(\frac{2\sqrt{K}}{1+K}\right)^{N}\left(\frac{1-K}{1+K}\right)^{2}\left[N - \frac{4NK}{(1+K)^2} + \left(\frac{1-K}{1+K}\right)^{2}\left(\frac{N^2+N}{2}\right)\right]\right\} P_o$

TABLE III

STRESS AMPLITUDES

(ODD VALUES OF N)

P	$= \sqrt{K}\left(\frac{2\sqrt{K}}{1+K}\right)^N P_0$
S_1	$= \sqrt{K}\left(\frac{N+1}{2}\right)\left(\frac{1-K}{1+K}\right)^2\left(\frac{2\sqrt{K}}{1+K}\right)^N P_0$
S_2	$= \sqrt{K}\left(\frac{N-1}{2}\right)\left(\frac{1-K}{1+K}\right)^2\left(\frac{2\sqrt{K}}{1+K}\right)^N P_0$
S_{11}	$= \sqrt{K}\left(\frac{N^2+4N+3}{8}\right)\left(\frac{2\sqrt{K}}{1+K}\right)^N\left(\frac{1-K}{1+K}\right)^4 P_0$
S_{22}	$= \sqrt{K}\left(\frac{N^2-1}{8}\right)\left(\frac{2\sqrt{K}}{1+K}\right)^N\left(\frac{1-K}{1+K}\right)^4 P_0$
S_{12}	$= -\sqrt{K}\left(\frac{2\sqrt{K}}{1+K}\right)^N\left(\frac{\sqrt{1-K}}{1+K}\right)^4\left[4NK-\frac{(1-K)^2(N^2-1)}{4}\right] P_0$
$S_1 + S_2$	$= N\sqrt{K}\left(\frac{1-K}{1+K}\right)^2\left(\frac{2\sqrt{K}}{1+K}\right)^N P_0$
$S_{11} + S_{12} + S_{22}$	$= N\sqrt{K}\left(\frac{\sqrt{1-K}}{1+K}\right)^4\left(\frac{2\sqrt{K}}{1+K}\right)^N\left[\frac{(1-K)^2(N+1)}{2}-4K\right] P_0$
$P + S_1 + S_{11}$	$= \left\{\sqrt{K}\left(\frac{2\sqrt{K}}{1+K}\right)^N + \sqrt{K}\left(\frac{1-K}{1+K}\right)^2\left(\frac{2\sqrt{K}}{1+K}\right)^N\left[\frac{N+1}{2}+\left(\frac{1-K}{1+K}\right)^2\left(\frac{N^2+4N+3}{8}\right)\right]\right\} P_0$
$P + S_2 + S_{22}$	$= \left\{\sqrt{K}\left(\frac{2\sqrt{K}}{1+K}\right)^N + \sqrt{K}\left(\frac{1-K}{1+K}\right)^2\left(\frac{2\sqrt{K}}{1+K}\right)^N\left[\frac{N-1}{2}+\left(\frac{1-K}{1+K}\right)^2\left(\frac{N^2-1}{8}\right)\right]\right\} P_0$
$S_1 + S_{12}$	$= \sqrt{K}\left(\frac{1-K}{1+K}\right)^2\left(\frac{2\sqrt{K}}{1+K}\right)^N\left[\frac{N+1}{2}-\frac{4NK}{(1+K)^2}+\frac{N^2-1}{4}\left(\frac{1-K}{1+K}\right)^2\right] P_0$
$S_2 + S_{12}$	$= \sqrt{K}\left(\frac{1-K}{1+K}\right)^2\left(\frac{2\sqrt{K}}{1+K}\right)^N\left[\frac{N-1}{2}-\frac{4NK}{(1+K)^2}+\frac{N^2-1}{4}\left(\frac{1-K}{1+K}\right)^2\right] P_0$
ΣS	$= \left\{\sqrt{K}\left(\frac{2\sqrt{K}}{1+K}\right)^N + \sqrt{K}\left(\frac{2\sqrt{K}}{1+K}\right)^N\left(\frac{1-K}{1+K}\right)^2\left[N-\frac{4NK}{(1+K)^2}+\left(\frac{1-K}{1+K}\right)^2\left(\frac{N^2+N}{2}\right)\right]\right\} P_0$

TABLE IV

LOCATION OF MAXIMUM AND MINIMUM STRESS

(EVEN VALUES OF N)

$S_1 = S_2$	$\ln\left(\frac{1+K}{2\sqrt{K}}\right) = \frac{1}{N}$
$S_{11} = S_{22}$	$\ln\left(\frac{1+K}{2\sqrt{K}}\right) = \frac{2(N+1)}{N(N+2)}$
S_{12}	$\ln\left(\frac{1+K}{2\sqrt{K}}\right) = \frac{-16K + 2N(1-K)^2}{-16NK + N^2(1-K)^2}$
$S_1 + S_2$	$\ln\left(\frac{1+K}{2\sqrt{K}}\right) = \frac{1}{N}$
$S_{11} + S_{12} + S_{22}$	$\ln\left(\frac{1+K}{2\sqrt{K}}\right) = \frac{(2N+1)(1-K)^2 - 8K}{N(N+1)(1-K)^2 - 8NK}$
$S_1 + S_{12} = S_2 + S_{12}$	$\ln\left(\frac{1+K}{2\sqrt{K}}\right) = \frac{2(1+K)^2 - 16K + 2N(1-K)^2}{2N(1+K)^2 - 16NK + N^2(1-K)^2}$
$P + S_1 + S_2$	$\ln\left(\frac{1+K}{2\sqrt{K}}\right) = \frac{(1-K)^2}{(1+K)^2 + N(1-K)^2}$
ΣS	$\ln\left(\frac{1+K}{2\sqrt{K}}\right) = \frac{2(1-K)^2(1+K)^2 + (2N+1)(1-K)^4 - 8K(1-K)^2}{2(1+K)^4 + 2N(1-K)^2(1+K)^2 + (N^2+N)(1-K)^4 - 8NK(1-K)^2}$

TABLE V

LOCATION OF MAXIMUM AND MINIMUM STRESS

(ODD VALUES OF N)

S_1	$\ln\left(\frac{1+K}{2\sqrt{K}}\right) = \frac{1}{N+1}$
S_2	$\ln\left(\frac{1+K}{2\sqrt{K}}\right) = \frac{1}{N-1}$
S_{11}	$\ln\left(\frac{1+K}{2\sqrt{K}}\right) = \frac{2(N+2)}{(N+3)(N+1)}$
S_{22}	$\ln\left(\frac{1+K}{2\sqrt{K}}\right) = \frac{2N}{(N+1)(N-1)}$
S_{12}	$\ln\left(\frac{1+K}{2\sqrt{K}}\right) = \frac{-16K + 2N(1-K)^2}{-16NK + (N^2-1)(1-K)^2}$
$S_1 + S_2$	$\ln\left(\frac{1+K}{2\sqrt{K}}\right) = \frac{1}{N}$
$S_{11} + S_{12} + S_{22}$	$\ln\left(\frac{1+K}{2\sqrt{K}}\right) = \frac{(2N+1)(1-K)^2 - 8K}{N(N+1)(1-K)^2 - 8NK}$
$S_1 + S_{12}$	$\ln\left(\frac{1+K}{2\sqrt{K}}\right) = \frac{2(1+K)^2 - 16K + 2N(1-K)^2}{2(N+1)(1+K)^2 - 16NK + (N^2-1)(1-K)^2}$
$S_2 + S_{12}$	$\ln\left(\frac{1+K}{2\sqrt{K}}\right) = \frac{2(1+K)^2 - 16K + 2N(1-K)^2}{2(N-1)(1+K)^2 - 16NK + (N^2-1)(1-K)^2}$
$P + S_1 + S_2$	$\ln\left(\frac{1+K}{2\sqrt{K}}\right) = \frac{(1-K)^2}{(1+K)^2 + N(1-K)^2}$
ΣS	$\ln\left(\frac{1+K}{2\sqrt{K}}\right) = \frac{2(1-K)^2(1+K)^2 + (2N+1)(1-K)^4 - 8K(1-K)^2}{2(1+K)^4 + 2N(1-K)^2(1+K)^2 + (N^2+N)(1-K)^4 - 8NK(1-K)^2}$

TABLE VI

DISTANCE TRAVELED AND TIME REQUIRED FOR PULSE TO REACH ANY LAMINATION

PULSE	N	DISTANCE	TIME
P	EVEN	$\frac{Nt_0}{2} + \frac{Nt_1}{2}$	$\frac{Nt_0}{2c_0} + \frac{Nt_1}{2c_1}$
	ODD	$\frac{(N+1)t_0}{2} + \frac{(N-1)t_1}{2}$	$\frac{(N+1)t_0}{2c_0} + \frac{(N-1)t_1}{2c_1}$
S_1 $(N \geq 2)$	EVEN	$\frac{Nt_0}{2} + \frac{(N+4)t_1}{2}$	$\frac{Nt_0}{2c_0} + \frac{(N+4)t_1}{2c_1}$
	ODD	$\frac{(N+1)t_0}{2} + \frac{(N+3)t_1}{2}$	$\frac{(N+1)t_0}{2c_0} + \frac{(N+3)t_1}{2c_1}$
S_2 $(N \geq 3)$	EVEN	$\frac{(N+4)t_0}{2} + \frac{Nt_1}{2}$	$\frac{(N+4)t_0}{2c_0} + \frac{Nt_1}{2c_1}$
	ODD	$\frac{(N+5)t_0}{2} + \frac{(N-1)t_1}{2}$	$\frac{(N+5)t_0}{2c_0} + \frac{(N-1)t_1}{2c_1}$
S_{11} $(N \geq 2)$	EVEN	$\frac{Nt_0}{2} + \frac{(N+8)t_1}{2}$	$\frac{Nt_0}{2c_0} + \frac{(N+8)t_1}{2c_1}$
	ODD	$\frac{(N+1)t_0}{2} + \frac{(N+7)t_1}{2}$	$\frac{(N+1)t_0}{2c_0} + \frac{(N+7)t_1}{2c_1}$
S_{22} $(N \geq 3)$	EVEN	$\frac{(N+8)t_0}{2} + \frac{Nt_1}{2}$	$\frac{(N+8)t_0}{2c_0} + \frac{Nt_1}{2c_1}$
	ODD	$\frac{(N+9)t_0}{2} + \frac{(N-1)t_1}{2}$	$\frac{(N+9)t_0}{2c_0} + \frac{(N-1)t_1}{2c_1}$
S_{12} $(N \geq 3)$	EVEN	$\frac{(N+4)t_0}{2} + \frac{(N+4)t_1}{2}$	$\frac{(N+4)t_0}{2c_0} + \frac{(N+4)t_1}{2c_1}$
	ODD	$\frac{(N+5)t_0}{2} + \frac{(N+3)t_1}{2}$	$\frac{(N+5)t_0}{2c_0} + \frac{(N+3)t_1}{2c_1}$

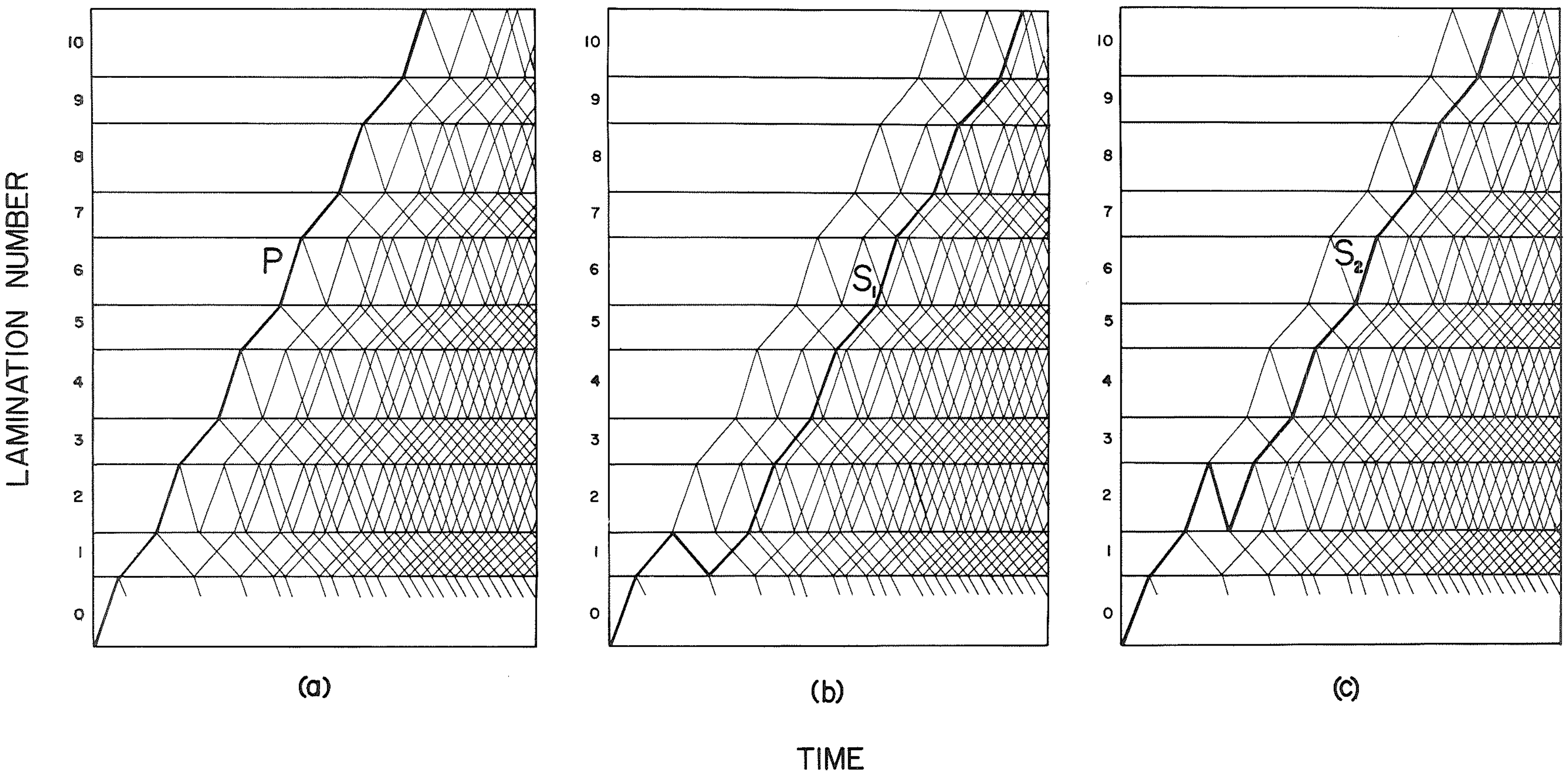


FIGURE I

PRIMARY AND SECONDARY WAVES

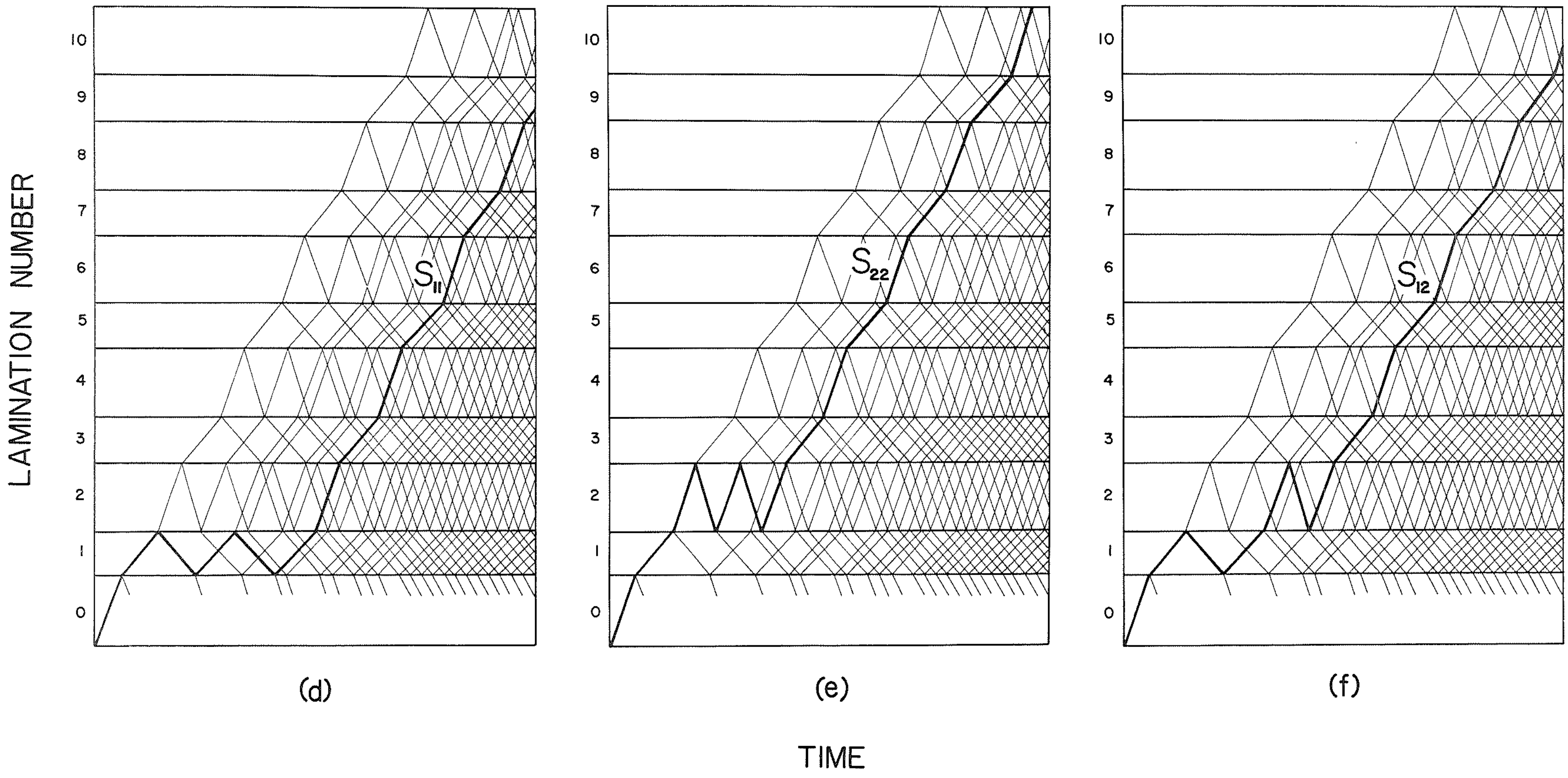


FIGURE 1 CONCLUDED

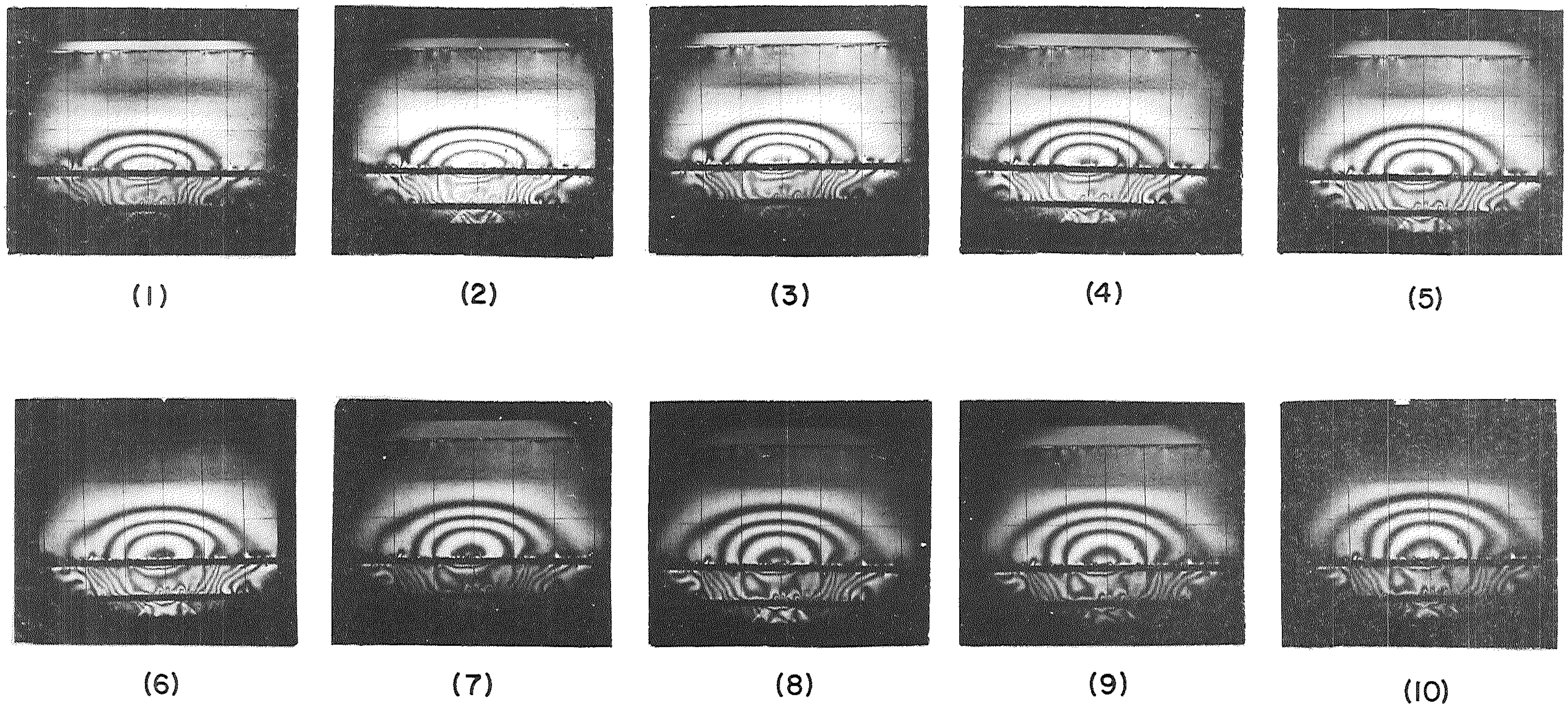


FIGURE 2

PHOTOGRAPHS OF STRESS WAVES IN LAMINATED MATERIAL

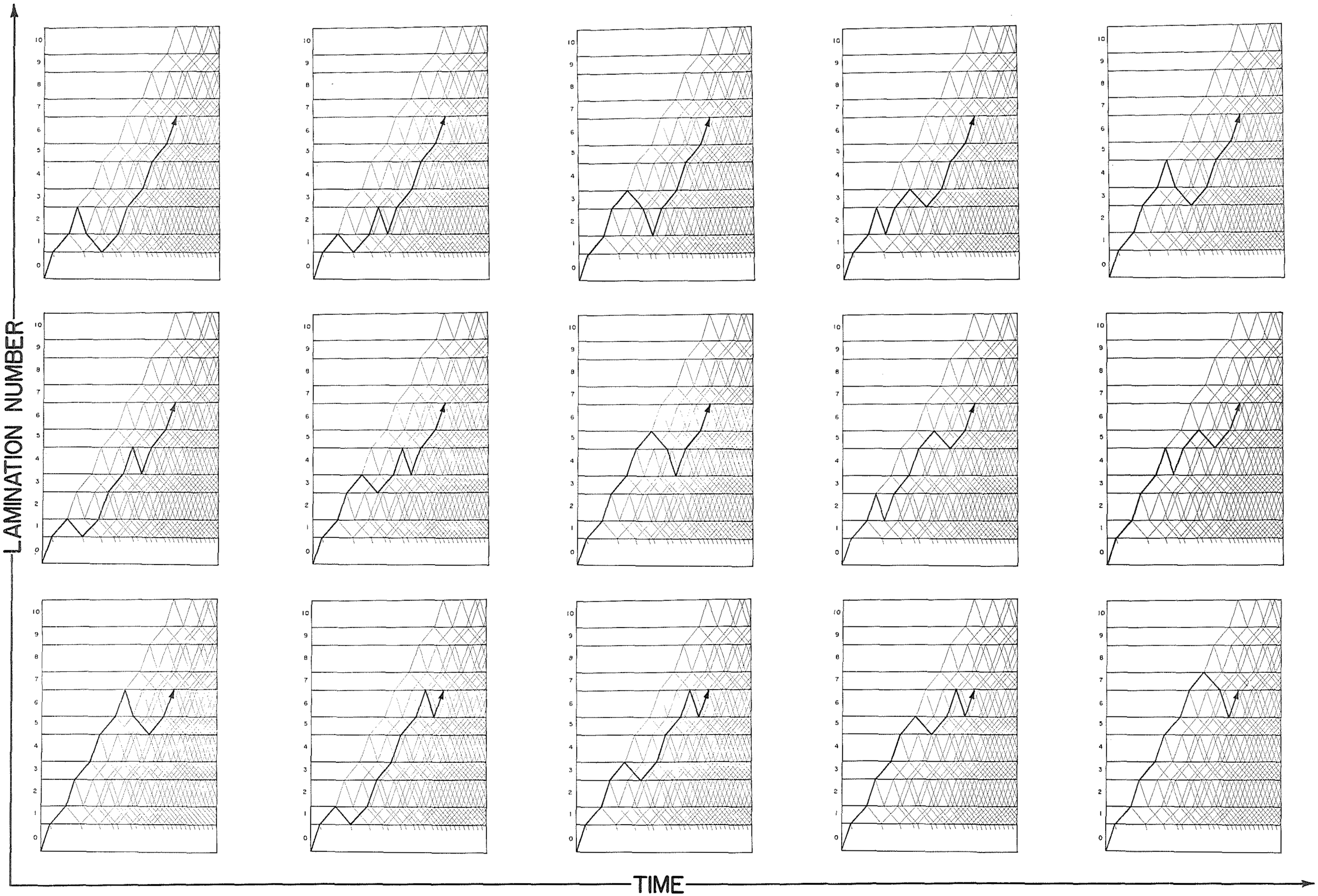


FIGURE 3

WAVES THAT CONTRIBUTE TO S_{12} WHEN N=6

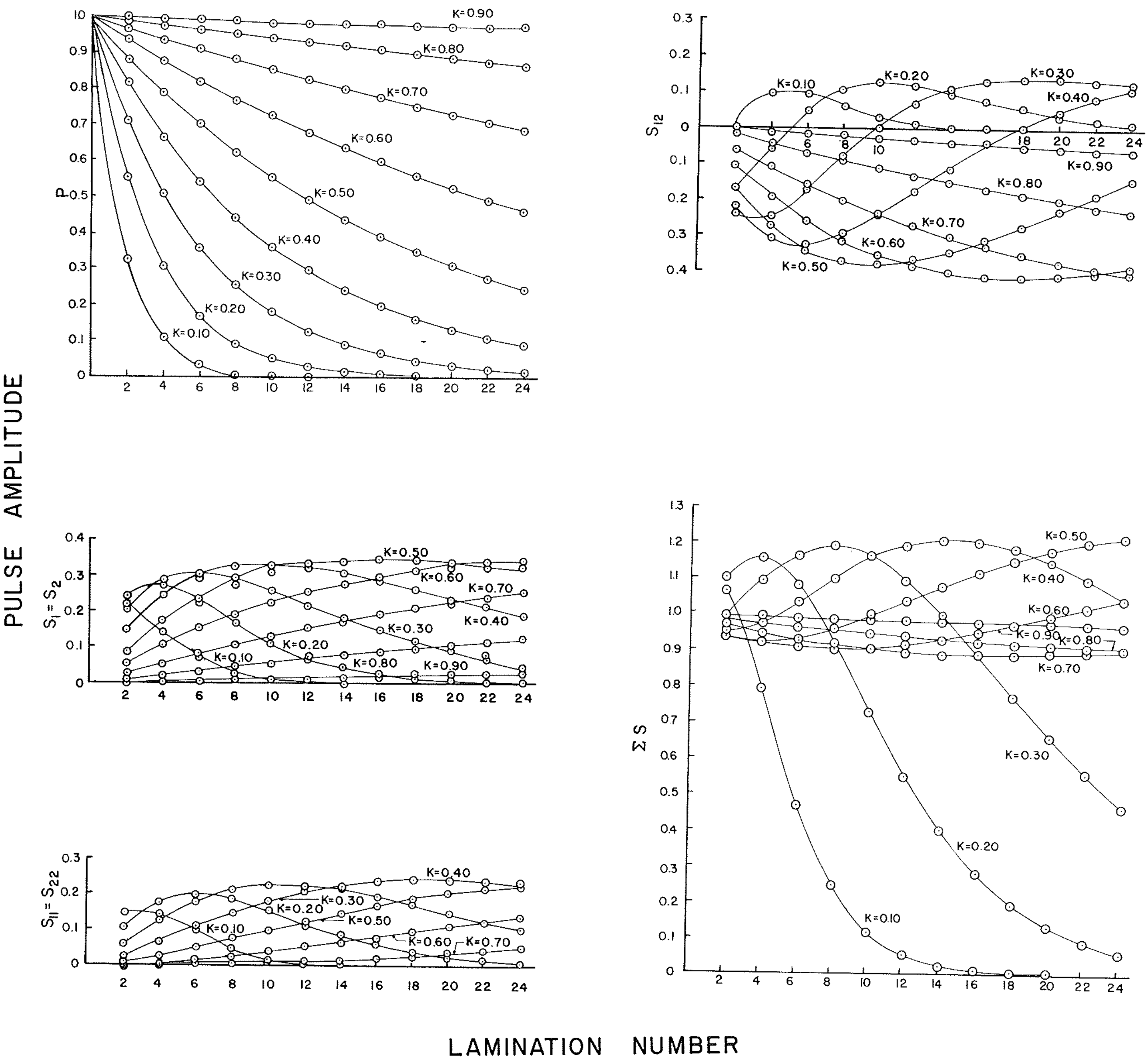


FIGURE 4

EFFECT OF IMPEDANCE MISMATCH UPON PULSE AMPLITUDE

(K < 1, N EVEN)

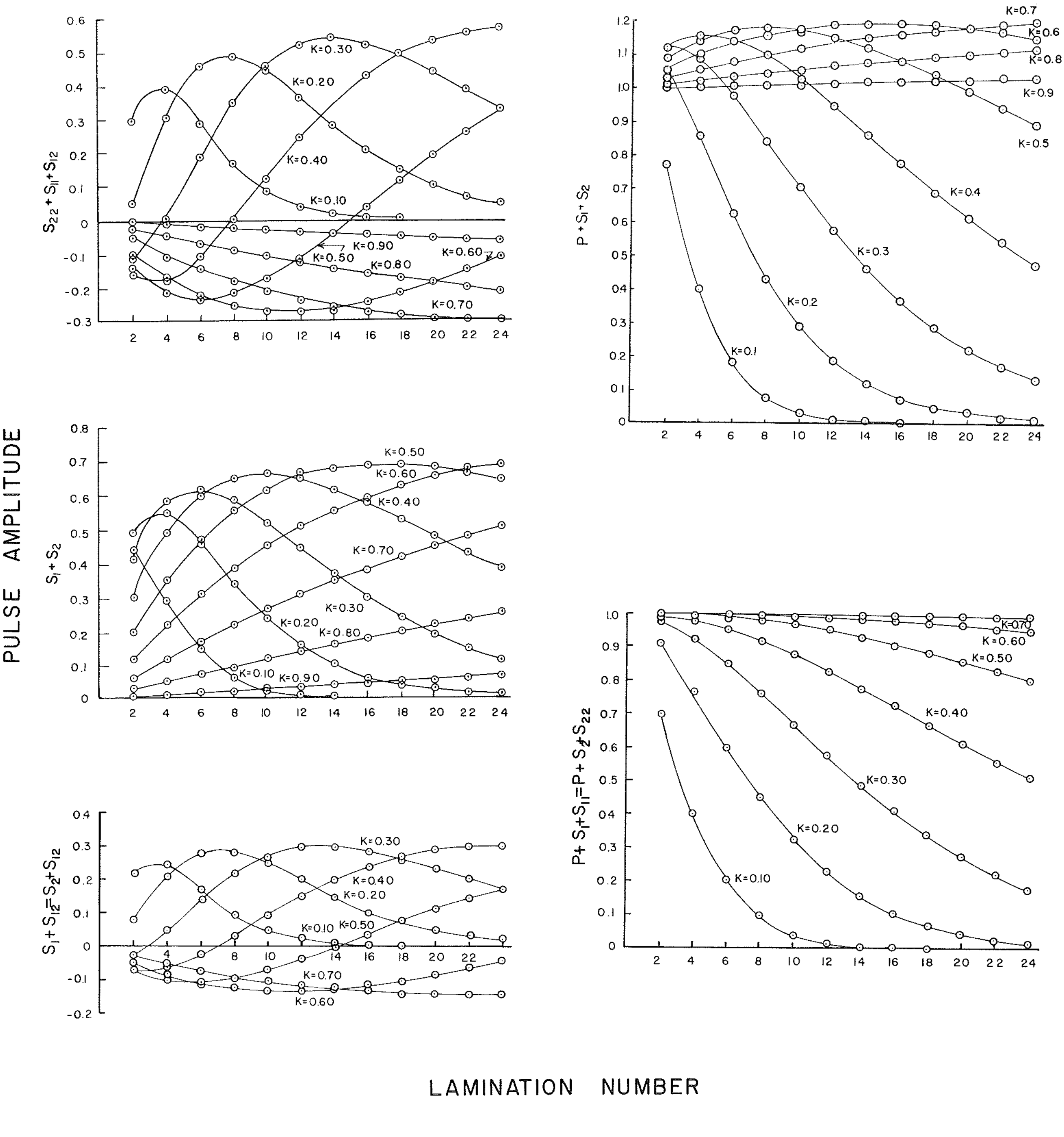


FIGURE 4 CONCLUDED

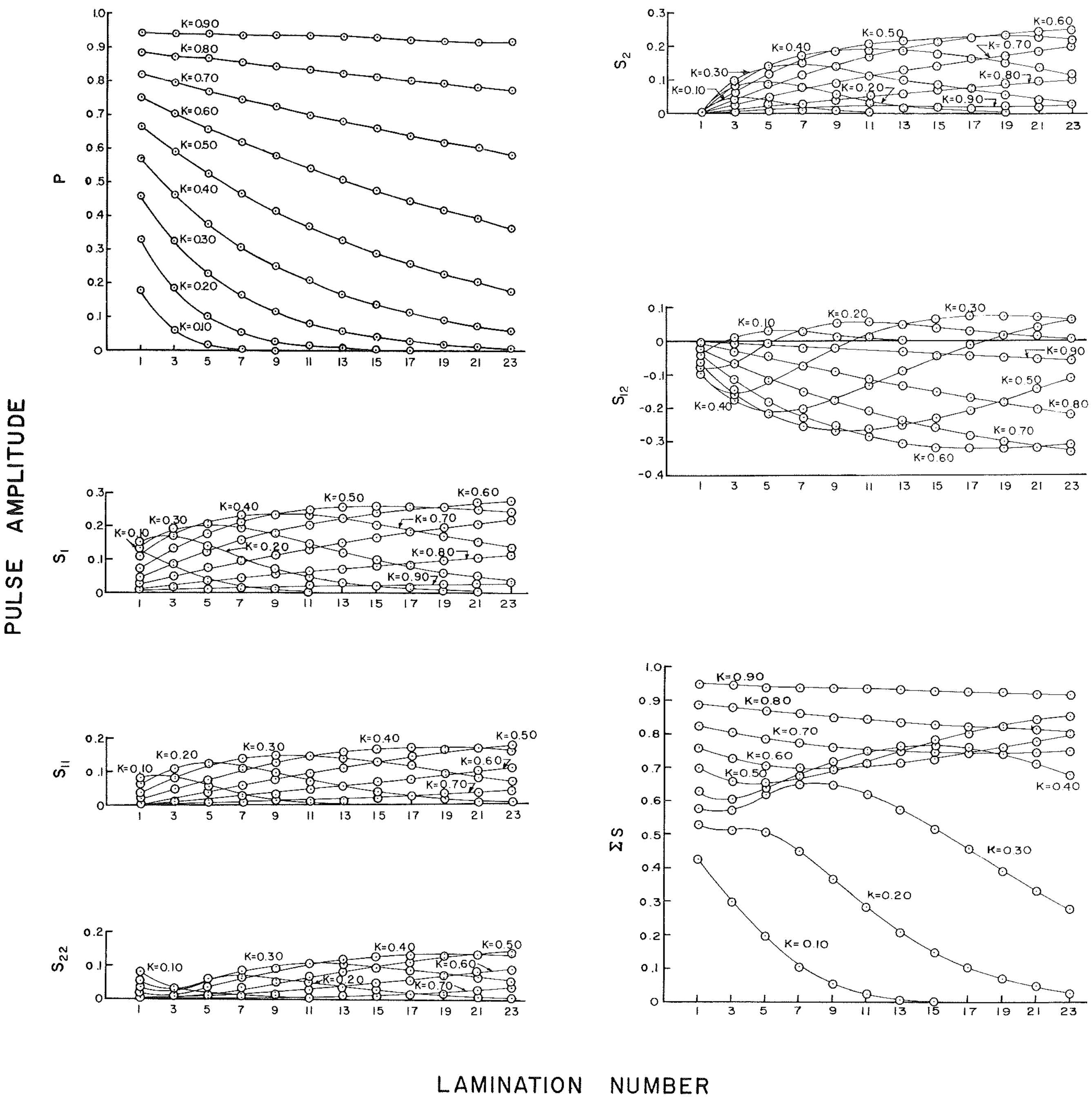


FIGURE 5

EFFECT OF IMPEDANCE MISMATCH UPON PULSE AMPLITUDE

(K<1, N ODD)

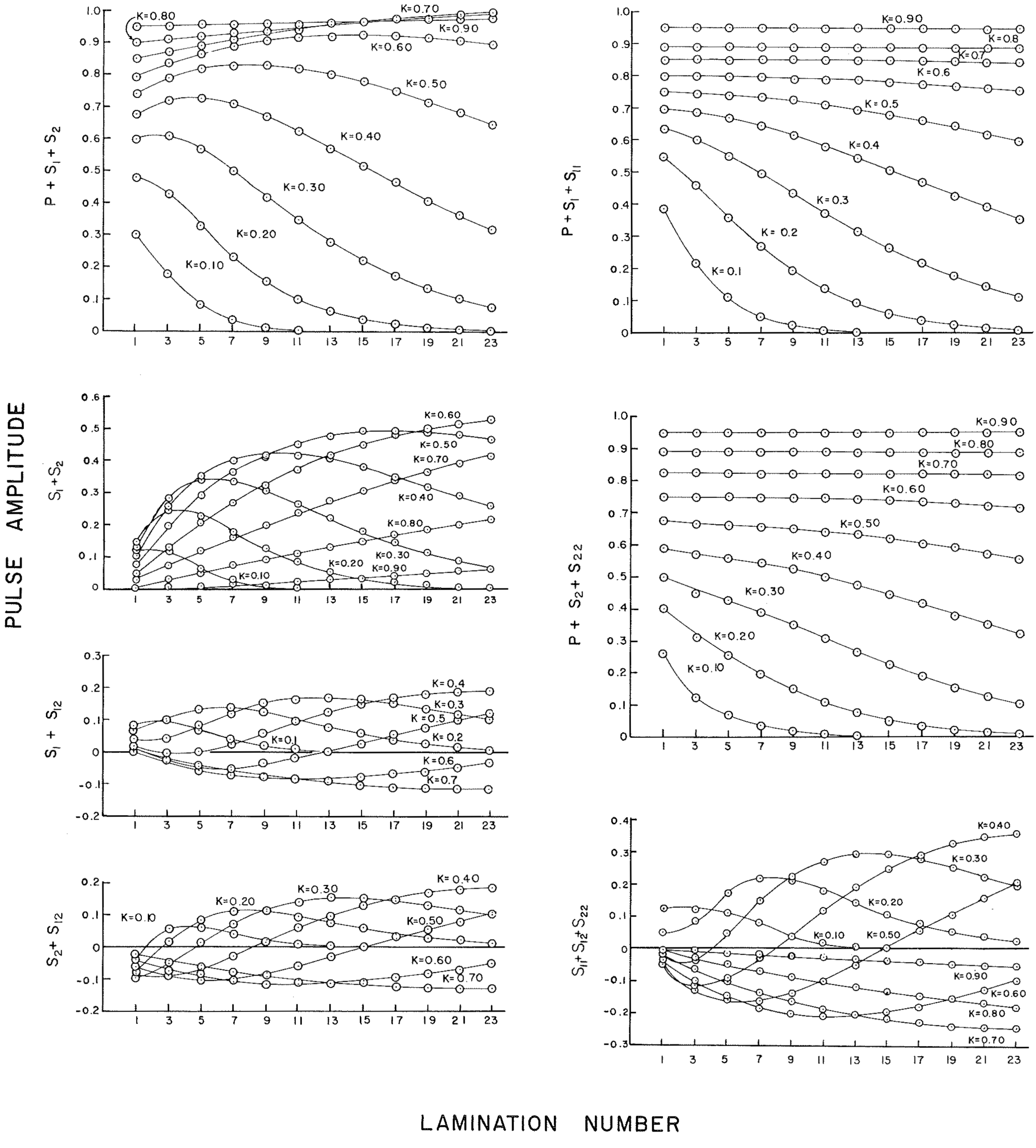


FIGURE 5 CONCLUDED

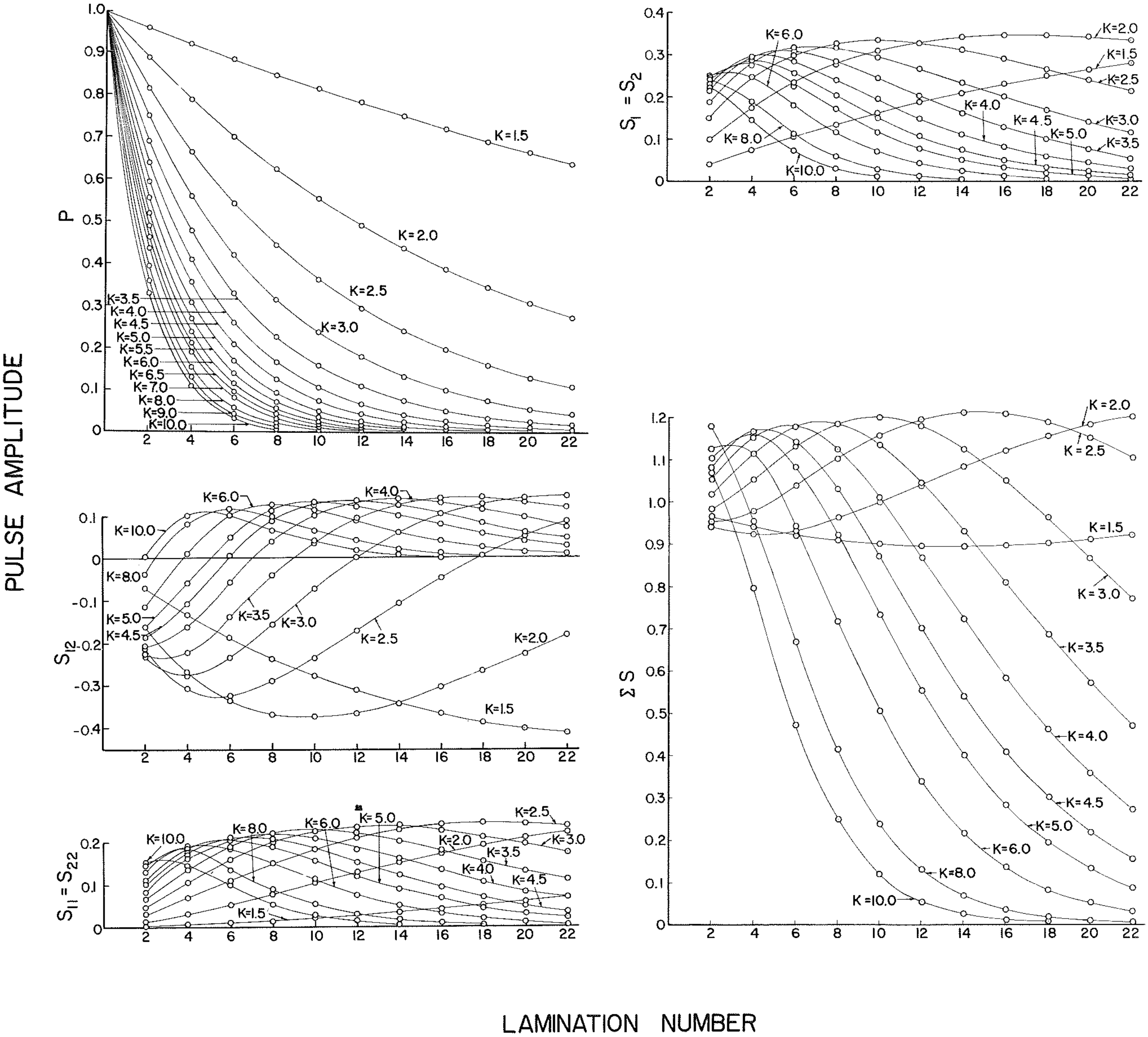


FIGURE 6

EFFECT OF IMPEDANCE MISMATCH UPON PULSE AMPLITUDE

(K > 1, N EVEN)

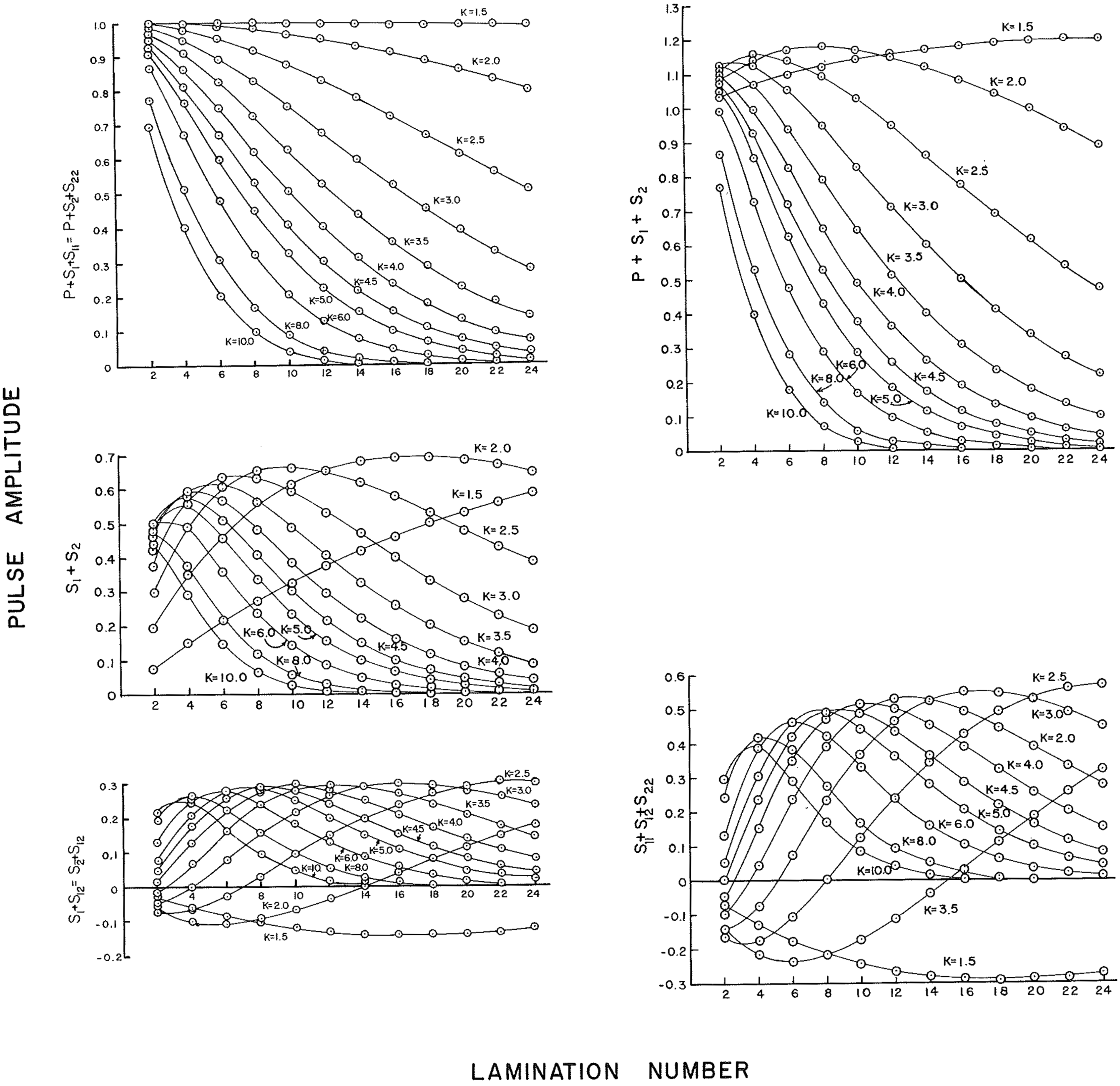


FIGURE 6 CONCLUDED

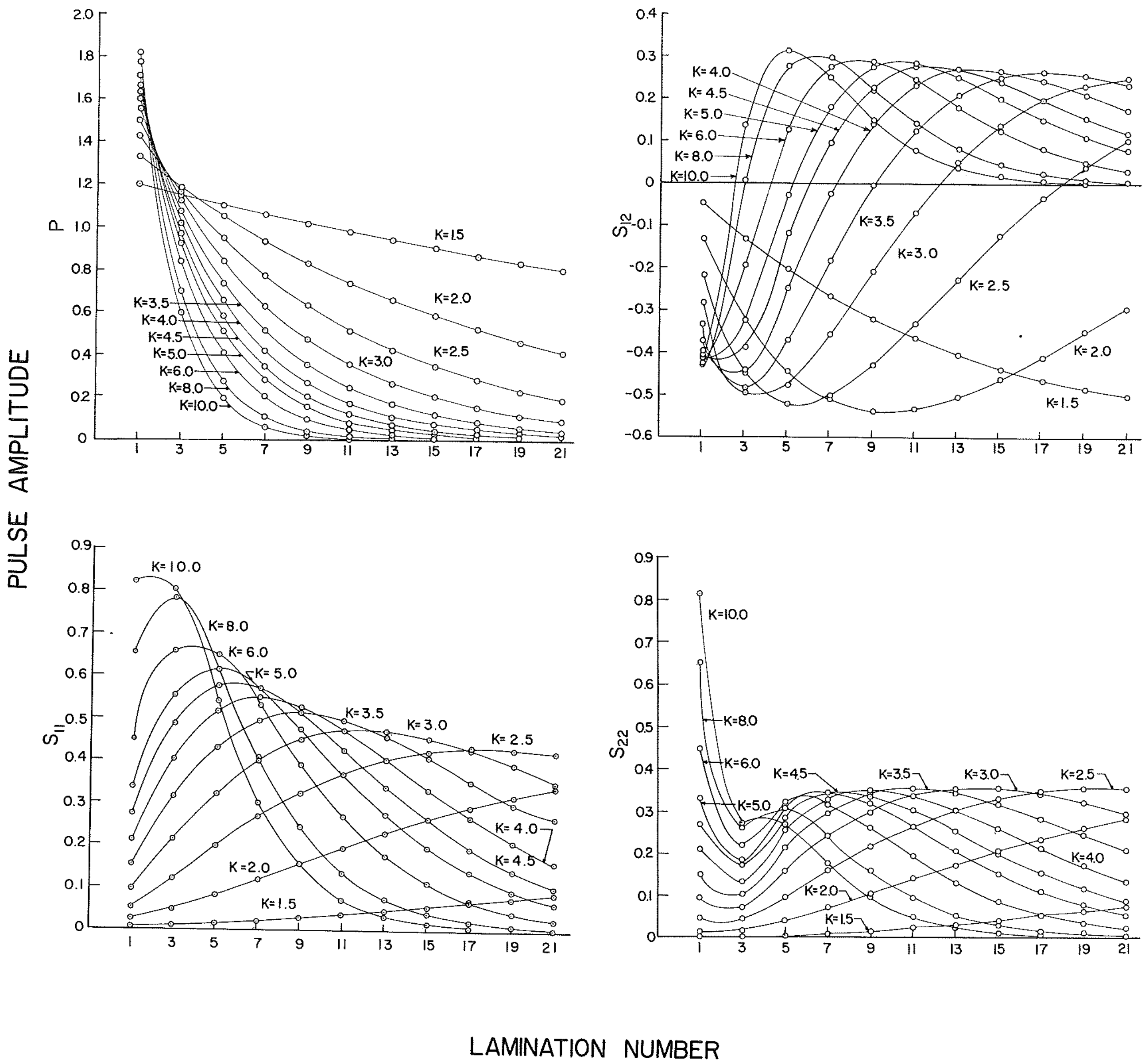


FIGURE 7

EFFECT OF IMPEDANCE MISMATCH UPON PULSE AMPLITUDE

(K > 1, N ODD)

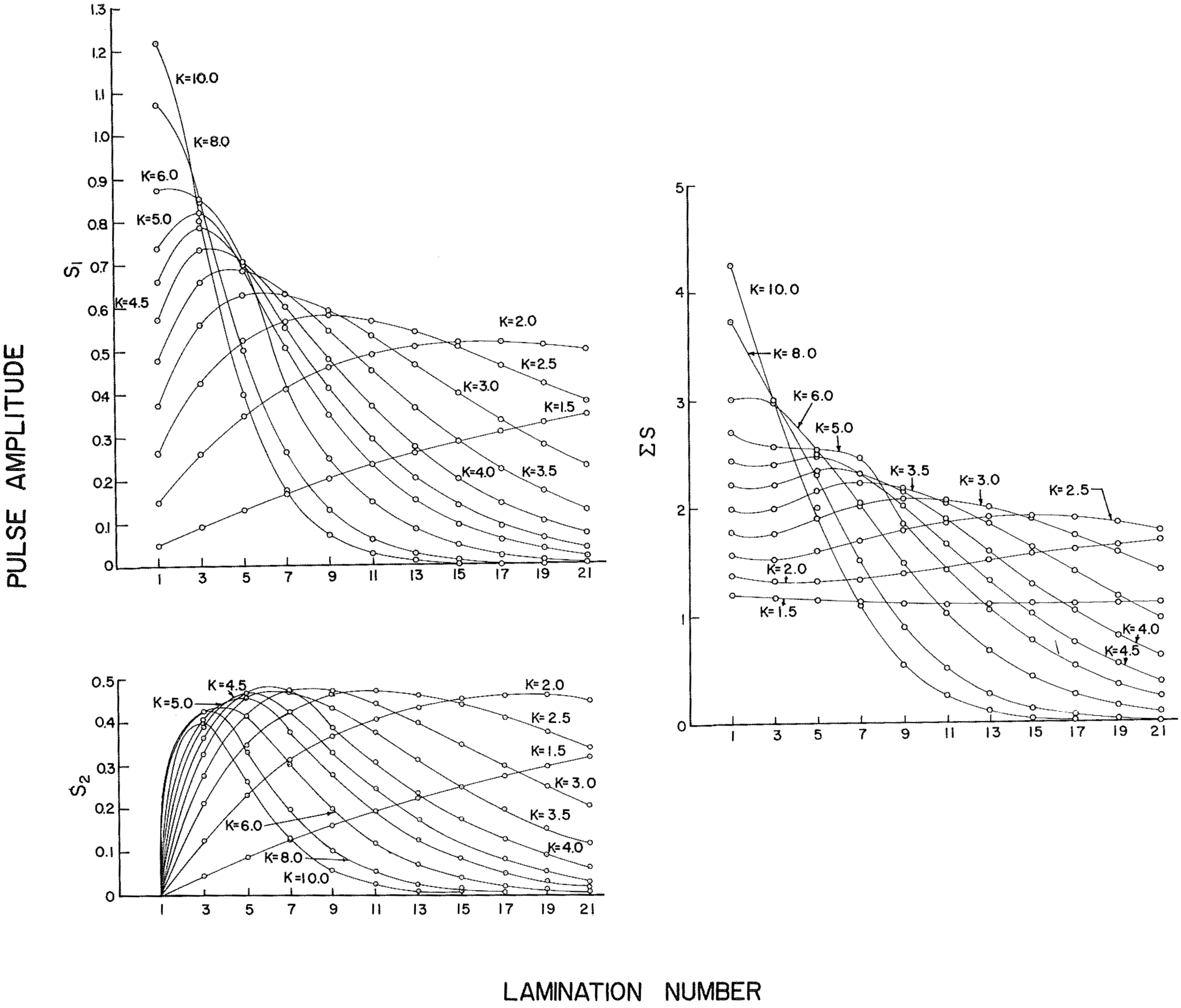


FIGURE 7 CONTINUED

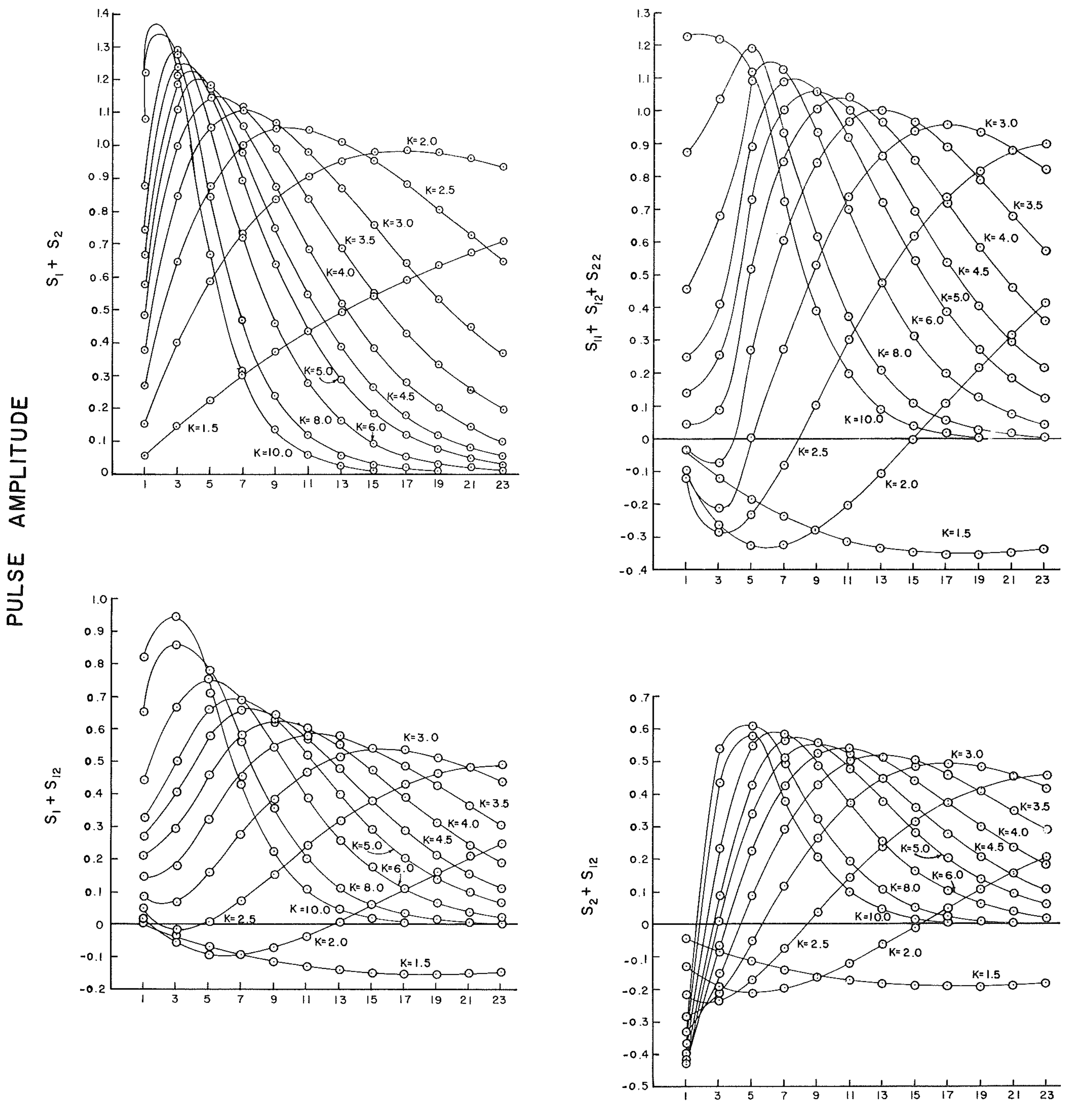


FIGURE 7 CONTINUED

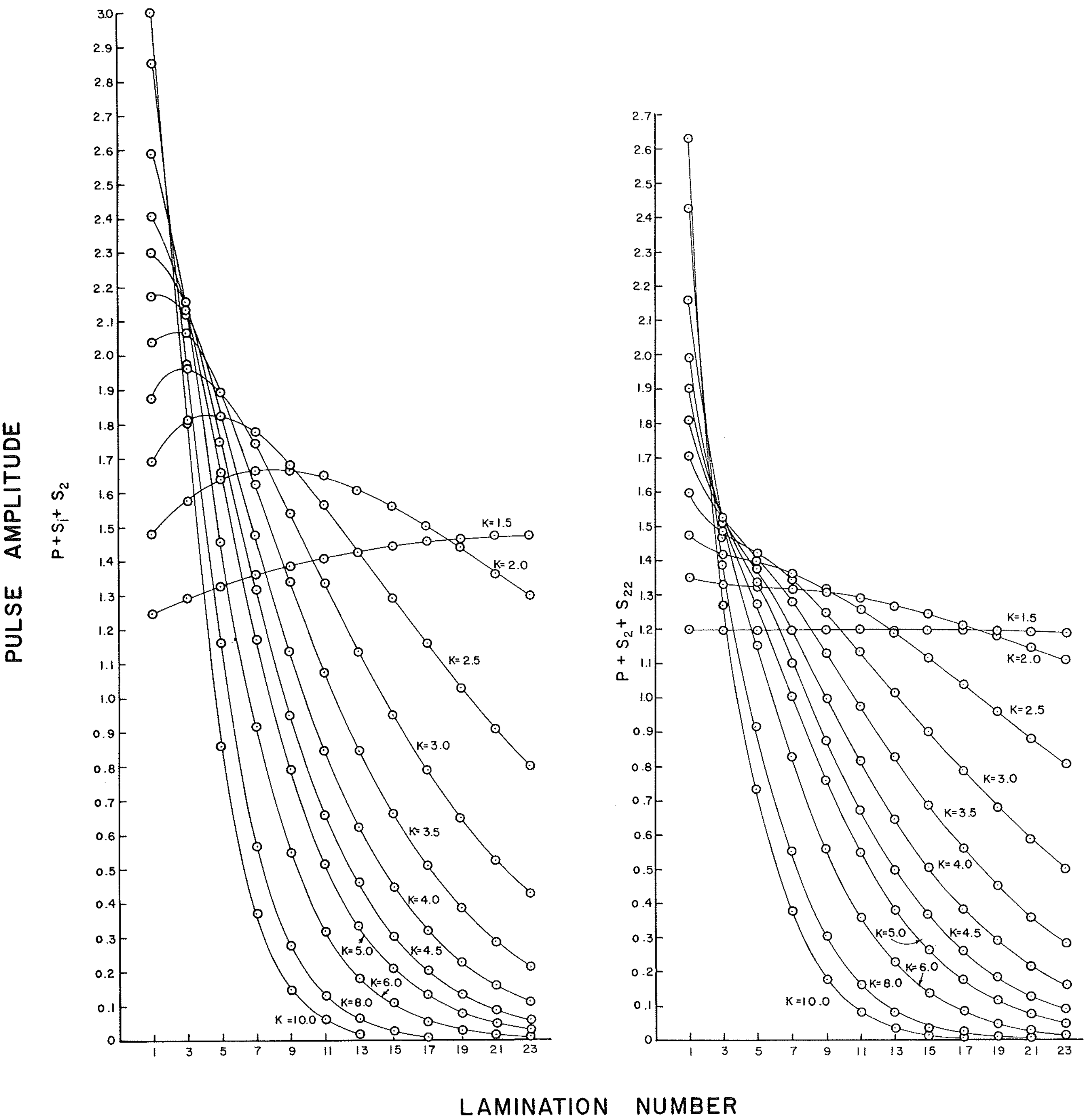


FIGURE 7 CONTINUED

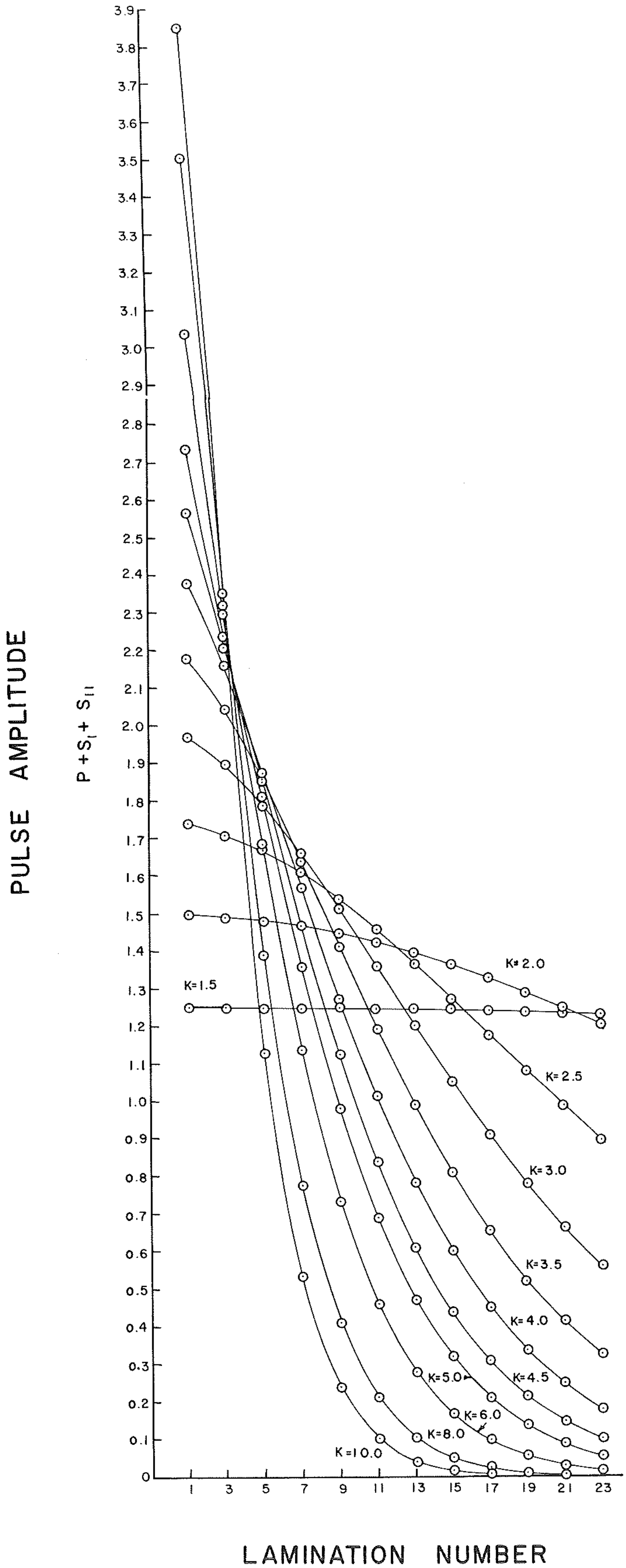


FIGURE 7 CONCLUDED

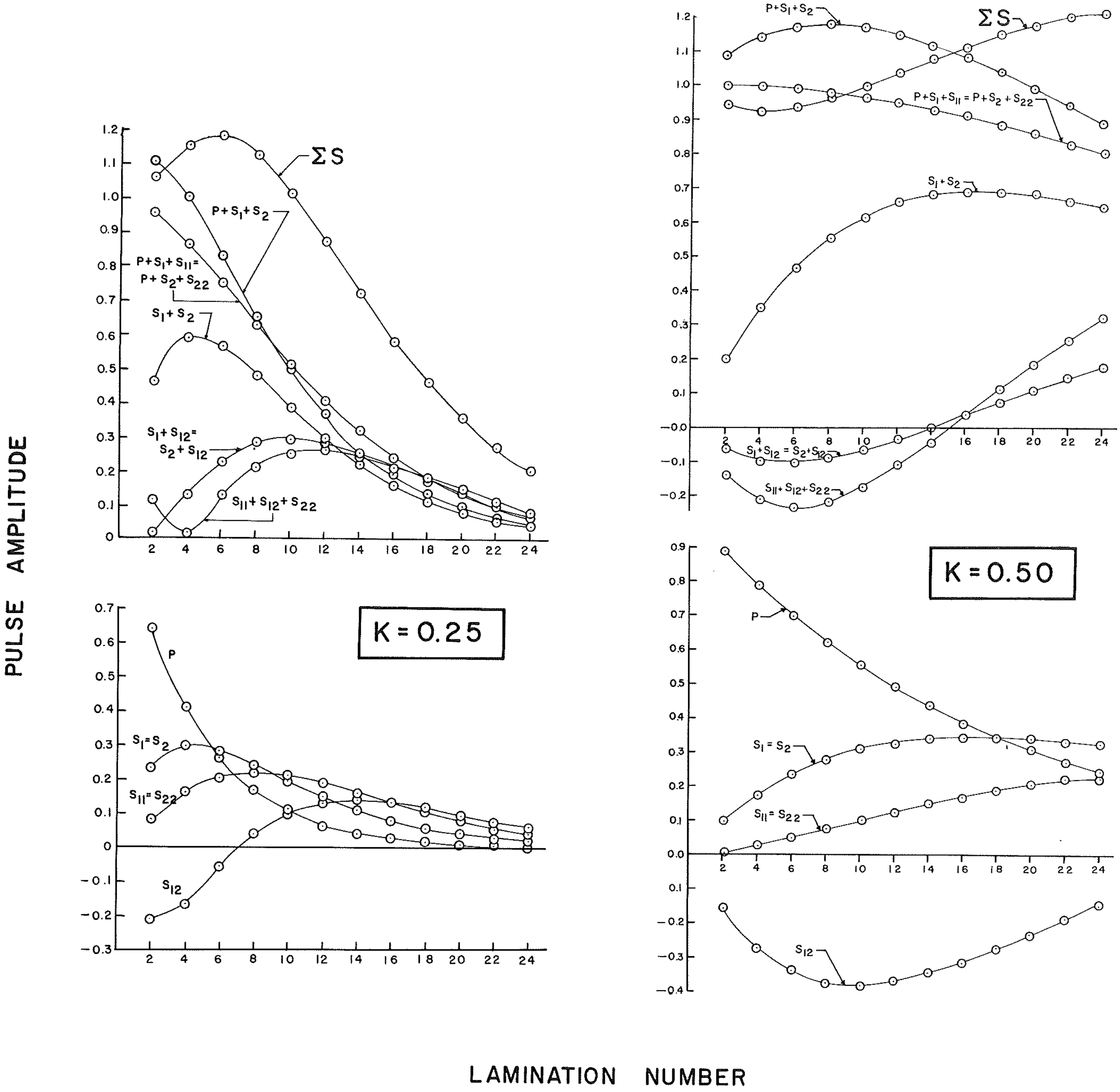


FIGURE 8

COMPARISON OF PRIMARY, SECONDARY, AND COMBINED WAVES

(N EVEN)

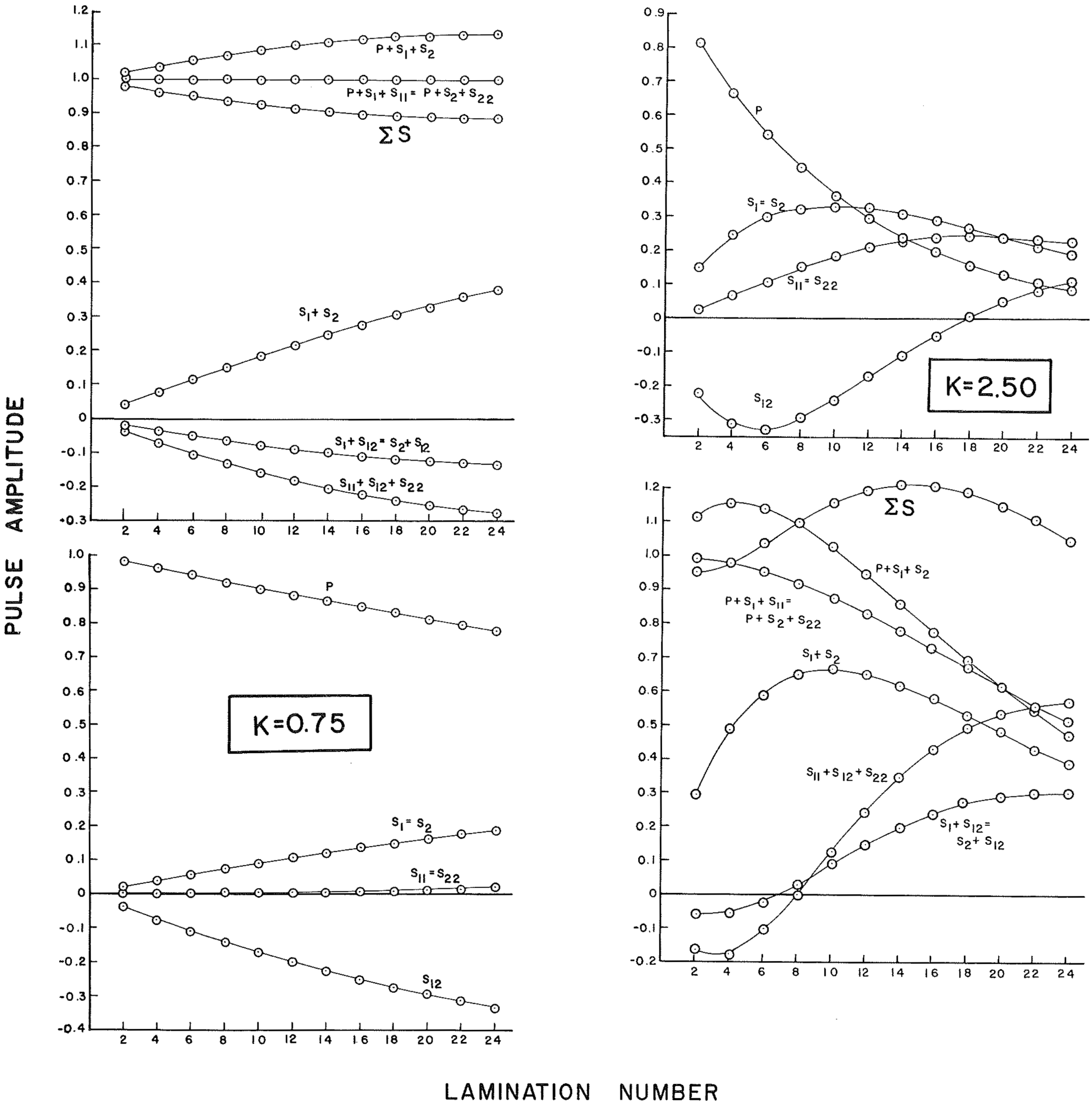


FIGURE 8 CONTINUED

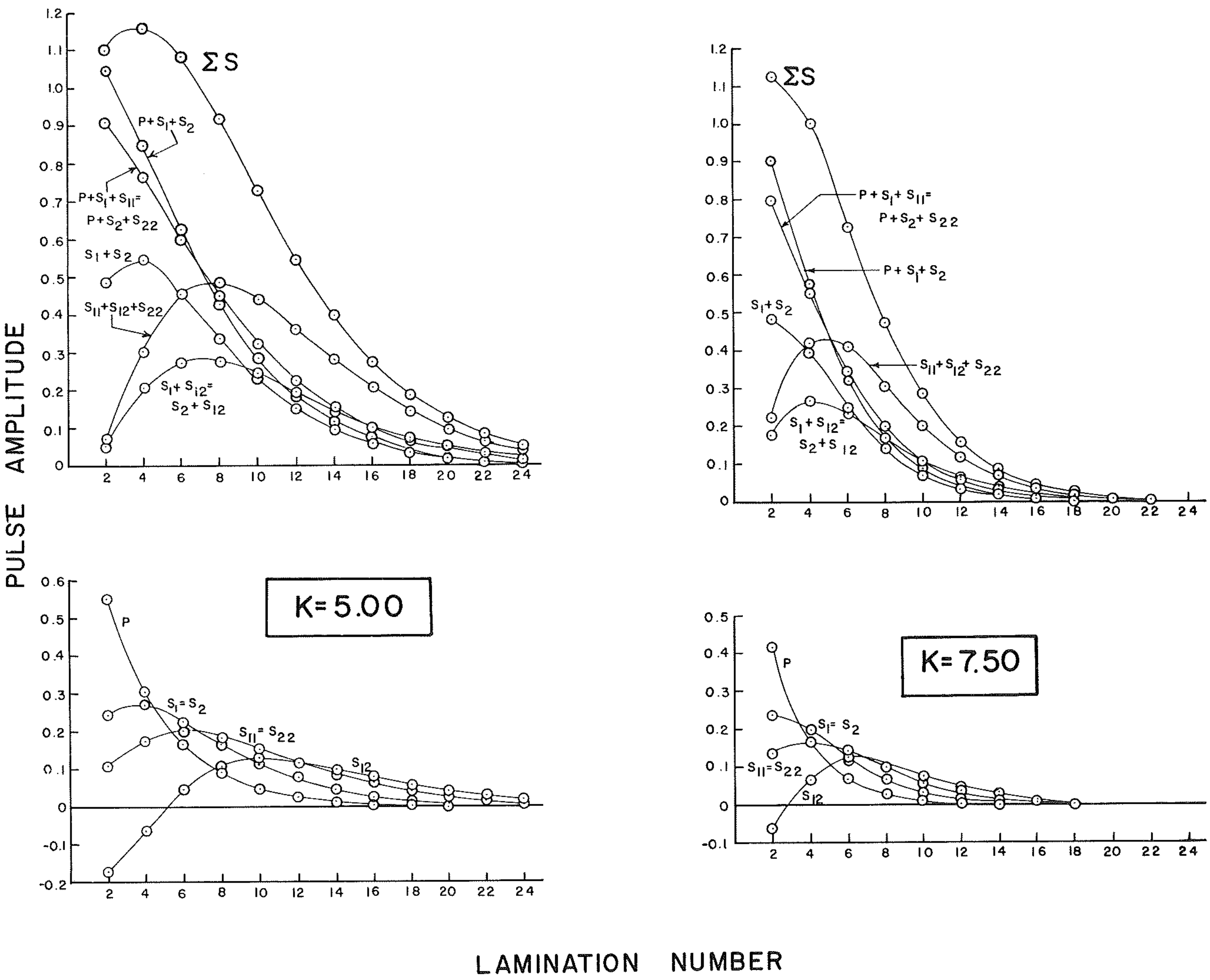


FIGURE 8 CONCLUDED

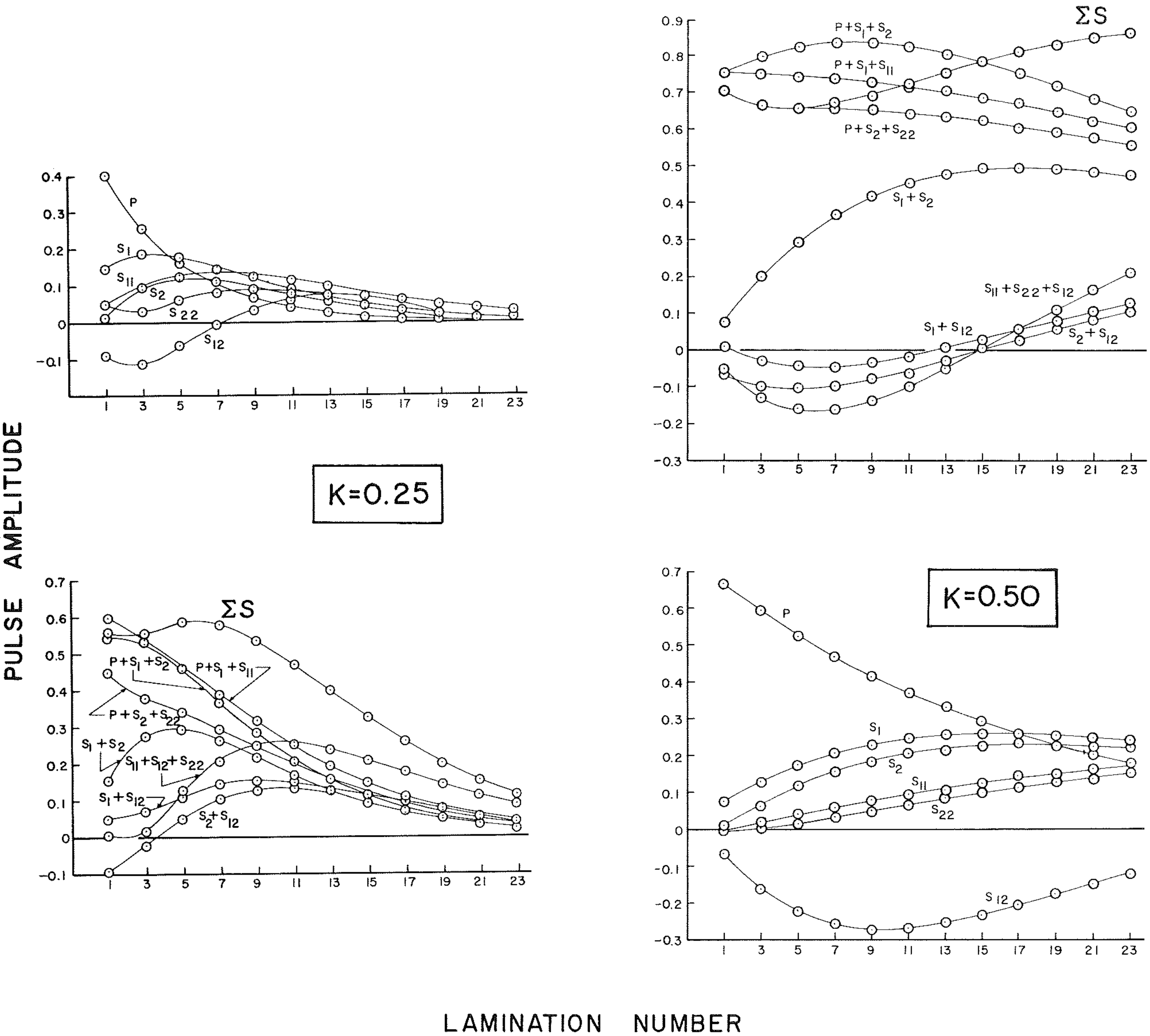


FIGURE 9

COMPARISON OF PRIMARY, SECONDARY, AND COMBINED WAVES

(N ODD)

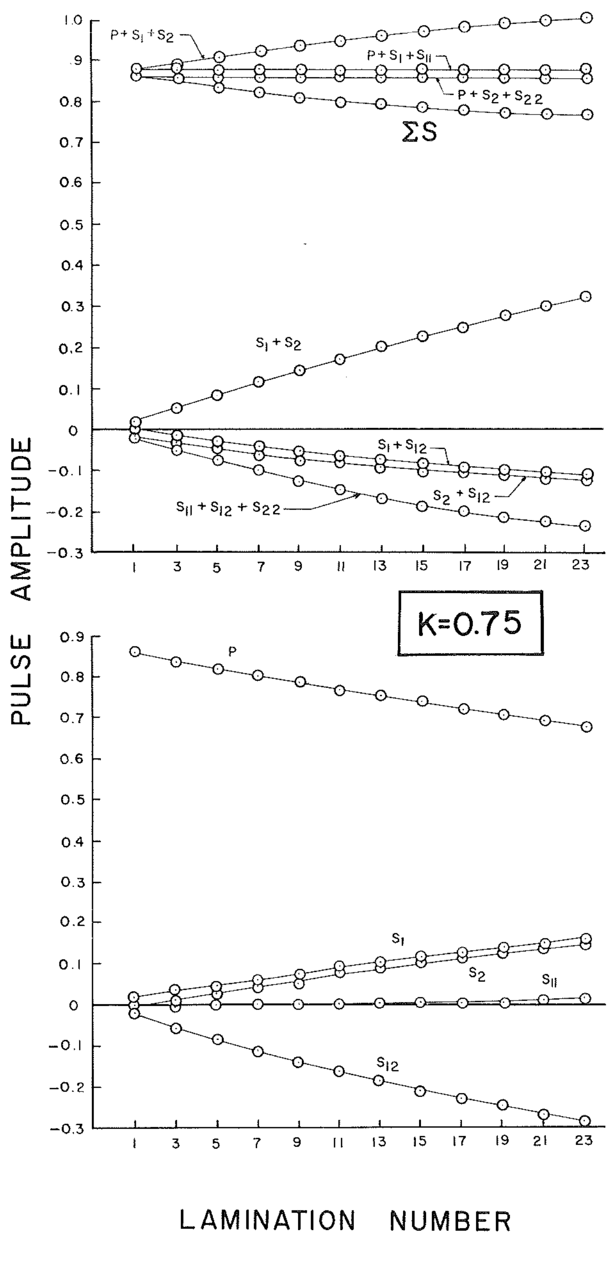


FIGURE 9 CONTINUED

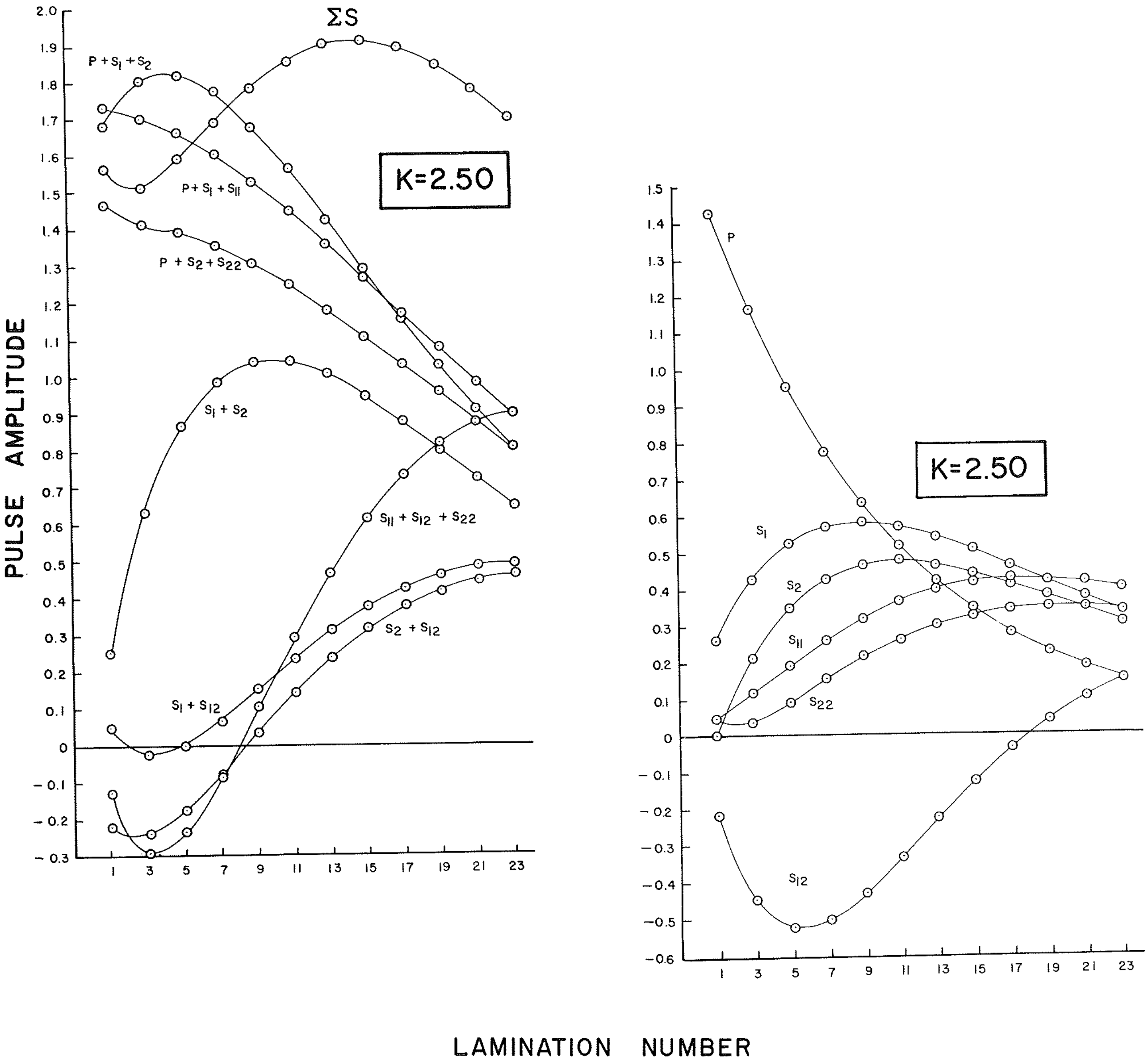


FIGURE 9 CONTINUED

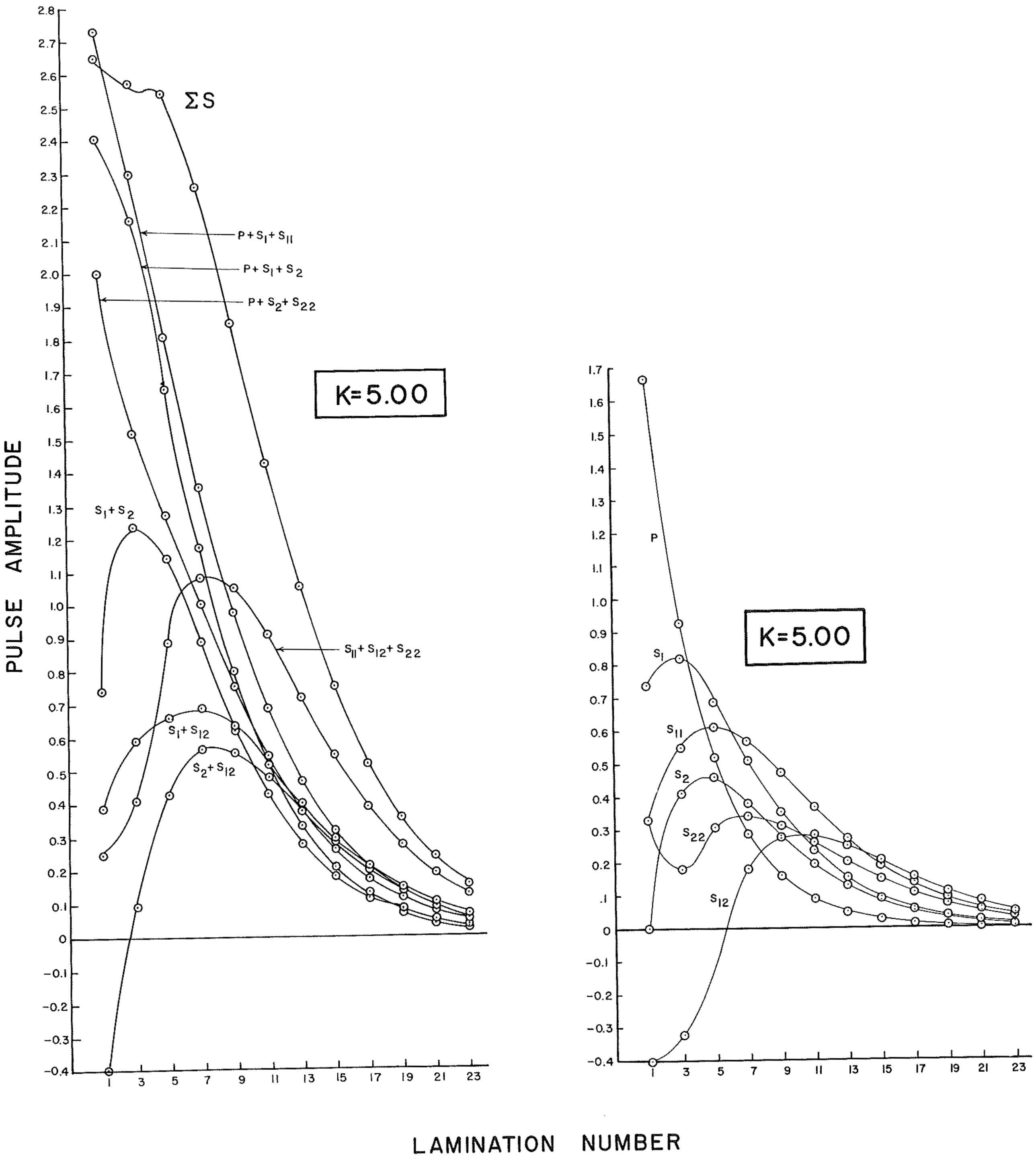


FIGURE 9 CONTINUED

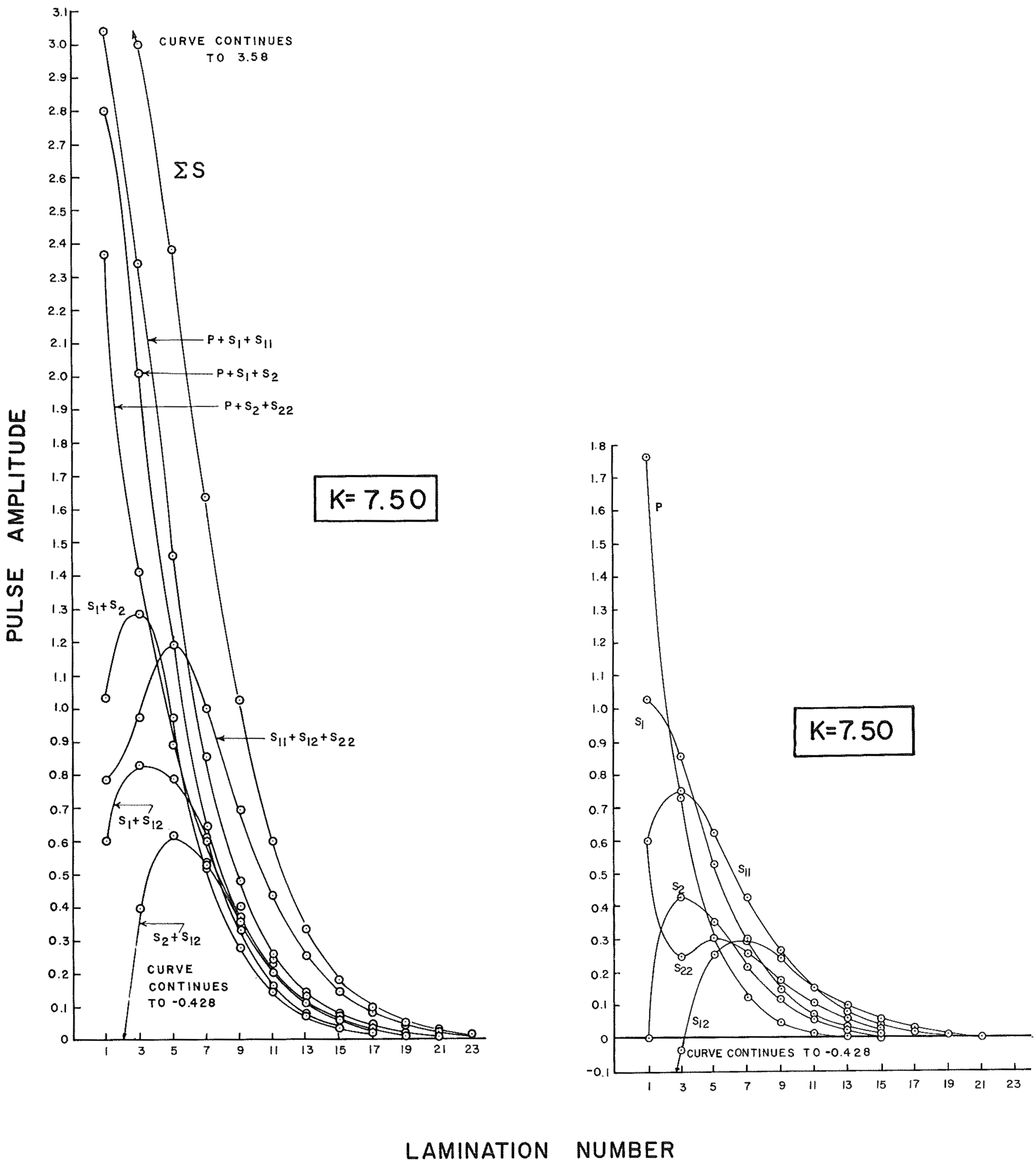


FIGURE 9 CONCLUDED

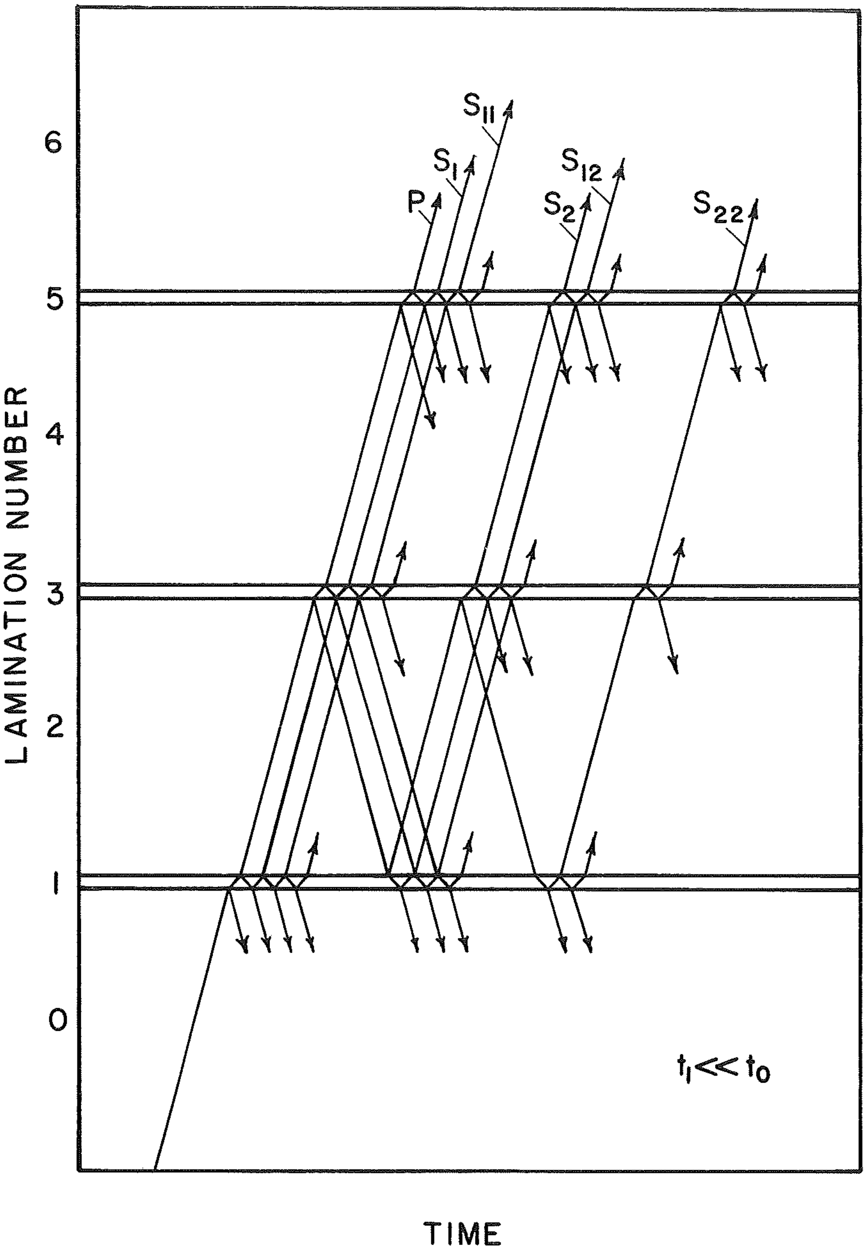


FIGURE 10

EFFECTS OF LAMINATION THICKNESS

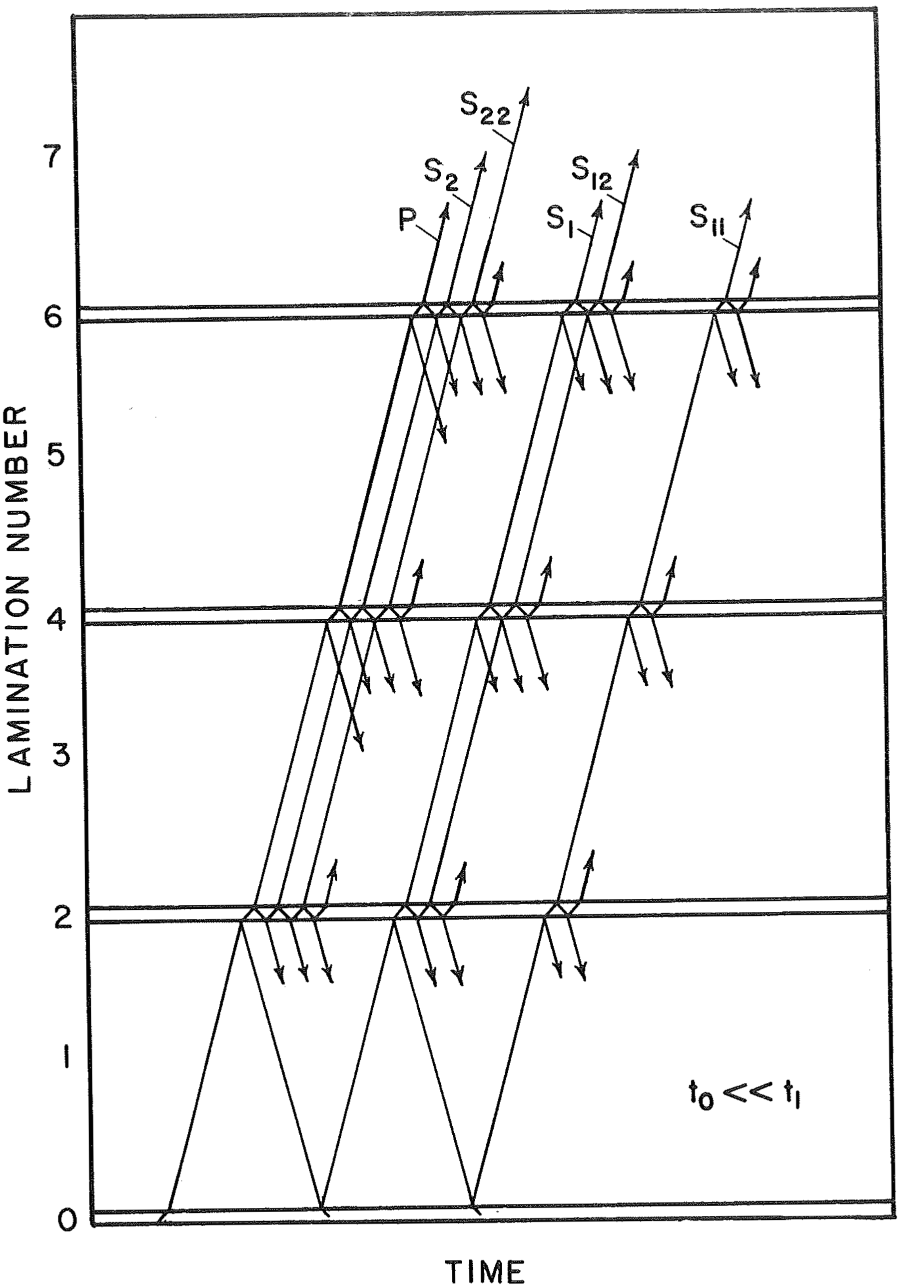


FIGURE 10 CONTINUED

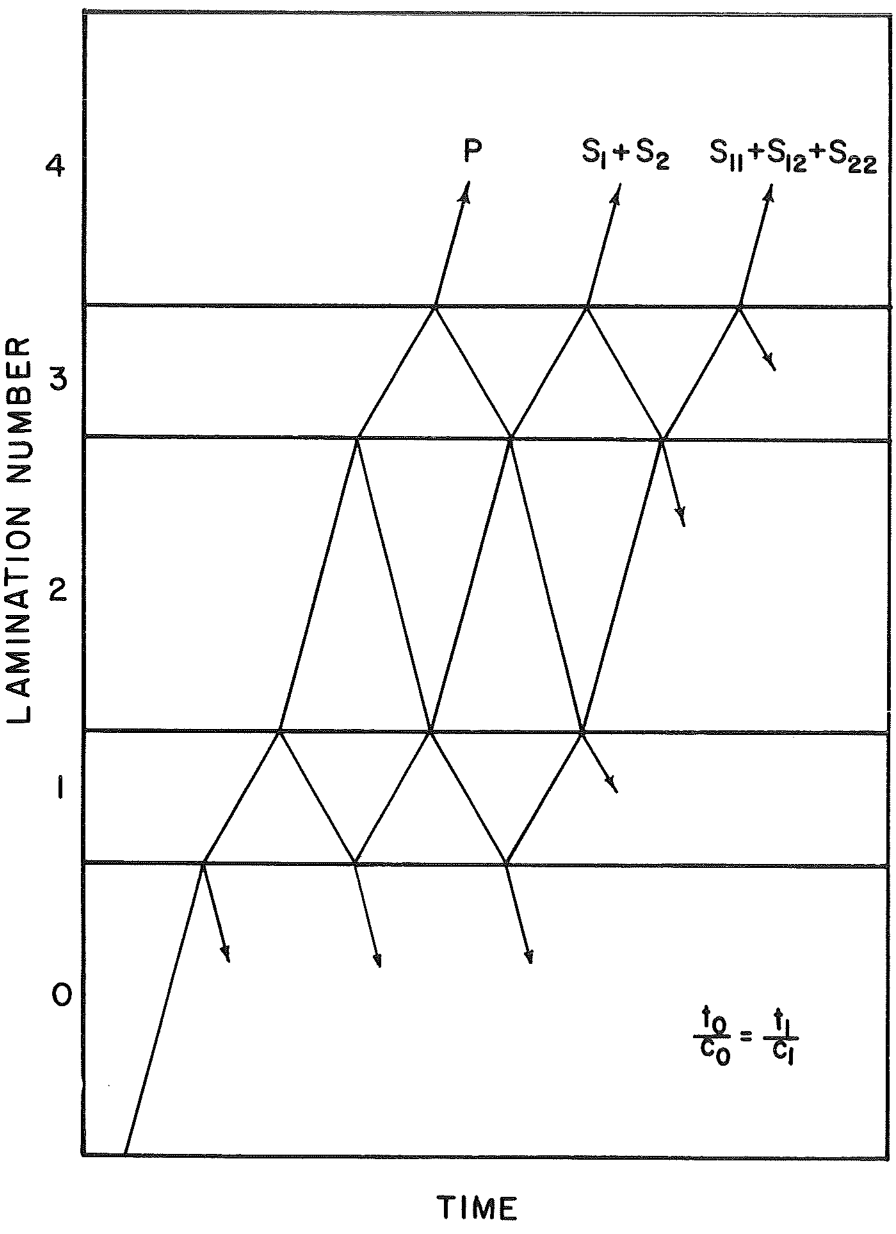


FIGURE 10 CONCLUDED

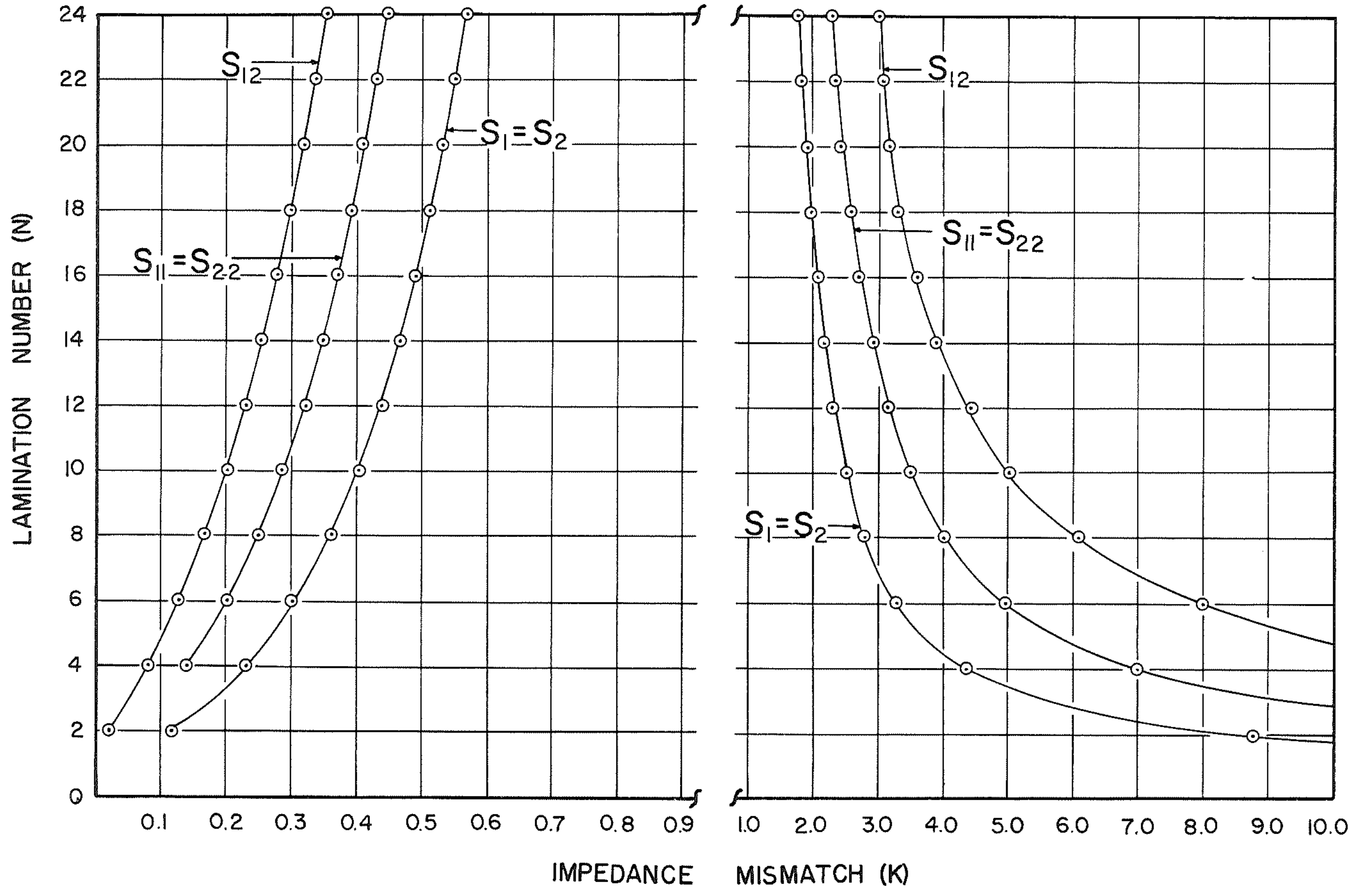


FIGURE 11

LOCATION OF MAXIMUM STRESS

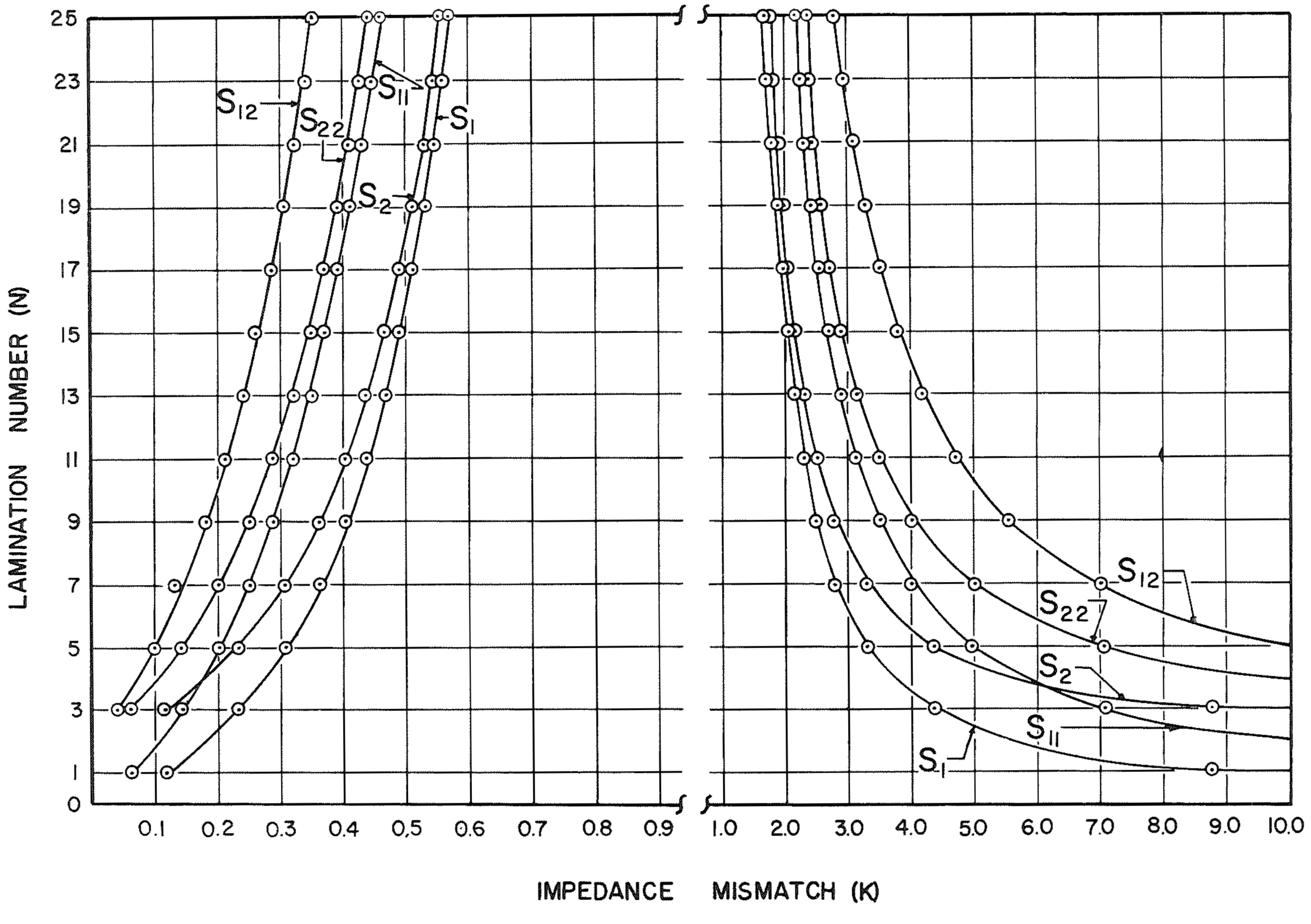


FIGURE 11 CONTINUED

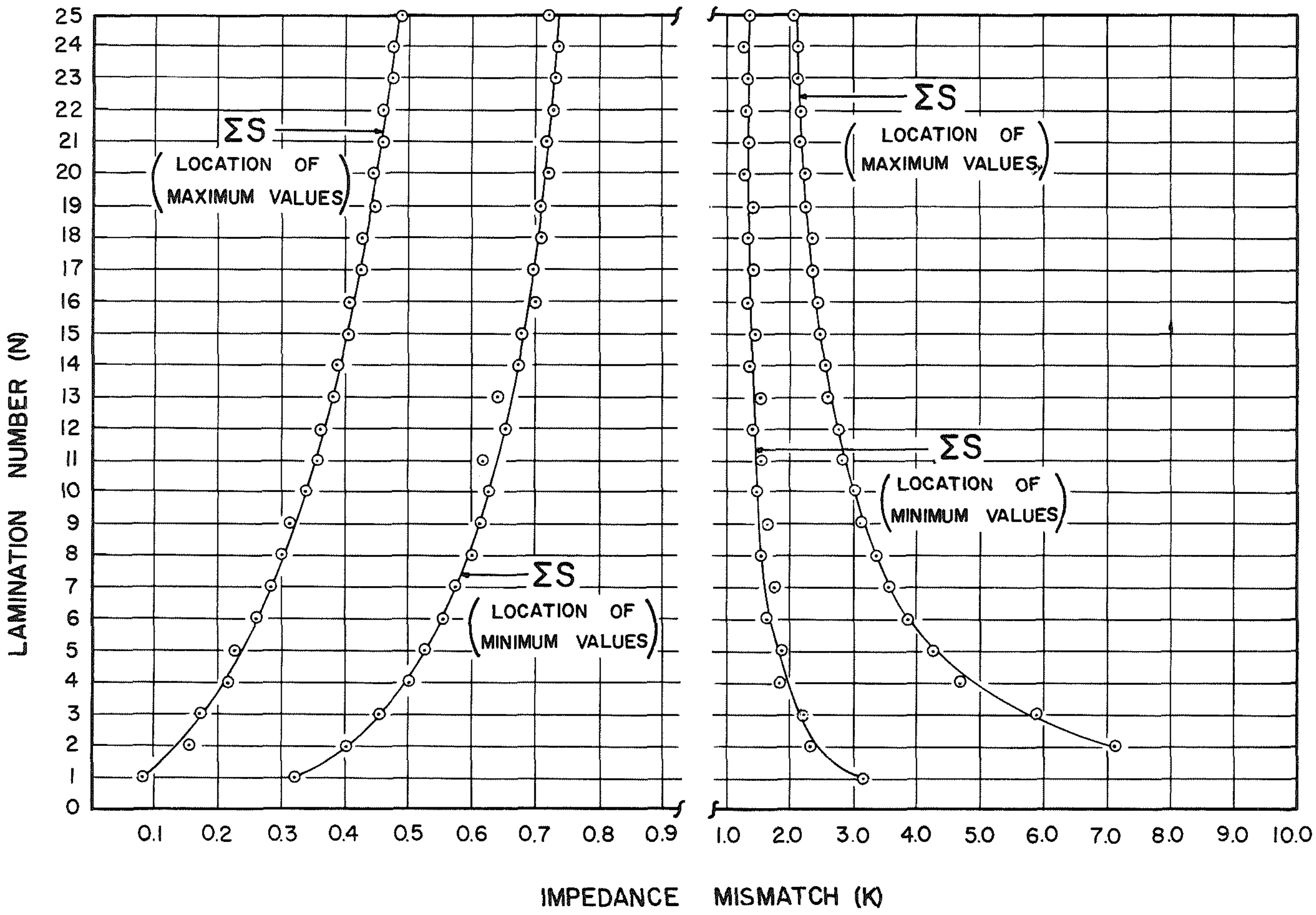
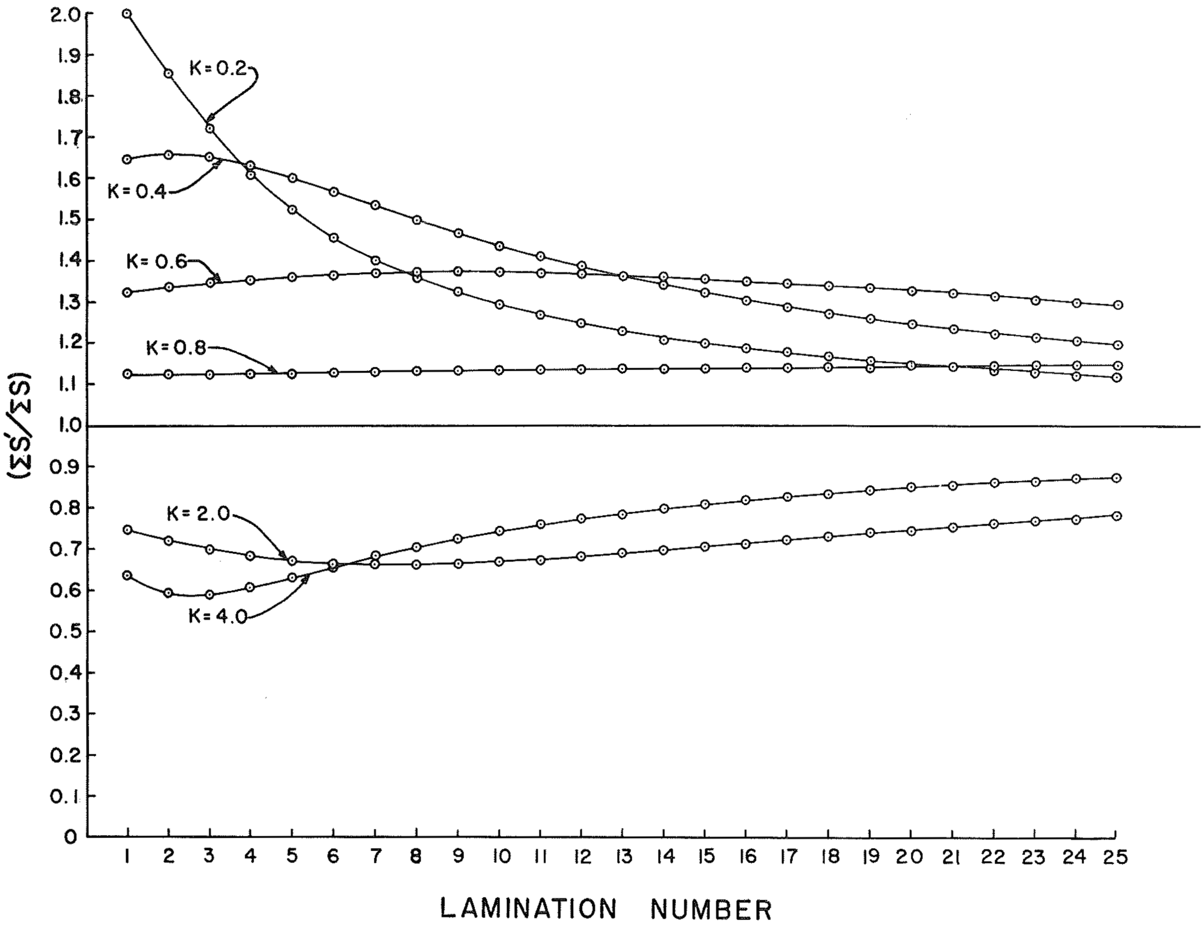


FIGURE 11 CONCLUDED



$$\frac{\Sigma S'}{\Sigma S} = 1 + \frac{2(1+K)^3(1-K) + 2(1-K^2)^2 + 2N(1+K)(1-K)^3}{2(1+K)^4 + 2N(1-K^2)^2 + N(N+1)(1-K)^4 - 8NK(1-K)^2}$$

FIGURE 12

EFFECT OF CONSIDERING REFLECTIONS FROM FRONT SURFACE