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## Preface

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## Abstract

Suboptimal filters based on erroneous models of system dynamics as well as on *a priori* statistics are discussed in this report. Suboptimal estimates for both continuous and discrete cases are derived and the error bounds are established under certain conditions. Two examples are provided to demonstrate the application of the theory developed in this report.

# Modeling Errors in Kalman Filters

## I. Introduction

In recent years the Kalman filter (Refs. 1 and 2) has been extensively used in such applications as the tracking of missiles or planes and the determination of spacecraft orbits. One of the problems that arise in these applications is that a precise knowledge of the *a priori* statistics of initial conditions and of the noise model (process noise and observation noise) are often unavailable. However, a knowledge of these conditions, as well as those of system models, is essential for the design of optimal filters.

For example, in the case of the orbit determination problem of spacecraft in deep space, observations are usually supplied in the form of doppler, counted doppler, or range data. These data are subject to oscillator instability, disturbances in the ionosphere, receiver noise, and quantization noise of the counter. These sources of interference constitute the observation noise. Additionally, the spacecraft is subject to various disturbances in the form of solar pressure, meteoroid impacts, and fuel leakage during transit to a planet (about 200 days for a typical mission to Mars). It is a rather difficult task to determine the statistics of any one of these noise sources. In addition to the uncertainty of the injection conditions of the spacecraft, the coordinates after the midcourse maneuver may enter into the filter design and influence the gain of the filter strongly during the initial period of estimation.

Errors are inevitable in assigning *a priori* covariance matrices of large dimensions because of the lack of sufficient experience or of the inability to analyze complex correlations among parameters. Lack of precise knowledge of system models is also a problem that practicing engineers frequently encounter in designing filters. This is closely related to the problem of identification, which is a major topic in control theory and applications.

The effect of incorrect *a priori* covariance matrices has been analyzed by Soong (Ref. 3) for the discrete case using the least-squares method. This analysis has been extended by the author (Refs. 4 and 5) to sequential filters and an error bound has been established for the performance of suboptimal filters. Heffes (Ref. 6) further extended this analysis to include the modeling errors of covariances of process and data noise, and the error bound for this model has been investigated by Sawaragi and Katayama (Ref. 7).

For continuous systems, an analysis similar to that for the discrete case was performed based on the *a priori* statistics (Ref. 8). Then Griffin and Sage (Ref. 9) extended this analysis to smoothing problems, including system modeling errors (errors in process and observation matrices). A general analysis was developed by the author for the *a priori* statistics as well as system modeling errors for discrete systems (Ref. 10).

This report presents a perspective of analyses on the suboptimal filter performance based on the aforementioned reference works. Both continuous and discrete systems are discussed and an effort is made to find the upper and lower bounds on the error covariances of these suboptimal filters. It is assumed that the systems are linear and the stochastic variables have gaussian distributions.

## II. Analysis for Continuous Systems

The basic process is described by a first order differential equation in vector form:

$$\frac{dx(t)}{dt} = F(t)x(t) + G(t)w(t) \tag{1}$$

The observation is

$$y(t) = H(t)x(t) + n(t) \tag{2}$$

where

$x(t)$ = an $n_x$ vector of states with

$$E[x(0)] = 0$$

$y(t)$ = an $n_y$ vector of observations

$w(t)$ = an $n_w$ vector of stochastic inputs to the process with

$$E[w(t)] = 0 \tag{3}$$

$$E[w(t)w'(\tau)] = Q(t)\delta(t-\tau) \tag{4}$$

$\delta(t)$ is the Dirac delta function.

$n(t)$ = an $n_y$ vector of the observation noise with

$$E[n(t)] = 0 \tag{5}$$

$$E[n(t)n'(\tau)] = R(t)\delta(t-\tau) \tag{6}$$

$F(t), G(t), H(t)$: $n_x \times n_x$, $n_x \times n_w$, $n_y \times n_x$ matrices respectively.

And $E[\ \ ]$ is an expected value operator on stochastic variables. Also it is assumed that the process noise $w$ and observation noise $n$ have no correlation to each other

$$E[w(t)n'(\tau)] = 0 \tag{7}$$

The optimal estimator $x^*(t)$ of $x(t)$ which minimizes $E[\|x^* - x\|^2]$ having the observation $y(t)$ from $t = 0$ to $t$ is described by the following differential equation (see Ref. 2),

$$\frac{dx^*(t)}{dt} = F(t)x^*(t) + K(t)[y(t) - H(t)x^*(t)] \tag{8}$$

where

$$K(t) = P(t)H'(t)R^{-1}(t) \tag{9}$$

It is assumed that $R(t)$ is positive definite for $t \geq 0$. The covariance matrix $P(t)$ is defined by

$$P(t) \triangleq E\{[x^*(t) - x(t)][x^*(t) - x(t)]'\} \tag{10}$$

and it is obtained as a solution of a matrix Riccati equation

$$\frac{dP(t)}{dt} = F(t)P(t) + P(t)F'(t) - P(t)H'(t)R^{-1}(t)H(t)P(t) + G(t)Q(t)G'(t) \tag{11}$$

The initial conditions for Eqs. (8) and (11) are, respectively,

$$x^*(0) = 0 \tag{12}$$

$$P(0) = E[x(0)x'(0)] \tag{13}$$

## III. Assumptions and Mathematical Derivations

The optimal estimator described in the previous section is based on the correct information of initial conditions, noise covariances, as well as coefficient matrices. Suppose one designs the estimator based on incorrect information with respect to these quantities:

(1) Incorrect $P_c(0)$ rather than the correct $P(0)$ (*a priori* covariance of states).

(2) Incorrect $Q_c(t)$ rather than the correct $Q(t)$ (covariance of the process noise).

(3) Incorrect $R_c(t)$ rather than the correct $R(t)$ (covariance of the observation noise).

(4) Incorrect $F_c(t)$ rather than the correct $F(t)$ (process matrix).

(5) Incorrect $G_c(t)$ rather than the correct $G(t)$ (coefficient matrix of the process noise).

(6) Incorrect $H_c(t)$ rather than the correct $H(t)$ (observation matrix).

The resultant estimator is no longer an optimal one, but becomes suboptimal. This suboptimal estimator is denoted $x_a^*(t)$ and is described by

$$\frac{dx_a^*(t)}{dt} = F_c(t)x_a^*(t) + K_c(t)[y(t) - H_c(t)x_a^*(t)] \tag{14}$$

where

$$x_a^*(0) = 0 \tag{15}$$

$$K_c(t) = P_c(t)H_c'(t)R_c^{-1}(t) \tag{16}$$

and the calculated covariance $P_c(t)$ is computed by the same Riccati equation as Eq. (11), but with the incorrect model specified by elements 1–6 in the above listing:

$$\begin{aligned}\frac{dP_c(t)}{dt} = F_c(t)P_c(t) + P_c(t)F_c'(t) \\ - P_c(t)H_c'(t)R_c^{-1}(t)H_c(t)P_c(t) \\ + G_c(t)Q_c(t)G_c'(t)\end{aligned} \tag{17}$$

The actual covariance $P_a(t)$ is defined as the error covariance associated with the suboptimal estimator Eq. (14), hence

$$P_a(t) \triangleq E\{[x_a^*(t) - x(t)]\,[x_a^*(t) - x(t)]\}' \tag{18}$$

This is the covariance to be expected in an estimator when insufficient design parameter data are available. The main objective of this section is to derive equations describing $P_a(t)$. For this purpose, it is easier to derive a differential equation governing the evolution of $P_a(t)$. Thus, differentiating $P_a(t)$ of Eq. (18), and exchanging the order of the differentiating operator and the expected value operator yield

$$\begin{aligned}\dot{P}_a(t) = E\{[\dot{x}_a^*(t) - \dot{x}(t)]\,[x^*(t) - x(t)]'\} \\ + E\{[x_a^*(t) - x(t)]\,[\dot{x}_a^*(t) - \dot{x}(t)]'\}\end{aligned} \tag{19}$$

However, from Eqs. (1) and (14),

$$\begin{aligned}\dot{x}_a^*(t) - \dot{x}^*(t) = [F_c(t) - K_c(t)H_c(t)]\,[x_a^*(t) - x(t)] \\ + \Delta F(t)x(t) - K_c(t)\Delta H(t)x(t) \\ + K_c(t)n(t) - G(t)w(t)\end{aligned} \tag{20}$$

where

$$\Delta F(t) = F_c(t) - F(t) \tag{21}$$

$$\Delta H(t) = H_c(t) - H(t) \tag{22}$$

Also, $x(t)$ is obtained from Eq. (1):

$$x(t) = U(t,0)x(0) + \int_0^t U(t,s)G(s)w(s)\,ds \tag{23}$$

where $U(t,s)$ is defined by

$$\frac{\partial U(t,s)}{\partial t} = F(t)U(t,s) \tag{24}$$

with

$$U(s,s) = I, \qquad t \geq s \geq 0 \tag{25}$$

and $I$ is an identity matrix.

Furthermore, $x_a^*(t)$ is derived from Eq. (14)

$$x_a^*(t) = \int_0^t V_c(t,s)K_c(s)y(s)\,ds \tag{26}$$

where $V_c(t,s)$ is defined by

$$\frac{\partial V_c(t,s)}{\partial t} = [F_c(t) - K_c(t)H_c(t)]\,V_c(t,s), \qquad t \geq s \geq 0 \tag{27}$$

When $\dot{x}_a^*(t)$ and $x_a^*(t)$ are substituted into Eq. (19) together with $\dot{x}(t)$ of Eq. (1) and its solution $x(t)$ in Eq. (23), paying attention to the fact that $w(t)$ and $n(t)$ are uncorrelated white noises, the following three differential equations were derived by Griffith and Sage (see Ref. 8):

$$\begin{aligned}\frac{dP_a(t)}{dt} = [F_c(t) - K_c(t)H_c(t)]P_a(t) + P_a(t)[F_c(t) \\ - K_c(t)H_c(t)]' \\ + [\Delta F(t) - K_c(t)\Delta H(t)]\Lambda(t) \\ + \Lambda'(t)[\Delta F(t) - K_c(t)\Delta H(t)]' \\ + K_c(t)R(t)K_c'(t) + G(t)Q(t)G'(t)\end{aligned} \tag{28}$$

$$\frac{d\Lambda(t)}{dt} = F(t)\Lambda(t) + \Lambda(t)[F_c(t) - K_c(t)H_c(t)]' + P_x(t)[\Delta F(t) - K_c(t)\Delta H(t)]' - G(t)Q(t)G'(t) \tag{29}$$

$$\frac{dP_x(t)}{dt} = F(t)P_x(t) + P_x(t)F'(t) + G(t)Q(t)G'(t) \tag{30}$$

where $\Lambda(t)$ and $P_x(t)$ are defined by

$$\Lambda(t) \triangleq E\{x(t)[x_a^*(t) - x(t)]'\} \tag{31}$$

$$P_x(t) \triangleq E[x(t)x'(t)] \tag{32}$$

The initial conditions for Eqs. (28–30) are given, respectively, by

$$P_a(0) = P(0) \tag{33}$$

$$\Lambda(0) = -P(0) \tag{34}$$

$$P_x(0) = P(0) \tag{35}$$

## IV. Error Bounds of Suboptimal Filters (Continuous Case)

When process matrices $F,G$ and observation matrix $H$ are known correctly, the results in Section III can be considerably simplified. Specifically, only the first differential equation need be solved (see Ref. 7):

$$\frac{dP_a(t)}{dt} = [F(t) - K_c(t)H(t)]P_a(t) + P_a(t)[F(t) - K_c(t)H(t)]' + K_c(t)R(t)K_c'(t) + G(t)Q(t)G'(t) \tag{36}$$

with

$$K_c(t) = P_c(t)H'(t)R_c^{-1}(t) \tag{37}$$

$$P_a(0) = P(0) \tag{38}$$

The differential equations associated with the error matrices $E_{ca}(t)$, $E_{ao}(t)$, and $E_{co}(t)$ defined as

$$E_{ca}(t) = P_c(t) - P_a(t) \tag{39}$$

$$E_{ao}(t) = P_a(t) - P(t) \tag{40}$$

$$E_{co}(t) = P_c(t) - P(t) \tag{41}$$

where $P_c(t)$, $P_a(t)$, and $P(t)$ are the computed, the actual, and the optimal covariance matrices, respectively, will be derived.

### A. Theorem 1 Development

Substitution of Eq. (36) into Eq. (39) yields the following differential equation of $E_{ca}(t)$, with the aid of Eq. (17):

$$\dot{E}_{ca}(t) = [F(t) - K_c(t)H(t)]E_{ca}(t) + E_{ca}(t)[F(t) - K_c(t)H(t)]' + K_c(t)\Delta R(t)K_c'(t) + G(t)\Delta Q(t)G'(t) \tag{42}$$

where $\Delta R(t)$ and $\Delta Q(t)$ are the differences between the incorrect and the correct noise covariances,

$$\Delta R(t) = R_c(t) - R(t) \tag{43}$$

$$\Delta Q(t) = Q_c(t) - Q(t) \tag{44}$$

Because Eq. (39) is a linear differential equation, an explicit analytic solution can be derived:

$$E_{ca}(t) = V_c(t,0)E_{ca}(0)V_c'(t,0) + \int_0^t V_c(t,s)K_c(s)\Delta R(s)K_c'(s)V_c'(t,s)\,ds + \int_0^t V_c(t,s)G(s)\Delta Q(s)G'(s)V_c'(t,s)\,ds \tag{45}$$

As observed from Eq. (45), $E_{ca}(t)$ is a sum of real symmetric matrices so that it is semipositive definite *provided* every term in the right-hand side of Eq. (45) is semipositive definite. Because of the specific (symmetric) configuration of these terms, every one will be respectively semipositive definite if every matrix at the center of each respective term, namely $E_{ca}(0)$, $\Delta R(s)$, and $\Delta Q(s)$ for $t \geq s \geq 0$, is semipositive definite.

Based on the above discussion, the following theorem is derived (see Ref. 7).

#### Theorem 1

*The difference $E_{ca}(t) \geq 0$, hence $P_c(t) \geq P_a(t)$ for $t \geq 0$ if the following condition, C-I, is satisfied.*

*C-I: $E_{ca}(0) \geq 0$, $\Delta Q(t) \geq 0$, and $\Delta R(t) \geq 0$.
Or equivalently,
$P_c(0) \geq P_a(0)$, $Q_c(t) \geq Q(t)$, and $R_c(t) \geq R(t)$ for $t \geq 0$.*

The implication of the greater than or equal to symbol is that the difference matrix $P_c(t) - P_a(t)$ is semipositive definite. Therefore, an upper bound for the variances of the suboptimal estimator $x_a^*(t)$ can be set that is equal to the diagonal components of the calculated covariance matrix $P_c(t)$ when the condition C-I is satisfied. The lower bound of these variances is, of course, zero. Let $p_{cii}(t)$ and $p_{aii}(t)$ be the respective diagonal components of $P_c(t)$ and $P_a(t)$; then

$$p_{cii}(t) \geq p_{aii}(t) \geq 0 \tag{46}$$

Though the *a priori* statistics are not known exactly, the suboptimal estimator can be expected to behave properly within the specified range provided that the conservative condition C-I is satisfied.

Though it is of less practical importance, the following corollary is derived from Eq. (45).

*Corollary 1*

*The difference $E_{ca}(t) \leq 0$, hence $P_c(t) \leq P_a(t)$ for $t \geq 0$ if the condition C-II is satisfied.*

*C-II: $E_{ca}(0) \leq 0$, $\Delta Q(t) \leq 0$, and $\Delta R(t) \leq 0$.
Or equivalently,
$P_c(0) \leq P_a(0)$, $Q_c(t) \leq Q(t)$, and $R_c(t) \leq R(t)$ for $t \geq 0$.*

### B. Theorem 2 Development

First, $\dot{E}_{ao}(t)$ can be obtained as a difference between $\dot{P}_a(t)$ and $\dot{P}(t)$ given by Eq. (36) and (11), respectively,

$$\begin{aligned}\dot{E}_{ao}(t) = {} & [F(t) - K_c(t)H(t)]E_{ao}(t) \\ & + E_{ao}(t)[F(t) - K_c(t)H(t)]' \\ & + [K_c(t)R(t) - P(t)H'(t)]\, R(t)^{-1}[K_c(t)R(t) \\ & - P(t)H'(t)]'\end{aligned} \tag{47}$$

When a similar discussion leading to Theorem 1 is applied to Eq. (47), it may be concluded that $E_{ao}(t)$ is always semipositive definite for all $t \geq 0$ because $R(t)$ is positive definite by assumption and $E_{ao}(0)$ is semipositive definite as deduced from the definition of $P(0)$.

#### Theorem 2

*The difference $E_{ao}(t) \geq 0$, hence $P_a(t) \geq P(t)$ for $t \geq 0$.*

This result is expected because $P(t)$ is the minimum variance by definition.

### C. Theorem 3 Development

Similarly, the differential equation for $E_{co}(t)$ is derived by subtracting $\dot{P}(t)$ of Eq. (11) from $\dot{P}_c(t)$ of Eq. (17)

$$\begin{aligned}E_{co}(t) = {} & [F(t) - K_c(t)H(t)]E_{co}(t) + E_{co}(t) \\ & \times [F(t) - K_c(t)H(t)]' + E_{co}(t)H'(t)R_c^{-1}(t)H(t)E_{co}(t) \\ & + G(t)\Delta Q(t)G'(t) + P(t)H'(t)\{R(t)[\Delta R(t)]^{-1}R(t) \\ & + R(t)\}^{-1}H(t)P(t) \qquad \text{for } \Delta R > 0\end{aligned} \tag{48}$$

It is clear from Eq. (48) that $E_{co}(t)$ is semipositive definite if condition C-I is satisfied. When $\Delta R = 0$ the same conclusion can be proved by taking a limit $\Delta R \rightarrow 0$.

#### Theorem 3

*The difference $E_{co} \geq 0$, hence $P_c(t) \geq P(t)$ for $t \geq 0$ if C-I is satisfied.*

## V. Analysis for Discrete Systems

The technique utilized in Section IV is applied to discrete systems and similar results are derived. Symbols are defined in the same manner as in the continuous systems, and similar assumptions are made concerning modeling errors and noise statistics.

The process and observation equations are respectively,

$$x(k+1) = \Phi(k)x(k) + G(k)w(k) \tag{49}$$

$$y(k) = H(k)x(k) + n(k) \tag{50}$$

The optimal estimate $x^*(k+1)$ with the information $Y(k) = [y(0), y(1), \cdots, y(k)]$ is given by (see Ref. 1)

$$x^*(k+1) = \Phi(k)x^*(k) + K(k)[y(k) - H(k)x^*(k)] \tag{51}$$

where

$$K(k) = \Phi(k)P(k)H'(k)[H(k)P(k)H'(k) + R(k)]^{-1} \tag{52}$$

$$x^*(0) = 0 \tag{53}$$

The covariance matrix $P(k)$ is defined by

$$P(k) \triangleq E\{[x^*(k) - x(k)]\,[x^*(k) - x(k)]'\} \tag{54}$$

and it is governed by the following nonlinear difference equation:

$$P(k+1) = [\Phi(k) - K(k)H(k)]P(k)[\Phi(k) - K(k)H(k)]' + K(k)R(k)K(k)' + G(k)Q(k)G'(k) \tag{55}$$

with

$$P(0) = E\,[x(0)x'(0)] \tag{56}$$

When the incorrect models that are the counterparts in a discrete system of those described in items (1–6) in Section III are used, the resultant suboptimal estimator $x_a^*(k)$ is computed by

$$x_a^*(k+1) = \Phi_c(k)x_a^*(k) + K_c(k)[y(k) - H_c(k)x_a^*(k)] \tag{57}$$

with

$$K_c(k) = \Phi_c(k)P_c(k)H_c'(k)\,[H_c(k)P_c(k)H_c'(k) + R_c(k)]^{-1} \tag{58}$$

$$x_a^*(0) = 0 \tag{59}$$

The calculated covariance $P_c(k)$ is

$$P_c(k+1) = [\Phi_c(k) - K_c(k)H_c(k)]P_c(k)[\Phi_c(k) - K_c(k)H_c(k)]' + K_c(k)R_c(k)K'(k) + G_c(k)Q_c(k)G_c'(k) \tag{60}$$

The actual covariance associated with this suboptimal estimator $x_a^*(k)$ is defined as

$$P_a(k) \triangleq E\,\{[x_a^*(k) - x(k)]\,[x_a^*(k) - x(k)]'\} \tag{61}$$

The recurrence equations describing $P_a(k)$ are derived similarly to the continuous case (Ref. 10),

$$\begin{aligned} P_a(k+1) = {} & [\Phi_c(k) - K_c(k)H_c(k)]P_a(k)\,[\Phi_c(k) - K_c(k)H_c(k)]' + [\Delta\Phi(k) - K_c(k)\Delta H(k)]\,\Lambda(k)[\Phi_c(k) - K_c(k)H_c(k)]' \\ & + [\Phi_c(k) - K_c(k)H_c(k)]\Lambda'(k)\,[\Delta\Phi(k) - K_c(k)\Delta H(k)]' + [\Delta\Phi(k) - K_c(k)\Delta H(k)]P_x(k)\,[\Delta\Phi(k) - K_c(k)\Delta H(k)]' \\ & + K_c(k)R(k)K_c'(k) + G(k)Q(k)G'(k) \end{aligned} \tag{62}$$

$$\Lambda(k+1) = \Phi(k)\Lambda(k)[\Phi_c(k) - K_c(k)H_c(k)]' + \Phi(k)P_x(k)\,[\Delta\Phi(k) - K_c(k)\Delta H(k)]' - G(k)Q(k)G'(k) \tag{63}$$

$$P_x(k+1) = \Phi(k)P_x(k)\Phi'(k) + G(k)Q(k)G'(k) \tag{64}$$

where $\Lambda(k)$ and $P_x(k)$ are defined by

$$\Lambda(k) \triangleq E\,\{x(k)[x_a^*(k) - x(k)]'\} \tag{65}$$

$$P_x(k) \triangleq E\,[x(k)x'(k)] \tag{66}$$

The initial conditions for the recurrence equations [Eqs. (62–64)] are respectively given by

$$P_a(0) = P(0) \tag{67}$$

$$\Lambda(0) = -P(0) \tag{68}$$

$$P_x(0) = P(0) \tag{69}$$

## VI. Error Bounds of Suboptimal Filters (Discrete Case)

When the process matrices $\Phi(k)$, $G(k)$ and the observation matrix $H(k)$ are perfectly known, only the first recurrence equation, Eq. (6), needs to be solved in order to find $P_a(k)$ (see Refs. 4–6, and 9):

$$P_a(k+1) = [\Phi(k) - K_c(k)H(k)]P_a(k)[\Phi(k) - K_c(k)H(k)]' + K_c(k)R(k)K_c'(k) + G(k)Q(k)G'(k) \tag{70}$$

### A. Theorem 4 Development

The difference matrix $E_{ca}(k+1)$ between $P_a(k+1)$ and $P_c(k+1)$ of Eqs. (60) and (78) respectively becomes

$$E_{ca}(k+1) = [\Phi(k) - K_c(k)H(k)]\, E_{ca}(k)[\Phi(k) - K_c(k)H(k)]' + K_c(k)\Delta R(k)K_c'(k) + G(k)\Delta Q(k)G'(k) \tag{71}$$

where

$$E_{ca}(k) = P_c(k) - P_a(k) \tag{72}$$

Following the same discussion used in the continuous case (Section IV) as well as the induction, the following theorem can be derived for discrete systems.

#### Theorem 4

*The difference $E_{ca}(k) \geq 0$, hence $P_c(k) \geq P_a(k)$ for $k \geq 0$ if the following condition, C-III, is satisfied.*

*C-III: $E_{ca}(0) \geq 0$, $\Delta Q(k) \geq 0$ and $\Delta R(k) \geq 0$.*
*Or equivalently,*
*$P_c(0) \geq P_a(0)$, $Q_c(k) \geq Q(k)$ and $R_c(k) \geq R(k)$ for $k \geq 0$.*

The counterpart of Corollary 1 is also derived which yields the lower bound of $P_a(k)$.

#### *Corollary 2*

*The difference $E_{ca}(k) \leq 0$, hence $P_c(k) \leq P_a(k)$ if condition C-IV is satisfied.*

*C-IV: $E_{ca}(0) \leq 0$, $\Delta Q(k) \leq 0$ and $\Delta R(k) \leq 0$ for $k \geq 0$.*
*Or equivalently,*
*$P_c(0) \leq P(0)$, $Q_c(k) \leq Q(k)$ and $R_c(k) \leq R(k)$ for $k \leq 0$.*

In the case of the other two differences,

$$E_{ao}(k) = P_a(k) - P(k) \tag{73}$$

$$E_{co}(k) = P_c(k) - P(k) \tag{74}$$

results similar to the continuous case can be proved.

### B. Theorem 5 Development

First, Eqs. (55) and (70) are substituted into Eq. (73) and after certain manipulation of matrices the following matrix form can be derived:

$$E_{ao}(k+1) = [\Phi(k) - K_c(k)H(k)]\, E_{ao}(k)[\Phi(k) - K_c(k)H(k)]' + [K_c(k) - K(k)]\, S(k)[K_c(k) - K(k)]' \tag{75}$$

where

$$S(k) = H(k)P(k)H'(k) + R(k) \tag{76}$$

$$S_c(k) = H_c(k)P_c(k)H_c'(k) + R_c(k) \tag{77}$$

The following relation is useful in the above derivation:

$$K_c(k) = K(k) + [\Phi(k) - K(k)H(k)]\, E_{co}(k)H'(k)S_c^{-1}(k) - K(k)\Delta R(k)S_c^{-1}(k) \tag{78}$$

The following theorem is derived from Eq. (75).

#### Theorem 5

*The difference $E_{ao}(k) \geq 0$, hence $P_a(k) \geq P(k)$ for $k \geq 0$.*

This is the logical conclusion because $P(k)$ is the optimum covariance by definition.

### C. Theorem 6 Development

For the third difference matrix $E_{co}(k)$ the following relation is derived:

$$E_{co}(k+1) = [\Phi(k) - K_c(k)H(k)]E_{co}(k)\,[\Phi(k) - K_c(k)H(k)]' + [K_c(k) - K(k)]S(k)[K_c(k) - K(k)]' + K_c(k)\Delta R(k)K_c'(k) + G(k)\Delta Q(k)G'(k) \tag{79}$$

and Theorem 6 is obtained.

#### Theorem 6

*The difference $E_{co}(k) \geq 0$, hence $P_c(k) \geq P(k)$ for $k \geq 0$, if condition C-III is satisfied.*

## VII. Examples

Two examples are presented to demonstrate the theoretical analysis. The first example is concerned with the modeling errors in the *a priori* statistics and the second example is concerned with the system modeling errors.

## A. Example 1: Analysis of the Effect of *A Priori* Statistics

Consider a spacecraft cruising with a constant speed along a straight line and the information is supplied by range data that are contaminated by white noise having a spectral density $\Phi_r$ and zero mean. Let $x_1$ and $x_2$ be deviations in speed and position of the spacecraft from the standard trajectory, respectively. The process equation then becomes

$$\dot{x}_1(t) = 0 \tag{80}$$

$$\dot{x}_2(t) = x_1(t) \tag{81}$$

and the observation equation is

$$y(t) = x_2(t) + n(t) \tag{82}$$

Therefore,

$$F(t) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{83}$$

$$H(t) = [0, 1] \tag{84}$$

$$Q(t) = 0 \tag{85}$$

$$R(t) = \Phi_r \tag{86}$$

The *a priori* covariance is chosen to be

$$P(0) = \begin{bmatrix} p_{11}(0) & 0 \\ 0 & p_{22}(0) \end{bmatrix} \tag{87}$$

Then the covariance $P(t)$ of the optimal estimator is derived from Eq. (11),

$$P(t) = \frac{1}{Z(t)} \begin{Bmatrix} p_{11}(0)\left[1 + \frac{p_{22}(0)t}{\Phi_r}\right] p_{11}(0)t & \left[1 + \frac{p_{22}(0)t}{2\Phi_r}\right] \\ p_{11}(0)t\left[1 + \frac{p_{22}(0)t}{2\Phi_r}\right] & p_{22}(0) + p_{11}(0)t^2 + \frac{p_{11}(0)tp_{22}(0)t^3}{3\Phi_r} \end{Bmatrix} \tag{88}$$

where

$$Z(t) = 1 + \left[ p_{22}(0) + \frac{p_{11}(0)t^2}{3} + \frac{p_{11}(0)p_{22}(0)t^3}{12\Phi_r} \right] \frac{t}{\Phi_r} \tag{89}$$

Suppose that the incorrect model actually used in the design of the suboptimal estimator is given by

$$P_c(0) = \begin{bmatrix} p_{c11}(0) & 0 \\ 0 & p_{c22}(0) \end{bmatrix} = \begin{bmatrix} p_{11}(0) + e_{11}(0) & 0 \\ 0 & p_{22}(0) + e_{22}(0) \end{bmatrix} \tag{90}$$

and

$$\Phi_{rc} = \Phi_r + \Delta R \tag{91}$$

Then the diagonal components of $E_{ca}(t)$ are computed by Eq. (36)

$$e_{ca11}(t) = e^{0}_{11}(t) + e^{R}_{11}(t) \tag{92}$$

where

$$e^{0}_{11}(t) = \frac{1}{Z_c(t)^2} \left\{ e_{11}(0)\left[1 + \frac{p_{c22}(0)t}{\Phi_{rc}}\right]^2 + \frac{e_{22}(0)p_{c11}(0)t^4}{4\Phi^2_{rc}} \right\} \tag{93}$$

$$e_{11}^{R}(t) = \frac{p_{c11}^{2}(0)\,\Delta R t^{3}}{12\Phi_{rc}^{4}\,Z_c(t)^{2}}\,[4\Phi_{rc}^{2} + 2p_{c22}(0)\,\Phi_{rc}t + p_{c22}^{2}\,(0)t^{2}] \tag{94}$$

$$Z_c(t) = 1 + \left[p_{c22}(0) + \frac{p_{c11}(0)t^{2}}{3} + \frac{p_{c11}(0)p_{c22}(0)t^{3}}{12\Phi_{rc}}\right]\frac{t}{\Phi_{rc}} \tag{95}$$

Also,

$$e_{ca22}(t) = e_{22}^{0}(t) + e_{22}^{R}(t) \tag{96}$$

where

$$e_{22}^{0}(t) = \frac{1}{Z_c(t)^{2}}\left\{e_{11}(0)\left[1 + \frac{p_{c22}(0)t}{2\Phi_{rc}}\right]^{2} t^{2} + e_{22}(0)\left[1 - \frac{p_{c11}(0)t^{3}}{6\Phi_{rc}}\right]^{2}\right\} \tag{97}$$

$$e_{22}^{R}(t) = \frac{\Delta R t}{\Phi_{rc}^{2}Z_c(t)^{2}}\left\{\left[p_{c22}(0) + \frac{p_{c11}(0)t^{2}}{2} + \frac{p_{c11}(0)p_{c22}(0)t^{3}}{12\Phi_{rc}}\right]^{2} + \frac{p_{c11}(0)^{2}t^{4}}{12}\left[1 + \frac{p_{c22}(0)t}{2\Phi_{rc}}\right]^{2}\right\} \tag{98}$$

In Fig. 1, the optimal variance $p_{22}(t)$ and actual variance $p_{a22}(t)$ of position of the spacecraft are depicted with $p_{c11}(0)$ as the variable parameter. [The values for $p_{11}(0)$, $p_{22}(0)$, and $\Phi_r$ are set at 1.0 m²/s², 10⁴ m², and 1.0 km²/s, respectively, in Figs. 1–5.] The optimal variance $p_{22}(t)$ is computed based on the true model. The suboptimal filter is designed such that

$$p_{c22}(0) > p_{22}(0)$$

$$\Phi_{rc} > \Phi_r$$

Figure 1 also indicates that the variance of the suboptimal filter is quite sensitive to variation of $p_{c11}(0)$ (i.e., the incorrect initial speed variance). Case (a) expresses an excessively large *a priori* uncertainty of speed [$p_{c11}(0) = 10p_{11}(0)$]. Conversely, case (e) expresses the *a priori* value taken as less than the true value [$p_{c11}(0) = p_{11}(0)/2$]. For both cases significant overshoots of the variance are observed. This is because the gain $K_c(t)$ was ill-conditioned for both extreme cases. In other words, sufficient weights had not been assigned to the information during the initial period so that the station did not track the spacecraft in a proper manner. Case (e) especially demonstrates how the estimator can behave poorly when an optimistic selection is made on the *a priori* covariance. In Fig. 2, the calculated variances $p_{c11}(t)$ used in Fig. 1 case (e) are plotted for the same parameters used in Fig. 1.

The variance of initial position $p_{c22}(0)$ is changed as a parameter in Fig. 3. It can be observed that the suboptimal filter is not as sensitive to the initial uncertainty of position as it is to that of speed. However, case (d) reveals a degraded performance of the filter when a smaller value is picked up for the positional uncertainty than the true value [$p_{c22}(0) = p_{22}(0)/10$].

In Fig. 4, the incorrect information $\Phi_{rc}$ of the power spectral density of the observation noise is employed as a parameter. The suboptimal filter behaves very poorly for $\Phi_{rc}$ which is either very large [case (a); $\Phi_{rc} = 10\,\Phi_r$] or very small [case (d) $\Phi_{rc} = \Phi_r/2$] compared to the true $\Phi_r$.

Figure 5 is one example of variances of speed of the spacecraft, corresponding to case (e) of Fig. 1.

### B. Example 2: Sensitivity Analysis of Noise Correlation Time

*1. Modeling error of data noise.* A spacecraft is assumed that is moving radially away from a fixed point and is tracked by doppler methods. These doppler data are contaminated by an exponentially correlated data noise. It is also assumed that the spacecraft is subject to a small, random, white-noise acceleration.

As a direct application of the results derived for the discrete case (Section V), the effect of modeling errors on an exponentially correlated data noise and on process noise are studied.

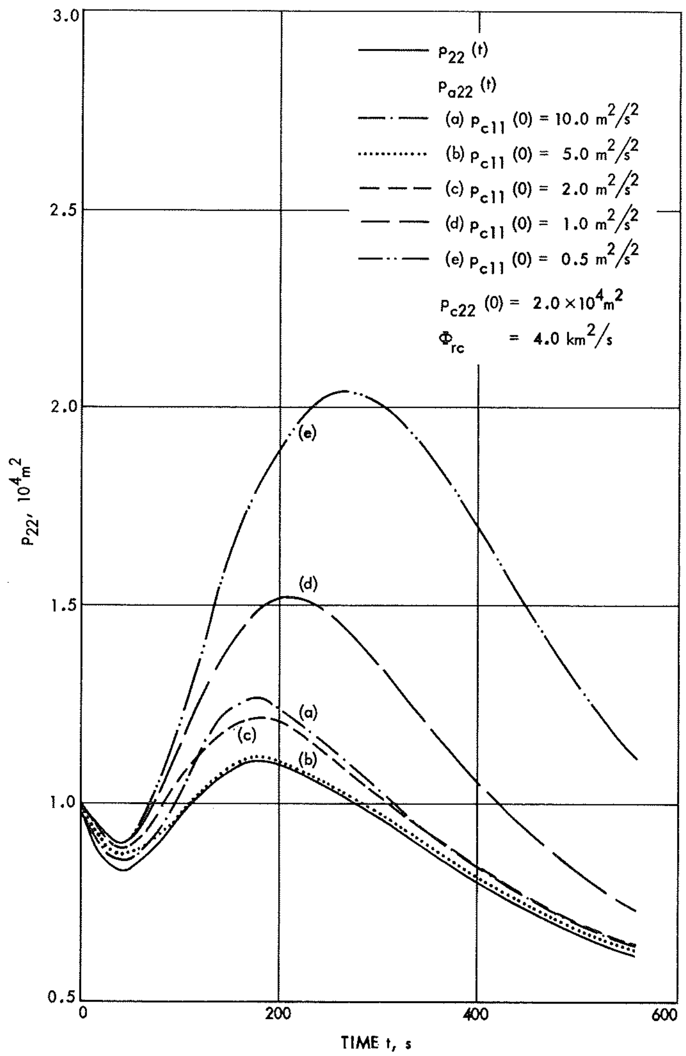


**Fig. 1. Actual and optimal variances of position of a spacecraft with range data for various initial suboptimal speed variances**

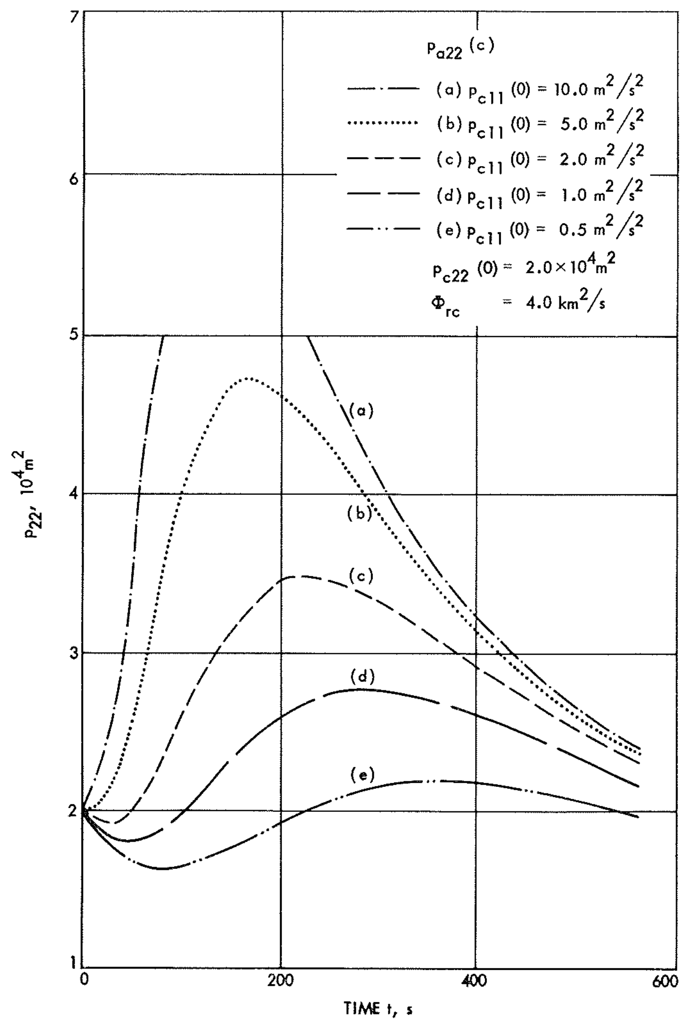


**Fig. 2. Computed variance of position of a spacecraft with range data for various initial suboptimal speed variances**

Let the speed of the spacecraft be $x_1$ and the data noise be $x_2$. Then the basic system equations are

$$x_1(k+1) = x_1(k) + w(k) \tag{99}$$

$$x_2(k+1) = bx_2(k) + v(k) \tag{100}$$

$$y(k) = x_1(k) + x_2(k) \tag{101}$$

where $w$ and $v$ are independent white noise with variances $q$ and $r$, respectively. The term $b$ is related to the correlation time $\tau$ by

$$b = e^{-T/\tau} \tag{102}$$

where $T$ is a discrete period. In this analysis, the steady-state solutions (as $k \rightarrow \infty$) are considered, mainly because simple analytic solutions can be obtained for this case. Also, this line of analysis is justifiable when the tracking period is much longer than the noise correlation time.

The elements $p_{ij}$ of the optimal covariance of Eq. (55) are computed.

$$p_{11} = \frac{1}{2(1-b)} \{(1-3b)q + [(1+b)^2 q^2 + 4qr]^{1/2}\} \tag{103}$$

$$p_{12} = -b(p_{11} - q) \tag{104}$$

$$p_{22} = b^2(p_{11} - q) + r \tag{105}$$

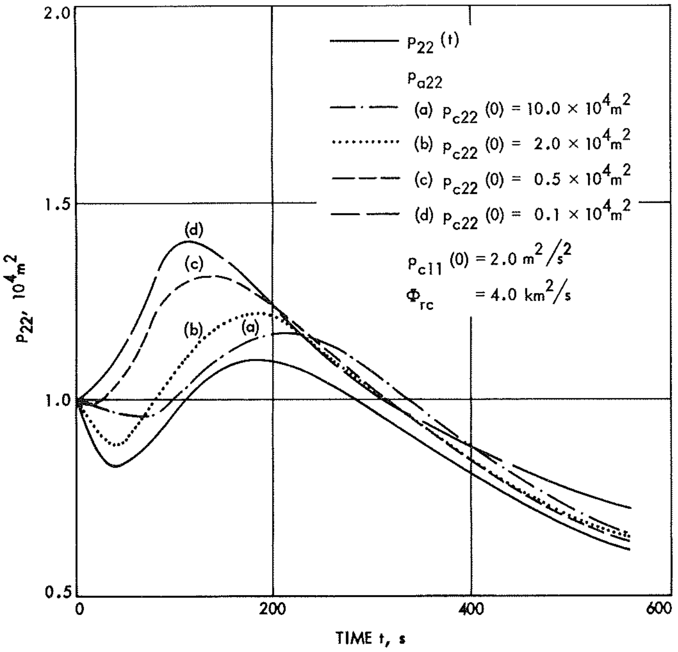


**Fig. 3. Actual and optimal variances of position of a spacecraft with range data for various initial suboptimal position variances**

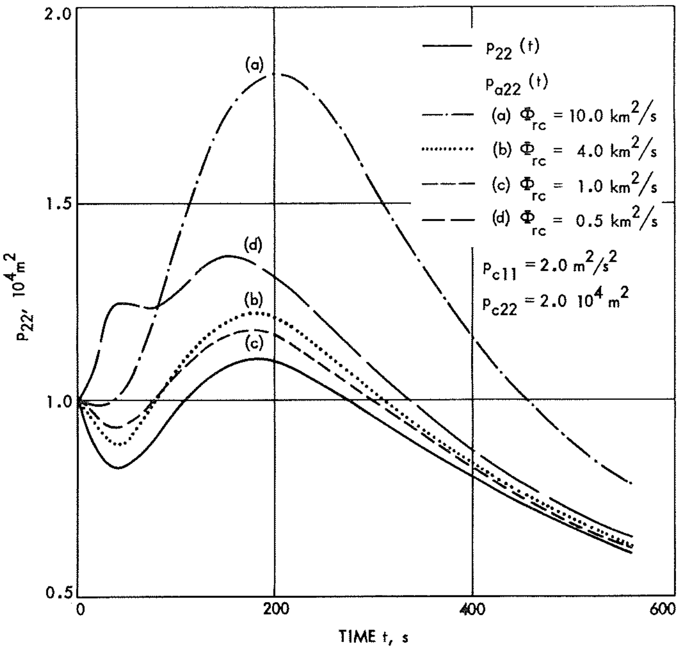


**Fig. 4. Actual and optimal variances of position of a spacecraft with range data for various suboptimal noise spectral densities**

When a different noise model $b_c$ (where $b_c = e^{-T/\tau_c}$) and $r_c$ is employed, the covariance $p_{cij}$ is computed by the above equations with $b_c$ and $r_c$ in the place of $b$ and $r$ respectively. The filter is designed which becomes inevitably suboptimal. The variances $p_{aij}$ associated with this suboptimal filter are computed as solutions of three sets of recurrence equations (Eqs. 62–64). Again only the steady state solutions are considered.

The related portions of $P_x$ and $\Lambda$ matrixes are ($\sigma_n$ = standard deviation of data noise)

$$p_{x22} = \frac{r}{1-b^2} = \sigma_n^2 \tag{106}$$

$$\lambda_{21} = -\frac{1}{D(\lambda)} b k_{c1} \sigma_n^2 \left[ b \Delta b - (1 - b^2) \right] \tag{107}$$

$$\lambda_{22} = \frac{1}{D(\lambda)} \left[ 1 - b(1 - k_{c1}) \right] \left[ b \Delta b - (1 - b^2) \right] \sigma_n^2 \tag{108}$$

where

$$D(\lambda) = (1 - b)(1 - bb_c) + b(k_{c1} + k_{c2} - bb_c) \tag{109}$$

$$\Delta b = b_c - b \tag{110}$$

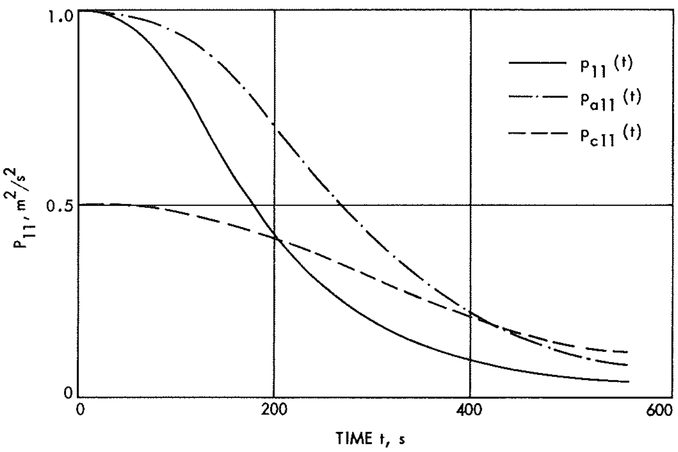


**Fig. 5. Computed, actual, and optimal variances of speed of a spacecraft with range data**

Also, the computed filter gains $k_{c1}$ and $k_{c2}$ are given by

$$k_{c1} = \frac{p_{c11} + p_{c12}}{D(p_c)} \tag{111}$$

$$k_{c2} = \frac{b_c (p_{c12} + p_{c22})}{D(p_c)} \tag{112}$$

with

$$D(p_c) = p_{c11} + 2p_{12} + p_{c22} \tag{113}$$

By the use of these elements of the $P_x$ and $\Lambda$ matrices, $p_{aij}$ are derived from Eq. (62) which are given in the form of the following three linear equations:

$$(2k_{c1} - k_{c1}^2)p_{a11} + 2k_{c1}(1 - k_{c1})p_{a12} - k_{c1}^2\, p_{a22} = q \tag{114}$$

$$k_{c2}(1 - k_{c1})p_{a11} + [1 - k_{c1}k_{c2} - (b_c - k_{c2})(1 - k_{c1})]p_{a12} + (b - k_{c2})k_{c1}\, p_{a22} = f_1 \tag{115}$$

$$-k_{c2}^2 p_{a11} + 2k_{c2}(b_c - k_{c2})p_{a12} + [1 - (b_c - k_{c2})^2]p_{a22} = f_2 + r \tag{116}$$

where

$$f_1 = (1 - k_{c1})\Delta b\, \lambda_{21} - \lambda_{22}k_{c1}\Delta b \tag{117}$$

$$f_2 = 2[\lambda_{22}(b_c - k_{c2})\Delta b - k_{c2}\lambda_{21}\Delta b] + (\Delta b)^2 \sigma_n^2 \tag{118}$$

Solving these linear equations provides the desired $p_{aij}$.

The standard deviations of the estimate of speed with the suboptimal filter are plotted in Figs. 6(a) and 7(a) against $\tau_c$ and $\sigma_{nc}$, respectively. The standard deviation of data noise is $\sigma_n = 1$ mm/s and that of acceleration noise is $\sigma_w = 5.771 \times 10^{-3}$ cm/s$^2$ and $T$ is taken as 1 min. The nominal correlation time of data noise is $\tau = 30$ s. In both

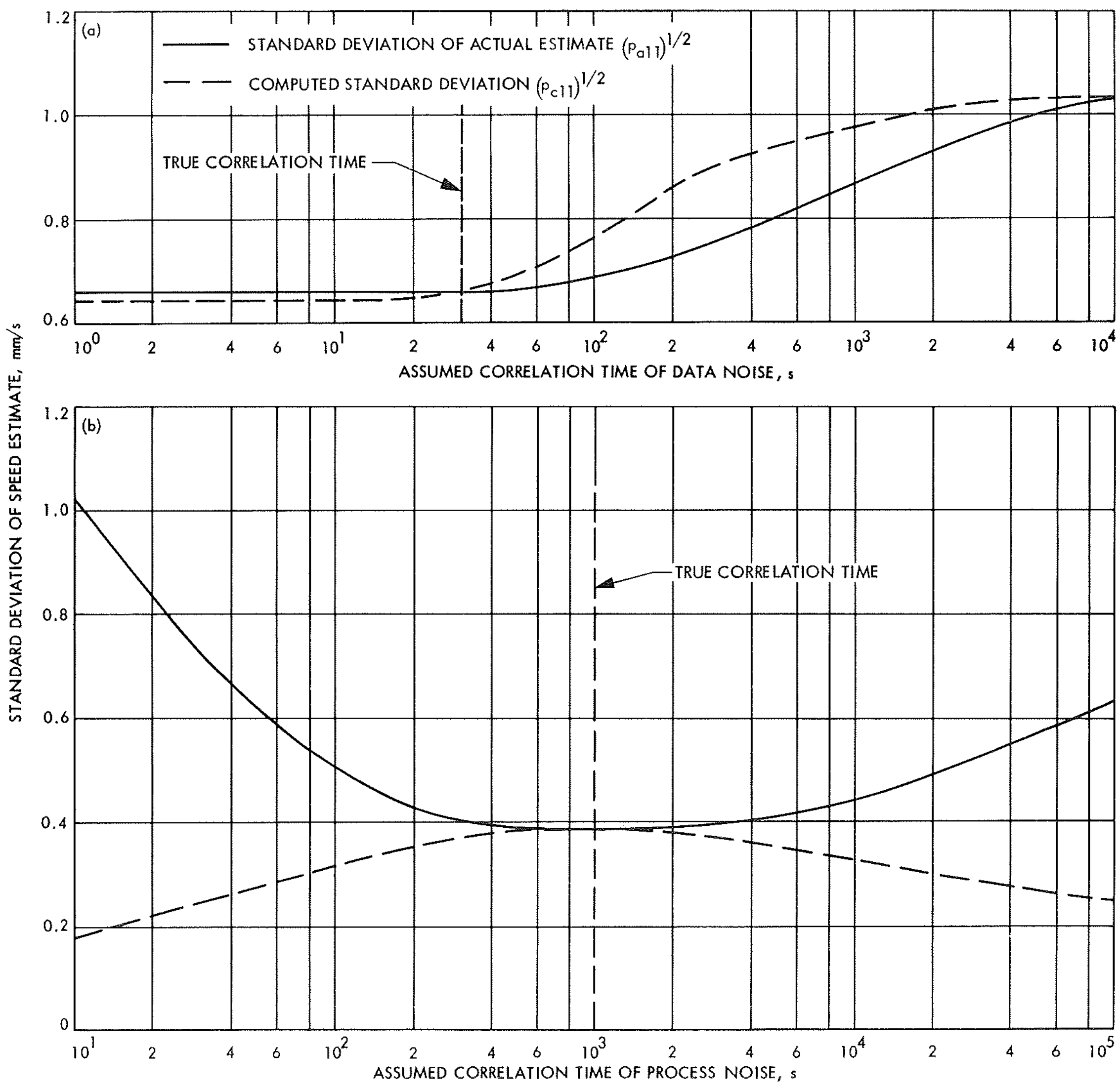


**Fig. 6. Actual and computed variances of suboptimal filters for assumed correlation time of data noise**

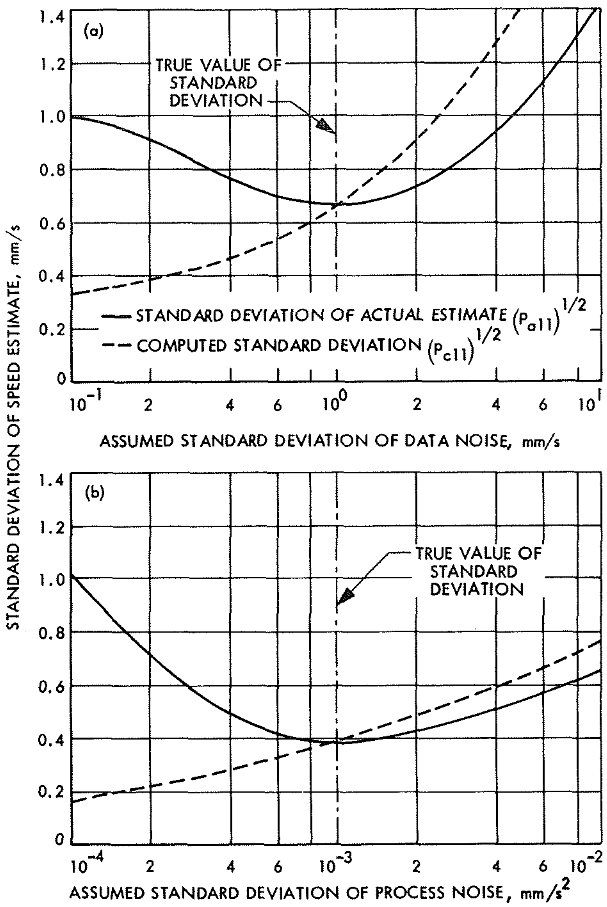


**Fig. 7. Actual and computed variances of suboptimal filters for assumed standard deviation of process noise**

cases the observed speed estimate is rather insensitive in magnitude of noise. They become minimum and equal to the optimal values when $\tau_c = \tau$ and $\sigma_{nc} = \sigma_n$.

*2. Modeling error of process noise.* The effect of modeling errors of process noise is analyzed in this subsection for the same rectilinear motion of the spacecraft. It is assumed that the spacecraft is subject to an exponentially correlated acceleration noise $x_2$ and its speed $x_1$ is estimated by the doppler data $y$ that are contaminated by the white data noise $n$. An analytical solution of the problem in discrete form as closed-form solutions, are not readily available. Therefore, solution in continuous form is attempted. The basic equations for the continuous case, described in Eqs. (28–30), are employed in the subsequent analysis. The process and observation equations are

$$\dot{x}_1(t) = x_2(t) \tag{119}$$

$$\dot{x}_2(t) = -\beta x_2(t) + w(t) \tag{120}$$

$$y(t) = x_1(t) + n(t) \tag{121}$$

where $w$ and $n$ are independent white noise with power spectral densities $q$ and $r$, respectively. Also,

$$\beta = 1/\tau \tag{122}$$

The steady-state solutions of the optimal variances $p_{ij}$ are computed as

$$p_{11} = -r\beta + [r^2\beta^2 + 2r(rq)^{1/2}]^{1/2} \tag{123}$$

$$p_{12} = \frac{p_{11}^2}{2r} \tag{124}$$

$$p_{22} = \frac{1}{2\beta}\left(q - \frac{p_{12}^2}{r}\right) \tag{125}$$

When an erroneous model $\beta_c(= 1/\tau_c)$ and $q_c$ is employed, the suboptimal filter is designed with $\beta_c$ and $q_c$ in place of $\beta$ and $q$, respectively, in the above equations. The related solutions of $\Lambda$ and $P_x$ matrices are computed as:

$$p_{x22} = \frac{q}{2\beta} \tag{126}$$

$$\lambda_{21} = -\frac{1}{D(\lambda)}(q + p_{x22}\Delta\beta) \tag{127}$$

$$\lambda_{22} = -\frac{1}{D(\lambda)}(\beta + k_{c1})(q + p_{x22}\Delta\beta) \tag{128}$$

where

$$\Delta\beta = \beta_c - \beta \tag{129}$$

$$k_{c1} = \frac{p_{c11}}{r} \tag{130}$$

$$k_{c2} = \frac{p_{c12}}{r} \tag{131}$$

$$D(\lambda) = (\beta + k_{c1})(\beta + \beta_c) + k_{c2} \tag{132}$$

Finally, the variance $p_{aij}$ associated with the suboptimal filter are derived as solutions of the following three linear equations:

$$2k_{c1}p_{a11} - 2p_{a12} = rk_{c1}^2 \tag{133}$$

$$k_{c22}p_{a11} + (\beta_c + k_{c1})p_{a12} - p_{a22} = rk_{c1}k_{c2} - \lambda_{21}\Delta\beta \tag{134}$$

$$2k_{c2}p_{a12} + 2\beta_c p_{a22} = rk_{c2}^2 - 2\lambda_{22}\Delta\beta \tag{135}$$

With the use of almost equivalent values for standard deviations of process and data noise to those of the preceding case ($\sigma_n = 1$ mm/s with 1 min count time, $\sigma_w = 10^{-4}$ cm/s$^2$, and the process noise correlation time $\tau = 1000$ s), the square root of $p_{a11}$ is plotted against $\tau_c$ and $\sigma_{wc}$ in Figs. 6(b) and 7(b), respectively. These numbers are typical for solar-electrically thrusted space vehicles (Ref. 11). The estimate of speed is considerably sensitive to the correlation time of process noise

$$\tau_c = \frac{1}{\beta_c}$$

as well as to its magnitude $\sigma_{wc}$.

## VIII. Conclusions

The algorithms for evaluating the effect of errors due to modeling errors in the Kalman filter have been presented in this report for both continuous and discrete systems.

The error bound of the Kalman filter has been studied when the incorrect *a priori* statistics of the initial conditions and system dynamic models as well as those of the noise models are employed. The conservative design criterion expressed in Theorems 1 and 4 (Sections IV and VI) guarantees that the suboptimal filter satisfying it remains within the specified range over the estimation period. Also, the formulas of Eqs. (28–30) for continuous systems and for those of Eqs. (62–64) for discrete systems supply the necessary information to evaluate the effect of errors qualitatively for parametric studies. Such parametric investigations are important to discover to what extent conservative assignment of *a priori* statistics and noise models can be made. Large covariances of initial coordinates and noise tend to increase the covariance of estimates and eventually to slow down its convergence. This degrades the sensitivity of the filter.

The first example of Section VII demonstrates the importance of preflight parametric studies when estimations are to be made in a short interval. An optimistic selection of the *a priori* statistics [smaller values of $P_c(0)$, $Q_c$, and $R_c$ than true values] is especially dangerous because it prevents the estimator from having a proper gain $K(t)$ during the initial period of estimation [case (e) of Fig. 1, case (d) of Fig. 3, and case (d) of Fig. 4]. It has been observed, however, that an excessively conservative choice may be harmful as well because it frequently results in a large offset of suboptimal covariances from the optimal ones at the end of the estimation period [case (a) of Figs. 1, 3, and 4].

The second example is given to study the influence of noise correlation time on the suboptimal filter performance. This is an important problem in space missions because it is often difficult to obtain the exact values of correlation time for stochastic variables such as fluctuations of solar pressure or of the low-thrust engine power. Therefore it is essential to carry out a sensitivity study of the filter.

## Nomenclature

$E[\ ]$ expected value operator on stochastic variables

$E_{ao}(t)$ $= P_a(t) - P(t)$ : difference between actual and true covariance matrix

$E_{ca}(t)$ $= P_c(t) - P_a(t)$ : difference between actual and computed covariance matrix

$E_{co}(t)$ $= P_c(t) - P(t)$ : difference between computed and true covariance matrices

$F'$ transpose of $F$

$F(t)$ $n_x \times n_x$ process matrix

$F_c(t)$ assumed process matrix

$\Delta F(t)$ $= F_c(t) - F(t)$ : difference between assumed and true process matrix

$G(t)$ $n_x \times n_w$ coefficient matrix of process noise

$G_c(t)$ assumed coefficient matrix of process noise

$H(t)$ $n_y \times n_x$ observation matrix

$H_c(t)$ assumed observation matrix

$\Delta H(t)$ $= H_c(t) - H(t)$ : difference between assumed and true observation matrix

$I$ identity matrix

$K(t)$ gain of optimal estimator

$K_c(t)$ computed gain

$n(t)$ $n_y$ vector of observation noise

$P(t)$ covariance matrix of optimal estimator

$P_a(t)$ actual covariance matrix of suboptimal estimator

$P_c(t)$ computed covariance matrix

$P_c(0)$ assumed *a priori* statistics

$Q(t)$ covariance matrix of process noise

$Q_c(t)$ assumed covariance matrix of process noise

$\Delta Q(t)$ $= Q_c(t) - Q(t)$ : difference between assumed and true covariance matrix of process noise

$R(t)$ covariance matrix of data noise

$R_c(t)$ assumed covariance matrix of data noise

$\Delta R(t)$ $= R_c(t) - R(t)$ : difference between assumed and true covariance matrix of data noise

$w(t)$ $n_w$ vector of process noise

$\dot{x}$ time derivative of $x$

$x(t)$ $n_x$ vector of parameters to be estimated

$x^*(t)$ optimal estimator

$x_a^*(t)$ suboptimal estimator

$y(t)$ $n_y$ vector of observation

$\delta(t)$ Dirac delta function

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