Antenna Laboratory Report No. 70-7

REMOTE PROBING METHODS FOR THE DETERMINATION OF THE PROFILE OF INHOMOGENEOUS MEDIA

by

D. Schaubert and R. Mittra

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ABSTRACT

The problem of determining the permittivity of a medium from a knowledge of the electromagnetic fields scattered by the medium is considered. A nonlinear integral equation is derived for the dielectric profile, which is a function of only one variable. The solution of this equation by an iterative scheme is investigated. An alternate formulation of the problem, which leads to a linear integral equation, is also studied. In both cases, the numerical solutions to these problems are unstable. That is, they are highly sensitive to small errors. The second method, however, leads to a uniqueness proof for the inversion process.

ACKNOWLEDGMENT

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1. INTRODUCTION

The purpose of this study is to develop a method by which the profile function of a dielectric can be determined from external measurements. Such a method would be of great use to engineers and physicists who wish to study plasmas. Remote probing could provide information about the properties of a plasma at every point within its volume. Such detailed information has not been available on laboratory plasmas because of the type of confinement vessels which are needed and because of the disruptive effects caused by the presence of a probe in the medium. A remote probing method would also prove useful in studying the ionosphere.

Geologists also have need of remote probing techniques. The echoes caused by seismic blasts have been used to determine the presence of various rock formations and mineral deposits within the earth, but the methods to date have relied greatly upon matching data to the results expected for some specific cases. This procedure is obviously very inefficient and lacks the very desirable quality of uniqueness of the result.

Remote probing might also be used by meteorologists and ecologists for studying the atmosphere. Astronomers might even find this technique applicable to studies of nebulas and other celestial structures.

The remote probing problem for a general medium is, however, a very difficult problem and little has been done on it to date. In this paper, the simpler case of a dielectric which is homogeneous in the y and z directions and which varies in the x direction is considered. This type of model is not greatly removed from the applications mentioned since the earth, the atmosphere, or a plasma may often be modeled this way for the analysis of local phenomena. It will be assumed throughout that the medium has the permeability of free space, μ_0, and that the permittivity is independent of frequency and is described by $\kappa(x)\ \varepsilon_0$, where $\kappa(x)$ is the relative dielectric constant and ε_0 is the permittivity of free space. The methods developed here will require data which can be obtained from external measurements, and from these data, the profile function $\kappa(x)$ will be calculated.

2. FORMULATION AS A SCALAR PROBLEM

2.1 The Geometry and Assumptions of the Problem

The geometry of the problem to be considered is shown in Figure 1. Note that there are to be no variations with respect to the y direction. The dielectric is also uniform in the z direction and varies in the x direction as $\kappa(x)\varepsilon_0$, where $\kappa(x)$ is real. It is assumed that a known electromagnetic wave of a single frequency is incident from the left on the air-dielectric interface at $x = 0$ and that the fields at the surface of the dielectric can be measured for all values of z. Of course, the actual measurement could be performed at some plane $x = \alpha$, $\alpha < 0$, and then the fields at the interface could be computed since the propagation in free space is completely known. To provide a termination in known boundary conditions, a perfect electric conducting plane is included at $x = L$.

2.2 Formulation of the Scalar Problem

Maxwell's equations for a source-free, lossless, charge-free region with constant permeability and varying permittivity $\varepsilon(x)$ are (using $\exp(j\omega t)$ time convention)

$$\nabla \times \bar{E} = -j\omega\mu_0\bar{H} \tag{1}$$

$$\nabla \times \bar{H} = j\omega\varepsilon(x)\bar{E} \tag{2}$$

$$\nabla \cdot \bar{D} = 0 \tag{3}$$

$$\nabla \cdot \bar{B} = 0. \tag{4}$$

Taking the curl of (1) and making use of Equation (2) and the relation

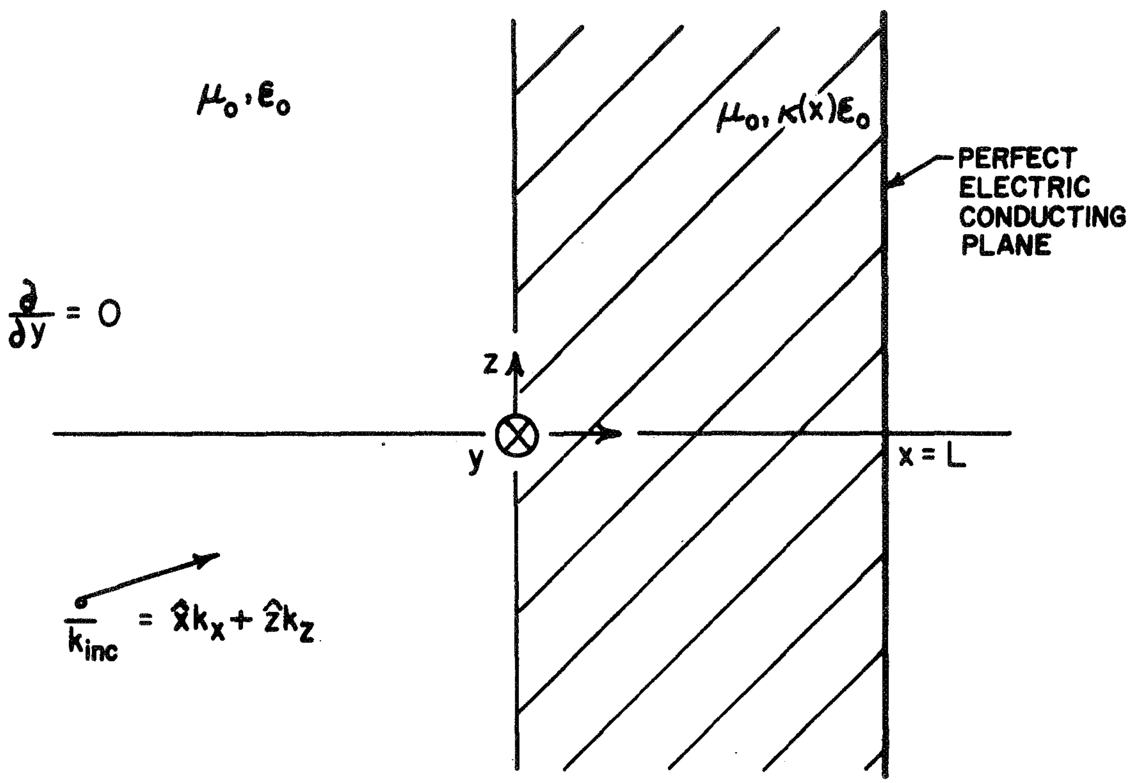


Figure 1. Geometry to be considered.

$$\nabla(\nabla\cdot\bar{E}) = \nabla(\nabla \cdot \frac{\bar{D}}{\varepsilon(x)})$$

$$= \nabla(\nabla \frac{1}{\varepsilon(x)} \cdot \bar{D} + \frac{1}{\varepsilon(x)} \nabla \cdot \bar{D})$$

$$= -\nabla(\frac{\nabla\varepsilon(x)\cdot\bar{E}}{\varepsilon(x)}) \tag{5}$$

we obtain

$$\nabla^2\bar{E} + \omega^2\mu_0\varepsilon(x)\bar{E} + \nabla(\frac{\nabla\varepsilon(x)\cdot\bar{E}}{\varepsilon(x)}) = 0. \tag{6}$$

Similarly, taking the curl of (2) and making use of (1) and (4), we obtain,

$$\nabla^2\bar{H} + \omega^2\mu_0\varepsilon(x)\bar{H} + j\omega(\nabla\varepsilon(x) \times \bar{E}) = 0. \tag{7}$$

At this point it is advantageous to consider two possible polarizations of the incident field.

<u>Case 1: Transverse Electric Type Waves</u>

$$(E_x = E_z = H_y = 0)$$

For the transverse electric type of waves, (6) and (7) can be written as

$$\nabla^2 E_y + \omega^2\mu_0\varepsilon(x)E_y = 0 \tag{8}$$

$$\nabla^2 H_x + \omega^2\mu_0\varepsilon(x)H_x = 0 \tag{9}$$

$$\nabla^2 H_z + \omega^2\mu_0\varepsilon(x)H_z + j\omega \frac{d\varepsilon(x)}{dx} E_y = 0 \tag{10}$$

where the subscripts denote the particular components of the field. The form of the solutions for (8) and (9) are the same and (remembering that $\frac{\partial}{\partial y} = 0$) can be written as

$$E_y(x,z) = e_y(x)\exp(jk_z z) \tag{11}$$

$$H_x(x,z) = h_x(x)\exp(jk_z z) \tag{12}$$

where k_z is a constant.

It is clear from Equation (1) that a knowledge of E_y is sufficient to determine H_x and H_z. However, in Chapter 4 it will be necessary to describe the fields in terms of one of the magnetic field components. Furthermore, the boundary conditions on this component should not depend explicitly on $\varepsilon(x)$.

Consider, then, the x component of (1) with (11) and (12) substituted.

$$-jk_z E_y = -j\omega\mu_0 Hx. \tag{13}$$

That is,

$$H_x = \frac{k_z}{\omega\mu_0} E_y. \tag{14}$$

The boundary conditions for the fields are that E_y and H_z must be continuous. Continuity of E_y simply implies continuity of H_x, and, since H_z is given by

$$H_z = \frac{j}{\omega\mu_0} \frac{\partial E_y}{\partial x} = \frac{j}{k_z} \frac{\partial H_x}{\partial x}, \tag{15}$$

continuity of H_z implies continuity of $\frac{\partial H_x}{\partial x}$.

Thus, it is possible to describe the transverse electric type of waves by the scalar quantity $H_x(x,z)$.

$$\nabla^2 H_x(x,z) + \omega^2 \mu_0 \varepsilon(x) H_x(x,z) = 0 \tag{16}$$

with $H_x(x,z)$ and $\frac{\partial H_x(x,z)}{\partial x}$ continuous.

Case 2: Transverse Magnetic Type Waves

$$(E_y = H_x = H_z = 0)$$

For the transverse magnetic type of waves, (6) and (7) can be written as

$$\nabla^2 E_x + \omega^2 \mu_0 \varepsilon(x) E_z + \frac{\partial}{\partial x} \left(\frac{\frac{d\varepsilon(x)}{dx} E_x}{\varepsilon(x)} \right) = 0 \tag{17}$$

$$\nabla^2 E_z + \omega^2 \mu_0 \varepsilon(x) E_z + \frac{\partial}{\partial z} \left(\frac{\frac{d\varepsilon(x)}{dx} E_x}{\varepsilon(x)} \right) = 0 \tag{18}$$

$$\nabla^2 H_y + \omega^2 \mu_0 \varepsilon(x) H_y - j\omega \frac{d\varepsilon(x)}{dx} E_z = 0. \tag{19}$$

The presence of the terms involving $\frac{d\varepsilon(x)}{dx}$ complicates these equations to such an extent that it is preferred to exclude transverse magnetic waves from consideration.

Therefore, the remainder of this paper will assume that transverse electric waves are incident on the dielectric and that these waves are completely described by the scalar wave function $\psi(x,z)$ where

$$\nabla^2 \psi(x,z) + \omega^2 \mu_0 \varepsilon(x) \psi(x,z) = 0 \tag{20}$$

with $\psi(x,z)$ and $\frac{\partial \psi(x,z)}{\partial x}$ continuous and $\psi(x,z) = 0$ at the perfect electric conductor.

3. AN ITERATION METHOD

3.1 The Approach to Be Taken

The approach to be taken in both this chapter and in Chapter 4 is to derive an integral equation which the unknown profile $\kappa(x)$ must satisfy. Unfortunately, this equation turns out to be nonlinear and, hence, its solution is not straightforward. In this chapter an iterative method is employed to linearize the problem, and in the next chapter, an asymptotic expansion is used to linearize the problem.

3.2 The Integral Equation for the Iterative Method

Figure 2a shows the geometry of the problem with a y-directed line source located at $x = x_0$, $z = 0$, with $\exp(j\omega t)$ variation. The scalar wave equation for this situation is

$$\nabla^2\psi(x,x_0,z) + \omega^2\mu_0\varepsilon(x)\psi(x,x_0,z) = \delta(x-x_0)\delta(z) \tag{21a}$$

with boundary conditions

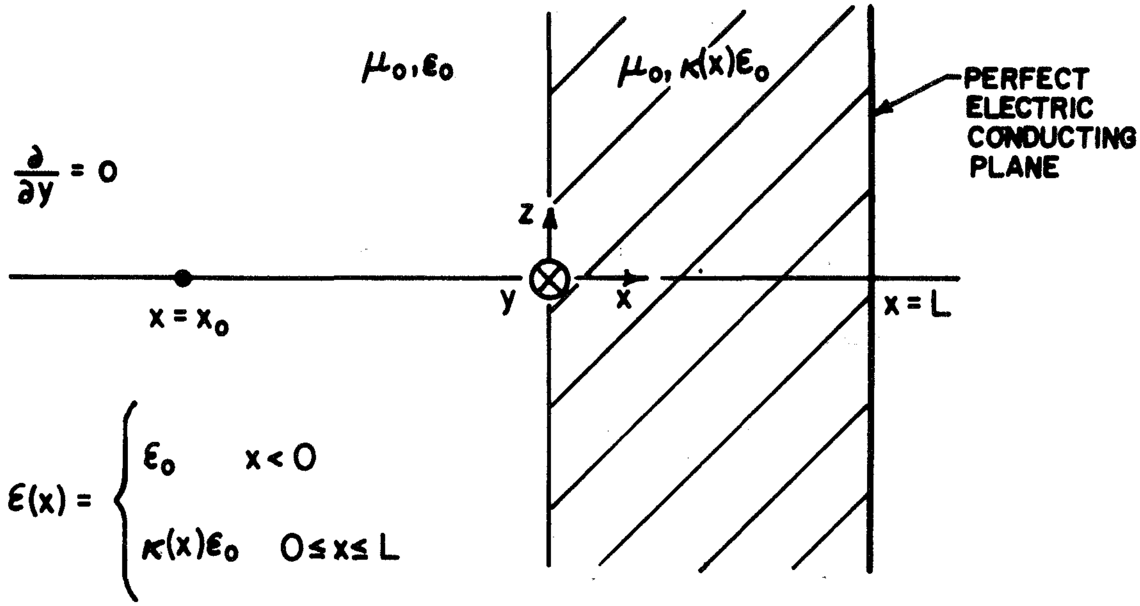
$$\psi(x,x_0,z) \text{ is continuous} \tag{21b}$$

$$\left.\frac{\partial\psi(x,x_0,z)}{\partial x}\right|_{x=x_0^+} - \left.\frac{\partial\psi(x,x_0,z)}{\partial x}\right|_{x=x_0^-} = 1 \tag{21c}$$

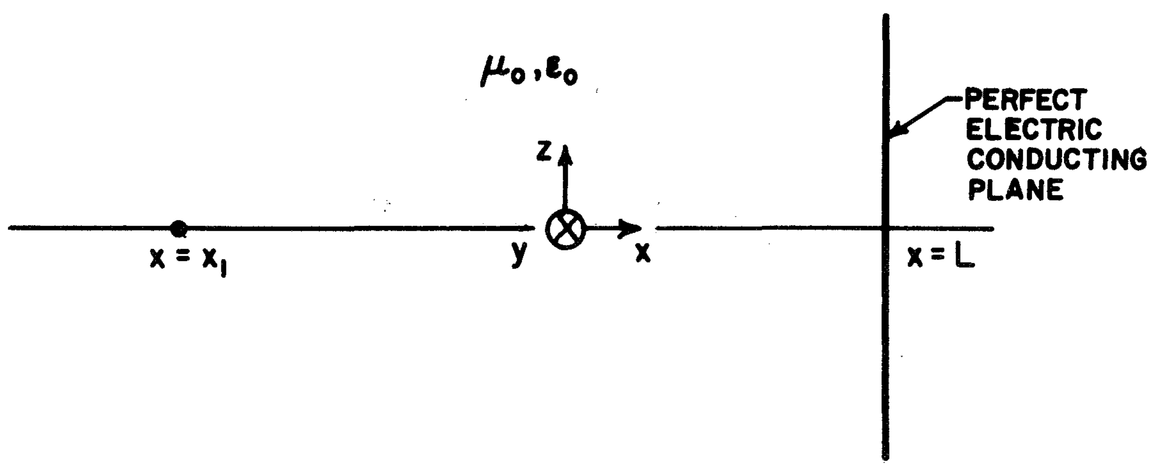
$$\psi(L,x,z) = 0 \tag{21d}$$

$$\psi(x,x_0,z) \text{ is outgoing for } x < x_0. \tag{21e}$$

Figure 2b shows the geometry for a similar problem, the Green's function problem, where free space exists for the entire region $x < L$.



(a)



(b)

Figure 2. Geometries used for deriving the integral equation.
(a). Geometry of the problem.
(b). Geometry for construction of the Green's function.

The Green's function source is a y-directed line source located at $x = x_1$, $z = 0$, with $\exp(j\omega t)$ time variation. The scalar Green's function equation is

$$\nabla^2 g(x,x_1,z) + \omega^2\mu_0\varepsilon_0 g(x,x_1,z) = \delta(x-x_1)\delta(z) \tag{22a}$$

with boundary conditions

$$g(x,x_1,z) \text{ is continuous} \tag{22b}$$

$$\left.\frac{\partial g(x,x_1,z)}{\partial x}\right|_{x=x_1^+} - \left.\frac{\partial g(x,x_1,z)}{\partial x}\right|_{x=x_1^-} = 1 \tag{22c}$$

$$g(L,x_1,z) = 0 \tag{22d}$$

$$g(x,x_1,z) \text{ is outgoing for } x < x_1. \tag{22e}$$

Before proceeding with the derivation, the z dependence is eliminated from (21) and (22) by means of the following Fourier transform relations:

$$T(x,x_0,\beta) = \int_{-\infty}^{\infty} t(x,x_0,z)e^{-j\beta z}dz \tag{23a}$$

$$t(x,x_0,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} T(x,x_0,\beta)e^{j\beta z}d\beta. \tag{23b}$$

With this, (21) becomes

$$\frac{d^2\Psi(x,x_0,\beta)}{dx^2} + (\omega^2\mu_0\varepsilon(x)-\beta^2)\Psi(x,x_0,\beta) = \delta(x-x_0) \tag{24a}$$

with boundary conditions

$$\Psi(x,x_0,\beta) \text{ is continuous} \tag{24b}$$

$$\left.\frac{\partial\Psi(x,x_0,\beta)}{\partial x}\right|_{x=x_0^+} - \left.\frac{\partial\Psi(x,x_0,\beta)}{\partial x}\right|_{x=x_0^-} = 1 \tag{24c}$$

$$\Psi(L,x_0,\beta) = 0 \tag{24d}$$

$$\Psi(x,x_0,\beta) \text{ is outgoing for } x < x_0 \tag{24e}$$

and (22) becomes

$$\frac{d^2G(x,x_1,\beta)}{dx^2} + (\omega^2\mu_0\varepsilon_0-\beta^2)G(x,x_1,\beta) = \delta(x-x_1) \tag{25a}$$

with boundary conditions

$$G(x,x_1,\beta) \text{ is continuous} \tag{25b}$$

$$\left.\frac{\partial G(x,x_1,\beta)}{\partial x}\right|_{x=x_1^+} - \left.\frac{\partial G(x,x_1,\beta)}{\partial x}\right|_{x=x_1^-} = 1 \tag{25c}$$

$$G(L,x_1,\beta) = 0 \tag{25d}$$

$$G(x,x_1,\beta) \text{ is outgoing for } x < x_1. \tag{25e}$$

Multiplying (24a) by $G(x,x_1,\beta)$ and subtracting from it $\Psi(x,x_0,\beta)$ times (25a) yields

$$G(x,x_1,\beta)\Psi''(x,x_0,\beta)-\Psi(x,x_0,\beta)G''(x,x_1,\beta)+(\omega^2\mu_0\varepsilon(x)-\omega^2\mu_0\varepsilon_0)\Psi(x,x_0,\beta)G(x,x_1,\beta)$$

$$= G(x,x_1,\beta)\delta(x-x_0)-\Psi(x,x_0,\beta)\delta(x-x_1) \tag{26}$$

where the primes denote differentiation with respect to x. Integrating (26) with respect to x from 0 to L gives the desired integral equation. (See Appendix A for the details of this operation.)

$$k^2 \int_0^L [\kappa(x)-1]\Psi(x,x_0,\beta)G(x,0,\beta)dx = -C(\beta)\{R(\beta)+\exp[-j\sqrt{(k^2-\beta^2)}2L]\} \quad (27)$$

where $R(\beta)$ is the reflection coefficient evaluated at the interface as a function of β, k is the free space wave number, $k = \omega\sqrt{(\mu_0\varepsilon_0)}$, and $C(\beta)$ is a complex scale constant. Note that since $\Psi(x,x_0,\beta)$ depends on $\kappa(x)$, (27) is a nonlinear integral equation for $\kappa(x)$.

3.3 The Iterative Method

Since Equation (27) is nonlinear, its solution is not straightforward. The approach to be discussed here involves solving (24) and (27) alternately in an iterative scheme. Figure 3 shows schematically the procedure.

(i) Assume a profile $\kappa_0(x)$ to start the process;

(ii) Solve (24) over the region $0 \leq x \leq L$ with $\varepsilon(x) = \varepsilon_0\kappa_0(x)$ to obtain $\Psi_0(x,x_0,\beta)$;

(iii) Solve (27) with $\Psi(x,x_0,\beta) = \Psi_0(x,x_0,\beta)$ to obtain $\kappa_1(x)$;

(iv) Repeat steps (ii) and (iii) with the new values of $\kappa(x)$ and $\Psi(x,x_0,\beta)$.

The questions of convergence and uniqueness are important whenever an iterative scheme is employed. To date, however, no method has been found to guarantee either convergence or uniqueness of this process. Therefore, the only results available on this iterative scheme come from calculations performed on the digital computer.

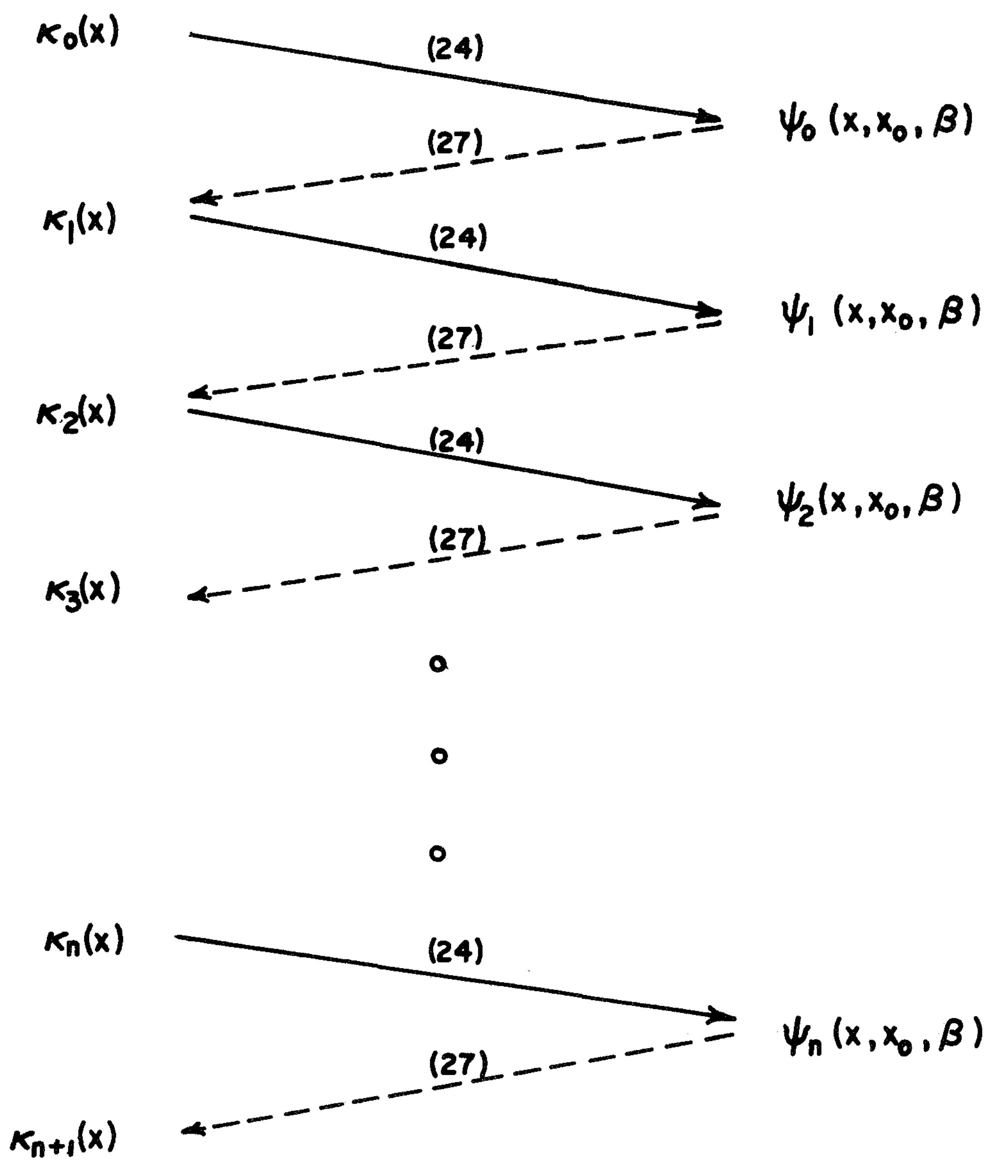


Figure 3. Schematic representation of the iterative process.

To perform the iterations on the computer, a matrix equation is formed. Beta is sampled at n points, say $\beta_1, \beta_2, \ldots, \beta_n$. Equation (24) is then solved n times, once for each value of β, to give n distinct wave functions. Note that the solution of (24) determines the scale constant $C(\beta)$. Substituting into (27) then gives n equations of the form

$$k^2 \int_0^L [\kappa(x)-1]\Psi(x,x_0,\beta_i)G(x,0,\beta_i)dx$$

$$= -C(\beta_i)\{R(\beta_i)+\exp[-j\sqrt{(k^2-\beta_i^2)}2L]\} \quad i = 1, 2, \ldots, n. \tag{28}$$

Two methods for transforming (28) into a matrix equation will be considered.

Method 1: Point Matching

In the point matching method, the unknown function, $[\kappa(x)-1]$, and the kernel function, $\Psi(x,x_0,\beta_i)G(x,0,\beta_i)$ in this case, are sampled at n points within the interval $0 \leq x \leq L$. The integral is then computed by the rectangular rule, or any of the other numerical forms for computing integrals. If the interval is uniformly subdivided as indicated in Figure 4, and if the rectangular rule is used to write the integral, then the matrix equation takes on an especially simple form. The height of each rectangle in Figure 4 is the value of the integrand of (28) at the corresponding point x^j. Therefore, (28) becomes

$$k^2 \sum_{j=1}^{n} h\Psi(x^j,x_0,\beta_i)G(x^j,0,\beta_i)[\kappa(x^j)-1] = -C(\beta_i)\{R(\beta_i)+\exp(-j\sqrt{(k^2-\beta_i^2)}2L]\}$$

$$i = 1, 2, \ldots, n. \tag{29}$$

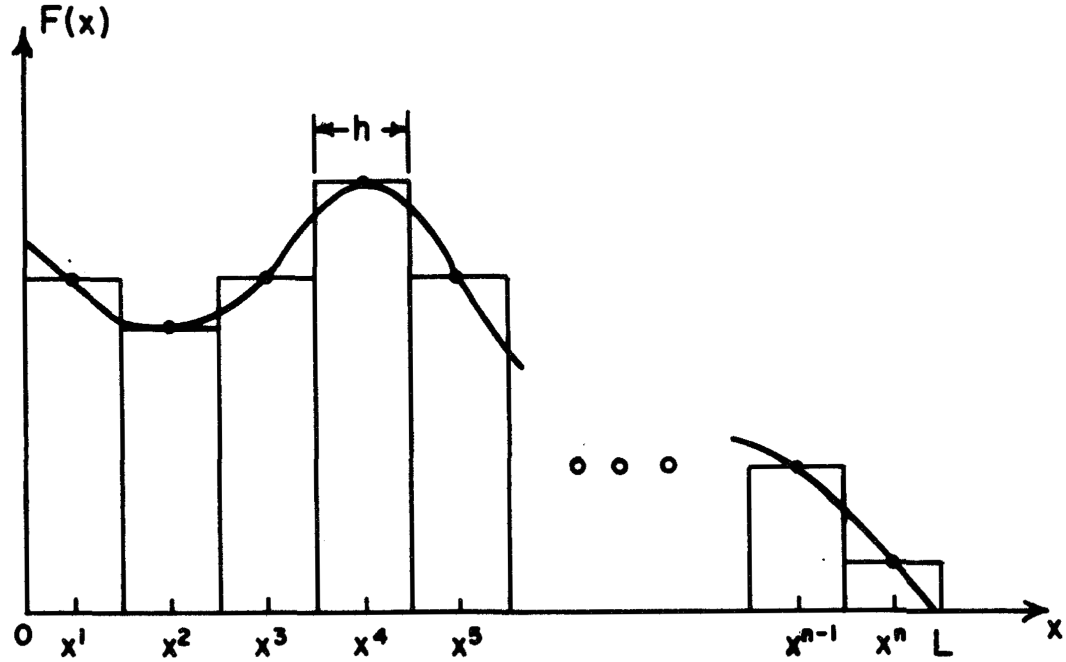


Figure 4. Rectangular approximation for point matching method.

Define the matrix A and the vectors F and B by

$$A = [a_{ij}] \qquad i = 1, 2, \ldots , n$$
$$j = 1, 2, \ldots , n \tag{30}$$

where

$$a_{ij} = k^2\Psi(x^j, x_0, \beta_i)G(x^j, 0, \beta_i)h$$

$$F = [f_j] \quad j = 1, 2, \ldots , n \tag{31}$$

where

$$f_j = \kappa(x^j) - 1$$

$$B = [b_i] \qquad i = 1, 2, \ldots , n \tag{32}$$

where

$$b_i = -C(\beta_i)\{R(\beta_i)+\exp[-j\sqrt{(k^2-\beta_i^2)}2L]\}.$$

Then (29) can be written simply as

$$AF = B. \tag{33}$$

Method 2: Expansion of $[\kappa(x)-1]$

In this method the unknown function, $[\kappa(x)-1]$, is written in the form

$$\kappa(x) - 1 = \sum_{j=1}^{\infty} \tau_j\phi_j(x) \tag{34}$$

$$\tau_j = \langle \kappa(x)-1,\ \phi_j(x) \rangle = \int_0^L [\kappa(x)-1]\phi_j(x)\,dx \tag{35}$$

where the $\{\phi_j\}$ form a real complete orthonormal set over the range $0 \leq x \leq L$. If the summation (34) is truncated to n terms, (28) takes the form

$$k^2 \sum_{j=1}^{n} \tau_j \int_0^L \phi_j(x)\Psi(x,x_0,\beta_i)G(x,0,\beta_i)\,dx = b_i \qquad i=1,\ 2,\ \ldots\ ,\ n \tag{36}$$

where b_i is defined by (32). Now define the matrix M and the vector T by

$$M = [m_{ij}] \qquad i = 1,\ 2,\ \ldots\ ,\ n$$

$$j = 1,\ 2,\ \ldots\ ,\ n \tag{37}$$

where

$$m_{ij} = \langle \phi_j(x),\ \Psi(x,x_0,\beta_i)G(x,0,\beta_i) \rangle$$

$$T = [\tau_j] \qquad j = 1,\ 2,\ \ldots,\ n. \tag{38}$$

Then (36) can be written as

$$MT = B. \tag{39}$$

3.4 Results of Numerical Calculations

A program was written to solve Equation (24) using the Runge-Kutta integration scheme. The results of this computation were then used to form A or M, depending on the method being used. Next, the square matrix A (or M) was inverted so that the unknown $\kappa(x)$ could be found. It was at this point that the iterative procedure failed. The reason for the failure is that the matrix A (or M) is ill-conditioned. That is, the matrix has eigenvalues which are very nearly zero. (See Appendix C for more discussion of ill-conditioned matrices.) Because the matrix is ill-conditioned, small errors in the vector B result in very large errors in the vector F (or T). In fact, upon investigating the matrix A^{-1} (or M^{-1}), it was found that the terms of this matrix were approximately of the order of 10^{2n} when A (or M) was n x n. Since the terms of F were of the order of 10^{+1}, the accuracy required of B for good results was of the order of 2n significant figures. Clearly, this makes the method impractical for use where experimental data are the input.

The character of the matrices seems to improve very little as the range of beta is varied. In fact, for the range much less than one or much greater than ten, the results deteriorate rapidly. Different choices of expansion functions for the second method likewise failed to improve the character of the matrix.

The ill-conditioned nature of the matrix was observed even when $\kappa(x)$ was constant so that an analytic expression could be found for $\Psi(x,x_0,\beta)$, thus avoiding computational errors arising in the solution of (24). It, therefore, appears that this formulation of the problem will not work. Chapter 4 discusses another formulation of the integral equation for $\kappa(x)$ and its solution.

4. AN ASYMPTOTIC EXPANSION METHOD

4.1 Formulating the Integral Equation

The geometries to be considered in this chapter are the same as those of Chapter 3. (See Figure 2.) Equation (24) again describes the fields for the problem. For this method, however, the wave function, $\Psi(x,x_0,\beta)$, is to be expanded in a power series of ω. It should be emphasized that the equation to be derived is exact, even though an asymptotic expansion is used for the derivation.

The Green's function geometry to be used is again illustrated by Figure 2(b). This time, however, the static Green's function will be used. That is, $\tilde{g}(x,x_1,\beta)$ satisfies

$$\nabla^2\tilde{g}(x,x_1,z) = \delta(x-x_1)\delta(z). \tag{40}$$

Transforming (40) by the relations (23) yields

$$\frac{d^2\tilde{G}(x,x_1,\beta)}{dx^2} - \beta^2\tilde{G}(x,x_1,\beta) = \delta(x-x_1). \tag{41a}$$

The boundary conditions on $\tilde{G}(x,x_1,\beta)$ are

$$\tilde{G}(x,x_1,\beta) \text{ is continuous} \tag{41b}$$

$$\left.\frac{d\tilde{G}(x,x_1,\beta)}{dx}\right|_{x=x_1^+} - \left.\frac{d\tilde{G}(x,x_1,\beta)}{dx}\right|_{x=x_1^-} = 1 \tag{41c}$$

$$\tilde{G}(L,x_1,\beta) = 0 \tag{41d}$$

$$\lim_{x\to-\infty} \tilde{G}(x,x_1,\beta) = 0. \tag{41e}$$

Multiplying (24a) by $\tilde{G}(x,x_1,\beta)$ and (41a) by $\Psi(x,x_0,\beta)$ and subtracting as in Chapter 3 gives the analog of (26).

$$\tilde{G}(x,x_1,\beta)\Psi''(x,x_0,\beta) - \Psi(x,x_0,\beta)\tilde{G}''(x,x_1,\beta) + \omega^2\mu_0\varepsilon(x)\Psi(x,x_0,\beta)\tilde{G}(x,x_1,\beta)$$

$$= \tilde{G}(x,x_1,\beta)\delta(x-x_0) - \Psi(x,x_0,\beta)\delta(x-x_1). \tag{42}$$

Equation (42) is integrated from $-\infty$ to L to yield

$$\omega^2\mu_0 \int_{-\infty}^{L} \varepsilon(x)\Psi(x,x_0,\beta)\tilde{G}(x,x_1,\beta)\,dx$$

$$= \tilde{G}(x_0,x_1,\beta) - \Psi(x_1,x_0,\beta) - [\tilde{G}(x,x_1,\beta)\Psi'(x,x_0,\beta)-\Psi(x,x_0,\beta)\tilde{G}'(x,x_1,\beta)]\Big|_{-\infty}^{L}. \tag{43}$$

The form of the solution to (41) gives the conditions that $\tilde{G}(x,x_1,\beta)$ and its derivation vanish at $-\infty$. (See Appendix B for details.) Therefore, since $\Psi(L,x_0,\beta) = \tilde{G}(L,x_1,\beta) = 0$, the last term of (43) vanishes. Then making use of

$$\varepsilon(x) = \begin{cases} \kappa(x)\,\varepsilon_0 & 0 \le x \le L \\ \varepsilon_0 & x < 0 \end{cases} \tag{44}$$

(43) becomes

$$\omega^2\mu_0\varepsilon_0 \int_{0}^{L} \kappa(x)\Psi(x,x_0,\beta,\omega)\tilde{G}(x,x_1,\beta)\,dx$$

$$= \tilde{G}(x_0,x_1,\beta) - \Psi(x_1,x_0,\beta,\omega) - \omega^2\mu_0\varepsilon_0 \int_{-\infty}^{0} \Psi(x,x_0,\beta,\omega)\tilde{G}(x,x_1,\beta)\,dx. \tag{45}$$

The ω has been explicitly included as a variable to emphasize that $\Psi(x,x_0,\beta,\omega)$ does depend on ω but that $\tilde{G}(x,x_0,\beta)$ does not.

Equation (45) is the nonlinear integral equation which is to be solved for $\kappa(x)$. Note that if $x_1 \leq 0$, then the right-hand side of (45) is known. (Recall that for $x \leq 0$, $\Psi(x,x_0,\beta,\omega)$ can be written in terms of $R(\beta)$ and solutions to the wave equation in free space.) The task is then to linearize (45) by replacing the unknown $\Psi(x,x_0,\beta,\omega)$ with an appropriate known function. To do this, $\Psi(x,x_0,\beta,\omega)$ is expanded as a power series in ω.

$$\Psi(x,x_0,\beta,\omega) = \sum_{n=0}^{\infty} \alpha_n(x,x_0,\beta)\omega^n \tag{46}$$

where

$$\alpha_n(x,x_0,\beta) = \frac{1}{n!} \left. \frac{\partial^n \Psi(x,x_0,\beta,\omega)}{\partial \omega^n} \right|_{\omega=0} .$$

The details of the development are included in Appendix B. The resulting linear equation which $\kappa(x)$ must satisfy is

$$\int_0^L \kappa(x) \sinh^2 [\beta(x-L)]dx = -\frac{1}{2}\{\frac{\beta^2}{\mu_0\varepsilon_0} \exp[2\beta L] \left. \frac{\partial^2 \Psi(0,0,\beta,\omega)}{\partial \omega^2} \right|_{\omega=0}$$

$$+ \frac{1}{\beta} \sinh^2 (\beta L)\}. \tag{47}$$

Note that for this method it is necessary to obtain data from which $\left. \frac{\partial^2 \Psi(0,0,\beta,\omega)}{\partial \omega^2} \right|_{\omega=0}$ can be calculated. This requires more measurements than the iterative scheme discussed in Chapter 3, but the advantage gained is that no iteration is required. Equation (47) is solved only once to obtain the result.

It should be noted that the derivation of (47) did not require that $\kappa(x)$ be real. Therefore, noting that only $\kappa(x)$ and $\left.\frac{\partial^2 \Psi(0,0,\beta,\omega)}{\partial \omega^2}\right|_{\omega=0}$ can be complex (β is real), Equation (47) can be separated into two equations, one for $\mathrm{Re}\{\kappa(x)\}$ and one for $\mathrm{Im}\{\kappa(x)\}$. It is then necessary that both the real and imaginary parts of $\Psi(0,0,\beta,\omega)$ be measured.

It should also be noted that (47) is valid only when $\Psi(x,x_0,\beta,\omega)$ is identified with H_x. This happens because the derivation of (47) requires that $\alpha_0(x,x_0,\beta) = \Psi(x,x_0,\beta,\omega)|_{\omega=0}$ be independent of $\kappa(x)$. Static magnetic fields are independent of the dielectric constant of the medium, but static electric fields are not.

4.2 Uniqueness of the Solution

As was indicated in Chapter 3, nothing could be said about the uniqueness of the solution obtained by iteration. For the asymptotic method, however, uniqueness can easily be proven. Suppose that $\kappa_1(x)$ and $\kappa_2(x)$ both satisfy (47). Then, subtracting the equations for the two solutions gives

$$\int_0^L [\kappa_1(x)-\kappa_2(x)] \sinh^2 [\beta(x-L)]dx = 0. \tag{48}$$

But this can be satisfied for all β only if $[\kappa_1(x)-\kappa_2(x)] = 0$. Therefore, $\kappa_1(x) = \kappa_2(x)$ and the solution is unique.

4.3 Results of Numerical Calculations

Just as for the iterative case, (47) can be converted to a matrix equation by point matching or by expansion of $\kappa(x)$ in a set of basis functions. The procedure is analogous to that of Chapter 3 and leads to an equation of the form

$$AF = B \tag{49}$$

where A is an n x n matrix and F and B are n x 1 vectors. Therefore, the problem has again come down to the inversion of the matrix A. Unfortunately, the matrix obtained by this approach is again ill-conditioned. The elements of A^{-1} are approximately of the order of 10^{2n} so that 2n significant figures are needed to get an accurate solution. Again the nature of A improves little for different ranges of β or for different choices of the basis functions in which $\kappa(x)$ is expanded.

In spite of the fact that the matrix A is ill-conditioned, the solution of (49) has been carried out for a few examples in order to test the method. The data used in these calculations were generated on the computer and, therefore, were accurate to 10 significant figures. With this extreme accuracy in the data, the solution to (49) gave results which agreed quite well with the true profile. However, with noise of less than one percent added, the solution contained oscillatory components which were orders of magnitude larger than the desired result.

Figure 5 shows the results of solving (49) for $\kappa(x) = \exp(2x)$, $0 \leq x \leq 1$. In Figure 5a are shown the results obtained using three, five, and seven terms of the expansion $\kappa(x) = \sum_{i=0}^{N} \tau_i x^i$. Note that the set $\{x^i\}$ is not orthogonal as was stated in the derivation of the matrix

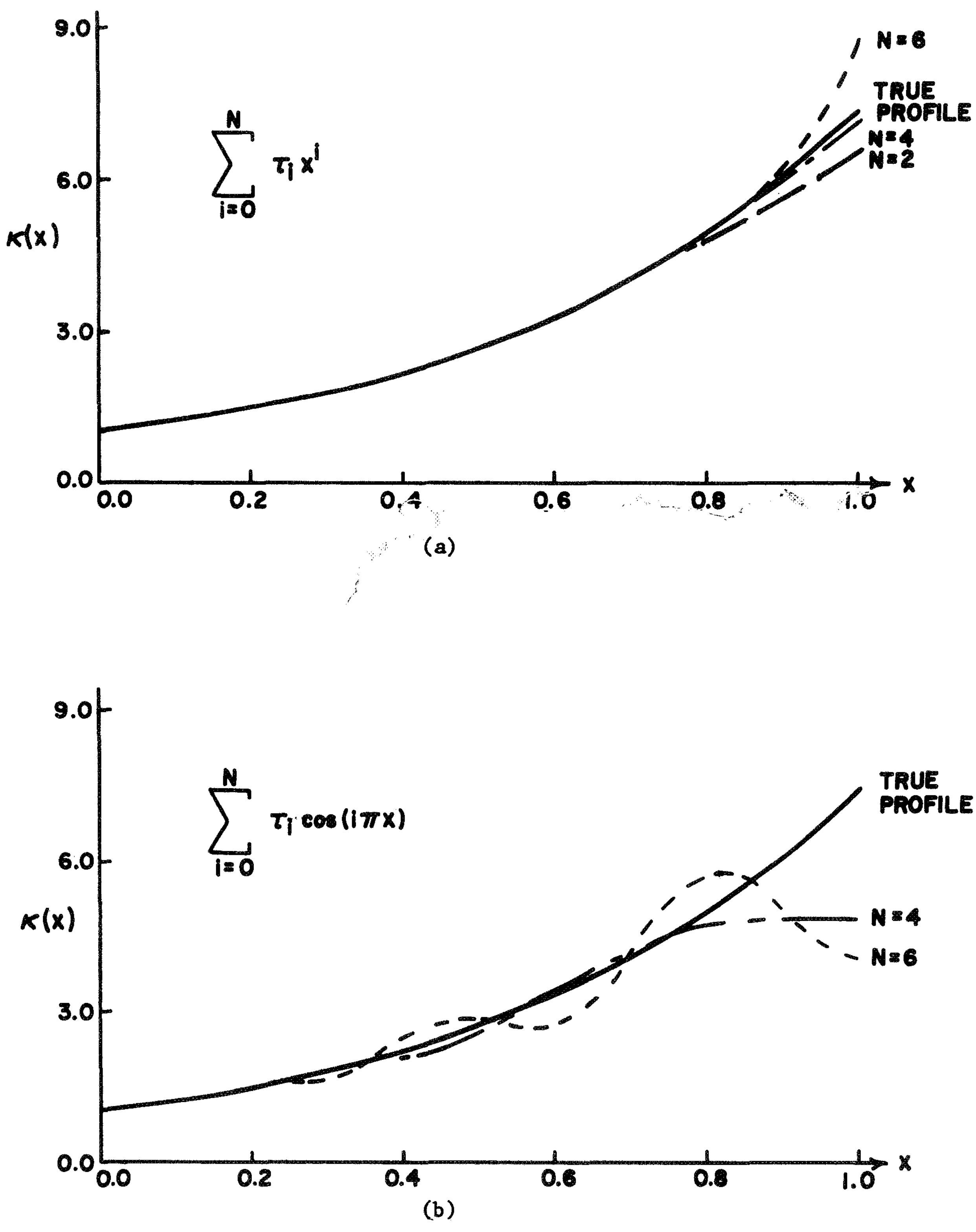


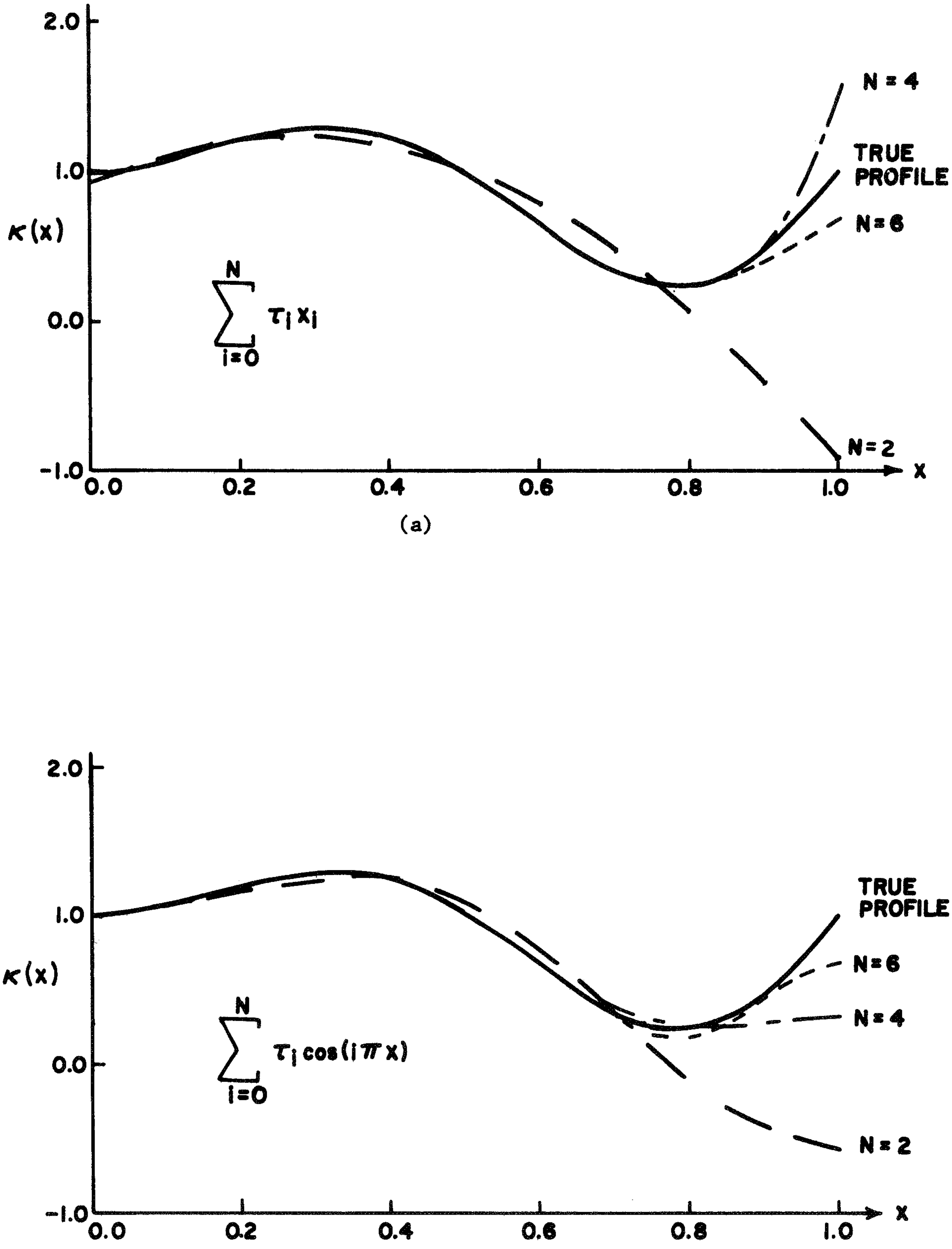
Figure 5. Calculated profiles for $\kappa(x) = \exp[2x]$.

(a) Power series expansion

(b) Cosine series expansion.

equation in Chapter 3, but the set is complete so that $\kappa(x)$ can be expanded as a power series of x. The only change which is required in the derivations of Chapter 3 is to replace Equation (35) for τ_i with a more complicated expression. Figure 5b shows the results obtained for five and seven terms of the series $\kappa(x) = \sum_{i=0}^{N} \tau_i \cos(i\pi x)$. The result for three terms was essentially the same as for five terms. As might be expected, the power series gives a better representation of the exponential function than the cosine series.

Figure 6 shows the results of using the same two series to expand the profile $\kappa(x) = 1 + x \sin 2\pi x$, $0 \leq x \leq 1$. The results for three, five, and seven terms are again shown. This time the profile is oscillating and so is more difficult to synthesize with only a few terms. The range of beta for both Figures 5 and 6 was from 0.2 to 1.4. One of the most significant features of both Figure 5 and Figure 6 is that all of the reconstructions were worse for the region near the conducting plane than for the region near the interface. Indeed, this was the case for every profile which was tried and for every method of forming the matrix equation from the integral equation. The explanation of this phenomenon seems to be that the fields in the vicinity of the perfect electric conductor are masked by the fields closer to the surface. That is, because the fields near the conductor are very small and because of the relatively large distance between the back of the dielectric and the surface where the fields are measured, the effect of moderate changes in the dielectric constant is not seen.



(a)

(b)

Figure 6. Calculated profiles for $\kappa(x) = 1 + x \sin 2\pi x$.

(a) Power series expansion
(b) Cosine series expansion.

5. CONCLUSIONS

The purpose of this study was to develop methods for reconstructing the profile function of a dielectric medium from external measurements. Two such methods were derived here. Both of these, however, led to ill-conditioned equations. Because of this, neither of these methods could be applied directly to experimental data.

The regularization method discussed in Appendix C was applied to Equations (33), (39), and (49), but the results were not satisfactory. The current literature contains other regularization schemes which might yield better results for this problem, but the investigation of these techniques has only been started and results are not yet available. Another approach which should be considered is parameter optimization. This technique has proven useful for solving nonlinear problems, and may also prove useful for solving ill-conditioned problems.

It is important to note that the equations derived here are ill-conditioned because of the problem which is being solved and not because of the method used to solve it. Indeed, many authors have found that problems involving indirect probing are ill-conditioned.

The important contribution of this work is to show that a unique solution can be obtained without the knowledge of the fields for all frequencies. In fact, as few as three frequencies might be used to calculate $\left.\dfrac{\partial^2\Psi(x,x_0,\beta,\omega)}{\partial\omega^2}\right|_{\omega=0}$ and from this a unique profile function can be obtained.

Future work will be directed toward methods of solving the ill-conditioned equations derived here in the hope that useful solutions can be obtained in the presence of experimental uncertainties.

APPENDIX A. OBTAINING THE INTEGRAL EQUATION FOR THE ITERATIVE METHOD

To carry out the steps linking Equations (26) and (27) of Chapter 3, it is necessary to consider the form of the solutions to Equations (24) and (25).

Considering Figure 2a, let $R(\beta)$ be the reflection coefficient evaluated at the interface as a function of β. That is,

$$R(\beta) = \frac{\Psi^{ref}(0,x_0,\beta)}{\Psi^{inc}(0,x_0,\beta)} \tag{A.1}$$

where Ψ^{inc} is the incident field and Ψ^{ref} is the reflected field. Then, the form of $\Psi(x,x_0,\beta)$ for $x_0 < x \leq 0^+$ is

$$\Psi(x,x_0,\beta) = C(\beta)\{\exp[-j\gamma x]+R(\beta)\exp[j\gamma x]\} \tag{A.2}$$

where

$$\gamma = \sqrt{(k^2-\beta^2)}$$

and

$$C(\beta) \text{ is a complex scale constant}$$

$$\Psi'(x,x_0,\beta) = \frac{d\Psi(x,x_0,\beta)}{dx} = -j\gamma C(\beta)\{\exp[-j\gamma x]-R(\beta)\exp[j\gamma x]\}. \tag{A.3}$$

The solution for Equation (25) is

$$G(x,x_1,\beta) = \begin{cases} \frac{j}{2\gamma}\{\exp[-j\gamma(x-x_1)]-\exp[j\gamma(x+x_1-2L]\} & x > x_1 \\ \\ \frac{j}{2\gamma}\{\exp[j\gamma(x-x_1)]-\exp[j\gamma(x+x_1-2L]\} & x < x_1 \end{cases} \tag{A.4}$$

$$G'(x,x_1,\beta) = \begin{cases} \frac{1}{2}\{\exp[-j\gamma(x-x_1)]+\exp[j\gamma(x+x_1-2L]\} & x > x_1 \\ \\ -\frac{1}{2}\{\exp[j\gamma(x-x_1)]-\exp[j\gamma(x+x_1-2L]\} & x < x_1 \end{cases} \tag{A.5}$$

Consider again Figure 2a. The expression $[\omega^2\mu_0\varepsilon(x)-\omega^2\mu_0\varepsilon_0]$ becomes

$$\omega^2\mu_0\varepsilon(x) - \omega^2\mu_0\varepsilon_0 = \begin{cases} k^2[\kappa(x)-1] & 0 \leq x \leq L \\ \\ 0 & x < 0 \end{cases} \tag{A.6}$$

where

$$k^2 = \omega^2\mu_0\varepsilon_0.$$

Equation (26) can then be written (for $0 \leq x \leq L$)

$$k^2[\kappa(x)-1]\Psi(x,x_0,\beta)G(x,x_1,\beta) = \Psi(x,x_0,\beta)G''(x,x_1,\beta) - G(x,x_1,\beta)\Psi''(x,x_0,\beta)$$

$$+ G(x,x_1,\beta)\delta(x-x_0) - \Psi(x,x_0,\beta)\delta(x-x_1). \tag{A.7}$$

Next, (A.7) is to be integrated with respect to x from 0 to L. After integration, it is necessary that the right hand side be a known quantity. Carrying out this integration for the case where x_0 and x_1 are both less than zero gives

$$k^2 \int_0^L [\kappa(x)-1)\Psi(x,x_0,\beta)G(x,x_1,\beta)dx$$

$$= [\Psi(x,x_0,\beta)G'(x,x_1,\beta)-G(x,x_1,\beta)\Psi'(x,x_0,\beta)]\Big|_0^L$$

$$= G(0,x_1,\beta)\Psi'(0,x_0,\beta) - \Psi(0,x_0,\beta)G'(0,x_1,\beta) \tag{A.8}$$

since $\Psi(L,x_0,\beta) = G(L,x_1,\beta) = 0$. Note that the form of $G(x,x_1,\beta)$ to be used in (A.8) is that corresponding to $x > x_1$. Substituting (A.2), (A.3), (A.4), and (A.5) into the Equation (A.8) gives

$$k^2 \int_0^L [\kappa(x)-1]\Psi(x,x_0,\beta)G(x,x_1,\beta)dx$$

$$= \frac{1}{2} C(\beta)[1-R(\beta)]\{\exp[j\gamma x_1]-\exp[j\gamma(x_1-2L)]\}$$

$$- \frac{1}{2} C(\beta)[1+R(\beta)]\{\exp[j\gamma x_1]+\exp[j\gamma(x_1-2L)]\}$$

$$= -C(\beta)\{R(\beta)\exp[j\gamma x_1]+\exp[j\gamma(x_1-2L)]\}. \tag{A.9}$$

Since x_1 is the source point for the Green's function, which was introduced in a purely analytic manner, it can be taken as 0^- with no loss of generality. Then (A.9) simplifies to

$$k^2 \int_0^L [\kappa(x)-1]\Psi(x,x_0,\beta)G(x,0^-,\beta)dx = -C(\beta)\{R(\beta)+\exp[-j2\gamma L]\}. \tag{A.10}$$

The minus superscript on the zero can actually be removed as can be seen by integrating (A.7) with $x_0 < 0$ and $x_1 = 0^+$. Then

$$k^2 \int_0^L [\kappa(x)-1]\Psi(x,x_0,\beta)G(x,0^+,\beta)dx$$

$$= -\Psi(0^+,x_0,\beta) + G(0,0^+,\beta)\Psi'(0,x_0,\beta) - \Psi(0,x_0,\beta)G'(0,0^+,\beta)$$

$$= -C(\beta)[1+R(\beta)] + \frac{1}{2} C(\beta)[1-R(\beta)]\{1-\exp[-j2\gamma L]$$

$$+ \frac{1}{2} C(\beta)[1+R(\beta)]\{1-\exp[-j2\gamma L]\}$$

$$= -C(\beta)\{R(\beta)+\exp[-j2\gamma L]\}. \tag{A.11}$$

Therefore, since $G(x,x_1,\beta)$ is continuous, the final result becomes

$$k^2 \int_0^L [\kappa(x)-1]\Psi(x,x_0,\beta)G(x,0,\beta)dx = -C(\beta)\{R(\beta)+\exp[-j\sqrt{(k^2-\beta^2)}2L]\}. \quad (A.12)$$

APPENDIX B. OBTAINING THE INTEGRAL EQUATION FOR THE ASYMPTOTIC EXPANSION METHOD

In order to simplify Equation (43) of Chapter 4, it is necessary to consider the form of $\tilde{G}(x,x_1,\beta)$. The solution of (41) is

$$\tilde{G}(x,x_1,\beta) = \begin{cases} \frac{1}{2|\beta|}\{\exp[2|\beta|(x_1-L)]-1\}\exp[|\beta|(x-x_1)] & x < x_1 < L \\ \frac{1}{2|\beta|}\{\exp[-2|\beta|L]\exp[|\beta|(x+x_1)]-\exp[-|\beta|(x-x_1)]\} & x_1 < x \leq L. \end{cases} \tag{B.1}$$

If only positive β are considered, the absolute value bars can be dropped from (B.1). Also, since the Green's function is a mathematical tool which was introduced only for analytic purposes, the source point x_1 can be taken at any convenient point. It turns out that $x_1 = 0$ is a convenient choice. Then (B.1) becomes

$$\tilde{G}(x,0,\beta) = \begin{cases} -\frac{1}{\beta}\exp[\beta(x-L)]\sinh \beta L & x < 0 \\ \frac{1}{\beta}\exp[-\beta L]\sinh \beta(x-L) & 0 < x \leq L \end{cases} . \tag{B.2}$$

From (B.2) it is clear that $\tilde{G}(x,0,\beta)$ and its derivative both vanish as $x \to -\infty$. Utilizing this result and the fact that $\tilde{G}(L,0,\beta) = \Psi(L,x_0,\beta,\omega) = 0$, the last term of (43) vanishes. Then, using (44), (43) gives (45) directly. To linearize (45), $\Psi(x,x_0,\beta,\omega)$ is expanded in a power series of ω as given by (46). Note that it is sufficient to consider only $n \geq 0$ in (46) since the wave function must be finite for zero frequency. Substituting (46) into (45) and interchanging the order of summation and integration, which can always be done in a neighborhood of $\omega = 0$, gives

$$\omega^2\mu_0\varepsilon_0 \sum_{n=0}^{\infty} \omega^n \int_0^L \kappa(x)\alpha_n(x,x_0,\beta)\tilde{G}(x,x_1,\beta)dx$$

$$= \tilde{G}(x_0,x_1,\beta) - \sum_{n=0}^{\infty} \alpha_n(x_1,x_0,\beta)\omega^n - \omega^2\mu_0\varepsilon_0 \sum_{n=0}^{\infty} \omega^n \int_{-\infty}^{0} \alpha_n(x,x_0,\beta)\tilde{G}(x,x_1,\beta)dx. \tag{B.3}$$

Matching like powers of ω gives the following equations:

$$0 = \tilde{G}(x_0,x_1,\beta) - \alpha_0(x_1,x_0,\beta) \tag{B.4}$$

$$0 = \alpha_1(x_1,x_0,\beta) \tag{B.5}$$

$$\mu_0\varepsilon_0 \int_0^L \kappa(x)\alpha_0(x,x_0,\beta)\tilde{G}(x,x_1,\beta)dx = -\alpha_2(x_1,x_0,\beta) - \mu_0\varepsilon_0 \int_{-\infty}^{0} \alpha_0(x,x_0,\beta)\tilde{G}(x,x_1,\beta)dx. \tag{B.6}$$

Consider (B.4) and note that

$$\alpha_0(x_1,x_0,\beta) = \Psi(x_1,x_0,\beta,\omega)\Big|_{\omega=0} = \tilde{G}(x_1,x_0,\beta). \tag{B.7}$$

This is easily verified by noting that (24) becomes identical to (41) when $\omega = 0$. Therefore, (B.4) states that the Green's function is symmetric in its two variables. Since the operator of (41) is self-adjoint, so this relation could be predicted from the theory of operators.

Considering (B.5), note that $\Psi(x,x_0,\beta,\omega)$ is an even function of ω. This occurs because ω only appears in Ψ through k^2 (which equals $\frac{\omega^2}{c^2}$).

Thus the expansion of $\Psi(x,x_0,\beta,\omega)$ must be an even function of ω which implies that all of the coefficients of the odd powers of ω must be zero.

Equation (B.6) is the linear equation which $\kappa(x)$ must satisfy. For the case of $x_1 = 0$, the integral on the right hand side becomes

$$\int_{-\infty}^{0} G(x,0,\beta)G(x,x_0,\beta)dx$$

$$= -\frac{1}{2\beta^3}\sinh \beta L\{\exp[2\beta(x_0-L)]\sinh \beta(x_0-L)$$

$$+\frac{1}{2}\exp[-3\beta L]\exp[\beta x_0][1-\exp(2\beta x_0)] + \beta x_0\exp[\beta(x_0-L)]\}. \quad \text{(B.8)}$$

For the purpose of simplifying the expressions, choose $x_0 = 0$. Then (B.6) becomes

$$\frac{\mu_0\varepsilon_0}{\beta^2}\exp[-2\beta L]\int_0^L \kappa(x)\sinh^2 \beta(x-L)dx$$

$$= -\frac{1}{2}\left.\frac{\partial^2\Psi(0,0,\beta,\omega)}{\partial\omega^2}\right|_{\omega=0} - \frac{\mu_0\varepsilon_0}{2\beta^3}\exp[-2\beta L]\sinh^2 \beta L. \quad \text{(B.9)}$$

This is the linear integral equation which $\kappa(x)$ must satisfy.

APPENDIX C. A BRIEF DISCUSSION OF ILL-CONDITIONED MATRICES AND THEIR SOLUTION

For the purposes of this paper, it is sufficient to consider the square matrix A to be ill-conditioned whenever it has an eigenvalue which is "nearly" zero. This somewhat ambiguous statement means that whenever the magnitude of the eigenvalue is comparable to the magnitude of the uncertainties in the problem, the matrix is ill-conditioned.

Twomey[1] provides an excellent discussion of ill-conditioned matrices and their role in physical problems. He also provides the following criterion for determining when a problem is ill-conditioned. Consider the matrix equation

$$Af = g + \varepsilon \qquad \text{(C.1)}$$

where A is an n x n matrix, f is an n x 1 unknown vector, g is an n x 1 given vector, and ε is an n x 1 error vector. The problem is well posed only if

$$C\,\lambda_{min} >> \sum_{i=1}^{N} \varepsilon_i^2 \qquad \text{(C.2)}$$

where λ_{min} is the minimum eigenvalue of A^*A (A^* is the transpose of A) and C is an upper bound for $\sum_{i=1}^{n} f_i^2$. If (C.2) is not satisfied, then (C.1) is ill-conditioned.

Ivanov[2] has presented one method for obtaining a solution to an ill-conditioned problem. His method of regularizing the problem involves restricting the solution to lie within a ball of radius R in the space of possible solutions. Applying his ideas to the matrix equation

$$Ax = b \tag{C.3}$$

means that the inner product $<Ax-b, Ax-b>$ is to be minimized under the constraint that $<x, x> \leq R^2$. For an ill-conditioned problem, the minimum will occur on the boundary so that the equality will hold. Using Lagrange multipliers, the variance of the expression

$$x^*A^*Ax - x^*A^*b - b^*Ax + b^*b + \lambda(x^*x-R^2)$$

is set to zero. This gives the equation

$$A^*Ax + \lambda x = A^*b. \tag{C.4}$$

The presence of the term λx can be thought of as a way of limiting the size of the vector x which satisfies the equation. The value of λ is determined by solving (C.4) for several values of λ and choosing the smallest one which gives reasonable results. If λ is too large, the solution will be too tightly constrained and will not be a good representation of the true solution. On the other hand, if λ is too small, the solution will contain large oscillatory components.

LIST OF REFERENCES

1. S. Twomey, "The Application of Numerical Filtering to the Solution of Integral Equations Encountered in Indirect Sensing Measurements," Journal of the Franklin Institute, Vol. 279, No. 2, February 1965.

2. V. K. Ivanov, "On Linear Problems Which Are Not Well-Posed," Soviet Math, Vol. 3, No. 4, July 1962.

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The problem of determining the permittivity of a medium from a knowledge of the electromagnetic fields scattered by the medium is considered. A nonlinear integral equation is derived for the dielectric profile, which is a function of only one variable. The solution of this equation by an iterative scheme is investigated. An alternate formulation of the problem, which leads to a linear integral equation, is also studied. In both cases, the numerical solutions to these problems are unstable. That is, they are highly sensitive to small errors. The second method, however, leads to a uniqueness proof for the inversion process.

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