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*for Langley Research Center*

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SUMMARY

The linear time optimal control problem is transformed into a Lagrange problem by means of a mapping which takes a closed control region into an open one. Then the problem and a particular example thereof are investigated in light of the classical necessary and sufficient conditions in the calculus of variations.

# INTRODUCTION

There are several different ways of approaching an optimal control problem by means of the calculus of variations. Perhaps the most well known technique, described in a paper by Berkovitz (1), involves adjoining new variables to the system, commonly called slack variables, in order to transform inequality constraints into differential equations. Another approach, described by Kalman (4), utilizes the Hamiltonian theory of the calculus of variations in a framework principally due to Caratheodory. These methods have been applied to various problems with some degree of success.

In this paper a method described by Park (5) will be illustrated by means of a simple example. However, whenever convenient, results for the general linear time optimal control problem will be exhibited. This technique involves the transformation of the closed control region in an optimal control problem to an open one by means of a suitable mapping, thereby making the problem accessible to techniques of the calculus of variations, including sufficiency criteria.

Section I describes in detail the problems to be considered and their transformation into classical Lagrange problems. Section II first lists the fundamental necessary conditions of the calculus of variations for the Lagrange problem, and then examines their implications for the particular transformed control problems being considered. Finally, Section III contains an investigation of field imbeddibility and the Weierstrass sufficiency condition, and also a global sufficient condition that does not require embeddability.

## ANALYSIS

### I. The Problem.

The problem to be considered, a linear time-optimal control problem, is the following:

Consider the equation $\ddot{x} = u$ where u is a real control restricted by the condition $|u| \leq 1$. To be found is a sectionally continuous function u(t) which yields a solution to the above differential equation such that one arrives at the origin from a given initial state in the shortest possible time. That is, u(t) is defined on some interval $[0,t_1]$ such that $x(0) = x_1^0$, $\dot{x}(0) = x_2^0$ and $x(t_1) = 0$, $\dot{x}(t_1) = 0$, and $t_1$ is as small as it can possibly be.

Notice that writing the above as a first order system we get:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned} \quad \text{with} \quad |u(t)| \leq 1$$

and

$$\begin{aligned} x_1(0) &= x_1^0 \qquad x_1(t_1) = 0 \\ x_2(0) &= x_2^0 \qquad x_2(t_1) = 0 \end{aligned}$$

such that $\int_0^{t_1} dt$ is a minimum.

In this case, the control region is the closed interval [-1,1]. However, if we replace u by cos $\dot{x}_3$, then the new variable $\dot{x}_3$ is not restricted in any sense since the cosine function maps the entire real line onto the closed interval [-1,1]. The reason that the new variable is introduced as a derivative is that the classical theory in the calculus of variations requires that each solution function $x_i(t)$ be sectionally smooth, and of course we want to allow the control variable to be possibly discontinuous at a finite number of points.

Thus the new problem, an equivalent Lagrange problem, may be stated as follows:

To be found is $(x_1(t), x_2(t), x_3(t))$ defined on $[0,t_1]$ such that these functions satisfy the differential equations:

$$\dot{x}_1 - x_2 = 0$$
$$\dot{x}_2 - \cos \dot{x}_3 = 0$$

and the boundary conditions:

$$x_1(0) = x_1^0 \qquad x_1(t_1) = 0$$
$$x_2(0) = x_2^0 \qquad x_2(t_1) = 0$$

such that

$$\int_0^{t_1} dt \quad \text{is a minimum.}$$

Whenever convenient, the following more general linear time optimal control problem will be considered: $\dot{x} = Ax + Bu$ where $x = (x_1, \ldots, x_n)$ and $u = (u_1, \ldots, u_m)$ with A and B being constant $n \times n$ and $n \times m$ matrices respectively. Here again the controls are restricted by the condition $|u_i(t)| \leq 1$ for $i = 1, \ldots, m$. We introduce $y = (y_1, \ldots, y_m)$ by setting $u_i = \cos \dot{y}_i$ for $i = 1, \ldots, m$. Note that for the previous problem $n = 2$, $m = 1$,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We have now stated the problem to be considered as a Lagrange problem which in its most general form is as follows:

To be found is a vector function $x = (x_1(t), \ldots, x_n(t))$ defined on $[t_0,t_1]$ which satisfies the constraining differential equations $\phi_i(t,x,\dot{x}) = 0$ $i = 1, 2, \ldots, \mu < n$ and initial and terminal conditions:

$$x_i(t_0) = x_i^0 \qquad i = 1, 2, \ldots, n$$
$$\psi_i(t_1, x(t_1)) = 0 \qquad i = 1, 2, \ldots, k \leq n$$

and is such that

$$\int_{t_0}^{t_1} f(t,x(t),\dot{x}(t))dt$$

takes on the smallest possible value. We also require that x(t) be sectionally smooth, and that the functions $\phi_i$, $\psi_i$ and f be continuously differentiable. A full treatment of the theory associated with this problem may be found in Sagan (7).

## II. Necessary Conditions

If $x(t) = (x_1(t), \ldots, x_n(t))$, a sectionally smooth vector valued function, is a solution of the Lagrange problem and if

$$\mathrm{rank}(\phi_{\dot{x}}) = \mu \ , \quad \mathrm{rank}(\psi_x) = k$$

where $\phi = (\phi_1, \ldots, \phi_\mu)$, $\psi = (\psi_1, \ldots, \psi_k)$ and $\phi_{\dot{x}}$ denotes the $\mu \times n$ matrix of partials composed of elements of the form $\frac{\partial \phi_i}{\partial \dot{x}_j}$ and $\psi_x$ denotes the $k \times n$ matrix composed of the $\frac{\partial \psi_i}{\partial x_j}$ then:

### 1. Lagrange Multiplier Rule

There exists a sectionally continuous vector function $\lambda = (\lambda_1, \ldots, \lambda_\mu)$ and a constant $\lambda_0$ such that $(\lambda_0, \lambda_1, \ldots, \lambda_\mu) \neq (0, 0, \ldots, 0)$ and

$$\phi(t,x,\dot{x}) = 0 \qquad \text{(Constraining equations)}$$

and

$$h_x(t,x,\dot{x},\lambda) - \frac{d}{dt} h_{\dot{x}}(t,x,\dot{x},\lambda) = 0 \qquad \text{(Mayer equations)}$$

are satisfied on every smooth portion of $x(t)$ where

$$h(t,x,\dot{x},\lambda) = \lambda_0 f(t,x,\dot{x}) + \lambda \cdot \phi(t,x,\dot{x}).$$

Note that by $\lambda \cdot \phi$ is meant the vector dot product of the $\mu$-vectors $\lambda$ and $\phi$. This will be commonly employed throughout the text of this paper.

### 2. Corner Conditions

At every point where $x$ has a jump discontinuity we must have that

$$h_{\dot{x}}(t,x(t),\dot{x}(t),\lambda(t))$$

and

$$h(t,x(t),\dot{x}(t),\lambda(t)) - \dot{x}(t) \cdot h_{\dot{x}}(t,x(t),\dot{x}(t),\lambda(t))$$

remain continuous.

3. Transversality Conditions

There exists a non-zero constant vector $\nu = (\nu_1, \ldots, \nu_k)$ such that

$$\psi(t_1, x(t_1)) = 0 \ , \quad \text{(terminal conditions)}$$

and

$$\nu \cdot \psi_t(t_1, x(t_1)) = -\lambda_0 f(t_1, x(t_1), \dot{x}(t_1)) - h_{\dot{x}}(t_1, x(t_1), \dot{x}(t_1), \lambda(t_1)) \cdot \dot{x}(t_1)$$

(time transversality condition)

and

$$\nu \ \psi_x(t_1, x(t_1)) = h_{\dot{x}}(t_1, x(t_1), \dot{x}(t_1), \lambda(t_1)). \quad \text{(state transversality conditions)}$$

If the initial state is not fixed, but expressed as above, we apply similar conditions at $t = t_0$.

4. Clebsch Condition

For every vector $\sigma = (\sigma_1, \ldots, \sigma_n)$ such that $\sigma$ is a solution to the linear system

$$\phi_{\dot{x}}(t, x(t), \dot{x}(t)) \cdot \sigma = 0$$

we must have

$$\sigma \cdot h_{\dot{x}\dot{x}}(t, x(t), \dot{x}(t), \lambda(t)) \sigma \geq 0$$

(see Bliss (8), p. 224).

5. Weierstrass Condition.

For all $(t, x, \dot{\bar{x}}) \neq (t, x, \dot{x})$ and satisfying the constraining equations we have

$$E(t, x, \dot{x}, \dot{\bar{x}}, \lambda) \geq 0$$

where

$$E = h(t, x, \dot{\bar{x}}, \lambda) - h(t, x, \dot{x}, \lambda) + (\dot{x} - \dot{\bar{x}}) \cdot h_{\dot{x}}(t, x, \dot{x}, \lambda)$$

(Weierstrass excess function).

Using these necessary conditions we shall now see what they mean in terms of our particular problem.

PROPOSITION 1

Let us assume that the rank of the matrix $(B^j, AB^j, \ldots, A^{n-1}B^j)$ is n for $1 \leq j \leq m$ where $B^j$ denotes the $j^{th}$ column of B, and that u(t) yields a solution to the general linear time optimal control problem. Then, u(t) is sectionally constant and takes on only the values +1 or -1. Moreover, the multipliers $\lambda(t)$ must satisfy the differential equations $\dot{\lambda} = -A^T\lambda$.

Proof:

According to the multiplier rule, $h = -\lambda_0 + \lambda\cdot(\dot{x} - Ax - B\cos\dot{y})$ where $\cos\dot{y} = (\cos\dot{y}_1, \ldots, \cos\dot{y}_m)$. Therefore the Mayer equations yield

$$\dot{\lambda} = -A^T\lambda \quad \text{and} \quad \lambda\cdot B^j \sin\dot{y}_j \equiv \text{constant}$$

for $1 \leq j \leq m$ on each smooth arc of our solution. Now the corner conditions tell us that $\lambda$ and the $\lambda\cdot B^j \sin\dot{y}_j$ must remain continuous at a switching point. Hence, the above equations must be satisfied throughout $[t_0,t_1]$. Looking at the state transversality conditions and using the fact that $y(t_1)$ is not fixed we see that $h_{\dot{y}_j}\Big|_{t = t_1} = \lambda\cdot B^j \sin\dot{y}_j\Big|_{t = t_1} = 0$ for $1 \leq j \leq m$. Hence $\lambda\cdot B^j \sin\dot{y}_j \equiv 0$ for all $t \in [t_0,t_1]$ and $1 \leq j \leq m$.

Now suppose $\sin\dot{y}_j \neq 0$ on some interval for some j, then $\lambda\cdot B^j = 0$ on that interval and we may differentiate inside that interval obtaining

$$0 = \frac{d}{dt}(\lambda\cdot B^j) = -A^T\lambda\cdot B^j = -\lambda\cdot AB^j.$$

In general we get by repeatedly differentiating that

$$0 = \lambda\cdot B^j = \lambda\cdot AB^j = \ldots = \lambda\cdot A^{n-1}B^j.$$

Since $\lambda \neq 0$ this means that $\{B^j,AB^j,\ldots,A^{n-1}B^j\}$ is a linearly dependent set of vectors or equivalently that the matrix $(B^j, AB^j, \ldots, A^{n-1}B^j)$ has rank less than n, a contradiction.

Thus we have that $\sin \dot{y}_j = 0$ except possibly at isolated points (the switching points) and so $u_j = \cos \dot{y}_j = +1$ or $-1$.
Note that in the example

$$(B,AB) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which clearly has rank 2, so the above proposition applies.

It is useful to form the expression $H = \lambda_0 + \lambda \cdot (Ax + Bu)$. We shall refer to this as the Hamiltonian of the system. Notice that

$$H_\lambda = Ax + Bu = \dot{x}$$

and

$$H_x = A^T\lambda = - \dot{\lambda}.$$

PROPOSITION 2

The Hamiltonian is constant and equal to 0 along a solution to the linear time-optimal control problem.

Proof:

If we differentiate H along a smooth arc of a solution we obtain

$$\frac{d}{dt}(H) = \dot{\lambda} \cdot (Ax + Bu) + \lambda \cdot A\dot{x}$$

since u(t) is constant on any smooth arc. Now using the fact that

$$\dot{\lambda} = -A^T\lambda \quad \text{and} \quad \dot{x} = Ax + Bu$$

we obtain

$$\frac{d}{dt}(H) = - \lambda \cdot A(Ax + Bu) + \lambda \cdot A(Ax + Bu) = 0$$

Thus the Hamiltonian is constant along any smooth arc of a solution. Moreover, looking at the second corner condition we see that

$$h - \dot{x} \cdot h_{\dot{x}} - \dot{y} \cdot h_{\dot{y}} = -H$$

along a solution must be continuous at the switching points, so H is constant everywhere. Finally the time transversality condition yields that

$$0 = -\lambda_0 - \dot{x}\cdot h_{\dot{x}} - \dot{y}\cdot h_{\dot{y}} = -H$$

at $t = t_1$. Therefore $H \equiv 0$ everywhere along our solution.

PROPOSITION 3

If u(t) yields a solution to the linear time-optimal control problem, then $\lambda\cdot Bu \geq \lambda\cdot B\bar{u}$ for all $\bar{u}$ such that $|\bar{u}_i| \leq 1$ for $1 \leq i \leq m$.

Proof:

For this problem the excess function is

$$E = -\lambda_0 + \lambda\cdot(\dot{\bar{x}} - Ax - B\cos\dot{\bar{y}}) + \lambda_0 - \lambda\cdot(\dot{x} - Ax - B\cos\dot{y})$$
$$+ (\dot{x} - \dot{\bar{x}})\cdot h_{\dot{x}} + (\dot{y} - \dot{\bar{y}})\cdot h_{\dot{y}}.$$

Now $h_{\dot{x}} = \lambda$ and $h_{\dot{y}_j} = \lambda\cdot B^j \sin\dot{y}_j = 0$ along a solution by Proposition 1. Therefore, when simplified the excess function becomes $E = \lambda\cdot B\cos\dot{y} - \lambda\cdot B\cos\dot{\bar{y}}$ which must be non-negative by the Weierstrass Condition. That is, since $u = \cos\dot{y}$, $\lambda\cdot Bu \geq \lambda\cdot B\bar{u}$.

Now let us see what we now know about the example. We have

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u = +1 \text{ or } -1$$

and

$$\dot{\lambda} = -A^T\lambda \quad \text{so} \begin{cases}\dot{\lambda}_1 = 0 \\ \dot{\lambda}_2 = -\lambda_1\end{cases} \quad \text{by Proposition 1;}$$
$$H = \lambda_0 + \lambda_1 x_2 + \lambda_2 u \equiv 0 \quad \text{by Proposition 2;}$$

and

$$\lambda\cdot Bu = \lambda_2 u \geq \lambda\cdot B\bar{u} = \lambda_2\bar{u} \quad \text{for all } \bar{u}$$

such that $|\bar{u}| \leq 1$ by Proposition 3. Notice that the last statement means, since $u = 1$ or $-1$, that $\text{sgn}(\lambda_2) = \text{sgn}(u)$. Moreover, since H must be continuous across a switching point, the product $\lambda_2 u$ must be continuous also because every other term in H is obviously continuous. Thus, when u switches we must have that $\lambda_2 = 0$.

Now, the adjoint equations yield

$$\begin{aligned} \lambda_1 &= k_1 \\ \lambda_2 &= -k_1 t + k_2 \end{aligned} \tag{1}$$

where $k_1$ and $k_2$ are constants. Moreover, from the proof of Proposition 1, we saw that we can only have that $B^T\lambda = 0$ at isolated points. But in this case $B^T\lambda = \lambda_2$ so, since $\lambda_2$ is a linear function, we see that $\lambda_2$ has at most one zero on any interval. Therefore, any solution will have at most one switching point.

<u>Case 1</u> (u = 1)

If we integrate the state equations we obtain

$$\begin{aligned} x_1 &= \frac{1}{2} t^2 + c_2 t + c_1 \\ x_2 &= t + c_2 \end{aligned} \tag{2}$$

and elimination of t from the above yields

$$x_1 = \frac{1}{2} x_2^2 + (c_1 - \frac{1}{2} c_2^2) = \frac{1}{2} x_2^2 + c. \tag{3}$$

In the phase plane this defines a one parameter family of parabolas with the $x_1$-axis as their axis of symmetry and all opening to the right. Moreover, since $\dot{x}_2 = u = +1$, the phase point moves from bottom to top along one of these parabolas as t increases.

By applying the boundary conditions to (2) we see that $c_1 = x_1^0$ and $c_2 = x_2^0$, and that it is possible to reach the origin without switching only if one starts out on the parabola $x_1 = \frac{1}{2} x_2^2$ with $x_2 \leq 0$.

<u>Case 2</u> (u = -1)

Here we obtain

$$\begin{aligned} x_1 &= -\frac{1}{2} t^2 + d_2 t + d_1 \\ x_2 &= t + d_2 \end{aligned} \tag{4}$$

and again eliminating t yields

$$x_1 = -\frac{1}{2}x_2^2 + (d_1 + \frac{1}{2}d_2^2) = -\frac{1}{2}x_2^2 + d. \qquad (5)$$

In the phase plane this defines a one parameter family of parabolas with the $x_1$-axis as their axis of symmetry and all opening to the left. Moreover, since $\dot{x}_2 = u = -1$, the phase point moves from top to bottom along one of these parabolas as t increases.

Again the boundary conditions yield that $d_1 = x_1^0$ and $d_2 = x_2^0$, and that it is possible to reach the origin without switching only if one starts out on the parabola $x_1 = -\frac{1}{2}x_2^2$ with $x_2 \geq 0$.

Now we are only allowed one switch from u = +1 to -1 or vice versa. So to get to the origin from an arbitrary initial point not lying on a parabola which leads to the origin, we must do the following. Through each $(x_1^0, x_2^0)$ there passes exactly one member of each of the families (3) and (5). However, only one of these parabolas leads (in the direction of increasing t) to a parabola which leads to the origin. So we must travel along that parabola and switch the value of u when the parabola leading to the origin is encountered. For a fuller discussion of this, see Sagan (7) pp. 295-301 and Pontryagin (6) pp. 23-27.

To summarize, if

1. $x_2 \leq 0$ and
   - $x_1 \leq \frac{1}{2}x_2^2$ use $u = +1$
   - $x_1 > \frac{1}{2}x_2^2$ use $u = -1$
2. $x_2 \geq 0$ and
   - $x_1 \geq -\frac{1}{2}x_2^2$ use $u = -1$
   - $x_1 < -\frac{1}{2}x_2^2$ use $u = +1$.

So we see that through each point in the phase plane there passes a unique trajectory satisfying all necessary conditions which leads to the origin.

Notice that <u>only</u> the above described trajectories can be optimal (solutions to our problem). Thus, if a solution to our problem exists, from a fixed initial point, then it must necessarily be the above unique trajectory passing through the given initial point and leading to the origin. To find out if the above trajectories are indeed optimal, we therefore shall examine them in the light of the classical sufficiency conditions of the calculus of variations.

## III. The Question of Sufficiency

We shall first consider a solution with no corners. In all that follows we assume that $\lambda_0 = -1$.

DEFINITION: A smooth vector valued function $x(t)$ defined on $[t_0,t_1]$ satisfying all necessary conditions is embeddable in a field if there exists in a neighborhood of $x(t)$ [considered as a curve in $n + 1$ space] a smooth vector valued function $\phi(t,x)$ such that $\dot{x}(t) = \phi(t,x(t))$ for $t \in [t_0,t_1]$, and a continuous vector valued function $\lambda(t,x)$ such that the solutions of $\dot{x} = \phi(t,x)$ and $\lambda$ evaluated along these solutions satisfy the Mayer equations, the constraining equations, the transversality conditions and the self-adjointness condition which states that

$$\frac{\partial}{\partial x_i} h_{\dot{x}_k}(t,x,\phi(t,x),\lambda(t,x)) = \frac{\partial}{\partial x_k} h_{\dot{x}_i}(t,x,\phi(t,x),\lambda(t,x))$$

for $i, k = 1, 2, \ldots, n$.

Now with this definition of field embeddibility, an invariant integral can be found and the following sufficiency theorem proved (see Sagan (7), pp. 371-375).

THEOREM 1

If $x(t)$ is embeddable in a field and if $E(t,x,\phi(t,x),\dot{\bar{x}},\lambda(t,x)) \geq 0$ (Weierstrass Excess Function) for all $(t,x)$ in a neighborhood of $x(t)$ and all $\dot{\bar{x}}$ for which $(t,x,\dot{\bar{x}})$ satisfies the constraining equations, then $x(t)$ yields a solution to the Lagrange Problem.

Note that in the case that the constraining equations are in the form $\dot{x} - f(t,x) = 0$, we have that $h_{\dot{x}} = \lambda$ and consequently the self adjointness condition in this case merely requires that the matrix $\frac{\partial}{\partial x}(\lambda(t,x))$ be symmetric. Also notice that while the solutions of $\dot{x} = \phi(t,x)$ must satisfy the transversality conditions, the boundary conditions need not be met.

Let us now apply this to our example in the case where we can get to the origin without switching. For example if we take $u = +1$ then we will have $x_1^0 = \frac{1}{2} x_2^{0\,2}$ with $x_2^0 \leq 0$ and

$$x_1 = \frac{1}{2} t^2 + c_2 t + c_1$$
$$x_2 = t + c_2$$
$$x_3 = c_3$$
$$\lambda_1 = k_1$$
$$\lambda_2 = - k_1 t + k_2$$

where $k_1$ and $k_2$ must be chosen so that $H \equiv 0$, $\mathrm{sgn}(\lambda_2) = \mathrm{sgn}(u)$ and the self adjointness condition is satisfied along our field.

It is easily seen that the following choice of a field suffices in this case.

$$\phi_1 = x_2 \qquad \lambda_1 = 0$$
$$\phi_2 = 1 \qquad \lambda_2 = 1$$
$$\phi_3 = 0$$

That is, with this choice of $\phi$ and $\lambda$, all conditions for field embeddability are satisfied. Moreover,

$$E(t,x,\phi(t,x),\, \dot{\bar{x}},\, \lambda(t,x)) = 1 - \cos \dot{\bar{x}}_3$$

which is clearly non-negative for all $\dot{\bar{x}}_3$. Hence the hypotheses of Theorem 1 are satisfied, so our proposed solution is indeed a solution to our problem in this case.

Now if we only require that $\phi$ be sectionally smooth in t and smooth in x, and that $\lambda$ be sectionally continuous in t and continuous in x, and require that solutions of $\dot{x} = \phi(t,x)$ also satisfy the corner conditions, then we obtain a generalization of Theorem 1 for the case where we allow solutions with corners. However, for the problem being considered if we attempt to embed a trajectory with corners in a field, trouble is encountered. It can be shown that no choice of $\lambda(t,x)$ can be made for which along solutions of $\dot{x} = \phi(t,x)$ the conditions $H \equiv 0$, $\mathrm{sgn}(\lambda_2) = \mathrm{sgn}(u)$, $\lambda_2 = 0$ at a corner, and the self adjointness condition are all met simultaneously. Thus, for this problem it is not possible to embed an extremal with corners in a field, so we must look elsewhere to establish sufficiency in this case.

The next approach that we shall consider is described in Ewing (2), p. 129-131. However, to facilitate the use of this theory, we must first transform our problem to one with a fixed time duration. This is done in the following way.

Introduce to the system in the general linear time optimal control problem the new variable $x_{n+1}$ so that $x_{n+1}$ is a constant whose square denotes the time duration of a trajectory, $t_1 - t_0$. Also introduce a new independent variable s by letting $t = t_0 + x_{n+1}^2 s$. Then for any function f we will have $\frac{df}{ds} = x_{n+1}^2 \frac{df}{dt}$ and when t is restricted to $[t_1, t_0]$, s will be restricted to the fixed interval $[0,1]$.

Then our new equivalent Lagrange problem is the following:

To be found are vector functions $x = (x_1(s), \ldots, x_n(s)), x_{n+1}(s)$, $y = (y_1(s), \ldots, y_m(s))$ defined on $[0,1]$ which satisfy the constraining equations

$$x' = x_{n+1}^2 \,(Ax + B\cos y')$$

$$x'_{n+1} = 0$$

and boundary conditions

$$x(0) = x^0, \ x(1) = 0$$

such that $\int_0^1 x_{n+1}^2 \, ds$ is minimal.

Note that here ' is used to denote $\frac{d}{ds}$ . It can be easily shown that this indeed constitutes an equivalent problem and that consequently the application of necessary conditions here yields the same trajectories as previously.

We are now in a position to apply the global condition in (2). For the Lagrange problem described in Section I define the function G to be

$$G(t,x,\dot{x},\overline{x},\dot{\overline{x}},\lambda) = h(t,\overline{x},\dot{\overline{x}},\lambda) - h(t,x,\dot{x},\lambda) + (x - \overline{x})\cdot h_x(t,x,\dot{x},\lambda) + (\dot{x} - \dot{\overline{x}})\cdot h_{\dot{x}}(t,x,\dot{x},\lambda).$$

Then we have the following theorem.

THEOREM 2

Given a Lagrange problem for which $t_0$ and $t_1$ are fixed and given a trajectory $x(t)$ which satisfies the necessary conditions with $\lambda_0 = -1$, then if

$$\begin{aligned}\int_{t_0}^{t_1} & G(t,x(t),\dot{x}(t),\overline{x}(t),\dot{\overline{x}}(t),\lambda(t))dt \\ & + [\overline{x}(t_1) - x(t_1)]\cdot h_{\dot{x}}(t_1,x(t_1),\dot{x}(t_1),\lambda(t_1)) \\ & + [x(t_0) - \overline{x}(t_0)]\cdot h_{\dot{x}}(t_0,x(t_0),\dot{x}(t_0),\lambda(t_0))\end{aligned} \qquad (6)$$

is non-negative for all $\overline{x}(t)$ satisfying the constraining equations and the boundary conditions, then $x(t)$ is a solution to this Lagrange problem.

For a discussion and proof of this theorem, see Ewing (2), pp. 129-134.

Now for our problem

$$h = x_{n+1}^2 + \lambda\cdot(x' - x_{n+1}^2(Ax + B\cos y')) + \lambda_{n+1}x'_{n+1}$$

so

$$G = \bar{x}_{n+1}^2 + \lambda\cdot(\bar{x}' - \bar{x}_{n+1}^2(A\bar{x} + B\cos\bar{y}')) + \lambda_{n+1}\,\bar{x}'_{n+1} - x_{n+1}^2$$

$$- \lambda\cdot(x' - x_{n+1}^2(Ax + B\cos y')) - \lambda_{n+1}x'_{n+1} - (x - \bar{x})\cdot x_{n+1}^2\,A^T\lambda$$

$$+ (x_{n+1} - \bar{x}_{n+1})(2x_{n+1} - 2x_{n+1}\lambda\cdot(Ax + B\cos y')) + (x' - \bar{x}')\cdot\lambda$$

$$+ (x'_{n+1} - \bar{x}'_{n+1})\lambda_{n+1} + (y' - \bar{y}')\,h_{y'}.$$

Now recall that the Mayer equations together with the transversality conditions (since y(1) is free) yield $h_{y'} \equiv 0$ along a solution. Thus the last term above is zero. Simplifying, we obtain

$$G = (\bar{x}_{n+1} - x_{n+1})^2 - \bar{x}_{n+1}^2\,\lambda\cdot(A\bar{x} - B\cos\bar{y}') + x_{n+1}^2\,\lambda\cdot(Ax + B\cos y')$$

$$+ x_{n+1}^2\,\lambda\cdot(A\bar{x} - Ax) + 2\bar{x}_{n+1}\,x_{n+1}\,\lambda\cdot(Ax + B\cos y').$$

Using Proposition 2, we have that $\lambda\cdot(Ax + B\cos y') = \lambda\cdot(Ax + Bu) = 1$ since $H \equiv 0$ along a trajectory which satisfies the necessary conditions. Therefore, after simplifying

$$G = \bar{x}_{n+1}^2(\lambda\cdot Bu - \lambda\cdot B\bar{u}) + (x_{n+1}^2 - \bar{x}_{n+1}^2)(\lambda\cdot A\bar{x} - \lambda\cdot Ax). \qquad (7)$$

PROPOSITION 4

Assume that u and x satisfy all necessary conditions and $\bar{u}$ is any other admissible control with $\bar{x}$ its corresponding trajectory such that $\bar{x}$ and $\bar{u}$ satisfy the constraining equations and boundary conditions. Then if

$$(\bar{x}_{n+1}^2 - x_{n+1}^2)\int_0^1 (\lambda\cdot Ax - \lambda\cdot A\bar{x})ds \geq 0 \qquad (8)$$

for all such $\bar{u}$, then u and x furnish a solution to the linear time optimal control problem

Proof:

We apply Theorem 2. First notice that the last two terms in (6) are zero for this problem since the boundary conditions fix x at 0 and 1 and $h_{y'} = 0$ for the free variables by the transversality conditions. Thus we need only consider the first term, that is, $\int_0^1 G\,ds$. Now (7) gives the expression for G, and by Proposition 3 we see immediately, that the first term in (7) is always non-negative. Moreover, condition (8) insures that the integral of the second term is always non-negative, so the hypotheses of Theorem 1 are satisfied, and therefore u and x must furnish a solution to our problem (i.e. be optimal).

Now let us apply this criteria to the trajectories obtained in Section II for our simple problem. Recalling that $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ we obtain that

$$\lambda\cdot Ax - \lambda\cdot A\bar{x} = \lambda_1 x_2 - \lambda_1 \bar{x}_2 .$$

Thus, (8) becomes

$$(\bar{x}_3^2 - x_3^2) \int_0^1 (\lambda_1 x_2 - \lambda_1 \bar{x}_2) ds .$$

Now $x_1' = x_3^2 x_2$ and $\bar{x}_1' = \bar{x}_3^2 \bar{x}_2$ and $\lambda_1$ is constant, so we can write the above as:

$$(\bar{x}_3^2 - x_3^2)\ \lambda_1 \left[ \frac{1}{x_3^2} \int_0^1 x_1'\ ds - \frac{1}{\bar{x}_3^2} \int_0^1 \bar{x}_1'\ ds \right]$$

$$= \frac{(\bar{x}_3^2 - x_3^2)\ \lambda_1 x_1(0)(x_3^2 - \bar{x}_3^2)}{x_3^2 \bar{x}_3^2} = \frac{-\lambda_1 x_1(0)(\bar{x}_3^2 - x_3^2)^2}{x_3^2 \bar{x}_3^2}$$

which is non-negative provided the product $\lambda_1 x_1(0)$ is non-positive.

Now $\lambda_1 = -\dot{\lambda}_2$ and where u = +1 initially then $\lambda_2(0) > 0$ so $\lambda_2$ must be a decreasing function. But this implies that $\dot{\lambda}_2 < 0$ and so $\lambda_1 > 0$. Similarly we can show that if u = -1 initially then $\lambda_1 < 0$. Thus the product $\lambda_1 x_1(0)$ will be non-positive for all extremals in which $u(0) = -1$ and $x_1(0) \geq 0$ or $u(0) = +1$ and $x_1(0) \leq 0$. Referring back to the summary on page 10 we see that this is satisfied in the second and fourth quadrants and also in the two regions

$$R_1 = \{(x_1,x_2) : x_1 \geq 0,\ x_2 < -\frac{1}{2} x_2^2\} \quad \text{and}$$

$$R_2 = \{(x_1,x_2) : x_2 \leq 0,\ x_1 > \frac{1}{2} x_2^2\} \ .$$

Therefore, we have established sufficiency for any extremal obtained in Section II which originates in any of these parts of the plane. Although, as is well known, the trajectories obtained in Section II originating at any point of the plane are optimal, this condition only establishes that fact in the above mentioned regions.

# CONCLUDING REMARKS

We have shown how an optimal control problem with the unit m-cube as control region may be transformed into a Lagrange problem. Then considering the linear time optimal control problem and a particular example thereof, we have investigated the implications for these problems of the classical necessary and sufficient conditions in the calculus of variations.

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