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SUMMARY

A general procedure for analyzing the nonlinear forced sloshing of an inviscid fluid in a cylindrical container is presented. The total energy and the frequency are related through a characteristic parameter in the corresponding linear problem. The application of the technique to circular container is presented.

INTRODUCTION

In this report we study the motion of an inviscid, incompressible fluid in a rigid cylindrical container that undergoes a prescribed periodic motion consisting of a sum of three translations in the directions parallel and perpendicular to the generators of the cylinder. The flat bottom of the container, perpendicular to the generators of the cylinder, remains horizontal during the motion. The vertical acceleration of the container is an arbitrary time periodic function; the horizontal acceleration components are small periodic functions with the same period. The elevation of the free surface is assumed to be a single-valued function of the horizontal variables.

We first formulate the fully nonlinear problem. By perturbing these equations about a trivial solution (occurring for zero transverse acceleration of the container), we reduce this problem to a sequence of linear problems on a fixed domain. These problems have an eigenvalue parameter in the free surface boundary conditions. By formulating these problems in an appropriate Hilbert space, we can efficiently obtain eigenfunction expansions for the solutions. From these solutions can be found a response diagram showing how the energy of the fluid depends upon the nature of the container motion. Resonant behavior can be determined from this diagram. The application to a circular container in lateral acceleration is handled in the last section.

1. Formulation of the Nonlinear Boundary Value Problem

We adopt the convention that Latin indices range over 1, 2, 3, and Greek indices range over 1, 2. We employ the summation convention. If $f - f(x,t)$ is any differentiable function of a three vector $x = (x_1, x_2, x_3)$ and a real variable t, then we use the notation

$$f,_t = \frac{\partial f}{\partial t} \qquad f,_k = \frac{\partial f}{\partial x_k} \quad , \quad k = 1,2,3.$$

Let x'_k be Cartesian coordinates of a frame of reference fixed in space. Let t' denote the time. Let $R(t')$ specify the position relative to the origin of this system of a reference point, fixed relative to the container, that lies on the plane of mean elevation of the free surface. Let p locate points on the container relative to this moving reference point.

The velocity potential is denoted by $\phi''(x',t')$ and the free surface elevation by $\zeta''(x',t')$. The outward normal to the container is denoted by n. The boundary value problem (Cf. /1/ , /2/) consists of the continuity equation

$$\Delta\phi'' \equiv \phi''_{,kk}(x',t') = 0 \tag{1.1}$$

in the region occupied by the fluid, subject to the boundary conditions

$$\nabla\phi''\cdot\underset{\sim}{n} = \phi''_{,k}n_k = \underset{\sim}{R},_{t'} \cdot\underset{\sim}{n} \tag{1.2}$$

on moving walls $\underset{\approx}{x'} = \underset{\approx}{R}(t) + \underset{\approx}{p}$,

$$\phi''_{,\alpha}\zeta''_{,\alpha} - \phi''_{,3} + \zeta''_{,t'} = 0 \tag{1.3}$$

on free surface $x'_3 = \zeta''(x'_\alpha, t')$

$$g\, x'_3 + \phi''_{,t'} + \frac{1}{2}\phi''_{,k}\phi''_{,k} + p/\rho = 0 \tag{1.4}$$

on free surface $x'_3 = \zeta''(x'_\alpha, t')$.

Equation (1.2) is the condition that the normal velocity of the fluid be equal to the normal velocity of the wall, (1.3) is the condition that the free surface be a material surface, and (1.4) is Bernoulli's law for the free surface with g the acceleration of gravity, p the prescribed surface pressure, and ρ the density. For simplicity we assume that $p = 0$.

We now introduce coordinates x_k fixed in the container by

$$x_k = x'_k - R_k\,(t') \quad , \tag{1.5}$$

$$\phi'(x_k, t') = \phi''(x_k'', t') - (x_k - R_k)\, R_{k,t'} \quad . \tag{1.6}$$

$$\zeta'(x_\alpha, t') = \zeta''(x'_\alpha, t') - R_3 \quad . \tag{1.7}$$

We denote the cross-sectional region of the cylinder by S and the average depth of the liquid by h . The region occupied by the liquid is $\{\underset{\sim}{x}:\; x_\alpha \in S \;,\; -h < x_3 < \zeta'(x_\alpha t')\}$. We substitute (1.5) - (1.7) into (1.1) - (1.4) and discard the purely time dependent terms that arise in (1.3) and (1.4) because they can be absorbed into ϕ' and they do not affect the velocity field. We thus obtain a boundary value problem for $\phi' = \phi'(\underset{\sim}{x}, t'),\ \zeta' = \zeta'(x_\alpha, t)$.

To study our problem on a fixed time interval, we introduce a frequency parameter λ by the transformation

$$t = \lambda t' \;,\; \phi(\underset{\sim}{x}, t) = \phi'(\underset{\sim}{x},\; t/\lambda) \;,\; \zeta(x_\alpha, t) = \zeta'\;(x_\alpha, t/\lambda) \;. \tag{1.8}$$

Substituting (1.8) into the last form of the boundary value problem we obtain

$$\Delta\phi = 0 \ , \ x_\alpha \in S \ , \ -h < x_3 < \zeta\ (x_\alpha, t) \ , \tag{1.9}$$

$$\phi,_{\alpha} n_\alpha = 0 \ , \ x_\alpha \in \partial S \ , \ -h < x_3 < 0 \tag{1.10}$$

$$\phi,_3 = 0 \ , \ x_3 = -h, \ x_\alpha \in S \tag{1.11}$$

$$\phi,_3 - \lambda\ \zeta,_t = \phi,_\alpha \zeta,_\alpha \ , \quad x_3 = \zeta\ (x_\alpha, t) \ , \ x_\alpha \in S \ , \tag{1.12}$$

$$\left[g + D_3(t)\right] \zeta + \lambda\ \phi,_t = -\frac{1}{2}\phi,_k \phi,_k - D_\alpha x_\alpha, \ x_3 = \zeta(x_\alpha, t), \ x_\alpha \in S . \tag{1.13}$$

Here D_k are proportional to the components of acceleration of the container. Note that this problem is specified on a time-varying domain.

2. The Perturbation Equations

We note that the problem (1.9) - (1.13) has the trivial solution $\phi = 0$, $\zeta = 0$, for $D_\alpha = 0$. (This solution represents the fluid moving as a rigid body with the container.) We study solutions in a neighborhood of this solution. Let ϵ be a small real number. We set

$$D_\alpha = \epsilon\ D_\alpha^{(1)} \ , \quad D_3 = D_3^{(o)} + \epsilon\ D_3^{(1)} \ . \tag{2.1}$$

We assume that the solution of (1.9) - (1.13) depends smoothly upon ϵ and may be represented by series in the form

$$\begin{aligned} \phi &= \epsilon\ \phi^{(1)} + (\epsilon^2/2\ !)\ \phi^{(2)} + \ldots . \\ \zeta &= \epsilon\ \zeta^{(1)} + (\epsilon^2/2\ !)\ \zeta^{(2)} + \ldots . \\ \lambda &= \lambda^{(o)} + \epsilon\ \lambda^{(1)} + (\epsilon^2/2\ !)\ \lambda^{(2)} + \ldots . \end{aligned} \tag{2.2}$$

Here

$$\phi^{(k)} \equiv \left.\frac{\partial^k \phi}{\partial \epsilon^k}\right|_{\epsilon = 0} \ , \quad \text{etc.}$$

We obtain the equations for these quantities by differentiating (1.9) - (1.13) with respect to ϵ and then setting $\epsilon = 0$. (Care must be exercised when treating the free surface boundary conditions since ϕ depends on ϵ both explicitly and implicitly through its dependence on x_3.) We get

$$\Delta \phi^{(k)} = 0 , \quad x_\alpha \in S , \ -h < x_3 < 0 , \quad k = 1,2, \ldots \tag{2.3}$$

$$\phi^{(k)}{}_{,\alpha} n_\alpha = 0 , \ x_\alpha \in \partial S , \ -h < x_3 < 0 , \quad k = 1,2, \ldots \tag{2.4}$$

$$\phi^{(k)}{}_{,3} = 0 , \ x_3 = -h, \ x_\alpha \in \partial S , \ k = 1,2, \ldots \tag{2.5}$$

$$\phi^{(1)}{}_{,3} - \lambda^{(o)} \zeta^{(1)}{}_{,t} = 0 , \tag{2.6}$$

$$\left[g + D_3^{(o)}(t)\right] \zeta^{(1)} + \lambda^{(o)} \phi^{(1)}{}_{,t} = - D_\alpha^{(1)} x_\alpha , \tag{2.7}$$

$$\phi^{(1)}{}_{,3} - \lambda^{(o)} \zeta^{(2)}{}_{,t} = 2 \lambda^{(1)} \zeta^{(1)}{}_{,t} - 2 \phi^{(1)}{}_{,33} \zeta^{(1)} + 2 \phi^{(1)}{}_{,\alpha} \zeta^{(1)}{}_{,\alpha} , \tag{2.8}$$

$$\begin{aligned}\left[g + D_3^{(o)}(t)\right] \zeta^{(2)} + \lambda^{(o)}\phi^{(2)}{}_{,t} = & -2\lambda^{(1)}\phi^{(1)}{}_{,t} - 2\lambda^{(o)}\phi^{(1)}{}_{,t3} \zeta^{(1)} \\ & - \phi^{(1)}{}_{,k}\phi^{(1)}{}_{,k} - 2D_3^{(1)} \zeta^{(1)} ,\end{aligned} \tag{2.9}$$

$$\begin{aligned}\phi^{(3)}{}_{,3} - \lambda^{(o)}\zeta^{(3)}{}_{,t} = & \ 3\lambda^{(1)}\zeta^{(2)}{}_{,t} + 3\lambda^{(2)}\zeta^{(1)}{}_{,t} - 3\phi^{(2)}{}_{,33}\zeta^{(1)} \\ & - 3\phi^{(1)}{}_{,333}\zeta^{(1)2} - 3\phi^{(1)}{}_{,33} \zeta^{(2)} \\ & + 3(\phi^{(2)}{}_{,\alpha} + 2\phi^{(1)}{}_{,\alpha 3}\zeta^{(1)}) \zeta^{(1)}{}_{,\alpha} ,\end{aligned} \tag{2.10}$$

$$\left[g + D_3^{(o)}(t)\right]\zeta^{(3)} + \lambda^{(o)}\phi^{(3)},_t = -3\lambda^{(2)}\phi^{(1)},_t$$

$$-3\lambda^{(1)}\left[\phi^{(2)},_t + 2\phi^{(1)},_{t3}\zeta^{(1)}\right]$$

$$-3\lambda^{(o)}\left[\phi^{(2)},_{t3}\zeta^{(1)} + \phi^{(1)},_{t,33}\zeta^{(1)2} + \phi,_{t3}^{(1)}\zeta^{(2)}\right]$$

$$-3D_3^{(1)}\zeta^{(2)} - 3\left[\phi^{(2)},_k + 2\phi,_{k3}^{(1)}\zeta^{(1)}\right]\phi^{(1)},_k \ , \qquad (2.11)$$

. . . . .

Note that these problems are defined on a fixed domain and that all the problems have the same homogeneous part.

Since we have introduced a parameter λ into the nonlinear problem we impose an additional condition. We specify the relative energy

$$\epsilon^2 E = \left[\int_{S\times[-h,\zeta]} \phi,_k\phi,_k \ dx_1dx_2dx_3 + g\int_S \zeta^2 dx_1dx_2\right]_{t=0} \ . \qquad (2.12)$$

By means of Green's theorem and (1.9) - (1.11) the volume integral of (2.12) can be written as

$$\int_{x_3=\zeta(x_{\alpha_1} t)} \phi\,\phi,_k\, n_k d\sigma = \int_S \phi\left[-\phi,_\alpha\zeta,_\alpha + \phi,_3\right] dx_1dx_2 \ .$$

Substituting this into (2.12) and differentiating the resulting expression with respect to ϵ , we obtain

$$E = \int_S \left[\phi^{(1)}\phi^{(1)},_3 + g\,\zeta^{(1)\,2}\right] dx_1dx_2 \Big|_{t=0} \ , \qquad (2.13)$$

$$0 = \int_S \Big\{\left[\phi^{(2)} + 2\phi^{(1)},_3\zeta^{(1)}\right]\phi^{(1)},_3 + \phi^{(1)}\left[\phi^{(2)},_3 + 2\phi^{(1)},_{33}\zeta^{(1)} - 2\phi,_\alpha^{(1)}\zeta,_\alpha^{(1)}\right]$$

$$+ 2g\zeta^{(1)}\zeta^{(2)}\Big\}\, dx_1dx_2 \Big|_{t=0} \qquad (2.14)$$

3. Solution of the Equations

We set $x_1 = x$, $x_2 = y$, $x_3 = z$: We assume that each D_k has period 2π in t. The boundary conditions (2.6) - (2.11) are of the form

$$\phi,_3 - \lambda^{(o)} \zeta,_t = f_1 \qquad \text{for } x, y \in S , \tag{3.1}$$

$$G(t)\zeta + \lambda^{(o)}\phi,_t = f_2 \qquad \text{for } x, y \in S . \tag{3.2}$$

where the prescribed functions $G(t)$, $f_1(x,y,t)$, $f_2(x,y,t)$ have period 2π in t. The elimination of ζ from (3.1), (3.2) yields

$$\phi,_3 + \lambda^{(o)2}\left[\phi,_t/G(t)\right],_t = f_1 + \lambda^{(o)}\left[f_2/G(t)\right],_t \equiv F . \tag{3.3}$$

We introduce the space H of functions $\varphi = \varphi(x,y,t)$ on $S \times [0,2\pi]$ having period 2π in t with $\varphi(x,y,t) = \phi(x,y,0,t)$ where

$$\Delta\phi = 0 , \quad x, y \in S , -h < z < 0 \tag{3.4}$$

$$\phi,_\alpha n_\alpha = 0 , \quad x, y \in \partial S, h < z < 0 \tag{3.5}$$

$$\phi,_3 = 0 , \quad z = -h, x, y \in S . \tag{3.6}$$

(H is a space of boundary values of harmonic function.) On H we define the inner product

$$< \varphi, \psi > \equiv \int_0^{2\pi} dt \int_S \varphi \bar{\psi} \, dx \, dy , \tag{3.7}$$

where the bar denotes the complex conjugate. By completing H with respect to this inner product, we obtain an appropriate Hilbert space for our problem which we continue to denote by H.

On H we define the linear operators

$$L\varphi = \phi,_3(x,y,o,t) , \tag{3.8}$$

$$M\varphi = -\left[\phi_{,t} / G(t)\right]_{,t} \quad . \tag{3.9}$$

Thus our boundary value problem reduces to the solution of

$$L\varphi - \lambda^{(o)2} M\varphi = F \tag{3.10}$$

on H .

We study the operators L and M . Let $D = S \times [-h, 0]$, let ∂D be the boundary surface of D , and let $\phi_{,n}$ denote the normal derivative of ϕ on ∂D . To find the operator L^* adjoint to L we use the identity

$$< L\varphi, \psi > = < \varphi, L^* \psi > \quad .$$

By Green's theorem and (3.4) - (3.6), we have

$$< L\varphi, \psi > = \int_0^{2\pi} dt \int_{S, z=0} \phi_{,3} \bar{\psi} \, dx\, dy = \int_0^{2\pi} dt \oint_{\partial D} \phi_{,n} \bar{\psi} \, d\sigma$$

$$= \int_0^{2\pi} dt \left\{ \oint_{\partial D} \phi \bar{\psi}_{,n} \, d\sigma - \int_D \phi \Delta \bar{\psi} \, dxdydx \right\} ,$$

which is to equal

$$\int_0^{2\pi} dt \int_{S, z=o} \phi \overline{L^* \psi} \, dxdy$$

Hence

$\Delta \psi = 0$ in D, $\psi_{,\alpha} n_\alpha = 0$ on $\partial S + [-h, o]$, $\psi_{,3} = 0$ for $z = -h$, $L^*\psi = \psi_{,3}(x,y,0,t)$.

Thus $L = L^*$, i.e., L is self-adjoint. Moreover, L is non-negative definite for

$$< L\varphi, \varphi > = \int_0^{2\pi} dt \oint_{\partial D} \phi_{,n} \bar{\phi}\, d\sigma = \int_0^{2\pi} dt \int_D \nabla \phi \cdot \nabla \bar{\phi}\, dxdydz \geq 0 .$$

L can be made positive definite by including the normalization

$$\int_0^{2\pi} dt \int_{S, z=o} \phi \, dxdy = 0$$

in the definition of H.

Turning to M we find

$$< M\varphi, \psi > = - \int_0^{2\pi} dt \int_{S, z=o} \left[\phi_{,t}/G\right]_{,t} \bar{\psi}\, dxdy$$

$$= - \int_{S,\ z=o} dxdy \left\{ \left[\frac{\phi_{,t}}{G}\bar{\psi}\right]_0^{2\pi} - \left[\frac{\phi}{G}\bar{\psi}_{,t}\right]_0^{2\pi} + \int_0^{2\pi} \phi \left[\bar{\psi}_{,t}/G\right]_{,t} dt\right\}$$

For this to equal $< \varphi, M^*\psi >$, the boundary terms must vanish implying that ψ must have period 2π and it then follows that $M^*\psi = M\psi$. Thus M is also self-adjoint. Since

$$< M\varphi, \varphi > = - \int_{S,z=o} dxdy \int_0^{2\pi} \left[\phi_{,t}/G\right]_{,t} \bar{\phi}\, dt = \int_{S,z=o} dxdy \int_0^{2\pi} \frac{\phi_{,t}\bar{\phi}_{,t}}{G}\, dt,$$

we see that M is positive definite if $G > 0$ (i.e., if the acceleration of gravity or the mean acceleration dominates the time varying acceleration), M is negative definite if $G < 0$, and M is indefinite if G changes sign.

We now turn to the problem of solving (3.10). We suspend the summation convention in the sequel. Let Ω_{kp} be the eigenfunctions of the homogeneous form of (3.10) and let μ_{kp} be the corresponding eigenvalues:

$$L\,\Omega_{kp} = \mu_{kp} M\,\Omega_{kp} \tag{3.11}$$

Thus $< L\,\Omega_{lq}, \Omega_{kp} > = \mu_{lq} < M\,\Omega_{lq}, \Omega_{kp} >$

By the self-adjointness of $x L$ and M

$$< \Omega_{lq}, L\Omega_{kp} > = \mu_{lq} < \Omega_{lq}, M\Omega_{kp} >$$

which, by virtue of (3.11), equals

$$\mu_{lq} < \Omega_{lq}, L\Omega_{kp} / \mu_{kp} >$$

Hence

$$(\mu_{kp} - \mu_{lq}) < \Omega_{lq}, L\Omega_{kp} > = 0 \quad . \tag{3.12}$$

We set

$$< \Omega_{lq}, L\Omega_{kp} > = \delta_{lk}\,\delta_{qp} \tag{3.13}$$

where δ_{ij} is the Kronecker delta.

We then have the Fourier expansion of a function h in H:

$$h = \sum_{k,p} < h, L\Omega_{kp} > \Omega_{kp} \quad . \tag{3.14}$$

From (3.10) we obtain

$$< L\Omega, \Omega_{kp} > - \lambda^{(o)2} < M\Omega, \Omega_{kp} > = < F, \Omega_{kp} > . \tag{3.15}$$

By (3.11) and the self-adjointness of L and M, this equation becomes

$$< \Omega, L\Omega_{kp} > - \lambda^{(o)2} < \Omega, \frac{L\Omega_{kp}}{\mu_{kp}} > = < F, \Omega_{kp} > . \tag{3.16}$$

If $\lambda^{(o)2} \neq \mu_{kp}$, (3.16) implies

$$< \Omega, L\Omega_{kp} > = < F, \Omega_{kp} > / \left[1 - \lambda^{(o)2}/\mu_{kp}\right] \tag{3.17}$$

whereas if $\lambda^{(o)2} = \mu_{kp}$, then (3.16) provides the compatibility condition

$$< F, \Omega_{kp} > = 0 . \tag{3.18}$$

From (3.14) and (3.17), we obtain the representation for the solution of (3.10):

$$\Omega = \sum_{k,p} \frac{< F, \Omega_{kp} >}{1 - \lambda^{(o)2}/\mu_{kp}} \Omega_{kp} \quad . \tag{3.19}$$

It is often convenient to obtain representations for the solution and related quantities in which F appears explicitly. We now obtain these:

$$\lambda^{(o)} M \Omega = - F + L \Omega$$

$$= - F + \sum_{k,p} \frac{< F, \Omega_{kp} >}{1 - \frac{\lambda^{(o)2}}{\mu_{kp}}} L \Omega_{kp} \tag{3.20}$$

$$= - F + \sum_{k,p} \mu_{kp}{}^2 \frac{< F, \Omega_{kp} >}{\mu_{kp} - \lambda^{(o)2}} M \Omega_{kp} \quad . \tag{3.21}$$

Thus,

$$\lambda^{(o)2} \Omega = - M^{-1} F + \sum_{k,p} \mu_{kp}{}^2 \frac{< F, \Omega_{kp} >}{\mu_{kp} - \lambda^{(o)2}} \Omega_{kp} \quad . \tag{3.22}$$

Also,

$$L \Omega = F + \lambda^{(o)2} M \Omega$$

$$= F + \lambda^{(o)2} \sum_{k,p} \frac{< F, \Omega_{kp} > M \Omega_{kp}}{1 - \lambda^{(o)2}/\mu_{kp}} \tag{3.23}$$

$$= F + \lambda^{(o)2} \sum_{k,p} \frac{< F, \Omega_{kp} >}{\mu_{kp} - \lambda^{(o)2}} L \Omega_{kp} \quad . \tag{3.24}$$

From (3.1) we then have

$$\lambda^{(o)} \zeta,_t = L \Omega - f_1 = \lambda^{(o)} (f_2/G),_t + \lambda^{(o)2} \sum \frac{< F, \Omega_{kp} >}{\mu_{kp} - \lambda^{(o)2}} L \Omega_{kp} \quad . \tag{3.25}$$

4. The Eigenfunctions

We seek nontrivial solutions of the problem

$$\Delta\phi = 0 \quad \text{in} \quad D: \quad x, y \in S, \; -h < z < 0, \tag{4.1}$$

$$\phi,_n = 0 \text{ on } \partial S, \; -h < z < 0, \tag{4.2}$$

$$\phi,_z = 0 \text{ on } z = -h, \; x\, y \in S, \tag{4.3}$$

$$\phi,_z = \mu \left[\phi,_t / G(t)\right],_t \quad \text{on} \quad z = 0, \; x, y \in S. \tag{4.4}$$

$$\phi(t) = \phi\,(t + 2\pi). \tag{4.5}$$

We employ separation of variables and assume the solution to be in the form

$$\phi = U(x,y)Z(z)T(t) \tag{4.6}$$

Substituting this expression into (4.1) and (4.2), we find that U must satisfy the Neumann problem

$$-(U,_{xx} + U,_{yy}) = \gamma^2 U, \; U,_n = 0 \quad \text{on} \quad \partial S. \tag{4.7}$$

We denote the eigenfunctions of this problem by U_k and the corresponding eigenvalues by γ_k. The functions Z have the form

$$Z = A\cosh \gamma_k z + B \sinh \gamma_k z \tag{4.8}$$

with condition (4.3) requiring the constants A and B to satisfy

$$A \sinh \gamma_k h = B \cosh \gamma_k h. \tag{4.9}$$

The substitution into (4.4) and (4.5) yields

$$Z'(0) = \mu\, \beta^2 Z(0), \tag{4.10}$$

$$(T'/G)' + \beta^2 T = 0, \; T(t) = T(t + 2\pi). \tag{4.11}$$

The eigenfunctions of (4.11) are denoted T_p and the corresponding eigenvalues are denoted β_p .

From (4.9) and (4.10) we find the eigenvalues for the original problem (4.1) - (4.5) to be

$$\mu_{kp} = \frac{\gamma_k \tanh \gamma_k h}{\beta_p^2} \tag{4.12}$$

The corresponding nontrivial solutions of (4.1) - (4.5) are proportional to

$$\phi_{kp} = U_k \cosh \gamma_k(z+h)\ T_p\ (t) \ . \tag{4.13}$$

The eigenfunctions Ω_{kp} employed in Section 3 are just the surface values of the function in (4.13), namely

$$\Omega_{kp} = \text{const}\ U_k(x,y)\ T_p(t) \quad . \tag{4.14}$$

The constant is found from the normalization (3.13).

The corresponding surface elevations ζ_{kp} are found from

$$\phi_{kp,z} - \sqrt{\mu_{kp}}\ \zeta_{kp,t} = 0 \ , \tag{4.15}$$

$$G(t)\ \zeta_{kp} + \sqrt{\mu_{kp}}\ \phi_{kp,t} = 0 \ . \tag{4.16}$$

To find the equation satisfied by ζ_{kp} , we observe that by (4.10) we have

$$\phi_{kp,z}\Big|_{z=0} = \mu_{kp}\,\beta_p^2\,\phi_{kp}\Big|_{z=0} = (\gamma_k \tanh \gamma_k h)\ \phi_{kp}\Big|_{z=0} \ . \tag{4.17}$$

Substituting this result into (4.15), differentiating the resulting equation with respect to t , and using (4.16), we obtain

$$\zeta_{kp,tt} + \beta_p^2\ G(t)\ \zeta_{kp} = 0 \tag{4.18}$$

The time dependent part $Q_{kp}(t)$ of ζ_{kp} satisfies

$$Q_{kp,tt} + \beta_p^2 G(t) Q_{kp} = 0 \quad . \tag{4.19}$$

whereas the time dependent part of ϕ_{kp} satisfies (4.11). We have the interesting result that if one of these equations does not have an easily recognizable solution, the other might and the solutions can be connected by (4.15) and (4.16). For example, if $G(t) = a + b \cos mt$, then (4.19) is Mathieu's equation whereas (4.11) is unnamed. One can find other examples of this phenomenon.

From (4.11) we have

$$M T_p = \beta_p^2 T_p \tag{4.20}$$

Hence

$$M \Omega_{kp} = \beta_p^2 \Omega_{kp} \quad . \tag{4.21}$$

Introducing this explicit result into (3.21) and (3.23), we get

$$\lambda^{(o)2} M \Omega = - F + \sum_{k,p} \frac{\beta_p^2 \mu_{kp}^2 < F, \Omega_{kp} > \Omega_{kp}}{\mu_{kp} - \lambda^{(o)2}} \quad . \tag{4.22}$$

$$L \Omega = F + \lambda^{(o)2} \sum \frac{\beta_p^2 \mu_{kp} < F, \Omega_{kp} > \Omega_{kp}}{\mu_{kp} - \lambda^{(o)2}} \quad . \tag{4.23}$$

As in (3.22), (4.22) can be integrated to give Ω .

5. An Example

To demonstrate the application of the general formulation, the nonlinear sloshing in a circular cylindrical container subjected to a periodic lateral acceleration will be analyzed.

The radius of the cylinder is a , the height h , and the direction of the lateral acceleration is chosen to be x_1 - axis , and the maximum amplitude is defined as ϵg , therefore

$$D_1 = g \cos t \quad , \quad D_2 = 0$$

The linearlized solutions of the velocity potential $\epsilon\,\Omega^{(1)}$ and the free surface elevation $\epsilon\,\zeta^{(1)}$ are given by

$$\Omega^{(1)} = \frac{ag}{\lambda(o)}\left\{ -\frac{r}{a} + H^{(1)}\right\} \cos\theta \sin t$$

$$\zeta^{(1)} = -\,a\,H^{(1)} \cos\theta\cos t$$

where $H^{(1)} = \sum_{n=1}^{\infty} C'_{1n1}\left(1 - \frac{\lambda^{(o)2}}{\mu_{1n1}}\right)^{-1} J_1(\gamma_{1n}r),\ \mu_{1n1} = g\,\gamma_{1n} \tanh \gamma_{1n} h$,

$$C'_{1n1} = (\gamma_{1n}{}^2a^2-1)\,[\gamma_{1n}a\,J_1(\gamma_{1n}a)]^{-2}\,C_{1n1}\ ,\quad C_{1n1} = a^{-3}\int_0^a J_1(\gamma_{1n}a)\,r^2dr,$$

$$J_1'(\gamma_{1n}a) = 0 \quad ,$$

$J_1(x)$ is the Bessel function of the first kind of order one. The first order frequency parameter $\lambda^{(o)}$ is related to the given energy E at $t = 0$ by the relationship

$$E = \pi a^4 g \sum_{n=1}^{\infty} C^2_{1n1}\left(1 - \frac{\lambda^{(o)2}}{\mu_{1n1}}\right)^{-2} \tag{5.1}$$

The second order solutions of the velocity potential $(\epsilon^2/2!)\Omega^{(2)}$ and the free surface elevation $(\epsilon^2/2!)\zeta^{(2)}$ are given by

$$\Omega^{(2)} = -\frac{2\lambda^{(1)}}{\lambda^{(o)}}\Omega^{(1)} + \frac{ag}{\lambda^{(o)}}\left\{\frac{4\lambda^{(1)}}{\lambda^{(o)}} H^{(2)} \cos\theta \sin t\right.$$

$$\left. + \left(H^{(2,o)} + H^{(2,2)} \cos 2\theta\right) \sin 2t\right\}$$

$$\zeta^{(2)} = -\frac{4\lambda^{(1)}}{\lambda^{(o)}}\,a\,H^{(2)} \cos\theta\cos t - 2a\left(H^{(2,o)} + H^{(2,2)}\cos 2\theta\right)\cos 2t$$

$$-\frac{2\lambda^{(o)}}{g}\Omega,_3^{(1)}\,\zeta^{(1)} - \frac{1}{g}\Omega,_k^{(1)}\,\Omega,_k^{(1)}$$

where $H^{(2)} = \sum_{n=1}^{\infty} < \frac{\lambda^{(o)}}{ag} \Omega,_3^{(1)}, \Omega_{1n1} > (1 - \frac{\lambda^{(o)2}}{\mu_{1n1}})^{-1} N_{1n1} J_1(\gamma_{1n} r)$,

$$H(2,j) = \sum_{n=1}^{\infty} < F^{(2)} , \Omega_{jn2} > (1 - \frac{\lambda^{(o)2}}{\mu_{jn2}})^{-1} N_{jn2} J_j (\gamma_{jn} r) ,$$

$$F^{(2)} = - 2 \Omega,_{33}^{(1)} \zeta^{(1)} + 2\Omega,_{\alpha}^{(1)} \zeta,_{\alpha}^{(1)} - \frac{\lambda^{(o)}}{g} (2\lambda^{(o)} \Omega,_{t3}^{(1)} \zeta,^{(1)} - \Omega,_k^{(1)} \Omega,_k^{(1)}),t,$$

$$<\Omega, \psi> \equiv a^{-1} \int_0^{2\pi} dt \int_0^{2\pi} d\theta \int_0^a \Omega \psi r \, d r, \; \Omega_{mnp} = N_{mnp} J_m(\gamma_{mn} r) \cos m \theta \sin pt ,$$

$$N_{mnp} = < \Omega_{mnp}, \Omega_{mnp,3} >^{-1/2} , \; \mu_{mnp} = p^{-2} \gamma_{mn} g \tanh \gamma_{mn} h .$$

The energy condition of $E^{(1)} = 0$ is given by

$$\int_0^{2\pi} d\theta \int_0^a \{(\Omega^{(2)} + 2\Omega_{13}^{(1)} \zeta^{(1)}) \Omega,_3^{(2)} + 2\Omega,_{33}^{(1)} \zeta^{(1)} - 2\Omega,_{\alpha}^{(1)} \zeta,_{\alpha}^{(1)})$$

$$+ 2 g \zeta^{(1)} \zeta^{(2)}\} r \, d r \Big|_{t=0} = 0$$

Due to the orthogonalities between 1 and $\cos \theta$, between $\cos \theta$ and $\cos 2\theta$, and by the fact that $\Omega^{(j)} = 0$ at $t = 0$, it becomes

$$\lambda^{(1)} \int_0^{2\pi} \cos^2\theta d\theta \int_0^a H^{(1)} H^{(2)} r \, d r = 0$$

which yields

$$\lambda^{(1)} = 0 .$$

The third order solutions of velocity potential $(\epsilon^3/3!) \Omega^{(3)}$, and the surface elevation $(\epsilon^3/3!) \zeta^{(3)}$ are given by

$$\eta^{(3)} = -\frac{3\lambda^{(2)}}{\lambda^{(o)}}\eta^{(1)} + \frac{ag}{\lambda^{(o)}}\left\{\frac{6\lambda^{(2)}}{\lambda^{(o)}} H^{(2)} \cos\theta \sin t + \sum_{i,j=0}^{3} H^{(3,i,j)} \cos i\theta \sin jt\right\}$$

$$\zeta^{(3)} = -\frac{6\lambda^{(2)}}{\lambda^{(o)}} a H^{(2)} \cos\theta \cos t - a \sum_{i,j=0}^{\infty} H^{(3i,j)} j \cos i\theta \sin jt$$

$$-\frac{3\lambda^{(o)}}{g}(\eta,_{t3}^{(2)}\zeta^{(1)} + \eta,_{t33}^{(1)}\zeta^{(2)}) - \frac{3}{g}(\eta,_k^{(2)} + 2\eta,_{k3}^{(1)}\zeta^{(1)})\eta,_k^{(1)}$$

where $H^{(3,i,j)} = \sum_{n=1}^{\infty} < \frac{\lambda^{(o)}}{ag}F^{(3)}, \eta_{inj} > \left(1 - \frac{\lambda^{(o)2}}{\mu_{inj}}\right)^{-1} N_{inj} J_j(\gamma_{jn} r)$,

$$F^{(3)} = -3\eta,_{t3}^{(2)}\zeta^{(1)} - 3\eta,_{333}^{(1)}\zeta^{(1)^2} - 3\eta,_{33}^{(1)}\zeta^{(2)} + 3(\eta,_\alpha^{(2)} + 2\eta,_{\alpha 3}^{(1)}\zeta^{(1)}, \zeta,_\alpha^{(1)}$$

$$-\frac{3\lambda^{(o)^2}}{g}(\eta,_{t3}^{(2)}\zeta^{(1)} + \eta,_{t33}^{(1)}\zeta^{(1)^2} + \eta,_{t3}^{(1)}\zeta^{(2)}),_t - \frac{3\lambda^{(o)}}{g}\left\{(\eta,_k^{(2)} + 2\eta,_{k3}^{(1)}\zeta^{(1)})\eta,_k^{(1)}\right\},_t \;.$$

The energy condition of $E^{(2)} = 0$ is given by

$$\int_0^{2\pi} d\theta \int_0^a \{2(\eta^{(3)} + 3\eta,_3^{(2)}\zeta^{(1)} + 3\eta,_{33}^{(1)}\zeta^{(1)^2})\eta,_3^{(1)} + 3(2\eta,_3^{(1)}\zeta^{(1)} + \eta^{(2)})(-2\eta,_\alpha^{(1)}\zeta,_\alpha^{(1)}$$

$$+ \eta,_3^{(2)} + 2\eta,_{33}^{(1)}\zeta^{(1)}) + 2\eta^{(1)}(\eta,_3^{(3)} + 3\eta,_{33}^{(2)}\zeta^{(2)} + 3\eta,_{333}^{(1)}\zeta^{(1)^2} - 3\eta,_\alpha^{(1)}\zeta,_\alpha^{(2)}$$

$$- 3\eta,_\alpha^{(2)}\zeta,_\alpha^{(1)} - 6\eta,_{\alpha 3}^{(1)}\zeta,_\alpha^{(1)}\zeta^{(1)}) + g(4\zeta^{(1)}\zeta^{(3)} + 3\zeta^{(2)^2})\} r\,dr\Big|_{t=0} = 0 \;.$$

Due to the orthogonality between any two of the functions 1, $\cos\theta$, $\cos 2\theta$ and $\cos 3\theta$, and by the fact that $\eta^{(j)} = 0$ at $t = 0$, it becomes

$$\frac{\lambda^{(2)}}{\lambda^{(o)}} \int_o^a H^{(1)} H^{(2)} r \, dr = \frac{1}{6} \sum_{j=0}^{3} j \int_o^a H^{(1)} H^{(3,1,j)} r \, dr + \frac{1}{8\pi a^2} \int_o^{2\pi} d\theta \int_o^a (\zeta^{(2)})^2 r \, dr \tag{5.2}$$

Equations (5.1) and (5.2) express E and $\lambda^{(o)} + (\epsilon^2/2!)\lambda^{(2)}$ as functions of $\lambda^{(o)}$. Note that when and only when $\lambda^{(o)} = \mu_{inj}$, any one of the natural frequencies of the system, $\lambda^{(2)}(\lambda^{(o)})$ and $E = E^{(o)}(\lambda^{(o)}) \to \infty$. For $\lambda^{(o)}$ varying from zero to the first natural frequency, or between two successive values of μ_{inj}, we can construct an energy $E(\lambda^{(o)},\epsilon)$ vs. frequency parameter $\lambda(\lambda^{(o)},\epsilon)$ curve for given forcing amplitude ϵg. Whenever $\lambda = \lambda^{(o)} + (\epsilon^2/2!) \; \lambda^{(2)}(\lambda^{(o)})$ is finite E is also finite.

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