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FINAL TECHNICAL REPORT

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ABSTRACT

Conical flow is obtained by a flow over a body profile which is generated by ruled surfaces from a vertex point or apex of the body. Many component parts of supersonic-hypersonic vehicles have such profile, in particular a Delta Wing is a typical example.

In theoretical analysis linearized theory has been widely used to predict most aerodynamic data for supersonic flight of relatively low Mach numbers and the validity is good for very slender body at small angle of attack. For high Mach numbers, especially in the hypersonic range, the non-linear effects of the flow properties become predominant and the results obtained by the use of linearized theory are of little value. In fact, at higher Mach numbers even second order theory is not adequate. Meaningful results can be only obtained by the analysis including non-linear effects. The present work is a study of conical flow of a Delta Wing which may serve as a representative analysis to distinguish the non-linear effects from the linearized theory.

The present work deals with the complete solution of the compression side of the Wing. The whole flow region is divided into a supersonic cross-flow region and a subsonic flow region near the center portion of the wing. A system of rotational flow equations is first transformed into dimensionless conical coordinates. In the supersonic cross-flow region method of characteristics is applied. Though compatibility equations have been derived in spherical coordinates by Maslem, to the author's knowledge the derivation and actual computation of the compatibility equations in conical coordinates for varying supersonic cross-flows are the first presentation by this work. In the subsonic region a successive solution of non-linear central difference equations by steepest descent from a zeroth solution is used.

ANALYSIS OF SUPERSONIC CONICAL FLOWS

I. Introduction

Conical flow is obtained by a flow over a body profile which is generated by ruled surfaces from a vertex point or apex of the body. Many component parts of supersonic-hypersonic vehicles have such profile, in particular a Delta Wing is a typical example. In conical flow the flow properties along any ray directed from the vertex point remain uniform. This nice feature enables us to transform a three-dimensional cartesian coordinate system into a two-dimensional conical coordinate system and thus computations become greatly simplified.

In theoretical analysis linearized theory has been widely used to predict most aerodynamic data for supersonic flight of relatively low Mach numbers and the validity is good for very slender body at small angle of attack. For high Mach numbers, especially in the hypersonic range, the non-linear effects of the flow properties become predominant and the results obtained by the use of linearized theory are of little value. In fact, at higher Mach numbers even second order theory is not adequate. Meaningful results can be only obtained by the analysis including non-linear effects. The present work is a study of conical flow of a Delta Wing which may serve as a representative analysis to distinguish the non-linear effects from the linearized theory.

For a Delta Wing, solutions have been obtained by the use of linearized theory and the application of non-linear analysis of Wings of zero-thickness. The only known exact solutions of Wings of finite thickness are those which treat slab or elliptical cross-sections and which have large leading edge thickness. In the case of large Wings for supersonic or hypersonic flow at high Mach numbers, the leading edge is only a fraction of Wing size, thus solutions are needed for a small sharp leading edge as a starting point.

The present work deals with the complete solution on the compression side of the Wing. The whole flow region is divided into a supersonic cross-flow region and a subsonic flow region near the center portion of the wing. A system of rotational flow equations is first transformed into dimensionless conical coordinates. In the supersonic cross-flow region method of characteristics is applied. Though compatibility equations have been derived in spherical coordinates by Maslem, to the author's knowledge the derivation and actual computation of the compatibility equations in conical coordinates for varying supersonic cross-flows are the first presentation by this work. In the subsonic region a successive solution of non-linear central difference equations by steepest descent from a zeroth solution is used.

This method has been used quite successfully for a Delta Wing of zero-thickness by Babaev.

Solutions for the supersonic cross-flow region have been reported by C. W. Chiang and Richard D. Wagner in a NASA TN (submitted in December 1969) which constitute the first part of this final report. In this final report only analaysis for the subsonic region are presented. All notations and references are kept the same as those reported in the first part. Because of lack of time, solutions for the subsonic region are not obtained. A computer program was developed but has yet to be completely debugged. It is hoped in the future, when the NASA budgets are loosened up, this project can be continued.

II. Fundamental Equations

A rectangular coordinate system associated with a Delta Wing is shown in Figure 1. The origin is set at the vertex point or the apex of the wing, the axis OX is directed along the root chord of the lower surface from the apex. The upper surface of the Wing has a lenticular cross-sectional profile while the lower surface is kept as a flat plane which coincides with the flat plane passing through the OX and OY axes. The axis OZ is perpendicular to the lower surface of the Wing.

Flow Equations in Conical Coordinates

In dimensionless conical coordinates the position of a straight-line directed from the apex of the Wing is determined by the quantities $\eta = y/x$ and $\zeta = z/x$. Partial derivative operator in x, y and z may be written in the conical coordinates as follows:

$$\frac{\partial}{\partial x} = -\frac{1}{x}\left(\eta\frac{\partial}{\partial \eta} + \zeta\frac{\partial}{\partial \zeta}\right)$$
$$\frac{\partial}{\partial y} = \frac{1}{x}\frac{\partial}{\partial \eta}$$
$$\frac{\partial}{\partial z} = \frac{1}{x}\frac{\partial}{\partial \zeta}$$

Continuity, momentum and energy equations in conical coordinates may be written in the form:

$$\begin{aligned}
&v(\eta v_\eta + \zeta v_\zeta + u_\eta) + w(\eta w_\eta + \zeta w_\zeta + u_\zeta) = -\tilde{a}(\eta \tilde{s}_\eta + \zeta \tilde{s}_\zeta)\\
&u(\eta v_\eta + \zeta v_\zeta + v_\eta) + w(w_\eta - v_\zeta) = -\tilde{a}^2 \tilde{s}_\eta \qquad (1)\\
&u(\eta w_\eta + \zeta w_\zeta + u_\zeta) + v(v_\zeta - w_\eta) = -\tilde{a}^2 \tilde{s}_\zeta\\
&(\eta u - v)\tilde{s}_\eta + (\zeta u - w)\tilde{s}_\zeta = 0
\end{aligned}$$

The continuity equation in conical coordinates may be shown in the form:

$$2\tilde{a}^2(\eta u_\eta + \xi u_\xi - v_\eta - \omega_\xi) - (\eta u - v)v_\eta^2 - (\xi u - \omega)v_\xi^2 = 0 \qquad (2)$$

In the subsonic region when finite central difference equations are needed for numerical grid method, the central difference equations may be obtained as shown in Appendix A in the form:

$$\begin{aligned}
A_{i,k} &\equiv \frac{h}{2}\left[(\xi u)_{i,k}(Q^2_{i,k+1} - Q^2_{i,k-1}) + X_{i,k}(u^2_{i,k-1} - u^2_{i,k+1}) + 2(\xi u\tilde{s})_{i,k}(\tilde{a}^2_{i,k-1} - \tilde{a}^2_{i,k+1})\right. \\
&\quad \left. + (\eta u)_{i,k}(Q^2_{i+1,k} - Q^2_{i-1,k}) + E_{i,k}(u^2_{i-1,k} - u^2_{i+1,k}) + 2(\eta u\tilde{s})_{i,k}(\tilde{a}^2_{i-1,k} - \tilde{a}^2_{i+1,k})\right] \equiv 0 \\
B_{i,k} &\equiv \frac{h}{2}X_{i,k}(v^2_{i,k+1} - v^2_{i,k-1}) + v_{i,k}(Q^2_{i+1,k} - Q^2_{i-1,k}) + E_{i,k}(v^2_{i+1,k} - v^2_{i-1,k}) + 2(v\tilde{s})_{i,k}(\tilde{a}^2_{i-1,k} - \tilde{a}^2_{i+1,k}) \equiv 0 \\
C_{i,k} &\equiv \frac{h}{2}\left[\omega_{i,k}(Q^2_{i,k+1} - Q^2_{i,k-1}) + X_{i,k}(\omega^2_{i,k+1} - \omega^2_{i,k-1}) + 2(\omega\tilde{s})_{i,k}(\tilde{a}^2_{i,k-1} - \tilde{a}^2_{i,k+1})\right] + E_{i,k}(\omega^2_{i+1,k} - \omega^2_{i-1,k}) \equiv 0 \qquad (3) \\
D_{i,k} &\equiv \frac{h}{2}\left[2\tilde{a}^2_{i,k}(X_{i,k+1} - X_{i,k-1}) + X_{i,k}(v^2_{i,k-1} - v^2_{i,k+1})\right] + 2\tilde{a}^2_{i,k}(E_{i+1,k} - E_{i-1,k}) \\
&\quad + E_{i,k}(v^2_{i-1,k} - v^2_{i+1,k}) - 4u_{i,k}\tilde{a}^2_{i,k} \equiv 0
\end{aligned}$$

where $L = \eta u - v$, $X = \xi u - \omega$, $v^2 = u^2 + v^2 + \omega^2$, $Q^2 = v^2 + 2\tilde{a}^2\tilde{s}$

i,k represent grid node point along η and ξ axis respectively.

ℓ, h - incremental length between neighboring node points along η and ξ respectively.

The Velocity Components and Entropy Charge Behind the Shock

The velocity components behind the shock are determined in the TN report as follows:

$$\begin{aligned}
u &= \cos\beta\sin\chi - \frac{\sin\chi\cos\delta\sin\beta\cos\epsilon}{\cos(\epsilon - \alpha_1 - \delta)} \\
v &= \cos\beta\cos\chi - \frac{\sin\chi\cos\delta\sin\beta\cos\epsilon}{\cos(\epsilon - \alpha_1 - \delta)} \qquad (4) \\
\omega &= \frac{\sin\delta\sin\beta\cos\epsilon}{\cos(\epsilon - \alpha_1 - \delta)}
\end{aligned}$$

where with reference to the Figure 2

χ - the angle between the fictitious leading edge and oy axis.

β - the angle between the direction of undisturbed flow and the fictitious leading edge.

$\epsilon, \delta, \alpha_1$ - in the plane normal to the fictitious leading edge and the plane xoy, the angle between the shock plane and the plane of the undisturbed flow passing through the fictitious leading edge, the deflection angle due to the thickness of the wing, and the angle formed due to the presence of the angle of attack, α .

The entropy change behind the shock is calculated by

$$\Delta \tilde{s} = \tilde{s} - \tilde{s}_\infty = \frac{1}{\gamma(\gamma-1)} \ln\left[\frac{2\gamma}{\gamma+1}\left(M_\infty^2 \sin^2\beta \sin^2\epsilon - \frac{\gamma-1}{\gamma+1}\right)\right] - \frac{1}{\gamma-1} \ln\left[\frac{(\gamma+1) M_\infty^2 \sin^2\beta \sin^2\epsilon}{2+(\gamma-1) M_\infty^2 \sin^2\beta \sin^2\epsilon}\right] \tag{5}$$

Subsonic Region

With reference to the Figure 3 the subsonic region is bounded by the approximate sonic surface AB, the shock BC, the plane of symmetry CD and the body profile DA. The fundamental differential equations are elliptic and the method of characteristics is no longer applicable. All differential equations are replaced by central finite difference equations and numerical calculations may be performed for finite grid points. An approximate zeroth solution and the shock shape is assumed, then successive corrections of the shock shape and flow fields are obtained by the method of the steepest descent. All boundary conditions are to be satisfied. This method is essentially the same as the one used by Babaev.

Velocity Components in Front of and Behind the Shock

Velocity components in front of the shock BC are the velocity components of the undisturbed stream, namely,

$$\begin{aligned} u_\infty &= \cos\alpha \\ v_\infty &= 0 \\ w_\infty &= -\sin\alpha \end{aligned} \tag{6}$$

Velocity components behind the shock BC are determined as shown in the Appendix B by the following equations.

$$\begin{aligned} u_c &= u_\infty - \left(\frac{f}{f'} - \eta\right) f' R \\ v_c &= v_\infty - f' R \\ w_c &= w_\infty + R \end{aligned} \tag{7}$$

where $f = f(\eta)$, the equation of the shock BC

$$f' = \frac{df}{d\eta}$$

$$R = \frac{\omega_1}{\omega_2} - \frac{2}{\gamma+1}\left(\frac{1}{M_\infty^2\omega_1} + \frac{\gamma-1}{2}\frac{\omega_1}{\omega_2}\right) = \frac{2}{\gamma+1}\left(\frac{\omega_1}{\omega_2} - \frac{1}{\omega_1 M_\infty^2}\right)$$

$$\omega_1 = u_\infty(f - \eta f') + v_\infty f' - w_\infty$$

$$\omega_2 = 1 + (f - \eta f')^2 + f'^2$$

Initial Shock Shape BC and Zeroth Solution

The initial shock shape may be given by

$$\xi = \xi_B + \frac{\eta_B}{1+m}\sin\chi \tan(\epsilon - \alpha_1 - \delta)\left[1 - \left(\frac{\eta}{\eta_B}\right)^{m+1}\right] \tag{8}$$

where ξ_B, η_B – location of the point B which may be located from the supersonic cross-flow calculation

m – constant value, varying from 1 to 2

The zeroth values of u, v and w may be written in the form

$$u = u_D - u_A\left(\frac{\eta}{\eta_A}\right)^m - \left[u_D - u_A\left(\frac{\eta}{\eta_A}\right)^m - u(\xi_i)\right]\left(\frac{\xi}{\xi_i}\right)^2$$

$$v = v_A\left(\frac{\eta}{\eta_A}\right)^m - \left[v_A\left(\frac{\eta}{\eta_A}\right)^m - v(\xi_i)\right]\left(\frac{\xi}{\xi_i}\right)^2 \tag{9}$$

$$w = w(\xi_i)\left(\frac{\xi}{\xi_i}\right)^2$$

where u_D, u_A – velocity components at points, A, D

ξ_i – values of ξ on the shock BC corresponding to $\eta = ih$

$u(\xi_i), v(\xi_i), w(\xi_i)$ – velocities calculated on the shock BC.

The initial entropy change $\Delta\tilde{S}$ at the point C and the surface may be calculated. The entropy change $\Delta\tilde{S}_1$ on the surface of the Wing is the same as the one calculated for the straight wedge section 1° from the leading edge of the Wing, it is calculated by the equation (5) for given values of β, ϵ and γ. The entropy change $\Delta\tilde{S}_2$ at the point C is determined by the shock shape BC, it is calculated from the equation (5) with $M_\infty^2\omega_1^2/\omega_2$ replacing $M_\infty^2\sin^2\beta\sin^2\epsilon$. The initial entropy change at different locations may be approximated by interpolation from a linear table of entropy change vs. ξ/η. The zeroth

solutions should satisfy the boundary conditions.

Method of Steepest Descent

The solution of the equations (3) corresponds to a minimum of a function which is the sum of squares of each individual equation of equations (3). The minimum of the function can be approached rapidly in the direction opposite to that of the gradient of the function. Let $\mathcal{F}$ be the function.

$$\mathcal{F} = \sum_{i,\kappa}^{N} \left(A_{i,\kappa}^{2} + B_{i,\kappa}^{2} + C_{i,\kappa}^{2} + D_{i,\kappa}^{2} \right) \tag{10}$$

where N - total number of node points

$$q_{i,\kappa}^{n+1} = q_{i,\kappa}^{n} - \lambda \frac{\delta \mathcal{F}^{n}}{\delta q_{i,\kappa}} \tag{11}$$

where q - variable, namely, u, v, w or $\tilde{S}$.

n - number of approximations or iterations

λ - constant

$\lambda = 0$ is started, for values of λ incremented at $\Delta\lambda$, calculations of $\mathcal{F}^{(n)}$ is continued until $\frac{\delta \mathcal{F}}{\delta \lambda} \geq 0$. This gives λ_{min} corresponding to a minimum of $\mathcal{F}^{(n)}$. For the value of λ_{min} using equation (11) corresponding to λ_{min} to compute the next approximation until after n^{th} approximations for which the rate of decrease of $\mathcal{F}(\lambda_{min})^{(n)}$ as n increases becomes negligible. The calculations of $\frac{\delta \mathcal{F}}{\delta u}$, $\frac{\delta \mathcal{F}}{\delta v}$, $\frac{\delta \mathcal{F}}{\delta w}$ and $\frac{\delta \mathcal{F}}{\delta \tilde{S}}$ are shown in the Appendix C.

Computation

All computations are programmed in Fortran IV language. The method of steepest descent seems to work but converges very slowly. For CDC 6400 computer one whole loop to get 1st approximation from the zeroth solution takes 2 minutes.

Conclusion

The method of steepest descent seems to work in the subsonic region of wings with thickness although further debugging of computer programming is needed.

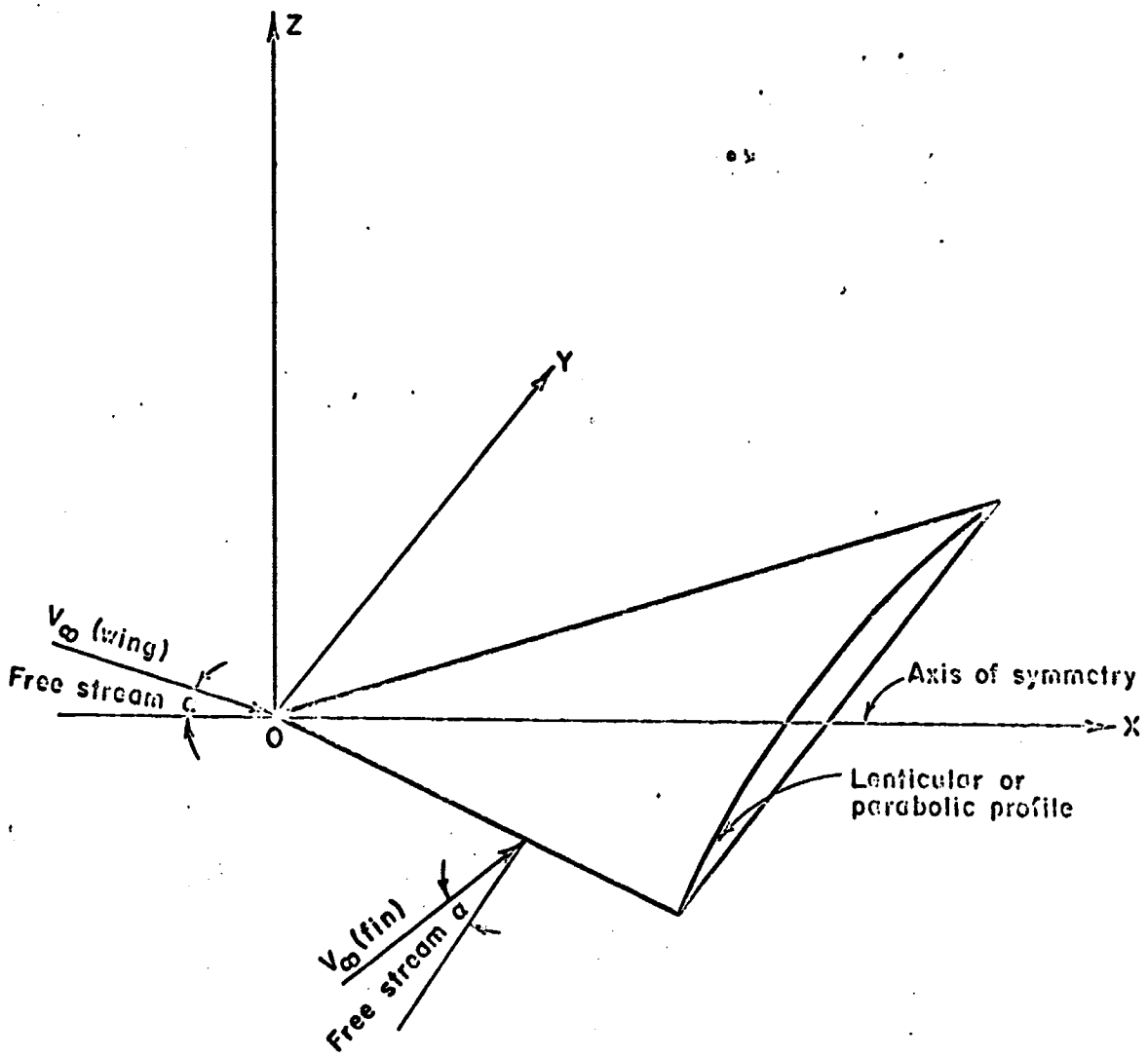

Figure 1. Configuration of a fin or a wing.

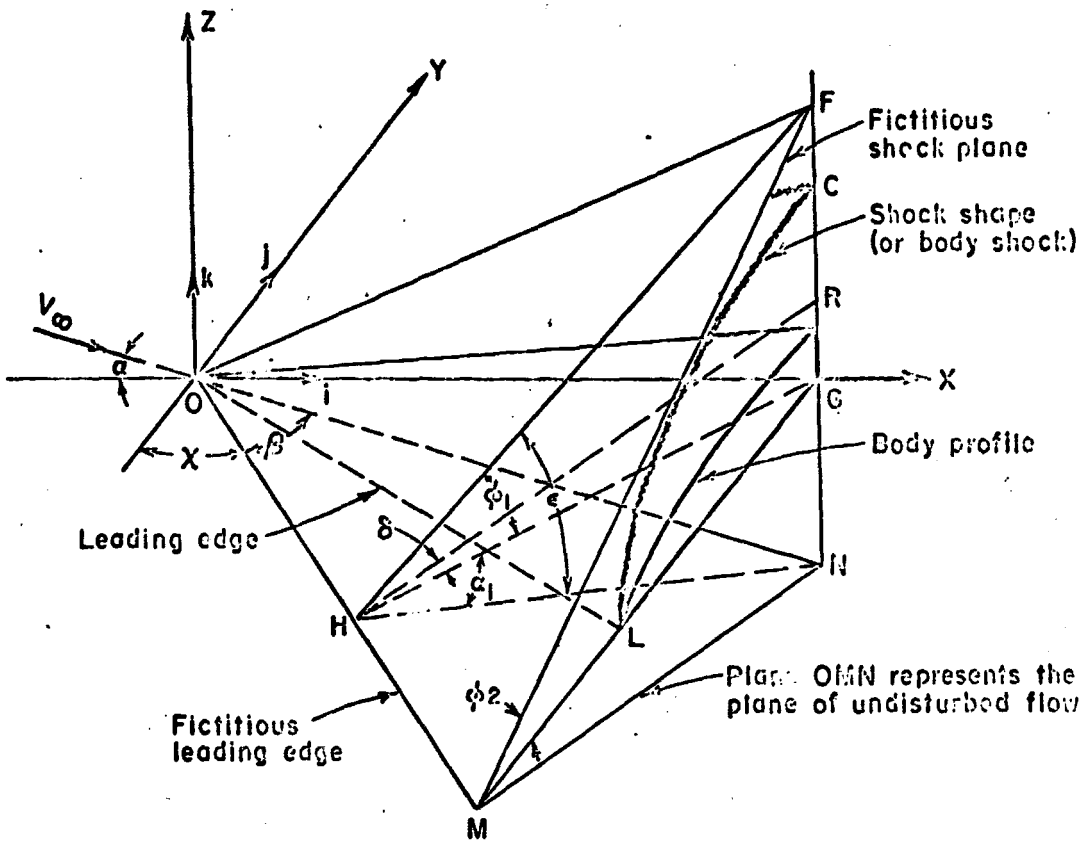

Figure 2. Geometrical layout.

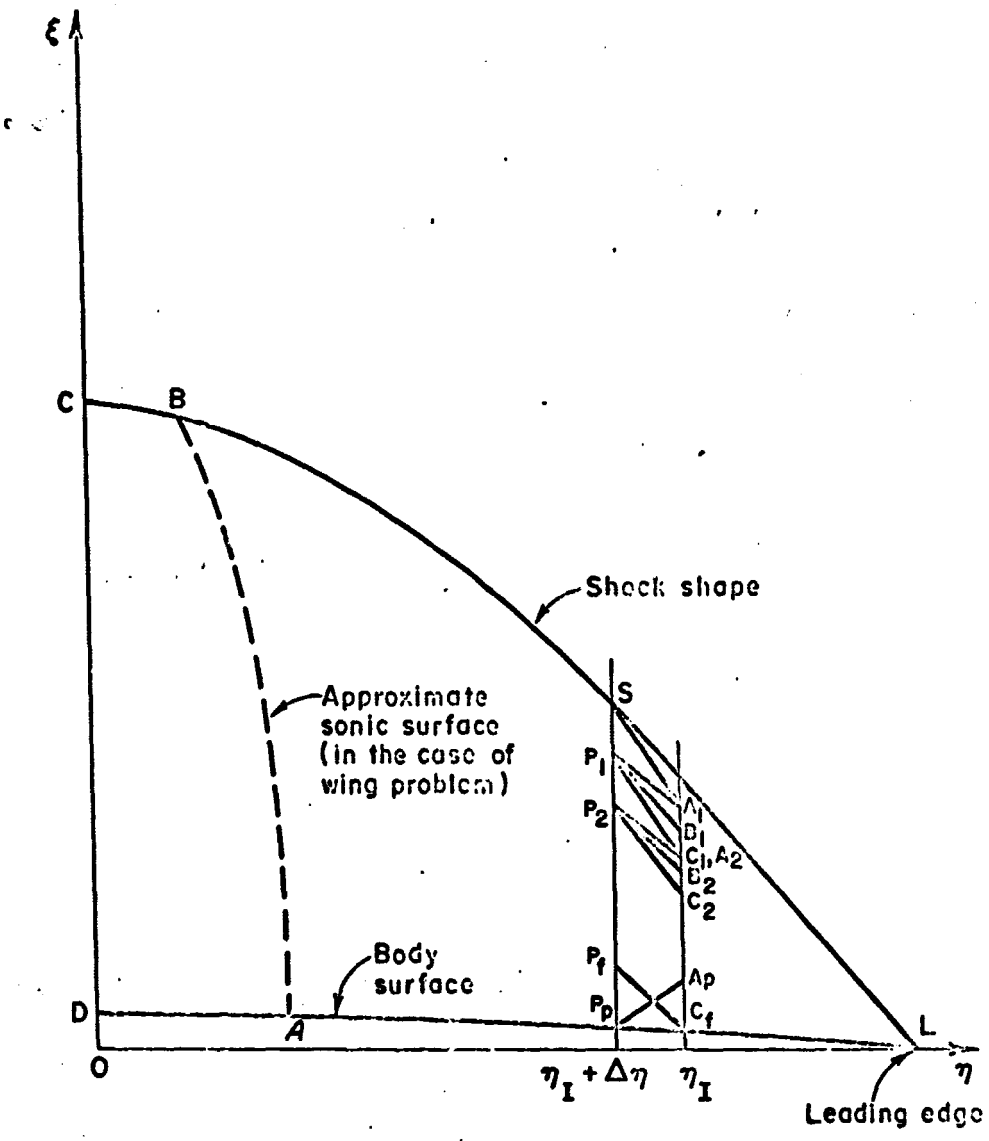

Figure 3. Sketch showing computing technique.

APPENDIX A

Continuity, momentum and energy equations may be written in the form:

$$\begin{aligned}
&\frac{w}{u}u_\xi^2+\frac{v}{u}u_\eta^2+\xi v_\xi^2+\eta v_\eta^2+\xi w_\xi^2+\eta w_\eta^2=-2\tilde{a}^2(\eta\tilde{s}_\eta+\xi\tilde{s}_\xi)\\
&u_\eta^2+\frac{\xi u-w}{v}v_\xi^2+\frac{\eta u}{v}v_\eta^2+w_\eta^2=-2\tilde{a}^2\tilde{s}_\eta\\
&u_\xi^2+v_\xi^2+\frac{\xi u}{w}w_\xi^2+\frac{\eta u-v}{w}w_\eta^2=-2\tilde{a}^2\tilde{s}_\xi
\end{aligned}\qquad \text{(B-1)}$$

or,

$$\begin{aligned}
&\xi v_\xi^2-\frac{\xi u-w}{u}u_\xi^2+\eta v_\eta^2-\frac{\eta u-v}{u}u_\eta^2+2\tilde{a}^2(\eta\tilde{s}_\eta+\xi\tilde{s}_\xi)=0\\
&v_\eta^2+\frac{\eta u-v}{v}v_\eta^2+\frac{\xi u-w}{v}v_\xi^2=-2\tilde{a}^2\tilde{s}_\eta\\
&v_\xi^2+\frac{\xi u-w}{w}w_\xi^2+\frac{\eta u-v}{w}w_\eta^2=-2\tilde{a}^2\tilde{s}_\xi
\end{aligned}\qquad \text{(B-2)}$$

The continuity equation (1a) may be written in the form

$$2\tilde{a}^2(\eta u-v)_\eta-(\eta u-v)v_\eta^2+2\tilde{a}^2(\xi u-w)_\xi-(\xi u-w)v_\xi^2-4\tilde{a}^2u=0 \qquad \text{(B-3)}$$

Equations (B-2) and B-3) may be further simplified in the form

$$\begin{aligned}
&\xi uq_\xi^2-Xu_\xi^2-2u\tilde{s}_\xi\tilde{a}_\xi^2+\eta uq_\eta^2-Eu_\eta^2-2u\tilde{s}_\eta\tilde{a}_\eta^2=0\\
&Xv_\xi^2+vq_\eta^2+Ev_\eta^2-2v\tilde{s}_\eta\tilde{a}_\eta^2=0\\
&wq_\xi^2+Xw_\xi^2-2w\tilde{s}_\xi\tilde{a}_\xi^2+Ew_\eta^2=0\\
&2\tilde{a}^2X_\xi-Xv_\xi^2+2\tilde{a}^2E_\eta-Ev_\eta^2-4u\tilde{a}^2=0
\end{aligned}\qquad \text{(B-4)}$$

where $E=\eta u-v$

$X=\xi u-w$

$v^2=u^2+v^2+w^2$

$q^2=v^2+2\tilde{a}^2\tilde{s}$

In order to solve the system of equations (B-4) by the numerical grid method, all first derivatives may be replaced by central finite difference, for the grid point (i,k)

$$\xi=k\ell$$

$$\eta=ih$$

$$f_\xi = \frac{1}{2\ell}(f_{i,k+1} - f_{i,k-1})$$
$$f_\eta = \frac{1}{2h}(f_{i+1,k} - f_{i-1,k}) \qquad \text{(B-5)}$$

where i,k - number of grid along η and ξ direction

f - any function

ℓ, h - the incremental length between neighboring node points along η and ξ respectively.

The system of equations (B-4) then becomes

$$A_{i,k} \equiv \frac{h}{\ell}\left[(\xi u)_{i,k}(Q^2_{i,k+1} - Q^2_{i,k-1}) + X_{i,k}(u^2_{i,k-1} - u^2_{i,k+1}) + 2(\xi u \tilde{s})_{i,k}(\tilde{a}^2_{i,k-1} - \tilde{a}^2_{i,k+1})\right]$$
$$+ (\eta u)_{i,k}(Q^2_{i+1,k} - Q^2_{i-1,k}) + E_{i,k}(u^2_{i-1,k} - u^2_{i+1,k}) + 2(\eta u \tilde{s})_{i,k}(\tilde{a}^2_{i-1,k} - \tilde{a}^2_{i+1,k}) \equiv 0 \qquad \text{(B-6)}$$

$$B_{i,k} \equiv \frac{h}{\ell} X_{i,k}(v^2_{i,k+1} - v^2_{i,k-1}) + v_{i,k}(Q^2_{i+1,k} - Q^2_{i-1,k}) + E_{i,k}(v^2_{i+1,k} - v^2_{i-1,k})$$
$$+ 2(v\tilde{s})_{i,k}(\tilde{a}^2_{i-1,k} - \tilde{a}^2_{i+1,k}) \equiv 0$$

$$C_{i,k} \equiv \frac{h}{\ell}\left[w_{i,k}(Q^2_{i,k+1} - Q^2_{i,k-1}) + X_{i,k}(w^2_{i,k+1} - w^2_{i,k-1}) + 2(w\tilde{s})_{i,k}(\tilde{a}^2_{i,k-1} - \tilde{a}^2_{i,k+1})\right]$$
$$+ E_{i,k}(w^2_{i+1,k} - w^2_{i-1,k}) \equiv 0$$

$$D_{i,k} \equiv \frac{h}{\ell}\left[2\tilde{a}^2_{i,k}(X_{i,k+1} - X_{i,k-1}) + X_{i,k}(v^2_{i,k-1} - v^2_{i,k+1})\right] + 2\tilde{a}^2_{i,k}(E_{i+1,k} - E_{i-1,k})$$
$$+ E_{i,k}(v^2_{i-1,k} - v^2_{i+1,k}) - 4u_{i,k}\tilde{a}^2_{i,k} \equiv 0$$

APPENDIX B

Let $\xi = f(\eta)$, it represents the equation of the shock BC, then $F = \xi - f(\eta)$, represents the shock surface whose normal is $\vec{n}$.

$$\vec{n} = \frac{\nabla F}{|\nabla F|} = -(\eta F_\eta + \xi F_\xi) i + F_\eta j + F_\xi k$$
$$= -\frac{1}{\sqrt{\omega_2}} [(f - \eta f') i + f' j - k] \qquad \text{(F-1)}$$

where $\omega_2 = 1 + (f - \eta f')^2 + f'^2$

Since $\vec{U}_\infty = u_\infty i + v_\infty j + w_\infty k$ (F-2)

where $|U_\infty| = 1 = \sqrt{u_\infty^2 + v_\infty^2 + w_\infty^2}$

The normal velocity component of $\vec{U}_\infty$ to the shock surface,

$$|U_{\perp\infty}| = \vec{U}_\infty \cdot \vec{n} = \frac{\omega_1}{\sqrt{\omega_2}} \qquad \text{(F-3)}$$

where $\omega_1 = u_\infty (f - \eta f') + v_\infty f' - w_\infty$

$$\vec{U}_{\perp\infty} = |U_{\perp\infty}| \vec{n} = \frac{\omega_1}{\omega_2} [(f - \eta f') i + f' j - k] \qquad \text{(F-4)}$$

The tangential velocity component of $\vec{U}_\infty$ to the shock surface,

$$\vec{U}_{t\infty} = \vec{U}_\infty - \vec{U}_{\perp\infty}$$
$$= [u_\infty - \frac{\omega_1}{\omega_2}(f - \eta f')] i + [v_\infty - \frac{\omega_1}{\omega_2} f'] j + [w_\infty + \frac{\omega_1}{\omega_2}] k \qquad \text{(F-5)}$$

$$U_{t\infty}^2 = 1 - \frac{\omega_1^2}{\omega_2} \qquad \text{(F-6)}$$

Since the shock wave must satisfy the relation.

$$U_1 U_{1n} = \tilde{a}^{2*} - \frac{\gamma-1}{\gamma+1} U_{tn}^2 \qquad \text{(F-7)}$$

where $\tilde{a}^{2*} = \frac{2}{\gamma+1}\left[\frac{1}{M_n^2} + \frac{\gamma-1}{2}\right]$

U_1 – Velocity component behind the shock.

Combining equations (F-6) and normal (F-7)

$$|U_1| = -\frac{\sqrt{\omega_2}}{\omega_1}\frac{2}{(\gamma+1)}\left(\frac{1}{M_n^2} + \frac{\gamma-1}{2}\frac{\omega_1^2}{\omega_2}\right) \qquad \text{(F-8)}$$

$$\vec{U}_1 = |U_1|\vec{n} = \frac{2}{\gamma+1}\left(\frac{1}{M_n^2\omega_1} + \frac{\gamma-1}{2}\frac{\omega_1}{\omega_2}\right)\left[(f-\eta f')i + f'j - k\right] \qquad \text{(F-9)}$$

Combining equations (F-5) and (F-9)

$$\vec{U} = \vec{U}_1 + \vec{U}_{tn} = \left[u_n - (f-\eta f')R\right]i + \left[v_n - f'R\right]j + (\omega_n + R)k$$

where $R = \frac{\omega_1}{\omega_2} - \frac{2}{\gamma+1}\left(\frac{1}{M_n^2\omega_1} + \frac{\gamma-1}{2}\frac{\omega_1}{\omega_2}\right)$

$$= \frac{2}{\gamma+1}\left(\frac{\omega_1}{\omega_2} - \frac{1}{\omega_1 M_n^2}\right)$$

APPENDIX C

$$A_{i,k} \equiv \frac{h}{2}\left[(\xi u)_{i,k} Q^2_{i,k+1} - X_{i,k} u^2_{i,k+1} - 2(\xi u\tilde{s})_{i,k} \tilde{a}^2_{i,k+1} - (\xi u)_{i,k} Q^2_{i,k-1} + X_{i,k} u^2_{i,k-1} + 2(\xi u\tilde{s})_{i,k} \tilde{a}^2_{i,k-1}\right]$$
$$+ (\eta u)_{i,k} Q^2_{i+1,k} - E_{i,k} u^2_{i+1,k} - 2(\eta u\tilde{s})_{i,k} \tilde{a}^2_{i+1,k} - (\eta u)_{i,k} Q^2_{i-1,k} + E_{i,k} u^2_{i-1,k} + 2(\eta u\tilde{s})_{i,k} \tilde{a}^2_{i-1,k} \equiv 0$$

$$\frac{\partial(\Sigma A^2_{i,k})}{\partial u_{m,n}} = \sum 2A_{i,k}\Big(\delta_{i,m}\delta_{k,n}\Big[\frac{h}{2}\tilde{s}_{i,k}(Ql_{i,k+1} - 2\tilde{s}_{i,k}\tilde{a}^2_{i,k+1} - Ql_{i,k-1} + 2\tilde{s}_{i,k}\tilde{a}^2_{i,k-1}) + \eta_{i,k}(Ql_{i+1,k} - 2\tilde{s}_{i,k}\tilde{a}^2_{i+1,k}$$
$$- Ql_{i-1,k} + 2\tilde{s}_{i,k}\tilde{a}^2_{i-1,k})\Big] + \delta_{i,m}\delta_{k+1,n}\frac{2h}{2}u_{i,k+1}\{[1-(\gamma-1)\tilde{s}]_{i,k+1}(\xi u)_{i,k} - X_{i,k} + (\gamma-1)(\xi u\tilde{s})_{i,k}\}$$
$$- \delta_{i,m}\delta_{k-1,n}\frac{2h}{2}u_{i,k-1}\{[1-(\gamma-1)\tilde{s}]_{i,k-1}(\xi u)_{i,k} - X_{i,k} + (\gamma-1)(\xi u\tilde{s})_{i,k}\} + \delta_{i+1,m}\delta_{k,n}2u_{i+1,k}\{[1-(\gamma-1)\tilde{s}]_{i+1,k}$$
$$(\eta u)_{i,k} - E_{i,k} + (\gamma-1)(\eta u\tilde{s})_{i,k}\} - \delta_{i-1,m}\delta_{k,n}2u_{i-1,k}\{[1-(\gamma-1)\tilde{s}]_{i-1,k}(\eta u)_{i,k} - E_{i,k} + (\gamma-1)(\eta u\tilde{s})_{i,k}\}\Big)$$
$$= 2\Big\{\frac{h}{2}(\xi A)_{m,n}[Ql_{m,n+1} - Ql_{m,n-1} + 2\tilde{s}_{m,n}(\tilde{a}^2_{m,n-1} - \tilde{a}^2_{m,n+1})] + (\eta A)_{m,n}[Ql_{m+1,n} - Ql_{m-1,n} + 2\tilde{s}_{m,n}(\tilde{a}^2_{m-1,n} - \tilde{a}^2_{m+1,n})]$$
$$+ 2u_{m,n}\Big[\frac{h}{2}(Wl_{m,n-1} - Wl_{m,n+1}) + Vl_{m+1,n} - Vl_{m-1,n}\Big] + 2(\gamma-1)(u\tilde{s})_{m,n}\Big[\frac{h}{2}(\xi uA)_{m,n+1} - \frac{h}{2}(\xi uA)_{m,n-1} + (\eta uA)_{m+1,n} - (\eta uA)_{m-1,n}\Big]\Big\}$$

where $Ql = Q^2 - u^2$, $Vl = [v + (\gamma-1)\eta u\tilde{s}]A$, $Wl = [w + (\gamma-1)\xi u\tilde{s}]A$

$$\frac{\partial(\Sigma A^2_{i,k})}{\partial v_{m,n}} = \sum 2A_{i,k}\Big(\delta_{i,m}\delta_{k,n}[u^2_{i+1,k} - u^2_{i-1,k}] + \delta_{i,m}\delta_{k+1,n}\frac{2h}{2}v_{i,k+1}\{[1-(\gamma-1)\tilde{s}]_{i,k+1}(\xi u)_{i,k} + (\gamma-1)(\xi u\tilde{s})_{i,k}\} - \delta_{i,m}\delta_{k-1,n}\frac{2h}{2}v_{i,k-1}\{[1-(\gamma-1)\tilde{s}]_{i,k-1}(\xi u)_{i,k}$$
$$+ (\gamma-1)(\xi u\tilde{s})_{i,k}\} + \delta_{i+1,m}\delta_{k,n}2v_{i+1,k}\{[1-(\gamma-1)\tilde{s}_{i+1,k}](\eta u)_{i,k} + (\gamma-1)(\eta u\tilde{s})_{i,k} - \delta_{i-1,m}\delta_{k,n}2v_{i-1,k}\{[1-(\gamma-1)\tilde{s}_{i-1,k}](\eta u)_{i,k} + (\gamma-1)(\eta u\tilde{s})_{i,k}\}\Big)$$
$$= -2\Big\{A_{m,n}(u^2_{m+1,n} - u^2_{m-1,n}) + 2v_{m,n}\Big[\frac{h}{2}(Xl_{m,n-1} - Xl_{m,n+1}) + El_{m,n} - El_{m,n}\Big] + 2(\gamma-1)(v\tilde{s})_{m,n}\Big[\frac{h}{2}(\xi uA)_{m,n+1} - \frac{h}{2}(\xi uA)_{m,n-1} + (\eta uA)_{m,n+1} - (\eta uA)_{m-1,n}\Big]\Big\}$$

where $Xl = [1 + (\gamma-1)\tilde{s}]\xi uA$, $El = [1 + (\gamma-1)\tilde{s}]\eta uA$

$$\frac{\partial(\Sigma A^2_{i,k})}{\partial w_{m,n}} = \sum 2A_{i,k}\Big(\delta_{i,m}\delta_{k,n}\Big[\frac{h}{2}(u^2_{i,k+1} - u^2_{i,k-1})\Big] + \delta_{i,m}\delta_{k+1,n}\frac{2h}{2}w_{i,k+1}\{[1-(\gamma-1)\tilde{s}_{i,k+1}](\xi u)_{i,k} + (\gamma-1)(\xi u\tilde{s})_{i,k}\}$$
$$- \delta_{i,m}\delta_{k-1,n}\frac{2h}{2}w_{i,k-1}\{[1-(\gamma-1)\tilde{s}_{i,k-1}](\xi u)_{i,k} + (\gamma-1)(\xi u\tilde{s})_{i,k}\} + \delta_{i+1,m}\delta_{k,n}2w_{i+1,k}\{[1-(\gamma-1)\tilde{s}_{i+1,k}](\eta u)_{i,k}$$
$$+ (\gamma-1)(\eta u\tilde{s})_{i,k}\} - \delta_{i-1,m}\delta_{k,n}2w_{i-1,k}\{[1-(\gamma-1)\tilde{s}_{i-1,k}](\eta u)_{i,k} + (\gamma-1)(\eta u\tilde{s})_{i,k}\}\Big)$$
$$= 2\Big\{\frac{h}{2}A_{m,n}(u^2_{m,n+1} - u^2_{m,n-1}) + 2w_{m,n}\Big[\frac{h}{2}(Xl_{m,n-1} - Xl_{m,n+1}) + El_{m,n} - El_{m+1,n}\Big] + 2(\gamma-1)(w\tilde{s})_{m,n}\Big[\frac{h}{2}(\xi uA)_{m,n+1}$$
$$- \frac{h}{2}(\xi uA)_{m,n-1} + (\eta uA)_{m+1,n} - (\eta uA)_{m-1,n}\Big]\Big\}$$

$$\frac{\partial(\Sigma A^2_{i,k})}{\partial \tilde{s}_{m,n}} = \sum 2A_{i,k}\Big\{\delta_{i,m}\delta_{k,n}\frac{h}{2}[-2(\xi u)_{i,k}\tilde{a}^2_{i,k+1} + 2(\xi u)_{i,k}\tilde{a}^2_{i,k-1}] + \delta_{i,m}\delta_{k,n}2(\eta u)_{i,k}(\tilde{a}^2_{i-1,k} - \tilde{a}^2_{i+1,k})$$
$$+ \delta_{i,m}\delta_{k+1,n}\frac{2h}{2}(\xi u)_{i,k}\tilde{a}^2_{i,k+1} - \delta_{i,m}\delta_{k-1,n}\frac{2h}{2}(\xi u)_{i,k}\tilde{a}^2_{i,k-1} + \delta_{i+1,m}\delta_{k,n}2(\eta u)_{i,k}\tilde{a}^2_{i+1,k} - \delta_{i-1,m}\delta_{k,n}2(\eta u)_{i,k}\tilde{a}^2_{i-1,k}\Big\}$$
$$= 4\Big\{\frac{h}{2}(\xi uA)_{m,n}(\tilde{a}^2_{m,n-1} - \tilde{a}^2_{m,n+1}) + (\eta uA)_{m,n}(\tilde{a}^2_{m-1,n} - \tilde{a}^2_{m+1,n})\Big] + \tilde{a}^2_{m,n}\Big[\frac{h}{2}(\xi uA)_{m,n+1} - \frac{h}{2}(\xi uA)_{m,n-1}$$
$$+ (\eta uA)_{m+1,n} - (\eta uA)_{m-1,n}\Big]\Big\}$$

$$B_{i,k} \equiv \frac{h}{2}\left[X_{i,k} v^2_{i,k+1} - X_{i,k} v^2_{i,k-1}\right] + v_{i,k} Q^2_{i+1,k} + E_{i,k} v^2_{i+1,k} - 2(v\tilde{s})_{i,k}\tilde{a}^2_{i+1,k}$$
$$- v_{i,k} Q^2_{i-1,k} - E_{i,k} v^2_{i-1,k} + 2(v\tilde{s})_{i,k}\tilde{a}^2_{i-1,k} \equiv 0$$

$$\frac{\partial(\Sigma B_{i,k}^2)}{\partial u_{m,n}} = \Sigma 2B_{i,k}\Big(\delta_{i,m}\delta_{k,n}\big[\tfrac{h}{L}\xi_{i,k}(v_{i,k+1}^2 - v_{i,k-1}^2) + \eta_{i,k}(v_{i+1,k}^2 - v_{i-1,k}^2)\big] + \delta_{i+1,m}\delta_{k,n}2u_{i+1,k}\{v_{i,k}[1-(\gamma-1)\tilde{\xi}_{i+1,k}$$
$$+(\gamma-1)(v\tilde{\xi})_{i,k}\} - \delta_{i-1,m}\delta_{k,n}2u_{i-1,k}\{v_{i,k}[1-(\gamma-1)\tilde{\xi}_{i-1,k}] + (\gamma-1)(v\tilde{\xi})_{i,k}\}\Big)$$
$$= 2\Big\{\tfrac{h}{L}(\xi B)_{m,n}(v_{m,n+1}^2 - v_{m,n-1}^2) + (\eta B)_{m,n}(v_{m+1,n}^2 - v_{m-1,n}^2) + 2u_{m,n}[V2_{m-1,n} - V2_{m+1,n}]$$
$$+2(\gamma-1)(u\tilde{\xi})_{m,n}[(vB)_{m+1,n} - (vB)_{m-1,n}]\Big\}$$

where $V2 = [1+(\gamma-1)\tilde{\xi}]vB$

$$\frac{\partial(\Sigma B_{i,k}^2)}{\partial \xi_{m,n}} = \Sigma 2B_{i,k}\Big(\delta_{i,m}\delta_{k,n}\big[Q2_{i+1,k}^2 - v_{i,k}^2 - 2\tilde{\xi}_{i,k}\tilde{a}_{i,k}^2 - Q2_{i-1,k}^2 + v_{i,k}^2 + 2\tilde{\xi}_{i,k}\tilde{a}_{i-1,k}^2\big] + \delta_{i,m}\delta_{k+1,n}\tfrac{2h}{L}v_{i,k+1}X_{i,k} - \delta_{i,m}\delta_{k-1,n}\tfrac{2h}{L}v_{i,k-1}X_{i,k}$$
$$+\delta_{i+1,m}\delta_{k,n}2v_{i+1,k}\{v_{i,k}[1-(\gamma-1)\tilde{\xi}_{i+1,k}] + E_{i,k} + (\gamma-1)(v\tilde{\xi})_{i,k}\} - \delta_{i-1,m}\delta_{k,n}2v_{i-1,k}\{v_{i,k}[1-(\gamma-1)\tilde{\xi}_{i-1,k}] + E_{i,k} + (\gamma-1)(v\tilde{\xi})_{i,k}\}\Big)$$
$$= 2\Big\{B_{m,n}[Q2_{m+1,n} - Q2_{m-1,n} + 2\tilde{\xi}_{m,n}(\tilde{a}_{m,n}^2 - \tilde{a}_{m-1,n}^2)] + 2v_{m,n}[\tfrac{h}{L}(XB)_{m+1,n} - \tfrac{h}{L}(XB)_{m-1,n} + U1_{m+1,n} - U1_{m-1,n}] + 2(\gamma-1)(v\tilde{\xi})_{m,n}[(vB)_{m+1,n} - (vB)_{m-1,n}]\Big\}$$

where $Q2 = Q^2 - v^2$, $U1 = [\eta u + (\gamma-1)v\tilde{\xi}]B$

$$\frac{\partial(\Sigma B_{i,k}^2)}{\partial w_{m,n}} = \Sigma 2B_{i,k}\Big(\delta_{i,m}\delta_{k,n}\big\{\tfrac{h}{L}(v_{i,k+1}^2 - v_{i,k-1}^2)\big\} + \delta_{i+1,m}\delta_{k,n}2w_{i+1,k}\{v_{i,k}[1-(\gamma-1)\tilde{\xi}_{i+1,k}] + (\gamma-1)(v\tilde{\xi})_{i,k}\} - \delta_{i-1,m}\delta_{k,n}2w_{i-1,k}\{v_{i,k}[1-(\gamma-1)\tilde{\xi}_{i-1,k}] + (\gamma-1)(v\tilde{\xi})_{i,k}\}\Big)$$
$$= 2\Big\{B_{m,n}\tfrac{h}{L}(v_{m,n+1}^2 - v_{m,n-1}^2) + 2w_{m,n}[V2_{m-1,n} - V2_{m+1,n}] + 2(\gamma-1)(w\tilde{\xi})_{m,n}[(vB)_{m+1,n} - (vB)_{m-1,n}]\Big\}$$

$$\frac{\partial(\Sigma B_{i,k}^2)}{\partial \tilde{\xi}_{m,n}} = \Sigma 2B_{i,k}\Big\{\delta_{i,m}\delta_{k,n}\big[2v_{i,k}(\tilde{a}_{i,k}^2 - \tilde{a}_{i-1,k}^2)\big] + \delta_{i+1,m}\delta_{k,n}2v_{i,k}\tilde{a}_{i,k}^2 - \delta_{i-1,m}\delta_{k,n}2v_{i,k}\tilde{a}_{i-1,k}^2\Big\} = 4\Big\{(vB)_{m,n}(\tilde{a}_{m+1,n}^2 - \tilde{a}_{m-1,n}^2) + \tilde{a}_{m,n}^2[(vB)_{m+1,n} - (vB)_{m-1,n}]\Big\}$$

$$C_{i,k} \equiv \tfrac{h}{L}[v_{i,k}Q_{i,k+1}^2 + X_{i,k}w_{i,k+1}^2 - 2(w\tilde{\xi})_{i,k}\tilde{a}_{i,k+1}^2 - w_{i,k}Q_{i,k-1}^2 - X_{i,k}w_{i,k-1}^2 + 2(w\tilde{\xi})_{i,k}\tilde{a}_{i,k-1}^2] + E_{i,k}v_{i,k+1}^2 - E_{i,k}w_{i,k-1}^2 \equiv 0$$

$$\frac{\partial(\Sigma C_{i,k}^2)}{\partial u_{m,n}} = \Sigma 2C_{i,k}\Big(\delta_{i,m}\delta_{k,n}\big[\tfrac{h}{L}\xi_{i,k}(w_{i,k+1}^2 - w_{i,k-1}^2) + \eta_{i,k}(w_{i+1,k}^2 - w_{i-1,k}^2)\big] + \delta_{i,m}\delta_{k+1,n}\tfrac{2h}{L}u_{i,k+1}\{v_{i,k}[1-(\gamma-1)\tilde{\xi}_{i,k+1}] + (\gamma-1)(w\tilde{\xi})_{i,k}\} - \delta_{i,m}\delta_{k-1,n}\tfrac{2h}{L}\{v_{i,k}[1-(\gamma-1)\tilde{\xi}_{i,k-1}] + (\gamma-1)(w\tilde{\xi})_{i,k}\}\Big)$$
$$= 2\Big\{\tfrac{h}{L}(\xi C)_{m,n}(w_{m,n+1}^2 - w_{m,n-1}^2) + (\eta C)_{m,n}(w_{m+1,n}^2 - w_{m-1,n}^2) + \tfrac{2h}{L}u_{m,n}[W2_{m,n+1} - W2_{m,n-1}] + \tfrac{2h}{L}(\gamma-1)(u\tilde{\xi})_{m,n}[(wC)_{m,n+1} - (wC)_{m,n-1}]\Big\}$$

where $W2 = [1+(\gamma-1)\tilde{\xi}]wC$

$$\frac{\partial(\Sigma C_{i,k}^2)}{\partial v_{m,n}} = \Sigma 2C_{i,k}\Big(\delta_{i,m}\delta_{k,n}\big[w_{i,k+1}^2 - w_{i,k-1}^2\big] + \delta_{i,m}\delta_{k+1,n}\tfrac{2h}{L}v_{i,k+1}\{w_{i,k}[1-(\gamma-1)\tilde{\xi}_{i,k+1}] + (\gamma-1)(w\tilde{\xi})_{i,k}\} - \delta_{i,m}\delta_{k-1,n}\tfrac{2h}{L}v_{i,k-1}\{w_{i,k}[1-(\gamma-1)\tilde{\xi}_{i,k-1}] + (\gamma-1)(w\tilde{\xi})_{i,k}\}\Big)$$
$$= 2\Big\{C_{m,n}(w_{m,n+1}^2 - w_{m,n-1}^2) + \tfrac{2h}{L}(v_{m,n})(W2_{m,n-1} - W2_{m,n+1}) + \tfrac{2h}{L}(\gamma-1)(v\tilde{\xi})_{m,n}[(wC)_{m,n+1} - (wC)_{m,n-1}]\Big\}$$

$$\frac{\partial(\Sigma C_{i,k}^2)}{\partial w_{i,k}} = \Sigma 2C_{i,k}\Big(\delta_{i,m}\delta_{k,n}\tfrac{h}{L}\big[Q_{i,k+1}^2 - v_{i,k+1}^2 - 2\tilde{\xi}_{i,k}\tilde{a}_{i,k+1}^2 - Q_{i,k-1}^2 + v_{i,k-1}^2 + 2\tilde{\xi}_{i,k}\tilde{a}_{i,k-1}^2\big] + \delta_{i,m}\delta_{k+1,n}\tfrac{2h}{L}w_{i,k+1}\{v_{i,k}[1-(\gamma-1)\tilde{\xi}_{i,k+1}] + X_{i,k} + (\gamma-1)(w\tilde{\xi})_{i,k}\}$$
$$-\delta_{i,m}\delta_{k-1,n}\tfrac{2h}{L}v_{i,k-1}\{v_{i,k}[1-(\gamma-1)\tilde{\xi}_{i,k}] + X_{i,k} + (\gamma-1)(w\tilde{\xi})_{i,k}\} + \delta_{i+1,m}\delta_{k,n}2v_{i+1,k}E_{i,k} - \delta_{i-1,m}\delta_{k,n}2w_{i-1,k}E_{i,k}\Big)$$
$$= 2\Big\{C_{m,n}[\tfrac{h}{L}(Q3_{m,n+1} - Q3_{m,n-1}) + 2\tilde{\xi}_{m,n}(\tilde{a}_{m,n+1}^2 - \tilde{a}_{m,n-1}^2)] + 2v_{m,n}[\tfrac{h}{L}(U2_{m,n+1} - U2_{m,n-1}) + (EC)_{m,n} - (EC)_{m,n}] + \tfrac{2h}{L}(\gamma-1)(w\tilde{\xi})_{m,n}[(wC)_{m,n+1} - (wC)_{m,n-1}]\Big\}$$

where $Q3 = Q^2 - w^2$, $U2 = [\xi u + (\gamma-1)w\tilde{\xi}]C$

$$\frac{\partial(\Sigma C_{i,k}^2)}{\partial \tilde{\xi}_{m,n}} = \Sigma 2C_{i,k}\Big\{\delta_{i,m}\delta_{k,n}\tfrac{2h}{L}v_{i,k}(\tilde{a}_{i,k+1}^2 - \tilde{a}_{i,k-1}^2) + \delta_{i,m}\delta_{k+1,n}\tfrac{2h}{L}w_{i,k}\tilde{a}_{i,k+1}^2 - \delta_{i,m}\delta_{k-1,n}\tfrac{2h}{L}w_{i,k}\tilde{a}_{i,k-1}^2$$
$$= 4\tfrac{h}{L}\Big\{(wC)_{m,n}(\tilde{a}_{m,n-1}^2 - \tilde{a}_{m,n+1}^2) + \tilde{a}_{m,n}^2[(wC)_{m,n-1} - (wC)_{m,n+1}]\Big\}$$

$$D_{i,k} \equiv \tfrac{h}{L}[2\tilde{a}_{i,k}^2 X_{i,k+1} - X_{i,k}U_{i,k+1}^2 - 2\tilde{a}_{i,k}^2 X_{i,k-1} + X_{i,k}U_{i,k-1}^2] + 2\tilde{a}_{i,k}^2 E_{i+1,k} - E_{i,k}U_{i+1,k}^2 - 2\tilde{a}_{i,k}^2 E_{i-1,k}$$
$$+E_{i,k}U_{i-1,k}^2 - 4u_{i,k}\tilde{a}_{i,k}^2 \equiv 0$$

$$\frac{\partial(\sum D_{i,\kappa}^2)}{\partial u_{m,n}} = \sum 2D_{i,\kappa}\{\delta_{i,m}\delta_{\kappa,n}[\tfrac{2h}{L}(\gamma-1)u_{i,\kappa}(X_{i,\kappa-1}-X_{i,\kappa+1}) + \tfrac{2h}{L}\xi_{i,\kappa}(U_{i,\kappa-1}^2 - U_{i,\kappa+1}^2) + 2(\gamma-1)u_{i,\kappa}(E_{i-1,\kappa}-E_{i+1,\kappa})$$
$$+ \eta_{i,\kappa}(U_{i-1,\kappa}^2 - U_{i+1,\kappa}^2) + 4(\gamma-1)u_{i,\kappa}^2 - 4\tilde{a}_{i,\kappa}^2] + \delta_{i,m}\delta_{\kappa+1,n}[\tfrac{2h}{L}(\tilde{a}_{i,\kappa}^2\xi_{i,\kappa+1} - X_{i,\kappa}u_{i,\kappa+1})]$$
$$- \delta_{i,m}\delta_{\kappa-1,n}[\tfrac{2h}{L}(\tilde{a}_{i,\kappa}^2\xi_{i,\kappa-1} - X_{i,\kappa}u_{i,\kappa-1})] + \delta_{i+1,m}\delta_{\kappa,n}[2(\tilde{a}_{i,\kappa}^2\eta_{i+1,\kappa} - E_{i,\kappa}u_{i+1,\kappa})]$$
$$- \delta_{i-1,m}\delta_{\kappa,n}2(\tilde{a}_{i,\kappa}^2\eta_{i-1,\kappa} - E_{i,\kappa}u_{i-1,\kappa})\}$$
$$= 2\{D_{m,n}[\tfrac{2h}{L}\xi_{m,n}(U_{m,n-1}^2 - U_{m,n+1}^2) + \eta_{m,n}(U_{m-1,n}^2 - U_{m+1,n}^2) - 4\tilde{a}_{m,n}^2] + 2(\gamma-1)(uD)_{m,n}[\tfrac{h}{L}(X_{m,n-1}-X_{m,n+1}) + E_{m-1,n} - E_{m+1,n} + 2u_{m,n}]$$
$$+ 2\tfrac{h}{L}\xi_{m,n}[(\tilde{a}^2D)_{m,n-1} - (\tilde{a}^2D)_{m,n+1}] + 2\eta_{m,n}[(\tilde{a}^2D)_{m-1,n} - (\tilde{a}^2D)_{m+1,n}] + 2u_{m,n}[\tfrac{h}{L}(XD)_{m,n+1} - \tfrac{h}{L}(XD)_{m,n-1} + (ED)_{m,n} - (ED)_{m-1,n}]\}$$

$$\frac{\partial(\sum D_{i,\kappa}^2)}{\partial v_{m,n}} = \sum 2D_{i,\kappa}\{\delta_{i,m}\delta_{\kappa,n}[\tfrac{2h}{L}(\gamma-1)v_{i,\kappa}(X_{i,\kappa-1}-X_{i,\kappa+1}) + 2(\gamma-1)v_{i,\kappa}(E_{i-1,\kappa}-E_{i+1,\kappa}+2u_{i,\kappa}) + (U_{i+1,\kappa}^2 - U_{i-1,\kappa}^2)]$$
$$+ \delta_{i,m}\delta_{\kappa+1,n}\tfrac{h}{L}(-2X_{i,\kappa}v_{i,\kappa+1}) + \delta_{i,m}\delta_{\kappa-1,n}\tfrac{h}{L}(2X_{i,\kappa}v_{i,\kappa-1}) + \delta_{i+1,m}\delta_{\kappa,n}(-2\tilde{a}_{i,\kappa}^2 - 2E_{i,\kappa}v_{i+1,\kappa}) + \delta_{i-1,m}\delta_{\kappa,n}(2\tilde{a}_{i,\kappa}^2 + 2E_{i,\kappa}v_{i-1,\kappa})\}$$
$$= 2\{2(\gamma-1)(vD)_{m,n}[\tfrac{h}{L}(X_{m,n-1}-X_{m,n+1}) + E_{m-1,n} - E_{m+1,n} + 2u_{m,n}] + D_{m,n}(U_{m+1,n}^2 - U_{m-1,n}^2)$$
$$+ 2v_{m,n}[\tfrac{h}{L}(XD)_{m,n+1} - \tfrac{h}{L}(XD)_{m,n-1} + (ED)_{m,n} - (ED)_{m-1,n}] + 2[(\tilde{a}^2D)_{m,n+1} - (\tilde{a}^2D)_{m,n-1}]\}$$

$$\frac{\partial(\sum D_{i,\kappa}^2)}{\partial w_{m,n}} = \sum 2D_{i,\kappa}\{\delta_{i,m}\delta_{\kappa,n}[\tfrac{2h}{L}(\gamma-1)w_{i,\kappa}(X_{i,\kappa-1}-X_{i,\kappa+1}) + U_{i,\kappa+1}^2 - U_{i,\kappa-1}^2 + 2(\gamma-1)w_{i,\kappa}(E_{i-1,\kappa}-E_{i+1,\kappa}) + 4u_{i,\kappa}(\gamma-1)w_{i,\kappa}]$$
$$+ \delta_{i,m}\delta_{\kappa+1,n}[-\tfrac{2h}{L}\tilde{a}_{i,\kappa}^2 - \tfrac{2h}{L}X_{i,\kappa}w_{i,\kappa+1}] + \delta_{i,m}\delta_{\kappa-1,n}[\tfrac{2h}{L}\tilde{a}_{i,\kappa}^2 + \tfrac{2h}{L}X_{i,\kappa}w_{i,\kappa-1}] - \delta_{i+1,m}\delta_{\kappa,n}[2E_{i,\kappa}w_{i+1,\kappa}] + \delta_{i-1,m}\delta_{\kappa,n}[2E_{i,\kappa}w_{i-1,\kappa}]\}$$
$$= 2\{2(\gamma-1)(wD)_{m,n}[\tfrac{h}{L}(X_{m,n-1}-X_{m,n+1}) + E_{m-1,n} - E_{m+1,n} + 2u_{m,n}] + \tfrac{h}{L}D_{m,n}(U_{m,n+1}^2 - U_{m,n-1}^2)$$
$$+ 2w_{m,n}[\tfrac{h}{L}(XD)_{m,n+1} - \tfrac{h}{L}(XD)_{m,n-1} + (DE)_{m,n} - (DE)_{m-1,n}] + \tfrac{2h}{L}[(\tilde{a}^2D)_{m,n+1} - (\tilde{a}^2D)_{m,n-1}]\}$$

$$\frac{\partial(\sum D_{i,\kappa}^2)}{\partial \xi_{m,n}} = 0$$

$$\frac{\partial \Phi}{\partial u_{m,n}} = \frac{\partial}{\partial u_{m,n}}\left[\sum (A_{i,\kappa}^2 + B_{i,\kappa}^2 + C_{i,\kappa}^2 + D_{i,\kappa}^2)\right]$$

$$\frac{\partial \Phi}{\partial v_{m,n}} = \frac{\partial}{\partial v_{m,n}}\left[\sum (A_{i,\kappa}^2 + B_{i,\kappa}^2 + C_{i,\kappa}^2 + D_{i,\kappa}^2)\right]$$

$$\frac{\partial \Phi}{\partial w_{m,n}} = \frac{\partial}{\partial w_{m,n}}\left[\sum (A_{i,\kappa}^2 + B_{i,\kappa}^2 + C_{i,\kappa}^2 + D_{i,\kappa}^2)\right]$$

$$\frac{\partial \Phi}{\partial \xi_{m,n}} = \frac{\partial}{\partial \xi_{m,n}}\left[\sum (A_{i,\kappa}^2 + B_{i,\kappa}^2 + C_{i,\kappa}^2 + D_{i,\kappa}^2)\right]$$