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Smoothness of Solutions of Volterra Integral Equations
with Weakly Singular Kernels

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## 1. Introduction

The purpose of this paper is to obtain some results on the differentiability properties of solutions of nonlinear integral equations of the form

$$x(t) = f(t) + \int_0^t a(t-s)g(s,x(s))ds \quad (0 \leq t \leq T) \tag{1}$$

when $f(t)$ and $g(t,x)$ are smooth functions, $a(t) \in C(0,T] \cap L^1(0,T)$ but $a(t)$ may become unbounded as $t \to 0$. Such results are necessary in order to estimate the error in numerical approximations of the solution of (1), c.f. e.g. Linz [1, Section II]. This type of result is also useful in proving the equivalence of certain nonlinear boundary value problems for the heat equation with a corresponding Volterra system, c.f. [2, Theorem 3 and its proof].

The general problem of determining the smoothness of solutions of (1) is rather complex. Suppose we fix a function $g$ and a kernel $a(t) \in L^1(0,T)$. Then as $f$ varies over the set $C[0,T]$ the solution $x(t) = x(t;f)$ will also vary over all possible functions in $C[0,T]$. To see this one has only to fix any $x^*(t) \in C[0,T]$ and then define

$$f^*(t) = x^*(t) - \int_0^t a(t-s)g(s,x^*(s))ds$$

on $0 \leq t \leq T$. Then $x^*$ is the solution of (1) corresponding to $f^* \in C[0,T]$.

This overabundance of solutions is caused by the overabundance of forcing functions -f can be any continuous function. One would expect that as f and g become smoother the solution of (1) must become smoother. This is true to some extent but intuition should not be trusted too far. To see this consider the equation

$$x(t) = f(t) - \int_0^t (t-s)^{-1/2} x(s)ds. \tag{2}$$

If $f(t) \equiv 1$ (an entire function), then a Laplace transform argument may be used to see that

$$x(t) = \exp(\pi t)\,\mathrm{erfc}(\sqrt{\pi t})$$

where

$$\mathrm{erfc}(x) = (2/\sqrt{\pi})\int_x^\infty \exp(-r^2)dr$$

is the complimentary error function. On the other hand if $f(t) = 1+2\sqrt{t}$, then it is easy to verify that $x(t) \equiv 1$ (entire). In particular this shows that fortuitous choices of the function f yield smoothness properties at $t = 0$ which are "inversely proportional" to intuition.

Since (2) is a linear equation or convolution type, then it is possible to analyze the behavior of solutions in some detail. Given any fixed continuous function f let $x_0(t,f)$ be the corresponding solution of (2). Integrating in (2) from zero to t and rearranging the double integral yields

$$x_1(t,f) = \int_0^t x_0(\tau,f)d\tau$$

$$= \int_0^t f(\tau)d\tau - \int_0^t (t-s)^{-1/2}\{\int_0^s x_0(\tau,f)d\tau\}ds$$

or

$$x_1(t,f) = \int_0^t f(\tau_1)d\tau_1 - \int_0^t (t-s)^{-1/2} x_1(s,f)ds.$$

This integration process can be continued indefinitely

$$x_{n+1}(t,f) = \int_0^t x_n(\tau,f)d\tau$$

$$= \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_n} f(\tau_{n+1})d\tau_{n+1} \cdots d\tau_1 -$$

$$\int_0^t (t-s)^{-1/2} x_{n+1}(s,f)ds$$

or

$$x_{n+1}(t,f) = \int_0^t \{f(\tau)(t-\tau)^n/n!\}d\tau - \int_0^t (t-s)^{-1/2} x_{n+1}(s,f)ds.$$

Given a function $f \in C^{N+1}[0,T]$ write $f$ in the form

$$f(t) = \Sigma_{j=0}^{N} f^{(j)}(0)t^j/j! + \int_0^t \{f^{(N+1)}(\tau)(t-\tau)^N/N!\}d\tau.$$

Let $u_j(t) = x_j(t,f)$ for the special choice $f(t) \equiv 1$. Then the solution of (2) may be written in the form

$$x_0(t,f) = \sum_{j=0}^{N} f^{(j)}(0)u_j(t)/j! + x_{N+1}(t,f^{(N+1)}).$$

The functions $u_j$ can be computed explicitly. They are of class $C^j[0,T]$, indeed analytic in the complex plane cut by the negative real axis. The function $x_{N+1}(t,f^{(N+1)})$ is at least of class $C^{N+1}[0,T]$.

The foregoing analysis of (2) shows that as $f$ becomes smoother, then $x(t,f)$ also becomes smoother but only for $t > 0$. In general there will be no increase in smoothness of the solution at $t = 0$. At the same time very special choices of $f$ (for example $f(0) = 0$ or $f(0) = f'(0) = 0$) may force smoother behavior at the origin. This general type of behavior seems to be typical of solutions of (1). The analysis given below is an attempt to prove this under reasonably general assumptions on the kernel $a(t)$.

In equation (2) let $u_0(t)$ be the solution when $f(t) \equiv 1$. Then one can easily and explicitly compute $u_0'(t) = t^{-1/2} + \pi u_0(t)$. In the more general case where $f \in C^{N+1}$ $(N \geq 1)$ then

$$\begin{aligned} x_0'(t,f) &= \sum_{j=0}^{N} f^{(j)}(0)u_j'(t)/j! + x_{N+1}'(t,f^{(N+1)}) \\ &= f(0)\{t^{-1/2} + \pi u_0(t)\} + \text{continuous terms.} \end{aligned}$$

More generally one could rewrite (1) in the form

$$(1') \qquad x(t) = f(t) + \int_0^t a(s)g(t-s,x(t-s))ds$$

and then formally differentiate to obtain

$$(3) \qquad x'(t) = f'(t) + a(t)g(0,x(0)) + \int_0^t a(s)\{g_1(t-s,x(t-s)) + g_2(t-s,x(t-s))x'(t-s)\}ds$$

where $g_1 = \partial g/\partial t$ and $g_2 = \partial g/\partial x$. One might expect that if $g(0,x(0)) \neq 0$, then $x'(t) = \mathcal{O}(a(t))$ as $t \to \infty$. We shall show that this is the case for a certain class of kernels $a(t)$. In general the nature of the singularity at $t = 0$ is hard to analyze since the integral

$$\int_0^t a(s)g_2(t-s,x(t-s))x'(t-s)ds$$

may also be singular at $t = 0$. We shall provide a detailed analysis for the case $a(t) = t^{-p}$, $0 < p < 1$. The existence and nature of possible singularities at $t = 0$ for arbitrary kernels $a(t)$ is open.

## 2. Preliminary Lemmas.

Consider a linear Volterra integral equation of the form

$$(4) \qquad X(t) = F(t) + \int_0^t a(t-s)h(s)X(s)ds.$$

Lemma 1. *For some* $T > 0$ *assume* $F$ *and* $a$ *are of class* $L^1(0,T)$ *and* $h \in L^\infty(0,T)$. *Then* (3) *has a unique solution* $X \in L^1(0,T)$. *If in addition* $h, F$ *and* $a$ *are scalars and are* a.e. *nonnegative, then* $X(t) \geq 0$ a.e.

*Proof:* The usual contraction map and translation argument works. Let $h_0 = \text{ess. sup}|h(t)|$ on $0 \leq t \leq T$. Pick an integer $J$ such that if $S = T/J$ then

$$h_0 \int_0^S |a(t)|\,dt = \alpha < 1.$$

Then existence of $X(t)$ on $0 \leq t \leq S$ follows immediately by the principle of contraction mappings on $L^1(0,S)$.

Replacing $t$ by $t+S$ in (4) one obtains

$$X_1(t) = F_1(t) + \int_0^t a(t-s)h(s+S)X_1(s)ds$$

where $X_1(t) = X(t+S)$ and

$$F_1(t) = F(t+S) + \int_0^S a(t+S-s)h(s)X(s)ds.$$

Since $X \in L^1(0,S)$ is known and $F_1 \in L^1(0,S)$ is known, then the contraction mapping argument yields $X_1(t) \in L^1(0,S)$. Define $X(t+S) = X_1(t)$ a.e. on $0 < t < S$. Continue by induction on the intervals $jS < t < (j+1)S$.

If $h$, $F$ and $a$ are nonnegative, then the argument is the same except that the contraction mappings are defined on the set $\{\varphi \in L^1(0,S)\colon \varphi(t) \geq 0 \text{ a.e.}\}$. Q.E.D.

Lemma 2. _Suppose_ $F$ _and_ $a \in L^1(0,T)$, $h \in L^\infty(0,T)$ _and_ $h_0 \geq \text{ess.sup}|h(t)|$. _Suppose there exists_ $r > 0$ _and a function_ $\beta \in L^1(0,r)$ _such that_

$$|F(t)| + h_0\int_0^t |a(t-s)|\beta(s)ds \leq \beta(t) \quad (0 \leq t \leq r).$$

_Then there exists_ $r_0 \leq \min\{r,T\}$ _such that the unique_ $L^1$ _solution of_ (4) _satisfies the estimate_ $|X(t)| \leq \beta(t)$ a.e. _on_ $0 < t < r_0$.

_Proof:_ Let $S$ be the number given in the proof of Lemma 1 and let $r_0 = \min\{r,S\}$. Define

$$A = \{\varphi \in L^1(0,r_0)\colon |\varphi(t)| \leq \beta(t) \text{ a.e.}\}.$$

Apply a contraction mapping on $A$. Q.E.D.

Corollary 1. _Suppose the hypotheses of Lemma_ 1 _are satisfied._ _Assume_ $|F(t)|$ _and_ $|a(t)| \leq K\beta(t)$ _a.e. on_ $0 \leq t \leq r$ _and that_

(H1) _For each_ $\varepsilon > 0$ _there exists_ $\delta(\varepsilon) > 0$ _such that_

$$\int_0^t \beta(t-s)\beta(s)ds \leq \varepsilon\beta(t) \quad (\text{a.e. on } 0 < t < \delta).$$

*Then there exists* $r_0 \leq \min\{T,r\}$ *such that the unique* $L^1$ *solution of* (4) *satisfies the estimate* $|X(t)| \leq (K+1)\beta(t)$ *a.e. on* $0 < t < r_0$.

*Proof:* Pick $\varepsilon > 0$ so small that $\varepsilon h_0 K(K+1) < 1$ where $h_0 \geq$ ess. $\sup|h(t)|$. Then pick $\delta = \delta(\varepsilon)$ using (H1). Let $r_0 = \min\{\delta,r,T\}$. For almost all $t$ in $0 < t < r_0$ one has

$$
\begin{aligned}
|F(t)| &+ h_0\int_0^t|a(t-s)|(K+1)\beta(s)ds \\
&\leq K\beta(t) + h_0K(K+1)\int_0^t\beta(t-s)\beta(s)ds \\
&\leq (K+1)\beta(t).
\end{aligned}
$$

Now apply Lemma 2. Q.E.D.

It is easy to find examples of functions which satisfy hypotheses (H1). For example if $0 < p < 1$ then

$$\int_0^t(t-s)^{-p}s^{-p}ds = Kt^{1-2p} = (Kt^{1-p})t^{-p} = o(t^{-p})$$

where $K = \Gamma(1-p)^2/\Gamma(2-p)$ and $\Gamma(z)$ is the gamma function. If $\beta(t) = -\log t$ and $0 < t \leq 1$, then

$$
\begin{aligned}
0 \leq \int_0^t\log(t-s)\log s\,ds &= \int_0^{t/2}\log(t-s)\log s\,ds + \int_{t/2}^t\log(t-s)\log s\,ds \\
&\leq \log(t/2)\int_0^{t/2}\log s\,ds + \log(t/2)\int_{t/2}^t\log(t-s)ds = 2\log(t/2)\int_0^{t/2}\log s\,ds
\end{aligned}
$$

$$= t \log(t/2)[\log(t/2)-1] = o(\log t) \quad (t \to 0^+).$$

If $\beta(t) = \sum_{n=1}^{\infty} e^{-n^2 t}$, then

$$\int_0^t \beta(t-s)\beta(s)ds = \sum_{\substack{n,m=1 \\ n\neq m}}^{\infty} \frac{e^{-n^2t} - e^{-m^2t}}{m^2-n^2} + t\sum_{n=1}^{\infty} e^{-n^2t}$$
$$= 2\gamma(t) + t\beta(t)$$

where $\gamma(t)$ is defined by

$$\gamma(t) = \sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty} \frac{e^{-n^2t}-e^{-m^2t}}{m^2-n^2} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{e^{-n^2t}-e^{-(m+n)^2t}}{m^2+2mn}$$
$$= \sum_{n=1}^{\infty} e^{-n^2t} \sum_{m=1}^{\infty} \frac{1-e^{-(m^2+2nm)t}}{m^2+2nm}.$$

Clearly $t\beta(t) = o(\beta(t))$ as $t \to 0$. To see that $\gamma(t) = o(\beta(t))$ fix any $\varepsilon > 0$ and then pick $M$ so large that

$$\sum_{m+1}^{\infty} 2m^{-2} < \varepsilon/2, \ \sum_{m=1}^{\infty}(m^2+2nm)^{-1} < \varepsilon \quad (n \geq m).$$

Now pick $\delta > 0$ so small that if $0 < t < \delta$, then

$$\sum_{m=1}^{M}(1 - e^{-(m^2+2mn)t}) < \varepsilon/2 \quad (1 \leq m \leq M).$$

Then one has

$$0 \leq \gamma(t) \leq \sum_{n=1}^{M} e^{-n^2 t}\left(\sum_{m=1}^{M}(1-e^{-(m^2+2mn)t}) + \sum_{M+1}^{\infty} m^{-2}\right)$$

$$+ \sum_{n=M+1}^{\infty} e^{-n^2 t} \sum_{m=1}^{\infty}(m^2+2mn)^{-1}$$

$$\leq \sum_{n=1}^{M} e^{-n^2 t}(\varepsilon/2 + \varepsilon/2) + \sum_{n=M+1}^{\infty} e^{-n^2 t}\varepsilon = \varepsilon\beta(t).$$

Using this result it is easy to show that the function $\beta(t) = K\sum_{n=0}^{\infty} e^{-n^2 t}$ also satisfies (H1).

The resolvent $R(t)$ associated with a given kernel function $a(t)$ is defined as the unique $L^1$ solution of the linear equation

$$R(t) = a(t) + \int_0^t a(t-s)R(s)ds. \tag{5}$$

It is well known (c.f. e.g. Tricomi [3, Chapter I]) that the solution of a linear equation

$$X(t) = f(t) + \int_0^t a(t-s)X(s)ds \tag{6}$$

may be represented in terms of $f$ and the resolvent:

$$X(t) = f(t) + \int_0^t R(t-s)f(s)ds. \tag{7}$$

Consider a pair of nonlinear equations

$$X_j(t) = F_j(t) + \int_0^t a(t-s)g_j(s,X_j(s))ds. \quad (j = 1,2). \tag{8}$$

Lemma 3. Assume

i. $a, F_1$ and $F_2 \in L^1(0,T)$,

ii. $g_1(t,x)$ and $g_2(t,x)$ are continuous in $(t,x)$ for $0 \leq t \leq T$ and all $x$,

iii. $g_1(t,x)$ is Lipschitz continuous in $x$ with Lipschitz constant $L$ (independent of $t$ and $x$), and

iv. $X_1$ and $X_2$ exist a.e. on $0 \leq t \leq T$ and are $L^1$.

Let $r(t)$ be the resolvent of the kernel $L|a(t)|$ and define

$$Q(t) = F_1(t)-F_2(t) + \int_0^t a(t-s)\{g_1(s,X_2(s))-g_2(s,X_2(s))\}ds.$$

Then a.e. on $0 < t < T$ one has

$$|X_1(t)-X_2(t)| \leq |Q(t)| + \int_0^t r(t-s)|Q(s)|ds.$$

Proof: Define $z(t) = X_1(t)-X_2(t)$, $F(t) = F_1(t)-F_2(t)$ and

$$G(t) = \{g_1(t,X_1(t)) - g_1(t,X_2(t))\}/z(t)$$

when $z(t) \neq 0$ and $G(t) = 0$ when $z(t) = 0$. Clearly $z$ and $F \in L^1(0,T)$, $G \in L^\infty(0,T)$ and $|G(t)| \leq L$ a.e. Using (8) and the definitions above it follows that

$$z(t) = F(t) + \int_0^t a(t-s)\{g_1(s,X_2(s))-g_2(s,X_2(s))\}ds$$

$$+ \int_0^t a(t-s)\{g_1(s,X_1(s)) - g_1(s,X_2(s))\}ds$$

$$= Q(t) + \int_0^t a(t-s)G(s)z(s)ds,$$

so that

$$|z(t)| \leq |Q(t)| + L\int_0^t |a(t-s)||z(s)|ds.$$

Let $p(t)$ be a nonnegative function such that

$$|z(t)| = \{Q(t)-p(t)\} + L\int_0^t |a(t-s)||z(s)|ds.$$

Since $r(t)$ is the resolvent of $L|a(t)|$, then formula (7) implies that

$$|z(t)| = Q(t) - p(t) + \int_0^t r(t-s)\{Q(s)-p(s)\}ds.$$

Since $r$ and $p$ are nonnegative the lemma follows. Q.E.D.

If $a(t) \equiv 1$ and both $F_1(t) \equiv F_1$ and $F_2(t) \equiv F_2$ are constants, then $r(t) = L\exp(Lt)$. In this case Lemma 3 reduces to a familiar estimate for ordinary differential equations.

In certain cases the resolvent associated with a kernel $a(t) \in C(0,T] \cap L^1(0,T)$ is not only $L^1(0,T)$ but also continuous for $t > 0$. This is trivial to see if $a \in L^2(0,T)$. Another instance is given by the following result:

Lemma 4. _Suppose_ $a(t) \in C(0,T] \cap L^1(0,T)$. _If_ $a(t)$ _is nonnegative and nonincreasing then its resolvent is continuous on_ $0 < t \leq T$.

_Proof:_ Let $r(t)$ be the resolvent of $a(t)$. By Lemma 1 $r(t) \in L^1(0,T)$ and $r(t) \geq 0$ a.e. Therefore the function $a(\delta-s)r(s) \in L^1(0,\delta)$ for almost all $\delta \in (0,T)$. Fix any such $\delta$. Then

$$(9) \qquad r(t+\delta) = \{a(t+\delta) + \int_0^\delta a(t+\delta-s)r(s)ds\} + \int_0^t a(t-s)r(s+h)ds.$$

The function $a(t+\delta) \in C[0,T-\delta]$ and the function

$$E(t) = \int_0^\delta a(t+\delta-s)r(s)ds$$

is easily seen to be continuous on $0 < t \leq T$. To see that $E(t)$ is continuous at $t = 0$ we must show that for any sequence $t_n$ tending monotonically to zero one has

$$\int_0^\delta a(t_n+\delta-s)r(s)ds \to \int_0^\delta a(\delta-s)r(s)ds.$$

But $a(t)$ is nonincreasing so that $a(t_n+\delta-s)r(s) \to a(\delta-s)r(s)$ monotonically. Now apply the dominated convergence theorem.

We have shown that (9) has the form

$$x(t) = f(t) + \int_0^t a(t-s)x(s)ds, \quad x(t) = r(t+\delta)$$

where $f \in C[0,T-\delta]$ and $a(t) \in L^1(0,T-\delta)$. Using an argument similar to the proof of Lemma 1 it follows that $x(t) = r(t+\delta) \in C[0,T-\delta]$. Since $\delta > 0$ can be made arbitrarily small, we are done. Q.E.D.

A similiar proof will establish the following result.

Lemma 5. Suppose $F$, $a$ and $\beta \in C(0,T] \cap L^1(0,T)$, $h \in L^\infty(0,T)$ and $|a(t)| \leq \beta(t)$ on $0 < t \leq T$. If $\beta$ is nonincreasing then the solution $X$ of (4) is continuous on $0 < t \leq T$.

3. Differentiability of Solutions

Consider the integral equation

$$(1') \qquad x(t) = f(t) + \int_0^t a(s)g(t-s,x(t-s))ds$$

and its formal derivative

$$(3) \qquad x'(t) = f'(t) + a(t)g(0,f(0)) + \int_0^t a(t-s)\{g_1(s,x(s)) + g_2(s,x(s))x'(s)\}ds.$$

This last equation may be written in the form

$$X(t) = F(t) + \int_0^t a(t-s)g_2(s,x(s))X(s)ds \tag{10}$$

where $X(t) = x'(t)$ and

$$F(t) = f'(t) + a(t)g(0,x(0)) + \int_0^t a(t-s)g_1(s,x(s))ds. \tag{11}$$

In the sequel we shall need some or all of the following hypotheses.

(A1) $f(t)$ and $g(t,x)$ are of class $C^1$ in $t$ and resp. $(t,x)$ for $0 \leq t \leq T$ and for all $x$.

(A2) The function $g_2(t,x) = \partial g(t,x)/\partial x$ is locally Lipschitz continuous in $x$.

(A3) $a(t) \in L^1(0,T) \cap C(0,T]$ and there exists a nonincreasing function $\alpha(t) \in L^1(0,T) \cap C(0,T]$ such that $|a(t)| \leq \alpha(t)$ on $0 < t \leq T$.

(A4) The unique continuous solution of (1) exists on the entire interval $0 \leq t \leq T$.

(A5) $f(t)$ and $g(t,x)$ are of class $C^{\nu+1}$ for some integer $\nu \geq 1$.

(A6) $a(t) \in C^{\nu-1}[0,T] \cap C^{\nu}(0,T]$ and $|a^{(\nu)}(t)| \leq \alpha(t)$ where $\alpha$ is nonincreasing and integrable on $0 < t < T$.

Theorem 1. *Suppose* (A1-4) *are true. Let* $X(t)$ *be the solution of* (10) *with* $F$ *defined by* (11). *Then the solution* $x(t)$ *of* (1) *is of class* $C[0,T] \cap C^1(0,T]$ *and* $x'(t) = X(t)$ *on the interval* $0 < t \leq T$.

Proof: Note that by Lemmas 1 and 5 it follows that $X \in C(0,T] \cap L^1(0,T)$. Let $M = \max|x(t)|$ on $0 \leq t \leq T$ and let $P(x)$ be a $C^\infty$ function such that $P(x) = 1$ if $|x| \leq M+1$ and $P(x) \equiv 0$ if $|x| \geq M+2$. If the function $g(t,x)$ in (1) is replaced by $g(t,x)P(x)$ then nothing is changed in the range of interest. Therefore we shall assume that $g$ has compact support. In particular then $g$, $g_1$ and $g_2$ are bounded and $g_2(t,x)$ is globally Lipschitz continuous in $x$.

Fix a number $\delta$ in the range $0 < \delta < T/2$. Define

$$Z(t,h) = \{x(t+h)-x(t)\}/h$$

for $0 < h \leq \delta$ and $0 < t \leq T-\delta$. Since $x(t)$ solves (1), then $Z$ solves an equation of the form

$$Z(t,h) = R(t,h) + \int_0^t a(t-s)g_2(s,x^*(s))Z(s,h)ds$$

where $x^*(s)$ is between $x(t)$ and $x(t+h)$, $0 < \theta(h) < h$ and

$$R(t,h) = (f(t+h)-f(t))/h + h^{-1}\int_t^{t+h} a(s)g(t+h-s,x(t+h-s))ds$$

$$+ \int_0^t a(s)g_1(t+\theta(h)-s,x(t-s))ds.$$

Let $r(t)$ be the resolvent of $L|a(t)|$. By Lemma 3 above

$$|Z(t,h)-X(t)| \leq Q(t,h) + \int_0^t r(t-s)Q(s,h)ds$$

on $0 < t \leq T-\delta$ where

$$Q(t,h) = |R(t,h)-F(t)| + \int_0^t |a(t-s)||g_2(s,x^*(s)) - g_2(s,x(s))|ds.$$

Let $K > 0$ be a bound for all of the functions $|f'(t)|$, $|g(t,x)|$, $|g_1(t,x)|$ and $|g_2(t,x)|$. Then the definitions of $Q$, $R$ and $F$ may be used to obtain the bound

$$\begin{aligned} |Q(s,h)| &\leq K + (K/h)\int_s^{s+h}|a(u)|du + 3K\int_0^T|a(u)|du + |F(s)| \\ &\leq 2K + (K/h)\int_s^{s+h}\alpha(u)du + 4K\int_0^T|a(u)|du + K\alpha(s) \\ &\leq 2K\{1 + 2\int_0^T|a(u)|du + \alpha(s)\} \qquad (0 < s < t). \end{aligned}$$

Write this bound in the form $|Q(t,h)| \leq K_0 + K_1\alpha(t)$.

Given $\varepsilon > 0$ let $K_2$ be a bound for $r(t)$ over $\delta \leq t \leq T-\delta$. Pick $\eta$ in the range $0 < \eta \leq \delta$ and so small that

$$\int_0^\eta K_2\{K_0 + K_1\alpha(t)\}dt < \varepsilon.$$

Now pick $h_0$ so small that whenever $0 < h \leq h_0$ then

$$|Q(t,h)| \leq \varepsilon\{\int_0^T r(s)ds + 1\}^{-1}$$

uniformly over $\eta \leq t \leq T-\eta$. Then for $h$ and $t$ in the range $0 < h \leq h_0$, $\delta \leq t \leq T-\delta$ one has

$$\begin{aligned}|Z(t,h)-X(t)| &\leq Q(t,h) + \int_0^\eta r(t-s)Q(s,h)ds \\ &\quad + \int_\eta^t r(t-s)Q(s,h)ds \\ &\leq \varepsilon + \int_0^\eta K_2\{K_0 + K_1\alpha(s)\}ds + \int_\eta^t r(t-s)\{\varepsilon/\int_0^T r(u)du\}ds \\ &< 3\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary this shows that $z(t,h) \to X(t)$ as $h \to 0^+$ uniformly in $\delta \leq t \leq T-\delta$. But $\delta > 0$ is also arbitrary so that $X(t)$ is the continuous right derivative of $x(t)$ on the interval $0 < t < T$.

By virtue of the uniform convergence to $X(t)$ it follows that on any interval $I = \{t: \delta \leq t \leq T-\delta\}$ the set $\{Z(\cdot,h): 0 < h < \delta\}$ is equicontinuous. Therefore

$$\lim_{h \to 0^+} z(t,h) = \lim_{h \to 0^+} z(t-h,h) = X(t)$$

uniformly on $I$. But $z(t-h,h)$ is a left difference. For $t = T$ a separate but similiar argument shows that $X(T)$ is the left derivative of $x(t)$ at $t = T$. Q.E.D.

Exactly the same proof will establish the following theorem.

Theorem 2. <u>Theorem</u> 1 <u>remains true if the assumptions on</u> $f$ <u>are weakened to</u> $f \in C[0,T] \cap C^1(0,T]$ <u>and</u> $\int_0^\eta |(f(t+h)-f(t))/h - f'(t)| dt \to 0$ <u>as</u> $\eta \to 0$ <u>uniformly in</u> $h$.

In case (A4-6) are true then one can formally differentiate (1') as follows

$$(12) \qquad x^{(n)}(t) = f^{(n)}(t) + \sum_{k=0}^{n-1} a^{(k)}(t)\{D^{n-k-1}g(u,x(u))\}_{u=0} + \int_0^t a(s)\{D^n g(u,x(u))\}_{u=t-s} ds$$

where $D^j = d^j/du^j$ denotes the $j^{th}$ derivative and $n = 0,1,\ldots,\nu+1$. We shall prove:

Theorem 3. <u>Suppose</u> (A4-6) <u>are true with</u> $\nu \geq 1$. <u>Then the solution of</u> (1) <u>satisfies the following</u>:

i. $x \in C^\nu[0,T] \cap C^{\nu+1}(0,T]$,

ii. $x^{(\nu+1)} \in L^1(0,T)$, <u>and</u>

iii. $x(t)$ <u>satisfies</u> (12) <u>for</u> $1 \leq n \leq \nu+1$ <u>and</u> $0 < t \leq T$.

<u>Proof</u>: Since the hypotheses of Theorem 1 are trivally satisfied then $x'(t) \in C(0,T] \cap L^1(0,T)$ and $x(t)$ satisfies (12) on $0 < t \leq T$ for $n = 1$. Since $a(t)$ is continuous at $t = 0$, it is clear that

$$\{x(h)-x(0)\}h^{-1} = \{f(h)-f(0)\}h^{-1} + h^{-1}\int_0^h a(s)g(h-s,x(h-s))ds$$

$$\to f'(0) + a(0)g(0,x(0))$$

as $h \to 0^+$. Therefore $x'(0)$ exists and satisfies (12).

Continuing by induction one can use Theorem 1 to establish (12) for $n = 1,2,\ldots,\nu$. Applying Theorem 2 to (12) with $n = \nu$ one then obtains (12) for $n = \nu+1$. Q.E.D.

4. Weakly Singular Kernels.

Definition 1. Suppose $\nu$ is a nonnegative integer and $F$ is a function defined on $(0,T]$ or on $[0,T]$. Then $F$ is called weakly singular of order $\nu$ if and only if

i. $F \in C(0,T]$ if $\nu = 0$ or $F \in C^{\nu-1}[0,T] \cap C^{\nu}(0,T]$ if $\nu > 0$,

ii. For each $\varepsilon > 0$, $F^{(\nu)}(t)$ is absolutely continuous on $\varepsilon \le t \le T$, and finally

iii. the function defined by

$$\alpha_\nu(t,F) = F(T) + \int_t^T |F^{(\nu+1)}(s)|\,ds \qquad (0 < t \le T)$$

is of class $L^1(0,T)$.

For any integer $\nu \geq 0$ let $WS(\nu)$ denote the set of all functions $F$ which are weakly singular of order $\nu$ ($T > 0$ is fixed). The function $\alpha_\nu(t,F)$ is a measure of the singularity of $F^{(\nu)}$ at $t = 0$. Indeed it is easy to see that $\alpha_\nu$ is nonnegative, nonincreasing and that $|F^{(\nu)}(t)| \leq \alpha_\nu(t,F)$ on $0 < t \leq T$. The precise value of $T$ is unimportant in the sense that if $T$ is replaced by another value $T'$ then $\alpha_\nu$ must be adjusted only by an additive constant.

Theorem 4. Suppose (A4) is true, (A5) is true with $\nu = 1$ and $a(t) \in WS(0)$. Then the solution $x(t)$ of (1) is of class $C^2(0,T]$ and there exists a constant $K^*$ such that

$$(13) \qquad \int_\tau^T |x''(t)|\,dt \leq K^*\Big(1 + \int_\tau^T |a'(t)|\,dt + \int_\tau^T |x'(t)|^2 dt + \int_0^\tau |x'(s)| \Big(\int_{\tau-s}^T |a'(t)|dt\Big)ds$$

on $0 < \tau \leq T$.

Proof: We shall assume that $g(t,x) = g(x)$ is independent of $t$. The only addition complications in the general case are notational. By Theorem 1 above $x \in C[0,T] \cap C^1(0,T]$ and $x'(t) \in L^1(0,T)$. For any $\tau \in (0,T)$ one has

$$(14) \quad x'(t+\tau) = \{f'(t+\tau)+a(t+\tau)g(x(0))+\int_0^\tau a(t+\tau-s)g'(x(s))x'(s)ds\} + \int_0^t a(t-s)g'(x(s+\tau))x'(s+\tau)ds$$

on $0 < t \leq T-\tau$. Note that $f'(t+\tau)$, $a(t+\tau)$ and $g'(x(s+\tau))$ are of class $C^1[0,T-\tau]$. Also note that the function defined by

$$E(t) = \int_0^\tau a(t+\tau-s)g'(x(s))x'(s)ds$$

is of class $C[0,T-\tau] \cap C^1(0,T-\tau]$ and that $E' \in L^1(0,T-\tau)$. Indeed one has

$$(15) \quad \int_0^{T-\tau}|E'(t)|dt = \int_0^{T-\tau}\left|\int_0^\tau a'(t+\tau-s)g'(x(s))x'(s)ds\right|dt$$

$$\leq \int_0^\tau\left(\int_C^{T-\tau}|a'(t+\tau-s)|dt\right)|g'(x(s))x'(s)|ds$$

$$\leq K\int_0^\tau|x'(s)|\left(\int_{\tau-s}^{T-s}|a'(t)|dt\right)ds < \infty$$

where $K$ is an a priori constant which bounds $|g'(x(s))|$. Moreover for all small $h$ one has

$$h^{-1}\int_0^\eta|E(t+h)-E(t)|dt \leq (K/h)\int_0^\eta\int_0^\tau\left|\int_0^h a'(u+t+\tau-s)du\right||x'(s)|dsdt$$

$$= (K/h)\int_0^h\left\{\int_0^\tau\left(\int_0^\eta|a'(u+t+\tau-s)|dt\right)|x'(s)|ds\right\}du$$

$$= (K/h)\int_0^h\left\{\int_0^\tau(\alpha_0(u+\eta+\tau-s)-\alpha_0(u+\tau-s))|x'(s)|ds\right\}du$$

where $K$ is the bound on $|g'(x(s))|$ and $\alpha_0(t) = \alpha_0(t,a)$. The expression inside the brackets in the last integral will tend to zero as $\eta \to 0$ uniformly for $0 \leq u \leq 1$. Indeed if this were not true, then there would exist sequences $\eta_n \to 0$ and $u_n \to u_0$ such that $0 \leq u_0 \leq 1$ and such that along this sequence the expression is larger than some preassigned $\epsilon > 0$. But $\alpha_0(t)$ is continuous for $t > 0$ and

$$0 \leq \alpha_0(t+\eta+\tau-s) - \alpha_0(u+\tau-s)$$
$$\leq \alpha_0(u+\eta+\tau-s) \leq \alpha_0(\tau-s)$$

when $0 \leq u \leq 1$, $\eta > 0$ and $0 \leq s \leq \tau$. Therefore the dominated convergence theorem implies that

$$\epsilon \leq \lim_{h\to\infty} \int_0^\tau (\alpha_0(u_n+\eta_n+\tau-s) - \alpha_0(u_n+\tau-s))|x'(s)|ds = 0.$$

These remarks show that

$$\lim_{\eta\to 0} \int_0^\eta |(E(t+h)-E(t))/h - E'(t)|dt$$
$$\leq \lim\{\int_0^\eta |E(t+h)-E(t)|/hdt + \int_0^\eta |E'(t)|dt\} = 0$$

uniformly for $0 < h \leq 1$. In particular Theorem 2 applies to (14). Therefore $x''(t+\tau)$ exists and is continuous on $0 < t \leq T-\tau$. Since $\tau > 0$ can be made arbitrarily small, it follows that $x' \in C^1(0,T]$.

For any $\tau \in (0,T)$ the function $x''(t+\tau)$ satisfies an equation of the form

$$x''(t+\tau) = f''(t+\tau) + a'(t+\tau)g(x(0)) + E'(t) + F_1(t) + F_2(t) + \int_0^t a(t-s)g'(x(s+\tau))x''(s+\tau)ds \tag{16}$$

where $E$ is the function defined above, $F_1(t) = a(t)g'(x(\tau))x'(\tau)$ and

$$F_2(t) = \int_0^t a(t-s)g''(x(\tau+s)x'(\tau+s)^2 ds.$$

Let $K$ be a bound on $0 \leq t \leq T$ for the functions $f''(t)$, $g(x(t))$, $g'(x(t))$ and $g''(x(t))$. There will be no loss of generality in assuming that $T$ is small enough so that

$$\alpha = K\int_0^T |a(t)| dt < 1/2.$$

Take absolute values in (16) and integrate:

$$\int_\tau^T |x''(t)| dt = \int_0^{T-\tau} |x''(t+\tau)| dt$$

$$\leq K(T-\tau)+K\int_\tau^T |a'(t)| dt + \int_0^{T-\tau} \{|E'(t)| + |F_1(t)| + |F_2(t)|\} dt$$

$$+ \int_0^{T-\tau} \left| \int_0^t a(t-s)g'(x(\tau+s))x''(\tau+s)\, ds \right| dt.$$

The last term in this inequality may be bounded as follows:

$$K\int_0^{T-\tau}\int_s^{T-\tau} |a(t-s)||x''(\tau+s)| dt ds$$

$$= K\int_\tau^T \left(\int_s^T |a(t-s)| dt\right) |x''(s)| ds \leq \alpha\int_\tau^T |x''(s)| ds.$$

Furthermore

$$\int_0^{T-\tau}|F_1(t)|dt \le K\int_\tau^T|a(t)|dt|x'(\tau)| \le \alpha|x'(\tau)|$$
$$\le \alpha\{|x'(T)| + \int_\tau^T|x''(t)|dt\},$$

and

$$\int_0^{T-\tau}|F_2(t)|dt \le K\int_0^{T-\tau}(\int_s^{T-\tau}|a(t-s)|dt)|x'(\tau+s)|^2ds$$
$$\le K\int_\tau^T(\int_0^T|a(t)|dt)|x'(s)|^2ds = \alpha\int_\tau^T|x'(s)|^2ds.$$

Combining these inequalities with (15) and rearranging yields

$$(1-2\alpha)\int_\tau^T|x''(s)|ds \le KT+K\int_\tau^T|a'(t)|dt + K\int_0^\tau|x'(s)|\int_{\tau-s}^{T-s}|a'(t)|dtds$$
$$+ \alpha|x'(T)| + \alpha\int_\tau^T|x'(s)|^2ds.$$

Since $1-2\alpha > 0$, then the proof is complete. Q.E.D.

Corollary 2. <u>Assume the hypotheses of Theorem</u> 4. <u>If in addition the function</u> $\alpha_0(t,a) \in L^2(0,T)$ <u>then</u> $x \in WS(1)$.

<u>Proof</u>: First note that

$$\int_0^T(\int_\tau^T|x'(t)|^2dt)d\tau = \int_0^T(\int_0^t|x'(t)|^2d\tau)dt = \int_0^T t|x'(t)|^2dt < \infty.$$

and

$$\int_\tau^T |a'(t)|\,dt \leq \alpha_0(t,a).$$

Finally note that $x'(t)$ and $\alpha_0(t,a) \in L^1(0,T)$. Therefore

$$\int_0^\tau |x'(s)| \left(\int_{\tau-s}^T |a'(t)|\,dt\right)ds \leq \int_0^\tau |x'(s)|\alpha_0(\tau-s)ds$$

with the last function of class $L^1$ in $\tau$, $0 \leq \tau \leq T$. Using these estimates in (13) it follows that

$$\alpha_1(t,x(\cdot)) = |x'(T)| + \int_t^T |x''(u)|\,du \in L^1(0,T). \qquad \text{Q.E.D.}$$

Corollary 3. Assume the hypotheses of Theorem 4. If in addition $\beta(t) = \alpha_0(t,a)$ satisfies assumption (H1), then $x(t) \in WS(1)$ and $\alpha_1(t,x(\cdot)) \leq K\beta(t)$ on $0 < t \leq T$ for some apriori constant $K$.

Proof: Theorem 1 and Corollary 1 imply that $|x'(t)| \leq K_0\beta(t)$ on $0 < t \leq T$ for some fixed constant $K_0 > 0$. Since $\beta$ is nonincreasing, then

$$\int_\tau^T |x'(t)|^2 dt \leq \int_\tau^T K_0^2\beta(t)^2 dt \leq K_0^2\beta(\tau)\int_0^T\beta(t)dt.$$

Moreover (H1) may be used to see that

$$\int_0^\tau |x'(s)| \left(\int_{\tau-s}^T |a'(t)|\,dt\right)ds \leq \int_0^\tau |x'(s)|\beta(\tau-s)ds$$

$$\leq K_0\int_0^\tau \beta(s)\beta(\tau-s)ds \leq K_1\beta(\tau).$$

Using these estimates in (13) one obtains

$$\int_\tau^T |x''(t)|dt \le K^*[1 + (1+K_0^2\int_0^T\beta(t)dt + K_1)]\beta(\tau). \qquad \text{Q.E.D.}$$

Theorem 5. Suppose (A4) is true, $a(t) \in WS(\nu)$ where $\nu \ge 1$ and both $f$ and $g$ are of class $C^{\nu+2}$. Then the solution of equation (1) is of class $WS(\nu+1)$ and $\alpha_{\nu+1}(t,x(\cdot)) \le K\alpha_\nu(t,a)$ on $0 < t \le T$ for some fixed constant $K > 0$.

Proof: Apply Theorem 3 to obtain (12). Replace $t$ by $t+\tau$ in (12) and proceed as in Theorem 4. Q.E.D.

## 5. Special Kernels - An example.

Suppose $x(t)$ is the solution of

$$x(t) = f(t) + \int_0^t (t-s)^{\nu-p} g(s,x(s))ds \tag{17}$$

on $0 \le t \le T$ where $\nu \ge 0$ is an integer and $0 < p < 1$. If $f$ and $g$ are sufficiently smooth, then the results in section 4 show that $x \in WS(\nu)$ and $x^{(\nu)}(t) = \mathcal{O}(t^{-p})$ as $t \to 0$. Further information may be obtained for this special kernel. Change variables in the integral to $s = t\sin^2\theta$.

$$x(t) = f(t) + 2\int_0^{\pi/2} t^{\nu+1-p} \cos^{\nu+1-p} \cos^{\nu+1-p}\theta \sin\theta g(t\sin^2\theta, x(t\sin^2\theta))d\theta. \tag{17'}$$

If $f$ and $g$ are of class $C^{\nu+2}$, then by differentiating $\nu+1$ times on both sides of (17') one obtains

$$x^{(\nu+1)}(t) = \{2(\nu+1-p)(\nu-p)\ldots(1-p)\int_0^{\pi/2} \cos^{\nu+1-p}\theta \sin\theta g(t \sin^2\theta,$$

$$x(t \sin^2\theta))d\theta\} t^{-p} + \text{continuous terms of order}$$

$$t^{1-p}\ (t^{1-2p} \text{ if } \nu = 0) \text{ or higher.}$$

In particular then not only is $x^{\nu+1}(t) = \mathcal{O}(t^{-p})$ but

$$x^{(\nu+1)}(t) = f^{(\nu+1)}(0) + K_1 t^{-p} + \mathcal{O}(t^{1-p}) \qquad (\nu \geq 1)$$

or

$$x'(t) = K_1 t^{-p} + \mathcal{O}(t^{1-2p}) \qquad (\nu = 0)$$

where

$$K_1 = 2(\nu+1-p)(\nu-p)\ldots(1-p)\int_0^{\pi/2}\cos^{\nu+1-p}\sin\theta g(0,x(0))d\theta$$

$$= 2g(0,f(0))(\nu+1-p)\ldots(1-p)/(\nu+2-p).$$

Even more information is available when f and g are analytic.

Theorem 6. <u>Assume</u> $\nu$ <u>is a nonnegative integer</u>, $0 < p < 1$ <u>and that</u> $f(t)$ <u>is real analytic in a neighborhood of</u> $0 \leq t \leq T$. <u>Suppose</u> $g(t,x)$ <u>is real analytic on an open set which contains all real ordered pairs</u> $(t,x)$, $0 \leq t \leq T$ <u>and</u> $|x| < \infty$. <u>Then</u> $x(t)$ <u>is real analytic in a neighborhood of the set</u> $0 < t \leq T$.

<u>Proof</u>: Let $\|x\| = \max|x(t)|$ on $0 \leq t \leq T$. Given $\varepsilon > 0$ define

$$D(\varepsilon) = \{z: 0 \leq \text{Re } z \leq T+\varepsilon \text{ and } |\text{Im } z| \leq \varepsilon\}$$

and

$$E(\varepsilon) = \{(z,w): \; z \in D(\varepsilon) \; \text{ and } \; |w| \le \|x\|+1\}.$$

Define

$$M = \max\{|g(z,w)|, \; |\tfrac{\partial g}{\partial w}(z,w)|: \; (z,w) \in E(\varepsilon)\},$$

$$K = \max\{|f(z)| + TM|(z+s)^{\nu+1-p}|: \; z \in D(\varepsilon) \; \text{ and } \; 0 \le s \le T\},$$

and

$$S(\varepsilon) = \{z \text{ complex}: \; 0 \le \operatorname{Re} z \le \varepsilon, \; |\operatorname{Im} z| \le \varepsilon/2\}.$$

Pick $\varepsilon_0$ so small that whenever $0 < \varepsilon \le \varepsilon_0$ then $f$ is analytic in $D(\varepsilon)$ and $g$ is analytic in $E(\varepsilon)$.

Let $F(\varepsilon)$ denote the set of all functions $\varphi$, real analytic in the interior of $S(\varepsilon)$, continuous on $S(\varepsilon)$ and satisfying the bound $|\varphi(z)| \le K+1$ for all $z \in S(\varepsilon)$. Given $\varphi$ in $F(\varepsilon)$ define

$$(R\varphi)(z) = f(z) + \int_0^{\pi/2} 2z^{\nu+1-p} \cos^{\nu+1-p}\theta \sin\theta \, g(z \sin^2\theta, \; \varphi(z \sin^2\theta))\, d\theta.$$

If $\varepsilon$ is chosen so that

$$\beta = (\sqrt{5}\,\varepsilon/2)^{\nu+1-p}\pi M < 1,$$

then for any $z \in S(\varepsilon)$

$$|(R\varphi)(z)| \le |f(z)| + \int_0^{\pi/2} 2(\sqrt{5}\,\varepsilon/2)^{\nu+1-p}|g(z \sin^2\theta, \varphi(z \sin^2\theta))|\,d\theta$$

$$\le K + \int_0^{\pi/2} 2(\sqrt{5}\,\varepsilon/2)^{\nu+1-p} M\,d\theta \le K+\beta < K+1.$$

Therefore $S\varphi \in F(\varepsilon)$ when $\varphi \in F(\varepsilon)$. Moreover if $\varphi_1$ and $\varphi_2 \in F(\varepsilon)$ then

$$|R\varphi_1(z)-R\varphi_2(z)| \le \int_0^{\pi/2} 2(\sqrt{5}\,\varepsilon/2)^{\nu+1-p} M|\varphi_1(z \sin^2\theta)-\varphi_2(z \sin^2\theta)|\,d\theta$$

$$\le \beta \max\{|\varphi_1(z)-\varphi_2(z)| : z \in S(\varepsilon)\}.$$

Therefore $R$ is a contraction mapping on $F(\varepsilon)$.

Let $x(z)$ be the unique fixed point of $R$. Then $x(z)$ is real analytic in the interior of $S(\varepsilon)$, continuous on all of $S(\varepsilon)$ and $x(t)$ solves (17) if $0 \le t \le \varepsilon$. This means that the solution of (17) is analytic in a neighborhood of $0 < t < \varepsilon$.

Suppose we know that $x(z)$ is analytic in a set $\{z: 0 < \text{real } z \le \tau, |\text{Im } z| \le \varepsilon/2\}$ where $\tau < T$.

Translation in (17) will show that

$$x(t+\tau) = f_\tau(t) + \int_0^t (t-s)^{\nu-p} g(s+\tau, x(s+\tau))\,ds$$

where

$$f_\tau(t) = f(t+\tau) + \int_0^\tau (t+\tau-s)^{\nu-p} g(s,x(s))ds.$$

Since $f_\tau$ is real analytic in $t$ and $|f_\tau(z)| \leq K$ if $z \in S(\mathcal{E})$, then the first part of the proof applies. This means that $x(z+\tau)$ is in the class $F(\mathcal{E})$. Since the number $\mathcal{E}$ has been fixed beforehand, one may step across the interval $0 \leq t \leq T$ in a finite number of steps. Q.E.D.

Corollary 4. *Assume the hypotheses of Theorem 6. If $p$ is a rational number, $p = r/q$ in lowest terms, then $x(t^q)$ is analytic in a neighborhood of $t = 0$.*

*Proof:* First replace $t$ by $z^q$ in (17'):

$$x(z^q) = f(z^q) + \int_0^{\pi/2} 2z^{(\nu+1-p)q} \cos^{\nu+1-p}\theta \sin t \quad g(z^q \sin^2\theta, x(z^q \sin^2\theta))d\theta. \tag{17''}$$

Let $F(\mathcal{E}) = \{\varphi: \varphi$ is real analytic in $|z| < \mathcal{E}$ and continuous on $|z| \leq \mathcal{E}\}$. Define

$$R\varphi(z) = f(z^q) + \int_0^{\pi/2} 2z^{(\nu+1-p)q} \cos^{\nu+1-p}\theta \sin \theta g(z^q \sin^2\theta, \varphi(z \sin^{2/q}\theta))d\theta$$

for $\varphi \in F(\mathcal{E})$ and $|z| \leq \mathcal{E}$. As in the proof of Theorem 6 one can show that if $\mathcal{E}$ is sufficiently small then $R$ is a contraction mapping on

$F(\varepsilon)$. Q.E.D.

It would be interesting to know whether or not Theorem 6 can be generalized to a large class of kernels $a(t)$ which are analytic for $\mathrm{Re}\, t > 0$. The proof of Theorem 6 cannot be generalized too much since it depends on the monotonicity and homogeneity of $a(t) = t^{\nu-p}$. Corollary 4, which establishes the exact nature of the singularity of $x(z)$ at $z = 0$, is even more firmly wedded to the particular properties of $t^{\nu-p}$.

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