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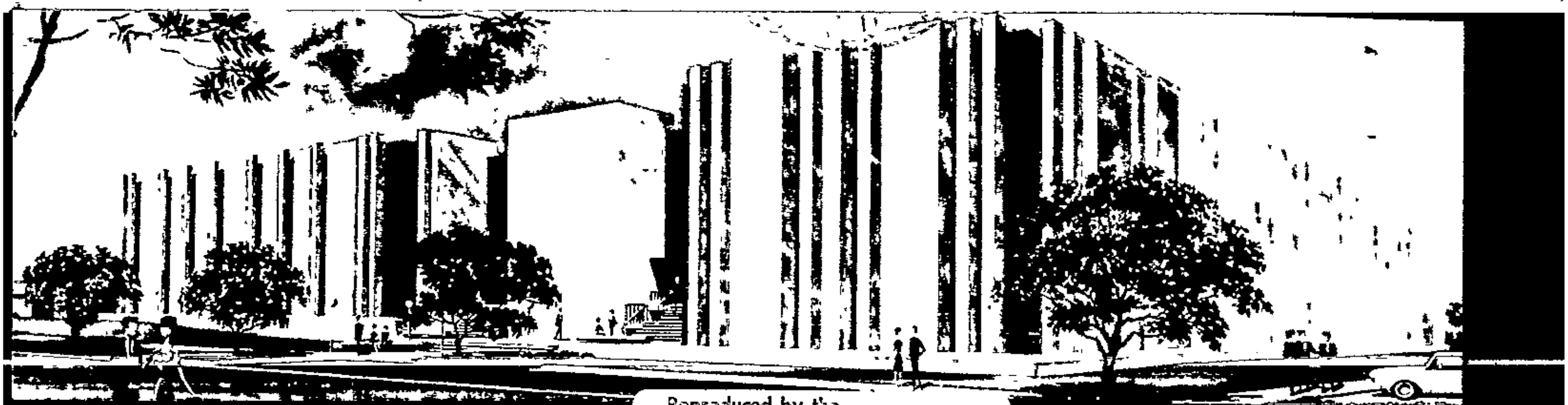

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ESTIMATION OF THE MEAN OF A DISCRETE PARAMETER, COVARIANCE STATIONARY, STOCHASTIC PROCESS IN ROTATION SAMPLING

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0. Introduction

The classical minimum variance unbiased linear estimator of the population mean in rotation sampling, derived by Yates (1949) and Patterson (1950), is invariant under alterations of the rotation scheme. Herein we derive by a constrained optimization procedure similar simultaneous linear equations. Patterson considered this technique but rejected it because of the lengthy estimator expressions, however by introducing matrix expressions and assuming a special rotation scheme this technique is found to be tractible. That is, we give an alternative way for deriving the estimator.

The merits of our procedure are as follows:

(1) Exact expression for the population mean estimator can be found for each occasion.

(2) The matrix notation leads to easier computer programming.

(3) The derivation of the exact expression for the variance-covariance matrix leads to ease in investigating the large sample properties of the estimators and their relation to maximum likelihood.

Yates-Patterson's method depends on a necessary and sufficient condition for minimum variance unbiased linear estimation of the population mean which is given in the following theorem and its corollary.

<u>Theorem 1</u>. When we have h classes of observations

$$\{Z_i(t),\ i = 1, 2, \ldots, n(t)\}\ ,\ t = 1, 2, \ldots, h$$

such that

$$E(Z_i(t)) = \mu(t) \text{ for any } i,$$

then a function of these observations, Y_τ, is minimum variance unbiased linear estimator of $\mu(\tau)$ if and only if

(1) Y_τ is an unbiased estimator of $\mu(\tau)$

(2) Y_τ may be expanded into a linear function of $Z_i(t)$'s, and

(3) $Cov(Z_i(t), Y_\tau) = C_{t\tau}$ for all i, t.

<u>Corollary</u>. An unbiased linear estimator Y_τ of $\mu(\tau)$ is of minimum variance if and only if it holds for any unbiased linear estimator U_s of $\mu(s)$ that

$$cov(U_s, Y_\tau) = C_{s\tau} = cov(Z_i(s), Y_\tau)$$

for any s. Specifically it must be held that

$$var(Y_\tau) = C_{\tau\tau} = cov(Z_i(\tau), Y_\tau).$$

Using the above relationships Yates-Patterson derived an expression for the minimum variance unbiased linear estimator of the population mean on the last observation occasion under the assumptions that:

(A1.) Correlation coefficient between observations on the same unit i occasions apart is ρ^i and known, $1 \leq i \leq h$.

(A2.) The variances are the same on each occasion and known (denoted by σ^2).

(A3.) Sample sizes on each occasion are equal to n.

(A4.) $n\phi$ units, $0 \leq \phi \leq 1$, are replaced by newly chosen units on each occasion.

(A5.) Sampling of each unit is done mutually independently from an infinite population, so that correlation coefficient between any two observations on different units is zero.

The estimator resulting is given by

$$Y_h = \varphi_h \bar{y}_h'' + (1 - \varphi_h)\left\{\bar{y}_h + \rho(Y_{h-1} - \bar{x}_{h-1}')\right\}$$

where $\bar{x}_{h-1}'$ is the mean of observations on occasion h-1 associated with $n(1-\phi)$ units common with occasion h, $\bar{y}_h'$ is the sample mean on occasion h associated with the same common units, $\bar{y}_h''$ is the mean on occasion h associated with the newly chosen uncommon units, and Y_{h-1} is the estimator of $\mu(h-1)$ based on the observations up to occasion h-1. Note Y_{h-1} is the minimum variance unbiased linear estimator of $\mu(h-1)$. Moreover, φ_h is determined through the following recursive relation

$$\varphi_h = \frac{\rho^2(1-\phi)\varphi_{h-1} + (1-\rho^2)\phi}{\rho^2(1-\phi)\varphi_{h-1} + (1-\rho^2)\phi + (1-\phi)},$$

with the initial condition $\varphi_1 = 1$, and the variance of Y_h is given by

$$\text{var}(Y_h) = \frac{\varphi_h \sigma^2}{n\phi}.$$

For proofs of the above results see Patterson's paper, as well as, Cochran (1963), Des Raj (1968), and Eckler (1955). It should be noted that each proof given for the above relationships is independent and invariant to the specification of the correlation between $Z_i(t)$ and $Z_j(s)$, that is, the specific rotation scheme can be changed without changing the formulation for Y_h, φ_h and the variances of Y_h. We state this result in the following theorem.

Theorem 2. Under the assumptions (A1.) - (A5.) above, the minimum variance unbiased linear estimator of population mean on the last occasion, Y_h, and its variance are both invariant to any further specification of rotation scheme.

Patterson further derived the minimum variance unbiased linear estimate of $\mu(h-k)$ when all observations up to the h^{th} occasion are available and denotes this by ${}_hY_{h-k}$. This estimation procedure is also independent of the particular rotation scheme used but the formulation was extremely lengthy and was not exhibited explicitly by Patterson.

By an alternative derivation the minimum variance unbiased linear estimator of each of the above population means is derived by a constrained minimization of the variance of the linear estimator with respect to its coefficients. That is, we can derive these estimators by minimizing the variance while guaranteeing unbiasedness. The result of this procedure, is nothing but Aitken's generalized least square estimator. But it is necessary to know the explicit representation of variance-covariance matrix in order to actually calculate this estimator. In doing so we will give a simplified specification of the estimator.

1. Assumptions and Rotation Scheme.

We retain all assumptions (A1.) - (A5.) described in the previous section, however for convenience we will exclude the cases where $\phi = 0$, and $\phi = 1$ from assumption (A4.) and will consider them later as special cases. For these special cases it can be easily shown that the minimum variance unbiased linear estimator of the population mean is given by the arithmetic mean of the sample observations only on each single occasion. We also exclude the cases where $\rho^2 = 0$ and $\rho^2 = 1$ as trivial cases, though the case where $\rho^2 = 0$ does not invalidate our results. We,however, make an additional assumption which specifies the particular rotation scheme to be employed.

(A6.) n sample units are partitioned into m+2 portions each of which contains $n/(m+2)$ sample units. For generality, any k, $1 \leq k \leq m+2$, portions may be replaced on each occasion from 0 to τ.

Denote the arithmetic mean of observations on newly chosen units which belong to i^{th} portion on t^{th} occasion by $\bar{x}_i(t)$, $i = 1, 2, \ldots, k$,and that of the oldest i^{th} portion which is to be replaced on $t + 1^{st}$ occasion by $\bar{z}_i(t)$, $i = 1, 2, \ldots, k$. Also denote the arithmetic means of observations on each of the remaining portions on t^{th} occasion by $\bar{y}_1(t), \bar{y}_2(t), \ldots, \bar{y}_\ell(t)$, where $\ell = m+2 - 2k$.

Let us represent the means described above in vector notation by

$$X_i' = (\bar{x}_i(0), \bar{x}_i(1), \ldots, \bar{x}_i(\tau)),\ 1 \leq i \leq k$$

$$Z_i' = (\bar{z}_i(0), \ldots, \bar{z}_i(\tau)),\ 1 \leq i \leq k$$

$$Y_i' = (\bar{y}_i(0), \ldots, \bar{y}_i(\tau)),\ 1 \leq i \leq \ell$$

and $X' = (X_1' \vdots \ldots \vdots X_k')$, $Z' = (Z_1' \vdots \ldots \vdots Z_k')$, $Y' = (Y_1' \vdots \ldots \vdots Y_\ell')$ with $(Y^*) = (X' \vdots Y' \vdots Z')$.

When ϕ is greater than 1/2 there are some units common to both a newly chosen portion and that to be replaced on the next occasion, that is, some units will be sampled only once. In this case, we will denote the mean vectors of those common portions by $\dot{Y}_i$ and when we use the notation Y^*, we will exclude those means of common portions out of the elements of X and Z. Under the rotation scheme specified by assumption (A6.), the variance-covariance matrix of the observations is derived below for $\phi \leq 1/2$.

$$V = E(Y^* - \mu^*)(Y^* - \mu^*)'$$

$$= \frac{(m+2)\sigma^2}{n} \begin{bmatrix} I_{k(\tau+1)} & \tilde{A}_1' & \tilde{A}_2' & \cdots & \tilde{A}_{\alpha-1}' & \dot{A}_\alpha' \\ & I_{k(\tau+1)} & \tilde{A}_1' & \cdots & \tilde{A}_{\alpha-2}' & \dot{A}_{\alpha-1}' \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & \tilde{A}_1' & \dot{A}_2' \\ & & & & I_{k(\tau+1)} & \dot{A}_1' \\ & & & & & I_{\ell'(\tau+1)} \end{bmatrix} \qquad (1.1)$$

given that V is symmetric and where $\mu^{*\prime} = (\mu' \vdots \cdots \vdots \mu')$, $\mu' = [\mu(0) \ \dots \ \mu(\tau)]$,

$$A_i = \rho^i \left[\begin{array}{c|c} 0 & 0 \\ \hline I_{\tau-1} & 0 \end{array}\right], \quad 1 \leq i \leq \alpha - 1,$$

$$\tilde{A}_i = \begin{bmatrix} A_i & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 0 \\ & & & A_i \end{bmatrix}$$

and A_i is an $\ell'(\tau+1)$ matrix defined as

$$\dot{A}_i = \begin{bmatrix} A_i & 0 \cdots 0 & & 0 \cdots 0 \\ 0 & \ddots & 0 & \vdots & \vdots \\ \vdots & \ddots & & & \\ 0 \cdots 0 & & A_i & 0 \cdots 0 \end{bmatrix}, \; 1 \leq i \leq \alpha ,$$

providing that $\alpha = [\frac{m+2}{k}]$, $\ell' = m+2-\alpha k$. As usual I_m is the m x m identity matrix, 0 is the null matrix of proper order.

Similarly, if $\phi > 1/2$, the variance covariance matrix becomes

$$V = \frac{(m+2)\sigma^2}{n} \begin{bmatrix} I_{\ell(\tau+1)} & 0 & \tilde{A}_1' \\ & I_{(k-\ell)(\tau+1)} & 0 \\ & & I_{\ell(\tau+1)} \end{bmatrix}$$

where $\ell = m+2-k$ and $\tilde{A}_1$ is a square matrix of order $\ell(\tau+1)$ defined by

$$\tilde{A}_1 = \begin{bmatrix} A_1 & 0 & \cdots 0 \\ & A_1 & \cdots 0 \\ & & \cdots A_1 \end{bmatrix}$$

Redefine

$$Y^* = \begin{bmatrix} X^* \\ \cdots \\ Y \\ \cdots \\ Z^* \end{bmatrix}, \qquad X^* = \begin{bmatrix} X_1 \\ \cdots \\ X_2 \\ \vdots \\ \cdots \\ X_\ell \end{bmatrix} \qquad Z^* = \begin{bmatrix} Z_{\ell'} \\ \cdots \\ Z_{\ell'+1} \\ \vdots \\ \cdots \\ Z_k \end{bmatrix}$$

where $\ell' = k-\ell+1$. That is, X* denotes the mean vector on such units as are observed on each occasion and also observed on the next occasion, and Z* denotes the mean vector on such units that were observed on the preceding occasion and are also to be observed on the current occasion though these units are to be replaced on the next occasion.

2. Inversion of Variance Covariance Matrix

In a special case when $k = 1$ (implying $\phi \leq 1/2$), we have the following expression of the variance covariance matrix

$$V = \frac{(m+2)\sigma^2}{n} \begin{bmatrix} I_{\tau+1} & A_1' & A_2' & \cdots & A_{m+1}' \\ & I_{\tau+1} & A_1' & \cdots & A_m' \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & A_1' \\ & & & & I_{\tau+1} \end{bmatrix}, \tag{2.1}$$

It can easily be shown that the inverse of this variance covariance matrix can be written in the following form:

$$V^{-1} = n\left\{(m+2)\sigma^2\,(1-\rho^2)\right\}^{-1} \begin{bmatrix} J_1 & B' & 0 & 0 & 0 & \cdots & 0 \\ & J_2 & B' & 0 & 0 & \cdots & 0 \\ & & J_2 & B' & 0 & \cdots & 0 \\ & & & \ddots & \ddots & \ddots & \vdots \\ & & & & \ddots & \ddots & 0 \\ & & & & & J_2 & B' \\ & & & & & & J_3 \end{bmatrix} \tag{2.2}$$

where J_1, J_2, J_3 and 0 are all square matrices of order $\tau+1$ and given by

$$J_1 = ({}_1g_{ij}),\quad {}_1g_{ij} = \begin{cases} 1, & 1 \leq i = j \leq \tau \\ 1-\rho^2, & i = j = \tau+1\,; \\ 0, & i \neq j \end{cases}$$

$$J_2 = ({}_2g_{ij}),\ {}_2g_{ij} = \begin{cases} 1, & i = j = 1 \text{ and } i = j = \tau + 1 \\ 1+\rho^2, & 2 \le i = j \le \tau \\ 0, & i \ne j \end{cases}$$

$$J_3 = ({}_3g_{ij}),\ {}_3g_{ij} = \begin{cases} 1-\rho^2, & i = j = 1 \\ 1, & 2 \le i = j \le \tau + 1; \text{ and } B = -A_1 \\ 0, & i = j \end{cases}$$

In (2.1), if it is the case that $\tau < m+1$, then we have

$$A_\tau = ({}_\tau a_{ij}),\ {}_\tau a_{ij} = \begin{cases} \rho^\tau, & i = \tau + 1,\ j = 1 \\ 0, & \text{otherwise} \end{cases}$$

and all A_i's, $i \ge \tau + 1$, are reduced to null matrices. Even in this case, however, it can be shown that the inverse of (2.1) is given by (2.2).

We have a similar result for the general case using the following lemma, which can be considered a generalization of the above result.

<u>Lemma 2.1</u>. Define square matrices $\tilde{A}_i$'s, $\tilde{J}_j$'s and $\tilde{B}$ all of order $k(\tau+1)$ given as follows:

$$\tilde{A}_i = \begin{bmatrix} A_i & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 0 \\ & & & A_i \end{bmatrix},\ 1 \le i \le \alpha-1,$$

$$\tilde{J}_j = \begin{bmatrix} J_j & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 0 \\ & & & J_j \end{bmatrix},\ j = 1, 2, 3,$$

$$\tilde{B} = \begin{bmatrix} B & & 0 & \cdots & 0 \\ & \ddots & & \ddots & \vdots \\ & & \ddots & & 0 \\ & & & \ddots & \\ & & & & B \end{bmatrix}$$

then

$$\begin{bmatrix} I_{k(\tau+1)} & \tilde{A}_1' & \tilde{A}_2' & \cdots\cdots & \tilde{A}_{\alpha-1}' \\ & I_{k(\tau+1)} & \tilde{A}_1' & \cdots\cdots & \tilde{A}_{\alpha-2}' \\ & & I_{k(\tau+1)} & \ddots & \vdots \\ & & & \ddots & \tilde{A}_1' \\ & & & & I_{k(\tau+1)} \end{bmatrix}$$

$$= \frac{1}{1-\rho^2} \begin{bmatrix} \tilde{J}_1 & \tilde{B}' & 0 & 0 & 0 & \cdots & 0 \\ & \tilde{J}_2 & \tilde{B}' & 0 & 0 & \cdots & 0 \\ & & \tilde{J}_2 & \tilde{B}' & 0 & \cdots & 0 \\ & & & \ddots & \ddots & \ddots & 0 \\ & & & & \tilde{J}_2 & & \tilde{B}' \\ & & & & & & \tilde{J}_3 \end{bmatrix}$$

With this lemma, using the inversion formula of partitioned matrix, we can find the inverse of (1.1), which is given by

$$V^{-1} = \frac{n}{(m+2)\sigma^2(1-\rho^2)} \begin{bmatrix} \tilde{J}_1 & \tilde{B}' & 0 & 0 \\ & \tilde{J}_2 & \tilde{B}' & 0 \\ & & \tilde{J}_2 & \tilde{B}' \\ & & & \tilde{J}_3 \end{bmatrix}$$

When $\phi > 1/2$, we have the following matrix as the inverse of (1.2):

$$V^{-1} = \frac{n}{(m+2)\sigma^2(1-\rho^2)} \begin{bmatrix} J_1 & 0 & \tilde{B}' \\ 0 & (1-\rho^2)\, I_{(k-\ell)(\tau+1)} & 0 \\ \tilde{B} & 0 & \tilde{J}_3 \end{bmatrix} \qquad (2.4)$$

where $\tilde{J}_1$, $\tilde{J}_3$ and $\tilde{B}$ are as given in Lemma 2.1 but of order ℓ, providing that $\ell = m+2-k$.

All these results can be confirmed by the direct multiplication, so that the detailed proofs are omitted here.

3. Minimum Variance Unbiased Linear (MVUL) Estimator of μ.

Define the $(m+2)(\tau+1) \times (\tau+1)$ matrix Λ as

$$\Lambda' = [I_{\tau+1} \vdots \cdots \vdots I_{\tau+1}]$$

then μ^* can be expressed as

$$\mu^* = \Lambda\mu.$$

Then, clearly, if V is known as is assumed in our formulation, the MVUL estimator of μ is given by Aitken's generalized (or weighted) least square estimate. That is,

$$\hat{\mu} = (\Lambda' V^{-1}\Lambda)^{-1}\Lambda' V^{-1} Y^*,$$

which is also the maximum likelihood estimate of μ when Y^* is assumed to be distributed as a multivariate normal with mean vector μ^* and variance-covariance matrix V.

Since a specific expression of V^{-1} has been derived in the preceding section under our special rotation scheme (A6.), using this expression of V^{-1}, we obtain a further specification of $(\Lambda' V^{-1}\Lambda)^{-1}$. That is,

$$(\Lambda' V^{-1}\Lambda)^{-1} = \left\{(m+2)\sigma^2(1-\rho^2)/n\right\} \Gamma^{-1} ,$$

where $\Gamma = (m+2)\Gamma^*$, and $\Gamma^* = (g_{ij})$

$$g_{ij} = \begin{cases} 1 - \phi\rho^2 \text{ for } i = j = 1 \text{ and } \tau+1 \\ 1 + (1 - 2\phi)\rho^2 \text{ for } 2 \leq i = j < \tau \\ -(1 - \phi)\rho^2 \text{ for } 2 \leq j = i+1 \leq \tau+1 \\ \qquad \text{and } 2 \leq i = j+1 \leq \tau+1 \\ 0, \text{ otherwise} \end{cases} \tag{3.1}$$

Also we have

$$\Lambda' V^{-1} Y^{*} = \left\{(m+2)\sigma^2(1-\rho^2)/n\right\}^{-1} \left\{k\Gamma_1\bar{X} + \ell\Gamma_2\bar{Y} + k\Gamma_3\bar{Z}\right\}$$

where $\ell = m+2-2k$,

$$\Gamma_1 = J_1 + b,\ \Gamma_2 = B' + J_2 + B,\ \Gamma_3 = B' + J_3,$$

and

$$\bar{x}' = [\bar{x}(0),\ \bar{x}(1),\ \ldots,\ \bar{x}(\tau)],\ \bar{y}' = [\bar{y}(0),\ \bar{y}(1),\ \ldots,\ \bar{y}(\tau)],$$

$$\bar{z}' = [\bar{z}(0),\ \bar{z}(1),\ \ldots,\ \bar{z}(\tau)],$$

with the definitions that

$$\bar{x}(t) = \sum_{i=1}^{k} \bar{x}_i(t)/k,\ \bar{y}(t) = \sum_{i=1}^{\ell} \bar{y}_i(t)/\ell,\ \bar{z}(t) = \sum_{i=1}^{k} \bar{z}_i(t)/k$$

for $t = 0, 1, 2, \ldots, \tau$.

Denoting as

$$u = k\Gamma_1\bar{x} + \ell\Gamma_2\bar{y} + k\Gamma_3\bar{z}$$

$$= (m+2)[\phi\Gamma_1 \vdots (1-2\phi)\Gamma_2 \vdots \phi\Gamma_3]\ [\bar{x}' \vdots \bar{y}' \vdots \bar{z}' \vdots\ ]', \tag{3.2}$$

we can reduce the estimator above to the following simple form:

$$\hat{\mu} = \Gamma^{-1}u.$$

Further, if we define u^* as

$$u^* = (m+2)^{-1}u, \tag{3.3}$$

we have an alternative expression of the estimator

$$\hat{\mu} = \Gamma^{*-1} u^{*}.$$

In passing, it might be interesting to point out that

$$\Gamma^{*} = \phi\Gamma_1 + (1-2\phi)\Gamma_2 + \phi\Gamma_3$$

and sometimes it might be useful for computational purposes to know the following representation of each component of u^{*}, that is,

$$u^{*} = \begin{bmatrix} \phi\bar{x}(0) + (1-2\phi)\bar{y}(0) + \phi(1-\rho^2)\bar{z}(0) - \rho\left\{(1-2\phi)\bar{y}(1) + \phi\bar{z}(1)\right\} \\ \phi\bar{x}(1) + (1-2\phi)\bar{y}(1) + \phi\bar{z}(1) - \rho\left\{\phi\bar{x}(0) + (1-2\phi)(\bar{y}(0) + \bar{y}(2)) + \phi\bar{z}(1)\right\} \\ \cdots \\ \phi\bar{x}(t) + (1-2\phi)\bar{y}(t) + \phi\bar{z}(t) - \rho\left\{\phi\bar{x}(t-1) + (1-2\phi)(\bar{y}(t-1) + \bar{y}(t+1)) + \phi\bar{z}(t+1)\right\} \\ \cdots \\ \phi\bar{x}(\tau-1) + (1-2\phi)\bar{y}(\tau-1) + \phi\bar{z}(\tau-1) - \rho\left\{\phi\bar{x}(\tau-2) + (1-2\phi)(\bar{y}(\tau-2) + \bar{y}(\tau)) + \phi\bar{z}(\tau)\right\} \\ \phi(1-\rho^2)\bar{x}(\tau) + (1-2\phi)\bar{y}(\tau) + \phi\bar{z}(\tau) - \rho\left\{\phi\bar{x}(\tau-1) + (1-2\phi)\bar{y}(\tau-1)\right\} \end{bmatrix}$$

The variance-covariance matrix of $\hat{\mu}$ is given by

$$E(\hat{\mu} - \mu)(\hat{\mu} - \mu)' = (\Lambda' V^{-1}\Lambda)^{-1} = \left\{(m+2)\sigma^2(1-\rho^2)/n\right\}\Gamma^{-1}$$
$$= \frac{\sigma^2}{n}(1-\rho^2)\Gamma^{*-1}.$$

These results are obtained commonly for both cases when $\phi > \frac{1}{2}$ and $\phi \leq \frac{1}{2}$.

According to Yates-Patterson's method, the variance of MVUL estimator on the last occasion τ is given by $\varphi_h \sigma^2/n\phi$, where $h = \tau+1$. Hence, from Theorem 2 in section 0 and the result just obtained above on variance-covariance matrix of $\hat{\mu}$, it is clear that φ_h is equivalent to the bottom diagonal element of $(1-\rho^2)\phi\Gamma^{*-1}$. It is also possible to ascertain this equivalence directly by mathematical induction.

We will summarize all these results by the following theorem.

Theorem 3. Under the assumptions (A1.) - (A5.), the minimum variance unbiased linear estimator of μ is given by $\hat{\mu}$ such that

$$\hat{\mu} = \Gamma^{*-1} u^{*}$$

and its variance-covariance matrix is given by $\frac{\sigma^2}{n}(1-\rho^2)\Gamma^{*-1}$, where Γ^{*} and u^{*} are defined by (3.1) and (3.3) derived from (3.2), respectively.

Further the bottom diagonal element of $(1-\rho^2)\phi\Gamma^{*-1}$ is equivalent to Yates-Patterson's φ_h with the provision that $h = \tau+1$. In passing, note that we do not have to assume (A6.) in Theorem 3 because of Theorem 2 in section 0. Hence the estimation formula stated in Theorem 3 is applicable for any rotation scheme as long as (A1.) - (A5.) are satisfied.

4. Explicit Representation of the Elements of Γ^{*-1}

Define $\Gamma^{(t)}$, a square matrix of order t + 1, by

$$\Gamma^{(t)} = (g_{ij}^{(t)}), \text{ where } g_{ij}^{(t)} = \begin{cases} 1 - \phi\rho^2, & i = j = 1 \text{ and } i = j = t+1 \\ 1 + (1-2\phi)\rho^2, & 2 \leq i = j \leq t \\ -(1 - \phi)\rho, & 1 \leq j = i+1 \leq t+1 \\ & \text{and } 1 \leq i = j+1 \leq t+1 \\ 0, & \text{otherwise.} \end{cases}$$

Also define $D^{(t-1)}$ to be the matrix of order t constructed from $\Gamma^{(t)}$ by deleting the last row and column. Similarly, $\Delta^{(t-1)}$ is a t-matrix constructed from $\Gamma^{(t)}$ by deleting the last row and the t^{th} column.

Then we have the following relations among the determinants of these matrices:

$$|\Delta^{(t)}| = -(1 - \phi)\rho \; |D^{(t-1)}| \quad \text{for any } t \geq 1$$

$$|D^{(t)}| = (1 - \phi)\rho \; |\Delta^{(t-1)}| + \left\{1 + (1 - 2\phi)\rho^2\right\}|D^{(t-1)}| \quad \text{for any } t \geq 2$$

$$|\Gamma^{(t)}| = (1 - \phi)\rho \; |\Delta^{(t-1)}| + (1 - \phi\rho^2) \; D^{(t-1)} \quad \text{for any } t \geq 2$$

which yield

$$|D^{(t)}| = -(1 - \phi)^2\rho^2 \; |D^{(t-2)}| + \left\{1 + (1 - 2\phi)\rho^2\right\} \; |D^{(t-1)}| \quad \text{for any } t \geq 2$$

$$|\Gamma^{(t)}| = -(1 - \phi)^2\rho^2 \; |D^{(t-2)}| + (1 - \phi\rho^2) \; |D^{(t-1)}| \quad \text{for any } t \geq 2.$$

Denote

$g(t) = |D^{(t)}|, \quad f(t) = |\Gamma^{(t)}|$, then the relations above are rewritten as

$$g(t) = \left\{1 + (1 - 2\phi)\rho^2\right\} g(t-1) - (1 - \phi)^2\rho^2 \, g(t-2) \;,$$

$$f(t) = (1 - \phi\rho^2) \, g(t-1) - (1 - \phi)^2\rho^2 \, g(t-2) \;,$$

subject to the conditions

$$g(0) = 1 - \phi\rho^2 = f(0) \;,$$

$$g(1) = (1 - \phi\rho^2)\left\{1 + (1 - 2\phi)\rho^2\right\} - (1 - \phi)^2\rho^2 \;,$$

$$f(1) = (1 - \phi\rho^2)^2 - (1 - \phi)^2\rho^2 \;.$$

These simultaneous homogeneous difference equations of order 2 have the following solution

$$g(t) = A_1C_1^t + A_2C_2^t$$

$$f(t) = (1 - \phi\rho^2)(A_1C_1^{t-1} + A_2C_2^{t-1}) - (1 - \phi)^2\rho^2(A_1C_1^{t-2} + A_2C_2^{t-2})$$

where

$$C_1 = \tfrac{1}{2}\left\{1 + (1 - 2\phi)\rho^2 + \left[\left\{1 + (1 - 2\phi)\rho^2\right\}^2 - 4(1 - \phi)^2\rho^2\right]^{\frac{1}{2}}\right\} ,$$

$$C_2 = \tfrac{1}{2}\left\{1 + (1 - 2\phi)\rho^2 \quad \left[\left\{1 + (1 - 2\phi)\rho^2\right\}^2 - 4(1 - \phi)^2\rho^2\right]^{\frac{1}{2}}\right\} ,$$

$$A_1 = -\frac{C_2g(0) - g(1)}{C_1 - C_2} \quad \text{and} \quad A_2 = \frac{C_1g(0) - g(1)}{C_1 - C_2}$$

Recalling that the (i, j)th element of $\Gamma^{*-1} = (\Gamma^{(\tau)})^{-1}$ is $g_{ij}^{(\tau)}$, then $g_{11}^{(\tau)} = g_{\tau+1,\ \tau+1}^{(\tau)} = |D^{(\tau-1)}|/|\Gamma^{(\tau)}| = g(\tau-1)/f(\tau)$.

Considering cofactors of each element of $\Gamma^{(\tau)}$, we have following representation of the bottom row elements of Γ^{*-1}:

$$g_{\tau+1,\ \tau-j}^{(\tau)} = (1 - \phi)^{j+1}\rho^{j+1}\ g(\tau-j-2)/f(\tau) \quad j = -1, \ldots, \tau-2$$

$$\text{and } g_{\tau+1,1}^{(\tau)} = (1 - \phi)^{\tau}\ \rho^{\tau}/f(\tau)\ .$$

The diagonal elements of Γ^{*-1} are represented by

$$\begin{aligned}
g_{11}^{(\tau)} &= g_{\tau+1,\tau+1}^{(\tau)} = g(\tau-1)/f(\tau) \\
g_{22}^{(\tau)} &= g_{\tau,\tau}^{(\tau)} \qquad = g(0)\ g(\tau-2)/f(\tau) \\
g_{33}^{(\tau)} &= g_{\tau-1,\tau-1}^{(\tau)} = g(1)\ g(\tau-3)/f(\tau) \\
&\vdots \\
g_{ii}^{(\tau)} &= g_{\tau-i+2,\tau-i+2}^{(\tau)} = g(i-2)\ g(\tau-i)/f(\tau) \\
&\vdots
\end{aligned}$$

and

$$\begin{cases}
g_{\frac{\tau+1}{2},\frac{\tau+1}{2}}^{(\tau)} = g_{\frac{\tau+1}{2}+1,\frac{\tau+1}{2}+1}^{(\tau)} = g(\frac{\tau-3}{2})\ g(\frac{\tau-1}{2})/f(\tau), & \text{if } \tau+1 \text{ is even} \\
g_{\frac{\tau}{2}+1,\frac{\tau}{2}+1}^{(\tau)} = \left\{g(\frac{\tau}{2} - 1)\right\}^2/f(\tau), & \text{if } \tau+1 \text{ is odd.}
\end{cases}$$

Define

$$k_{\tau+1,i} = g_{ii}^{(\tau)}/g_{\tau+1,i}^{(\tau)} = g(\tau-i)/\left\{(1-\emptyset)^{\tau-i+1}\ \rho^{\tau-i+1}\right\},\ 2 \le i \le \tau, \text{ providing}$$

that $\rho \neq 0$, then by Ukita's theorem (see Greenburg and Sarhan (1959) [4], Uppuluri and Carpentor (1969) [7], or Ukita (1955) [6]), we have

$$g_{ji}^{(\tau)} = g_{ij}^{(\tau)} = k_{\tau+1,i}\ g_{\tau+1,j}^{(\tau)} = (1-\emptyset)^{i-j}\ \rho^{i-j}\ g(j-2)\ g(\tau-i)/f(\tau)$$

where $2 \le j \le i$ and $2 \le i \le \tau$, and

$$g_{1i}^{(\tau)} = g_{i1}^{(\tau)} = (1-\emptyset)^{i-1}\ \rho^{i-1}\ g(\tau-i)/f(\tau)$$

Note that

$$g_{1,\tau+1}^{(\tau)} = g_{\tau+1,1}^{(\tau)} = (1-\emptyset)^{\tau}\ \rho^{\tau}/f(\tau),$$

which is already given.

These results, associated with the above solution of difference equations, will give explicit representation to every element of Γ^{*-1} The representation of Γ^{-1} directly follow from $\Gamma^{-1} = \frac{1}{m+2}\ \Gamma^{*-1}$

5. Convergency of Γ^{*-1}.

5.1 Definition of Convergency.

We will prove the convergency of Γ^{*-1} in the sense defined below.

Definition 1. (Matrix of variable order). A matrix, each element of which can be expressed as a function of its order, is referred to as a "matrix of variable order". Denoting the number of rows by s and the number of columns by t, we also call "matrix of variable order s by t". Specifically, if the matrix is square and its order is denoted by t, then we refer the matrix to a "square matrix of variable order t" or simply a "matrix of variable order t".

In the terminology of Definition 1, we can say that Γ^{*-1} is a matrix of variable order $\tau+1$.

Definition 2. (Convergency of a matrix of variable order). When G is a matrix of variable order s by t, and one of s and t, say s, tends to infinity, if every element of G converges to a unique function of t which remains finite for any finite t, then it is said that G is convergent with respect to s. Further if every element of G converges to a finite number when both s and t tend to infinity, G is convergent with respect to its order.

Specifically, if G is a square matrix of variable order t and every element of G converges to a finite number when t tends to infinity, then G is said to be convergent with respect to its order (t).

That is to say, if we denote (i, j)th element of a matrix G of variable order s by t by $g_{ij}(s, t)$, then the convergency of G means

$$\lim_{t\to\infty} \lim_{s\to\infty} g_{ij}(s, t) = g_{ij}$$

for any i, j such that $1 \leq i \leq s$, $1 \leq j \leq t$ and g_{ij} being finite.

In the case of square matrix, this reduces to

$$\lim_{t \to \infty} g_{ij}(t) = g_{ij}$$

for any i, j such that $1 \leq i \leq t$, $1 \leq j \leq t$ and g_{ij} being finite.

Our problem here is to prove this property concerning to Γ^{*-1}.

5.2 Properties of g(t) and f(t).

The elements of Γ^{*-1}, as we have seen in the preceding section, depend upon functions g(t) and f(t) by specifying t appropriately. It is necessary then to know some useful properties of g(t) and f(t) in order to prove the convergency of Γ^{*-1}.

Assuming $\rho^2 \neq 1$, we can prove following properties concerning to g(t) and f(t).

<u>Property</u> (1): $g(t) > 0$ for any $t \geq 0$.

Proof: Define

$$\alpha = \frac{1}{2} g(0), \quad \beta = (a\, g(0) - g(1))/2b$$

where

$$a = \frac{1}{2}\left\{1 + (1 - 2\phi)\rho^2\right\}, \quad b = \frac{1}{2}\sqrt{\left\{1 + (1 - 2\phi)\rho^2\right\}^2 - 4(1 - \phi)^2\rho^2},$$

then we have

$$A_1 = \alpha - \beta, \quad A_2 = \alpha + \beta$$

$$C_1 = a + b, \quad C_2 = a - b.$$

Since $\phi < 1$, it is clear that

$$\alpha > 0, \; a > 0.$$

Further we can show that b is a real number and hence $b \geq 0$ as follows:

$$\begin{aligned} 4b^2 &= \left\{1 + (1 - 2\phi)\rho^2\right\}^2 - 4(1 - \phi)^2\rho^2 \\ &= \left\{(1 - \phi)^2 + \phi^2\right\}(1 - \rho^2) + 1 - \left\{(1 - \phi)^2 + \phi^2\right\} - 2\phi(1 - \phi)\rho^4 \\ &\geq \left\{(1 - \phi)^2 + \phi^2\right\}(1 - \rho^2) + 1 - \left\{(1 - \phi)^2 + \phi^2\right\} - 2\phi(1 - \phi) \\ &= \left\{(1 - \phi)^2 + \phi^2\right\}(1 - \rho^2) \\ &\geq 0. \end{aligned}$$

This implies b is real and hence $b \geq 0$, but since we assumed $\rho^2 \neq 1$ and $(1 - \phi)^2 + \phi^2 > 0$ for any ϕ, we know that b is strictly positive.

We can also show that $\beta < 0$ as follows:

$$\begin{aligned} \text{the numerator of } \beta &= \left\{1 + (1 - 2\phi)\rho^2\right\}g(0) - 2g(1) \\ &= -(1 - \rho^2)\left\{1 + \phi(1 - 2\phi)\rho^2\right\}. \end{aligned}$$

Since $1 + \phi(1 - 2\phi)\rho^2 > 1 - \rho^2 > 0$, we thus know that the numerator of β is always negative under our assumptions. As the denominator of β is given by 4b and we already know that $b > 0$, $\beta < 0$ has now been established.

With these notations, we can write $g(t)$, $t \geq 0$, in the following way:

$$\begin{aligned} g(t) &= (\alpha - \beta)(a + b)^t + (\alpha + \beta)(a-b)^t \\ &= 2\alpha \sum_{t-x:\ \text{even}} \binom{t}{x} a^x b^{t-x} - 2\beta \sum_{t-x:\ \text{odd}} \binom{t}{x} a^x b^{t-x} \end{aligned}$$

where $\sum_{t-x:\ \text{even}}$ denotes the summation for all x such that t-x is even and

$\sum_{t-x:\ \text{odd}}$ denotes the summation for all x such that t-x is odd.

Now that $a > 0$, $b > 0$, $\alpha > 0$, and $\beta < 0$, it is clear $g(t) > 0$ for any t such that $t \geq 1$. Further, it is also clear from the definition that $g(0) > 0$. Thus we have $g(t) > 0$ for any t such that $t \geq 0$.

<u>Property</u>: (2): $\quad f(t) > 0$ for any $t \geq 0$.

<u>Proof</u>: By the notation above, we can write $g(t+1)$ for $t \geq 0$ as following way:

$$
\begin{aligned}
g(t+1) &= (\alpha-\beta)(a+b)^t(a+b) + (\alpha+\beta)(a+b)^t(a-b) \\
&= a\left\{2\alpha \sum_{t-x:\text{even}} \binom{t}{x} a^x b^{t-x} - 2\beta \sum_{t-x:\text{odd}} \binom{t}{x} a^x b^{t-x}\right\} \\
&\qquad + b\left\{2\alpha \sum_{t-x:\text{odd}} \binom{t}{x} a^x b^{t-x}\right. \\
&\qquad \left. - 2\beta \sum_{t-x:\text{even}} \binom{t}{x} a^x b^{t-x}\right\}
\end{aligned}
$$

and hence

$$
\begin{aligned}
f(t+1) &= g(0)\, g(t+1) - (1-\emptyset)^2\rho^2\, g(t) \\
&= 2\left[\alpha\left\{a\, g(0) - (1-\emptyset)^2\rho^2\right\} - \beta b\, g(0)\right] \sum_{t-x:\text{even}} \binom{t}{x} a^x b^{t-x} \\
&\qquad - 2\left[\beta\left\{a\, g(0) - (1-\emptyset)^2\rho^2\right\} - \alpha\, b\, g(0)\right] \sum_{t-x:\text{odd}} \binom{t}{x} a^x b^{t-x}
\end{aligned}
$$

We already know that $\alpha > 0$, $\beta < 0$, $a > 0$, $b > 0$, and $g(0) > 0$, hence we further know that $\beta b g(0) < 0$ and $\alpha b g(0) > 0$. We will show here $a g(0) - (1-\emptyset)^2\rho^2 > 0$ to yield $f(t+1) > 0$. Take

$$
\begin{aligned}
2\left\{a\, g(0) - (1-\emptyset)^2\rho^2\right\} &= \left\{1 + (1-2\emptyset)\rho^2\right\}(1-\emptyset\rho^2) - 2(1-\emptyset)^2\rho^2 \\
&= 2\left[\left\{1 + (1-2\emptyset)\rho^2\right\}(1-\emptyset\rho^2) - (1-\emptyset)^2\rho^2\right] \\
&\qquad - \left\{1 + (1-2\emptyset)\rho^2\right\}(1-\emptyset\rho^2)
\end{aligned}
$$

$$= 2\ g(1) - \left\{1 + (1 - 2\emptyset)\rho^2\right\} g(0)$$

$$= - \left\{\text{the numerator of } \beta\right\} .$$

Since the numerator of β is negative, we have

$$a\ g(0) - (1 - \emptyset)^2\rho^2 > 0$$

which implies $f(t+1) > 0$ for any $t \geq 0$. This establishes $f(t) > 0$ for any $t \geq 1$. We already know that $f(0) = g(0) > 0$. Thus finally we have

$$f(t) > 0 \text{ for any } t \geq 0.$$

From the properties (1) and (2) above, we have

Property (3): $g(t-1)/g(t) > 0$ and $g(t-1)/f(t) > 0$ for any $t \geq 1$.

We can also prove that

Property (4): $\frac{g(t-1)}{g(t)} - \frac{g(t-2)}{g(t-1)} < 0$ for any $t \geq 2$, that is to say, $g(t-1)/g(t)$, $t = 1, 2, 3, \ldots$ yield a monotonically decreasing progression.

Proof: Since

$$\frac{g(t-1)}{g(t)} - \frac{g(t-2)}{g(t-1)} = \frac{\left\{g(t-1)\right\}^2 - g(t)\ g(t-2)}{g(t)\ g(t-1)}\ ,\ t \geq 2$$

and $g(t)\ g(t-1) > 0$ by property (1), we have only to show that

$$\left\{g(t-1)\right\}^2 - g(t)\ g(t-2) < 0.$$

With the notations introduced before, we have

$$\left\{g(t-1)\right\}^2 - g(t)\ g(t-2) = \left\{(\alpha-\beta)(a+b)^{t-1} + (\alpha+\beta)(a-b)^{t-1}\right\}^2$$
$$- \left\{(\alpha-\beta)(a+b)^{t} + (\alpha+\beta)(a-b)^{t}\right\}$$
$$\cdot \left\{(\alpha-\beta)(a+b)^{t-2} + (\alpha+\beta)(a-b)^{t-2}\right\}$$

$$= -(\alpha^2 - \beta^2)(a^2 - b^2)^{t-2}\left\{(a+b) - (a-b)\right\}^2$$

$$= -4b^2\ (\alpha^2 - \beta^2)(a^2 - b^2)^{t-2}.$$

In this expression, we have

$$4(a^2 - b^2) = \left\{1 + (1 - 2\emptyset)\rho^2\right\}^2 - \left\{1 + (1 - 2\emptyset)\rho^2\right\}^2 + 4(1 - \emptyset)^2\rho^2$$

$$= 4(1 - \emptyset)^2\rho^2 > 0.$$

And further

$$16b^2(\alpha^2 - \beta^2) = 4[\left\{1 + (1 - 2\emptyset)\rho^2\right\}^2 - 4(1 - \emptyset)^2\rho^2](\alpha^2 - \beta^2)$$

$$= 4\emptyset(1 - \emptyset)^3\rho^4(1 - \rho^2) > 0.$$

Thus we have $\alpha^2 - \beta^2 > 0$. Hence,

$$\left\{g(t-1)\right\}^2 - g(t)\ g(t-2) = -\ 4b^2(\alpha^2 - \beta^2)(a^2 - b^2)^{t-2} < 0$$

for any $t \geq 2$, which implies the property (4).

From the property (4) it follows that

Property (5): $\dfrac{g(t-1)}{f(t)} - \dfrac{g(t-2)}{f(t-1)} < 0$, for any $t \geq 2$.

Proof:

$$\frac{g(t-1)}{f(t)} = \frac{g(t-1)}{(1 - \emptyset\rho^2)\ g(t-1) - (1 - \emptyset)^2\rho^2\ g(t-2)}$$

$$= 1/\left\{(1 - \emptyset\rho^2) - (1 - \emptyset)^2\rho^2 \cdot \frac{g(t-2)}{g(t-1)}\right\}$$

which implies that $g(t-1)/f(t)$ decreases as $g(t-2)/g(t-1)$ decreases. Meanwhile, by the property (4), we know that $g(t-1)/g(t-1)$ decreases as t increases. Hence $g(t-1)/f(t)$ decreases as t increases. Thus the property (5) must hold.

As is shown by the property (3), both $g(t-1)/g(t)$ and $g(t-1)/f(t)$ are bounded from below. Therefore, the properties (4) and (5) imply respectively that $g(t-1)/g(t)$ and $g(t+1)/f(t)$ converge to each limit value as t tends to infinity. And we can show

Property (6): $\lim_{t\to\infty} \frac{g(t-1)}{g(t)} = \frac{1}{C_1}$,

and hence

Property (7): $\lim_{t\to\infty} \frac{g(t-1)}{f(t)} = \frac{C_1}{C_1 g(0) - (1-\phi)^2 \rho^2}$

Proof: $\frac{g(t-1)}{g(t)} = \frac{(\alpha-\beta)(a+b)^{t-1} + (\alpha+\beta)(a-b)^{t-1}}{(\alpha-\beta)(a+b)^{t-1}(a+b) + (\alpha+\beta)(a-b)^{t-1}(a-b)}$

$$= [a + b \left\{1 - \frac{\alpha+\beta}{\alpha-\beta} \left(\frac{C_2}{C_1}\right)^{t-1}\right\} / \left\{1 + \frac{\alpha+\beta}{\alpha-\beta} \left(\frac{C_2}{C_1}\right)^{t-1}\right\}]^{-1} .$$

Since $C_1 > 0$, $C_2 > 0$ and $C_1 > C_2$, $C_2/C_1 < 1$ then $\lim_{t\to\infty} \left(\frac{C_2}{C_1}\right)^{t-1} = 0$.

Hence

$$\lim_{t\to\infty} \frac{g(t-1)}{g(t)} = \frac{1}{a+b} = \frac{1}{C_1} .$$

Property (7) directly follows from this because

$$\frac{g(t-1)}{f(t)} = \left\{g(0) - (1-\phi)^2\rho^2 \frac{g(t-2)}{g(t-1)}\right\}^{-1} .$$

Property (4) also implies that the maximum value of g(t-1)/g(t) is given by

$$\frac{g(0)}{g(1)} = \frac{1-\phi\rho^2}{(1-\phi\rho^2)\left\{1+(1-2\phi)\rho^2\right\} - (1-\phi)^2\rho^2} ,$$

which is finite, and hence, by Property (5), the maximum value of g(t-1)/f(t) is given by

$$\frac{g(0)}{f(1)} = \frac{1-\phi\rho^2}{(1-\phi\rho^2)^2 - (1-\phi)^2\rho^2} .$$

Thus, combining Properties (6) and (7) to these results, we have following:

<u>Property (8)</u>: $\frac{g(t-1)}{g(t)}$ and $\frac{g(t-1)}{f(t)}$ are both bounded,

$$\frac{1}{C_1} \leq \frac{g(t-1)}{g(t)} \leq \frac{g(0)}{g(1)} ,$$

and

$$\frac{C_1}{C_1\, g(0) - (1-\phi)^2\rho^2} \leq \frac{g(t-1)}{f(t)}$$

$$\leq \frac{g(0)}{\left\{g(0)\right\}^2 - (1-\phi)^2\rho^2} .$$

5.3 Proof of Convergency of Γ^{*-1}

Using the properties derived above on $g(t)$ and $f(t)$, we will first prove several lemmas which lead to the proof of convergency of Γ^{*-1}.

Lemma 5.1. (Monotonicity of $g(\tau-i)/f(\tau)$)

$g(\tau-i)f(\tau)$ is a monotonically decreasing function of τ for any i, $1 \leq i \leq \tau - 1$, where $\tau \geq 1$.

Proof:

$$\frac{g(\tau-i)}{f(\tau)} = \frac{g(\tau-i)}{g(0)\ g(\tau-1) - (1-\phi)^2 \rho^2 g(\tau-2)}$$

$$= \frac{g(\tau-i)/g(\tau-1)}{g(0) - (1-\phi)^2 \rho^2 g(\tau-2)/g(\tau-1)}$$

We already know that $g(\tau-2)/g(\tau-1)$ is a monotonically decreasing function of τ by Property (4). Hence, the denominator of this expression is an increasing function of τ. It will be enough, therefore, to show that $g(\tau-i)/g(\tau-1)$ is also a monotonically decreasing function of τ, in order to prove the lemma.

Since

$$\frac{g(\tau-i)}{g(\tau-1)} - \frac{g(\tau-i-1)}{g(\tau-2)} = \frac{g(\tau-2)\ g(\tau-i) - g(\tau-1)\ g(\tau-i-1)}{g(\tau-1)\ g(\tau-2)}$$

and $g(\tau-1)\ g(\tau-2) > 0$, it suffices to show that $g(\tau-2)\ g(\tau-i) - g(\tau-1)\,g(\tau-i-1) < 0$ for any $i \geq 2$.

Using the notation introduced in the preceding subsection, it can be written that

$$g(\tau-2)\,g(\tau-i) - g(\tau-1)\,g(\tau-i-1)$$

$$= \left\{(\alpha-\beta)(a+b)^{\tau-2} + (\alpha+\beta)(a-b)^{\tau-2}\right\}\left\{(\alpha-\beta)(a+b)^{\tau-i} + (\alpha+\beta)(a-b)^{\tau-i}\right\}$$

$$- \left\{(\alpha-\beta)(a+b)^{\tau-1} + (\alpha+\beta)(a-b)^{\tau-1}\right\}\left\{(\alpha-\beta)(a+b)^{\tau-i-1} + (\alpha+\beta)(a-b)^{\tau-i-1}\right\}$$

$$= 2b(\alpha^2 - \beta^2)(a^2 - b^2)^{\tau-i-1}\left\{(a-b)^{i-1} - (a+b)^{i-1}\right\}$$

We already know that $a > 0$, $b > 0$, $\alpha^2 - \beta^2 > 0$ and $a^2 - b^2 > 0$ and, hence, $a + b > a - b > 0$. Therefore

$$(a-b)^{i-1} - (a+b)^{i-1} \leq 0 ,$$

where equality holds only when $i = 1$. Thus we have

$$g(\tau-2)\,g(\tau-i) - g(\tau-1)\,g(\tau-i-1) = 2b(\alpha^2-\beta^2)(a^2-b^2)^{\tau-i-1}\left\{(a-b)^{i-1} - (a+b)^{i-1}\right\} < 0$$

for any $i \geq 2$.

In passing, note that the case when $i = 1$ reduces to Property (5) of the preceding subsection. This establishes the lemma.

Lemma 5.2. (Boundedness of the principal diagonal elements of Γ^{*-1}).

We have the following relations among the principal diagonal elements of Γ^{*-1}, that is, for each $\tau \geq 1$,

$$g_{11}^{(\tau)} = g_{\tau+1,\tau+1}^{(\tau)} > g_{22}^{(\tau)} = g_{\tau,\tau}^{(\tau)} > g_{33}^{(\tau)} = g_{\tau-1,\tau-1}^{(\tau)} > \dots > g_{ii}^{(\tau)} = g_{\tau-i-2,\tau-i-2}^{(\tau)} > \dots > 0.$$

Proof: Suppose $i \geq 3$, then from the results in section 4 we have

$$g_{ii}^{(\tau)} = g_{i-1,i-1}^{(\tau)} = \left\{g(i-2)\ g(\tau-i) - g(i-3)\ g(\tau-i+1)\right\} / f(\tau)$$

The numerator of the right hand side in this expression is equal to

$$\left\{(\alpha-\beta)(a+b)^{i-2} + (\alpha+\beta)(a-b)^{i-2}\right\} \left\{(\alpha-\beta)(a+b)^{\tau-i} + (\alpha+\beta)(a-b)^{\tau-i}\right\}$$

$$- \left\{(\alpha-\beta)(a+b)^{i-3} + (\alpha+\beta)(a-b)^{i-3}\right\}\left\{(\alpha+\beta)(a+b)^{\tau-i+1} + (\alpha+\beta)(a-b)^{\tau-i+1}\right\}$$

providing that $\tau-1 \geq i-2$.

Thus we have

$$f(\tau)(g_{ii}^{(\tau)} - g_{i-1,i-1}^{(\tau)}) = 2b(\alpha^2-\beta^2)(a^2-b^2)^{i-3} \left\{(a-b)^{(\tau-i)-(i-3)} - (a+b)^{(\tau-i)(i-3)}\right\}$$

Since we assumed $\tau-1 \geq i-2$, hence $(\tau-i) - (i-3) > 0$, and since $a > 0$, $b > 0$, $a^2 - b^2 > 0$, we have $a + b > 0$. Further, since we know that $\alpha^2 - \beta^2 > 0$,

$$f(\tau)\ (g_{ii}^{(\tau)} - g_{i-1,i-1}^{(\tau)}) < 0$$

and since $f(\tau) > 0$, this implies that

$$g_{ii}^{(\tau)} < g_{i-1,i-1}^{(\tau)} \quad \text{for } i \geq 3\ .$$

Combining this result to the results obtained in the section 4, we have

$$g_{22}^{(\tau)} = g_{\tau,\tau}^{(\tau)} > \ldots > g_{i-1,i-1}^{(\tau)} = g_{\tau-i-1,\tau-i-1}^{(\tau)} > g_{i,i}^{(\tau)} = g_{\tau-i-2,\tau-i-2}^{(\tau)} > \ldots > 0 \ .$$

Now consider

$$f(\tau)\ (g_{22}^{(\tau)} - g_{11}^{(\tau)}) = g(0)\ g(\tau-2) - g(\tau-1)$$

$$= g(\tau-1)\ \left\{g(0) \cdot \frac{g(\tau-2)}{g(\tau-1)} - 1\right\}$$

Since $g(\tau-1) > 0$ and $\frac{g(\tau-2)}{g(\tau-1)} \leq \frac{g(0)}{g(1)}$ by Property (8) in the preceding subsection we have

$$f(\tau)\ (g_{22}^{(\tau)} - g_{11}^{(\tau)}) \leq \frac{g(\tau-1)}{g(1)} \left[\{g(0)\}^2 - g(1) \right]$$

But since

$$\{g(0)\}^2 - g(1) = (1 - \phi\rho^2)^2 - (1 - \phi\rho^2)\ \left\{1 + (1 - 2\phi)\rho^2\right\} + (1 - \phi)^2\ \rho^2$$

$$= -\phi(1 - \phi)\ \rho^2\ (1 - \rho^2) < 0 \ .$$

and $g(\tau-1)/g(1) > 0$, we have

$$f(\tau)(g_{22}^{(\tau)} - g_{11}^{(\tau)}) < 0$$

which implies

$$g_{11}^{(\tau)} = g_{\tau+1,\tau+1}^{(\tau)} > g_{22}^{(\tau)} = g_{\tau,\tau}^{(\tau)} .$$

Furthermore, by Property (8) in the preceding subsection, we have

$$g_{11}^{(\tau)} = g_{\tau+1,\tau+1}^{(\tau)} = g(\tau-1)/f(\tau) \leq g(0)/[\ g(0)^{\ 2} - (1-\phi)^2\rho^2] ,$$

which implies the boundedness of the all principal diagonal elements of Γ^{*-1}.

Lemma 5.3. (Boundedness of $g_{ij}^{(\tau)}$)

$g_{ij}^{(\tau)}$ is bounded for any i, j and τ such that $1 \leq i \leq \tau + 1$, $1 \leq j \leq \tau + 1$, and $\tau \geq 1$.

Proof: Recall that

$$E(\hat{\mu} - \mu)(\hat{\mu} - \mu)' = \frac{\sigma^2}{n}(1-\rho^2)\Gamma^{*-1} .$$

From this, we know that the correlation coefficient between MVUL estimators $\hat{\mu}(i)$ and $\hat{\mu}(j)$ of $\mu(i)$ and $\mu(j)$, respectively is given by

$$\rho_{\hat{\mu}(i),\hat{\mu}(j)} = g_{ij}^{(\tau)} / \sqrt{g_{ii}^{(\tau)}\ g_{jj}^{(\tau)}} .$$

Since $0 \leq \rho^2_{\hat{\mu}(i),\hat{\mu}(j)}, \leq 1$, we have

$$0 \leq g_{ij}^{(\tau)^2} \leq g_{ii}^{(\tau)}\ g_{jj}^{(\tau)} .$$

But by Lemma 5.2., both $g_{ii}^{(\tau)}$ and $g_{jj}^{(\tau)}$ are bounded. Hence $g_{ij}^{(\tau)}$ is bounded as well.

<u>Lemma 5.4.</u> (Monotonicity of $g_{ii}^{(\cdot)}$)

For any i, such that $1 \leq i \leq \tau+1$, if i is fixed, $g_{ii}^{(\tau)}$ decreases monotonically as τ increases holds for any $\tau \geq 1$.

<u>Proof:</u> Since

$$g_{ii}^{(\tau)} = g(i-2)\ g(\tau-1)/f(\tau)$$

and $g(\tau-i)/f(\tau) > g(\tau-i+1)/f(\tau+1)$ by Lemma 5.1, it is clear that

$$g_{i,i}^{(\tau)} = g(i-2)\ g(\tau-i)/f(\tau) > g(i-2)(\tau-i+1)/f(\tau+1) \doteq g_{ii}^{(\tau+1)} .$$

Note that this argument can be applicable even to such element or elements that are located at the center of the principal diagonal of Γ^{*-1}.

Suppose $\tau+1$ is even, then there are two such central elements which can be written as

$$g^{(\tau)}_{\frac{\tau+1}{2},\, \frac{\tau+1}{2}} = g^{(\tau)}_{\frac{\tau+1}{2}+1,\, \frac{\tau+1}{2}+1} = g\left(\frac{\tau-3}{2}\right) g\left(\frac{\tau-1}{2}\right) / f(\tau)$$

Now let us denote $i = (\tau+1)/2$ and suppose i is fixed, then this can be rewritten as

$$g_{i,i}^{(\tau)} = g_{i+1,i+1}^{(\tau)} = g(i-2)g(i-1)/f(\tau) = g(i-2)g(\tau-i)/f(\tau).$$

Then let us compare $g_{ii}^{(\tau)}$ with $g_{ii}^{(\tau+1)}$ Since

$$g_{ii}^{(\tau+1)} = g(i-2)\ g(\tau+1-i)/f(\tau+1)$$

and $g(\tau+1-i)/f(\tau+1) < g(\tau-i)/f(\tau)$, we have

$$g_{ii}^{(\tau+1)} < g_{ii}^{(\tau)}$$

Next, compare $g^{(\tau)}_{i+1,i+1}$ with $g^{(\tau+1)}_{i+1,i+1}$. This time, $g^{(\tau+1)}_{i+1,i+1}$ comes to the unique central element of the principal diagonal of new matrix with increased order and is written as

$$g^{(\tau+1)}_{i+1,i+1} = g^{(\tau+1)}_{\frac{\tau+1}{2}+1,\ \frac{\tau+1}{2}+1} = \left\{g\left(\frac{\tau+1}{2} - 1\right)\right\}^2 / f(\tau+1)$$

$$= \left\{g(i-1)\right\}^2 / f(\tau+1).$$

Hence

$$g^{(\tau)}_{2+1,i+1} = g(i-2)g(i-1)/f(\tau) > \left\{g(i-1)\right\}^2/f(\tau+1) = g^{(\tau+1)}_{i+1,i+1}$$

because

$$g(i-2)/f(\tau) > g(i-1)/f(\tau+1).$$

Suppose $\tau+1$ is odd, then we have a unique central element which is given by

$$g^{(\tau)}_{\frac{\tau}{2}+1,\ \frac{\tau}{2}+1} = \left\{g\left(\frac{\tau}{2} - 1\right)\right\}^2 / f(\tau)$$

Now let us denote $i = \frac{\tau}{2} + 1$, then

$$g^{(\tau)}_{ii} = \left\{g(i-2)\right\}^2 / f(\tau)$$

Meanwhile

$$g^{(\tau+1)}_{ii} = g\left(\frac{\tau+1-3}{2}\right) g\left(\frac{\tau+1-1}{2}\right) \Big/ f(\tau+1)$$

$$= g\left(\frac{\tau}{2} - 1\right) g\left(\frac{\tau}{2}\right) \Big/ f(\tau+1)$$

$$= g(i-2)g(i-1)/f(\tau+1).$$

Hence we have

$$g_{ii}^{(\tau)} = \left\{g(i-2)\right\}^2/f(\tau) > g(i-2)g(i-1)/f(\tau+1) = g_{ii}^{(\tau+1)}$$

because

$$g(i-2)/f(\tau) > g(i-1)/f(\tau+1)$$

Thus $g_{ii}^{(\tau)} > g_{ii}^{(\tau+1)}$ holds for any i.

<u>Lemma 5.5.</u> (Convergency of $(1-\phi)^t\rho^t/g(t-1)$)

$(1-\phi)^t\rho^t/g(t-1)$ converges to zero as t tends to infinity.

<u>Proof:</u> Since

$$\begin{aligned}(1-\phi)^t\rho^t/g(t-1) &= (1-\phi)^t\rho^t/\left\{(\alpha-\beta)(a+b)^{t-1} + (\alpha+\beta)(a-b)^{t-1}\right\}\\ &= (\alpha-\beta)(1-\phi)^t\rho^t/\left\{(\alpha-\beta)^2(a+b)^{t-1} + (\alpha^2-\beta^2)(a-b)^{t-1}\right\}\end{aligned}$$

and, as we already know,

$$a+b > a-b > 0 \text{ and } \alpha^2 - \beta^2 > 0 ,$$

We have then

$$\begin{aligned}\left\{(1-\phi)^t\rho^t/g(t-1)\right\}^2 \quad & (\alpha-\beta)^2(1-\phi)^{2t}\rho^{2t}/\left\{(\alpha-\beta)^2(a+b)^{t-1} + (\alpha^2-\beta^2)(a-b)^{t-1}\right\}^2\\ &< (1-\phi)^{2t}\rho^{2t}/(\alpha-\beta)^2(a+b)^{2(t-1)}\\ &= \left\{(1-\phi)^2\rho^2/A_1^2\right\}\left\{(1-\phi)^2\rho^2/c_1^2\right\}^{t-1}\\ &< \left\{(1-\phi)^2\rho^2/A_1^2\right\}\left[4(1-\phi)^2\rho^2/\left\{1+(1-2\phi)\rho^2\right\}^2\right]^{t-1} \qquad (5.1)\end{aligned}$$

Meanwhile, since, as we already shown,

$$\left\{1+(1-2\phi)\rho^2\right\}^2 - 4(1-\phi)^2\rho^2 > 0$$

it holds that

$$4(1-\phi)^2\rho^2/\left\{1+(1-2\phi)\rho^2\right\}^2 < 1 .$$

Note that strict inequality holds here because of the assumption that $\rho^2 \neq 1$. Hence, the right hand side of (5.1) converges to zero as t tends to infinity, so the left hand side converges to zero as well.

Using these lemmas, we can prove the convergency of Γ^{*-1} stated by the following theorem.

Theorem 4. (Convergency of Γ^{*-1}) Γ^{*-1} is convergent with respect to its order.

Proof: From Lemma 5.2. and Lemma 5.4., it directly follows that each principal diagonal element, $g_{ii}^{(\tau)}$, i = 1, 2, ..., τ+1, converges to some limit value.

Now let us consider about the bottom row elements, $g_{\tau+1,j}^{(\tau)}$, j = 1, 2, ..., τ. For $j \geq 2$, we have

$$g_{\tau+1,j}^{(\tau)} = (1-\phi)^{\tau-j+1}\ \rho^{\tau-j+1}\ g(j-2)/f(\tau) .$$

Suppose j is fixed, then we have

$$\left\{g_{\tau+1,j}^{(\tau)}\right\}^2 = \left\{(1-\phi)^{\tau-j+1}\rho^{\tau-j+1}/g(\tau-j)\right\}^2\left\{g(j-2)g(\tau-j)/f(\tau)\right\}^2$$

$$= \left\{(1-\phi)^{\tau-j+1}\rho^{\tau-j+1}/g(\tau-j)\right\}^2\left\{g_{jj}^{(\tau)}\right\}^2 .$$

Denote $t = \tau-j+1$. Since $g_{jj}^{(\tau)}$ converges to some limit value when τ tends to infinity, applying Lemma 5.5., we have

$$\lim_{\tau\to\infty}\left\{g_{\tau+1,j}^{(\tau)}\right\}^2 = \lim_{\tau\to\infty}\left\{(1-\phi)^{\tau-j+1}\rho^{\tau-j+1}/g(\tau-j)\right\}^2\ \lim_{\tau\to\infty}\left\{g_{jj}^{(\tau)}\right\}^2$$

$$= \lim_{\tau\to\infty}\left\{(1-\phi)^t\rho^t/g(t-1)\right\}^2\ \lim_{\tau\to\infty}\left\{g_{jj}^{(\tau)}\right\}^2$$

$$= 0 .$$

For j = 1, we have

$$g^{(\tau)}_{\tau+1,1} = (1-\phi)^{\tau}\rho^{\tau}/f(\tau) = \left\{(1-\phi)^{\tau}\rho^{\tau}/g(\tau-1)\right\}\left\{g(\tau-1)/f(\tau)\right\}$$

$$= \left\{(1-\phi)^{\tau}\rho^{\tau}/g(\tau-1)\right\} g^{(\tau)}_{11}$$

Hence, in the same way, we obtain

$$\lim_{\tau\to\infty} \left\{g^{(\tau)}_{\tau+1,j}\right\}^2 = 0$$

Thus for any fixed j, $g^{(\tau)}_{\tau+1,j}$ converges to zero when τ goes to infinity.

Now suppose $\tau-j+1$ is fixed, that is, $g^{(\tau)}_{\tau+1,j}$ is such element that locates finitely away from the principal diagonal with fixed distance.

Denote $i = \tau-j$, which is fixed because $\tau-j+1$ is fixed, then we have

$$g^{(\tau)}_{\tau+1,j} = (1-\phi)^{i+1}\rho^{i+1}g(\tau-i-2)/f(\tau)$$

Now that i+1 is fixed and $g(\tau-i-2)/g(\tau)$ monotonically decreases as τ increases and is bounded from below, it is clear that $g^{(\tau)}_{\tau+1,j}$ converges to some limit value for any j such that $\tau-j$ is fixed. Thus, all elements of the last row of Γ^{*-1} converge to some limit values respectively.

As to the other off-diagonal elements, we already obtained the following representation:

$$g^{(\tau)}_{ij} = (1-\phi)^{i-j}\rho^{i-j}g(j-2)g(\tau-i)/f(\tau)$$

for such i, j that $2 \le j \le i$ and $2 \le i \le \tau$, and

$$g^{(\tau)}_{i1} = (1-\phi)^{i-1}\rho^{i-1}g(\tau-i)/f(\tau)$$

Let us first consider the case when both τ-i and τ-j are finitely fixed. Note that this implies that i-j is finitely fixed.

For any i, j such that $2 \leq j \leq i$ and $2 \leq i \leq \tau$,

$$g_{ij}^{(\tau)} = (1-\phi)^{i-j}\rho^{i-j}g(j-2)g(\tau-i)/f(\tau)$$

$$= (1-\phi)^{i-j}\rho^{i-j}\ g(\tau-i)g(\tau-k-2)/f(\tau)$$

where $k = \tau-j$. Since i-j, τ-i, and k are all fixed and $g(\tau-k-2)/f(\tau)$ is monotonically decreasing when τ increases and is bounded, it is clear that $g_{ij}^{(\tau)}$ converges to some limit value as τ goes to infinity.

Consider next the case when one of i and j, say j, is finitely fixed while τ goes to infinity and τ-i is finitely fixed if j is fixed. Define $k = \tau-i$, which is fixed, then we have

$$g_{ij}^{(\tau)} = M\left\{(1-\phi)^{\tau-k-1}\rho^{\tau-k-1}/g(\tau-k-2)\right\}\left\{g(k)g(\tau-k-1)/f(\tau)\right\}$$

where

$$M = (1-\phi)^{-(j-1)}\rho^{-(j-1)}g(j-2)$$

Since $(1-\phi)^{\tau-k-1}\rho^{\tau-k-1}/g(\tau-k-2)$ converges to zero, when τ tends to infinity by Lemma 5.5., and since

$$g(k)g(\tau-k-2)/f(\tau) = g(\ell-2)g(\tau-\ell)/f(\tau) - g_{\ell\ell}^{(\tau)}$$

where $\ell = k+2$, also converges to some limit value when τ goes to infinity, we have

$$\lim_{\tau \to \infty} g_{ij}^{(\tau)} = 0$$

Finally suppose both i and j are fixed, then since $(1-\phi)^{i-j}\rho^{i-j}g(j-2)$ is fixed and $g(\tau-i)/f(\tau)$ converges to some limit value when τ infinitely increases, $g_{ij}^{(\tau)}$ converges to some limit value. As to $g_{i1}^{(\tau)}$, we can write it as follows:

$$g_{i1}^{(\tau)} = (1-\phi)^{i-1}\rho^{i-1}g(\tau-i)/f(\tau)$$

$$= \left\{(1-\phi)^{i-1}\rho^{i-1}/g(i-2)\right\} \left\{g(i-2)g(\tau-i)/f(\tau)\right\}$$

If i is fixed, then it is clear that $g_{i1}^{(\tau)}$ is convergent from the convergency of $g(\tau-i)/f(\tau)$. If i is not fixed but τ-i is fixed, then define $k = \tau-i$, and we have

$$g_{i1}^{(\tau)} = \left\{(1-\phi)^{\tau-k-1}\rho^{\tau-k-1}/g(\tau-k-2)\right\} \left\{g(\tau-k-2)g(k)/f(\tau)\right\}$$

Since $(1-\phi)^{\tau-k-1}\rho^{\tau-k-1}/g(\tau-k-2)$ converges to zero as τ goes to infinity and

$$g(\tau-k-2)g(k)/f(\tau) = g(\tau-\ell)g(\ell-2)/f(\tau) = g_{\ell\ell}^{(\tau)}$$

where $\ell = k+2$, also converges to some limit value when τ tends to infinity, $g_{i1}^{(\tau)}$ converges to zero when τ goes to infinity.

Thus all off-diagonal elements converge to each limit value as τ tends to infinity and this completes the proof of Theorem 4.

Now that the convergency of Γ^{*-1} has been proven, we next evaluate the limit value of each element of Γ^{*-1}. Let us begin with the diagonal elements of Γ^{*-1} We already know that

$$g_{11}^{(\tau)} = g_{\tau+1,\tau+1}^{(\tau)} = g(\tau-1)/f(\tau)$$

Hence Property (7) in the preceding subsection directly yields the following limit value of $g_{11}^{(\tau)} = g_{\tau+1,\tau+1}^{(\tau)}$, that is,

$$\lim_{\tau\to\infty} g_{11}^{(\tau)} = \lim_{\tau\to\infty} g_{\tau+1,\tau+1}^{(\tau)} = \lim_{\tau\to\infty} g(\tau-1)/f(\tau) = C_1/\left\{C_1 g(0) - (1-\phi)^2\rho^2\right\}$$

$$= 1/\left\{g(0) - (1-\phi)^2\rho^2 C_1^{-1}\right\}$$

For i such that $2 \leq i \leq \tau$, the principal diagonal elements of Γ^{*-1} can be represented by

$$g_{ii}^{(\tau)} = g(i-2)g(\tau-i)/f(\tau)$$

If we put $k = \tau-i+2$, we have

$$g(i-2)g(\tau-i) = g(k-2)g(\tau-k)$$

Hence, without loss of generality, we can assume i is finitely fixed. Then

$$\lim_{\tau\to\infty} g_{ii}^{(\tau)} = g(i-2) \lim_{\tau\to\infty} g(\tau-i)/f(\tau)$$

Since

$$g(\tau-i)/f(\tau) = \frac{g(\tau-i)/g(\tau-1)}{g(0)-(1-\phi)^2\rho^2 g(\tau-2)/g(\tau-1)}$$

and we already know that $\lim_{\tau\to\infty} g(\tau-2)/g(\tau-1) = C_1^{-1}$, it suffices to evaluate the limit value of $g(\tau-i)/g(\tau-1)$. Since

$$g(\tau-i)/g(\tau-1) = \left\{(\alpha-\beta)(a+b)^{\tau-i}+(\alpha+\beta)(a-b)^{\tau-i}\right\}/\left\{(\alpha-\beta)(a+b)^{\tau-1}+(\alpha+\beta)(a-b)^{\tau-1}\right\}$$

$$= \frac{(\alpha-\beta)(a+b)^{\tau-i}}{(\alpha-\beta)(a+b)^{\tau-1}} \cdot \frac{1+\frac{\alpha+\beta}{\alpha-\beta}\left(\frac{a-b}{a+b}\right)^{\tau-i}}{1+\frac{\alpha+\beta}{\alpha-\beta}\left(\frac{a-b}{a+b}\right)^{\tau-1}}$$

and $\lim_{\tau\to\infty} \left(\frac{a-b}{a+b}\right)^{\tau-1} = \lim_{\tau\to\infty} \left(\frac{a-b}{a+b}\right)^{\tau-i} = 0$, because $a+b > a-b > 0$, we have

$$\lim_{\tau\to\infty} g(\tau-i)/g(\tau-1) = (a+b)^{-(i-1)} = C_1^{-(i-1)}$$

Hence we have

$$\lim_{\tau\to\infty} g_{ii}^{(\tau)} = g(i-2)\ C_1^{-(i-1)}/\left\{g(0) - (1-\phi)^2\rho^2C_1^{-1}\right\}$$

$$= \left\{A_1C_1^{-1} + A_2C_2^{-1}\left(\frac{C_2}{C_1}\right)^{i-1}\right\}/\left\{g(0) - (1-\phi)^2\rho^2C_1^{-1}\right\}$$

Further,

$$\lim_{i\to\infty}\lim_{\tau\to\infty} g_{ii}^{(\tau)} = A_1C_1^{-1}/\left\{g(0) - (1-\phi)^2\rho^2C_1^{-1}\right\}$$

It can be easily shown that this gives the limit value of the element which locates at the central position of the principal diagonal of Γ^{*-1}.

Let us consider about the bottom row elements of Γ^{*-1} next, which is represented as

$$g_{\tau+1,\tau-i}^{(\tau)} = (1-\phi)^{i+1}\rho^{i+1}g(\tau-i-2)/f(\tau)$$

for $2 \le \tau-i \le \tau$, or for $2 \le j \le \tau$,

$$g_{\tau+1,j}^{(\tau)} = (1-\phi)^{\tau-j+1}\rho^{\tau-j+1}g(j-2)/f(\tau).$$

If we assume $k = \tau - i$ is fixed, then we have

$$g^{(\tau)}_{\tau+1,\tau-i} = (1-\emptyset)^{\tau-k+1}\rho^{\tau-k+1}g(k-2)/f(\tau)$$

while

$$g^{(\tau)}_{\tau+1,j} = (1-\emptyset)^{\ell+1}\rho^{\ell+1}g(\tau-\ell-2)/f(\tau)$$

where $\ell = \tau - j$ which is fixed.

Hence, assuming i or j being fixed in each case, it suffices to consider $g^{(\tau)}_{\tau+1,i}$ and $g^{(\tau)}_{\tau+1,\tau-j}$, respectively.

From the result just obtained above, we have

$$\lim_{\tau\to\infty} g(\tau-i-2)/f(\tau) = C_1^{-(i+1)}/\left\{g(0) - (1-\emptyset)^2\rho^2 C_1^{-1}\right\}$$

Hence

$$\lim_{\tau\to\infty} g^{(\tau)}_{\tau+1,\tau-i} = (1-\emptyset)^{i+1}\rho^{i+1}C_1^{-(i+1)}/\left\{g(0) - (1-\emptyset)^2\rho^2 C_1^{-1}\right\}$$

Meanwhile, since

$$g^{(\tau)}_{\tau+1,j} = \left\{(1-\emptyset)^{\tau-j+1}\rho^{\tau-j+1}/g(\tau-j-2)\right\}\left\{g(j-2)g(\tau-j-2)/f(\tau)\right\}$$

$$= \left\{(1-\emptyset)^{\tau-j+1}\rho^{\tau-j+1}/g(\tau-j-2)\right\} \cdot \left\{g^{(\tau)}_{jj}\right\}$$

and $\lim_{\tau\to\infty} (1-\emptyset)^{\tau-j+1}\rho^{\tau-j+1}/g(\tau-j-2) = 0$ by Lemma 5.5, we have

$$\lim_{\tau\to\infty} g^{(\tau)}_{\tau+1,j} = 0$$

Further, since $C_1^2 > C_1C_2 = (1-\phi)^2\rho^2$,

$$\left\{(1-\phi)\rho/C_1\right\}^2 < 1$$

and so that

$$\lim_{i\to\infty}\lim_{\tau\to\infty} g^{(\tau)}_{\tau+1,\tau-i} = \lim_{\tau\to\infty} g^{(\tau)}_{\tau+1,j} = 0$$

because $\lim_{i\to\infty} (1-\phi)^{i+1}\rho^{i+1}C_1^{-(i+1)} = 0.$

If $j = 1$, from

$$g^{(\tau)}_{\tau+1,1} = (1-\phi)^2\rho^2/f(2) = \left\{(1-\phi)^2\rho^2/g(\tau-1)\right\}\left\{g(\tau-1)/f(\tau)\right\},$$

and $\lim_{\tau\to\infty}\left\{(1-\phi)^2\rho^2/g(\tau-1)\right\} = 0$, it directly follows that

$$\lim_{\tau\to\infty} g^{(\tau)}_{\tau+1,1} = 0 .$$

Finally let us consider about other off-diagonal elements. It is already shown in the proof of Theorem 4 that $g^{(\tau)}_{ij}$ converges to zero if i and $\tau-j$ are both fixed and τ goes to infinity, or j and $\tau-i$ are both fixed and τ goes to infinity. We also know that both $\tau-i$ and $\tau-j$ being fixed implies $i-j$ being fixed which is also implied by both i and j being fixed. Hence as long as $i-j$ is fixed, whether both i and j are fixed or both $\tau-i$ and $\tau-j$ are fixed is a matter of convenience. Because if we assume that $k = \tau-i+2$ and $\ell = \tau-j+2$ are both fixed, then we have

$$g_{ij}^{(\tau)} = (1-\phi)^{\ell-k}\ \rho^{\ell-k}\ g(k-2)\ g(\tau-\ell) = g_{\ell k}^{(\tau)}$$

Hence we will assume that both i and j are fixed. Then for $i > j \geq 2$, we have

$$\lim_{\tau\to\infty} g_{ij}^{(\tau)} = (1-\phi)^{i-j}\ \rho^{i-j}\ g(j-2) \lim_{\tau\to\infty} g(\tau-i)/f(\tau)$$

$$= (1-\phi)^{i-j}\ \rho^{i-j}\ g(j-2)C_1^{-(i-1)} / \left\{g(0) - (1-\phi)^2\ \rho^2 C_1^{-1}\right\}$$

$$= (1-\phi)^{i-j}\ \rho^{i-j}\ C_1^{-(i-j+1)} \left\{A_1 + A_2(C_2/C_1)^{j-2}\right\}/\left\{g(0) - (1-\phi)^2\rho^2 C_1^{-1}\right\} .$$

If $j = 1$, we have

$$g_{1i}^{(\tau)} = g_{i1}^{(\tau)} = (1-\phi)^{i-1}\ \rho^{i-1}\ g(\tau-i)/f(\tau)$$

Hence, for finitely fixed i, we have

$$\lim_{\tau\to\infty} g_{i1}^{(\tau)} = (1-\phi)^{i-1}\ \rho^{i-1}/\left\{g(0) - (1-\phi)^2\ \rho^2\ C_1^{-1}\right\}$$

Further, since $|(1-\phi)\rho| < 1$, we have

$$\lim_{i\to\infty} \lim_{\tau\to\infty} g_{i1}^{(\tau)} = 0.$$

Thus we have the following theorem.

<u>Theorem 5</u>. Each element of Γ^{*-1} has the following limit value respectively:

$$\lim_{\tau\to\infty} g_{11}^{(\tau)} = \lim_{\tau\to\infty} g_{\tau+1,\tau+1}^{(\tau)} = 1/\left\{g(0) - (1-\phi)^2\ \rho^2\ C_1^{-1}\right\} ,$$

$$\lim_{\tau\to\infty} g_{ii}^{(\tau)} = \lim_{\tau\to\infty} g_{\tau-i+2,\tau-i+2}^{(\tau)} = \left\{A_1 C_1^{-1} + A_2 C_2^{-1} \left(\frac{C_2}{C_1}\right)^{i-1}\right\}/\left\{g(0) - (1-\phi)^2\ \rho^2\ C_1^{-1}\right\} ,$$

where $2 \leq i \leq \tau$,

$$\lim_{\tau\to\infty} g^{(\tau)}_{\tau+1,\tau-i} = \lim_{\tau\to\infty} g^{(\tau)}_{\tau-i,\tau+1} = (1-\phi)^{i+1} \rho^{i+1} C_1^{-(i+1)}/\{g(0) - (1-\phi)^2 \rho^2 C_1^{-1}\} ,$$

where $0 \leq i \leq \tau-2$,

$$\lim_{\tau\to\infty} g^{(\tau)}_{\tau+1,j} = \lim_{i\to\infty} \lim_{\tau\to\infty} g^{(\tau)}_{\tau+1,\tau-i} = 0$$

where j is finitely fixed and $j \geq 1$ and $\tau > i$,

$$\lim_{\tau\to\infty} g^{(\tau)}_{1i} = \lim_{\tau\to\infty} g^{(\tau)}_{i1} = (1-\phi)^{i-1} \rho^{i-1}/\{g(0) - (1-\phi)^2 \rho^2 C_1^{-1}\}$$

$$\lim_{\tau\to\infty} g^{(\tau)}_{ij} = \lim_{\tau\to\infty} g^{(\tau)}_{ji} = (1-\phi)^{i-j} \rho^{i-j} C_1^{-(i-j+1)} \{A_1 + A_2(C_2/C_1)^{j-2}\}/$$

$$\{g(0) - (1-\phi)^2 \rho^2 C_1^{-1}\}$$

where $i > j \geq 2$, and

$$\lim_{i\to\infty} \lim_{\tau\to\infty} g^{(\tau)}_{ij} = \lim_{i\to\infty} \lim_{\tau\to\infty} g^{(\tau)}_{ji} = 0,$$

where $j \geq 1$ and j is finitely fixed.

So far, throughout this paper, we have assumed that $0 < \phi < 1$. We will consider now the limit value of each element of Γ^{*-1} when ϕ approaches zero or 1.

Directly substituting 0 or 1 for ϕ, we have the following results:

$$\lim_{\phi\to 0} C_1 = 1, \qquad \lim_{\phi\to 0} C_2 = \rho^2 ,$$

$$\lim_{\phi\to 1} C_1 = 1-\rho^2, \qquad \lim_{\phi\to 1} C_2 = 0,$$

$$\lim_{\phi\to 0} g(0) = 1, \qquad \lim_{\phi\to 1} g(0) = 1-\rho^2,$$

$$\lim_{\phi\to 0}\left\{g(0) - (1-\phi)^2 \rho^2 C_1^{-1}\right\} = \lim_{\phi\to 1}\left\{g(0) - (1-\phi)^2\rho^2 C_1^{-1}\right\} = 1-\rho^2,$$

$$\lim_{\phi\to 0} g(1) = 1, \qquad \lim_{\phi\to 1} g(1) = (1-\rho^2)^2,$$

$$\lim_{\phi\to 0} A_1 = 1, \qquad \lim_{\phi\to 1} A_1 = 1-\rho^2,$$

$$\lim_{\phi\to 0} A_2 = \lim_{\phi\to 1} A_2 = 0,$$

and, from these results, we also have

$$\lim_{\phi\to 0} f(\tau) = 1-\rho^2, \qquad \lim_{\phi\to 1} f(\tau) = (1-\rho^2)^{\tau+1},$$

and, for any t,

$$\lim_{\phi\to 0} g(t) = 1 \qquad \text{and} \qquad \lim_{\phi\to 1} g(t) = (1-\rho^2)^{t+1}.$$

From these results, it follows that

$$\lim_{\phi\to 0} g_{ii}^{(\tau)} = \lim_{\phi\to 1} g_{ii}^{(\tau)} = 1/(1-\rho^2)$$

for any i such that $1 \leq i \leq \tau+1$,

$$\lim_{\phi\to 0} g_{ij}^{(\tau)} = \rho^{i-j}/(1-\rho^2)$$

for any i,j such that $1 \leq j < i \leq \tau+1$, and

$$\lim_{\substack{\phi>1 \\ \phi\to 1}} g_{ij}^{(\tau)} = 0$$

for any i,j such that $1 \leq j < i \leq \tau+1$.

Note that Γ^* is non-singular even when $\phi = 0$ or $\phi = 1$ as long as $\rho^2 \neq 1$. So that the limit value of Γ^{*-1} just obtained above when ϕ goes to 0 or 1 actually gives the element of Γ^{*-1} when $\phi = 0$ or $\phi = 1$ respectively.

Since

$$E(\hat{\mu} - \mu)(\hat{\mu} - \mu)' = \frac{\sigma^2}{n}(1-\rho^2)\Gamma^{*-1}$$

our results imply that, when $\phi = 0$ or $\phi = 1$, the variance of MVUL estimator $\hat{\mu}(t)$ of $\mu(t)$ is equal to σ^2/n and the correlation coefficient between $\hat{\mu}(i)$ and $\hat{\mu}(j)$, where $i \neq j$, is equal to $\rho^{|i-j|}$ when $\phi = 0$ and equal to 0 when $\phi = 1$. And further, this implies that MVUL estimator of $\mu(t)$ is given by simple arithmetic mean of sample observations on t^{th} observations only, when $\phi = 0$ or $\phi = 1$. So our procedure derived in this paper can be applicable even when $\phi = 0$ or $\phi = 1$.

This fact can be used effectively to prove the existence of optimal ϕ value for given ρ and τ. Suppose $\phi = 1/2$. Let us derive $g_{11}^{(\tau)} = g_{\tau+1,\tau+1}^{(\tau)}$ for this value of ϕ. Note that

$$g_{11}^{(\tau)} = g_{\tau+1,\tau+1}^{(\tau)} = g(\tau-1)/f(\tau) = g(\tau-1)/\left\{(1-\phi\rho^2)\ g(\tau-1) - (1-\phi)^2\ \rho^2\ g(\tau-2)\right\}$$

$$= 1/\left\{(1-\phi\rho^2) - (1-\phi)^2\ \rho^2\ g(\tau-2)/g(\tau-1)\right\}$$

$$= 1/\left[(1-\phi\rho^2) - 2(1-\phi)^2\ \rho^2\left\{1 + \frac{A_2}{A_1}\left(\frac{C_2}{C_1}\right)^{\tau-2}\right\}/\left\{2C_1 + \frac{A_2}{A_1}\left(\frac{C_2}{C_1}\right)^{\tau-2} C_2\right\}\right]$$

and also note that $A_1 > A_2 > 0$ and $C_1 > C_2 > 0$. Thus we know that

$$1 + \frac{A_2}{A_1}\left(\frac{C_2}{C_1}\right)^{T-2} > 0 \text{ and } 2C_1 + \frac{A_2}{A_1}\left(\frac{C_2}{C_1}\right)^{T-2} \cdot 2C_2 > 0.$$

for $\phi = 1/2$, we have

$$C_1 = (1 + \sqrt{1 - \rho^2})/2 \text{ and } C_2 = (1 - \sqrt{1 - \rho^2}/2 \,.$$

Hence, from the fact that

$$\left\{2C_1 + 2\frac{A_2}{A_1}\left(\frac{C_2}{C_1}\right)^{T-2} C_2\right\} - \left\{1 + \frac{A_2}{A_1}\left(\frac{C_2}{C_1}\right)^{T-2}\right\} = (2C_1 - 1) + (2C_2 - 1) \cdot \frac{A_2}{A_1}\left(\frac{C_2}{C_1}\right)^{T-2}$$

$$= \sqrt{1-\rho^2}\left\{1 - \frac{A_2}{A_1}\left(\frac{C_2}{C_1}\right)^{T-2}\right\} > 0,$$

we obtain

$$\left\{1 + \frac{A_2}{A_1}\left(\frac{C_2}{C_1}\right)^{T-2}\right\} / \left\{2C_1 + 2\frac{A_2}{A_1}\left(\frac{C_2}{C_1}\right)^{T-2}\right\} < 1.$$

Meanwhile, since

$$(1-\phi\rho^2) = 1 - \rho^2/2 \quad\text{and}\quad 2(1-\phi)^2\,\rho^2 = \rho^2/2$$

for $\phi = 1/2$, we have

$$g_{11}^{(\tau)} = g_{\tau+1,\tau+1}^{(\tau)} = \left[(1 - \rho^2/2) - \rho^2/2 \cdot \left\{1 + \frac{A_2}{A_1}\left(\frac{C_2}{C_1}\right)^{\tau-2}\right\} \Big/ \left\{2C_1 + 2\frac{A_2}{A_1}\left(\frac{C_2}{C_1}\right)^{\tau-2} C_2\right\}\right]$$

$$< \left\{1 - \rho^2/2 - \rho^2/2\right\}^{-1}$$

$$= (1 - \rho^2)^{-1}.$$

This implies that the variance of $\hat{\mu}(z)$ is less than σ^2/n for $\phi = \frac{1}{2}$.

Since $f(\tau) > 0$ for any ϕ, $0 \leq \phi \leq 1$, providing that $\rho^2 \neq 1$ and $g(\tau-1)$ and $f(\tau)$ are both unique functions of ϕ, $g_{11}^{(\tau)} = g_{\tau+1,\tau+1}^{(\tau)}$ is a continuous function of ϕ in $0 \leq \phi \leq 1$. Further since $g_{11}^{(\tau)} = g_{\tau+1,\tau+1}^{(\tau)}$ is bounded in the same domain, the fact that

$$g_{11}^{(\tau)} = g_{\tau+1,\tau+1}^{(\tau)} < 1/(1 - \rho^2)$$

implies that there exists such a value of ϕ that minimizes the value of $g_{11}^{(\tau)} = g_{\tau+1,\tau+1}^{(\tau)}$.

Furthermore, since $g_{11}^{(\tau)} = g_{\tau+1,\tau+1}^{(\tau)} > g_{ii}^{(\tau)}$ for any i such that $2 \leq i \leq \tau$, and $g_{ii}^{(\tau)}$ is continuous and bounded, the optimal ϕ exists for any objective function which is an increasing function of every $g_{ii}^{(\tau)}$, for instance, for any weighted mean of $g_{ii}^{(\tau)}$'s.

Though it is difficult to derive a general or exact solution for optimal ϕ even when τ is infinitely large, it is always possible to derive the optimal value of ϕ numerically for any given ρ or τ. Because, now that we already know the explicit representation of $g_{ii}^{(\tau)}$, which represents $g_{ii}^{(\tau)}$ as a function of ϕ, ρ, and τ, and since the number of feasible ϕ values is finite because ϕ must satisfy such restrictions that $0 < \phi < 1$, and $k = \phi(m+2)$ is an integer such that $1 < k < m+1$, and further m+2 is less than or equal to n, we can actually calculate the value of the objective function for every feasible ϕ, assuming course, we have access to a high speed computing system.

6. Concluding Remarks

We derived the MVUL estimator of population mean vector in a rotation sampling design, which is an alternative representation of classical Yates - Patterson's estimator and is nothing but a specification of Aitken's generalized (or weighted) least squares estimator.

Now that it can be assumed that the variance-covariance matrix of observations is known, it is possibly well-known fact to most statisticians that the MVUL estimator will be given by Aitken's generalized least square estimator. Our essential contribution here, henceforth, is considered to be that a specific representation, $\hat{\mu} = \Gamma^{*-1}V^*$, was given to this estimator through the explicit specification of the inverse of variance-covariance matrix. This representation is simpler than the usual representation of Aitken's estimator in the sense that Γ^* has much more reduced order than the original variance-covariance matrix which must be used if we would work through the usual and general formula of Aitken's estimator.

Further, the explicit representation of each element of Γ^{*-1} which was also given in this paper makes it possible to give exact proof of convergency of Γ^{*-1} and hence convergency of the variance-covariance matrix of estimated mean vector as has actually been carried out in this paper. And it also provides a way to solve numerically the optimal replacement ratio, $\emptyset$, in a more general and more exact manner than those which has been advocated so far, because such optimal $\emptyset$ can be derived, according to our method, for more general classes of objective functions. This derivation can be done not only for the asymptotic variance of estimator when τ is infinitely large but also the exact variance formula for finite τ.

It is true, however, that Yates - Patterson's method still has great merits especially when estimating only the current population mean, in other words, population mean on the last occasion. Because in this case, according to Yates-Patterson's method, we do not need to keep records of all previous observations, as is required by our method, but we have only to keep records of observations, the MVUL estimator, and coefficient φ_{h-1} obtained on the last preceding occasion. The computation procedure is so simple that we hardly need to use a computer. Furthermore, through this simple procedure, we actually use all information in the past, which can be even the infinite past if we have really kept estimating since that infinitely past occasion. But there are many surveys which propose to provide a time series of estimates for some specified period. So the importance to improve the precision of estimates on previous occasions still exists in many practical situations. We believe that one of our method's merits is that it provides improved estimates on all previous occasions simultaneously when the MVUL estimate on the last occasion is obtained and this procedure, furthermore, looks even simpler than the one proposed by Patterson.

As a matter of fact, however, if the number of occasions is too large, the inversion of Γ^* becomes practically impossible. But the convergency of Γ^{*-1} makes it possible to restrict the number of occasions to some large but still finite number so as to be permissible to computer capacity. It is also possible to save records by using the formula of explicit representation of each element of Γ^{*-1}, though it might increase the computing hours.

Finally, derivation of the inverse of variance-covariance matrix, which was done in the course of derivation of our estimation formula, makes it possible to derive the maximum likelihood estimators of ρ and σ^2 (assumed known herein) assumir multivariate normal distribution. Another paper on this topic is under preparatio

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