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GRADIENT METHODS IN CONTROL THEORY
PART 6 - COMBINED GRADIENT-RESTORATION ALGORITHM

by

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Gradient Methods in Control Theory

Part 6 - Combined Gradient-Restoration Algorithm[1]

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Abstract. This paper considers the problem of minimizing a functional I which depends on the state x(t), the control u(t), and the parameter π. Here, I is a scalar, x an n-vector, u an m-vector, and π a p-vector. At the initial point, the state x is prescribed. At the final point, the state x and the parameter π are required to satisfy q scalar relations. Along the interval of integration, the state, the control, and the parameter are required to satisfy n scalar differential equations. A combined gradient-restoration algorithm is presented: this is an iterative algorithm characterized by variations $\Delta x(t)$, $\Delta u(t)$, $\Delta \pi$ leading toward the minimal condition while simultaneously leading toward constraint satisfaction. These variations are computed by minimizing the first-order change of the functional subject to the linearized differential equations, the linearized boundary conditions, and a quadratic constraint on the variations of the control and the parameter. The resulting linear, two-point boundary-value problem is solved via the method of particular solutions. The descent properties of the algorithm are studied, and schemes to determine the optimum stepsize are discussed.

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1. Introduction

In previous papers (Refs. 1-5), the problem of minimizing a functional subject to certain differential constraints and boundary conditions was considered. A sequential gradient-restoration algorithm was presented. This algorithm includes the alternate succession of gradient phases and restoration phases. In the gradient phase, one lowers the value of the functional while avoiding excessive violation of the differential constraints and the boundary conditions. In the restoration phase, one restores the differential constraints and the boundary conditions to a predetermined accuracy while avoiding excessive change in the value of the functional.

In this paper, the above problem is considered once more. The main idea is to develop a combined gradient-restoration algorithm, that is, an iterative algorithm in which the gradient phase and the restoration phase are combined in a single phase. Therefore, one generates variations leading toward the minimal condition while simultaneously leading toward constraint satisfaction.

2. Statement of the Problem

The purpose of this paper is to study the minimization of the functional

$$I = \int_0^1 f(x, u, \pi, t)dt + [g(x, \pi)]_1 \tag{1}$$

with respect to the functions x(t), u(t) and the parameter π which satisfy the differential constraint

$$\dot{x} - \varphi(x, u, \pi, t) = 0 \tag{2}$$

the initial condition

$$(x)_0 = \text{given} \tag{3}$$

and the final condition

$$[\psi(x, \pi)]_1 = 0 \tag{4}$$

In the above equations, the functions f and g are scalar, the function φ is an n-vector, and the function ψ is a q-vector. The symbol x, an n-vector, denotes the state variable; the symbol u, an m-vector, denotes the control variable; and the symbol π, a p-vector, denotes the parameter. The time t, a scalar, is the independent variable; without loss of generality, the prescribed initial time is $t = 0$ and the prescribed final time is $t = 1$.

At the initial point, all the components of the state vector are given, so that $(x)_0$ is known. At the final point, q scalar relations are specified, where $0 \leq q \leq n + p$. Problems where the final time is other than unity can be reduced to the form (1)-(4) by normalizing the time with respect to the final time and by regarding the final time, if it is free, as one of the components of the parameter π.

3. Exact First-Order Conditions

From calculus of variations (see, for instance, Refs. 6-7), it is known that the previous problem is one of the Bolza type. It can be recast as that of minimizing the augmented functional

$$J = \int_0^1 F dt + (G)_1 \tag{5}$$

subject to (2)-(4). In the above expression, the functions F and G are given by

$$F = f + \lambda^T(\dot{x} - \varphi) \quad , \quad G = g + \mu^T \psi \tag{6}$$

where λ, an n-vector, is a variable Lagrange multiplier and μ, a q-vector, is a constant Lagrange multiplier. The superscript T denotes the transpose of a matrix.

The optimum solutions x(t), u(t), π must satisfy (2)-(4), the Euler equations

$$(d/dt)F_{\dot{x}} = F_x \quad , \quad 0 = F_u \quad , \quad \int_0^1 F_\pi dt + (G_\pi)_1 = 0 \tag{7}$$

and the following natural condition arising from the transversality condition:

$$(F_{\dot{x}} + G_x)_1 = 0 \tag{8}$$

On account of (6), the explicit form of Eqs. (7)-(8) is the following:

$$\begin{aligned} \dot{\lambda} &= f_x - \varphi_x \lambda \\ 0 &= f_u - \varphi_u \lambda \\ 0 &= \int_0^1 (f_\pi - \varphi_\pi \lambda) dt + (g_\pi + \psi_\pi \mu)_1 \end{aligned} \tag{9}$$

and

$$(\lambda + g_x + \psi_x \mu)_1 = 0 \tag{10}$$

Summarizing, we seek the functions x(t), u(t), λ(t) and the parameters π, μ which satisfy Eqs. (2) and (9) subject to the boundary conditions (3), (4), (10).

4. Approximate Methods

In general, the differential system (2)-(4) and (9)-(10) is nonlinear; consequently, approximate methods must be employed. These methods are of two kinds: first-order methods and second-order methods.

Within the context of this paper, let the norm of a vector a be defined as

$$N(a) = a^T a \tag{11}$$

where the superscript T denotes the transpose of a matrix. Let the functionals P and Q be defined as

$$P = \int_0^1 N(\dot{x} - \varphi)dt + N(\psi)_1 \tag{12}$$

and

$$Q = \int_0^1 N(\dot{\lambda} - f_x + \varphi_x \lambda)dt + \int_0^1 N(f_u - \varphi_u \lambda)dt$$

$$+ N\left[\int_0^1 (f_\pi - \varphi_\pi \lambda)dt + (g_\pi + \psi_\pi \mu)_1\right] + N(\lambda + g_x + \psi_x \mu)_1 \tag{13}$$

These functionals measure the cumulative errors in the constraints and optimum conditions, respectively. We observe that $P = 0$ and $Q = 0$ for the exact variational solution, while $P > 0$ and $Q > 0$ for any approximation to the variational solution.

When approximate methods are used, they must ultimately lead to functions $x(t)$, $u(t)$, $\lambda(t)$ and parameters π, μ such that

$$P \leq \epsilon_1 \tag{14}$$

and

$$Q \leq \epsilon_2 \tag{15}$$

where ϵ_1 and ϵ_2 are small, preselected numbers.

5. Derivation of the Algorithm

Suppose that nominal functions x(t), u(t), π not satisfying the differential equation (2), the initial condition (3), and the final condition (4) are available. Let $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ denote varied functions satisfying Eqs. (2)-(4) to first order. These varied functions are related to the nominal functions as follows:

$$\tilde{x}(t) = x(t) + \Delta x(t) \quad , \quad \tilde{u}(t) = u(t) + \Delta u(t) \quad , \quad \tilde{\pi} = \pi + \Delta\pi \tag{16}$$

where $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ denote the perturbations of x, u, π about the nominal values.

To first order, the values of the varied functional $\tilde{I}$ and the nominal functional I are related by

$$\tilde{I} = I + \delta I \tag{17}$$

where the first variation δI is given by

$$\delta I = \int_0^1 (f_x^T \Delta x + f_u^T \Delta u + f_\pi^T \Delta\pi) dt + (g_x^T \Delta x + g_\pi^T \Delta\pi)_1 \tag{18}$$

Also to first order, Eq. (2) can be approximated by

$$\Delta\dot{x} - \varphi_x^T \Delta x - \varphi_u^T \Delta u - \varphi_\pi^T \Delta\pi + (\dot{x} - \varphi) = 0 \tag{19}$$

while the boundary conditions (3)-(4) are written as

$$(\Delta x)_0 = 0 \tag{20}$$

and

$$(\psi + \psi_x^T \Delta x + \psi_\pi^T \Delta\pi)_1 = 0 \tag{21}$$

For convenience, we introduce the scaling factor β such that

$$0 \leq \beta \leq 1 \tag{22}$$

and imbed Eqs. (19)-(21) in the more general family

$$\Delta\dot{x} - \varphi_x^T \Delta x - \varphi_u^T \Delta u - \varphi_\pi^T \Delta\pi + \beta(\dot{x} - \varphi) = 0$$

$$(\Delta x)_0 = 0 \tag{23}$$

$$(\beta\psi + \psi_x^T \Delta x + \psi_\pi^T \Delta\pi)_1 = 0$$

Next, we consider the following quadratic constraint on the variations $\Delta u(t)$, $\Delta\pi$:

$$K = \int_0^1 \Delta u^T \Delta u \, dt + \Delta\pi^T \Delta\pi \tag{24}$$

where K is a constant prescribed a priori. With this understanding, we formulate the following problem: Find the variations $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ which minimize (18) subject to (23)-(24).

5.1. Variational Approach. From calculus of variations (see, for instance, Refs. 6-7), it is known that the previous problem is one of the Bolza type with an added isoperimetric condition on the variations of the control and the parameter. It can be recast as that of minimizing the functional

$$J^* = \int_0^1 F^* dt + (G^*)_1 \tag{25}$$

subject to (23)-(24). In the above expression, the functions F^* and G^* are given by

$$F^* = f_x^T \Delta x + f_u^T \Delta u + f_\pi^T \Delta\pi + \lambda^T [\Delta\dot{x} - \varphi_x^T \Delta x - \varphi_u^T \Delta u - \varphi_\pi^T \Delta\pi + \beta(\dot{x} - \varphi)]$$
$$+ (1/2\alpha)\Delta u^T \Delta u$$

(26)

$$G^* = g_x^T \Delta x + g_\pi^T \Delta\pi + \mu^T(\beta\psi + \psi_x^T \Delta x + \psi_\pi^T \Delta\pi) + (1/2\alpha)\Delta\pi^T \Delta\pi$$

where the n-vector λ is a variable Lagrange multiplier, the q-vector μ is a constant Lagrange multiplier, and the scalar $1/2\alpha$ is a constant Lagrange multiplier. The quantity α is called the stepsize.

The optimum solutions $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ must satisfy Eqs. (23)-(24), the Euler equations

$$(d/dt)F^*_{\Delta\dot{x}} = F^*_{\Delta x} \quad , \quad 0 = F^*_{\Delta u} \quad , \quad \int_0^1 F^*_{\Delta\pi} dt + (G^*_{\Delta\pi})_1 = 0 \tag{27}$$

and the following natural condition arising from the transversality condition:

$$(F^*_{\Delta\dot{x}} + G^*_{\Delta x})_1 = 0 \tag{28}$$

On account of (26), the explicit form of Eqs. (27)-(28) is the following:

$$\dot{\lambda} = f_x - \varphi_x \lambda$$
$$0 = f_u - \varphi_u \lambda + \Delta u/\alpha \tag{29}$$
$$0 = \int_0^1 (f_\pi - \varphi_\pi \lambda) dt + (g_\pi + \psi_\pi \mu)_1 + \Delta\pi/\alpha$$

and

$$(\lambda + g_x + \psi_x \mu)_1 = 0 \tag{30}$$

5.2. <u>Coordinate Transformation.</u> To simplify the problem, we introduce the auxiliary variables

$$A = \Delta x/\alpha \quad , \quad B = \Delta u/\alpha \quad , \quad C = \Delta\pi/\alpha \tag{31}$$

where A denotes an n-vector proportional to the state change, B denotes an m-vector proportional to the control change, and C denotes a p-vector proportional to the parameter change. With these variables, Eqs. (23-1) and (29) become

$$\begin{aligned} \dot{A} &= \varphi_x^T A + \varphi_u^T B + \varphi_\pi^T C - \rho(\dot{x} - \varphi) \\ \dot{\lambda} &= f_x - \varphi_x \lambda \\ B &= - f_u + \varphi_u \lambda \\ C &= -\int_0^1 (f_\pi - \varphi_\pi \lambda) dt - (g_\pi + \psi_\pi \mu)_1 \end{aligned} \tag{32}$$

and the boundary conditions (23-2), (23-3), (30) are written as

$$(A)_0 = 0 \tag{33}$$

and

$$[\rho\psi + \psi_x^T A + \psi_\pi^T C]_1 = 0 \quad , \quad (\lambda + g_x + \psi_x \mu)_1 = 0 \tag{34}$$

where

$$\rho = \beta/\alpha \tag{35}$$

Finally, the isoperimetric condition (24) becomes

$$K = \alpha^2\left[\int_0^1 B^T B dt + C^T C\right] \tag{36}$$

Let β be proportional to α throughout the algorithm. That is, let the parameter ρ have a value assigned a priori. Under these conditions, the linear, nonhomogeneous differential system (32)-(34) can be solved without assigning a value to the stepsize α. Once the system (32)-(34) has been solved, the stepsize α can be determined from Eq. (36), since (36) establishes a correspondence between the values of the isoperimetric constant K and the values of the stepsize α. However, there is no way to determine a priori convenient values for the isoperimetric constant K; therefore, the implementation of the algorithm becomes simpler if one avoids evaluating α in terms of K and assigns values to α directly.

5.3. Integration Technique. We integrate the previous linear, nonhomogeneous differential system q + 1 times using a backward-forward integration scheme in combination with the method of particular solutions (Refs. 8-9). In each integration (subscript i), we assign a different set of values to the components of the multiplier μ, for instance,

$$\mu_i = \begin{bmatrix} \delta_{i1} \\ \delta_{i2} \\ \vdots \\ \delta_{iq} \end{bmatrix}, \qquad i = 1,2,\ldots,q+1 \tag{37}$$

where the Kronecker delta δ_{ij} is such that

$$\begin{aligned} \delta_{ij} &= 1 \ , \quad i = j \\ \delta_{ij} &= 0 \ , \quad i \neq j \end{aligned} \tag{38}$$

With μ_i specified, the corresponding multiplier λ_i at the final point is obtained from (34-2), that is, from

$$(\lambda_i + g_x + \psi_x \mu_i)_1 = 0 \ , \qquad i = 1, 2, \ldots, q+1 \tag{39}$$

Next, Eq. (32-2) is integrated backward q+1 times to yield the functions

$$\lambda_i = \lambda_i(t) \ , \qquad i = 1, 2, \ldots, q+1 \tag{40}$$

Then, the functions

$$B_i = B_i(t) \ , \qquad i = 1, 2, \ldots, q+1 \tag{41}$$

are computed from (32-3) and the parameters

$$C_i \ , \qquad i = 1, 2, \ldots, q+1 \tag{42}$$

are computed from (32-4). Subsequently, Eq. (32-1) is integrated forward q+1 times subject to the initial condition

$$(A_i)_0 = 0 \ , \qquad i = 1, 2, \ldots, q+1 \tag{43}$$

In this way, we obtain the functions

$$A_i = A_i(t) , \qquad i = 1, 2, \ldots, q+1 \tag{44}$$

which are characterized by final values generally not consistent with (34-1). Summarizing the q+1 particular solutions thus obtained satisfy Eqs. (32), (33), (34-2) but not (34-1).

Next, we introduce the q+1 undetermined, scalar constants k_i and form the linear combinations

$$A(t) = \sum_{i=1}^{q+1} k_i A_i(t) , \quad B(t) = \sum_{i=1}^{q+1} k_i B_i(t) , \quad \lambda(t) = \sum_{i=1}^{q+1} k_i \lambda_i(t) \tag{45}$$

and

$$C = \sum_{i=1}^{q+1} k_i C_i , \quad \mu = \sum_{i=1}^{q+1} k_i \mu_i \tag{46}$$

Then, we inquire whether, by an appropriate choice of the constants, these linear combinations can satisfy all the differential equations and boundary conditions. By simple substitution, it can be verified that (45)-(46) satisfy the differential equations (32), the initial condition (33), and the final condition (34-2) providing the constants k_i are such that

$$\sum_{i=1}^{q+1} k_i = 1 \tag{47}$$

Finally, the functions (45)-(46) satisfy the final condition (34-1) providing

$$\left[\rho\psi + \sum_{i=1}^{q+1} k_i(\psi_x^T A_i + \psi_\pi^T C_i)\right]_1 = 0 \tag{48}$$

The linear system (47)-(48) is equivalent to q+1 scalar equations: the unknowns are the q+1 constants k_i. In this way, the two-point boundary-value problem is solved.

After the quantities A(t), B(t), C have been determined and after a stepsize α has been selected (see Section 6), the variations $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ can be computed from (31) and the varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ from (16).

5.4. Descent Properties. After suitable manipulations, omitted for the sake of brevity, the first variation of the augmented functional (5) can be written in the form[3]

$$\delta J = -\alpha Q \tag{49}$$

where Q, given by Eq. (13), reduces to

$$Q = \int_0^1 B^T B dt + C^T C \tag{50}$$

Since $Q > 0$, Eq. (49) shows that the first variation δJ is negative for $\alpha > 0$. Therefore, if α is sufficiently small, the augmented functional J decreases during the gradient phase.

Next, consider the cumulative constraint error (12) and observe that the first variation of P is given by

$$\delta P = 2\int_0^1 (\dot{x} - \varphi)^T(\Delta\dot{x} - \varphi_x^T \Delta x - \varphi_u^T \Delta u - \varphi_\pi^T \Delta\pi)dt + 2[\psi^T(\psi_x^T \Delta x + \psi_\pi^T \Delta\pi)]_1 \tag{51}$$

In the light of (12), (23-1), and (23-3), Eq. (51) can be rewritten as

$$\delta P = -2\beta\left[\int_0^1 N(\dot{x} - \varphi)dt + N(\psi)_1\right] = -2\beta P \tag{52}$$

Equation (52) shows that the first variation of the cumulative constraint error is negative for $\beta > 0$. Note that β is proportional to α. Therefore, if α is sufficiently small, the cumulative constraint error decreases during any iteration.

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[3] In the computation of (49), the multipliers $\lambda(t)$ and μ are held constant.

Finally, define the augmented penalty functional (Ref. 10) as

$$W = J + kP \tag{53}$$

where $k \geq 0$ is the penalty constant. The first variation of (53) is given by

$$\delta W = \delta J + k\delta P \tag{54}$$

which, in the light of (49) and (52), can be written as

$$\delta W = -\alpha Q - 2\beta P \tag{55}$$

This equation shows that, for $\alpha > 0$ and $\beta > 0$, the first variation of the augmented penalty functional is negative. Note that β is proportional to α. Therefore, if α is sufficiently small, the augmented penalty functional decreases during any iteration.

6. Optimum Stepsize

If Eqs. (16) and (31) are combined, the varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ can be expressed as

$$\tilde{x}(t) = x(t) + \alpha A(t) , \quad \tilde{u}(t) = u(t) + \alpha B(t) , \quad \tilde{\pi} = \pi + \alpha C \tag{56}$$

Note that the nominal functions x(t), u(t), π are given and that the functions A(t), B(t), π are known from the solution of the linear, two-point boundary-value problem. Hence, Eq. (56) defines a one-parameter family of varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$, the parameter being the stepsize α. For this one parameter family, the functionals (5), (12), (53) yield the functions

$$\tilde{J} = \tilde{J}(\alpha) \quad , \quad \tilde{P} = \tilde{P}(\alpha) \quad , \quad \tilde{W} = \tilde{W}(\alpha) \tag{57}$$

Since these functionals have the descent properties (49), (52), (55), the functions (57) have a negative slope at $\alpha = 0$, specifically,

$$\tilde{J}_{\alpha}(0) = -\tilde{Q}(0) \quad , \quad \tilde{P}_{\alpha}(0) = -2\rho\tilde{P}(0), \quad \tilde{W}_{\alpha}(0) = -\tilde{Q}(0) - 2k\rho\tilde{P}(0) \tag{58}$$

Hence, any of the functions (57) can be employed in order to arrive at some desirable value of the stepsize. In this connection, several possible schemes are presented below.

Scheme I. Here, α is chosen to minimize the augmented functional $\tilde{J}(\alpha)$ subject to the inequalities

$$\tilde{P}(\alpha) < \tilde{P}(0) \quad \text{if} \quad \tilde{P}(0) \geq P_{*} \tag{59}$$

$$\tilde{P}(\alpha) < P_{*} \quad \text{if} \quad \tilde{P}(0) < P_{*} \tag{60}$$

where P_{*} is a preselected number.

Scheme II. Here, one searches alternatively on $\tilde{P}(\alpha)$ and $\tilde{J}(\alpha)$. The search on $\tilde{P}(\alpha)$ is subject to the inequality

$$\tilde{P}(\alpha) < \tilde{P}(0) \tag{61}$$

The search on $\tilde{J}(\alpha)$ is subject to Ineqs. (59)-(60).

Scheme III. In this scheme, which is a modification of Scheme II, one searches on $\tilde{P}(\alpha)$ or $\tilde{J}(\alpha)$ depending on the value of $\tilde{P}(0)$. Specifically, one searches on $\tilde{P}(\alpha)$ subject to (61) if

$$\tilde{P}(0) \geq P_{**} \tag{62}$$

And one searches on $\tilde{J}(\alpha)$ subject to (60) if

$$\tilde{P}(0) < P_{**} \tag{63}$$

Here, $P_{**} < P_{*}$ is a preselected number.

Scheme IV. In this scheme, which is a modification of Scheme II, one searches on $\tilde{P}(\alpha)$ or $\tilde{J}(\alpha)$ depending on the value of the ratio

$$\tilde{R}(0) = \tilde{P}(0)/\tilde{Q}(0) \tag{64}$$

Specifically, one searches on $\tilde{P}(\alpha)$ subject to (61) if

$$\tilde{R}(0) \geq R_{*} \tag{65}$$

And one searches on $\tilde{J}(\alpha)$ subject to (59)-(60) if

$$\tilde{R}(0) < R_{*} \tag{66}$$

Here, R_{*} is a preselected number.

Scheme V. Here, α chosen so as to minimize the augmented penalty functional,

$$\tilde{W}(\alpha) = \tilde{J}(\alpha) + k\tilde{P}(\alpha) \tag{67}$$

The symbol k denotes a preselected positive number which is constant throughout the algorithm. Optionally, the search can be subordinated to Ineqs. (59)-(60).

Scheme VI. In this scheme, which is a modification of Scheme V, one searches on the augmented penalty functional $\tilde{W}(\alpha)$, with this understanding: k is not constant throughout the algorithm but is updated with each step. For instance, one may choose

$$k = \tilde{P}(0) \tag{68}$$

or

$$k = |\tilde{P}(0)/\tilde{J}(0)| \tag{69}$$

or

$$k = |\tilde{J}(0)/\tilde{P}(0)| \tag{70}$$

The choice (70) produces values of $\tilde{J}(\alpha)$ and $k\tilde{P}(\alpha)$ having the same order of magnitude for small values of α.

Scheme VII. Here, the stepsize is set at the preselected value $\alpha = \alpha_*$. This value is accepted if any of the following inequalities is satisfied:

$$\tilde{P}(\alpha) < \tilde{P}(0) \quad \text{and} \quad \tilde{J}(\alpha) < \tilde{J}(0) \tag{71}$$

or

$$\tilde{P}(\alpha) < \tilde{P}(0) \quad \text{and} \quad |\tilde{J}(\alpha) - \tilde{J}(0)| \leq \epsilon_3 |\tilde{J}(0)| \tag{72}$$

or

$$\tilde{J}(\alpha) < \tilde{J}(0) \quad \text{and} \quad |\tilde{P}(\alpha) - \tilde{P}(0)| \leq \epsilon_4 \tilde{P}(0) \tag{73}$$

where ϵ_3 and ϵ_4 are small, preselected numbers. Otherwise, α is reduced (for example, with a bisection procedure) until any one of Ineqs. (71)-(73) is satisfied.

Remark. In Schemes I through VI, the search on the functionals $\tilde{J}(\alpha), \tilde{P}(\alpha), \tilde{W}(\alpha)$ can be performed via quasilinearization, quadratic interpolation, or cubic interpolation. Let $\tilde{Z}(\alpha)$ denote a generalized functional which can be $\tilde{J}(\alpha)$, $\tilde{P}(\alpha)$, $\tilde{W}(\alpha)$, depending on the scheme employed. The search is terminated when

$$\tilde{Z}_{\alpha}^{2}(\alpha) \leq \epsilon_5 \tag{74}$$

where ϵ_5 is a small, preselected number or when

$$|\alpha_2 - \alpha_1| \leq \epsilon_6 \alpha_2 \tag{75}$$

Here, ϵ_6 is a small, preselected number, and α_1, α_2 denote two consecutive optimum values of the stepsize.

Stopping Condition. The algorithm is terminated when the cumulative constraint error P satisfies Ineq. (14) and the cumulative error in the optimum conditions satisfies Ineq. (15).

7. Summary of the Combined Gradient-Restoration Algorithm

In this section, the combined gradient-restoration algorithm is summarized.

(a) Assume nominal functions $x(t)$, $u(t)$, π. Select a value for ρ.

(b) For the nominal functions, compute the vectors f_x, f_u, f_π and the matrices φ_x, φ_u, φ_π along the interval of integration.

(c) Integrate the differential system (32), (33), (34-2) q+1 times using a backward-forward integration scheme in combination with the method of particular solutions (see Section 5.3). Obtain the functions $A_i(t)$, $B_i(t)$, $\lambda_i(t)$ and the parameters C_i, μ_i, where $i = 1,2,\ldots,q+1$.

(d) Solve Eqs. (47)-(48) to obtain the constants k_i, where $i = 1,2,\ldots,q+1$.

(e) Using Eqs. (45)-(46), combine the particular solutions linearly and obtain the functions $A(t)$, $B(t)$, $\lambda(t)$ and the parameters C, μ.

(f) Once the functions $A(t)$, $B(t)$, $\lambda(t)$ and the parameters C, μ are known, compute the optimum stepsize α by a one-dimensional search on the generalized functional $\tilde{Z}(\alpha)$. This can be $\tilde{J}(\alpha)$, $\tilde{P}(\alpha)$, $\tilde{W}(\alpha)$, depending on the search scheme employed (see Section 6).

(g) Once the optimum stepsize α is known, compute the corrections $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ using Eqs. (31); then, obtain the varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ using Eqs. (16).

(h) With $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ known, the iteration is completed; the varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ become the nominal functions $x(t)$, $u(t)$, π for the next iteration; that is, return to (a), and iterate the algorithm.

(i) The algorithm is terminated when the stopping conditions (14)-(15) are satisfied.

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