170-00549

university of california • santa barbara

THE LIBRATIONAL DYNAMICS OF SATELLITES

Part I. The Motion of a Satellite With a Time-Dependent Inertia Tensor

Part II. The Librational Dynamics of a Composite Rigid-Elastic Satellite

by

J. E. Lingerfelt and T. P. Mitchell

UCSB-ME-70-9

College of Engineering

UCSB-ME-70-9

The Librational Dynamics of Satellites

Part I. The Motion of a Satellite With a Time-Dependent Inertia Tensor

Part II. The Librational Dynamics of a Composite Rigid-Elastic Satellite

by

J. E. Lingerfelt and T. P. Mitchell

A Technical Report to the
National Aeronautics and Space Administration

Grant No. NGR 05-010-020

January 30, 1970

Mechanical Engineering Department
University of California
Santa Barbara, California 93107

TABLE OF CONTENTS

PART I

Motion of a Satellite With Time Dependent Inertia Tensor

ABSTRACT

Conditions sufficient to guarantee the stability of the librational motion of a satellite with time varying inertia tensor are derived. These conditions involve the initial configuration, the rates of change of the principal moments of inertia and the total change in these parameters. Approximate solutions to the equations of motion are found in the special cases in which the inertial parameters vary rapidly or slowly relative to the librational period. The conditions for stability of these solutions are shown to be compatible with the criteria established in the stability analysis.

PART I

Motion of a Satellite With Time Dependent Inertia Tensor

Section 1. Introduction

The equation of planar libration for a satellite in orbit is

$$\frac{d}{dt}\,[C(\phi' + \omega)] + 3\omega^2(B-A)\sin\phi\cos\phi = 0 \qquad (1)$$

This equation assumes that the inertia tensor, whose elements are A, B and C, varies with time in such a way that the principal axes of inertia remain principal axes. The requirement that the z (or C) axis remains principal and, therefore, perpendicular to the orbit plane guarantees that an initially planar libration remains planar. The requirement that the x (or A) and the y (or B) axes retain their orientation relative to the orbit plane guarantees that the ϕ occuring in the first term of equation (1) is identical to that occurring in the second term.

Equation (1) is examined in some detail in this chapter. First, the stability of the solutions, where the motion is defined to be stable if it remains librational, is studied in section 2 where sufficient conditions for stability are derived. These conditions should prove useful as design criteria for gravity gradient stabilized satellites. In section 3 the case where deployment is rapid with respect to the orbital period is considered. In section 4, equation

(1) is studied for the case where the inertial parameters vary slowly with respect to an orbital period. Asmptotic techniques are utilized averaging in the first case about the linearized solution of the constant parameter case and in the second case, about the exact nonlinear solution of the constant parameter case. Closed form approximate solutions are presented for particular cases. In section 5 a power series solution is presented for the linearized equation.

Section 2: Stability of Motion of a Satellite With Time Dependent Inertia Tensor

The equation of planar libration is

$$\frac{d}{dt}\left[C\left(\frac{d}{dt}\varphi + \omega\right)\right] + 3\omega^2(B - A)\sin\varphi\cos\varphi = 0 \qquad (1)$$

which for a satellite in a circular orbit (ω = positive constant) may be written in the form

$$\ddot{x} + a(\tau)\dot{x} + b(\tau)\sin x = -2a(\tau) \qquad (2)$$

where

$$x = 2\varphi$$

$$a(\tau) = \frac{\dot{C}}{C}$$

$$b(\tau) = \frac{3(B - A)}{C}$$

$$\tau = \omega t$$

and the superscript dots denote differentiation with respect to the nondimensional time, τ. This equation is nonlinear, nonstationary (since the coefficients are nonconstant functions of time), and nonautonomous (since the right hand side is not identically zero). Very little is known concerning the solutions of such equations although G. Leitmann (1) has derived sufficient conditions for the stability of the nonlinear, nonstationary equation

$$a(t)\ddot{x} + b(t)\dot{x} + c(t)f(x) = 0 \qquad (3)$$

which includes the autonomous form of equation (2).

In this section the stability of the solutions of equation (2) are studied where stability is defined in a physical manner, viz.:

A solution to equation (2) is said to be stable if the corresponding motion of the satellite is librational as distinct from rotational.

Stability is examined about a stable equilibrium point and the coordinate axes are chosen so this equilibrium point is in the neighborhood of $x = 0$. This implies that the coordinate axes are chosen such that the stability requirement for three dimensional motion $C > B > A$ is satisfied. Therefore $(B - A) > 0$ and since C must be positive on physical grounds, $b(\tau) > 0$ for all τ.

Equation (2) may be written as a set of two first order differential equations

$$\dot{x} = z$$
$$\dot{z} = -a(\tau)z - b(\tau) \sin x - 2a(\tau) \qquad (4)$$

Consider the function $V(x,z,\tau)$ defined by

$$V = 2(1 - \cos x) + \frac{1}{b} z^2 + f(\tau) \qquad (5)$$

where $f(\tau)$ is to be defined later. Then $\frac{dV}{d\tau}$ following the motion is

$$\frac{dV}{d\tau} = 2z \sin x + \frac{2}{b} z \left[-az - b \sin x - 2a \right] - \frac{\dot{b}}{b^2} z^2 + \dot{f}$$
$$= -\left[\frac{2a}{b} + \frac{\dot{b}}{b^2} \right] z^2 - \frac{4a}{b} z + \dot{f} \qquad (6)$$

$\frac{dV}{d\tau} \leq 0$ for all z if the discriminant of the polynomial in z is less than or equal to zero and $\dot{f} < 0$. This implies that

$$\left[\frac{2a}{b} + \frac{\dot{b}}{b^2} \right] > 0 \qquad (7)$$

and that

$$\dot{f} \leq -\frac{4\left(\frac{a}{b}\right)^2}{\left[\frac{2a}{b} + \frac{\dot{b}}{b^2}\right]} \qquad (8)$$

Therefore, f will be defined by

$$f = \int_0^{\tau} -\frac{4\left(\frac{a}{b}\right)^2 dt}{\left[\frac{2a}{b} + \frac{\dot{b}}{b^2}\right]} \qquad (9)$$

Existence and finiteness are guaranteed by relation (7), the previously noted requirement that $b > 0$, and the physical requirement that $\frac{\dot{C}}{C} = a$ be finite.

Referring to equation (2) it is apparent that if $|2a| > b$, oscillatory solutions cannot be expected. Therefore, it will be assumed that

$$\left|\frac{2a}{b}\right| < 1 \qquad (10)$$

for all τ. If this condition is satisfied, the points at which

$$\sin x = -\frac{2a}{b} \qquad (11)$$

are instantaneous equilibrium points; i.e., if the values of a and b were frozen at any time τ, the constant values of x specified by equation (11) would satisfy the differential equation (2). Consider the equation

$$\ddot{y} + a\dot{y} + b\sin(y + x_o) = -2a \qquad (12)$$

obtained by substituting $(y + x_o)$ into equation (2) where x_o is a solution to equation (11). This equation may be written for small y in the form

$$\ddot{y} + a\dot{y} + b\cos x_o\, y = 0 \qquad (13)$$

Considering a and b constant, the solution to (13) will be oscillatory if $\cos x_o$ is positive. If $\cos x_o$ is negative, the solution will consist of an exponentially increasing function plus an exponentially decreasing function. Hence, the points $x = \pm 2(n-1)\pi + \sin^{-1}\frac{-2a}{b}$ are instantaneous stable equilibrium points and the points $x = \pm n\pi + \sin^{-1}\frac{-2a}{b}$ are unstable instantaneous equilibrium points.

Define

$$B_1 = -\pi + \sin^{-1}\frac{-2a}{b}$$
$$B_2 = +\pi + \sin^{-1}\frac{-2a}{b} \tag{14}$$

If x is originally bounded by $B_1(\tau)$ and $B_2(\tau)$, the motion remains bounded by B_1 and B_2 if the total energy of the motion at all times is less than the potential energy of the system at the unstable equilibrium points, B_1 and B_2. This condition may be expressed as

$$2(1 - \cos x) + \frac{1}{b} z^2 < \min\ 2(1 - \cos B_1) \tag{15}$$

since $\cos B_1 = \cos B_2$. But

$$V - f(\tau) = 2(1 - \cos x) + \frac{1}{b} z^2 \tag{16}$$

from equation (5) and $V \leq V(\tau = 0) = V_o$ if $\frac{dV}{d\tau} \leq 0$. Therefore, condition (15) may be expressed

$$V_o \leq \min\ 2(1 - \cos B_1) + f(\tau) \tag{17}$$

if conditions (7) and (10) hold and f is defined by equation (9). This demonstrates:

The solution of the equation

$$\ddot{x} + a(\tau)\,\dot{x} + b(\tau)\sin x = -2a(\tau)$$

is stable if a is finite, b > 0,

$$2(1 - \cos x_o) + \frac{1}{b_o}\,\dot{x}_o^2 \leq \min\; 2(1 - \cos B_1) + f(\tau)\,, \qquad (17)$$

$$\left[\frac{2a}{b} + \frac{\dot{b}}{b^2}\right] > 0\,, \qquad (7)$$

and

$$\left|\frac{2a}{b}\right| \leq 1\,, \qquad (10)$$

where

$$f = -\int_o^\tau \frac{4\left(\frac{a}{b}\right)^2}{\left[\frac{2a}{b} + \frac{\dot{b}}{b^2}\right]}\,dt\,, \qquad (9)$$

and

$$\cos B_1 = -\left(1 - \frac{4a^2}{b^2}\right)^{\frac{1}{2}}\,. \qquad (18)$$

In terms of the original parameters, these conditions may be expressed as

$$-\frac{d}{d\tau}\ln(B - A) < \frac{d}{d\tau}\ln C\,, \qquad (7)$$

$$\left|\frac{2\dot{C}}{3(B - A)}\right| \leq 1\,, \qquad (10)$$

$$f = -\int_0^{\tau} \frac{4\dot{C}^2}{3\frac{d}{dt}[C(B-A)]}\, dt \quad , \tag{9}$$

$$2(1 - \cos x_o) + \frac{C_o}{3(B_o - A_o)}\, \dot{x}_o^2 \leq \min\{2(1 - \cos B_1)\} + f(\tau) \ , \tag{17}$$

and

$$\cos B_1 = -\left[1 - \frac{4\dot{C}^2}{9(B-A)^2}\right]^{\frac{1}{2}} \quad . \tag{18}$$

Condition (17) can be satisfied only if $|f|$ is less than $2(1 - \cos B_1)$ since the left hand side is always positive. Since the integrand is always positive, this may be expressed as

$$\int_0^{\tau} \frac{4\dot{C}^2}{3\frac{d}{dt}[C(B-A)]} < 2(1 - \cos B_1) \text{ for all } \tau. \tag{19}$$

Relation (7) which may be expressed as

$$\frac{d}{dt}[C(B-A)] > 0 \tag{7}$$

must be interpreted in this light. In particular, although (7) does not preclude negative $\dot{C}$ (the retraction situation) or negative $\frac{d}{dt}(B-A)$, it must be understood that these conditions are reflected in the severity of restrictions imposed on the initial state. If both C and B-A are monotonically increasing functions, $|f| = 0\left(\ln \frac{C}{C_o}\right)$

so that conditions (17) and (19) are seen to impose conditions on the total change in inertial parameters. Equation (7) agrees with the condition derived in section 4, equation (15) for stability in a particular case where an approximate solution can be found.

If $\dot{C} = 0$, equation (2) becomes autonomous and the sufficient conditions for stability stated above may be compared with Leitmann's (8) results. According to Leitmann, sufficient conditions for stability are

$$-\frac{d}{dt} \ln C \leq \frac{d}{dt} \ln (B - A) \leq \frac{d}{dt} \ln C \qquad (20)$$

which, since C = constant, requires (B - A) to be constant also, and

$$\frac{C_o}{3(B_o - A_o)} \dot{x}_o^2 + 2(1 - \cos x_o) \leq 4 \qquad (21)$$

which is identical to relation (17) since $\dot{C} = 0$ implies $f(t) = 0$ and $\cos B_1 = -1$.

Comparing (20) with relation (7), the left hand inequality in (20) results from placing a bound on $\dot{x}$ in Leitmann's analysis, a bound which follows on physical grounds in this paper. Referring back to equation (6), the less than relation in relation (7) may obviously be changed to less than or equal to if $\dot{C} = 0$ (implies $\frac{4a}{b} = 0$ and $f = 0$). Thus, the general form of the results of this paper and that of Leitmann are compatible and similar in form.

The preceding analysis has been restricted by the condition $\left[\frac{2a}{b} + \frac{\dot{b}}{b^2}\right] > 0$, or, equivalently, $\frac{d}{dt}\left[C(B - A)\right] > 0$. This restriction is not necessary. Referring again to equation (6), consider the case where $\left[\frac{2a}{b} + \frac{\dot{b}}{b^2}\right] = 0$, or, equivalently, $\frac{d}{dt} C(B-A) = 0$. If $a = 0$, $\dot{f}$ and f may be chosen to be zero, $\frac{dV}{dt} \equiv 0$ and condition 17) suffices for stability, this being the case of constant parameters. If $a \neq 0$, $\frac{dV}{d\tau} \leq 0$ if $\frac{\dot{f}b}{4a} \leq z$ for all z assumed by the system, where $\dot{f} < 0$. Since z must be permitted to take on negative values, a must be greater than zero which implies $\dot{C} > 0$. From equation (5) z can be expressed as

$$z = \pm\left[V - 2(1 - \cos x) - f\right]^{\frac{1}{2}} b^{\frac{1}{2}} \tag{22}$$

Then the minimum value of z satisfies the inequality

$$z_{min} \geq - \left[V - f\right]_{max}^{\frac{1}{2}} b_{max}^{\frac{1}{2}} \tag{23}$$

If relation (15) is to be satisfied, equation (16) yields

$$z_{min} \geq - \left[\min \{2(1 - \cos B_1)\}\right]^{\frac{1}{2}} b_{max}^{\frac{1}{2}} \tag{24}$$

The condition for $\frac{dV}{d\tau} \leq 0$ may then be written

$$\frac{\dot{f}b}{4a} \leq - \left[\min \{2(1 - \cos B_1)\}\right]^{\frac{1}{2}} b_{max}^{\frac{1}{2}} \tag{25}$$

Thus, equation (2) is stable if $\left[\frac{2a}{b} + \frac{\dot{b}}{b^2}\right] = 0$, $a > 0$, relation (10) is satisfied, and there exists an $\dot{f}(\tau) < 0$ such that relations (17) and (25) are satisfied where $f(\tau) = \int_0^\tau \dot{f}(\tau)d\tau$. Such f exist for suitably chosen parameters a and b and initial conditions.

If $\left[\frac{2a}{b} + \frac{\dot{b}}{b^2}\right] < 0$ or $\frac{d}{dt}\left[C(B - A)\right] < 0$,

equation (6) yields $\frac{dV}{d\tau} \leq 0$ if $\dot{f} < 0$ and z is contained in the interval $\left[z_1 , z_2\right]$, where

$$z_1 = \frac{E + \sqrt{E^2 - 4D\dot{f}}}{2D} ,$$

$$z_2 = \frac{E - \sqrt{E^2 - 4D\dot{f}}}{2D} ,$$

$$D = \frac{2a}{b} + \frac{\dot{b}}{b^2} ,$$

and

$$E = \frac{4a}{b} . \tag{26}$$

Then the solution to equation (2) is stable if $D < 0$, relation (10) is satisfied and there exists an $\dot{f}(\tau) < 0$, such that

$$\left[\min \{2(1 - \cos B_1)\}\right]^{\frac{1}{2}} b_{max}^{\frac{1}{2}} \leq z_2 \tag{27}$$

and relation (17) are satisfied. The values of the parameters a and b and the initial conditions can be chosen such that a suitable f exists. However, for specified a and b and initial conditions, there may be no f.

Section 3. Rapid Change in Inertial Parameters

The equation of planar libration of a satellite is

$$\frac{d}{dt}\left[C(\dot{\varphi} + \omega)\right] + \frac{3\omega^2(B-A)}{2} \sin 2\varphi = 0 \qquad (1)$$

If the inertial parameters change rapidly with respect to $\frac{1}{\omega}$, the first term is of order ω and the second term is of order ω^2. Since ω is a small parameter relative to 1 sec^{-1}, the $\sin 2\varphi$ term may then be neglected during deployment or at least treated as a perturbation on the motion. Equation (1) becomes simply

$$\frac{d}{dt}\left[C(\dot{\varphi} + \omega)\right] = 0 \quad , \qquad (2)$$

an equation which expresses conservation of the angular momentum of the satellite. This integrates immediately to

$$C(\dot{\varphi} + \omega) = \text{constant} = h = C_o(\dot{\varphi}_o + \omega_o) \qquad (3)$$

or

$$\dot{\varphi} = \frac{h}{C} + \omega \qquad (4)$$

which in turn integrates to

$$\varphi = \int_o^t \left(\frac{h}{C} + \omega\right) dt + \varphi_o \quad . \qquad (5)$$

These equations hold during deployment permitting computation of an angular displacement, φ_f, and an angular velocity, $\dot{\varphi}_f$, at the end of the deployment period. These constitute initial conditions on the motion described now by equation (1) with C equal to the constant C_f. The stability condition for the resultant motion is

$$(1 - \cos 2\phi_f) + \frac{2C_f}{3\ {}^{2}(B_f - A_f)}\,\dot{\phi}_f^2 < 2$$

It is generally desirable to preserve orientation so that ϕ_f should be less than $\pi/2$.

The effect of the deployment is seen to be very much dependent upon the initial conditions, particularly upon ϕ_o and $\dot{\phi}_o$. If $\dot{\phi}_o = -\omega_o$, then $\phi_f = \omega$ and $\phi = \int_o^t \omega dt + \phi_o$. If ϕ_o and $\dot{\phi}_o$ have the same sign as ω, the destabilizing effects ($C < 0$, $\frac{d}{dt}(B-A) < 0$) will be more severe than if either or both of these conditions were not fulfilled. It is apparent that deployment and retraction can be utilized to either stabilize or destabilize librational motion by appropriate programming of the action.

Section 4. Planar Librations of a Satellite With Slowly Varying Inertial Parameters

In the situation where the inertial parameters in the planar libration equation vary slowly with respect to a librational (or orbital) period, asymptotic methods may be utilized to find approximate solutions (2,3). Three different asymptotic approaches are used to study the librational motion of a deploying satellite in this chapter. The first uses the Krylov-Bogoliubov method treating the full equation as a perturbation of the equivalent stationary linearized equation. The second also uses the Krylov-Bogoliubov method but treats the full equation as a perturbation on the equivalent stationary but fully nonlinear equation. The third method attempts to find information on the amplitude of the motion without attempting to follow the trajectory of the motion.

Consider the equation of motion in the form

$$\frac{d}{dt}(C\dot{x}) + 3\omega^2 (B-A) \sin x = -2\dot{C}\omega \qquad (1)$$

Suppose C and B-A are functions of $\tau = \epsilon t$ where ϵ is a small parameter. Define the transformation

$$y = x - \sin^{-1}\left(\frac{-2\dot{C}\omega}{3\omega^2(B-A)}\right) = x - \sin^{-1}\left(\epsilon \frac{-2C'}{3\omega(B-A)}\right)$$

$$= x - \sin^{-1}\epsilon u \qquad (2)$$

where

$$u = \frac{-2C'}{3\omega(B-A)} \qquad (3)$$

and the superscript prime indicates differentiation with respect to to τ. Then

$$\dot{x} = \dot{y} + \frac{\epsilon^2 u}{\sqrt{1-\epsilon^2 u^2}} = \dot{y} + \epsilon^2 u' \,(1 + \epsilon^2 u^2) + O(\epsilon^4) \qquad (4)$$

$$\ddot{x} = \ddot{y} + O(\epsilon^3) \qquad (5)$$

$$\sin x = \sin y \left[1 - \frac{\epsilon^2 u^2}{2} + \ldots\right] + \epsilon\, u \cos y \qquad (6)$$

To order ϵ^2 the differential equation (1) may then be written

$$\frac{d}{dt}\,(C\dot{y}) + 3\omega^2(B-A)\sin y = -\,\epsilon\left[2C'\omega(1-\cos y)\right] + O(\epsilon^2) \qquad (7)$$

Expanding sin y in a power series gives

$$\frac{d}{dt}\left[C\,\dot{y}\right] + 3\omega^2(B-A)\,y = -\,\epsilon\left[2C'\omega(1-\cos y) + \frac{1}{\epsilon}\,\frac{\omega^2(B-A)}{2}\,y^3 + \ldots\right]$$

$$+ O(\epsilon^2) \qquad (8)$$

Following the technique of Krylov-Bogoliubov define the change of variables

$$y = a\cos\Psi \quad ; \quad \dot{y} = -\,a\Omega(\tau)\sin\Psi \qquad (9)$$

where

$$\Omega^2 = \frac{3\omega^2(B-A)}{C} \qquad (10)$$

Then the first approximation (3) is given by

$$\frac{da}{dt} = -\,\frac{\epsilon a}{2C\Omega}\,\frac{d(C\Omega)}{dt} \qquad (11)$$

$$\frac{d\Psi}{dt} = \Omega(\tau) + \frac{\epsilon}{2\pi C \omega a}\int_0^{2\pi} \left\{2C'\Omega\left[1-\cos(a\cos\Psi)\right] + \frac{1}{\epsilon}\frac{\omega^2(B-A)}{2} \times a^3\cos^3\Psi\right\}\cos\Psi \, d\Psi \qquad (12)$$

Equation (11) immediately integrates to

$$a(\tau) = \frac{a_o}{(C\Omega)^{\frac{1}{2}}} = \frac{a_o}{\left[3\omega^2 C(B-A)\right]^{\frac{1}{4}}} \qquad (13)$$

Then, substituting for a from equation (13), equation (12) becomes

$$\frac{d\Psi}{dt} = \Omega(\tau) - \frac{3a_o^2\omega^2(B-A)}{16(C\Omega)} \qquad (14)$$

Therefore,

$$\Psi = \int_0^t \sqrt{\frac{3\omega^2(B-A)}{C}}\, dt - \int_0^t \frac{a_o^2}{16}\left[\frac{3\omega^2(B-A)}{C}\right]^{\frac{1}{2}} dt + \Psi_o \qquad (15)$$

It is apparent from equation (13) that the amplitude of the motion, a, increases (or decreases) as C(B-A) decreases (or increases). This compares with condition (2.17) in the stability analysis of section 2 and demonstrates that condition (2.17) is indeed a sufficient condition for stability if the initial conditions are satisfactorily bounded. It also indicates that condition (2.17) is not a necessary condition since increasing a does not necessarily imply instability. Note that the fact that a may be bounded in the first approximation does not imply stability of the full equation due to the approximation made for sin x.

The "forcing" term, $-2C\omega$, has no effect on the averaged equations, i.e. on a and Ψ, in the first approximation. However, it does produce a shift in the mean value of x in the amount $\sin^{-1} \frac{-2\dot{C}}{3\omega(B-A)}$, the same shift which would be produced in the linearized solution. This is to be expected from the assumption that C varies slowly with respect to a librational period and, therefore, the integrated effect of $\dot{C}$ over a period should be small.

Equation (15) can be integrated in closed form for certain cases and presents no difficulties with regard to numerical integration in physically reasonable cases compatible with the assumption of slow variation of inertial parameters. The original differential equation (1), however, presents serious problems for numerical integration, since the integration must follow the "rapid" oscillation with the attendant accumulation of error.

The preceding analysis is limited in usefulness to small values of y because of approximation of sin x by a truncated series. Although the inclusion of additional terms of the series presents no difficulty, a solution of the equation which does not approximate the nonlinearity is desirable, particularly if the transition to rotational motion is of interest. In the case of the planar librational equation an exact solution exists for the equation with constant coefficients in terms of elliptic functions. The full equation with varying coefficients is considered as a perturbation on the constant coefficient equation in what follows. The treatment is completely analogous to the Krylov-Bogoliubov method used previously, but the assumed form of the solution is the solution to the stationary nonlinear equation with time dependent amplitude and phase.

The starting point will be equation (7)

$$\frac{d}{dt}(C\dot{y}) + 3\omega^2(B-A)\sin y = -\epsilon\left[2C'\omega(1-\cos x)\right] + O(\epsilon^2) \quad (7)$$

Assume that

$$\Omega^2 = \frac{3\omega^2(B-A)}{C} = \Omega_o^2 + \epsilon\,\Omega_1^2\,(t) \tag{16}$$

and let

$$T = \Omega_o t \tag{17}$$

Then equation (7) may be written

$$y'' + \sin y = -\epsilon\left[\frac{2\omega}{C\Omega_o}\frac{\partial C}{\partial T}(1-\cos y) + \frac{\Omega_1^2}{\Omega_1^2}\sin y + \frac{C'}{C}y'\right]$$

$$= \epsilon\, f\,(y, y', T) \tag{18}$$

where the prime denotes differentiation with respect to T. The equation

$$y'' + \sin y = 0 \tag{19}$$

has known solutions in terms of elliptic functions dependent upon the initial conditions for exact form. The first integral of equation (19)

$$\mathcal{E} = 2\sin^2\frac{y}{2} + \frac{1}{2}y'^2 \tag{20}$$

expresses conservation of energy. The solution is librational if $E < 2$ or rotational if $E > 2$.

If $E < 2$, the solution of (19) may be written in the form

$$\begin{aligned} \sin \frac{y}{2} &= k \operatorname{sn} u \quad , \\ \cos \frac{y}{2} &= \operatorname{dn} u \quad , \\ \frac{y'}{2} &= k \operatorname{cn} u \quad , \end{aligned} \tag{21}$$

where sn, dn, cn are Jacobian Elliptic Functions and k is the modulus (4). To apply the Krylov-Bogoliubov method, the solution of equation (18) is assumed to be of the form of equations (21) where k and u are now unknown functions of T.

In terms of equations (21), equation (20) becomes

$$\mathcal{E} = 2k^2 \operatorname{sn}^2 u + 2k^2 \operatorname{cn}^2 u = 2k^2 \tag{22}$$

Thus

$$\begin{aligned} \frac{1}{2} \frac{d\mathcal{E}}{dT} = \frac{d(k^2)}{dT} &= \frac{1}{2} \frac{d}{dT} \left[2 \sin^2 \frac{y}{2} + \frac{1}{2} y'^2 \right] \\ &= \sin \frac{y}{2} \cos \frac{y}{2} y' + \frac{1}{2} y' y'' \\ &= \frac{y'}{2} \left[\sin y + \epsilon f - \sin y \right] \\ &= \epsilon \frac{y'}{2} f(y,y',T) \\ &= \epsilon f(y,y',T) k \operatorname{cn} u \quad . \end{aligned} \tag{23}$$

Also from equations (21).

$$\frac{d}{dt}(k \text{ sn } u) = \frac{\gamma'}{2} \cos \frac{\gamma}{2} = k \text{ cn } u \text{ dn } u \tag{24}$$

But (4)

$$\frac{d}{dT}(k \text{ sn } u) = k \text{ cn } u \text{ dn } u \left\{\frac{du}{dT} - \frac{1}{2} D \frac{dk^2}{dT}\right\} + \text{sn } u \frac{dk}{dT} \tag{25}$$

where

$$D = \frac{1}{kk'^2}\left[E(u) - k'^2 u - k^2 \text{ sn } u \text{ cd } u\right] = \int \frac{\text{sn}^2 u}{\text{dn}^2 u} du \quad . \tag{26}$$

Thus

$$\frac{du}{dT} = 1 + \frac{1}{2}\left\{D - \frac{\text{sn } u}{k^2 \text{cn } u \text{ dn } u}\right\}\frac{dk^2}{dT} \quad . \tag{27}$$

Therefore, equation (18) has been reduced to two first order differential equations, (25) and (27).

The function kf cn u appearing on the right hand side of equation (23) is periodic in u with period 4K where K(k) is the complete elliptic integral of the first kind. A periodic function F(k,u) can always be expressed as a sum of the form

$$F(k,u) = F_1(k) + \frac{\partial}{\partial u} F_2(k,u)$$

where $F_1(k)$ is the mean value of F(k,u) over a period and $F_2(k,u)$ is periodic in u with the same period as F and with zero mean value. By equation (23) $\frac{dk^2}{dT}$ is of order ϵ and $\frac{du}{dT}$ is of order 1 by equation

(27). Hence $\frac{d}{dT}$ may be represented by $\frac{\partial}{\partial u} + O(\epsilon)$. Therefore, equation (23) may be expressed as

$$\frac{dk^2}{dT} = \frac{\epsilon}{4K} \int_{-2K}^{2K} Kf \text{ cn } u \, du + \epsilon \frac{d}{dT} F_2(k,u) + O(\epsilon^2) \qquad (28)$$

where the average value of F_2, $< F_2 > = 0$. Equation (27) may be similarly treated

$$\frac{du}{dT} = 1 + \frac{\epsilon}{2} \left\{ D - \frac{\text{sn } u}{k^2 \text{cn } u \text{ dn } u} \right\} F$$

$$= 1 + \frac{\epsilon}{2} \left\{ \left[D(u) - \frac{u}{K} D(K) \right] + \frac{u}{K} D(K) - \frac{\text{sn } u}{k^2 \text{cn } u \text{ dn } u} \right\} F. \qquad (29)$$

Define

$$G(k,u) = D(u) - \frac{u}{K} D(K) \qquad (30)$$

so that G is periodic in u with period 4K. Then equation (29) becomes

$$\frac{du}{dT} = 1 + \frac{\epsilon}{2} \left\{ FG(k,u) + \frac{Fu}{K} D(k) - \frac{\text{sn } u \, F}{k^2 \text{cnu dn } u} \right\}. \qquad (31)$$

FG is also periodic in u with period 4K and, since it is odd in u, it has mean value zero. Similarly, the last term in equation (31) is periodic with period 4K and zero mean value. Also the identity

$$D(K) = 2 \frac{dK}{dk^2} \qquad (32)$$

holds (4). Therefore, equation (31) may be written

$$\frac{du}{dT} = 1 + \frac{u}{K}\frac{dK}{dT} + \epsilon \frac{d}{dT} H_2 + 0\ (\epsilon_2) \qquad (33)$$

where H_2 is periodic in u with period 4K and zero mean value. If equations (28) and (33) are averaged over a period, the terms in $F_2(k,u)$ and $H_2(k,u)$ drop out leaving the equations of the first approximation

$$\frac{d\overline{k}^2}{dT} = \frac{\epsilon}{4K} \int_{-2K}^{2K} kf\ cn\ u\ d\ u \qquad (34)$$

and

$$\frac{d\overline{u}}{dT} = 1 + \frac{\overline{u}}{K}\frac{dK}{dT} \qquad (35)$$

Equation (35) then yields

$$\overline{u} = K(T) \int_{o}^{T} \frac{dT}{K(T)} \qquad (36)$$

Consider the integral on the right hand side of equation (34). Substituting for f gives

$$- \frac{1}{4K} \int_{-2K}^{2K} k\ cn\ u \left[\frac{2\omega}{C\Omega_o} \frac{\partial C}{\partial \tau} (2k^2\ sn^2\ u) + \frac{\Omega_1^2}{\Omega_o^2} (2k\ sn\ u\ dn\ u) + \frac{\partial C}{\partial \tau} \left(\frac{2k\ cn\ u)}{C}\right)\right] du$$

Since the parameters C and Ω_1 are assumed to vary slowly with respect to a period, this integral may be written as

$$- \frac{\partial C}{\partial T} \frac{\omega k^3}{4KC\Omega_o} \int_{-2K}^{2K} \text{cn } u \text{ sn}^2 \, d\, u + \frac{\Omega_1^2}{2K\Omega_o^2} \int_{-2K}^{2K} \text{dn } u \, d\, (\text{dn } u) - \frac{\partial C}{\partial T} \frac{k^2}{2KC}$$

$$\int_{-2K}^{2K} \text{cn}^2 u \, d\, u \qquad (38)$$

The first integrand is even and antisymmetric with respect to K, and therefore the integral vanishes. The second integral is readily integrated and also vanishes. The total value of the integral $\frac{\varepsilon}{4K} \int_{-2K}^{2K} k \, f \, \text{cn } u \, du$ is then (4)

$$\frac{\varepsilon}{4K} \int_{-2K}^{2K} kf \text{ cn } u \, d\, u = - \frac{2C'}{KC} \left[E(k) - k'^2 K \right] . \qquad (39)$$

Therefore, equation (34) becomes

$$\frac{d\bar{k}^2}{dT} = - \frac{2C'}{KC} \left[E(k) - k'^2 K \right] \qquad (40)$$

where $k'^2 = 1 - k^2$ and E is the complete elliptic integral of the second kind.

Equation (40) can be evaluated numerically in a routine manner since the quantities involved are slowly varying and equations (36) may then also be evaluated numerically. This is considerably easier than numerical integrations of the differential equation (1). A closed form integration of equation (40) is possible if $\frac{C'}{C}$ is constant. The formula

$$\frac{d}{dk}(E - k'^2 K) = k K \tag{41}$$

is well known (4). Thus

$$\frac{d}{dk^2}(E - k'^2 K) = \frac{1}{2} K \tag{42}$$

and, therefore, equation (40) is immediately integrable to

$$E - k'^2 K = A_1 e^{-C'/C\, T} \tag{43}$$

where A_1 = constant. The function $E - k'^2 K$ goes from 0 to 1 as k goes from 0 to 1. Hence, by equation (40), $\mathcal{E} = \overline{2k}^2$ decreases as T increases if $C' > 0$. Then equation (43) implies that $\overline{k}^2$ goes to zero as T goes to infinity. Equations (21) then imply y is bound to the interval $(-\pi, \pi)$ and goes to zero as T goes to infinity. For large T

$$\overline{k}^2 \sim \frac{4A_1}{\pi} e^{-C'/C\, T} \tag{44}$$

and then equation (36) yields

$$\overline{u} = K\int_0^T \frac{dT}{K} = -\frac{1}{2}\frac{C'}{C} K \int \frac{dk^2}{E - k'^2 K}$$

$$\sim (T - T_o) + \frac{C_2}{\pi}\left(T - T_o + \frac{C}{C'}\right) e^{-\frac{C'}{C} T} \tag{45}$$

If $C' < 0$ equation (40) yields the result that $\overline{k}^2$ increases and equation (43) implies that the motion can be stable only if C' becomes zero or positive after a finite time dependent upon initial conditions and $\frac{C'}{C}$.

If $E > 2$, it is convenient to begin with equation (1). Assuming that

$$\Omega^2 = \Omega_o^2 + \epsilon\, \Omega_1^2\, (T) \tag{16}$$

and defining

$$T = \Omega_o t \tag{17}$$

gives equation (1) in the form

$$x'' + \sin x = -\epsilon \frac{\partial C}{\partial T}\left[\frac{2\omega}{C\Omega_o} + \frac{x'}{C}\right] - \epsilon \frac{\Omega_1^2}{\Omega_o^2} \sin x$$

$$= \epsilon\, f\, (x, x'; T) \tag{46}$$

Two cases are possible depending on the direction of rotation.

Suppose $E > 2$ and $x' > 0$. Then the solution of (46) for $\epsilon = 0$ may be written in the form

$$\sin \frac{y}{2} = \mathrm{sn}\, u$$

$$\cos y/2 = \mathrm{cn}\, u \tag{47}$$

$$\frac{y'}{2} = \frac{1}{k}\, \mathrm{dn}\, u$$

Proceeding as before

$$\mathcal{E} = 2\, \mathrm{sn}^2 u + \frac{2}{k^2} \mathrm{dn}^2 u = \frac{2}{k^2} , \tag{48}$$

$$\frac{d}{dT}\left(\frac{1}{k^2}\right) = \frac{\mathrm{dn}\, u}{k}\, \epsilon\, f , \tag{49}$$

and

$$\frac{du}{dT} = \frac{1}{k} + \frac{1}{2} D \frac{dk^2}{dt} \tag{50}$$

To order ϵ

$$\bar{u} = K(T) \int_0^T \frac{dT}{k(T)K(T)} \tag{51}$$

and

$$\frac{d\left(\frac{\overline{1}}{k^2}\right)}{dT} = \frac{\epsilon}{4K} \int_{-2K}^{2K} \frac{dn\, u}{k} f\, du \quad . \tag{52}$$

Evaluating the integral on the right hand side, equation (52) becomes

$$\frac{d\left(\frac{\overline{1}}{k^2}\right)}{dT} = \frac{\pi C'\omega}{kKC\Omega_o} - \frac{2C'}{k^2KC} E$$

$$= \frac{C'}{kKC}\left[\frac{\pi\omega}{\Omega_o} + \frac{2}{k} E\right] \quad . \tag{53}$$

Again, closed form solutions of (53) and (51) are possible only in special cases; for example, when C'/C = constant. In this circumstance, the solution to equation (53) is given by

$$\frac{E}{\bar{k}} = \frac{\pi C'\omega}{2C\Omega_o} + A_1 e^{+\frac{C'}{C}T} \tag{54}$$

where A_1 = constant. Considering the deployment case ($C' > 0$), if $\Omega_o > \frac{\pi\omega}{2}$ then the term in brackets in equation (53) is negative (assuming $\omega > 0$) so that $\mathcal{E} = \frac{2}{k^2}$ is decreasing and will eventually become 2, (Note that E decreases with increasing k reaching 1 when

$k = 1.)$ If $\Omega_o < \frac{\pi\omega}{2}$, the term in brackets has a single root k_o between 0 and 1 and $\dot{\mathcal{E}}$ increases or decreases depending on whether k is greater than or less than k_o. Therefore, k must converge monotonically to k_o the time required being infinite as is seen by expanding E in equation (54) in powers of $(\bar{k} - k_o)$ to give

$$\bar{k} - k_o \sim - \frac{k_o^2}{K_o} A_1 e^{\frac{C'}{C} T} \quad . \tag{55}$$

Then equation (51) gives

$$\bar{u} \sim \frac{T-T_o}{k_o} + (A_2 T + A_3) e^{\frac{C'}{C} T} \tag{56}$$

where A_2 and A_3 are constants.

If $\mathcal{E} > 2$ and $x' < 0$ the same procedure yields

$$\frac{E}{\bar{k}} = - \frac{\pi C'\omega}{2C\Omega_o} + A_1 e^{\frac{C'}{C} T} \tag{57}$$

in place of (54). Therefore, $\mathcal{E} = \frac{1}{k^2}$ decreases as T increases so that it reaches 2 after finite time.

In summary for the special case where $\frac{C'}{C}$ is constant and $C' > 0$, the libration is damped by the deployment if the total change in Ω^2 can be characterized as small. This result is compatible with the stability analysis because of the small change assumed for Ω^2. In the rotational cases, the rotational velocity diminishes and the rotation eventually becomes a libration after finite time if $x' < 0$, or if $x' > 0$ and $\Omega_o > \frac{\omega}{2}$. If $\Omega_o < \frac{\omega}{2}$ and $x' > 0$, the solution diverges with increasing T. The special case of $\frac{C'}{C}$ = constant and total change in Ω^2 small can be realized physically if C is described as a function of time by a real exponential and B and C are nearly equal and much larger than A or if deployment is primarily along the x and z axes, with C again a real exponential.

Some of the results of the preceding analysis may be obtained by means of a simpler analysis. For example, consider equation (7) in the form

$$\ddot{y} + \lambda^2(\tau) \sin y = -\epsilon \left[2\frac{C'}{C} \omega(1 - \cos y) + \frac{C'}{C} \dot{y}\right] + O(\epsilon^2) \quad (58)$$

$$= \epsilon f (y, y, T)$$

where

$$\lambda^2 = \frac{3\omega^2(B-A)}{C} \quad , \quad (59)$$

and the superscript prime and dot refer to differentiation with respect to t and τ respectively, where $\tau = \epsilon t$. A first integral of equation (58) is

$$\frac{1}{2} \dot{y}^2 + \int_{y_0}^{y} \lambda^2(\tau) \sin y \, d\, y = - \epsilon \int_{x_0}^{x} F (y, y, T) \, dy \quad . \quad (60)$$

Consider now equation (58) when $\epsilon = 0$

$$\ddot{u} + \lambda^2 \sin u = 0 \quad (61)$$

and λ^2 is constant. The integral corresponding to (60) is

$$\frac{1}{2}\dot{u}^2 + \lambda^2(1-\cos u) = \text{Const.} = \lambda^2(1-\cos a_1) = \lambda^2(1-\cos a_2) \qquad (62)$$

where a_1, a_2 are the minimum value and the maximum value taken on by u in li[illegible]ational motion. Then on the interval from a_1 to a_2

$$\dot{u} = \left[2\int_u^a \lambda^2 \sin u \, d\, u\right]^{\frac{1}{2}} = \sqrt{2}\ (\cos u - \cos a)^{\frac{1}{2}}\lambda \qquad (63)$$

and on the interval from a_2 to a_1

$$\dot{u} = -\left[2\int_u^a \lambda^2 \sin u \, d\, u\right]^{\frac{1}{2}} = -\sqrt{2}\ (\cos u - \cos a)^{\frac{1}{2}}\lambda \qquad (64)$$

Define y_1, y_2 to be extreme values taken on by y during the first period so that $\dot{y}(y_1) = \dot{y}(y_2) = 0$ and define

$$\Lambda(y) = \int_{y_0}^{y} \lambda^2(\tau)\sin y \, dy \qquad (65)$$

and

$$F(y) = \int_{y_0}^{y} f(\dot{y}, y, \tau)\, dy \qquad (66)$$

Then

$$\Lambda(y_1) = -\epsilon F(y_1) \qquad (67)$$

and

$$\wedge (y_2) = \wedge (y_1) - \epsilon \int_{y_1}^{y_2} f(\dot{y},y,\tau)\, dy \quad . \tag{68}$$

Now let u denote the solution to (61). Then

$$\wedge (y_2) \approx - \epsilon T < F(y) >_o \tag{69}$$

where $< F(y) >_o$ is the average of $F(y)$ taken over the first period of the unperturbed motion

$$< F(y) >_o = \frac{1}{T} \int_o^T f\ (\dot{u},u)\ \dot{u}dt \tag{70}$$

where $t = 0$ and y_o are chosen to correspond to an extreme point of the motion. The change in amplitude over a period is then

$$\Delta\ a = y_2 - y_1 \tag{71}$$

and for small ϵ

$$\wedge (x_2) = \Delta\ a\ \lambda_o^2 \sin x_o \quad . \tag{72}$$

Therefore,

$$\Delta\ a = - \epsilon \frac{< F(y) >_o}{\lambda_o^2 \sin x_o} T$$

Then

$$\frac{da}{dt} = - \epsilon \frac{< F(y) >}{\lambda^2 \sin a} \tag{74}$$

where

$$< F(y) > = \frac{C'}{CT}\int_{-a}^{a}\left[2\omega-2\omega\cos y+\sqrt{2}\,\lambda\,(\cos u-\cos a)^{\frac{1}{2}}\right]dy$$

$$+\frac{C'}{CT}\int_{a}^{-a}\left[2\omega-2\omega\cos y-\sqrt{2}\,\lambda\,(\cos u-\cos a)^{\frac{1}{2}}\right]dy \quad (75)$$

$$= \frac{4C'}{CK}\lambda^2\,\{E(k)-k'^2K(k)\}$$

and $k^2 = \frac{1-\cos a}{2}$. This agrees with equation (40) derived by the Krylov-Bogoliubov method.

Section 5: Linear Librations of a Satellite With Time Varying Inertia Tensor

The equation of planar librations may be linearized by assuming that the libration angle remains small. In this case, sin φ may be replaced by φ giving a linear equation of motion

$$\ddot{\varphi} + \frac{\dot{C}}{C}\dot{\varphi} + 3\omega^2 \frac{B - A}{C} \varphi = -\frac{\dot{C}}{C}\omega \qquad (1)$$

This equation has been studied in two particular cases previously (5). A general power series solution is derived here. Defining $T = \omega t$, equation (1) becomes

$$\varphi'' + \frac{C'}{C}\varphi' + 3 \frac{B - A}{C}\varphi = -\frac{C'}{C} \qquad (2)$$

where the superscript primes denote differentiation with respect to T. The first derivative term may be eliminated through use of the transformation $y = \sqrt{C}\,\varphi$ yielding

$$y'' + \lambda y = f \qquad (3)$$

where

$$\lambda = \frac{3(B - A)}{\sqrt{C}} + \frac{C'^2}{4C^{3/2}} - \frac{C''}{2\sqrt{C}} \qquad (4)$$

and

$$f = -\frac{2C'}{\sqrt{C}} \qquad (5)$$

The solution is sought in terms of a power series in T. That is, let

$$\varphi = \sum_{n=0}^{\infty} a_n T^n \qquad (6)$$

Then

$$y' = \sum_{n=1}^{\infty} n\, a_n T^{n-1} \quad . \qquad (7)$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n T^{n-2} \qquad (8)$$

Assume that λ and f are represented by power series in T, also, i.e.

$$\lambda = \sum_{n=0}^{\infty} b_n T^n \qquad (9)$$

and

$$f = \sum_{n=0}^{\infty} f_n T^n \qquad (10)$$

Substituting into the homogeneous equation corresponding to equation (3) gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n T^{n-2} + \sum_{n=0}^{\infty} b_n T^n \sum_{n=0}^{\infty} a_n T^n = 0 \quad . \qquad (11)$$

Equating coefficients of each power of T to zero gives the set of equations

$$2a_2 + b_o a_o = 0$$

$$3 \cdot 2a_3 + b_o a_1 + b_1 a_o = 0 \qquad (12)$$

$$4 \cdot 3a_4 + b_o a_2 + b_1 a_1 + b_2 a_o = 0 \quad ,$$

. . . .

$$k(k-1)a_k + b_o a_{k-2} + b_1 a_{k-3} + \cdots + b_{k-2}\, a_o = 0$$

. . . .

a_1 and a_o may be chosen arbitrarily (they will be determined, of course, by the initial conditions) so that two linearly independent solutions will result if a_1 and a_o are alternately equated to zero. If $a_1 = 0$, the set of equations (12) becomes

$$a_2 = -\frac{b_o a_o}{2}$$

$$a_3 = -\frac{b_1 a_o}{3 \cdot 2}$$

$$a_4 = -\frac{a_o}{4 \cdot 3}\left(b_2 - \frac{b_o^2}{2}\right) \qquad (13)$$

. . . .

$$(k+2)(k+1)a_{k+2} = -\sum_{j=0}^{k} b_{k-j}\, a_j$$

If $a_o = 0$, equations (12) become

$$a_2 = 0$$

$$a_3 = -\frac{b_o a_1}{3 \cdot 2}$$

$$a_4 = -\frac{b_1 a_1}{4 \cdot 3}$$

. . . .

$$(k+2)(k+1)a_{k+2} = -\sum_{j=0}^{k} b_{k-j}\, a_j$$

Let

$$y_1 = 1 - \frac{b_o}{2}T^2 - \frac{b_1}{3\cdot 2}T^3 + \left(\frac{b_o^{\,2}}{4\cdot 3\cdot 2} - \frac{b_2}{4\cdot 3}\right)T^4 + \ldots$$

(15)

$$y_2 = T - \frac{b_o}{3\cdot 2}\,T^3 - \frac{b_1}{4\cdot 3}\,T^4 + \ldots$$

Assume the particular solution is of the form

$$y_p = K_1(T)y_1 + K_2(T)y_2 \qquad (16)$$

Then

$$y_p'' = K_1''\,y_1 + K_2''\,y_2 + 2(K_1'\,y_1' + K_2'\,y_2') + K_1\,y_1'' + K_2\,y_2'' \qquad (17)$$

Impose a second condition on the K's, viz.:

$$K_1'\,y_1 + K_2'\,y_2 = 0 \qquad (18)$$

This implies

$$K_1''\,y_1 + K_2''\,y_2 = -K_1'\,y_1' - K_2'\,y_2' \qquad (19)$$

so that

$$y_p'' = K_1'\,y_1' + K_2'\,y_2' + K_1\,y_1'' + K_2\,y_2'' \qquad (20)$$

Substituting into the differential equation (3) gives

$$K_1' y_1' + K_2' y_2' + K_1 y_1'' + K_2 y_2'' + \lambda(K_1 y_1 + K_2 y_2) = f$$

or

$$K_1(y_1'' + \lambda y_1) + K_2(y_2'' + \lambda_2) + K_1' y_1' + K_2' y_2' = f$$

or

$$K_1' y_1' + K_2' y_2' = f \tag{21}$$

But equation (18) gives a second condition

$$K_1' y_1 + K_2' y_2 = 0 \tag{18}$$

Therefore,

$$K_1' = \frac{fy_2}{W} \tag{22}$$

$$K_2' = -\frac{fy_1}{W} \tag{23}$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{24}$$

The general solution is then

$$y = y_1\left[\int \frac{fy_2}{W} dt + y_o\right] + y_2\left[-\int \frac{fy_1}{W} dr + y_o'\right] \tag{25}$$

where y_o and y'_o refer to the values of y and y' at T = 0. The resulting expressions may be routinely evaluated for finite series. The rate of convergence of the series is related to the ratio of the total change in the inertial parameters to their initial values. The convergence of the series solution is guaranteed by the physical requirement that the series (9) and (10) converge.

Section 6: Conclusions

Sufficient conditions have been found to guarantee the stability of the librational motion of a satellite with time varying inertia tensor. These conditions involve the initial conditions, the rate of change of the principal moments of inertia, and the total change in these parameters. The conditions are easily applied in practical situations. Approximate solutions to the equations of motion are found in the special cases where the inertial parameters change rapidly or slowly relative to the librational period. The conditions for stability of the solutions to the differential equation of librations derived in these cases are compatible with the stability conditions established in the stability analysis. The approximate solutions may be displayed in closed form for particular cases or integrated numerically much more simply than the original differential equations.

REFERENCES

1. Leitmann, G.: "On the Stability of Solutions of a Nonlinear, Nonautonomous Equation," Int. J. Nonlinear Mechanics 1, pp. 291-295.

2. Bogoliubov, N. N., and Mitropol'skii, Yu. A.: Asymptotic Methods in the Theory of Nonlinear Oscillations, Gordon and Breach, Science Publishers, New York, 1961.

3. Minorsky, N.: Nonlinear Oscillations, D. Van Nostrand Co., Inc., Princeton, 1962.

4. Byrd, P. F., and Friedman, M. D.: Handbook of Elliptic Integrals for Engineers and Physicists, Springer-Verlag, Berlin, 1954.

5. Liu, H. S., and Mitchell, T. P.: "The Structural and Librational Dynamics of a Satellite Deploying Flexible Booms or Antennas," AIAA Paper No. 67-43, 1967.

N70-32119

PART II

The Librational Dynamics of a Composite Rigid-Elastic Satellite

ABSTRACT

The conditions under which librational and flexural resonances may be induced in a satellite consisting of a rigid central mass from which a lengthy flexible element extends are derived. The influence of the central body on the flexible element is specifically taken into account. The center of mass of the composite satellite is assumed to move in an elliptic planar orbit.

PART II

The Librational Dynamics of a Composite Rigid-Elastic Satellite

Introduction

The geometric configuration of the composite satellite envisaged in the following analysis is that consisting of a uniform flexible beam attached to a body which can be considered to be rigid in comparison with the beam. The length of the beam is assumed to be much greater than a characteristic length of the rigid body, and hence the action of this body on the beam is represented by a point load and a point moment. Both the load and the moment are considered to be localized at the center of mass of the rigid body which is taken to lie on the beam axis. The inertial and gravitational loading on the beam during its librational motion in addition to the loading exerted by the rigid element of the satellite contribute to the deformation energy stored in the beam. The purpose of this paper is to study the librational motion making allowance for this energy of deformation. The paper thus constitutes a generalization of earlier work, Liu and Mitchell[1], in which the influence of the rigid element of the composite was specifically ignored. The opportunity is also taken to correct an omission in the statement of the inertial reaction force used in reference 1. This oversight was pointed out to the authors by Mr. Eugene Cliff to whom we are greatly indebted.

The analysis to be presented is restricted to planar libration of the satellite assuming the orbital and librational motions to be uncoupled. The restrictions inherent in this latter assumption have been discussed by Kane[3] and Breakwell and Pringle[4]. If the satellite were completely rigid, the librational angle ϕ would be determined by the well-known equation

$$\frac{d^2\phi}{dt^2} + 3\,\frac{K}{R_c^3}\left(\frac{B-A}{C}\right)\sin\phi\,\cos\phi = -\frac{d^2\theta}{dt^2} \qquad (1)$$

where the orbit of the satellite center of mass is given by

$$R_c = p/(1 + e\cos\theta) \qquad (2)$$

p being the focal parameter and e the orbital eccentricity. In equation (1) K is the gravitational parameter; the elements of the inertia tensor of the entire satellite at its center of mass are A,B about principal axes in the orbital plane and C about the axis normal to it. See Figure 1 which also illustrates the coordinate system to be used.

It can be shown directly that the total body force per unit mass acting on the satellite is

$$\mathbf{f} = \mathbf{i}(xP_1 - yP_2) + \mathbf{j}(xP_3 + yP_4) \qquad (3)$$

where the notation

$$P_1 = (\dot\theta + \dot\phi)^2 + \frac{K}{R_c^3}(3\cos^2\phi - 1)$$

$$P_2 = \frac{3K}{R_c^3}(\alpha + 1)\sin\phi\cos\phi$$

$$P_3 = \frac{3K}{R_c^3} (\alpha - 1) \sin\phi \cos\phi$$

$$P_4 = (\dot{\phi} + \dot{\theta})^2 + \frac{K}{R_c^3} (3 \sin^2\phi + 1)$$

and

$$\alpha = (B - A)/C$$

is used. The deformation energy created in the beam by this force, the point load reaction F and moment M_r originating in the presence of the rigid body part of the composite satellite, will now be calculated following a method used previously[1,2].

Elastic Energies

Since the beam is in overall equilibrium, the force of reaction of the rigid body on the beam is, see Fig. 2,

$$F = -\rho \int_{-L_2}^{L_1} dx \int_{-a}^{a} dy \int_{-b}^{b} dz [\underset{\sim}{i}(xP_1 - yP_2) + \underset{\sim}{j}(xP_3 + yP_4)]$$

$$= -\frac{m}{2} (L_1 - L_2)(\underset{\sim}{i}P_1 + \underset{\sim}{j}P_3) \qquad (4)$$

in which $m = 4\rho ab(L_1+L_2)$ is the mass of the beam, its mass density being represented by ρ its rectangular cross section by 2a x 2b and the origin of coordinates coincides with the center of mass of the composite body. The bending moment in the beam is found to be

$$M(x) = 2ab\rho\sigma(L_1-x)[(\alpha-1)(2L_1^2-L_1x-x^2) + 2a^2(\alpha+1)] \tag{5}$$

if $x > x_o$, and

$$M(x) = 2ab\rho\sigma(L_1-x)[(\alpha-1)(2L_1^2-L_1x-x^2) + 2a^2(\alpha+1)]$$

$$+ \frac{3}{2} F(L_1-L_2)(\alpha-1)(x-x_o) + M_r \tag{6}$$

if $x \leq x_o$ where x_o represents the abscissa of the point of application of the rigid body reaction load and moment. For conciseness the symbol σ denotes the quantity $(K/R_c^3)\sin\phi \cos\phi$. The value of the point moment M_r is found by invoking the boundary condition $M(-L_2) = o$ which corresponds to the fact that the end $x = -L_2$ of the beam is free. Clearly $M(L_1) = o$ is automatically satisfied.

In physically interesting cases F will be zero if and only if $L_1 = L_2$ i.e. if the centers of mass of the rigid body and the beam coincide. Furthermore, M_r will vanish if

$$\alpha = \frac{L_1^2-L_1L_2 + 3(L_1+L_2)x_o/2 + L_2^2-a^2}{L_1^2-L_1L_2 + 3(L_1+L_2)x_o/2 + L_2^2+a^2}$$

Accordingly, the influence of the rigid body on the beam is exactly zero only if $L_1 = L_2 = L$ and $\alpha = (L^2-a^2)/(L^2+a^2)$. The shear S(x) and tension T(x) in the beam are

$$S(x) = -6ab\rho\sigma(\alpha-1)(L_1^2-x^2) \;, \; x > x_o$$

$$= 6ab\rho\sigma(\alpha-1)(L_2^2-x^2) \;, \; x \leq x_o$$

and

$$T(x) = 2ab\rho[(\dot{\phi} + \dot{\theta})^2 + \frac{K}{R_c^3}(3\cos^2\phi - 1)](L_1^2 - x^2), \quad x > x_o$$

$$= 2ab\rho[(\dot{\phi} + \dot{\theta})^2 + \frac{K}{R_c^3}(3\cos^2\phi - 1)](L_2^2 - x^2), \quad x \le x_o$$

One can now compute the deformation energy in the beam by determining the contributions of the bending, shear and tension energies respectively. Thus, the strain energy of bending is

$$U_B = \frac{1}{2EI}\int_{-L_2}^{L_1} [M(x)]^2 dx$$

$$= \frac{\lambda_1 \Omega^4}{EI}(1 + e\cos\theta)^6 \sin^2\phi \cos^2\phi \qquad (7)$$

the strain energy due to shear deformation is

$$U_S = \frac{1}{8abG}\int_{-L_2}^{L_1} [S(x)]^2\, dx$$

$$= \frac{\lambda_2 \Omega^4}{4abG}(1 + e\cos\theta)^6 \sin^2\phi \cos^2\phi \qquad (8)$$

and that produced by the tension is

$$U_T = \frac{1}{8abE}\int_{-L_2}^{L_1} [T(x) - T_o]^2\, dx$$

$$= \frac{\lambda_3}{8abE}[\dot{\phi}^2 + 2\dot{\phi}\dot{\theta} - 3\Omega^2(1+e\cos\theta)^3\sin^2\phi]^2 \quad (9)$$

In equations (7), (8) and (9) the λ's represent constants which are functions of L_1, L_2, a, b, x_o, α and ρ. Young's modulus is denoted by E and the modulus of rigidity by G. The values of the λ_i in the special case $L_1 = L_2 = L$, $\alpha = (L^2-a^2)/(L^2+a^2)$ are

$$\lambda_1 = 32b^4\rho^2L^7(1-\alpha)^2/105$$

$$\lambda_2 = 96a^2b^2\rho^2L^5(1-\alpha)^2/5$$

and

$$\lambda_3 = 64a^2b^2\rho^2L^5/15$$

The differential equation describing the librational motion is derived e.g. by writing the Lagrangian of the rigid body motion modified by the inclusion of the strain energy calculated above. Restricting the analysis to small values of ϕ and expressing the equation in terms of the true anomaly θ yields, recalling the assumed independence of the orbital and librational modes, a second order differential with constant coefficients. This equation will be written here in the case where the bending energy dominates the other elements of the strain energy. It is

$$C\frac{d^2\phi}{d\theta^2} - 2Ce\frac{d\phi}{d\theta}\left(\frac{\sin\theta}{1+e\cos\theta}\right) + \phi\left[\frac{3(B-A)}{1+e\cos\theta} - \frac{2\lambda_1\Omega^2}{EI}(1+e\cos\theta)^2\right]$$

$$= \frac{2e\,C\sin\theta}{1+e\cos\theta} \qquad (10)$$

which, when $e = o$, shows that the frequency, ω, of oscillation is given by

$$\omega^2 = \frac{\Omega^2[3(B-A) - 2\lambda_1\Omega^2/EI]}{C} \qquad (11)$$

When the orbit is elliptic i.e. $o < e < 1$ equation (10) can be treated as follows. Define

$$\Psi = (1 + e\cos\theta)\phi$$

to find

$$\frac{d^2\Psi}{d\theta^2} + \Psi\left[\frac{3\alpha + e\cos\theta}{1+e\cos\theta} - \frac{2\Omega^2\lambda_1}{CEI}(1+e\cos\theta)^2\right] = 2e\sin\theta$$

(12)

A general solution to equation (12) is sought in the form of a power series expansion in the orbital eccentricity

$$\Psi = \sum_{n=0} e^n \Psi_n \qquad (13)$$

in which Ψ_o is the solution for a circular orbit and the additive correction in powers of e represents the perturbational effect of the orbital eccentricity. To the first order in e there results

$$\frac{d^2\Psi_o}{d\theta^2} + (\mu_1 - \mu_2)\Psi_o = 0 \qquad (14)$$

and

$$\frac{d^2\Psi_1}{d\theta^2} + (\mu_1 - \mu_2)\Psi_1 = (\mu_1 + 2\mu_2 - 1)\cos\theta\ \Psi_o + 2\sin\theta \tag{15}$$

where $\mu_1 = 3\alpha$ and $\mu_2 = 2\lambda_1\Omega^2/CEI$.

The solution of equation (14) is

$$\Psi_o = A_o \sin(\mu_1 - \mu_2)^{1/2}\theta + B_o \cos(\mu_1 - \mu_2)^{1/2}\theta$$

where A_o and B_o are arbitrary constants. The complementary solution of equation (15) is of identical form and the particular solution found by the method of variation of parameters is

$$\begin{aligned}\Psi_1 = {} & \frac{2\sin\theta}{\mu_1-\mu_2-1} + \frac{\mu_1+2\mu_2-1}{4(\mu_1-\mu_2)^{1/2}}\left(-1 + \frac{1}{2(\mu_1-\mu_2)^{1/2}+1}\right)A_o\sin\left[(\mu_1-\mu_2)^{1/2}+1\right]\theta \\ & + \left(1 + \frac{1}{2(\mu_1-\mu_2)^{1/2}-1}\right)A_o\sin\left[(\mu_1-\mu_2)^{1/2}-1\right]\theta \\ & + \left(-1 + \frac{1}{2(\mu_1-\mu_2)^{1/2}+1}\right)B_o\cos\left[(\mu_1-\mu_2)^{1/2}+1\right]\theta \\ & + \left(1 + \frac{1}{2(\mu_1-\mu_2)^{1/2}-1}\right)B_o\cos\left[(\mu_1-\mu_2)^{1/2}-1\right]\theta\end{aligned} \tag{16}$$

Equation (16) clearly exhibits the possibilities of parametric resonance in the librational motion. Resonance may occur when the geometry and material properties of the satellite are such that either

$$\frac{3(B-A)}{C} - \frac{2\lambda_1 \Omega^2}{CE\,I} = 1 \qquad (17)$$

or

$$\frac{12(B-A)}{C} - \frac{8\lambda_1 \Omega^2}{CE\,I} = 1 \qquad (18)$$

In the special case where $L_1 = L_2 = L$ and $\alpha = \frac{L^2-a^2}{L^2+a^2}$, this reduces to

$$\frac{3(B-A)}{C} - \frac{16\rho^2 bL^7[C^2-(B-A)^2]\Omega^2}{35EaC^3} = 1$$

and

$$\frac{12(B-A)}{C} - \frac{64\rho^2 bL^7[C^2-(B-A)^2]\Omega^2}{35EaC^3} = 1$$

Equations (17) and (18) express resonance conditions correct to the first power in the eccentricity with other resonances to be expected from higher order corrective terms in equation (13).

Structural resonances may occur if the load distribution on the elastic elements contain periodic terms with frequencies close to the natural frequencies of the elastic elements. The eccentricity of the orbit provides a structural resonance condition

$$V_m = \frac{2\pi}{T} \qquad (19)$$

and the librational motion provides the additional conditions

$$(\mu_1 - \mu_2)^{\frac{1}{2}} = \frac{1}{n} \quad \text{or} \quad (\mu_1 - \mu_2)^{\frac{1}{2}} \pm 1 = \frac{1}{n} \qquad (20)$$

and

$$V_m = \frac{2\pi}{nT} \qquad (21)$$

where V_m is a natural frequency for the elastic elements. These structural resonance conditions are, of course, restricted to the first power of e and small values of ϕ and elastic-rigid coupling too small to appreciably alter the natural frequencies of the elastic elements.

References

1. Liu, H. S. and Mitchell, T. P., "Resonant Oscillations of a Flexible Satellite in an Elliptic Orbit," AIAA J. 5, 1269-1272 (1967).

2. Mitchell, T. P. and Liu, H. S., "The Influence of Material Elasticity on Satellite Motions," Proc. XVIth International Astronautical Federation Congress (Gauthiers-Villars, Paris, France, 1966).

3. Kane, T. R., "Attitude Stability of Earth Pointing Satellites," AIAA J. 3, 726-731 (1964).

4. Breakwell, J. V. and Pringle R., "Nonlinear Resonances Affecting Gravity-Gradient Stability," Proc. XVIth International Astronautical Federation Congress (Gauthier-Villars, Paris, France, 1966).

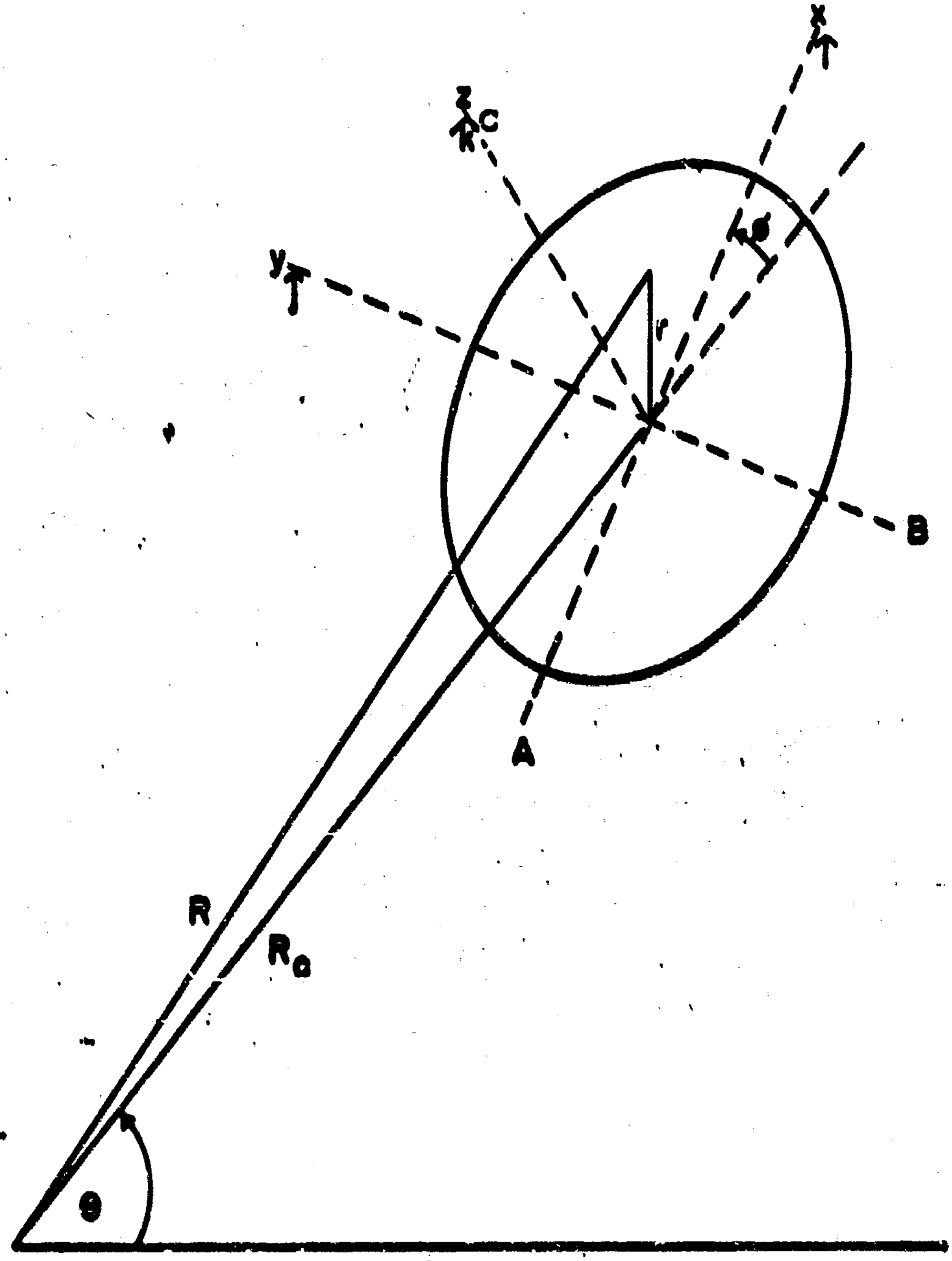

Figure 1. Coordinate Systems to Determine Body Force

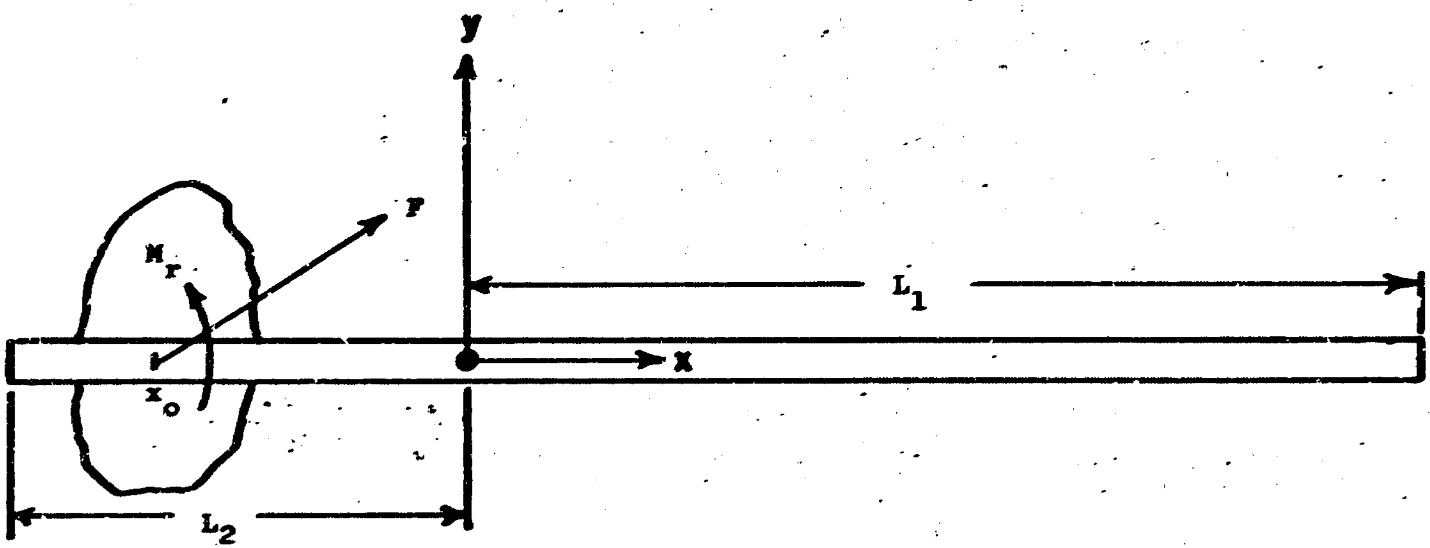

Figure 2. Rigid Body With Flexible Uniform Beam