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Slowly Varying Discrete System $x_{i+1} = A_i x_i$

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Abstract. If all eigenvalues of all matrices A_i $(i = 0, 1, 2, \ldots)$ lie in a disk of radius < 1 and if the matrices A_i vary sufficiently slowly, then the system $x_{i+1} = A_i x_i$ is exponentially stable. An explicit upper bound on the variation rate is given.

The purpose of this note is to prove for the discrete case a result that has been established for the continous case by Rosenbrock[1] and recently sharpened by Desoer[2].

Consider the system

$$x_{i+1} = A_i \, x_i \qquad i = 0, 1, 2, \ldots \tag{1}$$

where $x_i \in R^n$ and $A_i \in R^{n\times n}$ for all i. ($R^{n\times n}$ denotes the class of all n x n matrices with real elements). We assume that the sequence of matrices $\{A_i\}_o^\infty$ is bounded, i.e. there is some finite a_M such that

$$\sup_i \|A_i\| = a_M < \infty \, , \tag{2}$$

and that for some $\varepsilon > 0$,

$$\max_j |\lambda_j(A_i)| \le 1 - 2\,\varepsilon < 1 \qquad i = 0, 1, 2, \ldots \tag{3}$$

i.e. all eigenvalues of all the matrices A_i are in the disk of radius $1 - 2\,\varepsilon < 1$. Under these conditions, provided the sequence of matrices $\{A_i\}$ varies sufficiently slowly, i.e.

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$\sup_i \| A_{i+1} - A_i \|$ is sufficiently small,

the system (1) is exponentially stable.

It is well known that without the restriction on the rate of variation, the system may have exponentially increasing solutions.

As a first step in the proof, let us bound A_i^N where N is a positive integer. Calculate A^N by Dunford's integral[3] using the circle of radius $\rho = 1 - \varepsilon$ as contour,

$$A_i^N = \frac{1}{2\pi j} \oint_C s^N (sI - A_i)^{-1} ds \tag{4}$$

Now Kato[4] has shown that if one uses the euclidian norms in R^n and the induced norm for matrices, then $\|M^{-1}\| \leq \|M\|^{n-1} / |\det M|$. So

$$\|A_i^N\| \leq \frac{2\pi(1-\varepsilon)}{2\pi} (i-\varepsilon)^N \max_{|s|=1-\varepsilon} \left[\frac{\|sI - A_i\|^{n-1}}{|\det(sI - A_i)|} \right]$$

$$\leq (1-\varepsilon)^{N+1} \frac{(i-\varepsilon) + (a_M)^{n-1}}{\varepsilon^n}$$

where we used (2) in the last step. Thus we obtain the bound

$$\|A_i^N\| \leq m \rho^N \qquad \forall i, \forall N \tag{5}$$

where $\rho = 1 - \varepsilon$ and m depends on ε, n and a_M, but is independent of i.

Now, for the i^{th} sampling instant pick the Liapunov function

$$V_i(x) = x^T P_i x \tag{6}$$

Hence, by (1),

$$V_{i+1}(x_{i+1}) - V_i(x_i) = x_i^T [A_i^T P_{i+1} A_i - P_i] x_i \tag{7}$$

Pick P_{i+1} such that

$$A_i^T P_{i+1} A_i - P_{i+1} = - I \tag{8}$$

By direct substitution, the solution is the symmetric positive definite matrix

$$P_{i+1} = I + \sum_{k=1}^{\infty} (A_i^T)^k (A_i)^k \tag{9}$$

Note that the series (9) is absolutely convergent in view of (5), and

$$1 \le \|P_{i+1}\| \le \frac{m^2}{1-\rho^2} \qquad \forall i ,$$

where we observed that the induced enclidian norm of A^T was equal to that of A. Note that the upper bound does not depend on i; therefore

$$\|x\|^2 \le V_i(x) \le \frac{m^2}{1-\rho^2} \|x\|^2 \qquad \forall i \tag{10}$$

We are going to show that if $\sup_i \|A_i - A_{i-1}\|$ is sufficiently small, then there is an $\eta > 0$ such that

$$\|P_{i+1} - P_i\| \le 1 - \eta < 1 \quad \forall i \tag{11}$$

so that

$$V_i(x_i) - V_{i+1}(x_{i+1}) \le - \eta \|x_i\|^2 \quad \forall i . \tag{12}$$

If (12) holds, then, together with (10), they imply that (1) is exponentially stable.

By subtracting two consecutive instances of (8), we obtain

$$A_i^T (P_{i+1} - P_i) A_i - (P_{i+1} - P_i) = - [(A_i^T - A_{i-1}^T) P_i A_i + A_{i-1}^T P_i (A_i - A_{i-1})] \tag{13}$$

Call the right hand side $- M_i$ and note that by (2) and (5)

$$\|M_i\| \le 2 \|A_i - A_{i-1}\| \frac{m^2}{1-\rho^2} a_M$$

Solving (13),

$$P_{i+1} - P_i = M_i + \sum_{k=1}^{\infty} (A_i^T)^k M_i (A_i)^k$$

so

$$\|P_{i+1} - P_i\| \le \|A_i - A_{i-1}\| \frac{2m^2 a_M}{1-\rho^2} \frac{m^2}{1-\rho^2} . \tag{14}$$

Inequality (14) shows that if

$$\sup_i \|A_i - A_{i-1}\| \le \frac{(1-\rho^2)^2}{2m^4 a_M} (1-\eta)$$

then (11) follows. This concludes the proof.

References

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[3] Dunford, N. and J. T. Schwartz, Linear Operators, vol. I. Interscience, New York, 1958. (see p. 568).

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